Trends in Mathematics

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Editors

Geometric Methods in Physics

XXXIII Workshop, Białowieża, Poland, June 29 – July 5, 2014

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Participants of the XXXIII WGMP (Photo by Tomasz Goliński)

Preface

The Workshop on Geometric Methods in Physics, nowadays well known under the colloquial name of the “Białowieża Workshop”, is an annual conference organized by the Department of Mathematical Physics at the Faculty of Mathematics and Computer Science of the University of Białystok in Poland. The idea of the conference is to bring together mathematical physicists, mathematicians, and physicists to discuss in a mathematically precise way new developments and ideas which are of relevance to theoretical physics, both on its classical and quantum sides.

By its very nature, the scope of the topics discussed and the mathematical tools presented is very wide. It includes descriptions of non-commutative systems, completely integrable systems, quantization, groups, supergroups and supersymmetry, quantum groups, and many more.

The participation in the Workshop is open to every scientist and this makes it truly international. As every year, also this year, there were participants from almost all continents and many countries.

Białowieża, the traditional site of the Workshop, is a small village in the east of Poland at the border with Belarus. Białowieża is a place of remarkable unspoiled beauty. There the National Park with remnants of Europe’s last primeval forest and the European bison reserve are of unique character. The natural surroundings help to create a friendly atmosphere for formal and informal discussions and collaboration.

The Workshop in 2014, as in the previous years, was followed by the School on Geometry and Physics. It consisted of several mini-courses by top experts aimed mainly at young researchers and advanced students with the intention to help them to enter current research topics.

The organizers of the Workshop gratefully acknowledge the financial support from the University of Białystok and the Belgian Science Policy Office (BELSPO), IAP Grant P7/18 DYGEST.

Finally, we thank heartily the graduate students and young researchers from the University of Białystok for their indispensable help in the daily running of the Workshop.

The Editors February 2015

Part I: Higher Structures and Non-commutative Geometry

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Traces of Holomorphic Families of Operators on the Noncommutative Torus and on Hilbert Modules

Sara Azzali, Cyril Lévy, Carolina Neira-Jiménez and Sylvie Paycha

Abstract. We revisit traces of holomorphic families of pseudodifferential operators on a closed manifold in view of geometric applications. We then transpose the corresponding analytic constructions to two different geometric frameworks: the noncommutative torus and Hilbert modules. These traces are meromorphic functions whose residues at the poles as well as the constant term of the Laurent expansion at zero (the latter when the family at zero is a differential operator) can be expressed in terms of Wodzicki residues and extended Wodzicki residues involving logarithmic operators. They are therefore local and contain geometric information. For holomorphic families leading to zeta regularised traces, they relate to the heat-kernel asymptotic coefficients via an inverse Mellin mapping theorem. We revisit Atiyah's L^2-index theorem by means of the (extended) Wodzicki residue and interpret the scalar curvature on the noncommutative two torus as an (extended) Wodzicki residue.

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Keywords. Pseudodifferential operators, zeta-regularised traces, Wodzicki residue, index theory, noncommutative torus.

Introduction

The canonical trace $\mathrm{TR}(A(z))$ of a holomorphic family $\mathcal{A}\colon z \mapsto A(z)$ of classical pseudodifferential operators of affine order $-qz + a$, $q > 0$ acting on smooth sections of a vector bundle over an n-dimensional closed manifold ($n > 1$), defines a meromorphic function $z \mapsto \mathrm{TR}(A(z))$ with a discrete set of simple poles $\{d_j = \frac{a+n-j}{q}, j \in \mathbb{Z}_{\geq 0}\}$. The residue at a pole d_j is proportional to the Wodzicki residue of the operator $A(d_j)$. These are well known results due to Wodzicki whose residue [39] is the only[1] (up to a multiplicative factor) trace on classical pseudodifferential operators of integer order, Guillemin [17] who introduced the notion of gauged symbols and Kontsevich and Vishik [20] whose canonical trace TR corresponds to the unique linear extension to operators of non integer order of the ordinary trace defined on operators of order $< -n$.

When $A(0)$ is a differential operator, there is no pole at zero since the residue vanishes on differential operators. It was later observed in [29] that the limit $\lim_{z\to 0} \mathrm{TR}(A(z))$ is also proportional to a Wodzicki residue, namely to the extended Wodzicki residue $\mathrm{Res}\,(A'(0))$, *extended* since the operator $A'(0)$ given by the derivative at zero is typically not a classical operator any longer. So the regularised trace $\lim_{z\to 0} \mathrm{TR}(A(z))$ is then local as a consequence of the locality of the Wodzicki residue.

If the meromorphic map $\phi_{\mathcal{A}}\colon z \longmapsto \Gamma(z)\,\mathrm{TR}\,(A(z))$ corresponding to the holomorphic family $\mathcal{A}\colon z \longmapsto A(z)$ is the inverse Mellin transform of some function $f_{\mathcal{A}} : \mathbb{R}_+ \longrightarrow \mathbb{R}$, it follows from the inverse Mellin mapping theorem (see Proposition 19) that $f_{\mathcal{A}}$ admits an asymptotic expansion at 0 given by

$$f_{\mathcal{A}}(t) = \frac{1}{q} \sum_{j\geq 0} a_j\,(\mathcal{A})\, t^{-d_j} + O(t^{-\gamma}),$$

for some appropriate γ. Its singular coefficients are Wodzicki residues (Theorem 23)

$$a_j\,(\mathcal{A}) = -\frac{1}{q}\mathrm{Res}\,(A(d_j)) \quad \text{for} \quad d_j > 0, \tag{1}$$

and its constant term

$$a_j\,(\mathcal{A}) = -\frac{1}{q}\mathrm{Res}\,(A'(0)) \quad \text{for} \quad d_j = 0, \tag{2}$$

is an extended Wodzicki residue.

For holomorphic families $\mathcal{A}\colon z \longmapsto A(z) = A\,Q^{-z}$ built from complex powers Q^{-z} of some appropriate invertible elliptic operator Q and a classical pseudodifferential operator A, the canonical trace $\mathrm{TR}(A(z))$ is called the ζ-regularised trace $\zeta(A,Q)(z)$ of A with respect to the weight Q. It follows from the above discussion, that if $A(0) = A$ is a differential operator, then $\zeta(A,Q)(0)$ is a local quantity proportional to the extended Wodzicki residue

$$\mathrm{Res}\,(A \log Q) = -\frac{1}{q} \int_M \mathrm{res}_x(A\,\log Q)\,dx,$$

where $\mathrm{res}_x(A\,\log Q)\,dx$ is the pointwise extended residue density involving the logarithm of Q. When $A = I$ is the identity operator, this is the logarithmic residue $\mathrm{Res}\,(\log Q)$ investigated in [28] and [35].

When $Q = \Delta + \pi_\Delta$ with Δ an elliptic differential operator to which we add the orthogonal projection π_Δ onto its kernel, making Q invertible, then $A(z) = A\,Q^{-z}$ is the Mellin transform of the analytic family $\tilde{A}(t) = A\,e^{-t(\Delta+\pi_\Delta)}$. If Δ is

[1] when the dimension of the manifold is greater than one.

nonnegative, the inverse Mellin transform of the meromorphic map $z \mapsto \zeta(A,Q)(z)$ is the Schwartz function

$$f_A \colon t \longmapsto \operatorname{Tr}\left(A\, e^{-t(\Delta+\pi_\Delta)}\right) = \int_M A\, K_t\left(\Delta+\pi_\Delta\right)(x,x)\, \sqrt{\det(g)}(x)\, dx$$

on $]0,+\infty[$ where $K_t\left(A\right)(x,x)$ denotes the fibrewise trace at the point x of the kernel of e^{-tA} restricted to the diagonal and $\det(g)$ the determinant of a Riemannian metric g on the manifold M.

When applied to any multiplication operator $A=\phi$ given by a smooth function ϕ on M, this shows that both the constant *and* the singular coefficients in the time zero asymptotic expansion of the kernel $K_t(\Delta)(x,x)$[2] can be expressed as Wodzicki residues[3] (see Theorem 25):

$$K_t\left(\Delta\right)(x,x) \sim_{t\to 0} -\frac{(4\pi)^{\frac{n}{2}}}{2\sqrt{\det g(x)}} \Bigg[\operatorname{res}_x\left(\log\Delta\right)\, \delta_{\frac{n}{2}-\left[\frac{n}{2}\right]} + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\operatorname{res}_x\left(\Delta^{k-\frac{n}{2}}\right) t^{k-\frac{n}{2}}\Bigg]. \tag{3}$$

When integrated against $\phi\in C^\infty(M)$, the heat-kernel expansion (3) yields the following heat-operator trace expansion at zero

$$\operatorname{Tr}\left(\phi\, e^{-t\Delta}\right) \sim_{t\to 0} -\frac{(4\pi)^{\frac{n}{2}}}{2} \Bigg[\operatorname{Res}\left(\phi\,\log\Delta\right)\, \delta_{\frac{n}{2}-\left[\frac{n}{2}\right]} + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\operatorname{Res}\left(\phi\,\Delta^{k-\frac{n}{2}}\right) t^{k-\frac{n}{2}}\Bigg]. \tag{4}$$

That the singular coefficients of the heat-kernel (resp. heat-operator trace) expansion are proportional to the Wodzicki residue is a well-known (see, e.g., [1, 19]) fact often held for folklore knowledge. It has been extended to noncommutative geometry [5, Formula 1.5]) for the asymptotic expansion of the spectral action (take $f(\lambda)=e^{-t\lambda}$) whose non-constant coefficients arise as Dixmier traces. What is lesser known, is that

- not only the singular coefficients in the heat-kernel expansion (resp. heat-operator trace) but also the *constant coefficient* are (possibly extended) Wodzicki residues (and hence local), a property that can be easily transposed to other geometric frameworks in which the canonical trace of holomorphic families of pseudodifferential operators can be built.

[2]It follows from Duhamel's formula [4] that the time zero asymptotic expansions of $K_t\left(\Delta+R\right)(x,x)$ and $K_t(\Delta)(x,x)$ coincide for every smoothing operator R and hence in particular for $R=\pi_\Delta$, a fact that will be implicitly used throughout the paper.

[3]Similarly, since the Wodzicki residue vanishes on smoothing operators, a perturbation of the operator by a smoothing operator so in particular by the projection onto the kernel does not modify the residue (see Corollary 9).

- *no previous knowledge on the heat-operator trace asymptotics* is needed to express its coefficients in terms of Wodzicki residues, which is a purely analytic procedure.

Since the Wodzicki residue of a logarithm is an algebraic expression involving the jets of the first n homogeneous components of the symbol, we further recover other known facts (see [15, Lemma 1.8.2]), that the coefficients of the heat-operator trace expansion in (4) are

- functorial algebraic expressions in the jets of the homogeneous components of the symbol of the operator Δ as a consequence of the corresponding property of the Wodzicki residue.
- Consequently, if the operator Δ is of geometric nature, the coefficients are functorial algebraic expressions of the jets of the underlying metric and connection.

Thanks to the functoriality of the construction, we can transpose our approach via inverse Mellin transforms to two different geometric contexts, applying it to

1. Holomorphic families of pseudodifferential operators on Hilbert modules: Let Δ be an (essentially) nonnegative selfadjoint differential operator acting on a vector bundle $E \to M$. One builds the operator $\Delta_{\mathcal{H}}$ by twisting Δ by a flat connection on a bundle $\mathcal{H}$ of Hilbert modules over a finite von Neumann algebra. One can then implement the same constructions as above to $\Delta_{\mathcal{H}} + R$ with R a smoothing operator which makes the operator invertible (see Remark 43). The residue Res is then replaced by the τ-residue Res^{τ} (47), the L^2-trace (resp. canonical trace) Tr (resp. TR) by Tr^{τ} (resp. TR^{τ}), where τ is a finite trace on the von Neumann algebra.

The corresponding heat-kernel τ-trace $K_t^{\tau}(\Delta_{\mathcal{H}})(x,x)$ at a point x reads (see (63))

$$
\begin{aligned}
K_t^{\tau}\left(\Delta_{\mathcal{H}}\right)(x,x) \sim_{t\to 0} & -\frac{(4\pi)^{\frac{n}{2}}}{2\sqrt{\det g(x)}}\Bigg[\mathrm{res}_x^{\tau}\left(\log\Delta_{\mathcal{H}}\right)\,\delta_{\frac{n}{2}-\left[\frac{n}{2}\right]} \\
& + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\ \mathrm{res}_x^{\tau}\left(\Delta_{\mathcal{H}}^{k-\frac{n}{2}}\right)\,t^{k-\frac{n}{2}}\Bigg] \\
\sim_{t\to 0} & -\frac{(4\pi)^{\frac{n}{2}}}{2\sqrt{\det g(x)}}\Bigg[\mathrm{res}_x\left(\log\Delta\right)\,\delta_{\frac{n}{2}-\left[\frac{n}{2}\right]} \\
& + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\ \mathrm{res}_x\left(\Delta^{k-\frac{n}{2}}\right)\,t^{k-\frac{n}{2}}\Bigg].
\end{aligned}
\tag{5}
$$

The above formula (5) follows from the following more general property (61) relating the (extended) residue of locally equivalent differential operators A, A', respectively B, B' (with some admissibility condition on the latter),

$$\begin{aligned}\mathrm{Res}^\tau\left(A\log B\right) &= \mathrm{Res}^\tau(A'\log B')\\ \mathrm{Res}^\tau\left(AB^\alpha\right) &= \mathrm{Res}^\tau\left(A'B'^\alpha\right), \quad \alpha\in\mathbb{R},\end{aligned}$$

where for simplicity we have dropped the projections onto the kernels since they are smoothing operators which are not "seen" by the residue. Atiyah's L^2-index theorem then boils down to an easy consequence, see Corollary 50 of the first of these two identities, using a $\mathbb{Z}_2$ graded version of the (extended) residue. This alternative proof, which is equivalent to Roe's heat equation proof [31, Ch. 15], is of pseudodifferential analytic nature and relies on the locality of the Wodzicki residue.

2. Holomorphic families of pseudodifferential operators on the noncommutative torus: We consider a conformal perturbation $\mathbf{\Delta}_h$ (parameterised by a conformal factor h) of a Laplace-type operator $\mathbf{\Delta}$ acting on the noncommutative n-torus $\mathbb{T}^n_\theta$, where θ is an antisymmetric real matrix encoding the noncommutativity. The residue Res is then replaced by the θ-residue Res_θ (see [14]), the L^2-trace (resp. canonical trace) Tr (resp. TR) by Tr_θ (resp. TR_θ, see [24]). The coefficients of the Laurent expansions of traces of holomorphic families can be expressed in terms of Wodzicki residues (Theorem 53) and the heat-kernel expansion formula (4) reads for any $a\in\mathcal{A}_\theta$, the Fréchet algebra of "Schwartz functions" on the noncommutative torus $\mathbb{T}^n_\theta$ (see (68))

$$\begin{aligned}\mathrm{Tr}_\theta\left(a\,e^{-t\mathbf{\Delta}_h}\right)\sim_{t\to 0} -\frac{(4\pi)^{\frac{n}{2}}}{2}\Bigg[&\mathrm{Res}_\theta\left(a\,\log\mathbf{\Delta}_h\right)\,\delta_{\frac{n}{2}-\left[\frac{n}{2}\right]}\\ &+\sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}}\Gamma\left(\frac{n}{2}-k\right)\,\mathrm{Res}_\theta\left(a\,\mathbf{\Delta}_h^{k-\frac{n}{2}}\right)\,t^{k-\frac{n}{2}}\Bigg].\end{aligned} \tag{6}$$

Going back to the setup of closed manifolds, for certain geometric operators Δ, the coefficients of the heat-kernel expansion correspond to interesting geometric quantities; e.g., when Δ is the Laplace–Beltrami operator on a closed Riemannian manifold (M,g), the coefficient of t is proportional to the scalar curvature. The fact that the coefficient of t in (6) provides an analogue in the noncommutative setup of the scalar curvature on a noncommutative torus was exploited in [6], [7], [8], [11], [12], [13] to compute a noncommutative analogue of the scalar curvature on noncommutative tori. By means of the Wodzicki residue on the noncommutative torus [14], we use (6) to define the scalar curvature $\mathfrak{s}_h$ as a Wodzicki residue; for any $a\in\mathcal{A}_\theta$ we set (compare with (69)

$$\langle\mathfrak{s}_h,a\rangle_h=\begin{cases}-6\pi\,\mathrm{Res}_\theta\left(a\,\log\mathbf{\Delta}_h\right) & \text{if}\quad n=2\\ -\frac{3}{2}\,(4\pi)^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}-k\right)\,\mathrm{Res}_\theta\left(a\,\mathbf{\Delta}_h^{k-\frac{n}{2}}\right) & \text{otherwise},\end{cases}$$

with $\langle\cdot,\cdot\rangle_h$ an adequate inner product on $\mathcal{A}_\theta$.

The extensions (5) and (6) are possible thanks to the fact that Cauchy calculus extends to Hilbert modules (see [3]) and to the noncommutative torus (see [14], [24]). The analogies between the pseudodifferential calculi in these two frameworks lead to the question whether the pseudodifferential calculus on $\mathcal{N}\mathbb{Z}^n$-Hilbert module encompasses the $\mathbb{Z}^n$-invariant pseudodifferential calculus on $\mathbb{R}^n$ studied in the more general context of global pseudodifferential calculus by Ruzhanski and Turunen in [33]. The latter corresponds to the $\theta = 0$ case of the algebra $\Psi(\mathbb{T}^n_\theta)$ described in Section 4, which raises the further question, namely whether this issue can be carried out to the noncommutative setup.

1. Traces of holomorphic families on closed manifolds and their geometry

For the purpose of a later generalisation to the Hilbert module setting, we recall the well-known basic setup to define the Wodzicki residue on closed manifolds and extend it to logarithmic pseudodifferential operators.

1.1. The Wodzicki residue on pseudodifferential operators

Let (M, g) be a closed Riemannian manifold of dimension n, let $p\colon E \to M$ be a vector bundle and let $\mathrm{End}(E) = E^* \otimes E$ be the corresponding endomorphism bundle.

A linear operator $A\colon C^\infty(M, E) \to C^\infty(M, E)$ is a (classical) pseudodifferential operator of order $a \in \mathbb{C}$, denoted $A \in \Psi^a(M, E)$, if in some atlas of $E \to M$ it is of the form $A = \sum_{j=1}^{J} A_j + R$ where

- R is a smoothing operator, namely a linear operator $R\colon C^\infty(M, E) \to C^\infty(M, E)$ with Schwartz kernel given by a smooth section of the bundle $\mathcal{L}(M \times M, \mathrm{End}(E))$ of linear bounded operators whose fibre at $(x, y) \in M \times M$ is the Banach space of bounded linear operators from the fibre E_y to the fibre E_x,
- the operators A_j are properly supported operators (meaning that the canonical projections $M \times M \to M$ restricted to the support of the Schwartz kernel are proper maps) from $C^\infty(M, E)$ into itself such that in any coordinate chart in a neighborhood U of a point $x \in M$ the operators A_j are of the form
$$u \longmapsto \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_U dy\, e^{i\langle x-y,\xi\rangle}\, \sigma(x, \xi)\, u(y) \tag{7}$$
for some symbol $\sigma \in C^\infty(U \times \mathbb{R}^n, \mathrm{End}(E))$ which is asymptotically polyhomogeneous at infinity
$$\sigma(x, \xi) \sim \sum_{i=0}^{\infty} \omega(\xi)\, \sigma_{a-i}(x, \xi), \tag{8}$$
i.e., the components $\sigma_{a-i} \in C^\infty(T^*U \setminus (U \times \{0\}), \mathrm{End}(E))$ being positively homogeneous of degree $a - i$ with a the order of the operator. The asymptotic

behaviour is to be understood as[4]

$$\sigma^{(N)}(x,\xi) := \sigma(x,\xi) - \sum_{i=0}^{N-1} \omega(\xi)\, \sigma_{a-i}(x,\xi) \tag{9}$$

is a symbol of order $\Re(a) - N$ for any $N \in \mathbb{N}$. Also, ω is a smooth function on $\mathbb{R}^n$ which is zero in a neighborhood of zero and identically one outside the unit ball.

Let us denote by $\mathcal{S}^a\left(U, \mathrm{End}(E)\right)$ the set of such symbols and by

$$\mathcal{S}\left(U, \mathrm{End}(E)\right) = \langle \cup_{a\in\mathbb{C}} \mathcal{S}^a\left(U, \mathrm{End}(E)\right)\rangle$$

the algebra generated by all classical symbols of any complex order.

Finally, let us denote by $\Theta(M,E)$ the subalgebra in $\Psi(M,E)$ of differential operators.

Let tr_x denote the fibrewise trace on $\mathrm{End}(E)$ above a point $x \in M$. The **pointwise Wodzicki residue** of the operator A with symbol $\sigma(A)$ at the point x defined as

$$\mathrm{res}_x(A) := \int_{|\xi|_x=1} \mathrm{tr}_x\left(\sigma(A)_{-n}(x,\xi)\right) đ_S\xi, \tag{10}$$

vanishes on smoothing symbols so that it is independent of the choice of cut-off function ω chosen in (8). Here $d_S\xi$ is the measure on the unit cotangent sphere $S_x^*M = \{\xi \in T_x^*M,\ |\xi|_x := g_x(\xi,\xi) = 1\}$ induced by the one on the cotangent space T_x^*M to M at the point x given by a Riemannian metric g on M, $đ_S$ denotes the corresponding normalised measure $đ_S\xi = \frac{1}{(2\pi)^n} d_S\xi$.

Remark 1. Clearly the residue res_x vanishes on differential and non-integer-order operators.

An important result of Wodzicki is that $\mathrm{res}_x(A)\,dx$ defines a global density [39]. The **Wodzicki residue** of the operator A is then defined by

$$\mathrm{Res}(A) = \int_M \mathrm{res}_x(A)\,dx = \int_M dx \int_{|\xi|_x=1} \mathrm{tr}_x\left(\sigma_{-n}(A)(x,\xi)\right) đ_S\xi. \tag{11}$$

1.2. Logarithms of pseudodifferential operators on closed manifolds

Definition 2. (see, e.g., [37], [36]) Let A be an operator in $\Psi(M,E)$. A real number β is a principal angle of A if there exists a ray $R_\beta = \{re^{i\beta},\quad r \geq 0\}$ which is disjoint from the spectrum of the $\mathrm{End}(E_x)$-valued leading symbol $\sigma_L(A)(x,\xi)$ for any $x \in M$, $\xi \in T_x^*M \setminus \{0\}$.

Definition 3. We call an operator $A \in \Psi(M,E)$ **admissible** with **spectral cut** β if

- its order is positive,
- β is a principal angle for A.

Remark 4. An admissible operator is elliptic but not necessarily invertible. Admissibility is a covariant condition, that is to say it is preserved under diffeomorphisms.

[4]The remainder depends on the choice of the cut-off function ω which is not explicitly mentioned here to alleviate the notation.

Definition 5. We call **weight** an invertible admissible operator $Q \in \Psi(M,E)$.

As we shall see later, weights are used to regularise traces. Here is a useful lemma which strongly uses the theory of elliptic operators on closed manifolds.

Lemma 6. *If $A \in \Psi(M,E)$ is an admissible operator with spectral cut β then there is an angle β' arbitrarily close if not equal to β and a truncated solid angle $\Lambda_{\beta',\epsilon} := \{z \in \mathbb{C} \setminus 0 : |z| > \epsilon, \arg z \in (\beta' - \epsilon, \beta' + \epsilon)\}$ for some $\epsilon > 0$, outside of which lies the spectrum of A.*

If A is moreover invertible, i.e., if it is a weight, then its spectrum lies outside the solid angle $V_{\beta',\epsilon} := \{z \in \mathbb{C} \setminus 0 : \arg z \in (\beta' - \epsilon, \beta' + \epsilon)\}$ for some small $\epsilon > 0$. The angle β' is then called an **Agmon angle**.

Proof. Since A is admissible, it is elliptic. Since the order of A is positive, the manifold M being closed, the operator has a purely discrete spectrum, which consists of countably many eigenvalues with no accumulation point. So if the spectrum of A meets the ray R_β, there is a small perturbation β' of β such that the spectrum of A does not meet $R_{\beta'} \setminus \{0\}$. Being discrete, the spectrum of A actually lies outside a truncated solid angle $\{z \in \mathbb{C} \setminus 0 : |z| > \epsilon,\ \arg z \in (\beta' - \epsilon, \beta' + \epsilon)\}$ for some $\epsilon > 0$ chosen small enough so that 0 is the only eigenvalue in the ball of radius ϵ centered at zero. If moreover A is invertible, then its spectrum lies outside the solid angle $\{z \in \mathbb{C} \setminus 0 : \arg z \in (\beta' - \epsilon, \beta' + \epsilon)\}$ and β' is an Agmon angle for A. $\square$

Let now A be a weight in $\Psi^a(M,E)$ with spectral cut β. Then for $\Re(z) > 0$, its complex powers (see [37] for further details)

$$A_\beta^z = \frac{i}{2\pi} \int_{\Gamma_\beta} \lambda^z \, (\lambda - A)^{-1} \, d\lambda, \tag{12}$$

and respectively the operators [36, Par. 2.6.1.2.]

$$L_\beta(A,z) = \frac{i}{2\pi} \int_{\Gamma_\beta} \log_\beta \lambda \;\; \lambda^z \, (\lambda - A)^{-1} \, d\lambda, \tag{13}$$

are bounded linear maps from any Sobolev closure $H^s\,(M,E)$ of $C^\infty(M,E)$, $s \in \mathbb{R}$, with values in $H^{s-a\Re(z)}\,(M,E)$ respectively in $H^{s-a\Re(z)+\epsilon}\,(M,E)$, for any $\epsilon > 0$. Here Γ_β is a closed contour in $\mathbb{C} \setminus \{re^{i\beta},\, r \geq 0\}$ around the spectrum of A oriented clockwise. These definitions extend to the whole complex plane

$$A_\beta^z = A^k \, A_\beta^{z-k} \quad , \text{ resp. } \; L_\beta(A,z) = A^k \, L_\beta(A, z-k), \qquad \Re(z) < k$$

for any $k \in \mathbb{N}$, A_β^z is an operator in $\Psi(M,E)$ of order az for any complex number z, and the logarithm

$$\log_\beta(A) := L_\beta(A, 0)$$

of A is a bounded linear map from $H^s\,(M,E)$ to $H^{s+\epsilon}\,(M,E)$, $\forall \epsilon > 0$. One has by construction $\log_\beta A \, A_\beta^z = A_\beta^z \, \log_\beta A$, $\forall z \in \mathbb{C}$.

Remark 7. Just as a complex power does, the logarithm depends on the choice of a spectral cut β. However, in order to simplify the notation we shall often drop the explicit mention of β.

The logarithm of a classical pseudodifferential operator of positive order is not classical. Indeed, in a local trivialisation, the symbol of $\log_\beta A$ reads (see, e.g., [36])

$$\sigma(\log_\beta A)(x,\xi) = a\,\log|\xi| I + \sigma_{\rm cl}(\log_\beta A)(x,\xi) \tag{14}$$

where a denotes the order of A and $\sigma_{\rm cl}(\log_\beta A)$ is a classical symbol of order zero with homogeneous components $\sigma_{-j}(\log_\beta A)$ of degree $-j, j \in \mathbb{Z}_{\geq 0}$.
Moreover, the leading symbol $\sigma_{\rm cl}^L(\log_\beta A)$ of $\sigma_{\rm cl}(\log_\beta A)$ can be expressed in terms of the leading symbol $\sigma^L(A)$ of A as

$$\sigma_{\rm cl}^L(\log_\beta A)(x,\xi) = \log_\beta\left(\sigma^L(A)\left(x, \frac{\xi}{|\xi|}\right)\right) \quad \forall (x,\xi) \in T^*M \setminus M \times \{0\}. \tag{15}$$

1.3. The local Wodzicki residue extended to logarithms

In spite of the fact that logarithms are not classical, the Wodzicki residue on classical pseudodifferential operators does extend to logarithms.[5] As a first step we extend to logarithms the local residue res_x defined in (10). For a weight $Q \in \Psi(M,E)$ with spectral cut β we define the pointwise extended residue as

$$\mathrm{res}_x\left(\log_\beta Q\right) := \int_{|\xi|_x=1} \mathrm{tr}_x\left(\sigma_{-n}(\log_\beta Q)(x,\xi)\right)\, đ_S\xi.$$

By (14) this is a natural extension of the pointwise residue on classical symbols since the integral over the sphere vanishes on the logarithm of the norm. The fact that the n-form $\mathrm{res}_x\left(\log_\beta Q\right)dx$ defines a volume density will arise later as a consequence of a local formula for the ζ-regularised trace.

The subsequent proposition shows the locality of the extended residue of $A \log Q$ in so far as it only depends on a finite number of homogeneous components of the symbols of A and Q

Proposition 8. *For any weight $Q \in \Psi(M,E)$ of order q,*

1. (compare with [15, Lemma 1.8.2])
the local logarithmic residue at a point $x \in M$ which reads

$$\mathrm{res}_x(\log Q) = \frac{1}{2\pi i}\int_{|\xi|_x=1}\left(\int_\Gamma \log\lambda\, \sigma_{-q-n}(Q-\lambda)^{-1}(x,\xi)\,d\lambda\right) đ_S\xi, \tag{16}$$

where Γ is a contour around the spectrum of Q oriented clockwise, is an algebraic expression in the x-jets of the first n homogeneous components (taken in decreasing order of homogeneity) of the symbol $\sigma(Q)(x,\cdot)$ of Q at that point given by the integral over the unit cotangent sphere of an algebraic expression in the (x,ξ)-jets of the first n homogeneous components of the symbol $\sigma(Q)(x,\cdot)$ of Q at that point. Here $\sigma_{-q-n}(Q-\lambda)^{-1}$ is the $(-q-n)$th homogeneous component of the resolvent $(Q-\lambda)^{-1}$ of Q.

[5]This extended residue differs from the higher residue on log-polyhomogeneous operators introduced in [21].

2. (compare with [15, Lemma 1.9.1])
Given a differential operator $A = \sum_{|\alpha|\leq a} a_\alpha(x) D_x^\alpha \in \Theta(M,E)$ *of order* $a \in \mathbb{Z}_{\geq 0}$*, then* $\mathrm{res}_x\,(A \log Q)$ *is an algebraic expression in the coefficients* a_α *of* A *and in the* x*-jets of the first* $n+a$ *homogeneous components of the symbol* $\sigma(Q)(x,\cdot)$ *of* Q *at that point.*
In particular, if $Q = \sum_{|\beta|\leq q} b_\beta(x) D_x^\beta$ *is a differential operator, then the local residue* $\mathrm{res}_x(A\,\log(Q))$ *is an algebraic expression in the coefficients* a_α *and in the* x*-jets of the coefficients* b_β*.*

Proof. Since

$$\begin{aligned}\sigma_{-n}(A\,\log Q) &= \sum_{|\gamma|+j+k=a+n} \frac{(-i)^\gamma}{\gamma!}\partial_\xi^\gamma \sigma_{a-j}(A)\,\partial_x^\gamma \sigma_{-k}(\log Q)\\ &= \sum_{|\gamma|+k=a+n} (-i)^\gamma \binom{\alpha}{\gamma} a_\alpha(x)\xi^{a-\gamma}\,\partial_x^\gamma \sigma_{-k}(\log Q),\end{aligned}$$

the fact that $\mathrm{res}_x\,(A \log Q)$ is an algebraic expression in the coefficients a_α of A and in the x-jets of the first $n+a$ homogeneous components of the symbol $\sigma(Q)(x,\cdot)$ follows from a similar statement for the homogeneous components $\sigma_{-k}(\log Q)$ of order $-k \in \mathbb{Z}_{\leq 0}$ of the symbol of $\log Q$. Now, since $\log Q = \partial_z Q^z|_{z=0}$, the homogeneous component $\sigma_{-k}(\log Q)$ is derived by differentiating the homogeneous component $\sigma_{qz-k}(Q^z)$ of degree $qz-k$ at zero of the complex power Q^z. The latter is obtained from the homogeneous component σ_{-q-k} of the resolvent $(Q-\lambda)^{-1}$ by means of the Cauchy formula

$$Q^z = \frac{1}{2i\pi}\int_\Gamma \lambda^z (Q-\lambda)^{-1} d\lambda$$

where, as before, Γ is a contour around the spectrum of Q oriented clockwise. Thus we find

$$\sigma_{-k}(\log Q) = \frac{1}{2i\pi}\partial_z \left(\int_\Gamma \lambda^z \sigma_{-q-k}(Q-\lambda)^{-1}\, d\lambda\right)_{|z=0}.$$

In particular, setting $k=n$ and integrating over the unit cotangent sphere we find

$$\mathrm{res}_x(\log Q) = \frac{1}{2\pi i}\int_{|\xi|_x=1} \partial_z \left(\int_\Gamma \lambda^z \sigma_{-q-n}(Q-\lambda)^{-1}\, d\lambda\right)_{|z=0} đ_S\xi,$$

which yields (16). Since $\sigma_{-q-k}(Q-\lambda)^{-1}(x,\xi)$ is an algebraic expression in the (x,ξ)-jets of the first k homogeneous components

$$\sigma_{q-1}(Q)(x,\xi), \sigma_{q-2}(Q)(x,\xi), \ldots, \sigma_{q-k}(Q)(x,\xi)$$

of the symbol of Q, so is $\sigma_{-k}(\log Q)(x,\xi)$ an algebraic expression in the jets of the first k homogeneous components. This for $n=k$ yields the first assertion (compare with Formula (8) in [25]). The second assertion follows in a similar way after implementing the differential operator A and integrating over the unit cotangent sphere. $\square$

The locality of the extended residue can also be seen from the fact that it does not detect smoothing perturbations.

Corollary 9. *Let $A \in \Psi(M,E)$ be an admissible operator. Let R, S be two smoothing operators acting on $C^\infty(M,E)$ such that the perturbed operators $A+R$ and $A+S$ are invertible. They define weights and for any $x \in M$ we have*

$$\mathrm{res}_x\left(\log(A+R)\right) = \mathrm{res}_x\left(\log(A+S)\right). \tag{17}$$

Proof. This follows from Proposition 8 and the fact that a smoothing perturbation of an operator does not modify the homogeneous components of its symbol. □

Let $\Delta \in \Theta(M,E)$ be an admissible operator. Let E be equipped with a Hermitian metric, which combined with a Riemannian metric on M induces an inner product on $C^\infty(M,E)$. It follows from the theory of elliptic operators on a closed manifold (see, e.g., [15]) that the orthogonal projection π_Δ onto the kernel $\mathrm{Ker}\,(\Delta)$ is a finite rank operator and hence smoothing. Consequently, the operator $Q := \Delta + \pi_\Delta$ is a weight. On the grounds of Corollary 9, we define the **pointwise logarithmic residue** of Δ as

$$\mathrm{res}_x(\log\Delta) := \mathrm{res}_x(\log(\Delta+\pi_\Delta)) \quad \forall x \in M. \tag{18}$$

As we shall see below, the local density $\mathrm{res}_x(A\,\log\Delta)\,dx$ actually defines a global density on the manifold, confirming the known fact that the residue extends to logarithms [27–29].

1.4. The Wodzicki residue as a complex residue

Given a symbol $\sigma(x,\xi) \in \mathcal{S}^a\left(U, \mathrm{End}(E)\right)$ with x a point in M and $U \subset M$ an open neighborhood of x, the **cut-off integral**[6] is defined as the finite part

$$-\!\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}_x\sigma(x,\xi)\, đ\xi := \mathrm{fp}_{R\to\infty}\int_{B(0,R)} \mathrm{tr}_x\sigma(x,\xi)\, đ\xi \tag{19}$$

Remark 10. Whereas the residue vanishes on symbols whose order has real part smaller than $-n$, the cut-off integral coincides on those symbols with the ordinary integral on $\mathbb{R}^n$. A straightforward computation shows that, like the local residue, the cut-off integral also vanishes on polynomial symbols.

We need holomorphic families of classical pseudodifferential symbols first introduced by Guillemin in [17] and extensively used by Kontsevich and Vishik in [20]. The idea is to embed a symbol σ in a family $z \mapsto \sigma(z)$ depending holomorphically on a complex parameter z.

Definition 11. Let U be an open subset of M. We call a family $(\sigma(z))_{z\in\Omega}$ of symbols in $\mathcal{S}\left(U, \mathrm{End}(E)\right)$ parametrised by a domain Ω of $\mathbb{C}$ **holomorphic** at a point $z_0 \in \Omega$ if, with the notation of (8) and (9) we have

1. $\sigma(z)(x,\cdot)$ is uniformly in x on any compact subset of U, holomorphic at z_0 as a function of z with values in $C^\infty\left(U\times\mathbb{R}^n, \mathrm{End}(E)\right)$,

[6] also called **Hadamard finite part integral** see, e.g., [36, Example 2 Chapter II], and also [16].

2. for any z in a neighborhood of z_0 there is an asymptotic expansion of the type (8)
$$\sigma(z)(x,\cdot) \sim \sum_{j\geq 0} \sigma_{\alpha(z)-j}(z)(x,\cdot), \tag{20}$$
with $\alpha(z) := -qz+a$ for some positive number q and a the order of $\sigma := \sigma(0)$,
3. for any integer $N \geq 1$ the remainder
$$\sigma_{(N)}(z) := \sigma(z) - \sum_{j=0}^{N-1} \sigma_{\alpha(z)-j}(z)$$
is uniformly in x on any compact subset of U, holomorphic at z_0 as a function of z with values in $C^\infty(U \times \mathbb{R}^n, \mathrm{End}(E))$ with kth z-derivative
$$\sigma_{(N)}^{(k)}(z) := \partial_z^k(\sigma_{(N)}(z)) \tag{21}$$
a symbol on U of order $\alpha(z) - N + \epsilon$ for any $\epsilon > 0$ uniformly in x on any compact subset of U and locally uniformly in z around z_0, i.e., the kth derivative $\partial_z^k \sigma_{(N)}(z)$ satisfies a local uniform estimate in z around z_0
$$\|\partial_\xi^\beta \partial_z^k \sigma_{(N)}(z)(x,\xi)\| \leq C_\beta \langle\xi\rangle^{\Re(qz)-N-|\beta|} \quad \forall \xi \in \mathbb{R}^n, \tag{22}$$
where $\|A\| := \sqrt{\mathrm{tr}_x(A^*A)}$ is the norm on $\mathcal{L}(M \times M, \mathrm{End}(E))$ and where we have set $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ with $|\cdot|$ the Euclidean norm of ξ.

Example 12. If $\sigma \in S(U, \mathrm{End}(E))$ is a symbol of order $\alpha(0)$, then $\sigma(z)(x,\xi) = \sigma(x,\xi)\langle\xi\rangle^{-z}$ defines a holomorphic family of order $\alpha(z) = -z + \alpha(0)$.

The following assertion can be shown on direct inspection of the cut-off integral.

Proposition 13. *For any holomorphic family $\sigma(z)$ of classical symbols parametrised by $\mathbb{C}$ with affine order $\alpha(z) = -qz + a$ for some positive real number q and some real number a,*

1. *the map*
$$z \mapsto -\!\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(z)(x,\xi)\,đ\xi$$
is meromorphic with simple poles $d_j := \frac{a+n-j}{q}, \quad j \in \mathbb{Z}_{\geq 0}$.
2. [20] *The complex residue at the point d_j in $\mathbb{C}$ is given by:*
$$\mathrm{Res}_{z=d_j}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}_x\sigma(z)(x,\xi)\,đ\xi\right) = \frac{1}{q}\mathrm{res}_x(\sigma(d_j)). \tag{23}$$
3. [29] *The finite part at the pole d_j differs from the cut-off regularised integral $-\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}_x\sigma(d_j)(x,\xi)\,đ\xi$ by*
$$\mathrm{fp}_{z=d_j}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}_x\sigma(z)(x,\xi)\,đ\xi\right) - -\!\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}_x\sigma(d_j)(x,\xi)\,đ\xi = \frac{1}{q}\mathrm{res}_x(\sigma'(d_j)). \tag{24}$$

Here the noncommutative residue is extended to the possibly non-classical symbol [7] $\tau_j(x,\xi) := \sigma'(d_j)(x,\xi)$ *using the same formula as in Equation* (10)

$$\mathrm{res}_x(\tau_j) := \int_{|\xi|_x=1} \mathrm{tr}_x\,(\tau_j)_{-n}\,(x,\xi)\,đ_S\xi.$$

We are now ready to introduce holomorphic families of pseudodifferential operators.

Definition 14. Following [29, Definition 1.14] we call a family $(A(z))_{z\in\Omega}$ of operators in $\Psi\,(M,E)$ parametrised by a domain Ω of $\mathbb{C}$ holomorphic at a point $z_0\in\Omega$ if, with the notation of Section 1.1, in each local trivialisation U of $E\to M$ we have

$$A(z) = \sum_{j=1}^{J} A_j(z) + R(z)$$

with

1. $A_j(z) = \mathrm{Op}(\sigma_j(z))$, where $\sigma_j(z)$ is a holomorphic family of polyhomogeneous symbols on U,
2. $R(z)$ is a smoothing operator with Schwartz kernel $R(z,x,y)\in C^\infty(\Omega\times U\times U,\mathrm{End}(E))$ holomorphic in z.

Integrating the results of Proposition 13 over M yields the following theorem which we quote without proof, referring the reader to [20] and [29]. Let us however recall that the linear map TR introduced in the theorem below is the canonical trace popularised in [20], i.e., the unique linear form (up to a multiplicative factor) on the subset of $\Psi(M,E)$ consisting of non-integer-order classical pseudodifferential operators, which vanishes on commutators that lie in this set. It extends to differential operators where it vanishes and it coincides with the L^2-trace Tr on trace-class operators, i.e., the real part of the order is smaller than $-n$.

Theorem 15. *For any holomorphic family $A(z)\in\Psi(M,E)$ of classical operators parametrised by $\mathbb{C}$ with holomorphic order $-qz+a$ for some positive q and some real number a,*

1. *the meromorphic map $z\mapsto \mathrm{TR}_x(A(z)) := ⨍_{\mathbb{R}^n}\mathrm{tr}_x\sigma(z)(x,\xi)đ\xi$ integrates over M to the map*
$$z\mapsto \mathrm{TR}\,(A(z)) := \int_M \mathrm{TR}_x(A(z))\,dx$$
which is meromorphic with simple poles $d_j := \frac{a+n-j}{q}$, $\quad j\in\mathbb{Z}_{\geq 0}$.
2. [20] *The complex residue at the point d_j is given by:*
$$\mathrm{Res}_{z=d_j}\mathrm{TR}\,(A(z)) = \frac{1}{q}\mathrm{Res}(A(d_j)). \tag{25}$$

[7] The asymptotic expansion of $\tau_j(x,\xi)$ as $|\xi|\to\infty$ might present logarithmic terms $\log|\xi|$, which vanish on the unit sphere and therefore do not explicitly arise in the following definition.

3. [29] *If* $A(d_j)$ *differs from a differential operator*[8] *by a trace-class pseudodifferential operator* T_j, *then* $A(d_j)$ *has a well-defined canonical trace* $\mathrm{TR}(A(d_j)) = \mathrm{Tr}(T_j)$ *and* $A'(d_j)$ *has a well-defined Wodzicki residue*
$$\mathrm{Res}(A'(d_j)) := \int_M \mathrm{res}_x\left(A'(d_j)\right)\, dx$$
where
$$\mathrm{res}_x\left(A'(d_j)\right) := \int_{|\xi|_x=1} \sigma_{-n}\left(A'(d_j)\right)(x,\xi)\, đ_S\xi$$
at the point d_j *and we have*
$$\mathrm{fp}_{z=d_j}\mathrm{TR}\left(A(z)\right) = \mathrm{Tr}(T_j) - \frac{1}{q}\mathrm{Res}(A'(d_j)). \tag{26}$$

Remark 16. Formula (26) formally follows from (24) applied to the family $\tau_j(z) = \frac{\sigma(z)-\sigma(d_j)}{z-d_j}$ since $\tau_j(d_j) = \sigma'(d_j)$. However, $\tau_j(z)$ not strictly speaking being a holomorphic family of classical symbols since the two symbols $\sigma(z)$ and $\sigma(d_j)$ have different orders outside d_j, which do not differ by an integer, the proof is actually slightly more indirect.

Consequently, $\mathrm{Res}(A'(d_j))$ is the noncommutative residue extended to the typically non-classical operator $A'(d_j)$.

1.5. ζ-regularised traces

Given a weight $Q \in \Psi(M,E)$ of order $q \in \mathbb{R}_+$ and an operator $A \in \Psi(M,E)$ (not necessarily admissible) of order $a \in \mathbb{R}$, the map $z \mapsto A(z) := A\,Q^{-z}$ defines a holomorphic family. The subsequent theorem quoted from [29] follows from applying Theorem 15 to this family.

Theorem 17. *Given a weight* $Q \in \Psi(M,E)$ *of order* $q \in \mathbb{R}_+$ *and an operator* $A \in \Psi^a(M,E)$, *the map*
$$z \mapsto \zeta(A,Q)(z) := \mathrm{TR}(AQ^{-z}) \tag{27}$$
called the ζ*-regularised trace of* A *with respect to the weight* Q, *is holomorphic on a half-plane* $\Re(z) > \frac{n+a}{q}$, *meromorphic on the whole complex plane with poles at* $d_j = \frac{a+n-j}{q}$, $j \in \mathbb{Z}_{\geq 0}$ *and the complex residue at this pole can be expressed as a Wodzicki residue*
$$\mathrm{Res}_{z=d_j}\zeta(A,Q)(z) = \frac{1}{q}\mathrm{Res}(A\,Q^{-d_j}).$$
If A *differs from a differential operator by a trace-class pseudodifferential operator* T, *the* n*-form*
$$\mathrm{res}_x\left(A\,\log Q\right)\, dx := \int_{|\xi|_x=1} \sigma_{-n}\left(A\,\log Q\right)(x,\xi)\, đ_S\xi\, dx.$$

[8] This yields another application of the results of [29] since only the case $A(d_j)$ differential was considered in the examples given in that paper.

defines a global density on M which integrates to the extended Wodzicki residue

$$\mathrm{Res}(A\,\log Q) := \int_M \mathrm{res}_x\,(A\,\log Q)\;dx$$

of $A\,\log Q$.

The zeta function $\zeta(A,Q)$ is holomorphic at zero and we have

$$\zeta(A,Q)(0) = \lim_{z\to 0}\zeta(A,Q)(z) = \mathrm{Tr}(T) - \frac{1}{q}\mathrm{Res}(A\,\log Q). \tag{28}$$

When A is a differential operator, its Q-weighted ζ-regularised trace $\zeta(A,Q)(0)$ is therefore proportional to the extended Wodzicki residue $\mathrm{Res}(A\log Q)$. So in that case, the ζ-regularised trace $\zeta(A,Q)(0)$ is local since the extended Wodzicki residue is local in so far as it is expressed as an integral over M of the pointwise extended residue (Proposition 8), which only depends on finitely many (here one) homogeneous components of the symbol of A.

In particular, for $A = I$, we have as announced previously, that the **logarithmic residue** $\mathrm{Res}(\log Q)$ is well defined, moreover the zeta function of Q at zero is local

$$\zeta_Q(0) := \zeta(I,Q)(0) = -\frac{1}{q}\mathrm{Res}(\log Q).$$

2. Heat-kernel expansions revisited

2.1. The heat-kernel expansion in terms of Wodzicki residues

We recall the definition and some properties of the Mellin transform following [9] (see also [18]).

Definition 18. Let $f(t)$ be a locally Lebesgue integrable function over $]0,+\infty[$. The Mellin transform of $f(t)$ is defined as

$$\mathcal{M}(f)(z) := \int_0^\infty f(t)t^{z-1}\,dt.$$

The largest open strip $a < \Re(z) < b$ in which the integral converges is called the fundamental strip.

We recall the following well-known Inverse Mellin Mapping Theorem [9, Theorem 4].

Proposition 19. *Let $f(t)$ be continuous in $]0,+\infty[$ with Mellin transform $\phi(z)$ having a fundamental strip $a < \Re(z) < b$.*

1. *Provided*
   (a) *$\phi(z)$ admits a meromorphic continuation to the strip (γ, b) for some $\gamma < a$ with a finite number of poles in the strip, and is analytic on $\Re(z) = \gamma$,*
   (b) *there exists a real number c in (a,b) such that for some $r > 1$*

$$\phi(z) = O(|z|^{-r}), \quad \textit{when } |z|\to\infty \textit{ in } \gamma \le \Re(z) \le c,$$

(c) *ϕ admits the singular expansion for $z \in (\gamma, a)$*

$$\phi(z) \cong \sum_{i\geq 0}\sum_{j\geq 0} a_{i,j}\frac{(-1)^{k_j}k_j!}{(z-d_j)^{k_i+1}},$$

then f **admits an asymptotic expansion** *at* 0 *given by*

$$f(t) = \sum_{i\geq 0}\sum_{j\geq 0} a_{i,j}t^{-d_j}\log^{k_i} t + O(t^{-\gamma}).$$

2. *Provided*
   (a) *$\phi(z)$ admits a meromorphic continuation to the strip (a, γ) for some $\gamma > b$ with a finite number of poles in the strip, and is analytic on $\Re(z) = \gamma$,*
   (b) *there exists a real number c in (a, b) such that for some $r > 1$*

$$\phi(z) = O(|z|^{-r}), \quad \textit{when } |z| \to \infty \textit{ in } c \leq \Re(z) \leq \gamma,$$

   (c) *ϕ admits the singular expansion for $z \in (c, \gamma)$,*

$$\phi(z) \cong \sum_{i\geq 0}\sum_{j\geq 0} a_{i,j}\frac{(-1)^{k_i}k_i!}{(z-d_j)^{k_i+1}},$$

then f **admits an asymptotic expansion** *at ∞ given by*

$$f(t) = \sum_{i\geq 0}\sum_{j\geq 0} a_{i,j}t^{-d_j}\log^{k_i} t + O(t^{-\gamma}).$$

Remark 20. In particular, it follows from 2.c) that if ϕ is analytic in some half-plane $\Re(z) > c$ then $f(t) = O(t^{-\gamma})$ for any $\gamma > c$ and f is a Schwartz function.

Here is a useful example to keep in mind for what follows.

Example 21. For any $\lambda > 0$ the map $z \longmapsto \phi(z) = \Gamma(z)\,\lambda^{-z}$ satisfies the above assumptions as a result of the properties of the Gamma function. Indeed, on the one hand it is meromorphic on the complex plane with simple poles in $\mathbb{Z}_{\leq 0}$. On the other hand, it follows from Stirling's formula (see, e.g., [10, Proposition IV:1.14]) which expresses the Gamma function as $\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z}\, e^{H(z)}$ for some function H given by a series, that for any $0 < \gamma < \delta$ there is a positive constant $C_{\gamma,\delta}$ such that

$$\gamma \leq \Re(z) \leq \delta \Longrightarrow |\Gamma(z)| \leq C_{\gamma,\delta}\,|\Im(z)|^{\delta-\frac{1}{2}}\,e^{-\gamma}.$$

Since λ^{-z} is bounded from above by $\lambda^{-\gamma}$ this shows that Condition 1. b) is satisfied for any $r > 1$. The inverse Mellin transform is $f(t) = e^{-t\lambda}$ which defines a Schwartz function on $\mathbb{R}_+$.

Combining these properties of the Gamma function with the results of Theorem 15 yields the following useful properties.

Corollary 22. *Let $A(z) \in \Psi(M, E)$ be a holomorphic family of affine order $\alpha(z) = a - qz$ for some positive real number q; we set*

$$\phi(z) := \Gamma(z)\,\mathrm{TR}\,(A(z)) \quad \alpha := \mathrm{Max}\,\{0; (a+n)/q\}\,.$$

1. *The map ϕ is holomorphic on the half-plane $\Re(z) > \alpha$, it is analytic on any imaginary line $\Re(z) = \gamma$ for $\gamma > \alpha$, and (α, ∞) is the largest open strip where ϕ is defined.*
2. *ϕ admits a meromorphic extension to the whole complex plane with countably many simple poles $\{d_j := \frac{a+n-j}{q}, j \in \mathbb{Z}_{\geq 0}\} \cup \mathbb{Z}_{\leq 0}$*
3. *whenever $z \mapsto \mathrm{TR}\,(A(z))$ is uniformly bounded in closed strips $\frac{a+n}{q} < \gamma \leq \Re(z) \leq \delta$, then for some $r > 1$ we have the following asymptotic behaviour of ϕ at infinity along imaginary lines*
$$\phi(z) = O(|z|^{-r}), \quad \text{when } |z| \to \infty \text{ in } \frac{a+n}{q} < \gamma \leq \Re(z) \leq \delta,$$
as a consequence of the corresponding property of the Gamma function described above.
4. *ϕ admits a singular expansion on any horizontally bounded strip*
$$\phi(z) \cong \sum_{j=0} \frac{a_j}{z - d_j},$$
as a consequence of the second item.

Let as before, M be an n-dimensional closed manifold and E be a finite rank vector bundle over M. The subsequent theorem follows from Proposition 19 applied to $\phi(z) = \Gamma(z)\,\mathrm{TR}\,(A(z))$.

Theorem 23. *Let $\mathcal{A}\colon z \mapsto A(z) \in \Psi(M, E)$ be some holomorphic family of operators of affine order $\alpha(z) = a - qz$ with q some positive real number corresponding to the Mellin transform of some analytic family $\widetilde{A}(t), t > 0$ of trace-class operators in $\Psi(M, E)$. Whenever $z \mapsto \mathrm{TR}\,(A(z))$ is uniformly bounded on closed strips $\frac{a+n}{q} < \gamma \leq \Re(z) \leq \delta$, then $f_{\mathcal{A}}(t) = \mathrm{Tr}\left(\widetilde{A}(t)\right)$ admits an asymptotic expansion at 0 given by*
$$f_{\mathcal{A}}(t) \sim_0 \frac{1}{q} \sum_{j \geq 0} a_j t^{\frac{-a-n+j}{q}}, \tag{29}$$
with
$$a_j = -\frac{1}{q} \mathrm{Res}\,(A(d_j)) \quad \text{for} \quad j > a + n. \tag{30}$$
If $A(0)$ differs from a differential operator by a trace-class pseudodifferential operator T, the constant term in the expansion (29) *which coincides with the constant term in the Laurent expansion of* $\mathrm{TR}(A(z))$ *reads*
$$a_j = \mathrm{Tr}(T) - \frac{1}{q} \mathrm{Res}\,(A'(0)), \quad \text{for} \quad j = a + n. \tag{31}$$

Let $Q \in \Psi(M, E)$ be a weight of positive order q. We further assume that it is “close” to a positive operator, i.e., its spectrum is concentrated in a cone centered around the positive real line; in particular, it has spectral cut $\beta = \pi$.

The holomorphic family $\mathcal{A} : z \mapsto A(z) = A\,Q^{-z}$ is the Mellin transform of the analytic family $A\,e^{-tQ}, t > 0$. Since the map $z \mapsto \mathrm{TR}\,(A(z))$ is uniformly bounded on closed strips $\frac{a+n}{q} < \gamma \leq \Re(z) \leq \delta$ we can apply Theorem 23, which yields the following corollary.

Corollary 24. *The inverse Mellin transform*

$$f_{\mathcal{A}}(t) := \mathrm{Tr}\left(Ae^{-tQ}\right)$$

of $\phi_{\mathcal{A}}(z) := \Gamma(z)\mathrm{TR}\left(AQ^{-z}\right)$ *is a Schwartz function on* $]0, +\infty[$, *which admits an asymptotic expansion at* 0 *given by*

$$f_{\mathcal{A}}(t) \sim_0 t^{-\frac{a+n}{q}} \sum_{j \geq 0} a_j t^{\frac{j}{q}}, \tag{32}$$

where

$$a_j = -\frac{1}{q}\mathrm{Res}\left(AQ^{\frac{-a-n+j}{q}}\right) \quad \textit{for} \quad j < a+n$$

and

$$a_{a+n} = \zeta(A, Q)(0) = \mathrm{Tr}(T) - \frac{1}{q}\mathrm{Res}\left(A \log Q\right) \tag{33}$$

if $A \in \Psi(M, E)$ *differs from a differential operator by a trace-class operator* T.

Let $\Delta \in \Theta(M, E)$ be an admissible operator and let E be equipped with a Hermitian metric, which combined with a Riemannian metric on M induces a (weak) inner product on $C^\infty(M, E)$. We previously saw that the operator $Q := \Delta + \pi_\Delta$ defines a weight.

Notation convention: Integrating over M the pointwise residue in (16) gives rise to the **logarithmic residue of** Δ defined as

$$\mathrm{Res}(\log \Delta) := \mathrm{Res}(\log(\Delta + \pi_\Delta)). \tag{34}$$

Similarly, for any $x \in M$ and $\alpha \in \mathbb{R}$ we set

$$\mathrm{res}_x(\Delta^\alpha) := \mathrm{res}_x\left((\Delta + \pi_\Delta)^\alpha\right); \quad \mathrm{Res}(\phi\, \Delta^\alpha) := \mathrm{Res}\left(\phi\ (\Delta + \pi_\Delta)^\alpha\right).$$

We now specialise to the multiplication operator $A = \phi \in C^\infty(M)$ and apply Theorem 23 to the holomorphic family $A(z) = \phi\,(\Delta + \pi_\Delta)^{-z}$.

Theorem 25. *For any smooth function* $\phi \in C^\infty(M)$ *we have*

$$\mathrm{Tr}\left(\phi\, e^{-t\,\Delta}\right) \sim_{t \to 0} -\frac{(4\pi)^{\frac{n}{2}}}{2}\left[\mathrm{Res}\left(\phi\, \log \Delta\right)\, \delta_{\frac{n}{2} - \left[\frac{n}{2}\right]} \right.$$
$$\left. + \sum_{k \in \left[0, \frac{n}{2}\right[\cap \mathbb{Z}} \Gamma\left(\frac{n}{2} - k\right)\, \mathrm{Res}\left(\phi\, \Delta^{k - \frac{n}{2}}\right)\, t^{k - \frac{n}{2}}\right]. \tag{35}$$

The local heat-kernel trace $K_t(\Delta)(x, x)$ *of the operator* Δ *at the point* x *is defined by*

$$\mathrm{Tr}\left(\phi\, e^{-t\,\Delta}\right) = \int_M \phi(x) K_t\left(\Delta\right)(x, x)\, \sqrt{\det g}(x)\, dx \quad \forall \phi \in C^\infty(M)$$

and therefore it has the following asymptotic expansion

$$K_t(\Delta)(x,x) \sim_{t\to 0} -\frac{(4\pi)^{\frac{n}{2}}}{2\sqrt{\det g}(x)}\Bigg[\mathrm{res}_x\,(\log\Delta)\;\delta_{\frac{n}{2}-[\frac{n}{2}]} + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\,\mathrm{res}_x\left(\Delta^{k-\frac{n}{2}}\right)\,t^{k-\frac{n}{2}}\Bigg]. \tag{36}$$

Remark 26. Formula (35) compares with known formulae for the spectral action (take $f(\lambda)=e^{-t\lambda}$ in [5, Formula 1.5]), in which the non-constant coefficients arise as Dixmier traces.

Proof. It follows from (28) that the constant term in the heat-kernel expansion reads

$$\mathrm{fp}_{t=0}\mathrm{Tr}\left(\phi\,e^{-t(\Delta+\pi_\Delta)}\right)=\zeta\left(\phi,\Delta+\pi_\Delta\right)(0)=\frac{1}{2}\,\mathrm{Res}\left(\phi\,\log\Delta\right).$$

Formula (35) then follows from Corollary 24 with $q=2$ (cf. footnote in the introduction).

Since this holds for any smooth function ϕ, formula (36) follows. □

Remark 27. Combining (34) with (16) applied to Δ

$$\mathrm{res}_x(\log\Delta)=\frac{1}{2\pi i}\int_{|\xi|_x=1}\int_\Gamma \log\lambda\,\sigma_{-q-n}(\Delta+\pi_\Delta-\lambda)^{-1}(x,\xi)\,d\lambda\,đ_S\xi,$$

yields

$$a_{\frac{n}{2}}(x)=-\frac{1}{4i\,\pi^{\frac{n}{2}+1}\sqrt{\det g}(x)}\int_{|\xi|_x=1}\int_\Gamma \log\lambda\,\sigma_{-q-n}(\Delta+\pi_\Delta-\lambda)^{-1}(x,\xi)\,d\lambda\,d_S\xi.$$

This compares with similar formulae in the literature as for example [3, (2.11)]

$$a_{\frac{n}{2}}(x)=\frac{1}{2}\int_{|\xi|_x=1}\int_0^\infty \sigma_{-q-n}(\Delta+\pi_\Delta-\lambda)^{-1}(x,\xi)\,d\lambda\,đ_S\xi.$$

2.2. The case of geometric operators

We now single out a class of differential operators we call *geometric differential operators* (also considered in [25]), i.e., differential operators $A=\sum_{|\alpha|\le d} a_\alpha(x)D_x^\alpha \in \Theta(M,E)$ where d is the order of the operator and whose coefficients $a_\alpha(x)$ are given by an algebraic expression in terms of the jets at the point x of the metric on M and a connection on E.

The Laplace–Beltrami operator Δ_g on a closed Riemannian manifold (M,g) is a geometric differential operator of order 2. Another example is the Bochner-Laplacian

$$\Delta^\nabla := \mathrm{Tr}\left(\nabla^{T^*M\otimes E}\circ\nabla^E\right)\in\Psi(M,E)$$

built from a connection ∇^E on E and the induced connection $\nabla^{T^*M\otimes E}$ on the tensor product $T^*M\otimes E$. Here the trace is taken over the two factors of T^*M. The square of a Dirac operator on a spin manifold (M,g), which differs from the corresponding Bochner-Laplacian by a term proportional to the scalar curvature, is geometric. More generally, the square $A = D^2$ of a Dirac-type operator $D = \sum_{i=1}^n c(e_i)\nabla_i^E$ is a geometric operator; here ∇^E is a Clifford connection on a Clifford bundle E over M, c the Clifford multiplication and $e_i, i = 1, \ldots, n$ an orthonormal frame of the cotangent bundle at a point x. All these examples fall in the class of Laplace-type differential operators considered previously.

Proposition 28. *Let A and Δ be two geometric differential operators in $\Theta(M, E)$ and let Δ be admissible. The extended local residue* $\mathrm{res}_x(A \log \Delta)$ *at a point x and for any real number α, the local residue* $\mathrm{res}_x(A\,\Delta^\alpha)$, *are algebraic expressions of the jets of the metric and the connection.*

Proof. That $\mathrm{res}_x(A \log \Delta)$ is an algebraic expression in the x-jets of a finite number of homogeneous components of the symbols follows from Proposition 8, which tells us that the extended local residue is an algebraic expression in the x-jets of a finite number of homogeneous components of the symbols of A and Δ combined with the fact that these homogeneous components are themselves algebraic expressions in the jets of the metric and the connection.

That $\mathrm{res}_x(A\,\Delta^\alpha)$ is an algebraic expression in the x-jets of a finite number of homogeneous components of the symbols can be shown similarly. □

The case of the Laplace–Beltrami operator Δ_g on an n-dimensional Riemannian manifold (M, g) is particularly relevant for us since it enables to capture the scalar curvature as a Wodzicki (possibly extended, depending on the dimension) residue. Indeed, the scalar curvature $\mathfrak{s}_g$ is proportional to the coefficient $b_1(x)$ in the heat-expansion

$$K_t\left(\Delta_g\right)(x,x) \sim_{t\to 0} t^{-n\frac{n}{2}} \sum_{j\geq 0} a_k(x)\, t^{\frac{k}{2}} = t^{-\frac{n}{2}} \sum_{j\geq 0} b_j(x)\, t^j, \tag{37}$$

where we have set $b_j = a_{2j}$ since the coefficients a_{2j+1} vanish [15], [31].
We have [26, Formula (5a)]

$$\mathfrak{s}_g = 3\, b_1,$$

independently of the dimension n.

This combined with Theorem 25 leads to the following expressions of the scalar curvature in terms of Wodzicki residues.

Proposition 29. *When $n = 2$ we have*

$$\mathfrak{s}_g(x) = -\frac{6\pi}{\sqrt{\mathrm{det} g(x)}}\, \mathrm{res}_x\left(\log \Delta\right). \tag{38}$$

If $n > 2$ then,

$$\mathfrak{s}_g(x) = -\frac{3}{2}\,(4\pi)^{\frac{n}{2}}\, \Gamma\left(\frac{n}{2} - 1\right) \frac{\mathrm{res}_x\left((\Delta + \pi_\Delta)^{1-\frac{n}{2}}\right)}{\sqrt{\mathrm{det} g(x)}}. \tag{39}$$

In dimension 2, which is the case we are going to focus on in the sequel, we have

$$\langle \mathfrak{s}_g, \phi\rangle_g = -6\pi\, \mathrm{Res}\left(\phi\, \log \Delta_g\right) \quad \forall \phi \in C^\infty(M). \tag{40}$$

3. Holomorphic families on Hilbert modules

We carry out to pseudodifferential operators on bundles of von Neumann Hilbert modules the constructions of Section 1 on closed manifolds.

3.1. Finite type Hilbert modules

We start out recalling the setup of finite type Hilbert modules closely following [3] and [34][9].

Let $\mathcal{A}$ be a von Neumann algebra equipped with a **finite** trace $\tau\colon \mathcal{A} \to \mathbb{C}$. This means that $\mathcal{A}$ is a unital $\mathbb{C}$-algebra with a $\star$ operation, and the following properties are satisfied:

1. $(\cdot,\cdot)\colon \mathcal{A}\times\mathcal{A} \longrightarrow \mathbb{C}$, defined by $(a,b) := \tau(ab^\star)$ is a scalar product and the completion $\mathcal{A}_2$ with respect to this scalar product is a separable Hilbert space.
2. $\mathcal{A}$ is weakly closed when viewed as a subalgebra of the space $\mathcal{L}\left(\mathcal{A}_2\right)$ of linear, bounded operators on $\mathcal{A}_2$ (identifying elements of $\mathcal{A}$ with the corresponding left translations in $\mathcal{L}\left(\mathcal{A}_2\right)$).
3. The trace is normal, i.e., for any monotone increasing net, $(a_i)_{i\in I}$ such that $a_i \geq 0$ and $a = \sup_{i\in I} a_i$ exists in $\mathcal{A}$, one has $\mathrm{tr}_\tau(a) = \sup_{i\in I}\mathrm{tr}_\tau(a_i)$.

A right $\mathcal{A}$-Hilbert module is a Hilbert space $\mathcal{W}$ with a continuous right $\mathcal{A}$-action that admits an $\mathcal{A}$-linear isometric embedding into $\mathcal{A}_2 \otimes H$ for some Hilbert space H. The Hilbert module $\mathcal{W}$ is called of **finite type** if the space H can be chosen finite-dimensional. We denote by $\mathcal{L}_{\mathcal{A}}(\mathcal{W})$ the von Neumann algebra of bounded $\mathcal{A}$-linear operators on $\mathcal{W}$. The (unbounded) trace on $\mathcal{L}_{\mathcal{A}}(\mathcal{W})$ induced by τ and by the usual trace on $\mathcal{L}(H)$ is denoted tr_τ.

Example 30. Let Γ be a countable group and $\ell^2(\Gamma)$ be the Hilbert space of square integrable complex-valued functions on Γ. The von Neumann algebra $\mathcal{N}\Gamma$ consists by definition of all bounded operators on $\ell^2(\Gamma)$ that commute with the left convolution action of Γ. It contains $\mathbb{C}\Gamma$ as a weakly dense subset, and on $\mathbb{C}\Gamma$ the canonical trace τ is given by

$$\tau\left(\sum a_\gamma \gamma\right) = a_e$$

where e is the unit element in Γ. Then $\mathcal{W} = \ell^2(\Gamma)$ is a finite type $\mathcal{N}\Gamma$-module, indeed $\ell^2(\Gamma) \simeq (\mathcal{N}\Gamma)_2$.

Example 31. In particular, for $\Gamma = \mathbb{Z}^n$, the Fourier transform gives an isometric $\mathbb{Z}^n$-equivariant isomorphism $\ell^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n)$, where $\mathbb{T}^n$ is the n-dimensional torus. Therefore $\mathcal{N}\mathbb{Z}^n$ coincides with the commutant $\mathcal{L}(L^2(T^n))^{\mathbb{Z}_n}$ of the $\mathbb{Z}_n$-action on $L^2(\mathbb{T}^n)$, and one obtains an isomorphism $\mathcal{N}\mathbb{Z}^n \simeq L^\infty(\mathbb{T}^n)$. The canonical trace $\tau\colon L^\infty(\mathbb{T}^n) \to \mathbb{C}$ is

$$\tau(f) = \int_{\mathbb{T}^n} f d\mu\ , \tag{41}$$

where μ is the measure on $\mathbb{T}^n$ induced by the canonical Lebesgue measure on $\mathbb{R}^n$.

[9] Note that in Schick's paper these objects are called $\mathcal{A}$-Hilbert spaces to distinguish them from Hilbert C^*-modules.

On a manifold M, an **$\mathcal{A}$-Hilbert module bundle** $\mathcal{E} \to M$ is a locally trivial bundle with fibre a (finitely generated, projective) $\mathcal{A}$-Hilbert module $\mathcal{W}$, the transition functions being isometries of $\mathcal{A}$-Hilbert modules.

Remark 32. The space of L^2-sections of an $\mathcal{A}$-Hilbert module bundle $\mathcal{E} \to M$ is an $\mathcal{A}$-Hilbert module. In fact, if U is a subset of M such that $M \setminus U$ has measure zero and $\mathcal{E}_{|U} \simeq U \times \mathcal{W}$ (where $\mathcal{W}$ is the fibre), then $L^2(M, \mathcal{E}) \simeq L^2(U, \mathcal{E}_{|U}) \simeq L^2(U) \otimes \mathcal{W}$. The set U can be chosen for example as being the union of the interiors of the top-order cells of a triangulation of M.

Example 33. (Flat $\mathcal{N}\Gamma$-Hilbert module) Let M be a closed manifold with $\pi_1(M) = \Gamma$, let $\pi\colon \widetilde{M} \to M$ be a universal covering of M, with Γ acting on the right by deck transformations. Because the left Γ-action and the right $\mathcal{N}\Gamma$-action on $\ell^2(\Gamma)$ commute, $\mathcal{H} := \widetilde{M} \times_\Gamma \ell^2(\Gamma)$ is a finitely generated projective bundle of (right) $\mathcal{N}\Gamma$-Hilbert modules over M. Moreover, $\mathcal{H}$ is endowed with a flat structure since the transition functions are locally constant.

The flat bundle $\mathcal{H} \to M$ can be used to describe the analysis on the universal covering. In fact, there is a well-known correspondence between $L^2(M, \mathcal{H})$ and $L^2(\widetilde{M})$, which translates twisted differential operators on one hand with Γ-invariant differential operators on the other. We refer for example to [34, 7.5] or [30, Prop. E.6] for the complete dictionary for spaces, operators, and L^2-invariants. Let us illustrate this in the case of the n-torus discussed above.

Example 34. To the universal covering $\pi\colon \mathbb{R}^n \to \mathbb{T}^n$ with fundamental group $\mathbb{Z}^n$ corresponds the finitely generated projective bundle $\mathcal{H} := \mathbb{R}^n \times_{\mathbb{Z}^n} \ell^2(\mathbb{Z}^n)$ of (right) $L^\infty(\mathbb{T}^n)$-Hilbert modules over $\mathbb{T}^n$. In this correspondence L^2-functions on $\mathbb{R}^n$ are viewed as L^2-sections of the bundle $\mathcal{H}$ over $\mathbb{T}^n$ via the map which sends f to $\hat{f}\colon x \mapsto \sum_{\gamma \in \mathbb{Z}^n} f(\gamma \pi^{-1}(x)) \otimes [\pi^{-1}(x), \gamma]$. This induces an isometry

$$\Phi\colon L^2(\mathbb{R}^n) \simeq L^2(\mathbb{T}^n) \otimes \ell^2(\mathbb{Z}^n) \longrightarrow L^2(\mathbb{T}^n, \mathcal{H}) \tag{42}$$

which sends $\{f \in C^\infty(\mathbb{R}^n) : \sum_{\gamma \in \mathbb{Z}^n} |f(\gamma x)|^2 < \infty \ \forall x \in \mathbb{R}^n\}$ to $C^\infty(\mathbb{T}^n, \mathcal{H})$.

3.2. Pseudodifferential operators on bundles of Hilbert modules

We describe pseudodifferential operators on bundles of Hilbert modules, following [3, 2.2-4].

Let (M, g) be a closed Riemannian manifold of dimension n and let $p\colon \mathcal{E} \to M$ be a bundle of finitely generated projective $\mathcal{A}$-Hilbert modules with fibre $\mathcal{W}$.
A linear operator $A\colon C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ is a (classical) pseudodifferential $\mathcal{A}$-operator of order $a \in \mathbb{C}$ if in some atlas of $\mathcal{E} \to M$ it is of the form $A = \sum_{j=1}^J A_j + R$ where

- $R\colon C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ is a smoothing operator, namely with Schwartz kernel given by a smooth section of the bundle $\mathcal{L}_\mathcal{A} \to M \times M$ of $\mathcal{A}$-linear bounded operators whose fibre at $(x, y) \in M \times M$ is the Banach space of $\mathcal{A}$-linear operators from the fibre $\mathcal{E}_y$ to the fibre $\mathcal{E}_x$,

- the operators A_j are properly supported operators (meaning that the canonical projections $M \times M \to M$ restricted to the support of the Schwartz kernel are proper maps) from $C^\infty(M, \mathcal{E})$ into itself. This implies in particular that the A_j send $C_0^\infty(M, \mathcal{E})$ into itself. In any coordinate chart A_j is therefore of the form given in (7), for some symbol $\sigma \in C^\infty(U \times \mathbb{R}^n, \mathrm{End}_{\mathcal{A}}\mathcal{E})$ which is asymptotically polyhomogeneous at infinity.

For $a \in \mathbb{C}$, let $\Psi^a(M, \mathcal{E})$ be the set of classical pseudodifferential operators of order a; then $\Psi^{-\infty}(M, \mathcal{E}) := \cap_a \Psi^a(M, \mathcal{E})$ corresponds to the smoothing operators, and $\Psi(M, \mathcal{E}) := \cup_a \Psi^a(M, \mathcal{E})$. The subalgebra of differential operators will be denoted $\Theta(M, \mathcal{E})$.

Remark 35. Going back to the example of the n-torus $\mathbb{T}^n$ and its $\mathbb{Z}^n$-covering $\pi : \mathbb{R}^n \to \mathbb{T}^n$, we know that $\mathbb{Z}^n$-invariant (i.e., $A \circ t_a^* = t_a^* \circ A$ for any $a \in \mathbb{Z}^n$ with $t_a(x) = x + a$) differential operators on $\mathbb{R}^n$ give rise to elements of $\Theta(M, \mathcal{H})$. This raises the question how the algebra $\Psi(\mathbb{T}^n, \mathcal{H})$ relates to the algebra of $\mathbb{Z}^n$-invariant classical pseudodifferential operators on $\mathbb{R}^n$ studied by Ruzhanski and Turunen in [33] or equivalently the $\theta = 0$ instance of the algebra $\Psi(\mathbb{T}_\theta^n)$ described in Section 4.

For any $A \in \Psi^a(M, \mathcal{E})$, $a \in \mathbb{R}$, A defines a bounded linear map on the H^s-Sobolev closure $H^s(M, \mathcal{E})$ of $C^\infty(M, \mathcal{E})$ with values in $H^{s-a}(M, \mathcal{E})$, for any $s \in \mathbb{R}$. We refer the reader to [3, §2.2] for Sobolev closures and to [3, Prop. 2.7] for the property in the case $s = a$.

The notions of admissibility (Definition 3) and Agmon angle (Lemma 6) carry out from the closed manifold case to the setup of pseudodifferential operators on bundles of von Neumann Hilbert modules, taking into account that here the spectrum is not necessarily purely discrete.

Definition 36. Let A be an operator in $\Psi^a(M, \mathcal{E})$. For an angle β and for $\epsilon > 0$, denote $V_{\beta,\epsilon} := \{z \in \mathbb{C} \ : |z| < \epsilon\} \cup \{z \in \mathbb{C} \setminus 0 \ : \ \arg z \in (\beta - \epsilon, \beta + \epsilon)\}$. Then β is called an **Agmon angle** for A if there is some $\epsilon > 0$ such that $\mathrm{sp}(A) \cap V_{\beta,\epsilon} = \emptyset$, where $\mathrm{sp}(A)$ stands for the spectrum of A.

In particular an operator with Agmon angle is elliptic and invertible.

Definition 37. We call **weight** an operator in $\Psi(M, \mathcal{E})$ of positive order that admits an Agmon angle. We call **admissible** an operator $A \in \Psi(M, \mathcal{E})$ which is a weight modulo a smoothing perturbation, i.e., if it has positive order, and there exists a smoothing operator R such that $A + R$ has an Agmon angle.

The constructions of complex powers and the logarithm recalled in Section 1.2 extend word for word to the setting of Hilbert modules.

Let A be a weight in $\Psi^a(M, \mathcal{E})$ with spectral cut β. Then for $\Re(z) > 0$, its complex powers (see [3, (2.8)] for further details, and compare with (12))

$$A_\beta^z = \frac{i}{2\pi} \int_{\Gamma_\beta} \lambda^z (\lambda - A)^{-1} \, d\lambda, \tag{43}$$

and respectively the operators (as in (13))

$$L_\beta(A,z) = \frac{i}{2\pi}\int_{\Gamma_\beta} \log_\beta \lambda \;\; \lambda^z\,(\lambda - A)^{-1}\, d\lambda, \tag{44}$$

are well defined bounded linear maps from $H^s\,(M,\mathcal{E})$, $s \in \mathbb{R}$, with values in $H^{s-a\Re(z)}\,(M,\mathcal{E})$ respectively in $H^{s-a\Re(z)+\epsilon}\,(M,\mathcal{E})$, for any $\epsilon > 0$. Here Γ_β is a closed contour in $\mathbb{C}\setminus\{re^{i\beta},\ r\geq 0\}$ around the spectrum of A oriented clockwise. These definitions extend to the whole plane inductively on $k\in\mathbb{N}$ setting

$$A^z_\beta := A^k\ A^{z-k}_\beta \quad, \text{ respectively } \ L_\beta(A,z) := A^k\ L_\beta(A, z-k), \text{ for } \ \Re(z) < k.$$

A^z_β is an operator in $\Psi(M,\mathcal{E})$ of order az for any complex number z, and the logarithm

$$\log_\beta(A) := L_\beta(A,0)$$

of A is a bounded linear map from $H^s\,(M,\mathcal{E})$ to $H^{s+\epsilon}\,(M,\mathcal{E})$, $\forall\epsilon > 0$. One has by construction $\log_\beta A\, A^z_\beta = A^z_\beta\, \log_\beta A$, $\forall z\in\mathbb{C}$.

Remark 38. Using the same convention as in Remark 7, we usually drop the explicit mention of the dependence of β.

In a local trivialisation, the symbol of $\log_\beta A$ reads (see, e.g., [36])

$$\sigma(\log_\beta A)(x,\xi) = a\,\log|\xi| I + \sigma_{\mathrm{cl}}(\log_\beta A)(x,\xi) \tag{45}$$

where a denotes the order of A and $\sigma_{\mathrm{cl}}(\log A)$ is a classical symbol of order zero with homogeneous components $\sigma_{-j}(\log A)$ of degree $-j, j\in\mathbb{Z}_{\geq 0}$.

3.3. The extended τ-Wodzicki residue

Mimicking the definition of the pointwise residue (10) on closed manifolds, for any $A\in\Psi\,(M,\mathcal{E})$ with symbol $\sigma(A)$ in a local chart of M at a point x, we call

$$\mathrm{res}^\tau_x(A) := \int_{|\xi|_x=1} \mathrm{tr}^\tau(\sigma_{-n}(A))(x,\xi)\, đ_S\xi, \tag{46}$$

the pointwise τ-Wodzicki residue of A. As in the closed manifold case, one proves that $\mathrm{res}^\tau_x(\sigma(A))\,dx$ defines a global density on M [39], so we can define the τ-Wodzicki residue of A (compare with (11))

$$\mathrm{Res}^\tau(A) = \int_M \mathrm{res}^\tau_x(A)\,dx = \int_M dx \int_{|\xi|_x=1} \mathrm{tr}^\tau\left(\sigma_{-n}(A)(x,\xi)\right)\, đ_S\xi. \tag{47}$$

Remark 39. Definition (46) relates to the definitions of the Wodzicki residue by Benameur–Fack [2, Def. 9] and Vassout [38] in the context of measured foliations.

The **pointwise Wodzicki τ-residue** extends to the logarithm $\log A$ of an admissible invertible operator $A\in\Psi(M,\mathcal{E})$. At a point x we set

$$\mathrm{res}^\tau_x(\log A) := \int_{|\xi|_x=1} \mathrm{tr}^\tau(\sigma_{-n}(\log A))(x,\xi)\, đ_S\xi, \tag{48}$$

Mimicking the definition (18), for any admissible operator $A \in \Psi(M, \mathcal{E})$, we set

$$\mathrm{res}_x^\tau(\log A) := \mathrm{res}_x^\tau(\log(A+R)) \quad \forall x \in M, \tag{49}$$

where R is any smoothing operator such that $A+R$ has an Agmon angle.

In the subsequent paragraph, we show as in the closed manifold case, that $\mathrm{res}_x^\tau(\log A)\, dx$ defines a global density.

3.4. The τ-Wodzicki residue as a complex residue

For holomorphic families of operators in $\Psi(M, \mathcal{E})$, whose explicit definition we omit here since they are defined in the same manner as in the closed case (see Definition 14), we give the Hilbert-module counterpart of Theorem 15. For this we need as in the case of operators on closed manifolds, the cut-off integral $-\!\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}_x^\tau \sigma(x,\xi)\, đ\xi$ of a local symbol σ of a classical operator, which is defined in the same way as in (19), only replacing the fibrewise trace by tr^τ.

Proposition 40. *For any holomorphic family $A(z) \in \Psi(M, \mathcal{E})$ of classical operators parametrised by $\mathbb{C}$, with local symbols $\sigma(z)$ and holomorphic order $-qz + a$ for some positive q and some real number a,*

1. *the meromorphic map $z \mapsto \mathrm{TR}_x^\tau(A(z)) := -\!\!\!\!\!\!\int_{\mathbb{R}^n} \mathrm{tr}^\tau\left(\sigma(z)(x,\xi)\right) đ\xi$ integrates over M to the map*

$$z \mapsto \mathrm{TR}^\tau\left(A(z)\right) := \int_M \mathrm{TR}_x^\tau(A(z))\, dx$$

   which is meromorphic with simple poles $d_j := \frac{a+n-j}{q}, j \in \mathbb{Z}_{\geq 0}$.
2. [20] *The complex residue at the point d_j is given by:*

$$\mathrm{Res}_{z=d_j} \mathrm{TR}^\tau\left(A(z)\right) = \frac{1}{q} \mathrm{Res}^\tau(A(d_j)). \tag{50}$$

3. [29] *If $A(d_j)$ lies in $\Theta(M, E)$, i.e., if it is a differential operator, then $A'(d_j)$ which need not be a classical pseudodifferential operator, nevertheless has a well-defined Wodzicki residue*

$$\mathrm{Res}^\tau(A'(d_j)) := \int_M \mathrm{res}_x^\tau\left(A'(d_j)\right) dx$$

   where

$$\mathrm{res}_x^\tau\left(A'(d_j)\right) := \int_{|\xi|_x=1} \mathrm{tr}^\tau\left(\sigma_{-n}\left(A'(d_j)\right)\right)(x,\xi)\, đ_S\xi$$

   at the pole d_j and we have

$$\mathrm{fp}_{z=d_j} \mathrm{TR}^\tau\left(A(z)\right) = \frac{1}{q} \mathrm{Res}^\tau(A'(d_j)). \tag{51}$$

As in Theorem 23 we deduce the asymptotic expansion of the inverse Mellin transform of traces $\mathrm{TR}^\tau(A(z))$ of holomorphic families $A(z)$. Substituting TR^τ to TR in Theorem 23 tells us that, if f is continuous on $]0, +\infty[$ with Mellin transform $z \mapsto \mathrm{TR}^\tau\left(A(z)\right)$ for some holomorphic family $A(z) \in \Psi(M, \mathcal{E})$ of affine order $\alpha(z) = a - qz$ with q some positive real number, then f admits an asymptotic expansion at 0 given by

$$f(t) = \frac{1}{q} \sum_{j \geq 0} a_j t^{-d_j} + O(t^{-\gamma}),$$

with

$$a_j = -\frac{1}{q} \mathrm{Res}^\tau \left(A(d_j)\right) \quad \text{for} \quad d_j > 0, \tag{52}$$

and constant term

$$a_j = -\frac{1}{q} \mathrm{Res}^\tau \left(A'(0)\right) \quad \text{for} \quad d_j = 0. \tag{53}$$

Theorem 17 extends in a straightforward manner.

Theorem 41. *Given a weight $Q \in \Psi(M, \mathcal{E})$ of order $q \in \mathbb{R}_+$ and any operator $A \in \Psi(M, \mathcal{E})$ (so not necessarily admissible) of order $a \in \mathbb{R}$, the map*

$$z \mapsto \zeta^\tau(A, Q)(z) := \mathrm{TR}^\tau(AQ^{-z}) \tag{54}$$

is holomorphic on the half-plane $\Re(z) > \frac{n+a}{q}$ and defines a meromorphic map on $\mathbb{C}$ called the ζ^τ-regularised trace of A with respect to the weight Q, with poles at $d_j = \frac{a+n-j}{q}$, $j \in \mathbb{Z}_{\geq 0}$. The complex residue at such a pole is related to the Wodzicki τ-residue of AQ^{-d_j} by

$$\mathrm{Res}_{z=d_j} \zeta^\tau(A, Q)(z) = \frac{1}{q} \mathrm{Res}^\tau(AQ^{-d_j}).$$

For any differential operator $A \in \Theta(M, \mathcal{E})$, the n-form $\mathrm{res}_x^\tau(A \log Q)\, dx$ defines a global density on M which integrates to the extended Wodzicki τ-residue of $A \log Q$. The ζ^τ-regularised trace $\zeta^\tau(A, Q)(z)$ is holomorphic at zero and we have

$$\zeta^\tau(A, Q)(0) = \lim_{z \to 0} \zeta^\tau(A, Q)(z) = -\frac{1}{q} \mathrm{Res}^\tau(A \log Q). \tag{55}$$

Remark 42. Let A, Δ be differential operators in $\Theta(M, \mathcal{E})$ with Δ admissible. Then exactly as in Proposition 8 one can prove that the pointwise extended Wodzicki residue $\mathrm{res}_x^\tau \left(A \log \Delta\right)$ is an algebraic expression in the coefficients of A and in the x-jets of the coefficients of Δ at that point.

3.5. The τ-index as an extended τ-residue

The above constructions extend to $\mathbb{Z}_2$-graded vector bundles. Let $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ be a $\mathbb{Z}_2$-graded bundle of finite type $\mathcal{A}$-Hilbert modules over M and let

$$D_\pm \colon C^\infty\left(M, \mathcal{E}_\pm\right) \longrightarrow C^\infty\left(M, \mathcal{E}_\mp\right)$$

be two elliptic differential operators of positive order d. We assume that the operators D_+ and D_- are formally adjoint to each other which we write $D_- = D_+^*$. Hence

$$D := \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

is essentially selfadjoint. Let $\Delta := D^2 = \begin{bmatrix} D_-D_+ & 0 \\ 0 & D_+D_- \end{bmatrix} = \Delta_+ \oplus \Delta_-$. The ellipticity of D implies that the projection π_Δ onto $\operatorname{Ker}\Delta = \operatorname{Ker} D$ is a smoothing operator of finite τ-rank [3, 34]. Therefore one defines the τ-dimension of the $\mathcal{A}$-Hilbert modules $\operatorname{Ker} D_\pm$ as

$$\dim_\tau(\operatorname{Ker} D_\pm) := \tau(\pi_{\Delta_\pm}) \in \mathbb{R}$$

and the difference

$$\operatorname{ind}^\tau D_+ := \dim_\tau(\operatorname{Ker} D_+) - \dim_\tau(\operatorname{Ker} D_-) \in \mathbb{R} \tag{56}$$

is called the **τ-index** of the operator D. In the case $\mathcal{A} = \mathbb{C}$ this is the usual definition of the Fredholm index of the operator.

Remark 43. There exist smoothing perturbations R of the nonnegative selfadjoint operator D^2 such that D^2+R is invertible. Since we are in a von Neumann algebraic setting, this follows for example from [22, Proposition 2.10], see also [23].

Therefore there exist operators R, R' such that $D_-D_+ + R$ and $D_+D_- + R'$ are elliptic and nonnegative and hence so are their leading symbols nonnegative. Thus $\Delta, \Delta_+, \Delta_-$ define admissible operators with spectral cut π.
Consequently, we can define the **pointwise extended super τ-residue** of $\log\Delta$ as the difference of the pointwise extended residues of $\log\Delta_+$ and $\log\Delta_-$

$$\operatorname{sres}_x^\tau(\log\Delta)(x) := \operatorname{res}_x^\tau(\log(\Delta_+ + R')) - \operatorname{res}_x^\tau(\log(\Delta_- + R)).$$

It can be integrated over M to build the **extended super τ-residue**

$$\operatorname{sRes}^\tau(\log\Delta) := \frac{1}{(2\pi)^n}\int_M \operatorname{sres}_x^\tau(\log\Delta)(x)\,dx.$$

Corollary 44. *The τ-index of D_+ is a local expression proportional to the extended Wodzicki (super) τ-residue of the logarithm of Δ*

$$\operatorname{ind}^\tau(D_+) = -\frac{1}{2d}\operatorname{sRes}^\tau(\log(\Delta)). \tag{57}$$

Proof. The McKean–Singer formula combined with a Mellin transform yields for any positive real number t and for any complex number z

$$\begin{aligned}\operatorname{ind}^\tau(D_+) &= \operatorname{Tr}^\tau\left(e^{-t(\Delta_+ + \pi_{\Delta_+})}\right) - \operatorname{Tr}^\tau\left(e^{-t(\Delta_- + \pi_{\Delta_-})}\right)\\ &= \zeta^\tau_{\Delta_+ + \pi_{\Delta_+}}(z) - \zeta^\tau_{\Delta_- + \pi_{\Delta_-}}(z)\\ &= -\frac{1}{2d}\left(\operatorname{Res}^\tau\left(\log\left(\Delta_+ + \pi_{\Delta_+}\right)\right) - \operatorname{Res}^\tau\left(\log\left(\Delta_- + \pi_{\Delta_-}\right)\right)\right)\\ &= -\frac{1}{2d}\operatorname{sRes}^\tau(\log\Delta).\end{aligned}$$

□

3.6. The extended residue for locally equivalent operators and Atiyah's L^2-index theorem

In the following, denote by $(U;\mathcal{E},\mathcal{F})$ a triple where U is a manifold and $\mathcal{E},\mathcal{F}$ are bundles of finitely generated projective $\mathcal{A}$-Hilbert modules over U. Morphisms between these objects are of the form $\alpha=(f;r,s)\colon (U';\mathcal{E}',\mathcal{F}')\to(X;\mathcal{E},\mathcal{F})$ where $f\colon U'\to U$ is an **open embedding**, $r\in \mathrm{Hom}(\mathcal{E}',f^*\mathcal{E})$, $s\in\mathrm{Hom}(f^*\mathcal{F},\mathcal{F}')$.

Given any linear map $L\colon C_0^\infty(U,\mathcal{E})\to C^\infty(U,\mathcal{F})$, a morphism

$$\alpha\colon (U';\mathcal{E}',\mathcal{F}')\to(U;\mathcal{E},\mathcal{F})$$

defines a map $\alpha^\sharp L\colon C_0^\infty(U',\mathcal{E}')\to C^\infty(U',\mathcal{F}')$ that makes the following diagram commute

$$\begin{tikzcd}
C_0^\infty(U,\mathcal{E}) \arrow[r, "L"] & C^\infty(U,\mathcal{F}) \arrow[d, "\alpha^*"] \\
C_0^\infty(U',\mathcal{E}') \arrow[u, "\alpha_*"] \arrow[r, "\alpha^\sharp L"] & C^\infty(U',\mathcal{F}')
\end{tikzcd}$$

where α^* denotes the composition of s with the map induced by the pullback, and α_* denotes the composition of r with the push-forward.

Definition 45. Let M,M' be two manifolds and $A\in\Theta\left(M,\mathcal{E}\right), A'\in\Theta\left(M',\mathcal{E}'\right)$ be **differential operators** acting on the sections of $\mathcal{A}$-Hilbert modules bundles $\mathcal{E},\mathcal{E}'$ over M and M' respectively. The operators A and A' are said to be *locally equivalent* if

- there exists a local diffeomorphism $\phi\colon M'\to M$ meaning that for any x' in M' there is a neighborhood U' of x' such that $U=\phi(U')$ is open and $\phi_{U'}^U\colon U'\to U$ is a diffeomorphism
- correspondingly, there are morphisms

$$\alpha=(f;r,s)\colon (U',\mathcal{E}'_{|U'},\mathcal{E}'_{|U'})\to(U,\mathcal{E}_{|U},\mathcal{E}_{|U})$$

with $f=\phi_{U'}^U$, and with r,s isomorphisms such that

$$A'_{U'}=\alpha^\sharp(A_U) \tag{58}$$

where we have denoted by $A'_{U'}$ the restriction of A' to the open set U'.

Example 46 (Twists by flat bundles of Hilbert modules). Let M be closed, $E\to M$ be a vector bundle, and $B\in\Theta(M,E)$ be a differential operator. Let $\mathcal{F}\to M$ be a flat bundle of finitely generated projective $\mathcal{A}$-Hilbert modules, endowed with a flat connection $\nabla_{\mathcal{F}}$. Denote by $\mathcal{W}$ the fibre. Let $\underline{\mathcal{W}}=M\times\mathcal{W}$ denote the trivial bundle with fibre $\mathcal{W}$. Because B is differential, one can consider on the one hand $B_{\underline{\mathcal{W}}}$ to be the trivial extension of B to $E\otimes\underline{\mathcal{W}}$ and on the other hand $B_{\mathcal{F}}$ the operator B twisted by the flat connection on $\mathcal{F}$. Then $A=B_{\underline{\mathcal{W}}}$ on $\mathcal{E}'=E\otimes\underline{\mathcal{W}}$ and $A'=B_{\mathcal{F}}$ on $\mathcal{E}'=E\otimes\mathcal{F}$ are locally equivalent by taking π the identity map on M, and the local morphisms given by local trivialisations of $\mathcal{F}$ which are parallel with respect to $\nabla_{\mathcal{F}}$.

Example 47. As a particular case of the above (with $\mathcal{W} = \ell^2(\Gamma)$), let $\mathcal{H} = \widetilde{M} \times_\Gamma \ell^2(\Gamma)$ be the bundle defined in Example 33, and $\underline{\ell^2\Gamma} = M \times \ell^2(\Gamma)$. If $A \in \Theta(M, E)$ is a differential operator, $A_{\mathcal{H}}$ is locally equivalent to $A_{\underline{\ell^2\Gamma}}$.

Example 48. Specialising to the torus and using the notations of Example 34, for any differential operator A on $\mathbb{T}^n$, the local equivalence between $A_{\mathcal{H}}$ and $A_{\underline{\ell^2\Gamma}}$ translates to the well-known local equivalence of the lifted operator $\pi^\sharp A$ (with a slight abuse of notation) on $\mathbb{R}^n$ with A.

Proposition 49. *Let A, A' and B, B' be pairs of locally equivalent differential operators in the sense of Definition* 45*. Moreover assume that B, B' are admissible. Then with the notation of Definition* 45

$$f^* \left(\mathrm{res}_x^\tau \left(A \log B\right)\, dx\right) = \mathrm{res}_{x'}^\tau \left(A' \,\log\left(B'\right)\right)\, dx', \tag{59}$$

$$f^* \left(\mathrm{res}_x^\tau \left(A\, B^\alpha\right)\, dx\right) = \mathrm{res}_{x'}^\tau \left(A' \,\left(B'\right)^\alpha\right)\, dx' \ \ , \alpha \in \mathbb{R} \tag{60}$$

for any point x' and any local diffeomorphism $f = \phi_{U'}^U : U' \to U$ on an open subset U' containing x', which when integrated over M' yields

$$\mathrm{Res}^\tau \left(A \log B\right) = \mathrm{Res}^\tau (A' \log\left(B'\right)) \tag{61}$$

$$\mathrm{Res}^\tau \left(AB^\alpha\right) = \mathrm{Res}^\tau \left(A' \left(B'\right)^\alpha\right), \ \ \alpha \in \mathbb{R}. \tag{62}$$

Proof. Since B and B' are admissible, there are smoothing operators R, R' such that $Q = B + R$ and $Q = B' + R'$ are weights. From Theorem 41 applied to the operator A in $\Theta(M, \mathcal{E})$, we know that the n-form $x \mapsto \mathrm{res}_x^\tau \left(A \log B\right) dx$ defines a global density so it transforms covariantly under coordinate transformations. Thus, the pull-back of this density by the local diffeomorphism $f = \phi_{U'}^U : U' \to U$ reads

$$f^* \left(\mathrm{res}_x^\tau \left(A \log B\right)\, dx\right) = \mathrm{res}_{x'}^\tau (f^\sharp A \,\log f^\sharp B)\, dx',$$

where $x' = f^{-1}(x)$. Now $A' = f^\sharp A$ and $B' = f^\sharp B$, so we get the result. $\square$

Specialising to (essentially) **selfadjoint** elliptic differential operators provides an alternative proof of Atiyah's L^2-index theorem in the Hilbert module formulation.

Corollary 50 (Atiyah's L^2-index theorem). *Let D be an essentially selfadjoint differential operator of positive order d acting on a $\mathbb{Z}_2$-graded vector bundle $E^+ \oplus E^- \to M$, and assume D is odd with respect to the grading, i.e., $D := \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$. Let $D_{\mathcal{H}} := \begin{bmatrix} 0 & D_{\mathcal{H},-} \\ D_{\mathcal{H},+} & 0 \end{bmatrix}$ be the twisted operator defined in Example* 47 *acting on the bundle of $\mathcal{N}\Gamma$-Hilbert modules $E \otimes \mathcal{H}$. Then*

$$\mathrm{ind}^\tau (D_{\mathcal{H},+}) = \mathrm{ind}(D_+) \ .$$

Proof. Recall that by Example 46 the operator $D^2_{\mathcal{H}}$ is locally equivalent to the trivial extension $D_{\underline{\ell^2\Gamma}}$ acting on $\underline{\ell^2\Gamma} = M \times \ell^2(\Gamma)$. Using Remark 43, $D^2_{\mathcal{H}}$ is admissible (with π as a spectral cut), so we can apply Proposition 49 which yields[10]

$$\mathrm{sRes}^\tau(\log(D^2_{\mathcal{H}})) = \mathrm{sRes}^\tau(\log(D^2_{\underline{\ell^2\Gamma}}))\ .$$

By formula (57)

$$\mathrm{ind}^\tau(D_{\mathcal{H},+}) = -\frac{1}{2d}\mathrm{sRes}^\tau(\log(D^2_{\mathcal{H}}))\ .$$

Analogously,

$$\mathrm{ind}(D_+) = \mathrm{ind}^\tau(D_{\underline{\ell^2\Gamma},+}) = -\frac{1}{2d}\mathrm{sRes}^\tau(\log(D^2_{\underline{\ell^2\Gamma}}))$$

where we have used that $\mathrm{ind}^\tau(D_{\underline{\ell^2\Gamma},+}) = \mathrm{ind}(D_+)$, so that the equality follows. □

Remark 51. This is a self-contained pseudodifferential proof of Atiyah's result, which relies on the locality of the Wodzicki residue. It is very similar in spirit to John Roe's proof on coverings [31].

Let D be an essentially selfadjoint differential operator acting on a vector bundle $E \to M$. Let $D_{\mathcal{H}}$ be the twisted operator defined in Example 47.

Since $D^2_{\mathcal{H}}$ is admissible (see Remark 43), the operator $Q_{\mathcal{H}} := D^2_{\mathcal{H}} + R$ defines a weight. Let us consider for any positive t the associated heat-operator $e^{-tQ_{\mathcal{H}}}$. The corresponding heat-kernel τ-trace $K^\tau_t\left(Q_{\mathcal{H}}\right)(x,x)$ at a point x is defined by

$$\mathrm{Tr}^\tau\left(\phi\, e^{-tQ_{\mathcal{H}}}\right) := \int_M \phi(x)\, K^\tau_t(Q_{\mathcal{H}})(x,x)\, dx \quad \forall \phi \in C^\infty(M)\ .$$

Applying as in Section 2 an inverse Mellin transform to the holomorphic families $A(z) = \phi\, Q_{\mathcal{H}}^{-z}$ and using Proposition 49 we find that

$$\begin{aligned}
K^\tau_t\left(D^2_{\mathcal{H}}\right)(x,x) \sim\ & -\frac{(4\pi)^{\frac{n}{2}}}{2\sqrt{\det g}(x)}\Bigg[\mathrm{res}^\tau_x\left(\log D^2_{\mathcal{H}}\right)\,\delta_{\frac{n}{2}-\left[\frac{n}{2}\right]} \\
& + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\ \mathrm{res}^\tau_x\left((D^2_{\mathcal{H}})^{k-\frac{n}{2}}\right)\, t^{k-\frac{n}{2}}\Bigg] \\
\sim\ & -\frac{(4\pi)^{\frac{n}{2}}}{2\sqrt{\det g}(x)}\Bigg[\mathrm{res}_x\left(\log D^2\right)\,\delta_{\frac{n}{2}-\left[\frac{n}{2}\right]} \\
& + \sum_{k\in\left[0,\frac{n}{2}\right[\cap\mathbb{Z}} \Gamma\left(\frac{n}{2}-k\right)\ \mathrm{res}_x\left((D^2)^{k-\frac{n}{2}}\right)\, t^{k-\frac{n}{2}}\Bigg].
\end{aligned} \tag{63}$$

Remark 52. It follows from the Duhamel formula [4] that the time zero asymptotics of $K^\tau_t\left(D^2_{\mathcal{H}}\right)(x,x)$ coincide with that of $K^\tau_t\left(D^2_{\mathcal{H}}+R\right)(x,x)$. This is here confirmed by the fact that the residues involved in the asymptotics are invariant under perturbation by the smoothing operator R.

[10]Prop. 49 easily extends to the $\mathbb{Z}_2$-graded case replacing the τ-residue by the super τ-residue.

4. The scalar curvature on the noncommutative two-torus

We want to define the scalar curvature on the noncommutative two-torus by means of a Wodzicki residue in analogy to the formula (40) established for Riemannian surfaces. We need a noncommutative analogue of Theorem 17 on the noncommutative torus $\mathbb{T}^n_\theta$. Let us briefly recall the results of [24] we need for that purpose.

4.1. Pseudodifferential operators on the noncommutative torus

Let θ be a symmetric $n \times n$ real matrix. The noncommutative deformation $\mathbb{T}^n_\theta$ of the commutative torus $\mathbb{T}^n \sim \mathbb{R}^n/\mathbb{Z}^n$ is encoded in the C^*-algebra A_θ. An element $a \in A_\theta$ decomposes as the convergent series $a = \sum_{k\in\mathbb{Z}^n} a_k U_k$ where the (U_k) are unitaries in A_θ that satisfy $U_0 = 1$ and

$$U_k U_l = e^{-2\pi i\,\langle k,\theta l\rangle} U_l U_k.$$

Let $\mathcal{A}_\theta$ denote the algebra consisting of series of the form $\sum_{k\in\mathbb{Z}^n} a_k U_k$, where the sequence $(a_k)_k \in \mathcal{S}(\mathbb{Z}^n)$, the vector space of sequences $(a_k)_k$ that decay faster than the inverse of any polynomial in k. We shall also need the linear form $\mathbf{t}$ on A_θ which to an element $a = \sum_{k\in\mathbb{Z}^n} a_k U_k$ assigns the scalar term a_0, and the Laplace operator $\boldsymbol{\Delta} = \sum_j \delta_j^2$ defined in [24, Example 3.13] acting on A_θ with $\delta_j \left(\sum_{k\in\mathbb{Z}^n} a_k U_k\right) = \sum_{k\in\mathbb{Z}^n} k_j\, a_k U_k$.

We refer to [24] for the construction of the corresponding algebra $\Psi(\mathbb{T}^n_\theta)$ of classical toroidal pseudodifferential operators [24, Paragraph 3.2] on $\mathbb{T}^n_\theta$. When $\theta = 0$, the noncommutative torus $\mathbb{T}^n_\theta$ coincides with $\mathbb{T}^n$, $\mathcal{A}_\theta$ with $C^\infty(\mathbb{T}^n)$ and $\Psi(\mathbb{T}^n_\theta)$ with the algebra $\Psi(\mathbb{T}^n)$ of classical pseudodifferential operators on the closed manifold $\mathbb{T}^n$ considered in the first section.

4.2. Holomorphic families of operators on the noncommutative torus

In [24] we defined holomorphic families in $\Psi(\mathbb{T}^n_\theta)$ and extended the canonical trace to such families by

$$\mathrm{TR}_\theta(A(z)) := \sum_{\mathbb{Z}^n} \mathbf{t}\left(\mathrm{Op}_\theta^{-1}(A(z))\right)$$

where Op_θ is the one to one map which takes a toroidal symbol to a toroidal operator. As seen in [24, Proposition 6.2], the Wodzicki residue Res_θ on $\mathcal{A}_\theta$, is a noncommutative analogue of the classical Wodzicki residue and (up to a multiplicative factor) it is the only continuous linear form on $\Psi(\mathbb{T}^n_\theta)$ vanishing on smoothing operators [14] (see also [24] for a slightly different characterisation which does not require continuity). In contrast to this, the canonical trace TR_θ is (up to a multiplicative factor) the only linear form on non integer operators in $\Psi(\mathbb{T}^n_\theta)$ whose restriction to trace-class operators is continuous [24].

Theorem 53. *For any holomorphic family $A(z) \in \Psi(\mathbb{T}^n_\theta)$ of classical operators parametrised by $\mathbb{C}$ with holomorphic order $-qz + a$ for some positive q and some real number a,*

1. *the map $z \mapsto \mathrm{TR}_\theta\,(A(z))$ is meromorphic with simple poles $d_j := \frac{a+n-j}{q}$, $j \in \mathbb{Z}_{\geq 0}$,*

2. *the complex residue at the point* d_j *is given by:*

$$\mathrm{Res}_{z=d_j} \mathrm{TR}_\theta\,(A(z)) = \frac{1}{q}\mathrm{Res}_\theta(A(d_j)). \tag{64}$$

3. *If* $A(d_j)$ *is a differential operator, then* $A'(d_j)$ *has a well-defined Wodzicki residue* $\mathrm{Res}_\theta(A'(d_j))$ *and we have*

$$\mathrm{fp}_{z=d_j} \mathrm{TR}\,(A(z)) = \frac{1}{q}\mathrm{Res}_\theta(A'(d_j)). \tag{65}$$

It was shown in [24] how one can define via a Cauchy formula the logarithm [24, Paragraph 7] $\log(\mathbf{\Delta})$ of the Laplace operator and a noncommutative analogue $\zeta_\theta(A,Q)$ [24, Paragraph 7] of the Q-regularised ζ-trace of A with $Q := 1+\mathbf{\Delta}$. Applying Theorem 53 to the holomorphic family $A(z) = A\,Q^{-z}$ yields the following extension of Theorem 17 to the noncommutative torus.

Theorem 54. *Let* $Q := 1+\mathbf{\Delta}$ *and let* A *be a* differential operator *in* $\Psi(\mathbb{T}^n_\theta)$. *Then*

1. *the* ζ_θ*-regularised trace* $\zeta_\theta(A,Q)$ *of* A *is holomorphic at zero,*
2. *the residue* Res_θ *extends to* $A\log Q$ *and we have*

$$\zeta_\theta(A,Q)(0) = -\frac{1}{q}\mathrm{Res}_\theta(A\,\log Q). \tag{66}$$

4.3. The scalar curvature as an extended Wodzicki residue

We now want to define the scalar curvature on $\mathbb{T}^2_\theta$ by means of a noncommutative analogue of (40). We work on a conformal deformation of the complexified two torus using the notational conventions of [12].

Let $\tau = \tau_1 + i\tau_2$ with $\tau_1,\tau_2 \in \mathbb{R}$. Let $\partial = \delta_1 + \bar\tau\delta_2$. Then $\partial^* = \delta_1 + \tau\delta_2$. The operators ∂_τ and ∂_τ^* are the noncommutative counterparts of $-i\,(\partial_{x_1} + \bar\tau\partial_{x_2})$ and $-i\,(\partial_{x_1} + \tau\partial_{x_2})$ acting on $\mathcal{A}_0 = C^\infty(\mathbb{T}^2)$. Let $h \in \mathcal{A}_\theta$ be selfadjoint and set $k = e^{\frac{h}{2}}$. Let $\mathcal{H}_h$ be the completion of $\mathcal{A}_\theta$ for the inner product

$$\langle a,b\rangle_h = \mathfrak{t}(b^* a k^{-2}) = \langle a k^{-1}, b k^{-1}\rangle_0 = \langle R_{k^{-2}} a, b\rangle_0$$

on $\mathcal{A}_\theta$ where $R_b : a \mapsto a\,b$ stand for the right multiplication by b. The map $R_k : a \mapsto a\,k$ induces an isometry $U : \mathcal{H}_0 \longrightarrow \mathcal{H}_h$.

The analogue of the space of $(1,0)$-forms on the ordinary two-torus is defined to be the Hilbert space completion $\mathcal{H}_0^{(1,0)}$ of the space of finite sums $a\partial b,\, a,b \in \mathcal{A}_\theta$ for the inner product $\langle a,b\rangle_0$. We view ∂ as an unbounded operator $\partial_h : \mathcal{H}_h \longrightarrow \mathcal{H}_0^{(1,0)}$. Then

$$\langle \partial a, b\rangle_0 = \langle a, \partial^* b\rangle_0 = \langle R_{k^2} a, \partial^* b\rangle_h = \langle a, R_{k^2}\partial^* b\rangle_h.$$

Thus its formal adjoint is given by

$$\partial_h^* = R_{k^2}\partial^*.$$

We consider the operator

$$D_h = \begin{bmatrix} 0 & \partial_h^* \\ \partial_h & 0 \end{bmatrix} = \begin{bmatrix} 0 & R_{k^2}\partial^* \\ \partial & 0 \end{bmatrix}$$

acting on the $\mathbb{Z}_2$- graded space $\widetilde{\mathcal{H}_h} := \mathcal{H}_h \oplus \mathcal{H}_0^{(1,0)}$. Let

$$\boldsymbol{\Delta}_h := \partial_h^* \partial_h + \partial_h \partial_h^* = R_{k^2}\, \partial^* \partial + \partial R_{k^2} \partial^*,$$

which for τ_i and $h = 0$ coincides with $\boldsymbol{\Delta}$.

Here is a corollary of Theorem 17 extended to this slightly more general framework.

Corollary 55. *Let $Q := \boldsymbol{\Delta}_h + \pi_{\boldsymbol{\Delta}_h}$, where $\pi_{\boldsymbol{\Delta}_h}$ is the orthogonal projection onto the kernel of $\boldsymbol{\Delta}$ and let $a \in \mathcal{A}_\theta$. Then*

1. *the ζ_θ-regularised trace $\zeta_\theta(a, Q)$ of a is holomorphic at zero,*
2. *the residue Res_θ extends to $a \log Q$ and we have*

$$\zeta_\theta(a, Q)(0) = -\frac{1}{q} \mathrm{Res}_\theta(a\, \log Q). \tag{67}$$

Exactly as in the case of manifolds, if we specialise the holomorphic family to the case $A(z) = a\, Q^{-z}$, where a is an element of the algebra of the noncommutative torus, using the same arguments, we get the following result:

Theorem 56. *For any $a \in \mathcal{A}_\theta$ we have*

$$\begin{aligned} \mathrm{Tr}\left(a\, e^{-t\boldsymbol{\Delta}_h}\right) \sim_{t \to 0} -\frac{(4\pi)^{\frac{n}{2}}}{2} \Bigg[& \mathrm{Res}\,(a\, \log \boldsymbol{\Delta}_h)\ \delta_{\frac{n}{2} - \left[\frac{n}{2}\right]} \\ & + \sum_{k \in \left[0, \frac{n}{2}\right[\cap \mathbb{Z}} \Gamma\left(\frac{n}{2} - k\right)\, \mathrm{Res}\left(a\, \boldsymbol{\Delta}_h^{k - \frac{n}{2}}\right)\, t^{k - \frac{n}{2}} \Bigg]. \end{aligned} \tag{68}$$

The scalar curvature being associated to the a_1 coefficient of the heat kernel expansion, Theorem 56 motivates the following definition:

Definition 57. The "scalar curvature" $\mathfrak{s}_h$ on the noncommutative two torus $\mathbb{T}_\theta^2$ associated with the "metric" determined by the conformal factor h is defined as

$$\langle \mathfrak{s}_h, a \rangle_h = \begin{cases} -6\pi\, \mathrm{Res}_\theta\,(a\, \log \boldsymbol{\Delta}_h) & \text{if} \quad n = 2 \\ -\frac{3}{2}\,(4\pi)^{\frac{n}{2}}\, \Gamma\left(\frac{n}{2} - k\right)\, \mathrm{Res}_\theta \left(a\, \boldsymbol{\Delta}_h^{k - \frac{n}{2}}\right) & \text{otherwise,} \end{cases} \tag{69}$$

which compares with the definitions in [7], [8], [11], [12].

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r_∞-Matrices, Triangular L_∞-Bialgebras and Quantum$_\infty$ Groups

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Abstract. A homotopy analogue of the notion of a triangular Lie bialgebra is proposed with a goal of extending basic notions of the theory of quantum groups to the context of homotopy algebras and, in particular, introducing a homotopical generalization of the notion of a quantum group, or quantum$_\infty$-group.

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1. Introduction

1.1. Conventions and notation

We will work over a ground field k of characteristic zero. A differential graded (dg) vector space V will mean a complex of k-vector spaces with a differential of degree one. The degree of a homogeneous element $v \in V$ will be denoted by $|v|$. In the context of graded algebra, we will be using the Koszul rule of signs when talking about the graded version of notions involving symmetry, including commutators, brackets, symmetric algebras, derivations, *etc.*, often omitting the modifier *graded*. For any integer n, we define a *translation* (or *n-fold desuspension*) $V[n]$ of V: $V[n]^p := V^{n+p}$ for each $p \in \mathbb{Z}$. For two graded vector spaces V and W, we define grading on the space $\mathrm{Hom}(V, W)$ of k-linear maps $V \to W$ by $|f| := n - m$ for $f \in \mathrm{Hom}(V^m, W^n)$.

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1.2. Quantum groups

Recall that a *quantum group* in the sense of Drinfeld and Jimbo is an associative, coassociative Hopf algebra A subject to the condition of being *quasitriangular* [5]. The latter implies, in particular, the existence of a solution $\mathcal{R}$ to the *quantum Yang–Baxter equation* $\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}$ set up in A. More conceptually, the quasitriangularity condition provides data needed to put a braided structure on the monoidal category of left A-modules.

The most basic examples of quantum groups appear as *quantizations* or certain types of *deformations* (in the sense of Hopf algebras) of universal enveloping algebras and algebras of functions on groups. In the first case, starting with a Lie algebra $\mathfrak{g}$ and a Hopf-algebra deformation $U_h(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$, one passes to the "*(semi)classical limit*" $\delta(x) := \frac{\Delta_h(x)-\Delta_h^{op}(x)}{h}$ thus equipping $U(\mathfrak{g})$ with a *co-Poisson–Hopf structure* with $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ being the *co-Poisson cobracket*. In particular, the restriction $\delta|_{\mathfrak{g}}$ becomes a well-defined cobracket on $\mathfrak{g}$ turning it into a Lie bialgebra.

1.3. Quantization of triangular Lie bialgebras

The above process can be reversed: as it was shown in [13], any finite-dimensional Lie bialgebra $(\mathfrak{g}, [,], \delta)$ can be *quantized*, meaning that one can always come up with a Hopf-algebra deformation $U_h(\mathfrak{g})$ whose classical limit, in the sense of the above formula, agrees with δ. While a priori $U_h(\mathfrak{g})$ is just a Hopf algebra, one would really be interested in having a quasitriangular structure on it. As a special case, it was shown in [4] that such a structure exists, when $\mathfrak{g}$ is a *triangular* Lie bialgebra. This class of Lie bialgebras is defined as follows: let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$ ("*a classical r-matrix*") be a skew-symmetric element satisfying the *classical Yang–Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{23}, r_{13}] = 0,$$

which can be conveniently restated in the form of the *Maurer–Cartan equation*

$$[r, r] = 0$$

taking place in the graded Lie algebra $S(\mathfrak{g}[-1])[1]$ with respect to the *Schouten bracket* for elements r of degree one: $r \in (S(\mathfrak{g}[-1])[1])^1 = (S(\mathfrak{g}[-1]))^2 = S^2(\mathfrak{g}[-1])[2] = \mathfrak{g} \wedge \mathfrak{g}$. Such an element r, called a *Maurer–Cartan element*, gives rise to a Lie cobracket on $\mathfrak{g}$ in the form of the coboundary $\partial_{\mathrm{CE}}(r) : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ of r taken in the cochain *Chevalley–Eilenberg complex* of $\mathfrak{g}$ with coefficients in $\mathfrak{g} \wedge \mathfrak{g}$ (here, r is regarded as a 0-cocycle). The compatibility with the Lie-algebra structure on $\mathfrak{g}$ is packed into the relation $\partial^2_{\mathrm{CE}}(r) = 0$, thus guaranteeing that $\mathfrak{g}$ with such a cobracket is indeed a Lie bialgebra. The *co-Jacobi identity*, which could be rewritten as

$$[\partial_{\mathrm{CE}}(r), \partial_{\mathrm{CE}}(r)] = 0, \tag{1}$$

follows from the following statement, which is an odd version of the Hamiltonian correspondence in Poisson geometry, if one regards $S(\mathfrak{g}[-1])[1]$ as the shifted

Gerstenhaber algebra of functions on the odd Poisson manifold $(\mathfrak{g}[-1])^*$ and $\mathrm{Hom}(\mathfrak{g}[-1], S(\mathfrak{g}[-1]))$ as the graded Lie algebra of vector fields on $(\mathfrak{g}[-1])^*$.

Proposition 1. *The linear map*

$$\partial_{\mathrm{CE}}; S(\mathfrak{g}[-1])[1] \to \mathrm{Hom}(\mathfrak{g}[-1], S(\mathfrak{g}[-1]))$$

is a graded Lie-algebra morphism.

A *triangular Lie bialgebra* is a Lie algebra $\mathfrak{g}$ provided with a Lie cobracket $\partial_{\mathrm{CE}}(r)$ coming out of an r-matrix r. A basic statement concerning this class of Lie bialgebras is that $U(\mathfrak{g})$ can be quantized to a *triangular Hopf algebra* $U_h(\mathfrak{g})$. This condition is stronger than being quasitriangular, and in particular, the category of (left) modules over a triangular Hopf algebra turns out to be symmetric monoidal, as opposed to just being braided.

1.4. The homotopy quantization program

The upshot of the above construction is that there is a source of quantum groups coming from the data of a Lie algebra $\mathfrak{g}$ and a solution of the Maurer–Cartan equation in $S(\mathfrak{g}[-1])[1]$. The goal of our project is to promote this construction to the realm of homotopy Lie algebras. In particular, this would generalize the work [2] done for the case of Lie 2-bialgebras. Here is an outline of our program:

1. Develop the notion of a *triangular L_∞-bialgebra* extending the classical one. In analogy with the classical case, the input data for this construction consists of an L_∞-algebra $\mathfrak{g}$ and a solution r of a generalized Maurer–Cartan equation set up in an appropriate algebraic context;
2. Show that the universal enveloping algebra $U(\mathfrak{g})$ of a triangular L_∞-bialgebra $\mathfrak{g}$ admits a natural *homotopy co-Poisson–Hopf* structure;
3. Extend the *Drinfeld twist* construction [5], which equips a cocommutative Hopf algebra with a new, triangular coproduct, to the homotopical context. Apply it to the case of universal enveloping algebra $U(\mathfrak{g})$ of the previous step to obtain a *quantum$_\infty$ group.*

The current paper is dedicated to describing the first step of the construction, which we believe might be interesting on its own.

The second step is work in progress. While the universal enveloping algebra $U(\mathfrak{g})$ of an L_∞-algebra $\mathfrak{g}$ is a strongly homotopy associative (or A_∞-) algebra that also turns out to be a cocommutative, coassociative coalgebra object in Lada–Markl's, see [10], symmetric monoidal category of A_∞-algebras, we would be interested in verifying that $U(\mathfrak{g})$ is actually a *Hopf$_\infty$ algebra*, that is, possesses an antipodal map satisfying certain compatibility conditions. We would also need to translate an L_∞-bialgebra structure on an L_∞-algebra $\mathfrak{g}$ into a co-Poisson$_\infty$ structure on the Hopf$_\infty$ algebra $U(\mathfrak{g})$. This would result in providing $U(\mathfrak{g})$ with the structure of a cocommutative co-Poisson$_\infty$ coalgebra object in Lada–Markl's symmetric monoidal category of A_∞-algebras.

Furthermore, we would like to develop deformation theory of homotopy Hopf algebras and use it to quantize homotopy Lie bialgebras. A different approach to quantization of homotopy Lie bialgebras (using the framework of PROPs) was taken in [12], in which a different notion of a homotopy Hopf algebra (or rather, homotopy bialgebra) was used. That notion depends on the choice of a minimal resolution of the bialgebra properad. The notion we outline above appears to be more canonical.

In the future we would also be interested in investigating what this program produces for L_∞-algebras arising in the geometric context, such as generalized Poisson geometry, L_∞-algebroids, or BV_∞-geometry.

2. The big bracket and L_∞-bialgebras

Recall that the structure of a (strongly) homotopy Lie algebra (or *L_∞-algebra*) on a graded vector space $\mathfrak{g}$ may be given by a *codifferential*, *i.e.*, a degree-one, square-zero coderivation D such that $D(1) = 0$, on the graded cocommutative coalgebra $S(\mathfrak{g}[1])$ equipped with the shuffle comultiplication. The date given by D is equivalent to a collection of "higher Lie brackets" $l_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$, $k \geq 1$, of degree one obtained by restriction of D to the kth symmetric component of $S(\mathfrak{g}[1])$ followed by projection to the cogenerators. The condition $D^2 = 0$ is equivalent to the *higher Jacobi identities*, homotopy versions of the Jacobi identity. Outside of deformation theory, nontrivial examples of homotopy Lie algebras are known to arise in the context of multisymplectic geometry [1, 14], Courant algebroids [15], and closed string field theory [7, 11, 17].

In order to discuss the structure of an L_∞-bialgebra on a graded vector space $\mathfrak{g}$, we need to mix the graded Lie algebra $\mathrm{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1])$, used to define the structure of an L_∞-algebra on $\mathfrak{g}$, with the graded Lie algebra $\mathrm{Hom}(\mathfrak{g}[-1]), S(\mathfrak{g}[-1]))$, used to define the cobracket on $\mathfrak{g}$ in the classical, Lie algebra setting in Section 1.3. Consider the graded vector space

$$\mathcal{B} := \prod_{m,n \geq 0} \mathrm{Hom}(S^m(\mathfrak{g}[1]), S^n(\mathfrak{g}[-1]))[2]$$

and provide it with the structure of a graded Lie algebra given by the graded commutator, called the *big bracket*,

$$[f, g] := f \circ g - (-1)^{|f| \cdot |g|} g \circ f$$

under the *circle*, or $\cup_1$ *product*, cf. [6] and [16]:

$$\begin{aligned}&(f \circ g)(x_1 \cdots x_n)\\ &\quad := \sum_{\sigma \in \mathrm{Sh}_{k,l}} (-1)^\varepsilon f\big(x_{\sigma(1)} \cdots x_{\sigma(k)} g\big(x_{\sigma(k+1)} \cdots x_{\sigma(n)}\big)_{(1)}\big)\, g\big(x_{\sigma(k+1)} \cdots x_{\sigma(n)}\big)_{(2)},\end{aligned}$$

where $x_1, \dots, x_n \in \mathfrak{g}[1]$,

$$f \in \prod_{m \geq 0} \operatorname{Hom}(S^{k+1}(\mathfrak{g}[1]), S^m(\mathfrak{g}[-1]))[2],$$
$$g \in \prod_{m \geq 0} \operatorname{Hom}(S^{l}(\mathfrak{g}[1]), S^m(\mathfrak{g}[-1]))[2],$$

$n = k + l$ – otherwise we set $(f \circ g)(x_1 \cdots x_n) = 0$, $\mathrm{Sh}_{k,l}$ is the set of (k,l) *shuffles*: permutations σ of $\{1, 2, \dots, n\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+1) < \cdots < \sigma(n)$, $\varepsilon = |x_\sigma| + |g|(|x_{\sigma(1)}| + \cdots + |x_{\sigma(k)}|)$, $(-1)^{|x_\sigma|}$ is the *Koszul sign* of the permutation of $x_1 \cdots x_n$ to $x_{\sigma(1)} \cdots x_{\sigma(n)}$ in $S(\mathfrak{g}[1])$, and we use Sweedler's notation to denote the result $g_{(1)} \otimes g_{(2)}$ of applying to $g \in S(\mathfrak{g}[-1])[2]$ the (shifted) comultiplication $S(\mathfrak{g}[-1])[2] \to S(\mathfrak{g}[-1])[2] \otimes S(\mathfrak{g}[-1])$ followed by the projection $S(\mathfrak{g}[-1])[2] \to \mathfrak{g}[1]$ onto the cogenerators in the first tensor factor. This graded Lie algebra $\mathcal{B}$, under the assumption that $\dim \mathfrak{g} < \infty$ and in a slightly different incarnation, was introduced by Y. Kosmann-Schwarzbach [8] in relation to Lie bialgebras and later used by O. Kravchenko [9] in relation to L_∞-bialgebras. The graded Lie algebra $\mathcal{B}$ has the property that its Maurer–Cartan elements represent L_∞ brackets and cobrackets on $\mathfrak{g}$, as well as mixed operations, comprising the structure of an L_∞-bialgebra on $\mathfrak{g}$. Here we adopt Kravchenko's approach and define an *L_∞-bialgebra structure* on $\mathfrak{g}$ as a Maurer–Cartan element μ in the subalgebra

$$\mathcal{B}^+ := \prod_{m,n \geq 1} \operatorname{Hom}(S^m(\mathfrak{g}[1]), S^n(\mathfrak{g}[-1]))[2]$$

of the graded Lie algebra $\mathcal{B}$.[1] This means

$$\begin{gathered} \mu = \sum_{m,n \geq 1} \mu_{mn}, \\ \mu_{mn} : S^m(\mathfrak{g}[1]) \to S^n(\mathfrak{g}[-1])[2] \quad \text{of degree } 1, \\ [\mu, \mu] = 0. \end{gathered} \tag{2}$$

3. r_∞-matrices and triangular L_∞-bialgebras

For an L_∞-algebra $\mathfrak{g}$, one can generalize the Schouten bracket to an L_∞ structure on $S(\mathfrak{g}[-1])[1]$ by extending the higher brackets l_k on $\mathfrak{g}$ as graded multiderivations of the graded commutative algebra $S(\mathfrak{g}[-1])$. This L_∞ structure may also be described via *higher derived brackets* (in the semiclassical limit) on the BV$_\infty$-algebra $S(\mathfrak{g}[-1])$, see [3, Example 3.4]. The L_∞ structure can be naturally extended to the completion

$$\widehat{S}(\mathfrak{g}[-1])[1] := \prod_{n \geq 0} S^n(\mathfrak{g}[-1])[1].$$

[1] Maurer–Cartan elements in $\mathcal{B}$ would correspond to more general, *curved* L_∞-bialgebras.

While investigating the deformation-theoretic meaning of solutions $r = r(\lambda) \in \lambda\widehat{S}(\mathfrak{g}[-1])[1][[\lambda]]$, where λ is the *deformation parameter*, that is to say, a (degree-zero) formal variable, of the *generalized Maurer–Cartan equation*

$$l_1(r) + \frac{1}{2!} l_2(r \odot r) + \frac{1}{3!} l_3(r \odot r \odot r) + \cdots = 0, \tag{3}$$

where $\odot$ refers to multiplication in $S(V)$ for $V = \widehat{S}(\mathfrak{g}[-1])[2]$, certain analogies can be drawn with basic constructions of the theory of quantum groups.

Note that an L_∞-algebra structure on $\mathfrak{g}$ endows the graded Lie algebras $\mathcal{B}$ and $\mathcal{B}^+$ with the structure of a dg Lie algebra. Namely, bracketing with the Maurer–Cartan element $l_1 + l_2 + \cdots \in \prod_{m \geq 1} \operatorname{Hom}(S^m(\mathfrak{g}[1]), \mathfrak{g}[-1])[2]$, representing the L_∞-algebra structure on $\mathfrak{g}$, creates a differential

$$d\gamma := [l_1 + l_2 + \cdots, \gamma]$$

on $\mathcal{B}$ and $\mathcal{B}^+$, compatible with the "big" bracket.

An *L_∞-morphism* φ from the L_∞-algebra $\widehat{S}(\mathfrak{g}[-1])[1]$ to the dg Lie algebra $\mathcal{B}^+$, that is to say, a morphism $S(\widehat{S}(\mathfrak{g}[-1])[2]) \to S(\mathcal{B}^+[1])$ of dg coalgebras mapping 1 to 1, amounts to defining a series of degree-zero linear maps $\varphi_n : S^n(\widehat{S}(\mathfrak{g}[-1])[2]) \to \mathcal{B}^+[1]$ for all $n \geq 1$ satisfying the following compatibility conditions:

$$\begin{aligned} & d\varphi_n(x_1 \odot \cdots \odot x_n) \\ & \quad + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}_{k,n-k}} (-1)^{\varepsilon} [\varphi_k(x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k)}), \varphi_{n-k}(x_{\sigma(k+1)} \odot \cdots \odot x_{\sigma(n)})] \\ & = \sum_{m=1}^{n} \sum_{\tau \in \mathrm{Sh}_{m,n-m}} (-1)^{|x_\tau|} \varphi_{n-m+1}(l_m(x_{\tau(1)} \odot \cdots \odot x_{\tau(m)}) \odot x_{\tau(m+1)} \odot \cdots \odot x_{\tau(n)}), \end{aligned}$$

where $x_1, \dots, x_n \in \widehat{S}(\mathfrak{g}[-1])[2]$ and $\varepsilon = |x_\sigma| + |x_{\sigma(1)}| + \cdots + |x_{\sigma(k)}|$. There is a *canonical L_∞-morphism* $\varphi : \widehat{S}(\mathfrak{g}[-1])[1] \to \mathcal{B}^+$, which may be defined by the maps

$$\varphi_n(x_1 \odot \cdots \odot x_n)(y) := l_{n+p}(x_1 \odot \cdots \odot x_n \odot N(y)),$$

where $x_1, \dots, x_n \in \widehat{S}(\mathfrak{g}[-1])[2]$, $y \in S^p(\mathfrak{g}[1])$, $N(y) \in S^p(\widehat{S}(\mathfrak{g}[-1])[2])$, $p \geq 1$, and $N : S(\mathfrak{g}[1]) \to S(\widehat{S}(\mathfrak{g}[-1])[2])$ is the graded-algebra morphism induced by the obvious linear map $\mathfrak{g}[1] \hookrightarrow \widehat{S}(\mathfrak{g}[-1])[2] \hookrightarrow S(\widehat{S}(\mathfrak{g}[-1])[2])$. The following theorem generalizes Proposition 1 to the L_∞ setting.

Theorem 2. *The above maps φ_n, $n \geq 1$, define an L_∞-morphism*

$$\varphi : \widehat{S}(\mathfrak{g}[-1])[1] \to \mathcal{B}^+$$

from the L_∞-algebra $\widehat{S}(\mathfrak{g}[-1])[1]$ to the dg Lie algebra $\mathcal{B}^+$.

The proof of the theorem is a straightforward checkup that reduces the statement to the higher Jacobi identities for the L_∞ brackets on $\widehat{S}(\mathfrak{g}[-1])[1]$. This theorem also generalizes Kravchenko's result [9, Theorem 19], which provides an L_∞-morphism from an L_∞-algebra $\mathfrak{g}$ to the graded Lie algebra $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$.

An *r_∞-matrix* r is a (*generalized*) *Maurer–Cartan element* $r = r(\lambda)$ in the L_∞-algebra $\lambda\widehat{S}(\mathfrak{g}[-1])[1][[\lambda]]$, *i.e.*, a degree-one solution of the generalized Maurer–Cartan equation (3). Sending an r_∞-matrix to the subalgebra $\mathcal{B}^+$ of the big-bracket dg Lie algebra $\mathcal{B}$ under an L_∞-morphism $\varphi : \widehat{S}(\mathfrak{g}[-1])[1] \to \mathcal{B}^+$ would yield a Maurer–Cartan element $\mu' = \mu'(\lambda)$:

$$\mu' := \varphi(e^r) = \varphi_1(r) + \frac{1}{2!}\varphi_2(r \odot r) + \frac{1}{3!}\varphi_3(r \odot r \odot r) + \cdots ,$$

depending on the deformation parameter λ, in the dg Lie algebra $\lambda\mathcal{B}^+[[\lambda]]$:

$$d\mu' + \frac{1}{2}[\mu', \mu'] = 0,$$

or, equivalently, a Maurer–Cartan element

$$\mu = \mu' + l_1 + l_2 + \cdots$$

in the graded Lie algebra $\lambda\mathcal{B}^+[[\lambda]]$:

$$[\mu, \mu] = 0,$$

giving rise to an L_∞-bialgebra structure on $\mathfrak{g}$, as per Section 2, in analogy with $\partial_{\mathrm{CE}}(r)$ giving rise to an ordinary Lie cobracket in the classical, nonhomotopical case, see also Example 3 below.

We call the L_∞-bialgebra $(\mathfrak{g}, \mu)$ produced out of an L_∞-algebra $\mathfrak{g}$ and an r_∞-matrix r by transferring it to a Maurer–Cartan element $\mu' = \varphi(e^r)$ in $\mathcal{B}^+$ via the canonical L_∞-morphism φ, as above, a *triangular L_∞-bialgebra.*

Example 3. In the case of a classical Lie algebra $\mathfrak{g}$, the graded Lie algebra $\widehat{S}(\mathfrak{g}[-1])[1]$ is just the completed graded Lie algebra of (right-)invariant multivector fields on the corresponding local Lie group with the Schouten bracket (or, equivalently, up to degree shift, functions on the formal odd Poisson manifold $(\mathfrak{g}[-1])^*$), and the canonical L_∞-morphism φ is just linear: $\varphi = \varphi_1 = \partial_{\mathrm{CE}}$, equal to the Chevalley–Eilenberg differential of 0-cochains of the Lie algebra $\mathfrak{g}$ with coefficients in the graded $\mathfrak{g}$-module $\widehat{S}(\mathfrak{g}[-1])[2]$. The fact that φ is an L_∞-morphism translates into being a dg Lie morphism, *i.e.*, satisfying two compatibility conditions

$$d\varphi(x) = 0,$$
$$[\varphi(x), \varphi(y)] = \varphi([x, y]),$$

where d is the operator taking the big bracket with the Lie structure $l_2 \in \mathrm{Hom}(S^2(\mathfrak{g}[1]), \mathfrak{g}[-1])[2] \subset \mathcal{B}^+$:

$$d\alpha := [l_2, \alpha].$$

The first condition means that $\partial_{\mathrm{CE}}(\varphi(x)) = \partial^2_{\mathrm{CE}}(x) = 0$, and the second states that ∂_{CE} is a Lie-algebra morphism, which is the assertion of Proposition 1. An r_∞-matrix is a solution $r \in (\widehat{S}(\mathfrak{g}[-1])[1])^1 = (S^2(\mathfrak{g}[-1]))^2 = \mathfrak{g} \wedge \mathfrak{g}$ of the generalized Maurer–Cartan equation (3), which turns into the classical Maurer–Cartan equation $[r, r] = 0$ in this case, and thereby r is just a classical r-matrix. Thus, the transfer $\varphi(e^r) = \varphi(r)$ of an r-matrix is a Lie cobracket $\varphi(r)$ on $\mathfrak{g}$, satisfying the compatibility condition with the Lie bracket and the co-Jacobi identity (1), resulting in the structure of a triangular Lie bialgebra on $\mathfrak{g}$. Here we ignored the deformation parameter λ, because $\widehat{S}(\mathfrak{g}[-1])[1]$ is just a graded Lie algebra and the generalized Maurer–Cartan equation in $\widehat{S}(\mathfrak{g}[-1])[1]$ and the morphism φ have only finitely many terms.

Example 4. When $\mathfrak{g}$ is a dg Lie algebra, the picture is dramatically different from the classical picture of Example 3. Now an r_∞-matrix r may have many more components than just one in $\mathfrak{g} \wedge g$:

$$r \in (\widehat{S}(\mathfrak{g}[-1])[1])^1 = (\widehat{S}(\mathfrak{g}[-1]))^2 = \prod\nolimits_{n\geq 0}(S^n(\mathfrak{g}[-1]))^2 = \prod\nolimits_{n\geq 1}(S^n(\mathfrak{g}[-1]))^2.$$

However, $\widehat{S}(\mathfrak{g}[-1])[1]$ is just a dg Lie algebra, and the generalized Maurer–Cartan equation (3) is still classical,

$$dr + \tfrac{1}{2}[r, r] = 0.$$

The L_∞-morphism φ is still linear $\varphi = \varphi_1$, *i.e.*, φ is a dg Lie-algebra morphism. The transfer $\mu' = \varphi(e^r) = \varphi(r)$ of r via φ will result in a L_∞-bialgebra structure on $\mathfrak{g}$ with higher operations μ_{mn}, see (2), which are trivial for all pairs (m, n) but those with $m = 1$, $n \geq 1$ and $m = 2$, $n = 1$. Thus, even in the case of a dg Lie algebra $\mathfrak{g}$, our construction creates a triangular L_∞-bialgebra, rather than just a (dg) triangular Lie algebra.

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Normal Forms and Lie Groupoid Theory

Rui Loja Fernandes

Abstract. In these lectures I discuss the Linearization Theorem for Lie groupoids, and its relation to the various classical linearization theorems for submersions, foliations and group actions. In particular, I explain in some detail the recent metric approach to this problem proposed in [6].

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Lecture 1: Linearization and normal forms

In Differential Geometry one finds many different normal forms results which share the same flavor. In the last few years we have come to realize that there is more than a shared flavor to many of these results: they are actually instances of the same general result. The result in question is a linearization result for Lie groupoids, first conjectured by Alan Weinstein in [17, 18]. The first complete proof of the linearization theorem was obtained by Nguyen Tien Zung in [19]. Since then several clarifications and simplifications of the proof, as well as more general versions of this result, were obtained (see [3, 6]). In these lectures notes we give an overview of the current status of the theory.

The point of view followed here, which was greatly influenced by an ongoing collaboration with Matias del Hoyo [6–8], is that the linearization theorem can be thought of as an Ehresmann Theorem for a submersion onto a stack. Hence, its proof should follow more or less the same steps as the proof of the classical Ehresmann Theorem, which can be reduced to a simple argument using the exponential map of a metric that makes the submersion Riemannian. Although I will not go at all into geometric stacks (see the upcoming paper [8]), I will adhere to the metric approach introduced in [6].

Let us recall the kind of linearization theorems that we have in mind. The most basic is precisely the following version of Ehresmann's Theorem:

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Theorem 1 (Ehresmann). *Let $\pi : M \to N$ be a* **proper** *surjective submersion. Then π is locally trivial: for every $y \in N$ there is a neighborhood $y \in U \subset N$, a neighborhood $0 \in V \subset T_yN$, and diffeomorphism:*

$$\begin{tikzcd} V \times \pi^{-1}(y) \arrow[r, "\cong"] \arrow[d, "pr"'] & \pi^{-1}(U) \subset M \arrow[d, "\pi"] \\ V \arrow[r, "\cong"] & U\,. \end{tikzcd}$$

One can also assume that there is some extra geometric structure behaving well with respect to the submersion, and then ask if one can achieve "linearization" of both the submersion and the extra geometric structure. For example, if one assumes that $\omega \in \Omega^2(M)$ is a closed 2-form such that the pullback of ω to each fiber is non-degenerate, then one can show that π is a locally trivial symplectic fibration (see, e.g., [13]). We will come back to this later, for now we recall another basic linearization theorem:

Theorem 2 (Reeb). *Let $\mathcal{F}$ be a foliation of M and let L_0 be a compact leaf of $\mathcal{F}$ with* **finite holonomy***. Then there exists a saturated neighborhood $L_0 \subset U \subset M$, a* $\mathrm{hol}(L_0)$*-invariant neighborhood $0 \in V \subset \nu_{x_0}(L_0)$, and a diffeomorphism:*

$$\widetilde{L_0}^h \underset{\mathrm{hol}(L_0)}{\times} V \xrightarrow{\cong} U \subset M$$

sending the linear foliation to $\mathcal{F}|_U$.

Here, $\widetilde{L_0}^h \to L$ denotes the holonomy cover, a $\mathrm{hol}(L_0)$-principal bundle, and the holonomy group $\mathrm{hol}(L_0)$ acts on the normal space $\nu_{x_0}(L_0)$ via the linear holonomy representation. By "linear foliation" we mean the quotient of the horizontal foliation $\{\widetilde{L_0}^h \times \{t\}, t \in \nu_{x_0}(L)\}$.

Notice that this result generalizes Ehresmann's Theorem, at least when the fibers of the submersion are connected: any leaf of the foliation by the fibers of π has trivial holonomy so $\mathrm{hol}(L_0)$ acts trivially on the transversal, and then Reeb's theorem immediately yields Ehresmann's Theorem. For this reason, maybe it is not so surprising that the two results are related.

Let us turn to a third linearization result which, in general, looks to be of a different nature from the previous results. It is a classical result from Equivariant Geometry often referred to as the Slice Theorem (or Tube Theorem):

Theorem 3 (Slice Theorem). *Let K be a Lie group acting in a* **proper** *fashion on M. Around any orbit $\mathcal{O}_{x_0} \subset M$ the action can be linearized: there exist K-invariant neighborhoods $\mathcal{O}_{x_0} \subset U \subset M$ and $0_{x_0} \in V \subset \nu_{x_0}(\mathcal{O}_{x_0})$ and a K-equivariant diffeomorphism:*

$$K \times_{K_{x_0}} V \xrightarrow{\cong} U \subset M$$

Here K_{x_0} acts on the normal space $\nu_{x_0}(\mathcal{O}_{x_0})$ via the normal isotropy representation. If the action is locally free then the orbits form a foliation, the isotropy groups K_x are finite and $\mathrm{hol}(\mathcal{O}_x)$ is a quotient of K_x. Moreover, the action of K_x on a slice descends to the linear holonomy action of $\mathrm{hol}(\mathcal{O}_x)$. The slice theorem is then a special case of the Reeb stability theorem. However, in general, the isotropy groups can have positive dimension and the two results look apparently quite different.

Again, both in the case of foliations and in the case of group actions, we could consider extra geometric structures (e.g., a metric or a symplectic form) and ask for linearization taking into account this extra geometric structure. One can find such linearization theorems in the literature (e.g., the local normal form theorem for Hamiltonian actions [10]). Let us mention one such recent result from Poisson geometry, due to Crainic and Marcut [4]:

Theorem 4 (Local normal form around symplectic leaves). *Let (M, π) be a Poisson manifold and let $S \subset M$ be a compact symplectic leaf. If the Poisson homotopy bundle $G \curvearrowright P \to S$ is a smooth* **compact** *manifold with vanishing second de Rham cohomology group, then there is a neighborhood $S \subset U \subset M$, and a Poisson diffeomorphism:*

$$\phi : (U, \pi|_U) \to (P \times_G \mathfrak{g}, \pi^{\mathrm{lin}}).$$

We will not discuss here the various terms appearing in the statement of this theorem, referring the reader to the original work [4]. However, it should be clear that this result has the same flavor as the previous ones: some compactness type assumption around a leaf/orbit leads to linearization or a normal form of the geometric structure in a neighborhood of the leaf/orbit.

Although all these results have the same flavor, they do look quite different. Moreover, the proofs that one can find in the literature of these linearization results are also very distinct. So it may come as a surprise that they are actually just special cases of a very general linearization theorem.

In order to relate all these linearization theorems, and to understand the significance of the assumptions one can find in their statements, one needs a language where all these results fit into the same geometric setup. This language exists and it is a generalization of the usual Lie theory from groups to groupoids. We will recall it in the next lecture. After that, we will be in shape to state the general linearization theorem and explain how the results stated before are special instances of it.

Lecture 2: Lie groupoids

In this Lecture we provide a quick introduction to Lie groupoids and Lie algebroids. We will focus mostly on some examples which have special relevance to us. A more detailed discussion, along with proofs, can be found in [1, 12, 14]. Let us start by recalling:

Definition 1. A **groupoid** is a small category where all arrows are invertible.

Let us spell out this definition. We have a **set of objects** M, and a **set of arrows** G. For each arrow $g \in G$ we can associate its source $\mathbf{s}(g)$ and its target $\mathbf{t}(g)$, resulting in two maps $\mathbf{s}, \mathbf{t} : G \to M$. We also write $g : x \longrightarrow y$ for an arrow with source x and target y.

For any pair of composable arrows we have a **product** or composition map:

$$m : G^{(2)} \to G, (g, h) \mapsto gh.$$

In general, we will denote by $G^{(n)}$ the set of n strings of composable arrows:

$$G^{(n)} := \{(g_1, \dots, g_n) : \mathbf{s}(g_i) = \mathbf{t}(g_{i+1})\}.$$

The multiplication satisfies the *associativity property*:

$$(gh)k = g(hk), \quad \forall (g, h, k) \in G^{(3)}.$$

For each object $x \in M$ there is an identity arrow 1_x and the *identity property* holds:

$$1_{\mathbf{t}(g)} g = g = g 1_{\mathbf{s}(g)}, \quad \forall g \in G.$$

It gives rise to an **identity section** $u : M \to G$, $x \mapsto 1_x$.

For each arrow $g \in G$ there is an inverse arrow $g^{-1} \in G$, for which the *inverse property* holds:

$$g g^{-1} = 1_{\mathbf{t}(g)}, \quad g^{-1} g = 1_{\mathbf{s}(g)}, \quad \forall g \in G.$$

This gives rise to the **inverse** map $\iota : G \to G$, $g \mapsto g^{-1}$.

Definition 2. A **morphism of groupoids** is a functor $\mathcal{F} : G \to H$.

This means that we have a map $\mathcal{F} : G \to H$ between the sets of arrows and a map $f : M \to N$ between the sets of objects, making the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "\mathcal{F}"] \arrow[d, shift right, "\mathbf{s}"'] \arrow[d, shift left, "\mathbf{t}"] & H \arrow[d, shift right, "\mathbf{s}"'] \arrow[d, shift left, "\mathbf{t}"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

such that $\mathcal{F}(gh) = \mathcal{F}(g)\mathcal{F}(h)$ if $g, h \in G$ are composable, and $\mathcal{F}(1_x) = 1_{f(x)}$ for all $x \in M$.

We are interested in groupoids and morphisms of groupoids in the smooth category:

Definition 3. A **Lie groupoid** is a groupoid $G \rightrightarrows M$ whose spaces of arrows and objects are both manifolds, the structure maps $\mathbf{s}, \mathbf{t}, u, m, i$ are all smooth maps and such that $\mathbf{s}$ and $\mathbf{t}$ are submersions. A **morphism of Lie groupoids** is a morphism of groupoids for which the underlying map $\mathcal{F} : G \to H$ is smooth.

Before we give some examples of Lie groupoids, let us list a few basic properties.

- the unit map $u : M \to G$ is an embedding and the inverse $\iota : G \to G$ is a diffeomorphism.

- The source fibers are embedded submanifolds of G and **right multiplication by** $g : x \longrightarrow y$ is a diffeomorphism between $\mathbf{s}$-fibers
$$R_g : \mathbf{s}^{-1}(y) \longrightarrow \mathbf{s}^{-1}(x), \quad h \mapsto hg.$$
- The target fibers are embedded submanifolds of G and **left multiplication by** $g : x \longrightarrow y$ is a diffeomorphism between $\mathbf{t}$-fibers:
$$L_g : \mathbf{t}^{-1}(x) \longrightarrow \mathbf{t}^{-1}(y), \quad h \mapsto gh.$$
- The **isotropy group at** x:
$$G_x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x).$$
is a Lie group.
- The **orbit through** x:
$$\mathcal{O}_x := \mathbf{t}(\mathbf{s}^{-1}(x)) = \{y \in M : \exists\, g : x \longrightarrow y\}$$
is a regular immersed (possibly disconnected) submanifold.
- The map $\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathcal{O}_x$ is a principal G_x-bundle.

The connected components of the orbits of a groupoid $G \rightrightarrows M$ form a (possibly singular) foliation of M. The set of orbits is called the **orbit space** and denoted by M/G. The quotient topology makes the natural map $\pi : M \to M/G$ into an open, continuous map. In general, there is no smooth structure on M/G compatible with the quotient topology, so M/G is a singular space. One may think of a groupoid as a kind of atlas for this orbit space.

There are several classes of Lie groupoids which will be important for our purposes. A Lie groupoid $G \rightrightarrows M$ is called **source** k**-connected** if the s-fibers $\mathbf{s}^{-1}(x)$ are k-connected for every $x \in M$. When $k = 0$ we say that G is a s**-connected groupoid**, and when $k = 1$ we say that G is a s**-simply connected groupoid**. We call the groupoid **étale** if $\dim G = \dim M$, which is equivalent to requiring that the source or target map be a local diffeomorphism. The map $(\mathbf{s}, \mathbf{t}) : G \to M \times M$ is sometimes called the **anchor of the groupoid** and a groupoid is called **proper** if the anchor $(\mathbf{s}, \mathbf{t}) : G \to M \times M$ is a proper map. In particular, when the source map is proper, we call the groupoid **s-proper**. We will see later that proper and s-proper groupoids are, in some sense, analogues of compact Lie groups.

Example 1. Perhaps the most elementary example of a Lie groupoid is the **unit groupoid** $M \rightrightarrows M$ of any manifold M: for each object $x \in M$, there is exactly one arrow, namely the identity arrow. More generally, given an open cover $\{U_i\}$ of M one constructs a **cover groupoid** $\bigsqcup_{i,j \in I} U_i \cap U_j \rightrightarrows \bigsqcup_{i \in I} U_i$: we picture an arrow as $(x, i) \longrightarrow (x, j)$ if $x \in U_i \cap U_j$. The structure maps should be obvious. In both these examples, the orbit space coincides with the original manifold M and the isotropy groups are all trivial. These are all examples of proper, étale, Lie groupoids. If the open cover is not second countable, then the spaces of arrows and objects are not second countable manifolds. In Lie groupoid theory one sometimes allows manifolds which are not second countable. However, we will always assume manifolds to be second countable.

Example 2. Another groupoid that one can associate to a manifold is the **pair groupoid** $M \times M \rightrightarrows M$: an ordered pair (x, y) determines an arrow $x \longrightarrow y$. This is an example of **transitive groupoid**, i.e., a groupoid with only one orbit. More generally, if $K \curvearrowright P \to M$ is a principal K-bundle, then K acts on the pair groupoid $P \times P \rightrightarrows P$ by groupoid automorphisms and the quotient $P \times_K P \rightrightarrows M$ is a transitive groupoid called the **gauge groupoid**. Note that for any $x \in M$ the isotropy group G_x is isomorphic to K and the principal G_x-bundle $\mathbf{t} : \mathbf{s}^{-1}(x) \to M$ is isomorphic to the original principal bundle. Conversely, it is easy to see that any transitive groupoid $G \rightrightarrows M$ is isomorphic to the gauge groupoid of any of the principal G_x-bundles $\mathbf{t} : \mathbf{s}^{-1}(x) \to M$. It is easy to check that that the gauge groupoid associated with a principal K-bundle $P \to M$ is proper if and only if K is a compact Lie group. Moreover, it is s-proper (respectively, source k-connected) if and only if P is compact (respectively, k-connected).

Example 3. To any surjective submersion $\pi : M \to N$ one can associate the **submersion groupoid** $M \times_N M \rightrightarrows M$. This is a subgroupoid of the pair groupoid $M \times M \rightrightarrows M$, but it fails to be transitive if N has more than one point: the orbits are the fibers of $\pi : M \to N$, so the orbit space is precisely N. The isotropy groups are all trivial. The submersion groupoid is always proper and it is s-proper if and only if π is a proper map.

Example 4. Let $\mathcal{F}$ be a foliation of a manifold M. We can associate to it the **fundamental groupoid** $\Pi_1(\mathcal{F}) \rightrightarrows M$, whose arrows correspond to foliated homotopy classes of paths (relative to the end-points). Given such an arrow $[\gamma]$, the source and target maps are $\mathbf{s}([\gamma]) = \gamma(0)$ and $\mathbf{t}([\gamma]) = \gamma(1)$, while multiplication corresponds to concatenation of paths. One can show that $\Pi_1(\mathcal{F})$ is indeed a manifold, but it may fail to be Hausdorff. In fact, this groupoid is Hausdorff precisely when $\mathcal{F}$ has no vanishing cycles. In Lie groupoid theory, one often allows the total space of a groupoid to be non-Hausdorff, while M and the source/target fibers are always assumed to be Hausdorff. On the other hand, we will always assume manifolds to be second countable.

Another groupoid one can associate to a foliation is the **holonomy groupoid** $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$, whose arrows correspond to holonomy classes of paths. Again, one can show that $\mathrm{Hol}(\mathcal{F})$ is a manifold, but it may fail to be Hausdorff. In general, the fundamental groupoid and the holonomy groupoid are distinct, but there is an obvious groupoid morphism $\mathcal{F} : \Pi_1(\mathcal{F}) \to \mathrm{Hol}(\mathcal{F})$, which to a homotopy class of a path associates the holonomy class of the path (recall that the holonomy only depends on the homotopy class of the path). This map is a local diffeomorphism.

It should be clear that the leaves of these groupoids coincide with the leaves of $\mathcal{F}$ and that the isotropy groups of $\Pi_1(\mathcal{F})$ (respectively, $\mathrm{Hol}(\mathcal{F})$) coincide with the fundamental groups (respectively, holonomy groups) of the leaves. It is easy to check also that $\Pi_1(\mathcal{F})$ is always s-connected. One can show that $\Pi_1(\mathcal{F})$ (respectively, $\mathrm{Hol}(\mathcal{F})$) is s-proper if and only if the leaves of $\mathcal{F}$ are compact and have finite fundamental group (respectively, finite holonomy group). In general, it is not so easy to give a characterization in terms of $\mathcal{F}$ of when these groupoids are proper.

Example 5. Let $K \curvearrowright M$ be a smooth action of a Lie group K on a manifold M. The associated **action groupoid** $K \ltimes M \rightrightarrows M$ has arrows the pairs $(k, x) \in K \times M$, source/target maps given by $\mathbf{s}(k, x) = x$ and $\mathbf{t}(k, x) = kx$, and composition:

$$(k_1, y)(k_2, x) = (k_1 k_2, x), \quad \text{if } y = k_1 x.$$

The isotropy groups of this action are the stabilizers K_x and the orbits coincide with the orbits of the action. The action groupoid is proper (respectively, s-proper) precisely when the action is proper (respectively, K is compact). Moreover, this groupoid is source k-connected if and only if K is k-connected.

Lie groupoids, just like Lie groups, have associated infinitesimal objects known as *Lie algebroids*:

Definition 4. A **Lie algebroid** is a vector bundle $A \to M$ together with a **Lie bracket** $[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ on the space of sections and a bundle map $\rho_A : A \to TM$, called the **anchor**, such that the Leibniz identity:

$$[X, fY]_A = f[X, Y]_A + \rho_A(X)(f)Y, \tag{1}$$

holds for all $f \in C^\infty(M)$ and $X, Y \in \Gamma(A)$.

Given a Lie groupoid $G \rightrightarrows M$ the associated Lie algebroid is obtained as follows. One lets $A := \ker \mathrm{d}_M \mathbf{s}$ be the vector bundle whose fibers consists of the tangent spaces to the s-fibers along the identity section. Then sections of A can be identified with right-invariant vector fields on the Lie groupoid, so the usual Lie bracket of vector fields induces a Lie bracket on the sections. The anchor is obtained by restricting the differential of the target, i.e., $\rho_A := \mathrm{d}\mathbf{t}|_A$.

There is a Lie theory for Lie groupoids/algebroids analogous to the usual Lie theory for Lie groups/algebras. There is however one big difference: Lie's Third Theorem fails and there are examples of Lie algebroids which are not associated with a Lie groupoid ([2]). We shall not give any more details about this correspondence since we will be working almost exclusively at the level of Lie groupoids. We refer the reader to [1] for a detailed discussion of Lie theory in the context of Lie groupoids and algebroids.

Lecture 3: The linearization theorem

For a Lie groupoid $G \rightrightarrows M$ a submanifold $N \subset M$ is called **saturated** if it is a union of orbits. Our aim now is to state the linearization theorem which, under appropriate assumptions, gives a normal form for the Lie groupoid in a neighborhood of a saturated submanifold. First we will describe this local normal form, which depends on some standard constructions in Lie groupoid theory.

Let $G \rightrightarrows M$ be a Lie groupoid. The **tangent Lie groupoid** $TG \rightrightarrows TM$ is obtained by applying the tangent functor: hence, the spaces of arrows and objects are the tangent bundles to G and to M, the source and target maps are the differentials $\mathrm{d}\mathbf{s}, \mathrm{d}\mathbf{t} : TG \to TM$, the multiplication is the differential $\mathrm{d}m : (TG)^2 \equiv TG^{(2)} \to TG$, etc.

Assume now that $S \subset M$ is a saturated submanifold. An important special case to keep in mind is when S consists of a single orbit of G. Then we can restrict the groupoid G to S:

$$G_S := \mathbf{t}^{-1}(S) = \mathbf{s}^{-1}(S),$$

obtaining a Lie subgroupoid $G_S \rightrightarrows S$ of $G \rightrightarrows M$. If we apply the tangent functor, we obtain a Lie subgroupoid $TG_S \rightrightarrows TS$ of $TG \rightrightarrows TM$.

Proposition 5. *There is a short exact sequence of Lie groupoids:*

$$\begin{tikzcd}
1 \arrow[r] & TG_S \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & TG \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \nu(G_S) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 1 \\
0 \arrow[r] & TS \arrow[r] & TM \arrow[r] & \nu(S) \arrow[r] & 0\,.
\end{tikzcd}$$

There is an alternative description of the groupoid $\nu(G_S) \rightrightarrows \nu(S)$ which sheds some light on its nature. For each arrow $g : x \longrightarrow y$ in G_S one can define the linear transformation:

$$T_g : \nu_x(S) \to \nu_y(S), [v] \mapsto [\mathrm{d}_g\mathbf{t}(\tilde{v})],$$

where $\tilde{v} \in T_gG$ is such that $\mathrm{d}_g s(\tilde{v}) = v$. One checks that this map is independent of the choice of lifting $\widetilde{v}$. Moreover, for any identity arrow one has $T_{1_x} = \mathrm{id}_{\nu_x(S)}$ and for any pair of composable arrows $(g,h) \in G^{(2)}$ one finds:

$$T_{gh} = T_g \circ T_h.$$

This means that $G_S \rightrightarrows S$ acts linearly on the normal bundle $\nu(S) \to S$, and one can form the **action groupoid**:

$$G_S \ltimes \nu(S) \rightrightarrows \nu(S).$$

The space of arrows of this groupoid is the fiber product $G_S \times_S \nu(S)$, with source and target maps: $\mathbf{s}(g,v) = v$ and $\mathbf{t}(g,v) = T_g v$. The product of two composable arrows (g,v) and (h,w) is then given by:

$$(g,v)(h,w) = (gh,w).$$

One then checks that:

Proposition 6. *The map $\mathcal{F} : \nu(G_S) \to G_S \ltimes \nu(S)$, $v_g \mapsto (g, [\mathrm{d}_g\mathbf{s}(v_g)])$, is an isomorphism of Lie groupoids.*

Finally, let us observe that the restricted groupoid $G_S \rightrightarrows S$ sits (as the zero section) inside the Lie groupoid $\nu(G_S) \rightrightarrows \nu(S)$ as a Lie subgroupoid. This justifies introducing the following definition:

Definition 5. For a saturated submanifold S of a Lie groupoid $G \rightrightarrows M$ the local **linear model** around S is the groupoid $\nu(G_S) \rightrightarrows \nu(S)$.

Example 6 (Submersions). For the submersion groupoid $G = M \times_N M \rightrightarrows M$ associated with a submersion $\pi : M \to N$, a fiber $S = \pi^{-1}(z)$ is a saturated submanifold (actually, an orbit) and we find that

$$G_S := \{(x, y) \in M \times M : \pi(x) = \pi(y) = z\} = S \times S \rightrightarrows S.$$

is the pair groupoid. Notice that the normal bundle $\nu_M(S)$ is naturally isomorphic to the trivial bundle $S \times T_z N$. It follows that the local linear model is the groupoid:

$$S \times S \times T_z N \rightrightarrows S \times T_z N,$$

which is the direct product of the pair groupoid $S \times S \to S$ and the identity groupoid $T_z N \rightrightarrows T_z N$. More importantly, we can view this groupoid as the submersion groupoid of the projection : $S \times T_z N \to T_z N$.

Example 7 (Foliations). Let $\mathcal{F}$ be a foliation of M and let $L \subset M$ be a leaf. The normal bundle $\nu(\mathcal{F})$ has a natural flat $\mathcal{F}$-connection, which can be described as follows. Given a vector field $Y \in \mathfrak{X}(M)$ let us write $\overline{Y} \in \Gamma(\nu(\mathcal{F}))$ for the corresponding section of the normal bundle. Then, if $X \in \mathfrak{X}(\mathcal{F})$ is a foliated vector field, one sets:

$$\nabla_X \overline{Y} := \overline{[X, Y]}.$$

One checks that this definition is independent of the choice of representative, and defines a connection $\nabla : \mathfrak{X}(\mathcal{F}) \times \Gamma(\nu(\mathcal{F})) \to \Gamma(\nu(\mathcal{F}))$, called the *Bott connection* (also *partial connection*). The Jacobi identity shows that this connection is flat.

Given a path $\gamma : I \to L$ in some leaf L of $\mathcal{F}$, parallel transport along ∇ defines a linear map

$$\tau_\gamma : \nu_{\gamma(0)}(L) \to \nu_{\gamma(1)}(L).$$

Since the connection is flat, it is clear that this linear map only depends on the homotopy class $[\gamma]$. One obtains a linear action of the fundamental groupoid $\Pi_1(\mathcal{F})$ on $\nu(\mathcal{F})$. The restriction of $\Pi_1(\mathcal{F})$ to the leaf L is the fundamental groupoid of the leaf L and we obtain the linear model for $\Pi_1(\mathcal{F})$ along the leaf L as the action groupoid:

$$\Pi_1(L) \ltimes \nu(L) \rightrightarrows \nu(L).$$

The fundamental groupoid $\Pi_1(L)$ is isomorphic to the gauge goupoid of the universal covering space $\widetilde{L} \to L$, viewed as a principal $\pi_1(L)$-bundle. Moreover, $\nu(L)$ is isomorphic to the associated bundle $\widetilde{L} \times_{\pi_1(L,x)} \nu_x(L)$. It follows that the linear model coincides with the fundamental groupoid of the linear foliation of $\nu(L) = \widetilde{L} \times_{\pi_1(L,x)} \nu_x(L)$.

The linear map τ_γ only depends on the holonomy class of γ, since this maps coincides with the linearization of the holonomy action along γ. For this reason, there is a similar description of the linear model of the holonomy groupoid $\mathrm{Hol}(\mathcal{F})$ along the leaf L as an action groupoid:

$$\mathrm{Hol}(\mathcal{F})_L \ltimes \nu(L) \rightrightarrows \nu(L).$$

Also, the holonomy groupoid is isomorphic to the gauge goupoid of the holonomy cover $\widetilde{L}^h \to L$, viewed as a principal $\mathrm{hol}(L)$-bundle and we can also describe the normal bundle $\nu(L)$ as the associated bundle $\widetilde{L}^h \times_{\mathrm{hol}(L,x)} \nu_x(L)$. The underlying foliation of this linear model is still the linear foliation of $\nu(L)$. However, the linear model now depends on the germ of the foliation around L, i.e., on the non-linear holonomy. Unlike the case of the fundamental groupoid, the knowledge of the Bott connection is not enough to build this linear model.

Example 8 (Group actions). Let K be a Lie group that acts on a manifold M and let K_x be the isotropy group of some $x \in M$. For each $k \in K_x$, the map $\Phi_k : M \to M$, $y \mapsto ky$, fixes x and maps the orbit $\mathcal{O}_x$ to itself. Hence, $\mathrm{d}_x\Phi_k$ induces a linear action of K_x on the normal space $\nu_x(\mathcal{O}_x)$, called the *normal isotropy representation*. Using this representation, it is not hard to check that we have a vector bundle isomorphism:

$$\begin{tikzcd} \nu(\mathcal{O}_x) \arrow[rr, "\simeq"] \arrow[dr] & & K \times_{K_x} \nu_x(\mathcal{O}_x) \arrow[dl] \\ & \mathcal{O}_x & \end{tikzcd}$$

where the action $K_x \curvearrowright K \times \nu_x(O_x)$ is given by $k(g,v) := (gk^{-1}, kv)$. Moreover, one has an action of K on $\nu(\mathcal{O}_x)$, which under this isomorphism corresponds to the action:

$$K \curvearrowright K \times_{K_x} \nu_x(O_x), \quad k[(k',v)] = [(kk',v)].$$

Now consider the action Lie groupoid $K \ltimes M \to M$. One checks that the local linear model around the orbit $\mathcal{O}_x$ is just the action Lie groupoid $K \ltimes \nu(\mathcal{O}_x) \rightrightarrows \nu(\mathcal{O}_x)$, which under the isomorphism above corresponds to the action groupoid:

$$K \ltimes (K \times_{K_x} \nu_x(O_x)) \rightrightarrows K \times_{K_x} \nu_x(O_x).$$

As we have already mentioned above, the Linearization Theorem states that, under appropriate conditions, the groupoid is locally isomorphic around a saturated submanifold to its local model. In order to make precise the expression "locally isomorphic" we introduce the following definition:

Definition 6. Let $G \rightrightarrows M$ be a Lie groupoid and $S \subset M$ a saturated submanifold. A **groupoid neighborhood** of $G_S \rightrightarrows S$ is a pair of open sets $U \supset S$ and $\widetilde{U} \supset G_S$ such that $\widetilde{U} \rightrightarrows U$ is a subgroupoid of $G \rightrightarrows M$. A groupoid neighborhood $\widetilde{U} \rightrightarrows U$ is said to be **full** if $\widetilde{U} = G_U$.

Our first version of the linearization theorem reads as follows:

Theorem 7 (Weak linearization). *Let $G \rightrightarrows M$ be a Lie groupoid with a 2-metric $\eta^{(2)}$. Then G is* **weakly linearizable** *around any saturated submanifold $S \subset M$: there are groupoid neighborhoods $\widetilde{U} \rightrightarrows U$ of $G_S \rightrightarrows S$ in $G \rightrightarrows M$ and $\widetilde{V} \rightrightarrows V$ of $G_S \to S$ in the local model $\nu(G_S) \rightrightarrows \nu(S)$, and an isomorphism of Lie groupoids:*

$$(\widetilde{U} \rightrightarrows U) \overset{\phi}{\cong} (\widetilde{V} \rightrightarrows V),$$

which is the identity on G_S.

A 2-metric is a special Riemannian metric in the space of composable arrows $G^{(2)}$. We will discuss them in detail in the next lecture. For now we remark that for a *proper* groupoid 2-metrics exist and, moreover, every groupoid neighborhood contains a full groupoid neighborhood. Hence:

Corollary 8 (Linearization of proper groupoids). *Let $G \rightrightarrows M$ be a proper Lie groupoid. Then G is* **linearizable** *around any saturated submanifold $S \subset M$: there exist open neighborhoods $S \subset U \subset M$ and $S \subset V \subset \nu(S)$ and an isomorphism of Lie groupoids*

$$(G_U \rightrightarrows U) \overset{\phi}{\cong} (\nu(G_S)_V \rightrightarrows V),$$

which is the identity on G_S.

This corollary does not yet yield the various linearization results stated in Lecture 1. The reason is that the assumptions do not guarantee the existence of a saturated neighborhood $S \subset U \subset M$. This can be realized for s-proper groupoids, since for such groupoids every neighborhood U of a saturated embedded submanifold S, contains a saturated neighborhood of S.

Corollary 9 (Invariant linearization of s-proper groupoids). *Let $G \rightrightarrows M$ be an s-proper Lie groupoid. Then G is* **invariantly linearizable** *around any saturated submanifold $S \subset M$: there exist saturated open neighborhoods $S \subset U \subset M$ and $S \subset V \subset \nu(S)$ and an isomorphism of Lie groupoids*

$$(G_U \rightrightarrows U) \overset{\phi}{\cong} (\nu(G_S)_V \rightrightarrows V),$$

which is the identity on G_S.

Example 9. For a proper submersion $\pi : M \to N$ the associated submersion groupoid $M \times_N M \rightrightarrows M$ is s-proper. Using the description of the local model that we gave before, it follows that for any fiber $\pi^{-1}(z)$ there is a saturated open neighborhood $\pi^{-1}(z) \subset U \subset M$ where the submersion if locally isomorphic to the trivial submersion $\pi^{-1}(z) \times V \to V$, for some open neighborhood $0 \in V \subset T_zN$. Hence, we recover the classical Ehresmann Theorem.

Example 10. Let $\mathcal{F}$ be a foliation of M whose leaves are compact with finite holonomy. Then the holonomy groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ is s-proper. Using the description of the local model given before, it follows that for any leaf L there is a saturated neighborhood $L \subset U \subset M$ where the canonical foliation is isomorphic to the linear foliation of $\nu(L) = \widetilde{L}^h \times_{\mathrm{hol}(L,x)} V$, for some $\mathrm{hol}(L,x)$-invariant, neighborhood $0 \in V \subset \nu_x(L)$. Hence, we recover the Local Reeb Stability Theorem.

Example 11. Let $K \times M \to M$ be an action of a *compact* Lie group. Then the the action groupoid is s-proper and we obtain invariant linearization. From the description of the local model, it follows that for any orbit $\mathcal{O}_x$ there is a saturated open neighborhood $\mathcal{O}_x \subset U \subset M$ and a K-equivariant isomorphism $U \simeq K \times_{K_x} V$, where $0 \in V \subset \nu_x(\mathcal{O}_x)$ is a K_x-invariant neighborhood. Hence, we recover the slice theorem for actions of compact groups. For a general proper action, the results above only give weak linearization, which does not allow to deduce the slice theorem. However, due to the particular structure of the action groupoid, every orbit has a saturated neighborhood and one has a uniform bound for the injectivity radius of the 2-metric. This gives invariant linearization and leads to the Slice Theorem for *any* proper action.

Example 12. Let (M, π) be a Poisson manifold. The cotangent bundle T^*M has a natural Lie algebroid structure with anchor $\rho : T^*M \to TM$ given by contraction by π and Lie bracket on sections (i.e., 1-forms):

$$[\alpha, \beta] := \mathcal{L}_{\rho(\alpha)}\beta - \mathcal{L}_{\rho(\beta)}\alpha - \mathrm{d}\pi(\alpha, \beta).$$

In general, this Lie algebroid fails to be integrable. However, under the assumptions of the local normal form theorem stated in Lecture 1, this algebroid is integrable, and then its source 1-connected integration is an s-proper Lie groupoid whose orbits are the symplectic leaves. This groupoid can then be linearized around a symplectic leaf, but this linearization *does not* yet yield the local canonical form for the Poisson structure.

It turns out that the source 1-connected integration is a symplectic Lie groupoid, i.e., there is a symplectic structure on its space of arrows which is compatible with multiplication. One can apply a Moser type trick to further bring the symplectic structure on the local normal form to a canonical form, which then yields the canonical form of the Poisson structure. The details of this approach, which differ from the original proof of the canonical form due to Crainic and Marcut, can be found in [4].

Lecture 4: Groupoid metrics and linearization

Let us recall that a submersion $\pi : (M, \eta) \to N$ is called a **Riemannian submersion** if the fibers are equidistant. The base N gets an induced metric $\pi_*\eta$ for which the linear maps $\mathrm{d}_x\pi : (\ker \mathrm{d}_x\pi)^\perp \to T_{\pi(x)}N$, $x \in M$, are all isometries. More generally, a (possibly singular) foliation in a Riemannian manifold M is called a **Riemannian foliation** if the leaves are equidistant. This is equivalent to the following property: any geodesic which is perpendicular to one leaf at some point stays perpendicular to all leaves that it intersects.

A simple proof of Ehreshman's Theorem can be obtained by choosing a metric on the total space of the submersion $\pi : M \to N$, that makes it into a Riemannian submersion. A partition of unity argument shows that this is always possible. Then the linearization map is just the exponential map of the normal bundle of a fiber, which maps onto a saturated neighborhood of the fiber, provided the submersion is proper. We will see that a proof similar in spirit also works for the general linearization theorem (Theorem 7).

Let $G \rightrightarrows M$ be a Lie groupoid. Like any category, G has a **simplicial model**:

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} G^{(n)} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} G^{(2)} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} G^{(1)} \rightrightarrows G^{(0)} \quad [\longrightarrow G^{(0)}/G^{(1)}]$$

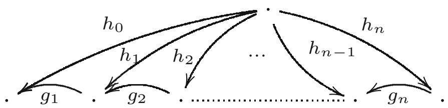
where, for each $n \in \mathbb{N}$, the **face maps** $\varepsilon_i : G^{(n)} \to G^{(n-1)}$, $i = 0, \ldots, n$ and the **degeneracy maps** $\delta_i : G^{(n)} \to G^{(n+1)}$, $i = 1, \ldots, n$, are defined as follows: for an n-string of composable arrows the ith face map associates the $(n-1)$-string of composable arrows obtained by omitting the i-object:

$$\varepsilon_i \left(\cdot \xleftarrow{g_1} \cdot \cdots\cdots \cdot \xleftarrow{g_n} \cdot \right) = \cdot \xleftarrow{g_1} \cdot \cdots\cdots \xleftarrow{g_i g_{i+1}} \cdot \cdots\cdots \cdot \xleftarrow{g_n} \cdot$$

while the ith degeneracy map inserts into an n-string of composable arrows an identity at the ith entry:

$$\delta_i \left(\cdot \xleftarrow{g_1} \cdot \cdots\cdots \cdot \xleftarrow{g_n} \cdot \right) = \cdot \xleftarrow{g_1} \cdot \cdots\cdots \cdot \overset{1_{x_i}}{\circlearrowleft} \cdot \xleftarrow{g_n} \cdot$$

For a Lie groupoid there is, additionally, a natural action of S_{n+1} on $G^{(n)}$: for a string of n-composable arrows $(g_1, \ldots, g_n)$ choose $(n+1)$ arrows $(h_0, \ldots, h_n)$, all with the same source, so that:



Then the S_{n+1}-action on the arrows $(h_0, \ldots, h_n)$ by permutation gives a well-defined S_{n+1}-action on $G^{(n)}$. Notice that this action permutes the face maps ε_i, since there are maps $\phi_i : S_{n+1} \to S_n$ such that:

$$\varepsilon_i \circ \sigma = \phi_i(\sigma) \circ \varepsilon_{\sigma(i)}.$$

Definition 7. An ***n*-metric** ($n \in \mathbb{N}$) on a groupoid $G \to M$ is a Riemannian metric $\eta^{(n)}$ on $G^{(n)}$ which is S_{n+1}-invariant and for which all the face maps $\varepsilon_i : G^{(n)} \to G^{(n-1)}$ are Riemmanian submersions.

Actually, it is enough in this definition to ask that one of the face maps is a Riemannian submersion: the assumption that the action of S_{n+1} is by isometries implies that if one face map is a Riemannian submersion then all the face maps are Riemannian submersions.

For any $n \geq 1$, the metrics induced on $G^{(n-1)}$ by the different face maps $\varepsilon_i : G^{(n)} \to G^{(n-1)}$ coincide, giving a well-defined metric $\eta^{(n-1)}$, which is a $(n-1)$-metric. Obviously, one can repeat this process, so that an n-metric on $G^{(n)}$ determines a k-metric on $G^{(k)}$ for all $0 \leq k \leq n$.

Example 13 (0-metrics). When $n = 0$ we adopt the convention that $\eta^{(0)}$ is a metric on $M = G^{(0)}$ which makes the orbit space a Riemannian foliation and is invariant under the action of G on the normal space to the orbits, i.e., such that each arrow T_g acts by isometries on $\nu(\mathcal{O})$. Such 0-metrics were studied by Pflaum, Posthuma and Tang in [16], in the case of proper groupoids. One can think of such metrics as determining a metric on the (possibly singular) orbit space M/G. Indeed, it is proved in [16] that the distance on the orbit space determined by $\eta^{(0)}$ enjoys some nice properties.

Example 14 (1-metrics). A 1-metric is just a metric $\eta^{(1)}$ on the space of arrows $G = G^{(1)}$ for which the source and target maps are Riemannian submersions and inversion is an isometry. These metrics were studied by Gallego *et al.* in [9]. A 1-metric induces a 0-metric $\eta^{(0)}$ on $M = G^{(0)}$, for which the orbit foliation is Riemannian.

Example 15 (2-metrics). A 2-metric is a metric $\eta^{(2)}$ in the space of composable arrows $G^{(2)}$, which is invariant under the S_3-action generated by the involution $(g_1, g_2) \mapsto (g_2^{-1}, g_1^{-1})$ and the 3-cycle $(g_1, g_2) \mapsto ((g_1 g_2)^{-1}, g_1)$, and for which multiplication is a Riemannian submersion. Hence, a 2-metric on $G^{(2)}$ induces a 1-metric on $G^{(1)}$. These metrics were introduced in [6].

Notice that the first three stages of the nerve

$$\begin{tikzcd} G^{(2)} \arrow[r, shift left=2, "\pi_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_2"'] & G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \end{tikzcd}$$

completely determine the remaining $G^{(n)}$, for $n \geq 3$. Hence, one should expect that n-metrics, for $n \geq 3$, are determined by their 2-metrics. In fact, one has the following properties, whose proof can be found in [7, 8]:

- there is at most one 3-metric inducing a given 2-metric and every 3-metric has a unique extension to an n-metric for every $n \geq 3$.
- there are examples of groupoids which admit an n-metric, but do not admit a $n+1$-metric, for $n = 0, 1, 2$.
- uniqueness fails in low degrees: one can have, e.g., two different 2-metrics on $G^{(2)}$ inducing the same 1-metric on $G^{(1)}$.

The geometric realization of the nerve of a groupoid $G \rightrightarrows M$ is usually denoted by BG, can be seen as the classifying space of principal G-bundles (see [11]). Two Morita equivalent groupoids $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ (see [5] or [14]), give rise to homotopy equivalent spaces BG_1 and BG_2. An alternative point of view, is to think of BG as a geometric stack with atlas $G \rightrightarrows M$, and two atlas represent the same stack if they are Morita equivalent groupoids. The following result shows that one may think of an n-metric as a metric in BG:

Proposition 10 ([7, 8]). *If $G \rightrightarrows M$ and $H \rightrightarrows N$ are Morita equivalent groupoids, then G admits an n-metric if and only if H admits an n-metric.*

In fact, it is shown in [7, 8] that it is possible to "transport" an n-metric via a Morita equivalence. This construction depends on some choices, but the *transversal component* of the n-metric is preserved. We refer to those references for a proof and more details.

Since an n-metric determines a k-metric, for all $0 \leq k \leq n$, a necessary condition for a groupoid $G \rightrightarrows M$ to admit an n-metric is that the orbit foliation can be made Riemannian. This places already some strong restrictions on the class of groupoids admitting an n-metric. So one may wonder when can one find such metrics. One important result in this direction is that any proper groupoid admits such a metric:

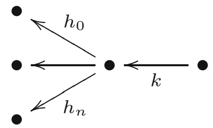
Theorem 11. *A proper Lie groupoid $G \rightrightarrows M$ admits n-metrics for all $n \geq 0$.*

Proof. The proof uses the following trick, called in [6] the **gauge trick**. For each n, consider the manifold $G^{[n]} \subset G^n$ of n-tuples of arrows with the same source, and the map

$$\pi^{(n)} : G^{[n+1]} \to G^{(n)}, \quad (h_0, h_1, \ldots, h_n) \longmapsto (h_0 h_1^{-1}, h_1 h_2^{-1}, \ldots, h_{n-1} h_n^{-1}).$$

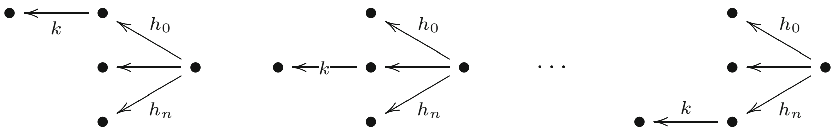
The fibers of $\pi^{(n)}$ coincide with the orbits of the right-multiplication action,

$$G^{[n+1]} \curvearrowleft G$$
$$(h_0, \ldots, h_n) \cdot k = (h_0 k, \ldots, h_n k).$$



and this action is free and proper, hence defining a principal G-bundle. The strategy is to define a metric on $G^{[n+1]}$ in such a way that $\pi^{(n)}$ becomes a Riemannian submersion, and that the resulting metric on $G^{(n)}$ is an n-metric.

The group S_{n+1} acts on the manifold $G^{[n+1]}$ by permuting its coordinates, and this action covers the action $S_{n+1} \curvearrowright G^{(n)}$, so the map $\pi^{(n)}$ is S_{n+1} equivariant. On the other hand, there are $(n+1)$ left groupoid actions $G \curvearrowright G^{[n+1]}$, each consisting in left multiplication on a given coordinate.



These left actions commute with the above right action and cover $(n+1)$-principal actions $G \curvearrowright G^{(n)}$, with projection the face maps $\varepsilon_i : G^{(n)} \to G^{(n-1)}$.

Now for a proper groupoid one can use averaging to construct a metric on G which is invariant under the left action of G on itself by left translations. The product metric on $G^{[n+1]}$ is invariant both under the $(n+1)$ left G-actions above and the S_{n+1}-action. In general, it will not be invariant under the right G-action $G \curvearrowright G^{[n+1]} \to G^{(n)}$, but we can average it to obtain a new metric which is invariant under all actions. It follows that the resulting metric on $G^{(n)}$ is an n-metric. $\square$

Remark 12. The maps $\pi^{(n)} : G^{[n+1]} \to G^{(n)}$, $n = 0, 1, \ldots$ that appear in this proof form the simplicial model for the universal principal G-bundle $\pi : EG \to BG$. This bundle plays a key role in many different constructions associated with the groupoid (see [7]).

Still, there are many examples of groupoids, which are not necessarily proper, but admit an n-metric. Some are given in the next set of examples.

Example 16. The unit groupoid $M \rightrightarrows M$ obviously admits n-metrics. The pair groupoid $M \times M \rightrightarrows M$ and, more generally, the submersion groupoid $M \times_N M \rightrightarrows M$ associated with a submersion $\pi : M \to N$, are proper so also admit n-metrics. For example, if η is a metric on M which makes π a Riemannian submersion, then one can take $\eta^{(0)} = \eta$, $\eta^{(1)} = p_1^*\eta + p_2^*\eta - p_N^*\eta_N$, $\eta^{(2)} = p_1^*\eta + p_2^*\eta + p_3^*\eta - 2p_N^*\eta_N$, etc.

Example 17. Let $\mathcal{F}$ be a foliation of M. Then $\mathrm{Hol}(\mathcal{F})$ and $\Pi_1(\mathcal{F})$ admit an n-metric if and only if $\mathcal{F}$ can be made into a Riemannian foliation. We already know that if these groupoids admit an n-metric, then the underlying foliation, i.e., $\mathcal{F}$, must be Riemannian. Conversely, if $\mathcal{F}$ is Riemannian then $\mathrm{Hol}(\mathcal{F})$ is a proper groupoid (see, e.g., [15]), so it carries an n-metric. Since $\Pi_1(\mathcal{F})$ is a covering of $\mathrm{Hol}(\mathcal{F})$, it also admits an n-metric. Not that, in general, $\Pi_1(\mathcal{F})$ does not need to be proper (e.g., if the fundamental group of some leaf is not finite).

Example 18. Any Lie group admits an n-metric. More generally, the action Lie groupoid associated with any isometric action of a Lie group on a Riemannian manifold admits an n-metric. This can be shown using a gauge trick, similar to the one used in the proof of existence of an n-metric for a proper groupoid sketched above (see [6]). Note that an isometric action does not need not be proper (e.g., if some isotropy group is not compact).

Finally, we can sketch the proof of the main Linearization Theorem, which we restate now in the following way:

Theorem 13. *Let $G \rightrightarrows M$ be a Lie groupoid endowed with a 2-metric $\eta^{(2)}$, and let $S \subset M$ be a saturated embedded submanifold. Then the exponential map defines a weak linearization of G around S.*

Proof. One choses a neighborhood $S \subset V \subset \nu(S)$ where the exponential map of $\eta^{(0)}$ is a diffeomorphism onto its image, and set

$$\widetilde{V} = (\mathrm{d}s)^{-1}(V) \cap (\mathrm{d}t)^{-1}(V) \cap \text{Domain of } \exp_{\eta^{(1)}} \subset \nu(G_S).$$

One shows that $\widetilde{V} \rightrightarrows V$ is a groupoid neighborhood of $G_S \rightrightarrows S$ in the linear model $\nu(G_S) \rightrightarrows \nu(S)$ and that we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{V} \times_V \widetilde{V} \arrow[r, "\exp_{\eta^{(2)}}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & G^{(2)} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\widetilde{V} \arrow[r, "\exp_{\eta^{(1)}}"] \arrow[d, shift left] \arrow[d, shift right] & G \arrow[d, shift left] \arrow[d, shift right] \\
V \arrow[r, "\exp_{\eta^{(0)}}"] & M
\end{tikzcd}$$

It follows that the exponential maps of $\eta^{(1)}$ and $\eta^{(0)}$ give the desired weak linearization, i.e., a groupoid isomorphism

$$(\widetilde{V} \rightrightarrows V) \overset{\exp}{\cong} (\exp(\widetilde{V}) \rightrightarrows \exp(V)). \qquad \square$$

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Higher Braces Via Formal (Non)Commutative Geometry

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Abstract. We translate the main result of [11] to the language of formal geometry. In this new setting we prove directly that the Koszul resp. Börjeson braces are pullbacks of linear vector fields over the formal automorphism $\varphi(a) = \exp(a) - 1$ in the Koszul, resp. $\varphi(a) = a(1-a)^{-1}$ in the Börjeson case. We then argue that both braces are versions of the *same* object, once materialized in the world of formal commutative geometry, once in the noncommutative one.

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Introduction

Let A be a graded commutative associative algebra with a degree $+1$ differential $\nabla : A \to A$ which need not be a derivation. *Koszul braces* are linear degree $+1$ maps $\Phi_n^\nabla : A^{\otimes n} \to A$, $n \geq 1$, defined by the formulas

$$\begin{aligned}
\Phi_1^\nabla(a) &= \nabla(a),\\
\Phi_2^\nabla(a_1,a_2) &= \nabla(a_1a_2) - \nabla(a_1)a_2 - (-1)^{a_1a_2}\nabla(a_2)a_1,\\
\Phi_3^\nabla(a_1,a_2,a_3) &= \nabla(a_1a_2a_3) - \nabla(a_1a_2)a_3 - (-1)^{a_1(a_2+a_3)}\nabla(a_2a_3)a_1\\
&\quad - (-1)^{a_3(a_1+a_2)}\nabla(a_3a_1)a_2 + \nabla(a_1)a_2a_3\\
&\quad + (-1)^{a_1(a_2+a_3)}\nabla(a_2)a_3a_1 + (-1)^{a_3(a_1+a_2)}\nabla(a_3)a_1a_2,\\
&\ \ \vdots\\
\Phi_n^\nabla(a_1,\dots,a_n) &= \sum_{1\leq i\leq n}(-1)^{n-i}\sum_\sigma \varepsilon(\sigma)\nabla(a_{\sigma(1)}\cdots a_{\sigma(i)})a_{\sigma(i+1)}\cdots a_{\sigma(n)},
\end{aligned}$$

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for $a, a_1, a_2, a_3, \ldots \in A$. The sum in the last line runs over all $(i, n-i)$-unshuffles σ, and $\varepsilon(\sigma) = \varepsilon(\sigma; a_1, \ldots, a_n)$ is the Koszul sign (recalled below). As proved for instance in [3], these operations form an L_∞-algebra.

The above construction, attributed to Koszul [6], is sometimes called the *Koszul hierarchy*, cf. also [1–4, 14]. It is, among other things, used to define higher-order derivations of commutative associative algebras: the operator $\nabla : A \to A$ is an order r derivation, $r \geq 1$, if Φ^∇_{r+1} vanishes. Higher-order derivations are crucial for the author's approach to the BRST complex of the closed string field theory [10, Section 4]; for this reason we believe that the braces might be interesting also for physicists.

The assumption of commutativity of A is crucial. Although the operations Φ^∇_n make sense for general A, they do not form any reasonable structure if A is not commutative.

The noncommutative analog of the Koszul hierarchy was found in April 2013 by Kay Börjeson [5] who also proved that the result is an A_∞-algebra; we recall his braces $\{b^\nabla_n\}_{n\geq 1}$ in Subsection 2.2. Amazingly, Börjeson's braces are very different from the Koszul ones. While Φ^∇_n consists of $2^n - 1$ terms, b^∇_n is for each $n \geq 3$ the sum of 4 terms only!

In [11] we proved that both the Koszul and Börjeson braces are the twistings of a linear L_∞- (resp. A_∞-) algebra determined by ∇. In this note we translate this result to the language of formal (non)commutative geometry where L_∞- resp. A_∞-algebras appear as homological vector fields. Namely, we show in Theorems 7 and 14 that both braces are pullbacks of the linear vector field ∇ over a formal automorphism φ of the formal graded affine pointed manifold A. This automorphism equals

$$\varphi(a) := \exp(a) - 1 \tag{1a}$$

in the commutative (Koszul) case, and

$$\varphi(a) := \frac{a}{1-a} \tag{1b}$$

in the non-commutative (Börjeson) case. An important fact is that the Taylor coefficients of both automorphisms are encapsulated in the <u>same</u> formal sum

$$\mathbb{1}_A + \mu^{[2]} + \mu^{[3]} + \cdots$$

where $\mu^{[n]} : A^{\otimes n} \to A$, $n \geq 2$, is the iterated multiplication in A. When interpreted in formal commutative geometry, the associated formal automorphism is (1a) while non-commutative geometry it is (1b). The Koszul resp. Börjeson braces are thus versions of the <u>same</u> object!

As a pleasant by-product of our calculations, we get an important general formula (23) stated without proof in [11].

Conventions. All algebraic objects will be considered over a field $\mathbb{k}$ of characteristic zero. By $\mathbb{1}_U$ or simply by $\mathbb{1}$ when U is understood we denote the identity automorphism of a vector space U. We will reserve the symbol $\mu : A \otimes A \to A$

for the multiplication in an associative algebra A. The product $\mu(a,b)$ of elements $a, b \in A$ will usually be abbreviated as ab.

A *grading* will always mean a $\mathbb{Z}$-grading though most of our results easily translate to the $\mathbb{Z}_2$-graded setting, i.e., to the super world. To avoid problems with dualization, we will assume that all graded vector spaces are of *finite type*.[1] For graded vector spaces U and V we denote by $U \otimes V$ their tensor product over $\mathbb{k}$, and by $Lin(U,V)$ the space of degree 0 $\mathbb{k}$-linear maps $U \to V$.

A permutation $\sigma \in \Sigma_n$ and graded variables $\nu_1, \dots, \nu_n$ determine the *Koszul sign* $\varepsilon(\sigma; \nu_1, \dots, \nu_n) \in \{-1, +1\}$ via the equation

$$\nu_1 \wedge \cdots \wedge \nu_n = \varepsilon(\sigma; \nu_1, \dots, \nu_n) \cdot \nu_{\sigma(1)} \wedge \cdots \wedge \nu_{\sigma(n)},$$

in the free graded commutative associative algebra $\mathbb{S}(\nu_1, \dots, \nu_n)$ generated by $\nu_1, \dots, \nu_n$. We usually write $\varepsilon(\sigma)$ instead of $\varepsilon(\sigma; \nu_1, \dots, \nu_n)$ when the meaning of $\nu_1, \dots, \nu_n$ is clear. For integers $u, v \geq 0$, an (u,v)-*unshuffle* is a permutation $\sigma \in \Sigma_{u+v}$ satisfying

$$\sigma(1) < \cdots < \sigma(u) \ \text{ and } \ \sigma(u+1) < \cdots < \sigma(u+v).$$

1. Recollection of formal geometry

In this section we recall basic concepts of formal (commutative) geometry. We start with polynomial algebras, their completions and duals, and explain how they are related to Taylor series of formal maps. We then interpret L_∞-algebras as homological vector fields on formal graded pointed affine manifolds. All notions recalled here are standard and appeared in various forms in the literature. Their non-commutative variants are briefly addressed in Subsection 2.2.

1.1. Algebras and coalgebras of symmetric tensors

For a finite-dimensional vector space X denote by

$$\mathbb{S}(X) = \bigoplus_{k \geq 1} \mathbb{S}^k(X)$$

the symmetric (polynomial) algebra generated by X. To distinguish the multiplication in $\mathbb{S}(X)$ from other products that may occur in this note we denote the product of two polynomials $p, q \in \mathbb{S}(X)$ by $p \odot q$.

Let $A = X^*$ be the linear dual of X. One has, for each $n \geq 1$, a non-degenerate pairing

$$\langle - \,|\, - \rangle : \mathbb{S}^n(X) \otimes \mathbb{S}^n(A) \to \mathbb{k} \tag{2}$$

given by

$$\langle x_1 \odot \cdots \odot x_n \,|\, a_1 \odot \cdots \odot a_n \rangle := \sum_{\sigma} x_1(a_{\sigma(1)}) \cdots x_n(a_{\sigma(n)}), \tag{3}$$

[1] The general case can be controlled by a linear topology, but it would reach beyond the scope of the present paper.

where $x_1, \ldots, x_n \in X$, $a_1, \ldots, a_n \in A$, and the summation ranges over all permutations $\sigma \in \Sigma_n$. As a particular case of (3) we get

$$\langle x_1 \odot \cdots \odot x_n \,|\, a \odot \cdots \odot a \rangle := n!\; x_1(a) \cdots x_n(a). \tag{4}$$

Notice the factorial $n!$ emerging there.

We will need also the completion

$$\widehat{\mathbb{S}}(X) = \prod_k \mathbb{S}^k(X)$$

of $\mathbb{S}(X)$; $\widehat{\mathbb{S}}(X)$ is the algebra of power series in X. The pairing (3) extends to a non-degenerate pairing

$$\langle - \,|\, - \rangle : \widehat{\mathbb{S}}(X) \otimes \mathbb{S}(A) \to \mathbb{k} \tag{5}$$

that identifies $\widehat{\mathbb{S}}(X)$ with the linear dual of $\mathbb{S}(A)$. This identification is such that the coalgebra structure on $\mathbb{S}(A)$ induced from the algebra structure of $\widehat{\mathbb{S}}(X)$ is given by the deconcatenation coproduct $\Delta : \mathbb{S}(A) \to \mathbb{S}(A) \otimes \mathbb{S}(A)$ defined as

$$\Delta(a_1 \odot \cdots \odot a_n) = \sum_{1 \le j \le n-1} \sum_{\sigma} (a_{\sigma(1)} \odot \cdots \odot a_{\sigma(j)}) \otimes (a_{\sigma(j+1)} \odot \cdots \odot a_{\sigma(n)}), \tag{6}$$

where $a_1, \ldots, a_n \in A$ and σ runs through all $(j, n-j)$ unshuffles. We denote $\mathbb{S}(A)$ viewed as a coalgebra with this coproduct by $\mathbb{S}^c(A)$.

The algebra $\widehat{\mathbb{S}}(X)$ is the free complete commutative associative algebra generated by X. Therefore, for each complete commutative associative algebra B and a linear map $\omega : X \to B$, there exists a unique morphism $h : \widehat{\mathbb{S}}(X) \to B$ of complete algebras such that the diagram

$$\begin{tikzcd} \widehat{\mathbb{S}}(X) \arrow[r, dashed, "h"] & B \\ X \arrow[u, hook, "\iota"] \arrow[ur, bend right, "\omega"'] & \end{tikzcd}$$

in which $\iota : X \hookrightarrow \widehat{\mathbb{S}}(X)$ is the obvious inclusion, commutes. In particular, an endomorphism $\phi : \widehat{\mathbb{S}}(X) \to \widehat{\mathbb{S}}(X)$ is determined by the composition

$$X \xhookrightarrow{\iota} \widehat{\mathbb{S}}(X) \xrightarrow{\phi} \widehat{\mathbb{S}}(X).$$

Likewise, the coalgebra $\mathbb{S}^c(A)$ is the cofree nilpotent cocommutative coassociative coalgebra cogenerated by A [12, Definition II.3.72]. This means the following. Let $\pi : \mathbb{S}^c(A) \twoheadrightarrow A$ be the obvious projection. Then for any nilpotent cocommutative coassociative coalgebra C and for any linear map $\rho : C \to A$, there exists exactly one coalgebra morphism $g : C \to \mathbb{S}^c(A)$ making the diagram

$$\begin{tikzcd} C \arrow[r, dashed, "g"] \arrow[dr, bend right, "\rho"'] & \mathbb{S}^c(A) \arrow[d, two heads, "\pi"] \\ & A \end{tikzcd}$$

commutative. In particular, any endomorphism $\phi : \mathbb{S}^c(A) \to \mathbb{S}^c(A)$ is determined by the composition

$$\mathbb{S}^c(A) \xrightarrow{\phi} \mathbb{S}^c(A) \overset{\pi}{\twoheadrightarrow} A.$$

1.2. Morphisms

Let A be a finite-dimensional vector space, A^* its linear dual, $\mathbb{S}(A^*)$ the symmetric algebra generated by A^*, and V another vector space. Elements of $\mathbb{S}(A^*) \otimes V$ are polynomials with coefficients in V. Each $p \in \mathbb{S}(A^*) \otimes V$ determines a (non-linear) map $f : A \to V$ as follows.

The pairing (2) in the obvious manner extends to a linear map

$$\langle - \,|\, - \rangle : \mathbb{S}^n(A^*) \otimes V \otimes \mathbb{S}^n(A) \longrightarrow V.$$

We associate to every homogeneous polynomial $p_n \in \mathbb{S}^n(A^*) \otimes V$, $n \geq 1$, a map $f_n : A \to V$ defined as

$$f_n(a) := \big\langle p_n \,|\, D^{[n]}(a) \big\rangle \in V, \ \ a \in A, \tag{7}$$

where $D^{[n]} : A \to \mathbb{S}^n(A)$ is the 'diagonal' given by

$$D^{[n]}(a) := \frac{1}{n!}(\underbrace{a \odot \cdots \odot a}_{n \text{ times}}).$$

The factorial $n!$ in the definition of $D^{[n]}$ compensates the factorial in (4). Every polynomial $p \in \mathbb{S}(A^*) \otimes V$ is a finite sum

$$p = p_1 + p_2 + p_3 + \cdots, \ \ p_n \in \mathbb{S}^n(A^*) \otimes V, \ \ n \geq 1,$$

of its homogeneous components. We define $f : A \to V$ corresponding to p as the finite sum

$$f = f_1 + f_2 + f_3 + \cdots, \tag{8}$$

where f_n's are, for $n \geq 1$, as in (7). If the vector space V equals the ground field $\Bbbk$, $\mathbb{S}(A^*) \otimes \Bbbk$ is isomorphic to $\mathbb{S}(A^*)$ and we obtain the standard identification of the polynomial ring $\mathbb{S}(A^*)$ with the algebra of regular functions on the affine space A.

We may formally extend the above construction to the space $\widehat{\mathbb{S}}(A^*) \otimes V$ of power series with coefficients in V. The sum in (8) corresponding to $p \in \widehat{\mathbb{S}}(A^*) \otimes V$ then may be infinite. In this way we interpret power series in $\widehat{\mathbb{S}}(A^*) \otimes V$ as *Taylor coefficients* of maps $A \to V$.[2]

Example 1. Let $\Bbbk = \mathbb{R}$, A be the real affine line $\mathbb{R}$ with the basic vector $e := 1 \in A$, and $x \in A^*$ be such that $x(e) = 1$. For a smooth map $\varphi : A \to A$ with $\varphi(0) = 0$ consider the power series

$$p := \varphi'(0)\, x + \frac{\varphi''(0)}{2!}\, x^2 + \frac{\varphi'''(0)}{3!}\, x^3 + \cdots \ \in \widehat{\mathbb{S}}(A^*)$$

[2]Since we work formally, we do not pay any attention to the convergence issue.

in x. One then easily verifies that $p \in \mathbb{S}^c(A^*)$ determines, via the above construction, a formal map $f : A \to A$ given by

$$f(\alpha e) = \varphi'(0)\,\alpha + \frac{\varphi''(0)}{2!}\,\alpha^2 + \frac{\varphi'''(0)}{3!}\,\alpha^3 + \cdots\ ,\ \alpha \in \mathbb{R}, \tag{9}$$

i.e., the Taylor expansion at 0 of the smooth map φ.

An equivalent representation of formal maps $A \to V$ is based on the identification

$$\widehat{\mathbb{S}}(A^*) \otimes V \cong Lin\big(\mathbb{S}^c(A), V\big) \tag{10}$$

induced by the extended pairing (5). If $q : \mathbb{S}^c(A) \to V$ corresponds, under (10), to $p \in \widehat{\mathbb{S}}(A^*) \otimes V$ then f in (8) is, for $n \geq 1$, simply the composition

$$f = q \circ (1\!\!1 + D^{[2]} + D^{[3]} + \cdots). \tag{11}$$

Example 2 (Continuation of Example 1). Let A be again the real affine line $\mathbb{R}$ with the basic vector $e = 1 \in \mathbb{R}$. Define $q : \widehat{\mathbb{S}}(A) \to A$ by

$$q(\underbrace{e \odot \cdots \odot e}_{n \text{ times}}) := \varphi^{(n)}(0),$$

where $\varphi^{(n)}(0)$ is, for $n \geq 1$, the nth derivative of φ at 0. Clearly

$$q = \varphi'(0)\,1\!\!1 + \varphi''(0)\,\mu^{[2]} + \varphi'''(0)\,\mu^{[3]} + \cdots, \tag{12}$$

where $\mu^{[n]} : \mathbb{S}^n(A) \to A$ denotes the iterated multiplication in $A = \mathbb{R}$. With this choice, (11) gives the Taylor expansion (9). Notice that (12) does not contain any factorials.

Example 3. Suppose that A is a commutative associative algebra and $V = A$. For each $n \geq 1$ one has the map

$$\mu^{[n]} : \mathbb{S}^n(A) \to A$$

that takes $a_1 \odot \cdots \odot a_n \in \mathbb{S}^n(A)$ to the iterated product $a_1 \cdots a_n \in A$. Consider the morphism

$$q := 1\!\!1 + \mu^{[2]} + \mu^{[3]} + \cdots : \mathbb{S}^c(A) \to A. \tag{13}$$

Summation (11) is then the formal map

$$\exp(a) - 1 = a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots\ ,\ a \in A, \tag{14}$$

whose inverse clearly equals

$$\ln(a+1) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k}\ a^k = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots. \tag{15}$$

1.3. Formal manifolds

The previous constructions generalize to graded vector spaces. Then A is interpreted as a formal graded pointed affine manifold,[3] with $\widehat{\mathbb{S}}(A^*)$ playing the rôle of its ring of functions. The power series in $\widehat{\mathbb{S}}(A^*) \otimes V$ represent morphism from A to V, where V is interpreted as another formal graded manifold.

Since we work with graded objects, some expressions should acquire signs dictated by the Koszul sign convention. For instance, the terms of the sum in the right-hand side of (3) must be multiplied by the Koszul sign $\varepsilon(\rho)$ of the permutation

$$\rho : x_1, \ldots, x_n, a_1, \ldots, a_n \longmapsto x_1, a_{\sigma(1)}, \ldots, x_n, a_{\sigma(n)}.$$

This is standard so we will pay no special attention to it. Some issues related to dualization may also arise, but they are not the subject of this note.

Recall that a *vector field* χ on a (classical) smooth manifold M is a section of its tangent bundle. The crucial property of vector fields is that the assignment $f \mapsto \chi(f)$ is a derivation of the ring of smooth functions $f : M \to \mathbb{R}$. There is a contravariant action $\chi \mapsto \varphi^*\chi$ of diffeomorphisms $\varphi : M \to M$ on vector fields characterized by the formula

$$\varphi^*\chi(f)\big(\varphi(a)\big) = \chi(f \circ \varphi)(a) \tag{16}$$

in which $f : M \to \mathbb{R}$ is a smooth function and $a \in M$ a point. We call $\varphi^*\chi$ the *pullback* of the vector field χ *over* the diffeomorphism φ. Let us translate the pullbacks to the situation when we consider instead of M a formal affine graded manifold A as follows.

Vector fields on A are defined as derivations of the algebra of formal functions on A, i.e., derivations χ of the algebra $\widehat{\mathbb{S}}(A^*)$. An automorphism $\varphi : A \to A$ is given by a power series $p \in \widehat{\mathbb{S}}(A^*) \otimes A$, conveniently represented using the duality by a map $\bar{\phi} : A^* \to \widehat{\mathbb{S}}(A^*)$. The invertibility of φ is equivalent to the invertibility of the composition

$$A^* \xrightarrow{\bar{\phi}} \widehat{\mathbb{S}}(A^*) \overset{\pi}{\twoheadrightarrow} A^*.$$

Using the universal property of $\widehat{\mathbb{S}}(A^*)$, we uniquely extend $\bar{\phi}$ to an algebra automorphism $\phi : \widehat{\mathbb{S}}(A^*) \to \widehat{\mathbb{S}}(A^*)$. If χ is a vector field on A, i.e., a derivation of $\widehat{\mathbb{S}}(A^*)$, the action (16) is translated to the adjunction

$$\varphi^*\chi := \phi^{-1} \circ \chi \circ \phi.$$

1.4. L_∞-algebras as homological vector fields

Recall that an L_∞-algebra, also called sh or strongly homotopy Lie algebra, is a graded vector space L equipped with linear graded antisymmetric maps $\lambda_n : L^{\otimes n} \to L$, $n \geq 1$, $\deg(\lambda_n) = 2 - n$, that satisfy a set of axioms saying that λ_1 is a differential, λ_2 obeys the Jacobi identity up to the homotopy λ_3, etc., see for instance [7] or [8].

[3] We will sometimes omit 'pointed' or/and 'formal' to simplify the terminology.

It will be convenient for the purposes of this note to work with the version transferred to the desuspension $A := \downarrow L$. The transferred λ_k's have degree $+1$, are graded symmetric, and satisfy, for each $a_1, \ldots, a_n \in A$, $n \geq 1$, the axiom

$$\sum_{i+j=n+1} \sum_{\sigma} \varepsilon(\sigma) \lambda_j \big(\lambda_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), a_{\sigma(i+1)}, \ldots, a_{\sigma(n)}\big) = 0, \qquad (17)$$

where σ runs over $(i, n-i)$-unshuffles. In this guise, L_∞-algebra is an object $\mathcal{L} = (A, \lambda_1, \lambda_2, \lambda_3, \ldots)$ formed by a graded vector space A and degree $+1$ graded symmetric linear maps $\lambda_n : A^{\otimes n} \to A$, $n \geq 1$, satisfying (17) for each $n \geq 1$.

Let $\mathbb{S}^c(A)$ be, as before, the symmetric coalgebra with the deconcatenation coproduct (6). Thanks to the cofreeness of $\mathbb{S}^c(A)$, each coderivation ϱ of $\mathbb{S}^c(A)$ is determined by its components $\varrho_n := \pi \circ \varrho \circ \iota_n$, $n \geq 1$, where $\pi : \mathbb{S}^c(A) \twoheadrightarrow A$ is the projection and $\iota_n : \mathbb{S}^k(A) \hookrightarrow \mathbb{S}^c(A)$ the inclusion. We write $\varrho = (\varrho_1, \varrho_2, \varrho_3, \ldots)$.

In particular, let $\lambda := (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a degree 1 coderivation of $\mathbb{S}^c(A)$ determined by the linear maps λ_n. By a classical result [7, Theorem 2.3], (17) shrinks to a single equation $\lambda^2 = 0$. Thus an L_∞-algebra is a degree $+1$ coderivation of $\mathbb{S}^c(A)$ that squares to zero.

The linear dual of $\lambda : \mathbb{S}^c(A) \to \mathbb{S}^c(A)$ is a degree -1 derivation ϑ of the algebra $\widehat{\mathbb{S}}(A^*)$ such that $\vartheta^2 = 0$. In the language of formal geometry, ϑ is a degree -1 vector field on the formal affine manifold A that squares to zero. Such an object is called a *homological vector field*. This is expressed by

Definition 4. L_∞-algebras are homological vector fields on formal pointed graded affine manifolds.

Example 5. Linear vector fields on the classical affine pointed manifold $\mathbb{R}^k$ are those of the form

$$f_1 \frac{\partial}{\partial x^1} + \cdots + f_k \frac{\partial}{\partial x^k},$$

where $f_1, \ldots, f_k : \mathbb{R}^k \to \mathbb{R}$ are linear functions. The formal analog of linear vector fields are derivations χ of $\widehat{\mathbb{S}}(A^*)$ such that $\chi(A^*) \subset A^*$.

In the dual setting, linear vector fields are represented by coderivations ∇ of $\mathbb{S}^c(A)$ for which the composition

$$\mathbb{S}^{\geq 2}(A^*) \hookrightarrow \mathbb{S}^c(A^*) \xrightarrow{\chi} \mathbb{S}^c(A^*) \overset{\pi}{\twoheadrightarrow} A$$

where $\mathbb{S}^{\geq 2}(A^*)$ is the subspace of $\mathbb{S}^c(A^*)$ of polynomials with vanishing linear term, is the zero map. Such a coderivation is determined by its restriction to $A \subset \mathbb{S}^c(A)$. This restriction is a linear map $A \to A$ which we denote ∇ again.

Example 6. An important particular case is a linear homological vector field given by a degree $+1$ differential $\nabla : A \to A$, or equivalently, by a 'trivial' L_∞-algebra $(A, \nabla, 0, 0, \ldots)$.

2. The braces

We prove that the Koszul braces, as well as their non-commutative analogs recalled in Subsection 2.2 below, are pullbacks of linear vector fields over specific automorphisms. Throughout this section, A will be a graded associative, in Subsection 2.1 also commutative, algebra, and $\nabla : A \to A$ a degree $+1$ differential which is not required to be a derivation. In Subsection 2.1 we interpret ∇ as a homological vector field on a (commutative) formal manifold or, equivalently, as a trivial L_∞-algebra. In Subsection 2.2 we interpret ∇ as a vector field on a non-commutative formal manifold, or as a trivial A_∞-algebra, cf. Examples 6 resp. 13 below.

2.1. Koszul braces

The aim of this subsection is to prove

Theorem 7. *The Koszul braces recalled in the introduction are given by the pullback of the linear homological field ∇ over the formal diffeomorphism* $\exp -1 : A \to A$.

We will study pullbacks of ∇ over more general diffeomorphisms and derive Theorem 7 as a particular case. This generality would make the calculations more transparent. As a bonus we obtain formula (23) given without proof in [11]. Consider therefore a formal diffeomorphism

$$A \ni a \longmapsto \varphi(a) = f_1 a + \frac{f_2}{2!}a^2 + \frac{f_3}{3!}a^3 + \cdots, \ f_1, f_2, f_3 \in \Bbbk, \ f_1 \neq 0, \tag{18}$$

having the inverse of the form

$$A \ni a \longmapsto \psi(a) = g_1 a + \frac{g_2}{2!}a^2 + \frac{g_3}{3!}a^3 + \cdots, \ g_1, g_2, g_3, \ldots \in \Bbbk.$$

Notice that $g_1 f_1 = 1$.

As in Example 3, one can easily check that φ is associated to the map $q : \mathbb{S}^c(A) \to A$ given by

$$q := f_1 + f_2\mu^{[2]} + f_3\mu^{[3]} + \cdots \ : \mathbb{S}^c(A) \to A. \tag{19}$$

Let us specify which map $\bar\phi : A^* \to \widehat{\mathbb{S}}(A^*)$ corresponds to q under the extended pairing (5).

Since q is an (infinite) sum of $f_n\mu^{[n]}$'s, we determine first which map $\bar\phi_n : A^* \to \mathbb{S}^n(A^*)$ corresponds to $f_n\mu^{[n]} : \mathbb{S}^n(A) \to A$. Such a map $\bar\phi_n$ is characterized by the duality

$$\big\langle x \,|\, f_n\mu^{[k]}(a_1 \odot \cdots \odot a_n)\big\rangle = \langle \bar\phi_n(x) \,|\, a_1 \odot \cdots \odot a_n\rangle$$

which has to hold for any $x \in A^*$ and $a_1, \ldots, a_n \in A$. It is obvious that $\bar\phi_n$ must equal $f_n\Delta^{[n]}$, with $\Delta^{[n]}$ the iterated deconcatenation diagonal (6),

$$\Delta^{[n]} := (\Delta \otimes \mathbb{1}^{\otimes(n-2)}) \circ (\Delta \otimes \mathbb{1}^{\otimes(n-3)}) \circ \cdots \circ \Delta$$

where we put by definition $\Delta^{[1]} := \mathbb{1}_{A^*}$. We therefore have by linearity

$$\bar\phi = f_1 + f_2\Delta + f_3\Delta^{[3]} + f_4\Delta^{[4]} + \cdots.$$

The formula for the map $\bar{\psi} : A^* \to \mathbb{S}^c(A^*)$ associated to the inverse ψ of φ is analogous, namely

$$\bar{\psi} = g_1 + g_2\Delta + g_3\Delta^{[3]} + g_4\Delta^{[4]} + \cdots .$$

The map $\bar{\phi} : A^* \to \widehat{\mathbb{S}}(A^*)$ extends to a unique automorphism ϕ of $\widehat{\mathbb{S}}(A^*)$ given by

$$\phi(x_1 \odot \cdots \odot x_n) = \sum_{i_1,\ldots,i_n \geq 1} f_{i_1} \cdots f_{i_n} \nabla^{[i_1]}(x_1) \odot \cdots \odot \nabla^{[i_n]}(x_n), \tag{20}$$

$x_1, \ldots, x_n \in A^*$. There is a similar obvious formula for the extension of $\bar{\psi}$, but we will not need it.

It is however easier to work in the dual setting when vector fields appear as *co*derivations of $\mathbb{S}^c(A)$ or, equivalently, as linear maps $\mathbb{S}^c(A) \to A$. To see how φ acts on vector fields in this setup, we need to co-extend the map $q : \mathbb{S}^c(A) \to A$ in (19) to a coalgebra morphism $\mathbb{S}^c(A) \to \mathbb{S}^c(A)$. We denote this co-extension by ϕ again, believing this ambiguity will not confuse the reader. It acts on $a_1 \odot \cdots \odot a_n \in \mathbb{S}^n(A)$ by

$$\begin{aligned} &\phi(a_1 \odot \cdots \odot a_n) \\ &\quad = \sum \epsilon(\sigma) \frac{f_{i_1} \cdots f_{i_k}}{k!} (a_{\sigma(1)} \cdots a_{\sigma(i_1)}) \odot \cdots \odot (a_{\sigma(i_1+\cdots+i_{k-1}+1)} \cdots a_{\sigma(n)}), \end{aligned} \tag{21}$$

with the sum taken over all $1 \leq k \leq n$, all $i_1, \ldots, i_k \geq 1$ such that $i_1 + \cdots + i_k = n$, and all permutations $\sigma \in \Sigma_n$ such that

$$\sigma(1) < \cdots < \sigma(i_1),\ \ldots, \sigma(i_1 + \cdots + i_{k-1} + 1) < \cdots < \sigma(n).$$

Formula (21) can be obtained either by dualizing (20) or directly, using the rule

$$\Delta\phi = (\phi \otimes \phi)\Delta$$

describing the interplay between morphisms of coalgebras and coproducts.[4] The rôle of $k!$ in (21) is explained in Example 8 below.

Convention. To shorten the expressions, we will <u>not</u> write the Koszul signs explicitly as they can always be easily filled in. We will also use the shorter $\phi(a_1, \ldots, a_n)$ instead of $\phi(a_1 \odot \cdots \odot a_n)$, etc.

Example 8. Formula (21) for $a, b, c \in A$ gives

$$\begin{aligned} \phi(a, b, c) = f_3(abc) &+ \frac{f_1 f_2}{2!}(ab \odot c + bc \odot a + ca \odot b + a \odot bc + b \odot ca + c \odot ab) \\ &+ \frac{f_1^3}{3!}(a \odot b \odot c + b \odot c \odot a + c \odot a \odot b + a \odot c \odot b + c \odot b \odot a + b \odot a \odot c) \\ &= f_3(abc) + f_1 f_2(ab \odot c + bc \odot a + ca \odot b) + f_1^3(a \odot b \odot c). \end{aligned}$$

The factorial in (21) therefore removes the multiplicities, so that each type of a term appears only once.

[4]Formula (21) must be well known, but we were unable to locate a reference.

Example 9. Let us compute some initial terms of the composition $\bar{\psi}\nabla\phi : \mathbb{S}^c(A) \to A$. For $a \in A$ we have

$$\bar{\psi}\nabla\phi(a) = g_1 f_1 \nabla(a) = \nabla(a)$$

where we used that $g_1 f_1 = 1$. For $a, b \in A$,

$$\begin{aligned}\bar{\psi}\nabla\phi(a,b) &= \bar{\psi}\nabla(f_2\ ab + f_1^2 a \odot b)\\ &= \bar{\psi}\big[f_2\nabla(ab) + f_1^2\big(\nabla(a)\odot b + a \odot \nabla(b)\big)\big]\\ &= g_1 f_2 \nabla(ab) + g_2 f_1^2\big(\nabla(a)b + a\nabla(b)\big),\end{aligned}$$

and, finally, for $a, b, c \in A$ one has

$$\begin{aligned}\bar{\psi}\nabla\phi(a,b,c) &= \bar{\psi}\nabla\big(f_3\, abc + f_2 f_1(ab\odot c + bc \odot a + ca \odot b) + f_1^3\, a\odot b\odot c\big)\\ &= \bar{\psi}\Big[f_3\nabla(abc) + f_2 f_1\big(\nabla(ab)\odot c + \nabla(bc)\odot a + \nabla(ca)\odot b\big)\\ &\quad + f_2 f_1\big(ab\odot\nabla(c) + bc\odot\nabla(a) + ca\odot\nabla(b)\big)\\ &\quad + f_1^3\big(\nabla(a)\odot b\odot c + a\odot\nabla(b)\odot c + a\odot b\odot\nabla(c)\big)\Big]\\ &= g_1 f_3\nabla(abc) + g_2 f_2 f_1\big(\nabla(ab)c + \nabla(bc)a + \nabla(ca)b\big)\\ &\quad + (g_3 f_1^3 + g_2 f_2 f_1)\big(\nabla(a)bc + a\nabla(b)c + ab\nabla(c)\big).\end{aligned}$$

Example 10. For $\varphi(a) = \exp(a) - 1$, $\psi = \ln(a+1)$ one has

$$f_1 = f_2 = f_3 = \cdots = 1$$

and

$$g_1 = 1,\ g_2 = -1,\ g_3 = 2,\ \ldots,\ g_n = (-1)^{n-1}(n-1)!$$

With this particular choice, the calculations of Example 9 lead, up to implicit Koszul signs, to the Koszul braces Φ_1^∇, Φ_2^∇ and Φ_3^∇ recalled in the introduction, i.e.,

$$\bar{\psi}\nabla\phi(a_1\odot\cdots\odot a_n) = \Phi_n^\nabla(a_1\odot\cdots\odot a_n) \tag{22}$$

for $n = 1, 2, 3$.

Let us derive a general formula for $\bar{\psi}\nabla\phi : \mathbb{S}^c(A) \to A$. Using (21), one obtains

$$\begin{aligned}&\bar{\psi}\nabla\phi(a_1,\ldots,a_n)\\ &\quad= \bar{\psi}\nabla\sum\frac{f_{i_1}\cdots f_{i_k}}{k!}(a_{\sigma(1)}\cdots a_{\sigma(i_1)})\odot\cdots\odot(a_{\sigma(i_1+\cdots+i_{k-1}+1)}\cdots a_{\sigma(n)})\\ &\quad= \bar{\psi}\sum\frac{f_{i_1}\cdots f_{i_k}}{(k-1)!}\nabla(a_{\sigma(1)}\cdots a_{\sigma(i_1)})\odot\cdots\odot(a_{\sigma(i_1+\cdots+i_{k-1}+1)}\cdots a_{\sigma(n)})\\ &\quad= \sum g_k\frac{f_{i_1}\cdots f_{i_k}}{(k-1)!}\nabla(a_{\sigma(1)}\cdots a_{\sigma(i_1)})a_{\sigma(i_1+1)}\cdots a_{\sigma(n)}.\end{aligned}$$

The summations in the above display run over the same data as in (21). The substitution $i_1 \mapsto i$ converts the last line into

$$\bar{\psi}\nabla\phi(a_1,\ldots,a_n) = \sum_{1\le i\le n}\sum_{\sigma} c_r f_i \nabla(a_{\sigma(1)},\ldots,a_{\sigma(i)})a_{\sigma(i+1)}\cdots a_{\sigma(n)} \tag{23}$$

where σ runs over all $(i, n-i)$-unshuffles, $r = n - i + 1$, and

$$c_r := \sum_{k\geq 2} \sum_{\substack{i_2+\cdots+i_k=r \\ i_2,\ldots,i_k\geq 1}} g_k \frac{f_{i_2}\cdots f_{i_k}}{(k-1)!} \frac{r!}{i_2!\cdots i_k!}. \tag{24}$$

The integer

$$\frac{r!}{i_2!\cdots i_k!}$$

is, for $r = (i_2 + \cdots + i_k)$, the number of permutations $\sigma \in \Sigma_n$ as in the sum (21) with fixed values $\sigma(1), \ldots, \sigma(i_1)$.

It is a simple exercise on manipulations with power series that

$$c_r := \frac{d^r \psi'\big(\varphi(a)\big)}{da^r}\Big|_{t=0}.$$

Formula (23) is the one we gave without proof in [11, Section 2.4].

In the situation of Theorem 7, $\varphi(a) = \exp(a) - 1$, $\psi(a) = \ln(a+1)$, so $\psi'(a) = (1+a)^{-1}$, thus $\psi'\big(\varphi(a)\big) = e^{-a}$, therefore $c_r = (-1)^r$ for $r \geq 1$. Since $f_s = 1$ for each $s \geq 1$, formula (23) reproduces the Koszul braces in the introduction as direct inspection shows, i.e., (22) holds for every $n \geq 1$. A combinatorial by-product of our calculations is the equality

$$\sum_{k\geq 2} \sum_{\substack{i_2+\cdots+i_k=r \\ i_2,\ldots,i_k\geq 1}} (-1)^{k-1} \frac{r!}{i_2!\cdots i_k!} = (-1)^r.$$

We do not know any elementary proof of this fact.

2.2. Börjeson braces

A non-commutative analog of Koszul braces was constructed by K. Börjeson. For a graded associative, not necessarily commutative, algebra A and a degree $+1$ differential $\nabla : A \to A$, he defined in [5] linear degree $+1$ operators $b_n^\nabla : A^{\otimes n} \to A$ by

$$\begin{aligned}
b_1^\nabla(a) &= \nabla(a),\\
b_2^\nabla(a_1,a_2) &= \nabla(a_1a_2) - \nabla(a_1)a_2 - (-1)^{a_1}a_1\nabla(a_2),\\
b_3^\nabla(a_1,a_2,a_3) &= \nabla(a_1a_2a_3) - \nabla(a_1a_2)a_3\\
&\quad - (-1)^{a_1}a_1\nabla(a_2a_3) + (-1)^{a_1}a_1\nabla(a_2)a_3,\\
b_4^\nabla(a_1,a_2,a_3,a_4) &= \nabla(a_1a_2a_3a_4) - \nabla(a_1a_2a_3)a_4\\
&\quad - (-1)^{a_1}a_1\nabla(a_2a_3a_4) + (-1)^{a_1}a_1\nabla(a_2a_3)a_4,\\
&\ \ \vdots\\
b_k^\nabla(a_1,\ldots,a_k) &= \nabla(a_1\cdots a_k) - \nabla(a_1\cdots a_{k-1})a_k\\
&\quad - (-1)^{a_1}a_1\nabla(a_2\cdots a_k) + (-1)^{a_1}a_1\nabla(a_2\cdots a_{k-1})a_k.
\end{aligned}$$

for $a, a_1, a_2, a_3, \ldots \in A$. He also proved that these operators form an A_∞-algebra. In [11] we showed that Börjeson braces are, as their commutative counterparts, a twisting of ∇ interpreted as a trivial A_∞-algebra. In this subsection we put these results into the context of non-commutative geometry.

Everything in fact translates verbatim with only minor modifications from the commutative case analyzed in the previous parts of this note. For a finite-dimensional vector space A we denote by

$$\mathbb{T}(A^*) = \bigoplus_{k\geq 1} \mathbb{T}^k(A^*),\ \mathbb{T}^k(A^*) := \underbrace{A^* \otimes \cdots \otimes A^*}_{k \text{ times}}$$

the tensor algebra of its dual. If V is another finite-dimensional vector space, then every homogeneous non-commutative polynomial $p_n \in \mathbb{T}^n(A^*) \otimes V$ for $n \geq 1$ determines a map $f_n : A \to V$ defined as in (7), with the only difference that the 'diagonal'

$$D^{[n]}(a) := (\underbrace{a \odot \cdots \odot a}_{n \text{ times}}),$$

now does not involve the factorial. Since every $p \in \mathbb{T}(A^*) \otimes V$ is a finite sum of its homogeneous components, we can linearly extend the above construction and assign to each p a map $f : A \to V$. Passing to the completion

$$\widehat{\mathbb{T}}(A^*) := \prod_{k\geq 1} \mathbb{T}^k(A^*),$$

we interpret non-commutative power series in $\widehat{\mathbb{T}}(A^*) \otimes V$ as non-commutative Taylor coefficients of maps $A \to V$. Since we have, by duality, an isomorphism

$$\widehat{\mathbb{T}}(A^*) \otimes V \cong Lin\big(\mathbb{T}^c(A), V\big)$$

where $\mathbb{T}^c(A)$ is the tensor coalgebra with the deconcatenation coproduct, we may equivalently represent non-commutative Taylor coefficients of maps $f : A \to V$ by linear morphisms $q : \mathbb{T}^c(A) \to V$. Let us give a non-commutative analog of Example 3.

Example 11. Let A be an associative algebra. Then the morphism

$$q := \mathbb{1} + \mu^{[2]} + \mu^{[3]} + \cdots : \mathbb{T}^c(A) \to A \tag{25}$$

represents the non-commutative Taylor series

$$\frac{a}{1-a} = a + a^2 + a^3 + \cdots,\ a \in A \tag{26}$$

whose inverse equals

$$\frac{a}{1+a} = a - a^2 + a^3 - \cdots,\ a \in A.$$

Formal non-commutative differential geometry is build upon the above classical non-graded finite-dimensional affine spaces analogously as explained in Subsection 1.3 for the commutative case. That is, A is now a graded vector space interpreted as a non-commutative formal affine pointed manifold with $\mathbb{T}(A^*)$ its non-commutative ring of regular functions. Given another formal non-commutative affine manifold V, the non-commutative power series in $\widehat{\mathbb{T}}(A^*) \otimes V$ represent formal maps $f : A \to V$. By duality, the same formal maps can equivalently be represented by linear morphisms $q : \mathbb{T}^c(A) \to A$.

Vector fields on a formal non-commutative manifold A are derivations of the complete algebra $\widehat{\mathbb{T}}(A^*)$ or, equivalently, coderivations of the tensor coalgebra $\mathbb{T}^c(A)$. As in the commutative case, automorphisms act on vector fields by adjunction.

A_∞-algebras are non-commutative versions of L_∞-algebras recalled in Subsection 1.4, and their historical precursors [13]. They consist of a graded vector space U together with degree $2-n$ linear maps $m_n : U^{\otimes n} \to U$, $n \geq 1$, such that m_1 is a differential, m_2 is associative up to the homotopy m_3, etc.

As in the case of L_∞-algebras, we transfer the maps $m_n : U^{\otimes n} \to U$ to the desuspension $A := \downarrow U$. These transferred m_n's are all of degree $+1$ and satisfy

$$\sum_{u+v=n+1} \sum_{1\leq i\leq u} m_u(\mathbb{1}_A^{\otimes i-1} \otimes m_v \otimes \mathbb{1}_A^{\otimes u-i}) = 0 \tag{27}$$

for each $n \geq 1$.

Since the tensor coalgebra $\mathbb{T}^c(A)$ is the cofree nilpotent coassociative coalgebra cogenerated by A, each coderivation ϱ of $\mathbb{T}^c(A)$ is uniquely determined by its components $\varrho_n : A^{\otimes n} \to A$, $n \geq 1$, defined as $\varrho_n := \pi \circ \varrho \circ \iota_n$, where $\pi : \mathbb{T}^c(A) \to A$ is the projection and $\iota_n : A^{\otimes n} \hookrightarrow \mathbb{T}^c(A)$ the obvious inclusion. We express this situation by $\varrho = (\varrho_1, \varrho_2, \varrho_3, \ldots)$.

One in particular has a degree $+1$ coderivation $m = (m_1, m_2, m_3, \ldots)$ of $\mathbb{T}^c(A)$ determined by the A_∞-algebra above. As in the L_∞-case, the system (27) is equivalent to a single equation $m^2 = 0$. In other words, an A_∞-algebra is a degree $+1$ coderivation of the tensor coalgebra $\mathbb{T}^c(A^*)$ that squares to zero. Its linear dual $\varpi : \widehat{\mathbb{T}}(A) \to \widehat{\mathbb{T}}(A)$ is a degree -1 derivation that squares to zero, i.e., a *homological vector field*. We may therefore give the following analog of Definition 4:

Definition 12. A_∞-algebras are homological vector fields on formal non-commutative graded pointed affine manifolds.

The notion of linear vector fields translate verbatim from the commutative case. Here is an analog of Example 6:

Example 13. A degree $+1$ differential $\nabla : A \to A$ extends to a degree $+1$ derivation of $\mathbb{T}^c(A)$, i.e., to a linear homological vector field on the formal non-commutative affine pointed manifold A. In other words, it determines a 'trivial' A_∞-algebra $(A, \nabla, 0, 0, \ldots)$.

We finally formulate a non-commutative analog of Theorem 7:

Theorem 14. *The Börjeson braces are given by the pullback of the linear homological field ∇ over the formal diffeomorphism*

$$\frac{a}{1-a} : A \to A$$

of the formal non-commutative pointed affine manifold A.

The proof of this theorem is simpler than that of Theorem 7, since no symmetry enters. We leave it to the reader, as well as the non-commutative analog of (24) as appeared in [11, Section 2.2].

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Conformally Rescaled Noncommutative Geometries

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Abstract. Noncommutative geometry aims to provide a set of mathematical tools to describe spacetime, gravity and quantum field theory at small scales. The paper reviews the idea that noncommutative spaces are described in terms of algebras and their geometry, which is encoded as spectral triples. The latter are basic ingredients of the new notion of Riemannian spin geometry adapted to the language of operator algebras. Using this background we propose a new idea of conformally rescaled and curved spectral triples, which are obtained from a real spectral triple by a nontrivial scaling of the Dirac operator. The obtained family is shown to share many properties with the original spectral triple. We compute the Wodzicki residue and the Einstein–Hilbert functional for such family on the four-dimensional noncommutative torus.

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1. Introduction

It is commonly believed that when approaching the smallest scale of physics, Planck length, current image of space (or space-time) as a differentiable manifold should break down. Still, is not clear whether this would call for a new notion of space or whether we will only need a better consistent description of quantum theory including the theory of quantum gravity. The latter, which is the long-awaited dream of theoretical physics, is still unattainable despite various attempts and huge efforts.

One of possible hints where to look for solutions is coming from simple physical considerations. Even though at the moment our limits of measurement are still quite low when compared to the Planck length – we measure distances of $10^{-15}m$, which is roughly the size of a proton, and time difference of the order

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of $10^{-18}s$, while the Planck length is $10^{-35}m$ – the question about possible limits of our measurement accuracy remains valid. Already in 1958 Salecker and Wigner [29] suggested that quantum mechanics implies:

$$\delta l \geq \left(\frac{\hbar l}{mc}\right)^{\frac{1}{2}},$$

which combined with the general relativity and Schwarzschild radius gives a rough estimate

$$\delta l \geq (l l_P^2)^{\frac{1}{3}},$$

where l_p is Planck length.

As the uncertainty relations are linked to noncommutativity of the observables in quantum theory we should expect that positions itself should be a noncommutative algebra. To link such a description with the classical tools of Riemannian geometry we need to look for a more general mathematical theory, which would imply both *geometry* as well a *noncommutativity* like in a quantum theory. The *Noncommutative Geometry* is a proposition, which goes into this direction.

The paper is organised as follows: we briefly review basic ideas and dictionary of noncommutative geometry and spectral triples (for a more comprehensive introduction with details, examples and references see [26, 27]). Based on the introduced notion we then propose a family of conformally rescaled geometries and study their fundamental properties. As a particular example we present the result of pure gravity functional (Einstein–Hilbert action) for the four-dimensional noncommutative torus, computed first with the use of the Wodzicki residue on the algebra of pseudodifferential symbols as well as using a "naive" approach based on the formalism of moving frames adapted to the noncommutative setting.

2. From spaces and algebras

Classical geometry is based on the principle of describing spaces, which are sets of points equipped with some additional structures. However, the notion of a function (in particular a continuous function, if we have a topological space) appears to be more fundamental. In quantum theory this is even more important, since the classical phase space (space of possible positions and momenta) of a physical object is no longer a space. Moreover, what we usually describe as a state of a physical object corresponds to the expectations values of these operators for a given state (a normalized vector) in the Hilbert space. However, the above picture lacks one significant ingredient, the metric, the ability to *measure* the noncommutative space. Noncommutative Geometry is the first sound mathematical concept, which proposes a consistent way of creating *a geometry* of quantum-like spaces. Its long term goal is to provide a meaningful definition of geometry, which would describe both the fundamental interactions as we know them together with the notion of quantised space (for some arguments and models see for instance [13]).

2.1. The theorems behind it: Gel'fand–Naimark

The basic ideas of noncommutative geometry lie in the theorems, which demonstrated that one can describe topological spaces using the algebra of continuous functions. Such functions form an algebra, more precisely a C^*-algebra. The latter is an involutive Banach algebra, that is, a complex normed algebra, which is complete as a topological space in the norm, and for every element $a \in A$:

$$||aa^*|| = ||a||^2.$$

It is easy to see that with the supremum norm on the space of continuous functions $C(X)$ for some topological space we have:

Remark 1. If X is a (locally) compact Hausdorff space and $C(X)$ is the algebra of continuous functions on X, then $C(X)$ is a commutative (non) unital C^*-algebra.

However, a typical example of a C^*-algebra comes from linear bounded operators on a Hilbert space:

Remark 2. Take a separable Hilbert space $\mathcal{H}$ and $B(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$ (with the operator norm). It is a C^* algebra. Any norm closed subalgebra of $B(\mathcal{H})$ is a C^*-algebra.

What makes these two remarks interesting is the following pair of theorems:

Theorem 3 (Gel'fand–Naimark–Segal, [20]). *Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.*

Theorem 4 (Gel'fand–Naimark [19]). *If a C^* algebra is commutative then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

So, shortly speaking – all C^* algebras are subalgebras of bounded operators on a Hilbert space and the commutative ones correspond $1:1$ to locally compact Hausdorff spaces. This makes all noncommutative C^*-algebras perfect candidates for noncommutative spaces, or spaces with singularities.

2.2. Dictionaries and examples

So far we had just given an idea that there exists a natural way to consider some *objects*, which have no counterparts as topological spaces yet still share a lot of common features with them. A simple example is given by a finite-dimensional matrices (like $M_n(\mathbb{C})$), which for $n > 1$ form a noncommutative algebra, or also, their direct sums. A different, more sophisticated example (one of the best-known ones), is, for instance the so-called noncommutative torus.

Example 1 (Irrational Rotation Algebra aka Noncommutative Torus).
Consider the Hilbert space $L^2(S^1)$ and the following operators:

$$(Uf)(z) = zf(z), \quad (Vf)(z) = f(e^{2\pi i\theta}z),$$

where $0 < \theta < 1$ is an irrational real number. We define $\mathbb{T}^2_\theta$ as a C^*-algebra generated by the unitary operators U, V, U^*, V^*. We easily check that:

$$UV = e^{2\pi i\theta} VU.$$

In fact one just takes the above relation as the defining relation of the noncommutative torus. Although there is no geometric picture what this algebra corresponds to (as there is no space) a good intuition is that the algebra describes the space of all possible leaves of Kronecker foliation (with the parameter θ) of the usual torus. If θ is irrational then all leaves are homeomorphic to the real line and the set of all leaves is not even Hausdorff. Yet passing to the algebra (one can understand it as a certain groupoid algebra) we have a much better description and can study it as a noncommutative manifold.

Remark 5. Let us note that although many of the "noncommutative spaces" (like the noncommutative torus above) are described in terms of deformations of manifolds (families of algebras, which for a certain value of a parameter give a commutative algebra of functions on a manifold) this is not always the case.

In the previous sections we indicated an equivalence between commutative C^*-algebras and spaces. Following the standard literature we just want to point out that this correspondence could be promoted to other topological constructions, like continuous maps between spaces, Cartesian products etc. The following dictionary provides the necessary links:

TOPOLOGY	ALGEBRA
locally compact Hausdorff topological space	nonunital C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact Hausdorff topological space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
Cartesian product	tensor product (completed)

Of course, the above notions are (almost) purely topological and we would like to extend them to more geometric objects. The noncommutative geometry is a programme to establish such correspondence and use it to study objects in the same way differential geometry is used to study spaces. Below is an approximate version of the extended version of the dictionary of noncommutative geometry.

DIFFERENTIAL GEOMETRY	NONCOMMUTATIVE GEOMETRY
vector bundle	finitely generated projective module
differential forms	differential forms
differential forms	Hochschild homology
de Rham cohomology	cyclic cohomology
vector fields	operators
group	Hopf algebra
Lie algebra	Hopf algebra
principal fibre bundle	Hopf–Galois extension
measurable functions	von Neumann algebra
infinitesimals	compact operators
metric	Dirac operator
spin^c geometry	spectral triple
spin geometry	real spectral triple
integrals	exotic traces

3. Spectral triples and the Dirac operator

In differential geometry the recipe to construct the Dirac operator over a spin manifold is rather easy. You start with a compact, closed Riemannian manifold with a fixed metric g. Then you find the Clifford algebra bundle, choose your favourite spinor bundle, then lift the Levi-Civita metric connection to the spinor bundle. If you compose it with the Clifford map then you obtain a first-order differential operator on smooth sections of the spinor bundle. A nontrivial statements can then be proven – that D is, in fact, an elliptic operator, extends to selfadjoint operator on the square-summable sections of the spinor bundle, has compact resolvent and hence a discrete spectrum.

However, a different approach is to use the operational definition. Take an algebra of smooth functions $C^\infty(M)$ represented on a Hilbert space of some sections of a suitable vector bundle over M and look for operators, which behave like the Dirac operators. The crucial point is, of course, in the work "like". What we require is that D needs to be a first-order differential operator with compact resolvent. Having that assured, one recovers the differential calculus (the bimodule of differential one-forms) by setting:

$$df := [D, f], \quad f \in C^\infty(M),$$

understood further as an operator on the Hilbert space. An arbitrary one-form will be $\sum f[D, g]$. Moreover, the following formula gives a nice way to recover the metric on your manifold:

$$d(x, y) = \sup_{||[D,f]|| \leq 1, f \in C^\infty(M)} |f(x) - f(y)|, \quad \forall x, y \in M.$$

These are just the examples – as from the spectral information about the Dirac operator we can recover a lot of information about the additional structures on the manifold. Apart from the differential calculus and the metric we can construct the measure and discover the dimension of the manifold.

3.1. The noncommutative generalisation

We are ready to define what is expected to replace Riemannian spin geometry in the realm of noncommutative algebras. The idea of spectral triples is based on the properties of Dirac operators and constructions we discussed earlier.

Definition 6 (see [3, 4]). A real (even) spectral triple is given by the data ($\mathcal{A}$, π, $\mathcal{H}$, D, J, (γ)), where $\mathcal{A}$ is an involutive algebra, π its faithful bounded star representation on a Hilbert space $\mathcal{H}$, D an selfadjoint operator with compact resolvent, such that $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$, γ is (in the even case) a Hermitian $\mathbb{Z}_2$ grading, $D\gamma = -\gamma D$, and J is an antilinear isometry such that:

$$[J\pi(a)J^{-1}, \pi(b)] = 0, \quad \forall a, b \in \mathcal{A},$$

and

$$\left[J\pi(a)J^{-1}, [D, \pi(b)]\right] = 0, \quad \forall a, b \in \mathcal{A}.$$

The latter requirement is called *the order-one condition*. The dimension of the real spectral triple is defined as the integer n, such that there exists an n-Hochschild cycle with coefficients in the bimodule $\mathcal{A} \otimes \mathcal{A}^{op}$,

$$a_0 \otimes b_0 \otimes a_1 \otimes \cdots \otimes a_n = c \in Z_0(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op}),$$

for which

$$\pi(c) = \pi(a_0)\left(J\pi(b_0)J^{-1}\right)[D, \pi(a_1)] \cdots [D, \pi(a_n)] = \gamma.$$

Moreover, one assumes further relations:

$$DJ = \epsilon JD, \quad J^2 = \epsilon', \quad J\gamma = \epsilon'' \gamma J.$$

where $\epsilon, \epsilon', \epsilon''$ are ± 1 depending on n modulo 8 according to the following rules:

n mod 8	0	1	2	3	4	5	6	7
ϵ	+	−	+	+	+	−	+	+
ϵ'	+	+	−	−	−	−	+	+
ϵ''	+		−		+		−	

If we do not assume existence of J, we have a spectral triple without real structure. If the spectral triple is odd then γ as described above does not exist and the cycle condition reduces to $\pi_D(c) = 1$.

It is reasonable to assume always that the subalgebra of elements of $\mathcal{A}$ which commute with D is $\mathbb{C}$ (in case of the unital algebra $\mathcal{A}$). Otherwise the differential algebra defined by D shall be degenerate, that is there shall be a nontrivial kernel of d in $\mathcal{A}$. We call spectral triples such that $[D, \pi(a)] = 0$ implies $a \in \mathbb{C}$ *non-degenerate*, we always consider only such triples.

The following tells us that the motivating example of spin geometry with Dirac operator is indeed described in this language:

Remark 7. If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ is the Hilbert space of summable sections of the spinor bundle and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

The above definition (which was shown more or less in this form) was proposed by Connes in [2] then developed later by many authors. Details of the proof of the above theorem could be found in [1].

In fact, spectral triples over commutative algebras (which satisfy some additional requirements) are only of that type, thanks to Connes' reconstruction theorem [8]. In other words, commutative spectral triples are equivalent (in the sense of 1:1 correspondence) to compact spin manifolds.

3.2. Examples of spectral triples

Several examples of genuinely noncommutative spectral geometries have already been constructed. The list includes the noncommutative torus [2, 23], more general θ-deformations of manifolds (of which the NC Torus is a special case) [5], Moyal deformation [18], finite matrix algebras: $\oplus_i M_{n_i}(\mathbb{C})$ [22] as well as some specific examples of quantum groups and quantum spaces [9, 12].

We shall review here very briefly the example of the spectral triple over the noncommutative tori, which shall be later used to modify the Dirac operator and introduce a new family of noncommutative metrics.

Example 2. We use the standard presentation of the algebra of d-dimensional noncommutative torus as generated by d unitary elements U_i, $i = 1, \ldots, d$, with the relations

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \quad 1 \leq j, k \leq d,$$

where $0 < \theta_{jk} < 1$ is real and antisymmetric. The smooth algebra $\mathcal{A}(\mathbb{T}^d_\theta)$ is then taken as an algebra of elements

$$a = \sum_{\beta \in \mathbb{Z}^d} a_\beta U^\beta,,$$

where a_β is a rapidly decreasing sequence and

$$U^\beta = U_1^{\beta_1} \cdots U_d^{\beta_d}, \quad \beta \in \mathbb{Z}^d.$$

The natural action of $U(1)^d$ by automorphisms, gives, in its infinitesimal form, d linearly independent derivations on the algebra, which are determined by the action on the generators:

$$\delta_k(U_j) = \delta_{jk} U_j, \quad \forall j, k = 1, \ldots, d,$$

here δ_{jk} denotes the Kronecker delta.

The algebra of the noncommutative torus $\mathcal{A}(\mathbb{T}^d_\theta)$ has a canonical trace:

$$\mathfrak{t}(a) = a_{\mathbf{0}},$$

where $\mathbf{0} = \{0, 0, \ldots, 0\} \in \mathbb{Z}^d$. The trace is invariant with respect to the action of $U(1)^d$, hence it is closed,

$$\mathfrak{t}(\delta_j(a)) = 0, \quad \forall j = 1, \ldots, d, \forall a \in \mathcal{A}(\mathbb{T}^d_\theta).$$

By $\mathcal{H}$ we denote the Hilbert space of the GNS construction with respect to the trace $\mathfrak{t}$ on the C^* completion of $\mathcal{A}(\mathbb{T}^d_\theta)$ and π the associated faithful representation. The elements of the smooth algebra $\mathcal{A}(\mathbb{T}^d_\theta)$ act on $\mathcal{H}$ as bounded operators by left multiplication, whereas the derivations δ_i extend to densely defined selfadjoint operators on $\mathcal{H}$ with the smooth elements of the Hilbert space, $\mathcal{A}(\mathbb{T}^d_\theta)$, in their common domain.

To construct a Dirac operator one usually restricts to the equivariant case [25] postulating that the spectral triple has $U(1)^d$ as the global isometry group. The equivariant Dirac operator (which we can also call flat) is defined over $\mathcal{H} \otimes \mathbb{C}^r$, where $r = 2^{[\frac{n}{2}]}$, as:

$$D = \sum_{i=1}^{n} \gamma_i \delta_i,$$

and γ^i are selfadjoint generators of the Clifford algebra:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}.$$

3.3. Getting numbers out of spectral triples

Having a spectral triple we have very little information on its geometry apart from the data hidden in the properties of the Dirac operator. To recover this information we use the spectrum of D.

Let T be a compact positive operator on a separable Hilbert space such that for sufficiently large $r > 0$ the operator T^r is trace class. Therefore, the function:

$$\zeta_T(z) := \operatorname{Tr} |T|^z,$$

is well defined and holomorphic for $\Re(z) > r$. Taking the analytic continuation of $\zeta_T(z)$ to the rest of the complex plane we obtain a function, which has (possibly) some poles. We may then set for any $d \in \mathbb{R}$:

$$\tau(T) := \operatorname{Res}_{z=d} \zeta_T(z).$$

It appears that for a genuine Dirac-type or Laplace-type operator and most of the known operators for spectral triples the residue is nonzero only for some discrete subset of $\mathbb{R}$. In fact, if D is the Dirac operator on a spin manifold of dimension n then the function $\zeta_{|D|^{-1}}$ (if D has a kernel it is certainly finite dimensional and we can neglect it) may have only first-order poles only at integers on the real axis not exceeding n and, in particular, has a nonzero residue at $z = n$ (which is proportional to the volume of the manifold). One usually shortens the notation writing ζ_D (meaning $\zeta_{|D|^{-1}}$).

Note, that the zeta function we may look at its poles which are located generally in a half of the complex plane and are not necessarily real. The collection of all points, which are the poles of the zeta function of the operator D from a given spectral triple we call *the dimension spectrum*. So, the dimension is not a number – it is a discrete set in a complex plane!

Remark 8. The dimension spectrum of a compact spin manifold M, given by its spectral triple $(C^\infty(M), L^2(S), D)$ is contained in a set: $\{n, n-1, n-2, \ldots\}$ where n is the classical dimension of M. In fact, $z = n$ is always in the dimension spectrum, whereas not all of other the points of the set may belong to the dimension spectrum.

Remark 9. The dimension spectrum may contain complex numbers (with nonzero imaginary part) and any real numbers (for instance, if one considers fractals) see [14], for an example.

3.4. Families of Dirac operators

A single Dirac operator is an interesting object in itself but it corresponds exclusively (in the classical case) to one fixed metric and one chosen spin structure. However, once we have such Dirac operator for a given spectral triple, we might construct an entire family of them by taking all *inner fluctuations* of Dirac operators:

$$\mathsf{D}_A = \{D' : D' = D + A\},$$

where A is a self-adjoint one-form $A = \sum_i a_i[D, b_i]$ and $A = A^*$. Classically this corresponds to the twisting of the Dirac operator by a (trivial in this case) complex line bundle, or – using physics terminology – adding the $U(1)$ gauge field. A generalisation, which involves twisting by nontrivial line bundle is also possible.

Of course, one could ask a question whether the family we get depends on the starting point (that is whether the family is the same if we start with the Dirac already perturbed by a one-form) and it is very convenient that indeed the inner fluctuation of inner fluctuation are inner fluctuations so the family we obtain is not dependent on the initial choice. If we restrict ourselves to real spectral triples then there is a huge difference between the classical (commutative situation) when we have:

Lemma 10. *Commutative real spectral triples (Dirac-type operators over spin manifolds) admit no fluctuations of the type $A = \sum_i a_i[D, b_i]$, however, might admit higher-order fluctuations if their dimension $d > 2$.*

The proof is based on the relations from Definition 6 and the fact, that the real structure for commutative spectral triples over spin manifolds is related to complex conjugation: $JfJ^{-1} = f^*$. Therefore on the one hand side, a real fluctuation of the Dirac operator must be:

$$D_A = D + A + \varepsilon JAJ^{-1},$$

as only then $JD_A = \varepsilon D_A J$. But since A is a selfadjoint one-form and the algebra is commutative:

$$Ja[D,b]J^{-1} = \varepsilon a^*[D,b^*] = -\varepsilon(a[D,b])^* = -\varepsilon a[D,b],$$

hence the fluctuation term identically vanishes. However, observe that this shall be different once the "fluctuations" are allowed to be (more generally) higher-order forms – as then the nontrivial commutation between one-forms will be significant. In particular, in any odd dimensions one can "fluctuate" the Dirac operator of a commutative real spectral triple by a scalar term Φ:

$$D_\Phi = D + \Phi, \quad \Phi = \Phi^* \in \mathcal{A},$$

but only in dimension 3 this has a geometric interpretation of a torsion.

4. Conformally rescaled spectral triples

A completely different family of Dirac operators and spectral triples has been suggested recently for noncommutative tori [6]. While originally the proposed setup used twisted spectral triples, it has a natural formulation in the language of spectral triples. In fact, the rephrasing of the original construction in the language, which we present below fits amazingly well into the entire picture of spectral geometry.

Our starting point is a real spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ and a positive element $h > 0$, $h \in \mathcal{A}$.

Definition 11. A conformally rescaled Dirac operator $D_h = h^o D h^o$ where $h^o = JhJ^{-1}$ defines a conformally rescaled spectral triple over $\mathcal{A}$: $(\mathcal{A}, \mathcal{H}, D_h)$.

Note that the triple is not real. Below we verify that all crucial conditions are satisfied. First of all, since h^o is in the commutant of $\mathcal{A}$, for every $a \in \mathcal{A}$:

$$[D_h, \pi(a)] = h^o[D, \pi(a)]h^o \in B(\mathcal{H}).$$

Since h commutes with γ so does H^o and therefore if the spectral triple was even γ still provides the $\mathbb{Z}_2$ grading for the conformally rescaled triple. The cocycle condition is also satisfied. If $c = a_0 \otimes b_0 \otimes a_1 \otimes \cdots \otimes a_n = c$ is the desired cycle for D then $c_h = a_0 \otimes b_0(h^o)^{-2n} \otimes a_1 \otimes \cdots \otimes a_n$ is good for D_h:

$$\begin{aligned}
\pi_{D_h}(c_h) &= \pi(a_0)\left(J\pi(b_0)J^{-1}\right)\left(J\pi(h^{2n})J^{-1}\right)[D_h, \pi(a_1)]\cdots[D_h, \pi(a_n)] \\
&= \pi(a_0)\left(J\pi(b_0)J^{-1}\right)(h^o)^{-2n}(h^o[D, \pi(a_1)]h^o)\cdots(h^o[D, \pi(a_n)]h^o) \\
&= \pi_D(c).
\end{aligned}$$

Furthermore let us compute the resolvent:

$$(D_h - \lambda)^{-1} = (h^o)^{-1}(D - \lambda(h^o)^{-2})^{-1}(h^o)^{-1}.$$

But $(h^o)^{-2}$ is also a positive bounded operator so the entire expression is compact for $\lambda = \pm i$, for instance (which is sufficient).

It is more complicated to check specific spectral properties of the conformally rescaled Dirac operator, in particular the dimension spectrum. One may only state the following:

Lemma 12. *Let $(\mathcal{H}, A, D, J)$ be a real spectral triple of metric dimension n, that is $|D|^{-(n+\epsilon)}$ is trace class for any $\epsilon > 0$. Then the conformally rescaled spectra triple $(\mathcal{A}, D_h, \mathcal{H})$ has the same metric dimension.*

The proof follows from a simple inequality between positive operators (we assumed that h^o is bounded positive and has a bounded inverse):

$$||(h^o)^{-1}||^{-2}|D| \leq |h^o D h^o| \leq ||h^o||^2|D|.$$

Extending it to respective powers and taking trace we see that trace of $(|h^o D h^o|)^\alpha$ will be estimated by a multiple of trace of $|D|^\alpha$ from both sides.

4.1. The Fredholm module and K-homology class

A spectral triple is an object, which has a significant topological importance when considered as an unbounded Fredholm module. Let us recall the definition of a Fredholm module ofer an algebra $\mathcal{A}$:

Definition 13. A triple $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module iff $F = F^*$, $F^2 = 1$ on $\mathcal{H}$ and for every $a \in \mathcal{A}$ the commutator $[F, \pi(a)]$ is compact on $\mathcal{H}$. If there exists a grading $\gamma = \gamma*$ such that $\gamma^2 = 1$ and $F\gamma = -\gamma F$ on $\mathcal{H}$ then we have an even Fredholm module, otherwise we have an odd Fredholm module.

A properly defined relation based on homotopy between Fredholm modules allows to introduce equivalence classes and show that these classes form an abelian group with respect to natural operations. These groups are, in a sense, corresponding dual objects to K-theory groups of the algebra $\mathcal{A}$: $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. The Chern character (expressed easily for finitely summable Fredholm modules) provides the standard pairing between the K-theory and K-homology groups and factorizes through the classes in cyclic cohomology of the algebra $\mathcal{A}$.

An unbounded Fredholm module (a spectral triple) immediately gives a Fredholm module by an assignment $F = \operatorname{sign}(D)$. Having constructed a family of conformally rescaled triples we might want to check how it affects the topological properties of the triple. Certainly the Fredholm module might not be the same, however what matters is its class in K-homology. We have:

Lemma 14. *The K-homology class of the Fredholm module obtained from the spectral triple of a conformally rescaled Dirac operator D_h does not depend on h.*

As $h > 0$ we define $s = \log h$ by continuous functional calculus. Then $h(t) = e^{ts}$ is a continuous path in $B(\mathcal{H})$ and $F_t = \operatorname{sign}(h(t)^o D h(t)^o)$ will be a continuous path of operators giving us the homotopy between the $(\mathcal{A}, \mathcal{H}, \operatorname{sign}(D))$ and $(\mathcal{A}, \mathcal{H}, \operatorname{sign}(D_h))$.

4.2. The differential calculus

Assume that we have a real spectral triple and a conformally rescaled one. Sind D establishes the first-order differential calculus we may ask a question whether the calculus defined by D_h is isomorphic to the original one.

Lemma 15. *Let $(\mathcal{A}, D, \mathcal{H}, J)$ be a real spectral triple and $D_h = h^o D h^o$ be a conformally rescaled Dirac operator. Then $\Omega^1_D(\mathcal{A}) \cong \Omega^1_{D_h}(\mathcal{A})$.*

We define for any one-form ω in $\Omega^1_D(\mathcal{A})$ the map ϕ_h:

$$\phi_h(\omega) = h^o \omega h^o.$$

Since h is invertible it is a bijective, and since h^o is in the commutant of $\mathcal{A}$ it clearly is a bimodule isomorphism. It remains to verify that:

$$\phi_h(da) = d_h a, \quad \forall a \in \mathcal{A},$$

where $d_h(a) = [D_h, \pi(a)]$. But:

$$\phi_h(da) = h^o[D, \pi(a)]h^o = [h^o D h^o, \pi(a)] = [D_h, \pi(a)] = d_h a.$$

4.3. Partial conformal rescaling

In special cases, where the Dirac operator can be presented as a sum of two (or more) operators, which alone satisfy most of the spectral triple conditions we can repeat the conformal rescaling but only partially. A typical example will be the case of the product of two spectral triples.

Let us assume that $(\mathcal{A}, D, \mathcal{H}, J)$ is a real spectral triple and $D = D_1 + D_2$ and D_1, D_2 are Dirac operators for real spectral triples for the largest subalgebras of $\mathcal{A}$, which is not annihilated (respectively) by them.

Then we can have for $h, k \in \mathcal{A}$ positive and such that inverses are bounded, $D_{h,k} = h^o D_1 h^o + k^0 D_2 k^o$. This is an operator, which has again bounded commutators and compact resolvent. Similar arguments as in the conformal case show that again the metric dimension does not change.

An example of partial conformal rescaling with h arbitrary positive and $k = 1$ was studied for the noncommutative torus in [11].

Remark 16. Note that to obtain the isomorphisms between the respective bimodules of one-forms one needs some additional requirement that the bimodule of one-forms split as a direct sum of two bimodules.

4.4. Fluctuations of conformally rescaled geometries

As a next problem we look into the fluctuations – of the type described earlier but this time with the operator D_h. We have:

Lemma 17. *Fluctuations of the conformally rescaled Dirac operator are conformally rescaled fluctuation of the original Dirac operator.*

Proof. Let us take $D + A$, where $A = \sum_i \pi(a_i)[D, \pi(b_i)]$. Then:

$$\begin{aligned} h^o(D+A)h^o &= D_h + h^o \left(\sum_i \pi(a_i)[D, \pi(b_i)] \right) h^0 \\ &= D_h + \sum_i \pi(a_i)[D_h, \pi(b_i)]. \end{aligned}$$ □

So, conformal rescaling does not change the family of possible fluctuactions.

5. The curvature and Einstein–Hilbert functional

One of the most interesting problems in all these examples appers to be the computation of some "local" geometric objects like scalar curvature. So far, the only approach that allowed to have an insight into such objects depends on the specific form of spectral triple for the noncommutative torus and explicit heat-trace computations using the generalized version of the pseudodifferential calculus for th noncommutative torus.

We shall review here some alternative approach, which is adapted to the case of dimension 4 (most interesting from the point of view of physical applications) and based on Wodzicki residue. We sketch the basic definitions and results below.

5.1. Wodzicki residue on noncommutative tori

In a series of papers [6, 7] and [15, 16] a conformally rescaled metric for the noncommutative two and four-tori was studied. This led to the expressions of Gauss–Bonnet theorem and formulae for the noncommutative counterpart of "dressed" scalar curvature.

The computations used explicitly the possibility to write the Laplace-type operators as pseudodifferential operators on the noncommutative torus with their symbol expansion and the possibility to construct a parametrix for a given elliptic operator.

As it has been shown [17] and more generally in [21] Wodzicki residue exists also in the case of the pseudodifferential calculus over noncommutative tori. This has been shown in full generality (in an explicit way, which follows directly from the classical situation) to the d-dimensional case [28]. The symbol calculus defined in [6] and developed further in [7] (see also [21]) is easily generalised to the d-dimensional case and to the operators defined above. Let us recall that a differential operator of order at most n is of the form

$$P = \sum_{0 \le k \le n} \sum_{|\beta_k| = k} a_{\beta_k} \delta^{\beta_k},$$

where a_{β_k} are assumed to be in the algebra $\mathcal{A}(\mathbb{T}^d_\theta)$, $\beta_k \in \mathbb{Z}^d$ and:

$$|\beta_k| = \beta_1 + \cdots + \beta_d, \qquad \delta^{\beta_k} = \delta_1^{\beta_{k,1}} \cdots \delta_d^{\beta_{k,d}}.$$

Its symbol is:

$$\rho(P) = \sum_{0 \le k \le n} \sum_{|\beta_k| = k} a_{\beta_k} \xi^{\beta_k}, \quad \text{where} \quad \xi^\beta = \xi_1^{\beta_1} \cdots \xi_d^{\beta_d}.$$

On the other hand, let ρ be a symbol of order n, which is assumed to be a C^∞ function from $\mathbb{R}^d$ to $\mathcal{A}(\mathbb{T}^d_\theta)$, which is homogeneous of order n, satisfying certain bounds (see [6] for details). With every such symbol ρ there is associated an operator P_ρ on a dense subset of $\mathbb{H}$ spanned by elements $a \in \mathcal{A}(\mathbb{T}^d_\theta)$:

$$P_\rho(a) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-i\sigma \cdot \xi} \rho(\xi) \alpha_\sigma(a) \, d\sigma d\xi,$$

where

$$\alpha_\sigma(U^\alpha) = e^{i\sigma \cdot \alpha} U^\alpha, \quad \sigma \in \mathbb{R}^d, \alpha \in \mathbb{Z}^d.$$

For two operators P, Q with symbols:

$$\rho(P) = \sum p_\alpha \xi^\alpha, \quad \rho(Q) = \sum q_\beta \xi^\beta,$$

we use the formula, which follows directly from the same computations as in the case of classical calculus of pseudodifferential operators:

$$\rho(PQ) = \sum_{\gamma \in \mathbb{N}^d} \frac{1}{\gamma!} \partial_\xi^\gamma (\rho(P)) \delta^\gamma (\rho(Q)), \tag{1}$$

where $\gamma! = \gamma_1! \cdots \gamma_d!$. In [28] we have shown that

Proposition 18. *Let $\rho = \sum_{j \le k} \rho_j(\xi)$ be a symbol over the noncommutative torus $\mathcal{A}(\mathbb{T}^d_\theta)$. Then the functional:*

$$\rho \mapsto \mathrm{Wres}(\rho) = \int_{S^{d-1}} \mathfrak{t}\left(\rho_{-d}(\xi)\right) d\xi,$$

is a trace over the algebra of symbols.

Then for family of conformally rescaled Laplace-type operators we have computed the following functional:

$$S(h) = \Lambda \ \mathrm{Wres}(D_h^{-4}) + \mathrm{Wres}(D_h^{-2}).$$

and demonstrated that it is not a minimal operator. That signifies that there is no single operator, which minimizes for a fixed h the second term (Einstein–Hilbert functional). Classically the minimal point corresponds to the Laplace operator obtained from the Levi-Civita (torsion free) connection.

5.2. Einstein–Hilbert functional for conformally rescaled Dirac in 4D

Let $h \in J\mathcal{A}(\mathbb{T}^4_\theta)J$, $h > 0$ from the commutatnt of the algebra. We know that for the standard Dirac operator D the conformally rescaled Dirac $D_h = h^{-1} D h^{-1}$ defines a spectral triple with the same metric dimension. For simplicity we denote by h already the element from the commutant. Fixing the dimension of the NC torus to be $d = 4$ and using the above-defined calculus of symbols of pseudodifferential operators on the NC Torus we obtain:

Lemma 19. *The action functional for the conformally rescaled Dirac over four-dimensional noncommutative torus is*

$$S(h) = \Lambda\, \mathfrak{t}(h^8) + \mathfrak{t}\left(h\delta_i(h)\delta_i(h)h + h\delta_i(h)h\delta_i(h)\right).$$

The proof is a straightforward but tedious computation, which is a part of computation made in [28]. It is interesting to compare it with the classical result. Since the Dirac operator is rescaled by h^{-1} from both sides in the commutative case this means that its principal symbol is rescaled by h^{-2} and the metric rescaled by h^4. The curvature scalar of such metric is:

$$R(h) = -12h^{-6}\left((\partial_i h)(\partial_i h) + h(\Delta h)\right),$$

whereas the volume form is h^8. It is easy to see that if h and its derivations commute with each other the result is the classical one, as:

$$h^3(\Delta h) = \partial_a(h^3\partial_a h) - 3h^2(\partial_a h)(\partial_a h),$$

and since on the torus the integral is a closed trace with respect to derivations, we have:

$$\int_{\mathbb{T}^4} \sqrt{g}R = 24h^2(\partial_i h)(\partial_i h).$$

Hence we might consider the operator D_h as truly the correct Dirac for a conformally rescaled noncommutative geometry.

5.3. Derivations and moving frame formalism

Apart from the classical limit there is also another possibility to check whether the above result makes sense. In [24] Rosenberg observed that conformal rescaling of the metric could be translated into the rescaling of derivations, since one can write the conformally rescaled metric tensor in the basis of dual space to derivations (forms) as:

$$g_h = \eta_{ab}(ke^a) \otimes (ke^b).$$

In his paper he studies the geometry and curvature tensors following standard recipe, which can be naturally adapted to this case. Reformulating slightly his approach and using the spin connection rather than Levi-Civita connection one can repeat the computations in arbitrary dimensions.

We introduce, similarly as in the classical case, the spin connection:

$$\omega^a_b = \omega^a_{bc}(ke^c).$$

Assuming metric compatibility and vanishing of the torsion:

$$0 = d(ke^a) + \omega^a_b(ke^b) = (\delta_i k)e^i e^a + \omega^a_{bc}k^2 e^c e^b.$$

we obtain the solution,

$$\omega^a_{bc} = \delta^a_c \delta_b(k)k^{-2}.$$

As the difference from the classical situation is only in the order of terms (as they do not commute with each other) one can easily compute the two-form of the curvature tensor:

$$R^a_b = \delta^a_c\delta_{br}(k)k^{-1}e^r e^c + \delta^a_c\delta_b(k)\delta_r(k^{-1})e^r e^c + \delta^a_c\delta_p(k)k^{-1}\delta_b(k)k^{-1}e^c e^p,$$

and its contraction to the Ricci tensor:

$$\mathrm{Ric}_{bc} = -\delta_{bc}(k)h^{-1} - \delta_b(k)\delta_c(k^{-1}) + \delta_c(k)k^{-1}\delta_b(k)k^{-1}.$$

Finally, one obtains a "naive" expression for the scalar curvature:

$$r = k^{-2}\left(-\delta_{aa}(k)k^{-1} + 2\delta_a(k)k^{-1}\delta_a(k)k^{-1}\right).$$

We call the expression "naive generalization" of the classical scalar curvature as we multiply the Ricci tensor by the conformal factor from the left when contracting it with the metric. We could have done it symmetrically or we could have multiplied from the right. Since later we compute the functional, which involves the trace, this does not play any significant role. On the other hand we always need to remember that this is not a curvature in the classical sense.

To compare with the result for the Dirac operator we need to set $k = h^{-2}$, then,

$$\delta_a(k) = -h^{-1}\delta_a(h)h^{-2} - h^{-2}\delta_a(h)h^{-1}$$

and

$$\begin{aligned}
\delta_{aa}(k) =& - h^{-1}\delta_{aa}(h)h^{-2} - h^{-2}\delta_{aa}(h)h^{-1} \\
&+ h^{-1}\delta_a(h)h^{-1}\delta_a(h)h^{-2} + h^{-2}\delta_a(h)h^{-1}\delta_a(h)h^{-1} \\
&+ h^{-1}\delta_a(h)\left(h^{-1}\delta_a(h)h^{-2} + h^{-2}\delta_a(h)h^{-1}\right) \\
&+ \left(h^{-1}\delta_a(h)h^{-2} + h^{-2}\delta_a(h)h^{-1}\right)\delta_a(h)h^{-1} \\
=& - h^{-1}\delta_{aa}(h)h^{-2} - h^{-2}\delta_{aa}(h)h^{-1} + 2\left(h^{-1}\delta_a(h)h^{-1}\delta_a(h)h^{-2}\right. \\
&\left. + h^{-1}\delta_a(h)h^{-2}\delta_a(h)h^{-1} + h^{-2}\delta_a(h)h^{-1}\delta_a(h)h^{-1}\right).
\end{aligned}$$

Therefore, at the end we have:

$$r(h) = 2h^{-6}\delta_a(h)\delta_a(h) + h^{-6}\delta_{aa}(h)h + h^{-5}\delta_{aa}(h).$$

Finally, we can compute the Einstein–Hilbert functional,

$$\mathfrak{t}(h^8 r) = -2\mathfrak{t}\left(h^2\delta_a(h)\delta_a(h) + h\delta_a(h)h\delta_a(h)\right).$$

where we have used the cyclicity and closedeness of the trace:

$$\mathfrak{t}\left(h^3\delta_{aa}(h)\right) = \mathfrak{t}\left(\delta_a\left(h^3\delta_a(h)\right) - 2h^2\delta_a(h)\delta_a(h) - h\delta_a(h)h\delta_a(h)\right).$$

It is surprising that (up to trivial rescaling) we obtain the same formula as this arising from the Wodzicki residue of the Dirac operator.

6. Conclusions

Noncommutative geometry is a sound and well-motivated theory, which can provide excellent tools to study and describe the geometry of the world. At its current stage, it still is focusing on some simple examples. The presented class of conformally rescaled spectral triples is one of the first steps to go beyond single Dirac operator or fluctuations of Dirac operator and study geometries in a similar manner as we study classical manifold.

As we have mentioned there are different approaches, which should merge to provide a comprehensive picture of the geometries we study. First, we have purely algebraic approach, when we could work with algebraic objects (at least in some cases) like symmetries (also in the Hopf algebra sense), derivations or twisted derivations, differential calculi etc. The second approach is based purely on spectral properties of the Dirac and computation of some geometric quantities using heat-trace expansion or natural traces like Wodzicki residue on an appropriate algebra of (generalized) pseudodifferential operators.

Surprisingly, as shown in the above paper, there might be a link, even in the noncommutative case between these two approaches. It is worth mentioning that a spectacular link between the notion of connection for noncommutative $U(1)$ principal bundles and a new families of Dirac operators was established by the author and L. Dąbrowski in [10]. All these examples and further might be a sound basis for a better understanding of geometry of quantum spaces.

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On Some Quaternionic Coherent States and Wavelets

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Abstract. As a natural extension of the affine groups of the reals and the complexes, leading to one and two-dimensional wavelets, we look at the quaternionic affine group and its representations on both complex and quaternionic Hilbert spaces. We then study the problem of coherent states and wavelets built from these representations.

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1. Coherent states and wavelets

The aim of this note is to outline a recent construction of wavelets and coherent states on both complex and quaternionic Hilbert spaces, using the recently introduced quaternionic affine group [2]. Precursors of this work, involving quaternionic coherent states and wavelets over quaternionic fields, may be be found in [7–9, 11].

We start with some group theoretical preliminaries (see, for example, [3, 4]). Let G be a locally compact group $d\mu$ the left Haar measure on G and $G \ni g \longmapsto U(g)$ a unitary irreducible representation of G on a complex (separable) Hilbert space $\mathfrak{H}$. The representation $U(g)$ is called *square integrable* if there exists a nonzero vector $\phi \in \mathfrak{H}$ such that

$$I(\phi) = \int_G |\langle U(g)\phi \mid \phi\rangle|^2 \, d\mu(g) < \infty.$$

In this case the vector ϕ is said to be *admissible.*

Given an admissible vector ϕ, the set of vectors

$$\eta_g = [I(\phi)]^{-\frac{1}{2}} U(g)\phi, \qquad g \in G \tag{1}$$

are called *coherent states* of the group G for the representation $U(g)$.

The coherent states have many useful and interesting properties. In particular they satisfy the *resolution of the identity*

$$\int_G |\eta_g\rangle\langle\eta_g|\, d\mu(g) = I_{\mathfrak{H}}. \tag{2}$$

Furthermore,

- if the group G is unimodular and $U(g)$ is square integrable, then every vector in $\mathfrak{H}$ is admissible;
- if G is non-unimodular and $U(g)$ is square integrable, then the set of admissible vectors is only dense in $\mathfrak{H}$;
- in this case, the dense set is the domain of an *unbounded* operator C, called the *Duflo–Moore* operator, i.e., if ϕ is admissible then

$$\|C\phi\|^2 < \infty.$$

For a square integrable representation $U(g)$ one also has an *orthogonality relation.* If η_1, η_2 are admissible vectors and $\phi_1, \phi_2 \in \mathfrak{H}$ are arbitrary vectors then

$$\int_G \overline{\langle U(g)\eta_1 \mid \phi_1\rangle}\langle U(g)\eta_2 \mid \phi_2\rangle\, d\mu(g) = \langle C\eta_2 \mid C\eta_1\rangle\langle\phi_1 \mid \phi_2\rangle. \tag{3}$$

Suppose now that $G = \mathbb{R} \rtimes \mathbb{R}^*$, i.e., the *affine group of the line.* A generic element in it has the matrix representation

$$g := (b, a) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a \neq 0, \ \ b \in \mathbb{R}. \tag{4}$$

On the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}, dx)$ this group has the unitary irreducible representation

$$(U(b, a)\phi)(x) = |a|^{-\frac{1}{2}}\phi\left(\frac{x - b}{a}\right). \tag{5}$$

This representation is irreducible and square integrable. Admissible vectors ϕ of this representation are such that their Fourier transforms $\widetilde{\phi}$ satisfy the condition

$$\int_{\mathbb{R}} \frac{|\widetilde{\phi}(k)|^2}{|k|}\, dk < \infty. \tag{6}$$

In the signal analysis and wavelet literature, these vectors are called *mother wavelets* and the corresponding coherent states $\eta_{b,a}$ are called *wavelets.*

A *signal* s is then is then identified with an element of the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}, dx)$ and the function

$$S(b, a) = \langle\eta_{b,a}|s\rangle, \tag{7}$$

is called its *wavelet transform.* The wavelet transform is a *time-frequency* transform. One can also build two-dimensional wavelets and wavelet transforms using the affine group of the complex plane

$$G = \mathbb{C} \rtimes \mathbb{C}^*.$$

Looked upon as a real group this group consists of all translations, dilations and rotations of the two-dimensional plane. Our aim here is to look at the quaternionic affine group

$$G_{\text{aff}}^{\mathbb{H}} = \mathbb{H} \rtimes \mathbb{H}^*, \tag{8}$$

and try to construct coherent states and wavelets for it. It is expected that these wavelets could, for example, be employed in the analysis of *sterophonic or stereographic signals*.

2. Some quaternionic facts

Quaternionic analysis is an active area of current research (see, for example, [5, 6, 10, 12] and references cited therein). Here we shall be using Hilbert spaces over the quaternions and some elements of group representation theory on them. We first list some useful facts about quaternions and the matrix representation of quaternions, that we shall use.

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of $\mathfrak{q}$ is

$$\overline{\mathfrak{q}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3.$$

We shall use the 2×2 matrix representation of the quaternions, in which

$$\mathbf{i} = \sqrt{-1}\sigma_1, \quad \mathbf{j} = -\sqrt{-1}\sigma_2, \quad \mathbf{k} = \sqrt{-1}\sigma_3,$$

and the σ's are the three Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

to which we add

$$\sigma_0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We shall also use the matrix-valued vector $\boldsymbol{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$. Thus, in this representation,

$$\mathfrak{q} = q_0\sigma_0 + i\mathbf{q} \cdot \boldsymbol{\sigma} = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix}, \quad \mathbf{q} = (q_1, q_2, q_3), \tag{9}$$

with the quaternionic conjugate of $\mathfrak{q}$ being given by $\mathfrak{q}^\dagger$.

Introducing two complex variables, which we write as

$$z_1 = q_0 + iq_3, \qquad z_2 = q_2 + iq_1,$$

we may also write

$$\mathfrak{q} = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix}. \tag{10}$$

This representation of a quaternion by two complex numbers will turn out to be the most useful for our purposes.

3. The quaternionic affine group

Let us analyze the *quaternionic affine group* $G^{\mathbb{H}}_{\text{aff}}$ or, what we shall call the *quaternionic wavelet group*. In the 2×2 matrix representation of the quaternions we shall represent an element of $G^{\mathbb{H}}_{\text{aff}}$ as the 3×3 matrix with quaternionic entries

$$g := (\mathfrak{b}, \mathfrak{a}) = \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad \mathfrak{a} \in \mathbb{H}^*, \quad \mathfrak{b} \in \mathbb{H}, \quad \mathbf{0}^T = (0, 0). \tag{11}$$

It can be shown that this group also has has only one irreducible unitary representation on a complex Hilbert space, exactly as the real or complex affine groups. The reason for this is that there is only one nontrivial orbit in the dual of $\mathbb{H}$, under the action of $\mathbb{H}^*$ by right multiplication. In addition to this representation on a complex Hilbert space, we shall also compute the UIR of $G^{\mathbb{H}}_{\text{aff}}$ on a quaternionic Hilbert space.

The group $G^{\mathbb{H}}_{\text{aff}}$ is non-unimodular. The left invariant measure is easily computed to be

$$d\mu(\mathfrak{b}, \mathfrak{a}) = \frac{d\mathfrak{b}\ d\mathfrak{a}}{(\det[\mathfrak{a}])^4}, \tag{12}$$

and similarly, the right Haar measure is given by

$$d\mu_r(\mathfrak{b}, \mathfrak{a}) = \frac{d\mathfrak{b}\ d\mathfrak{a}}{(\det[\mathfrak{a}])^2}. \tag{13}$$

The modular function Δ, such that $d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \Delta(\mathfrak{b}, \mathfrak{a})\ d\mu_r(\mathfrak{b}, \mathfrak{a})$, is

$$\Delta(\mathfrak{b}, \mathfrak{a}) = \frac{1}{(\det[\mathfrak{a}])^2}. \tag{14}$$

As noted earlier, from the general theory of semi-direct products of the type $\mathbb{R}^n \rtimes H$, where H is a subgroup of $GL(n, \mathbb{R})$, and which has open free orbits in the dual of $\mathbb{R}^n$, we know that $G^{\mathbb{H}}_{\text{aff}}$ has exactly one unitary irreducible representation on a complex Hilbert space and moreover, this representation is square-integrable (see, for example, [3, 4] for a detailed discussion).

Thus, using standard techniques, this representation can be realized on the Hilbert space $\mathfrak{K}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$ over the quaternions. We define the representation $G^{\mathbb{H}}_{\text{aff}} \ni (\mathfrak{b}, \mathfrak{a}) \longmapsto U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$,

$$(U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f)(\mathfrak{x}) = \frac{1}{\det[\mathfrak{a}]} f(\mathfrak{a}^{-1}(\mathfrak{x} - \mathfrak{b})), \qquad f \in \mathfrak{K}_{\mathbb{C}}. \tag{15}$$

The *Duflo–Moore operator* C is given in the Fourier domain as the multiplication operator

$$(\widehat{C}\widehat{f})(\mathfrak{k}) = \mathcal{C}(\mathfrak{k})\widehat{f}(\mathfrak{k}), \quad \text{where} \quad \mathcal{C}(\mathfrak{k}) = \left[\frac{2\pi}{|\mathfrak{k}|}\right]^2. \tag{16}$$

The admissibility condition for a mother wavelet is now

$$(2\pi)^4 \int_{\mathbb{R}^4} \frac{|\widehat{f}(\mathfrak{k})|^2}{|\mathfrak{k}|^4}\, d\mathfrak{k} < \infty,$$

and for any two vectors η_1, η_2 in the domain of C and arbitrary $f_1, f_2 \in \mathfrak{K}_{\mathbb{C}}$, one has the orthogonality relation:

$$\int_{G^{\mathbb{H}}_{\text{aff}}} \langle f_1 \mid U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})\eta_1\rangle\langle \eta_2 \mid U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})^* f_2\rangle \; d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \langle C\eta_2 \mid C\eta_1\rangle\langle f_1 \mid f_2\rangle. \tag{17}$$

It is useful to write the orthogonality relation in its operator version:

$$\int_{G^{\mathbb{H}}_{\text{aff}}} U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})|\eta_1\rangle\langle \eta_2| U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})^* \; d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \langle C\eta_2 \mid C\eta_1\rangle I_{\mathfrak{K}_{\mathbb{C}}} \;, \tag{18}$$

which in fact, expresses a generalized resolution of the identity. Indeed, for $\langle C\eta_2 \mid C\eta_1\rangle \neq 0$,

$$\frac{1}{\langle C\eta_2 \mid C\eta_1\rangle} \int_{G^{\mathbb{H}}_{\text{aff}}} U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})|\eta_1\rangle\langle \eta_2| U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})^* \; d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{K}_{\mathbb{C}}} \;. \tag{19}$$

Normalizing η so that $\|C\eta\|^2 = 1$, the family of *complex coherent states or wavelets of the quaternionic group* are

$$\mathfrak{S}_{\mathbb{C}} = \{\eta_{\mathfrak{b},\mathfrak{a}} = U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})\eta \mid (\mathfrak{b}, \mathfrak{a}) \in G^{\mathbb{H}}_{\text{aff}}\}, \tag{20}$$

and we have the associated resolution of the identity,

$$\int_{G^{\mathbb{H}}_{\text{aff}}} |\eta_{\mathfrak{b},\mathfrak{a}}\rangle\langle \eta_{\mathfrak{b},\mathfrak{a}}| \; d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{K}_{\mathbb{C}}} \;. \tag{21}$$

4. UIR of $G^{\mathbb{H}}_{\text{aff}}$ in a quaternionic Hilbert space

As the next step, we construct a unitary irreducible representation of the quaternionic affine group $G^{\mathbb{H}}_{\text{aff}}$ on a quaternionic Hilbert space following [2]. As might be expected, this representation has an intimate connection with the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ in (15) on $\mathfrak{K}_{\mathbb{C}}$.

We consider the Hilbert space $\mathfrak{H}_{\mathbb{H}}$, of quaternionic-valued functions over the quaternions. An element $\mathfrak{f} \in \mathfrak{H}_{\mathbb{H}}$ consists of two complex functions f_1 and f_2 and has the form

$$\mathfrak{f}(\mathfrak{x}) = \begin{pmatrix} f_1(\mathfrak{x}) & -\overline{f_2(\mathfrak{x})} \\ f_2(\mathfrak{x}) & \overline{f_1(\mathfrak{x})} \end{pmatrix}, \quad \mathfrak{x} \in \mathbb{H}, \tag{22}$$

with the norm being given by

$$\|\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}} = \int_{\mathbb{H}} \mathfrak{f}(\mathfrak{x})^\dagger \mathfrak{f}(\mathfrak{x}) \; d\mathfrak{x} = \int_{\mathbb{H}} |\mathfrak{f}(\mathfrak{x})|^2 \; d\mathfrak{x} = \left[\int_{\mathbb{H}} \left(\, |f_1(\mathfrak{x})|^2 + |f_2(\mathfrak{x})|^2 \, \right) \; d\mathfrak{x}\right] \sigma_0. \tag{23}$$

It is clear that the finiteness of this norm implies that both f_1 and f_2 have to be elements of $\mathfrak{K}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$, so that we may write

$$\|\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}} = \left(\, \|f_1\|^2_{\mathfrak{K}_{\mathbb{C}}} + \|f_2\|^2_{\mathfrak{K}_{\mathbb{C}}} \, \right) \sigma_0.$$

In view of this, we may also write $\mathfrak{H}_{\mathbb{H}} = L^2_{\mathbb{H}}(\mathbb{H}, d\mathfrak{x})$.

It is possible to use the usual "bra-ket" notation of physicists. We adopt the convention:

$$(\mathfrak{f}\mid \;=\begin{pmatrix}\langle f_1| & \langle f_2| \\ -\langle \overline{f}_2| & \langle \overline{f}_1|\end{pmatrix},\quad\text{and}\quad \mid \mathfrak{f}) = \begin{pmatrix}|f_1\rangle & -|\overline{f}_2\rangle \\ |f_2\rangle & |\overline{f}_1\rangle\end{pmatrix}, \tag{24}$$

The scalar product of two vectors $\mathfrak{f}, \mathfrak{f}' \in \mathfrak{H}_{\mathbb{H}}$ is

$$\begin{aligned}(\mathfrak{f}\mid \mathfrak{f}') &= \int_{\mathbb{H}} \mathfrak{f}(\mathfrak{x})^\dagger \mathfrak{f}'(\mathfrak{x})\, d\mathfrak{x} \\ &= \begin{pmatrix}\langle f_1 \mid f_1'\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle f_2 \mid f_2'\rangle_{\mathfrak{H}_{\mathbb{C}}} & -\langle f_2' \mid \overline{f}_1\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle f_1' \mid \overline{f}_2\rangle_{\mathfrak{H}_{\mathbb{C}}} \\ \langle \overline{f}_2' \mid f_1\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle \overline{f}_1' \mid f_2\rangle_{\mathfrak{H}_{\mathbb{C}}} & \langle f_1' \mid f_1\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle f_2' \mid f_2\rangle_{\mathfrak{H}_{\mathbb{C}}},\end{pmatrix}\end{aligned} \tag{25}$$

and thus,

$$(\mathfrak{f}\mid \mathfrak{f}')^\dagger = (\mathfrak{f}' \mid \mathfrak{f}).$$

Multiplication by quaternions on $\mathfrak{H}_{\mathbb{H}}$ is defined from the right:

$$(\mathfrak{H}_{\mathbb{H}} \times \mathbb{H}) \ni (\mathfrak{f}, \mathfrak{q}) \longmapsto \mathfrak{f}\mathfrak{q}, \quad \text{such that} \quad (\mathfrak{f}\mathfrak{q})(\mathfrak{x}) = \mathfrak{f}(\mathfrak{x})\mathfrak{q},$$

i.e., we take $\mathfrak{H}_{\mathbb{H}}$ to be a *right quaternionic Hilbert space*.

Note that this convention is consistent with the scalar product (25) in the sense that

$$(\mathfrak{f}\mid \mathfrak{f}'\mathfrak{q}) = (\mathfrak{f}\mid \mathfrak{f}')\mathfrak{q} \quad \text{and} \quad (\mathfrak{f}\mathfrak{q} \mid \mathfrak{f}') = \mathfrak{q}^\dagger(\mathfrak{f}\mid \mathfrak{f}').$$

However, the action of operators $\mathbf{A}$ on vectors $\mathfrak{f} \in \mathfrak{H}_{\mathbb{H}}$ is taken to be from the left $(\mathbf{A}, \mathfrak{q}) \longmapsto \mathbf{A}\mathfrak{f}$. In particular, an operator A on $\mathfrak{K}_{\mathbb{C}}$ defines an operator $\mathbf{A}$ on $\mathfrak{H}_{\mathbb{H}}$ in the manner,

$$(\mathbf{A}\mathfrak{f})(\mathfrak{x}) = \begin{pmatrix}(Af_1)(\mathfrak{x}) & -\overline{(Af_2)(\mathfrak{x})} \\ (Af_2)(\mathfrak{x}) & \overline{(Af_1)(\mathfrak{x})}\end{pmatrix}.$$

Multiplication of operators by quaternions will also be from the left. Thus, $\mathfrak{q}\mathbf{A}$ acts on the vector $\mathfrak{f}$ in the manner

$$(\mathfrak{q}\mathbf{A}\mathfrak{f})(\mathfrak{x}) = \mathfrak{q}(\mathbf{A}\mathfrak{f})(\mathfrak{x}).$$

We shall also need the "rank-one operator"

$$\begin{aligned}\mid \mathfrak{f})(\mathfrak{f}' \mid \; &= \begin{pmatrix}|f_1\rangle & -|\overline{f}_2\rangle \\ |f_2\rangle & |\overline{f}_1\rangle\end{pmatrix}\begin{pmatrix}\langle f_1'| & \langle f_2'| \\ -\langle \overline{f}_2'| & \langle \overline{f}_1'|\end{pmatrix} \\ &= \begin{pmatrix}|f_1\rangle\langle f_1'| + |\overline{f}_2\rangle\langle \overline{f}_2'| & |f_1\rangle\langle f_2'| - |\overline{f}_2\rangle\langle \overline{f}_1'| \\ -|\overline{f}_1\rangle\langle \overline{f}_2'| + |f_2\rangle\langle f_1'| & |\overline{f}_1\rangle\langle \overline{f}_1'| + |f_2\rangle\langle f_2'|\end{pmatrix}.\end{aligned} \tag{26}$$

Orthonormal bases in $\mathfrak{H}_{\mathbb{H}}$ can be built using orthonormal bases in $\mathfrak{K}_{\mathbb{C}}$. We indicated one simple way to do this. Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathfrak{K}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$ and define the vectors

$$\mid \mathbf{\Phi}_n) = \frac{1}{\sqrt{2}}\begin{pmatrix}|\phi_n\rangle & |\phi_n\rangle \\ -|\overline{\phi}_n\rangle & |\overline{\phi}_n\rangle\end{pmatrix}, \quad n = 0, 1, 2, \ldots, \tag{27}$$

in $\mathfrak{H}_{\mathbb{H}}$. It is easy to check that these vectors are orthonormal in $\mathfrak{H}_{\mathbb{H}}$. The fact that they form a basis follows from the fact that the vectors $\{\phi_n\}_{n=0}^{\infty}$ are a basis of $L^2_{\mathbb{C}}(\mathbb{H}, d\mathfrak{x})$. Indeed, writing

$$\mid \mathfrak{f}) = \begin{pmatrix} |f_1\rangle & -|\overline{f}_2\rangle \\ |f_2\rangle & |\overline{f}_1\rangle \end{pmatrix} \in L^2_{\mathbb{H}}(\mathbb{H}, d\mathfrak{x}),$$

we easily verify that

$$\mid \mathfrak{f}) = \sum_{n=0}^{\infty} \mid \mathbf{\Phi}_n)\mathfrak{q}_n,$$

where

$$\mathfrak{q}_n = (\mathbf{\Phi}_n \mid \mathfrak{f}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \langle \phi_n \mid f_1\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle \overline{\phi}_n \mid f_2\rangle_{\mathfrak{H}_{\mathbb{C}}} & -\langle f_2 \mid \overline{\phi}_n\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle f_1 \mid \phi_n\rangle_{\mathfrak{H}_{\mathbb{C}}} \\ \langle \overline{f}_2 \mid \phi_n\rangle_{\mathfrak{H}_{\mathbb{C}}} + \langle \overline{f}_1 \mid \overline{\phi}_n\rangle_{\mathfrak{H}_{\mathbb{C}}} & \langle f_1 \mid \phi_n\rangle_{\mathfrak{H}_{\mathbb{C}}} - \langle f_2 \mid \overline{\phi}_n\rangle_{\mathfrak{H}_{\mathbb{C}}} \end{pmatrix}.$$

A representation of $G^{\mathbb{H}}_{\text{aff}}$ on $\mathfrak{H}_{\mathbb{H}}$ can be obtained by simply transcribing (15) into the present context. We define the operators $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$ on $\mathfrak{H}_{\mathbb{H}}$:

$$(\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f})(\mathfrak{x}) = \frac{1}{\det[\mathfrak{a}]}\mathfrak{f}(\mathfrak{a}^{-1}(\mathfrak{x} - \mathfrak{b})), \qquad \mathfrak{f} \in \mathfrak{H}_{\mathbb{H}}, \tag{28}$$

which by (15) and (24) can also be written as

$$\mid \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}) = \begin{pmatrix} |U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_1\rangle & -|\overline{U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f}_2\rangle \\ U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})|f_2\rangle & |\overline{U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f}_1\rangle \end{pmatrix}. \tag{29}$$

This representation is both unitary and irreducible. The unitarity is easy to verify, since

$$\|\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}\|^2 == \int_{\mathbb{H}} \left(|(U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_1)(\mathfrak{x})|^2 + |(U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})f_2)(\mathfrak{x})|^2\right) \, d\mathfrak{x} \, \sigma_0,$$

which, by the unitarity of the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ on $\mathfrak{K}_{\mathbb{C}}$ gives

$$\|\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}|^2_{\mathfrak{H}_{\mathbb{H}}} = \left(\|f_1\|^2_{\mathfrak{K}_{\mathbb{C}}} + \|f_2\|^2_{\mathfrak{K}_{\mathbb{C}}}\right)\sigma_0 = \|\mathfrak{f}\|^2_{\mathfrak{H}_{\mathbb{H}}}.$$

Similarly, a straightforward argument, based on the irreducibility of $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ on $\mathfrak{K}_{\mathbb{C}}$ leads to the irreducibility of $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$.

Using the Duflo–Moore operator C in (16) for the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$ (see (15)), we define the Duflo–Moore operator $\mathbf{C}$ for the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$:

$$(\mathbf{C}\mathfrak{f})(\mathfrak{x}) = \begin{pmatrix} (Cf_1)(\mathfrak{x}) & -\overline{(Cf_2)(\mathfrak{x})} \\ (Cf_2)(\mathfrak{x}) & \overline{(Cf_1)(\mathfrak{x})} \end{pmatrix}.$$

We say that the vector $\mathfrak{f}$ is *admissible for the representation* $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$ if it is in the domain of $\mathbf{C}$, i.e., if both f_1 and f_2 are admissible for the representation $U_{\mathbb{C}}(\mathfrak{b}, \mathfrak{a})$. It is then easy to see that the set of admissible vectors is dense in $\mathfrak{H}_{\mathbb{H}}$.

Let $\mathfrak{f}$ and $\mathfrak{f}'$ be two admissible vectors. Then from (29), 26) and (18) we get

$$\int_{G^{\mathbb{H}}_{\text{aff}}} \mid \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f})(\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f}' \mid d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = \mathfrak{q} \; I_{\mathfrak{H}_{\mathbb{H}}}, \tag{30}$$

where $\mathfrak{q}$ denotes the operator of multiplication from the left, on the Hilbert space $\mathfrak{H}_{\mathbb{H}}$, by the quaternion

$$\mathfrak{q} = \begin{pmatrix} \langle Cf_1' \mid Cf_1\rangle_{\mathfrak{H}_\mathbb{C}} + \langle \overline{Cf'}_2 \mid \overline{Cf}_2\rangle_{\mathfrak{H}_\mathbb{C}} & \langle Cf_2' \mid Cf_1\rangle_{\mathfrak{H}_\mathbb{C}} - \langle \overline{Cf'}_1 \mid \overline{Cf}_2\rangle_{\mathfrak{H}_\mathbb{C}} \\ \langle Cf_1' \mid Cf_2\rangle_{\mathfrak{H}_\mathbb{C}} - \langle \overline{Cf'}_2 \mid \overline{Cf}_1\rangle_{\mathfrak{H}_\mathbb{C}} & \langle \overline{Cf'}_1 \mid \overline{Cf}_1\rangle_{\mathfrak{H}_\mathbb{C}} + \langle Cf_2' \mid Cf_2\rangle_{\mathfrak{H}_\mathbb{C}} \end{pmatrix}. \tag{31}$$

Equation (30) expresses the square-integrability condition for the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$.

In particular, with $\mathfrak{f} = \mathfrak{f}'$, we get the resolution of the identity,

$$\left[\|\mathbf{C}\mathfrak{f}\|^2_{\mathfrak{H}_\mathbb{H}}\right]^{-1} \int_{G^{\mathbb{H}}_{\text{aff}}} \mid \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f})(\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\mathfrak{f} \mid d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{H}_\mathbb{H}}. \tag{32}$$

5. Wavelets and reproducing kernels

Let $\boldsymbol{\eta} \in \mathfrak{H}_{\mathbb{H}}$ be an admissible vector for the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$, normalized so that

$$\|\mathbf{C}\boldsymbol{\eta}\|^2 = 1.$$

We define the *quaternionic wavelets or coherent states of the quaternionic affine group* to be the vectors

$$\mathfrak{S}_{\mathbb{H}} = \{\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}} = \mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})\boldsymbol{\eta} \mid (\mathfrak{b}, \mathfrak{a}) \in G^{\mathbb{H}}_{\text{aff}}\}, \tag{33}$$

By virtue of (30) they satisfy the resolution of the identity

$$\int_{G^{\mathbb{H}}_{\text{aff}}} |\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}})(\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}}| \, d\mu_\ell(\mathfrak{b}, \mathfrak{a}) = I_{\mathfrak{H}_\mathbb{H}} \,. \tag{34}$$

There is the associated reproducing kernel $\mathbf{K} : G^{\mathbb{H}}_{\text{aff}} \times G^{\mathbb{H}}_{\text{aff}} \longrightarrow \mathbb{H}$,

$$\mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}', \mathfrak{a}') = (\boldsymbol{\eta}_{\mathfrak{b},\mathfrak{a}} \mid \boldsymbol{\eta}_{\mathfrak{b}',\mathfrak{a}'})_{\mathfrak{H}_\mathbb{H}} \,, \tag{35}$$

with the usual properties,

$$\mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}', \mathfrak{a}') = \overline{\mathbf{K}(\overline{\mathfrak{b}'}, \overline{\mathfrak{a}'};\ \mathfrak{b}, \mathfrak{a})}, \qquad \mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}, \mathfrak{a}) > 0,$$

$$\int_{G^{\mathbb{H}}_{\text{aff}}} \mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}'', \mathfrak{a}'')\, \mathbf{K}(\overline{\mathfrak{b}''}, \overline{\mathfrak{a}''};\ \mathfrak{b}', \mathfrak{a}')\, d\mu_\ell(\mathfrak{b}'', \mathfrak{a}'') \quad = \quad \mathbf{K}(\overline{\mathfrak{b}}, \overline{\mathfrak{a}};\ \mathfrak{b}', \mathfrak{a}').$$

This kernel then defines a Hilbert space with properties very similar to reproducing kernel Hilbert spaces over the complexes. In particular, the entire theory of the wavelets and coherent states presented here may be formulated on such a Hilbert space.

6. Conclusion

The idea of formulating quantum mechanics and other physical theories using quaternions and quaternionic Hilbert spaces is an old one [1]. The present work may be thought of as being an attempt to develop a signal analytic counterpart of such theories. Interesting questions to pursue next could be:

1. The problem of a unitary embedding of $\mathfrak{H}_{\mathbb{H}}$ into $L^2_{\mathbb{H}}(G^{\mathbb{H}}_{\text{aff}}, d\mu_\ell)$.
2. Extensions of the representation $\mathbf{U}_{\mathbb{H}}(\mathfrak{b}, \mathfrak{a})$, e.g., by multiplying from the right by the $SU(2)$ part of $\mathfrak{a}$.
3. Physical applications of the wavelets and coherent states described here, particularly in the analysis of stereophonic and stereoscopic signals.
4. The problem of discretizing the continuous family of wavelets and of building quaternionic wavelet frames.

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Supersymmetric Vector Coherent States for Systems with Zeeman Coupling and Spin-Orbit Interactions

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Abstract. This work addresses a method for constructing supersymmetric vector coherent states (VCS) for a two-dimensional electron gas in a perpendicular magnetic field in the presence of both Rashba and Dresselhaus spin-orbit (SO) interactions, with an effective Zeeman coupling. The model Hamiltonian, decomposed into two conveniently defined operators, acts on a tensor product of two Hilbert spaces associated with corresponding chiral sectors. Supersymmetric (SUSY) VCS, related to a SUSY pair of Hamiltonians, are built. A generalized oscillator algebra is provided using quadrature operators. Mean-values of position and momentum operators and uncertainty relation in the SUSY VCS are obtained and discussed. Besides, the SUSY VCS time evolution is also given.

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1. Introduction

Standard and nonlinear vector coherent states (VCS) were constructed in quantum optics for nonlinear spin-Hall effect and for the Jaynes–Cummings models [1, 2]. In the context of supersymmetric (SUSY) quantum mechanics, CS were studied in some previous works, see for example [3–5] and references therein.

Our work deals with the construction of VCS for a Hamiltonian operator describing a two-dimensional electron gas in a perpendicular magnetic field in the presence of both Rashba and Dresselhaus spin-orbit (SO) interactions, with effective Zeeman coupling [6].

2. Preliminaries

The CS [7, 8] can be defined over complex domains in the Hilbert space $\mathfrak{H} = \text{span}\{\phi_m, m \in \mathbb{N}\}$ as [9, 10]

$$|z\rangle = (\mathcal{N}(|z|))^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} |\phi_m\rangle, \qquad z = re^{i\theta}, \tag{1}$$

where $\{\rho(m)\}_{m=0}^{\infty}$ is a sequence of non-zero positive numbers chosen so as to ensure the convergence of the sum in a non-empty open subset $\mathcal{D}$ of the complex plane, $\mathcal{N}(|z|)$ is the normalization factor ensuring that $\langle z|z\rangle = 1$.

The resolution of the identity is given by

$$\int_{\mathcal{D}} |z\rangle\langle z| d\mu = I, \tag{2}$$

where $d\mu$ is an appropriate chosen measure and I the identity operator on the Hilbert space $\mathfrak{H}$.

For $z_1 \neq z_2$, the overlap of two states $|z_1\rangle$ and $|z_2\rangle$ gives

$$\langle z_1|z_2\rangle = e^{iz_1 \wedge z_2} e^{-\frac{|z_1 - z_2|^2}{2}} \neq 0. \tag{3}$$

From (2) and (3), the CS $|z\rangle$ constitute an overcomplete family of vectors in the Hilbert space $\mathfrak{H}$.

From (2), for all $\Phi, \Psi \in \mathfrak{H}$,

$$\int_{\mathcal{D}} \langle \Phi|z\rangle\langle z|\Psi\rangle d\mu = \langle \Phi|\Psi\rangle. \tag{4}$$

Matrix vector coherent states (MVCS) are defined [11, 12] with

$$\mathcal{Z} = A(r)e^{i\zeta\Theta(k)}, \ (r, k, \zeta) \in D = R \times K \times [0, 2\pi),$$

where A, Θ are two $n \times n$ matrix-valued functions, as

$$|\mathcal{Z}, j\rangle = (\mathcal{N}(|\mathcal{Z}|))^{-1/2} \sum_{m=0}^{\infty} \frac{\mathcal{Z}^m}{\sqrt{\rho(m)}} |\chi^j \otimes \phi_m\rangle, \qquad j = 1, 2, \ldots, n \tag{5}$$

on the Hilbert space $\hat{\mathfrak{H}} = \mathbb{C}^n \otimes \mathfrak{H}$ with orthonormal basis $\{\chi^j \otimes \phi_m\}$, $j = 1, 2, \ldots, n, m = 0, 1, 2, \ldots, \infty$.

$$\Theta \quad \text{is Hermitian}, \quad \Theta = \Theta^\dagger, \tag{6}$$

$$\Theta^2 = \mathbb{I}_n, \qquad [A, \Theta] = 0, \qquad [A, A^\dagger] = 0. \tag{7}$$

The MVCS (5) satisfy on $\mathbb{C}^n \otimes \mathfrak{H}$ the following resolution of the identity

$$\sum_{j=1}^{n} \int_D W(|\mathcal{Z}|) |\mathcal{Z}, j\rangle\langle \mathcal{Z}, j| d\mu = \mathbb{I}_n \otimes \mathbb{I}_{\mathfrak{H}}. \tag{8}$$

$d\mu(r, k, \zeta) = dK(k)dR(r)d\zeta$, where the density function $W(|\mathcal{Z}|)$ is provided by

$$\int_R \frac{2\pi W(|\mathcal{Z}|)|A(r)|^{2m}}{\mathcal{N}(|\mathcal{Z}|)} dR = \rho(m)\mathbb{I}_n. \tag{9}$$

3. Hamiltonian operators with Zeeman coupling and spin-orbit interactions

3.1. Hamiltonian operator with Zeeman coupling

The Hamiltonian operator of a particle system with mass M and charge e with Zeeman coupling g, in a magnetic field can be defined by

$$\mathcal{H} = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 - g\frac{\mu}{2}\sigma \cdot \mathbf{B}. \tag{10}$$

The symbol $\mu = e\hbar/2M$ denotes the Bohr magneton, g the coupling factor related to the particle and $\sigma = (\sigma_x, \sigma_y, \sigma_z) \equiv (\sigma_1, \sigma_2, \sigma_3)$ the vector of Pauli spin matrices, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Explicitly,

$$\mathcal{H} = \frac{1}{2M}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2\right] - \frac{g\mu B}{2}\sigma_z \tag{12}$$

or, equivalently, in terms of annihilation and creation operators b' and $b'^\dagger$:

$$H = \hbar\omega_c\left(b'^\dagger b' + \frac{1}{2}\right)\sigma_0 + \hbar\omega_c\xi\sigma_z, \quad \xi = -g\mu Mc/2\hbar e, \tag{13}$$

where

$$b' = \frac{1}{\sqrt{2M\hbar\omega_c}}b, \qquad b'^\dagger = \frac{1}{\sqrt{2M\hbar\omega_c}}b^\dagger, \qquad [b', b'^\dagger] = \mathbb{I} \tag{14}$$

with

$$b = 2iP_z + \frac{eB}{2c}\bar{Z}, \qquad b^\dagger = -2iP_{\bar{z}} + \frac{eB}{2c}Z, \qquad [b, b^\dagger] = 2M\hbar\omega_c. \tag{15}$$

The eigenvalues of H are given by

$$E_n^{\pm} = \hbar\omega_c\left(n + \frac{1}{2}\right) \mp \hbar\omega_c\xi. \tag{16}$$

The corresponding eigenstates, denoted by $\begin{pmatrix} |\Phi_n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |\Phi_n\rangle \end{pmatrix}$, span the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$, with $\mathfrak{H} := \text{span}\{|\Phi_n\rangle\}_{n=0}^{\infty}\}$, $|\Phi_n\rangle$ corresponding to the number operator $N = b'^\dagger b'$ orthonormalized eigenstates $|n\rangle = (1/\sqrt{n!})(b'^\dagger)^n|0\rangle$, and on $\mathbb{C}^2 \otimes \mathfrak{H}$:

$$|\chi^+ \otimes \Phi_n\rangle = \begin{pmatrix} |\Phi_n\rangle \\ 0 \end{pmatrix}, \ |\chi^- \otimes \Phi_n\rangle = \begin{pmatrix} 0 \\ |\Phi_n\rangle \end{pmatrix}, n = 0, 1, 2, \ldots, \tag{17}$$

where χ^+ and χ^-, given by $\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, form an orthonormal basis of $\mathbb{C}^2$.

3.2. Hamiltonian operator with Zeeman coupling and spin-orbit interactions

The generalized Hamiltonian describing a $2D$ electron gas mixing with full Rashba and Dresselhaus SO interactions, and effective Zeeman coupling in a perpendicular magnetic field background is given by

$$H_{SO} = \frac{1}{2M}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2\right] - \frac{g\mu B}{2}\sigma_z + V_{SO}, \tag{18}$$

where V_{SO} stands for the generalized Rashba or Dresselhaus spin-orbit terms given by [13]

$$H_R = \hbar\omega_c\gamma(b'^\dagger\sigma_- + b'\sigma_+) = \hbar\omega_c\gamma\begin{pmatrix} 0 & b' \\ b'^\dagger & 0 \end{pmatrix}, \tag{19}$$

$$H_D = \hbar\omega_c\beta(ib'^\dagger\sigma_+ - ib'\sigma_-) = \hbar\omega_c\beta\begin{pmatrix} 0 & ib'^\dagger \\ -ib' & 0 \end{pmatrix}, \ \sigma_\pm = \frac{\sigma_x \pm i\sigma_y}{2} \tag{20}$$

where γ and β are dimensionless parameters defined by $\gamma = \gamma_0\sqrt{\frac{2M}{\hbar^3\omega_c}}$ and $\beta = \beta_0\sqrt{\frac{2M}{\hbar^3\omega_c}}$ with γ_0, β_0 being the Rashba and Dresselhaus coupling constants, respectively.

For $\beta = 0$ and non vanishing Rashba SO interaction term [14], which couples the up-spin level $(\Phi_{n-1}, 0)$ to the down-spin state $(0, \Phi_n), n \geq 1$

$$\psi_n^+ = \begin{pmatrix} \cos\theta_n \Phi_{n-1} \\ \sin\theta_n \Phi_n \end{pmatrix}, \quad \psi_n^- = \begin{pmatrix} -\sin\theta_n \Phi_{n-1} \\ \cos\theta_n \Phi_n \end{pmatrix}, \tag{21}$$

$$E_n^\pm = \hbar\omega_c n \mp \frac{\delta}{2}\sqrt{1 + 4\gamma^2 n\hbar^2\omega_c^2/\delta^2}, \tag{22}$$

in which $\delta = \hbar\omega_c(1 + 2\xi)$.

In the weak Rashba SO coupling limit case (γ negligible), we can expand $\mathcal{E}_n^\pm = E_n^\pm - E_0^\pm$ using the approximation $(1+\varepsilon)^{1/2} \simeq 1 + \varepsilon/2, \varepsilon \ll 1$ and, by taking $\hbar = 1$, get from (21)

$$\mathcal{E}_n^\pm = \omega_\pm(\gamma)n, \quad \omega_\pm(\gamma) = \omega_c\left(1 \mp \frac{2\gamma^2}{2 - gc}\right). \tag{23}$$

Define

$$\rho_\pm(n) := \mathcal{E}_1^\pm\mathcal{E}_2^\pm \cdots \mathcal{E}_n^\pm \tag{24}$$

with the increasing order $\mathcal{E}_1^\pm < \mathcal{E}_2^\pm < \cdots < \mathcal{E}_n^\pm$, such that

$$\rho_\pm(n) = \prod_{q=1}^{n} \omega_\pm(\gamma)q = n!(\omega_\pm(\gamma))^n. \tag{25}$$

Provided the quantities in (25), before constructing the VCS, let us first perform a change of basis as follows.

Introduce the passage operators from the Hilbert space $\mathcal{S}$ with eigenbasis $\{|\psi_n^\pm\rangle\}$ in (21) to $\{|\chi^\pm \otimes \Phi_n\rangle\}$ and vice versa, $\chi^\pm$ being the natural basis of $\mathbb{C}^2$, given by

$$\mathcal{U}|\chi^\pm \otimes \Phi_n\rangle = |\psi_n^\pm\rangle, \qquad \mathcal{U}^\dagger|\psi_n^\pm\rangle = |\chi^\pm \otimes \Phi_n\rangle, \tag{26}$$

where $\mathcal{U}, \mathcal{U}^\dagger$ expand as

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} |\psi_n^\pm\rangle\langle\chi^\pm \otimes \Phi_n|, \qquad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle\langle\psi_n^\pm| \tag{27}$$

such that

$$\mathcal{U}^\dagger\mathcal{U} = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle\langle\chi^\pm \otimes \Phi_n| = \mathbb{I}_2 \otimes I_{\mathfrak{H}}, \quad \mathcal{U}\mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |\psi_n^\pm\rangle\langle\psi_n^\pm| = \mathbb{I}_{\mathcal{S}}, \tag{28}$$

where $\mathbb{I}_2 \otimes I_{\mathfrak{H}}$ and $\mathbb{I}_{\mathcal{S}}$ are the identity operators on $\mathbb{C}^2 \otimes \mathfrak{H}$ and $\mathcal{S}$, respectively.

Then, the Hamiltonian H_{SO} takes the following diagonal form:

$$\mathbb{H}_{SO}^{\mathrm{red}} = \mathcal{U}^\dagger H_{SO}\mathcal{U} = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle\mathcal{E}_n^\pm\langle\chi^\pm \otimes \Phi_n|. \tag{29}$$

4. Supersymmetric VCS construction

In this section, we construct VCS by considering the physical Hamiltonian $\mathbb{H}_{SO}^{\mathrm{red}}$ (29). A method similar to this latter has been developed in [15] generalizing some previous constructions [16, 17]. Then, we provide a scheme of VCS definition for the SUSY Hamiltonians.

4.1. $\mathbb{H}_{SO}^{\mathrm{red}}$ VCS construction

Using the supersymmetric expression of the harmonic oscillator part $H_{OSC} = \hbar\omega_c\left(b'^\dagger b' + \frac{\mathbb{I}}{2}\right)$ of the spin-orbit Hamiltonian H_{SO}, we get

$$H^{\mathrm{SUSY}} = \begin{pmatrix} H^b & 0 \\ 0 & H^f \end{pmatrix} = \hbar\omega_c \begin{pmatrix} b'^\dagger b' & 0 \\ 0 & b' b'^\dagger \end{pmatrix}, \tag{30}$$

where the SUSY partner Hamiltonians, H^b and H^f, corresponding to the bosonic and fermionic Hamiltonians, respectively, are given by

$$H^b = H_{OSC} - \frac{\hbar\omega_c}{2}\mathbb{I}, \qquad H^f = H_{OSC} + \frac{\hbar\omega_c}{2}\mathbb{I}. \tag{31}$$

We have $H^{\mathrm{SUSY}} = \{Q^\dagger, Q\}$, with the supercharges

$$Q = \sqrt{\hbar\omega_c}\begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} = \sqrt{\hbar\omega_c}\, b'\sigma_-, \; Q^\dagger = \sqrt{\hbar\omega_c}\begin{pmatrix} 0 & b'^\dagger \\ 0 & 0 \end{pmatrix} = \sqrt{\hbar\omega_c}\, b'^\dagger\sigma_+. \tag{32}$$

On the Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \otimes \mathfrak{H}_- = \mathrm{span}\{|n_+, n_-\rangle, n_\pm \in \mathbb{N}\}$, assume that $\mathbb{H}_{SO}^{\mathrm{red}} = \hbar\omega_+(\gamma)N'_+ + \hbar\omega_-(\gamma)N'_-$ can be written as

$$\mathbb{H}_{SO}^{\mathrm{red}} = \tilde{H}_+ \otimes I_{\mathfrak{H}_-} + I_{\mathfrak{H}_+} \otimes \tilde{H}_- = \left[\hbar\omega_+(\gamma)N'_+\right] \otimes I_{\mathfrak{H}_-} + I_{\mathfrak{H}_+} \otimes \left[\hbar\omega_-(\gamma)N'_-\right] \tag{33}$$

with

$$|n_+, n_-\rangle = |n_+\rangle \otimes |n_-\rangle = \frac{1}{\sqrt{n_+!n_-!}}(b'^{\dagger}_+)^{n_+}(b'^{\dagger}_-)^{n_-}|0,0\rangle. \tag{34}$$

Let $\mathbb{C}^4 \otimes \mathfrak{H}$, $\mathfrak{H} = \mathfrak{H}^1_+ \otimes \mathfrak{H}^2_+ \otimes \mathfrak{H}^1_- \otimes \mathfrak{H}^2_-$ be the Hilbert space with orthonormal basis $\{|\chi^j, n^1_+, n^2_+, n^1_-, n^2_-\rangle\}_{j,n^1_+,n^2_+,n^1_-,n^2_-}$ where $n^i_\pm \in \mathbb{N}, i = 1,2$ and $\{\chi^j\}^4_{j=1}$ is the natural basis of $\mathbb{C}^4$. Each Hilbert subspace $\mathfrak{H}^i_\pm, i = 1,2$ corresponds to the Fock Hilbert space span$\{|n^i_\pm\rangle, n^i_\pm \in \mathbb{N}\}$ where by use of (33) we have $\tilde{H}^i_\pm|n^i_\pm\rangle = \hbar\omega_\pm(\gamma)N'^i_\pm|n^i_\pm\rangle = \hbar\omega_\pm(\gamma)n^i_\pm|n^i_\pm\rangle, \quad i = 1,2.$

In this case, we consider the following diagonal Hamiltonian

$$H_D = \begin{pmatrix} H^{\mathrm{SUSY}}_+ & 0 \\ 0 & H^{\mathrm{SUSY}}_- \end{pmatrix} \tag{35}$$

with

$$H^{\mathrm{SUSY}}_\pm = \{Q^\dagger_\pm, Q_\pm\}, \quad Q_\pm = \sqrt{\hbar\omega_\pm(\gamma)}\, b'_\pm\sigma_-, \tag{36}$$

where the annihilation and creation operators, $b'_\pm$, $b'^{\dagger}_\pm$ for the chiral sectors have been introduced.

On the Hilbert subspace $\mathfrak{H}^1_+ \otimes \mathfrak{H}^1_-$, the eigenvalues with associated eigenstates $|n^1_+, n^1_-\rangle$ of the Hamiltonian $\mathbb{H}^{\mathrm{red}}_{SO}$ expressed as in (33) are given by

$$\mathcal{E}_{n^1_+,n^1_-} = \omega_+(\gamma)\left[n^1_+ + \frac{\omega_-(\gamma)}{\omega_+(\gamma)}n^1_-\right], \qquad \hbar = 1. \tag{37}$$

Setting $\varpi = 1 + \frac{\omega_-(\gamma)n^1_-}{\omega_+(\gamma)}$, we define by fixing the Landau sector level n^1_- the quantity

$$\rho(n^1_+) = \prod_{k=1}^{n^1_+}\omega_+(\gamma)\left[k + \frac{\omega_-(\gamma)n^1_-}{\omega_+(\gamma)}\right] = (\omega_+(\gamma))^{n^1_+}(\varpi)_{n^1_+}, \tag{38}$$

where $(\varpi)_{n^1_+} = \frac{\Gamma(n^1_+ + \varpi)}{\Gamma(\varpi)}$ is the Pochhammer symbol.

Let $x_{n^1_+} := \frac{\rho(n^1_+)}{\rho(n^1_+ - 1)}$, $\forall n^1_+ \geq 1$; $x_0 = 0$ by convention. Then,

$$x_{n^1_+}! = \rho(n^1_+),\ n^1_+ \geq 1 \quad \text{and} \quad x_0! = 1.$$

Let $z_j \in \mathbb{C}$, $z_j = r_j e^{\imath\theta_j}$, where we have $r_j \geq 0, \theta_j \in [0, 2\pi)$, $j = 1,2,3,4$, $Z_{1_+} = \mathrm{diag}(z_1^{1_+}, z_2^{1_+}, z_3^{1_+}, z_4^{1_+})$.

With the above setup, we define the VCS given by

$$\begin{aligned} &|Z_{1_+}, n^2_+, n^1_-, n^2_-, j\rangle \\ &= \mathcal{N}(Z_{1_+}, n^2_+, n^1_-, n^2_-)^{-\frac{1}{2}}\sum_{n^1_+=0}^{\infty} R(n^1_+)^{-\frac{1}{2}} Z_{1_+}^{n^1_+}|\chi^j, n^1_+, n^2_+, n^1_-, n^2_-\rangle, \end{aligned} \tag{39}$$

where $R(n^1_+) = (\omega_+(\gamma))^{n^1_+}(\varpi)_{n^1_+}\mathbb{I}_4$.

Proposition 1. *These vectors are normalized to unity as*

$$\sum_{j=1}^{4}\langle Z_{1_+}, n_+^2, n_-^1, n_-^2, j|Z_{1_+}, n_+^2, n_-^1, n_-^2, j\rangle = 1 \tag{40}$$

with the normalization factor

$$\mathcal{N}(Z_{1_+}, n_+^2, n_-^1, n_-^2) = \sum_{j=1}^{4} {}_1F_1\left(1; \varpi; \frac{\left(r_j^{1+}\right)^2}{\omega_+(\gamma)}\right) \tag{41}$$

where ${}_1F_1$ are the confluent hypergeometric functions.

Proof. We have

$$\begin{aligned}\sum_{j=1}^{4}&\langle Z_{1_+}, n_+^2, n_-^1, n_-^2, j|Z_{1_+}, n_+^2, n_-^1, n_-^2, j\rangle\\ &= \mathcal{N}(Z_{1_+}, n_+^2, n_-^1, n_-^2)^{-1}\sum_{j=1}^{4}\sum_{m_+^1=0}^{\infty}\sum_{n_+^1=0}^{\infty}[R(m_+^1)R(n_+^1)]^{-1/2}\\ &\quad\times\langle Z_{1_+}^{m_+^1}\chi^j, m_+^1, n_+^2, n_-^1, n_-^2|Z_{1_+}^{n_+^1}\chi^j, n_+^1, n_+^2, n_-^1, n_-^2\rangle_{\mathbb{C}^4\otimes\mathfrak{H}},\end{aligned} \tag{42}$$

where

$$\begin{aligned}&\langle\chi^j, m_+^1, n_+^2, n_-^1, n_-^2|\chi^j, n_+^1, n_+^2, n_-^1, n_-^2\rangle_{\mathbb{C}^4\otimes\mathfrak{H}}\\ &\quad= \langle\chi^j|\chi^j\rangle_{\mathbb{C}^4}\langle m_+^1, n_+^2, n_-^1, n_-^2|n_+^1, n_+^2, n_-^1, n_-^2\rangle_{\mathfrak{H}}.\end{aligned} \tag{43}$$

Then for all $j = 1, 2, 3, 4$, we have

$$\begin{aligned}\sum_{j=1}^{4}&\sum_{m_+^1=0}^{\infty}\sum_{n_+^1=0}^{\infty}[R(m_+^1)R(n_+^1)]^{-1/2}\\ &\times\langle Z_{1_+}^{m_+^1}\chi^j, m_+^1, n_+^2, n_-^1, n_-^2|Z_{1_+}^{n_+^1}\chi^j, n_+^1, n_+^2, n_-^1, n_-^2\rangle_{\mathbb{C}^4\otimes\mathfrak{H}}\\ &= \sum_{n_+^1=0}^{\infty}[R(n_+^1)]^{-1}\,\mathrm{Tr}(|Z_{1_+}|^{2n_+^1}) = \sum_{j=1}^{4} {}_1F_1\left(1; \varpi; \frac{\left(r_j^{1+}\right)^2}{\omega_+(\gamma)}\right)\end{aligned} \tag{44}$$

because for all $j = 1, 2, 3, 4$, we get

$$\langle Z_{1_+}^{n_+^1}\chi^j|Z_{1_+}^{n_+^1}\chi^j\rangle_{\mathbb{C}^4} = \langle|Z_{1_+}|^{2n_+^1}\chi^j|\chi^j\rangle_{\mathbb{C}^4} \tag{45}$$

such that (41) is obtained iff (40) holds. □

Proposition 2. *They satisfy the partial resolution of the identity given by*

$$\sum_{j=1}^{4}\int_{D_{1_+}} |Z_{1_+}, n_+^2, n_-^1, n_-^2, j\rangle\langle Z_{1_+}, n_+^2, n_-^1, n_-^2, j| d\mu(Z_{1_+}, n_+^2, n_-^1, n_-^2) = \mathbb{I}_4 \otimes I_{n_+^2, n_-^1, n_-^2}, \qquad (46)$$

where $I_{n_+^2, n_-^1, n_-^2}$ *is the projector onto the subspaces indexed by Landau sectors levels* n_-^1, n_+^2, n_-^2*, with* $D_{1_+} = \{(z_1^{1+}, z_2^{1+}, z_3^{1+}, z_4^{1+}) \in \mathbb{C}^4,\ |z_j^{1+}| < \infty, j = 1, 2, 3, 4\}$ *and the measure given by*

$$d\mu(Z_{1_+}, n_+^2, n_-^1, n_-^2) = \frac{1}{(2\pi)^4}\prod_{j=1}^{4}\mathcal{N}(Z_{1_+}, n_+^2, n_-^1, n_-^2)\varrho(r_j^{1+})dr_j^{1+}d\theta_j^{1+}. \qquad (47)$$

The densities

$$\varrho(r_j^{1+}) = 2\exp\left\{-\frac{\left(r_j^{1+}\right)^2}{\omega_+(\gamma)}\right\}\left(r_j^{1+}\right)^{2\eta-1}(\Gamma(\varpi))^{-1}(\omega_+(\gamma))^{-\eta} \qquad (48)$$

solve the moment problems

$$\int_0^\infty (r_j^{1+})^{2n_+^1}\varrho(r_j^{1+})dr_j^{1+} = \rho(n_+^1), \qquad j = 1, 2, 3, 4. \qquad (49)$$

Proof. The left-hand side of (46), after some algebra, leads to the following Stieltjes moment problem:

$$\int_0^\infty (r_j^{1+})^{2n_+^1}\varrho(r_j^{1+})dr_j^{1+} = \left[\int_0^\infty e^{-u_j^{1+}}\left(\Omega_+ u_j^{1+}\right)^{n_+^1+\gamma-1} du_j^{1+}\right] \times \Gamma(\gamma)^{-1}(\Omega_+)^{-\gamma+1},\ u_j^{1+} = \frac{\left(r_j^{1+}\right)^2}{\Omega_+} \qquad (50)$$

which provides

$$\int_0^\infty (r_j^{1+})^{2n_+^1+1}\varrho(r_j^{1+})dr_j^{1+} = (\Omega_+)^{n_+^1}\Gamma(n_+^1+\gamma)\Gamma(\gamma)^{-1} = (\Omega_+)^{n_+^1}(\gamma)_{n_+^1}. \qquad (51)$$

□

Setting $b'_\pm = b^1_\pm$, the VCS $|\mathcal{Z}, n_{1_+}, n_{1_+}, n_{1_-}, n_{1_-}\rangle$ as defined in (39) coincide with the SUSY VCS of

$$\begin{pmatrix} H_+^{\mathrm{SUSY}} & 0 \\ 0 & H_-^{\mathrm{SUSY}} \end{pmatrix}, \quad H_\pm^{\mathrm{SUSY}} = \hbar\omega_\pm(\gamma)\begin{pmatrix} N'_\pm & 0 \\ 0 & N'_\pm + \mathbb{I} \end{pmatrix} \qquad (52)$$

denoted by $|\mathfrak{Z}_\pm, n_+, n_-\rangle$ where $\mathcal{Z}_\pm = \mathrm{diag}(z_\pm, \bar{z}_\pm)$, given on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}_+ \otimes \mathfrak{H}_-$ as a two-component vector

$$|\mathfrak{Z}_\pm, n_+, n_-\rangle = |\mathcal{Z}_\pm, n_+, n_-, \pm\rangle + |\mathcal{Z}_\pm, n_+, n_-, \mp\rangle, \qquad (53)$$

where

$$|\mathcal{Z}_\pm, n_+, n_-, \pm\rangle = \mathcal{N}(\mathcal{Z}_\pm, n_\mp)^{-\frac{1}{2}} \sum_{n_\pm=0}^{\infty} R(n_\pm)^{-\frac{1}{2}} \mathcal{Z}_\pm^{n_\pm} |n_+, n_-, \pm\rangle \tag{54}$$

and

$$R(n_\pm) = \begin{pmatrix} (\omega_\pm(\gamma))^{n_\pm}(\varpi)_{n_\pm} & 0 \\ 0 & (\omega_\pm(\gamma))^{n_\pm+1}(\varpi)_{n_\pm+1} \end{pmatrix} \tag{55}$$

implying

$$|\mathcal{Z}_+, n_+, n_-, +\rangle = \mathcal{N}(\mathcal{Z}_+, n_-)^{-\frac{1}{2}} \sum_{n_+=0}^{\infty} \begin{pmatrix} \frac{r_+^{n_+} e^{in_+\theta_+}}{\sqrt{(\omega_+(\gamma))^{n_+}(\varpi)_{n_+}}} \\ 0 \end{pmatrix} \otimes |n_+, n_-\rangle \tag{56}$$

and

$$|\mathcal{Z}_+, n_+, n_-, -\rangle = \mathcal{N}(\mathcal{Z}_+, n_-)^{-\frac{1}{2}} \sum_{n_+=0}^{\infty} \begin{pmatrix} 0 \\ \frac{r_+^{n_++1} e^{-i(n_++1)\theta_+}}{\sqrt{(\omega_+(\gamma))^{n_++1}(\varpi)_{n_++1}}} \end{pmatrix} \otimes |n_+, n_-\rangle \tag{57}$$

with n_- labelling the degeneracy levels. The VCS $|\mathcal{Z}_-, n_+, n_-, \pm\rangle$ can also be defined in the same way with the degeneracy levels labeled by n_+.

Proposition 3. *The vectors $|\mathcal{Z}_\pm, n_+, n_-, \pm\rangle$ are normalized to unity as*

$$\sum_\pm \langle \mathcal{Z}_\pm, n_+, n_-, \pm | \mathcal{Z}_\pm, n_+, n_-, \pm\rangle = 1 \tag{58}$$

with the normalization factor

$$\mathcal{N}(\mathcal{Z}_\pm, n_\mp) = 2\,{}_1F_1\left(1; \varpi; \frac{(r_\pm)^2}{\omega_\pm(\gamma)}\right) - 1. \tag{59}$$

Proof. See that of Proposition 1. □

Proposition 4. *The following partial resolution of the identity*

$$\sum_\pm \int_{D_\pm} |\mathcal{Z}_\pm, n_+, n_-, \pm\rangle\langle \mathcal{Z}_\pm, n_+, n_-, \pm| d\mu(\mathcal{Z}_\pm, n_\mp) = \mathbb{I}_2 \otimes I_{n_\mp} \tag{60}$$

is satisfied. On the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}_+ \otimes \mathfrak{H}_-$, the resolution of the identity is provided by

$$\sum_\pm \sum_{n_\mp} \int_{D_\pm} |\mathcal{Z}_\pm, n_+, n_-, \pm\rangle\langle \mathcal{Z}_\pm, n_+, n_-, \pm| d\mu(\mathcal{Z}_\pm, n_\mp) = \mathbb{I}_2 \otimes I_{\mathfrak{H}_+\otimes\mathfrak{H}_-}. \tag{61}$$

The densities $\varrho(r_\pm) = 2\exp\left\{-\frac{(r_\pm)^2}{\omega_\pm(\gamma)}\right\}(r_\pm)^{2\eta-1}(\Gamma(\varpi))^{-1}(\omega_\pm(\gamma))^{-\eta}$ solve

$$\int_0^\infty (r_\pm)^{2n_\pm} \varrho(r_\pm) dr_\pm = \rho(n_\pm) = (\omega_\pm(\gamma))^{n_\pm}(\varpi)_{n_\pm}. \tag{62}$$

Proof. Dealing with the definitions (56) and (57), we obtain

$$\begin{aligned}
&\sum_{\pm} |\mathcal{Z}_\pm, n_+, n_-, \pm\rangle\langle \mathcal{Z}_\pm, n_+, n_-, \pm| \\
&\quad = \mathcal{N}(\mathcal{Z}_\pm, n_\mp)^{-\frac{1}{2}} \mathcal{N}(\mathcal{Z}_\pm, m_\mp)^{-\frac{1}{2}} \sum_{\pm} \sum_{m_\pm=0}^{\infty} \sum_{n_\pm=0}^{\infty} R(m_\pm)^{-\frac{1}{2}} R(n_\pm)^{-\frac{1}{2}} \\
&\qquad \times |\mathcal{Z}_\pm^{n_\pm}, n_+, n_-, \pm\rangle\langle \mathcal{Z}_\pm^{m_\pm}, m_+, n_-, \pm| \\
&\quad = \mathcal{N}(\mathcal{Z}_\pm, n_\mp)^{-\frac{1}{2}} \mathcal{N}(\mathcal{Z}_\pm, m_\mp)^{-\frac{1}{2}} \sum_{m_\pm=0}^{\infty} \sum_{n_\pm=0}^{\infty} \bar{\mathcal{Z}}_\pm^{m_\pm} R(m_\pm)^{-\frac{1}{2}} \mathcal{Z}_\pm^{n_\pm} R(n_\pm)^{-\frac{1}{2}} \\
&\qquad \times \sum_{\pm} |\pm\rangle\langle\pm| \otimes |n_+, n_-\rangle\langle m_+, n_-|. \qquad (63)
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{\pm} \sum_{n_\mp=0}^{\infty} \int_{D_\pm} |\mathcal{Z}_\pm, n_+, n_-, \pm\rangle\langle \mathcal{Z}_\pm, n_+, n_-, \pm| d\mu(\mathcal{Z}_\pm, n_\mp) \\
&\quad = \sum_{n_\mp=0}^{\infty} \sum_{n_\pm=0}^{\infty} \\
&\begin{pmatrix} \int_0^\infty \frac{r_\pm^{2n_\pm}}{(\omega_\pm(\gamma))^{n_\pm} (\varpi)_{n_\pm}} \varrho(r_\pm) dr_\pm & 0 \\ 0 & \int_0^\infty \frac{r_\pm^{2(n_\pm+1)}}{(\omega_\pm(\gamma))^{n_\pm+1} (\varpi)_{n_\pm+1}} \varrho(r_\pm) dr_\pm \end{pmatrix} \\
&\qquad \times \mathbb{I}_2 \otimes |n_+, n_-\rangle\langle n_+, n_-|. \qquad (64)
\end{aligned}$$

Taking the densities

$$\varrho(r_\pm) = 2\exp\left\{-\frac{(r_\pm)^2}{\omega_\pm(\gamma)}\right\} (r_\pm)^{2\eta-1} \left(\Gamma(\varpi)\right)^{-1} (\omega_\pm(\gamma))^{-\eta} \qquad (65)$$

which lead to the Stieltjes moment problems

$$\int_0^\infty (r_\pm)^{2n_\pm} \varrho(r_\pm) dr_\pm = \rho(n_\pm) = (\omega_\pm(\gamma))^{n_\pm} (\varpi)_{n_\pm} \qquad (66)$$

we get

$$\begin{aligned}
&\sum_{\pm} \sum_{n_\mp=0}^{\infty} \int_{D_\pm} |\mathcal{Z}_\pm, n_+, n_-, \pm\rangle\langle \mathcal{Z}_\pm, n_+, n_-, \pm| d\mu(\mathcal{Z}_\pm, n_\mp) \\
&\quad = \mathbb{I}_2 \otimes \sum_{n_\mp=0}^{\infty} \sum_{n_\pm=0}^{\infty} |n_+, n_-\rangle\langle n_+, n_-| = \mathbb{I}_2 \otimes I_{\mathfrak{H}_+ \otimes \mathfrak{H}_-}. \qquad (67)
\end{aligned}$$

□

For the Hamiltonian

$$H_+^{\mathrm{SUSY}} = \hbar\omega_+(\gamma) \begin{pmatrix} N'_+ & 0 \\ 0 & N'_+ + \mathbb{I} \end{pmatrix} \tag{68}$$

the SUSY VCS are obtained from (53) as

$$|\mathfrak{z}_+, n_+, n_-\rangle = \mathcal{N}(\mathcal{Z}_+, n_-)^{-\frac{1}{2}} \sum_{n_+=0}^{\infty} \begin{pmatrix} \dfrac{r_+^{n_+} e^{in_+\theta_+}}{\sqrt{(\omega_+(\gamma))^{n_+}(\varpi)_{n_+}}} \\ \\ \dfrac{r_+^{n_++1} e^{-i(n_++1)\theta_+}}{\sqrt{(\omega_+(\gamma))^{n_++1}(\varpi)_{n_++1}}} \end{pmatrix} \otimes |n_+, n_-\rangle \tag{69}$$

with the normalization factor $\mathcal{N}(\mathcal{Z}_+, n_-)$ given in (59), where $n_- = 0, 1, \ldots$ labels the degeneracy levels.

Proposition 5. *These states satisfy, on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}_+ \otimes \mathfrak{H}_-$, the resolution of the identity given by*

$$\sum_{\pm} \int_D d\mu(\mathcal{Z}_+, n_-) d\mu(\mathcal{Z}_-, n_+) |\mathfrak{z}_+, n_+, n_-\rangle\langle\mathfrak{z}_+, n_+, n_-| = \mathbb{I}_2 \otimes I_{\mathfrak{H}_+\otimes\mathfrak{H}_-} \tag{70}$$

where $D = D_+ \times D_-$, $D_\pm = \{(r_\pm, \theta_\pm) | 0 \leq r_\pm < \infty,\ 0 \leq \theta_\pm < 2\pi\}$.

Proof. See that of Proposition 4. □

5. Generalized annihilation, creation and number operators

5.1. Generalized annihilation, creation and number operators

Consider $\mathfrak{a}_\pm, \mathfrak{a}_\pm^\dagger, \mathfrak{n}_\pm$ on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}_+ \otimes \mathfrak{H}_-$ by

$$\mathfrak{a}_+|0, n_-, \pm\rangle = 0, \ \ \mathfrak{a}_-|n_+, 0, \pm\rangle = 0 \ (x_0 = 0 \ \text{ by convention}) \tag{71}$$

and for $n \geq 1$,

$$\begin{aligned} \mathfrak{a}_+|n_+, n_-, \pm\rangle = \sqrt{x_{n_+}}|n_+ - 1, n_-, \pm\rangle,\ \mathfrak{a}_-|n_+, n_-, \pm\rangle \\ = \sqrt{x_{n_-}}|n_+, n_- - 1, \pm\rangle, \\ \mathfrak{a}_+^\dagger|n_+, n_-, \pm\rangle = \sqrt{x_{n_++1}}|n_+ + 1, n_-, \pm\rangle, \\ \mathfrak{a}_-^\dagger|n_+, n_-, \pm\rangle = \sqrt{x_{n_-+1}}|n_+, n_- + 1, \pm\rangle, \\ \mathfrak{n}_+|n_+, n_-, \pm\rangle = x_{n_+}|n_+, n_-, \pm\rangle,\ \mathfrak{n}_-|n_+, n_-, \pm\rangle = x_{n_-}|n_+, n_-, \pm\rangle \end{aligned} \tag{72}$$

with the following commutators actions:

$$\begin{aligned} [\mathfrak{a}_\pm, \mathfrak{a}_\pm^\dagger]|n_+, n_-, \pm\rangle &= (x_{n_\pm+1} - x_{n_\pm})\mathbb{I}\,|n_+, n_-, \pm\rangle, \\ [\mathfrak{n}_\pm, \mathfrak{a}_\pm]|n_+, n_-, \pm\rangle &= -(x_{n_\pm} - x_{n_\pm-1})\mathfrak{a}_\pm|n_+, n_-, \pm\rangle, \\ [\mathfrak{n}_\pm, \mathfrak{a}_\pm^\dagger]|n_+, n_-, \pm\rangle &= (x_{n_\pm+1} - x_{n_\pm})\mathfrak{a}_\pm^\dagger|n_+, n_-, \pm\rangle. \end{aligned} \tag{73}$$

The following relations

$$\begin{aligned}
\mathfrak{a}_+|\mathcal{Z}_+,n_+,n_-,-\rangle &= \mathcal{N}(\mathcal{Z}_+,n_-)^{-\frac{1}{2}}\sum_{n_+=0}^{\infty}\frac{\mathcal{Z}_+^{n_+}}{\sqrt{(\omega_+(\gamma))^{n_+}(\varpi+n_+)(\varpi)_{n_+-1}}} \\
&\qquad\times|n_+-1,n_-,-\rangle, \\
\mathfrak{a}_+^{\dagger}|\mathcal{Z}_+,n_+,n_-,-\rangle &= \mathcal{N}(\mathcal{Z}_+,n_-)^{-\frac{1}{2}}\sum_{n_+=0}^{\infty}\frac{\mathcal{Z}_+^{n_+}}{\sqrt{(\omega_+(\gamma))^{n_+}(\varpi)_{n_+}}}|n_++1,n_-,-\rangle, \\
\mathfrak{n}_+|\mathcal{Z}_+,n_+,n_-,-\rangle &= \mathcal{N}(\mathcal{Z}_+,n_-)^{-\frac{1}{2}}\sum_{n_+=0}^{\infty}\frac{\mathcal{Z}_+^{n_+}\omega_+(\gamma)(\varpi+n_+-1)}{\sqrt{(\omega_+(\gamma))^{n_++1}(\varpi)_{n_++1}}}|n_+,n_-,-\rangle
\end{aligned} \tag{74}$$

hold.

5.2. Operator mean values and uncertainty relation

Let the quadrature operators $\mathcal{Q}_\pm$ and $\mathcal{P}_\pm$ be given on $\mathbb{C}^2\otimes\mathfrak{H}_+\otimes\mathfrak{H}_-$ as

$$\mathcal{Q}_\pm = \frac{1}{\sqrt{2}}(\mathfrak{a}_\pm+\mathfrak{a}_\pm^{\dagger}), \qquad \mathcal{P}_\pm = \frac{1}{i\sqrt{2}}(\mathfrak{a}_\pm-\mathfrak{a}_\pm^{\dagger}). \tag{75}$$

The mean values of the squared operators $\mathcal{Q}_+^2$ and $\mathcal{P}_+^2$ are obtained as

$$\langle\mathcal{Q}_+^2\rangle_{|\mathcal{Z}_+,n_+,n_-,-\rangle} = r_+^2\left(\mathcal{G}_{2,+}(r_+)\cos 2\theta_+ + \mathcal{G}_+(r_+) - \frac{(\omega_+(\gamma))}{2r_+^2}\tilde{\mathcal{G}}_+(r_+)\right), \tag{76}$$

$$\langle\mathcal{P}_+^2\rangle_{|\mathcal{Z}_+,n_+,n_-,-\rangle} = r_+^2\left(-\mathcal{G}_{2,+}(r_+)\cos 2\theta_+ + \mathcal{G}_+(r_+) - \frac{(\omega_+(\gamma))}{2r_+^2}\tilde{\mathcal{G}}_+(r_+)\right) \tag{77}$$

and the dispersions are derived as

$$\begin{aligned}
(\Delta\mathcal{Q}_+)^2_{|\mathcal{Z}_+,n_+,n_-,-\rangle} &= r_+^2\left(\mathcal{G}_{2,+}(r_+)\cos 2\theta_+ - 2\mathcal{G}_{1,+}^2(r_+)\cos^2\theta_+\right. \\
&\quad\left. + \mathcal{G}_+(r_+) - \frac{(\omega_+(\gamma))}{2r_+^2}\tilde{\mathcal{G}}_+(r_+)\right), \\
(\Delta\mathcal{P}_+)^2_{|\mathcal{Z}_+,n_+,n_-,-\rangle} &= r_+^2\left(-\mathcal{G}_{2,+}(r_+)\cos 2\theta_+ - 2\mathcal{G}_{1,+}^2(r_+)\sin^2\theta_+\right. \\
&\quad\left. + \mathcal{G}_+(r_+) - \frac{(\omega_+(\gamma))}{2r_+^2}\tilde{\mathcal{G}}_+(r_+)\right)
\end{aligned} \tag{78}$$

where

$$\mathcal{G}_+(r_+) = \frac{{}_1F_1\left(1;\varpi;\frac{(r_+)^2}{\omega_+(\gamma)}\right)}{2{}_1F_1\left(1;\varpi;\frac{(r_+)^2}{\omega_+(\gamma)}\right)-1}, \qquad \tilde{\mathcal{G}}_+(r_+) = \frac{{}_1F_1\left(1;\varpi;\frac{(r_+)^2}{\omega_+(\gamma)}\right)-1}{2{}_1F_1\left(1;\varpi;\frac{(r_+)^2}{\omega_+(\gamma)}\right)-1}, \tag{79}$$

$$\mathcal{G}_{1,+}(r_+) = \frac{1}{2{}_1F_1\left(1;\varpi;\frac{(r_+)^2}{\omega_+(\gamma)}\right)-1}\sum_{n_+=0}^{\infty}\sqrt{\frac{\varpi+n_+}{\varpi+n_++1}}\frac{r_+^{2(n_++1)}}{(\omega_+(\gamma))^{n_++1}(\varpi)_{n_++1}}, \tag{80}$$

$$\mathcal{G}_{2,+}(r_+) = \frac{1}{{}_2F_1\left(1;\varpi;\frac{(r_+)^2}{\omega_+(\gamma)}\right)-1}\sum_{n_+=0}^{\infty}\sqrt{\frac{\varpi+n_+}{\varpi+n_++2}}\frac{r_+^{2(n_++1)}}{(\omega_+(\gamma))^{n_++1}(\varpi)_{n_++1}}. \tag{81}$$

If we take $\mathcal{G}_+(r_+) = \mathcal{G}_{1,+}(r_+) = \mathcal{G}_{2,+}(r_+) = \tilde{\mathcal{G}}_+(r_+) = \frac{1}{2}$, with $r_+^2\cos^2\theta_+ \geq \frac{1}{2}$ and $r_+^2\sin^2\theta_+ \geq \frac{1}{2}$, the above relations can be simplified to yield:

$$(\Delta\mathcal{Q}_+)^2_{|\mathcal{Z}_+,n_+,n_-,-\rangle} = \frac{r_+^2}{2}\cos^2\theta_+ - \frac{1}{4}\omega_+(\gamma) \geq \frac{1}{4}\left(1-\omega_+(\gamma)\right), \tag{82}$$

$$(\Delta\mathcal{P}_+)^2_{|\mathcal{Z}_+,n_+,n_-,-\rangle} = \frac{r_+^2}{2}\sin^2\theta_+ - \frac{1}{4}\omega_+(\gamma) \geq \frac{1}{4}\left(1-\omega_+(\gamma)\right). \tag{83}$$

Thereby, with $\omega_+(\gamma) \ll 1$, we arrive at

$$(\Delta\mathcal{Q}_+)^2_{|\mathcal{Z}_+,n_+,n_-,-\rangle}(\Delta\mathcal{P}_+)^2_{|\mathcal{Z}_+,n_+,n_-,-\rangle} \geq \frac{1}{16}\left(1-\omega_+(\gamma)\right)^2 \geq \frac{1}{16}. \tag{84}$$

5.3. Time evolution

The VCS $|\mathcal{Z}_\pm, n_+, n_-, \pm\rangle$ can be equipped with a parameter τ expressing their time evolution as follows:

$$|\mathcal{Z}_\pm, n_+, n_-, \pm, \tau\rangle = \mathcal{N}(\mathcal{Z}_\pm, n_\mp)^{-\frac{1}{2}}\sum_{n_\pm=0}^{\infty} R(n_\pm)^{-\frac{1}{2}}\mathcal{Z}_\pm^{n_\pm} e^{-i\tau\mathcal{E}^\pm_{n_\pm}}|n_+, n_-, \pm\rangle, \tag{85}$$

where $\mathcal{E}^\pm_{n_\pm} = \omega_\pm(\gamma)n_\pm$, relatively to the time evolution operator $\mathbb{U}(t) := e^{-it\mathbb{H}^{\mathrm{red}}_{SO}}$ that fulfills the property:

$$\mathbb{U}(t)|\mathcal{Z}_\pm, n_+, n_-, \pm, \tau\rangle = |\mathcal{Z}_\pm(t), n_+, n_-, \pm, \tau\rangle, \;\; \mathcal{Z}_\pm(t) := e^{-it\omega_\pm(\gamma)}\mathcal{Z}_\pm. \tag{86}$$

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Energy and Stability of the Pais–Uhlenbeck Oscillator

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Abstract. We study stability of higher-derivative dynamics from the viewpoint of more general correspondence between symmetries and conservation laws established by the Lagrange anchor. We show that classical and quantum stability may be provided if a higher-derivative model admits a bounded from below integral of motion and the Lagrange anchor that relates this integral to the time translation.

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Keywords. Higher derivative-theories; Pais–Uhlenbeck oscillator; stability; Lagrange anchor.

Introduction

A notorious trouble appears when the Noether theorem [1] is applied to the theories with higher derivatives, the models whose Lagrangians depend on accelerations and higher derivatives of generalized coordinates. In contrast to the lower-order theories, where unboundedness of the canonical energy from below usually indicates the presence of ghost states and instability of the model, in higher-derivative theories the unboundedness of canonical energy is not necessary to have negative impact on classical dynamics [2, 3]. The relationship between (un)boundedness of canonical energy from below and (in)stability of higher-derivative theory was subject of many works [4–8].

In this note, we consider a stability of higher-derivative dynamics from the viewpoint of more general correspondence between symmetries and conservation laws which is established by the Lagrange anchor. Following the ideas of [9, 10], we show that the stability of higher-derivative theory may be provided if the model

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admits a bounded from below integral of motion and a Lagrange anchor that associates the integral of motion with translation in time. The general construction is illustrated by the example of the Pais–Uhlenbeck oscillator.

The paper is organized as follows. In Section 1, we recall some basic facts about the conservation laws of the Pais–Uhlenbeck (PU) oscillator. In Section 2, we introduce the Lagrange anchor for the PU oscillator and establish a correspondence between symmetries and conservation laws. The bounded integral of motion and the Lagrange anchor that associates it to time translations are identified. Section 3 is devoted to the Pais–Uhlenbeck oscillator with equal frequencies. We show that in unstable theory the bounded integral of motion exists, but it appears to be unrelated to the time-translation symmetry.

1. Conservation laws of the PU oscillator

We consider the one-dimensional Pais–Uhlenbeck oscillator of order $2n$ [11], whose action functional has the form

$$S[x(t)] = \int dtL\,, \qquad L = \frac{1}{2\Omega}x(t)\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)x(t)\,; \tag{1}$$

here

$$0<\omega_1<\omega_2<\cdots<\omega_n$$

are the frequencies of oscillations. We assume that there is no resonance, so that all the frequencies are different. We also introduced the dimensional constant $\Omega>0$ to provide the correct dimension of the action (1).

The corresponding equation of motion reads

$$\frac{\delta S}{\delta x} \equiv \frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)x = 0\,. \tag{2}$$

It is convenient to introduce the wave operator that defines r.h.s. of the equation of motion

$$T = \frac{1}{\Omega}\prod_{i=1}^{n}\Big(\frac{d^2}{dt^2}+\omega_i^2\Big)\,, \qquad \frac{\delta S}{\delta x} = T(x)\,. \tag{3}$$

We will also use notation

$$\dot{x} = \frac{dx}{dt}, \qquad \ddot{x} = \frac{d^2x}{dt^2}, \qquad \dots, \qquad \overset{(n)}{x} = \frac{d^nx}{dt^n}$$

for the time derivatives of x.

The general solution of equation (2) is given by the sum of n oscillations with frequencies ω_i with different amplitudes A_i and phases φ_i

$$x(t) = \sum_{i=1}^{n}x_i(t)\,, \qquad x_i(t) \equiv A_i\sin(\omega_i t+\varphi_i) = \mathcal{P}_i x(t)\,. \tag{4}$$

Here, the operators

$$\mathcal{P}_i = \prod_{j\neq i} \frac{1}{\omega_j^2 - \omega_i^2}\Big(\frac{d^2}{dt^2} + \omega_j^2\Big)$$

have the sense of projectors to the subspaces of solutions with frequencies ω_i, respectively. There are two obvious properties

$$\begin{gathered}\sum_{i=1}^{n} \mathcal{P}_i = 1\,,\\ \mathcal{P}_i^2 = \mathcal{P}_i - \Omega \sum_{j\neq i}\prod_{r\neq i}\frac{1}{\omega_i^2 - \omega_r^2}\prod_{s\neq j}\frac{1}{\omega_j^2-\omega_s^2}\prod_{k\neq j,i}\Big(\frac{d^2}{dt^2}+\omega_k^2\Big)T\,.\end{gathered} \tag{5}$$

In order to prove the first relation one can apply to it the Fourier transform:

$$F(\sum_{i=1}^{n}\mathcal{P}_i - 1) = \sum_{i=1}^{n}\prod_{j\neq i}\frac{1}{\omega_j^2-\omega_i^2}\Big(\omega_j^2 - p^2\Big) - 1\,.$$

Then the r.h.s. of the relation is a polynomial of degree $2n-2$ that has $2n$ roots $p = \pm\omega_i$ and thus has to be equal to zero identically. The second relation follows from identity

$$\mathcal{P}_i^2 = \mathcal{P}_i\Big(1 - \sum_{j\neq i}\mathcal{P}_j\Big)$$

with account of notation (3).

Due to relations (5), formula (4) establishes a correspondence between the solutions to the PU oscillator equation and the system of n independent harmonic oscillators

$$\frac{\delta S}{\delta x(t)} = 0 \qquad \Leftrightarrow \qquad \Big(\frac{d^2}{dt^2} + \omega_i^2\Big)x_i(t) = 0, \qquad i = 1, 2, \dots, n. \tag{6}$$

From (6) it immediately follows that the PU oscillator has n independent integrals of motion

$$I_i \equiv \frac{1}{2}\Big[\dot{x}_i^2 + \omega_i^2 x_i^2\Big] = \frac{1}{2}\Big[\Big(\mathcal{P}_i\dot{x}\Big)^2 + \omega_i^2\Big(\mathcal{P}_i x\Big)^2\Big]\,. \tag{7}$$

The general quadratic integral of motion is given by the linear combination of integrals (7). Namely,

$$I = -\sum_{i=1}^{n}\Big[\prod_{j\neq i}\Big(\omega_j^2 - \omega_i^2\Big)\Big]\frac{\alpha_i I_i}{\Omega}\,, \tag{8}$$

with α_i being arbitrary real constants.

It is easy to see that expression I in (8) is conserved

$$\frac{dI}{dt} = Q\frac{\delta S}{\delta x}\,, \qquad Q(t, x, \dot{x}, \dots, \overset{(2n-1)}{x}) = -\sum_{i=1}^{n}\Big(\alpha_i \mathcal{P}_i \dot{x}\Big)\,, \tag{9}$$

where the coefficient Q is called the *characteristic* associated with the conservation law I. It is known [12] that there is a one-to-one correspondence between integrals of motion and characteristics. The last fact allows one to use characteristics for establishing a correspondence between the symmetries and conservation laws. The simplest example is provided by the Noether theorem. The Noether theorem identifies the characteristic Q with the infinitesimal symmetry transformation of action functional:

$$\delta_\varepsilon x = \varepsilon Q\,, \quad \delta_\varepsilon S = 0 \qquad \Leftrightarrow \qquad \frac{dI}{dt} = Q\frac{\delta S}{\delta x}\,. \tag{10}$$

The problem appears when a conservation law bounded form below is associated with the time translations. The general bounded conservation law (8) with $(-1)^i\alpha_i > 0$ corresponds to some symmetry of the action (1)

$$\delta_\varepsilon x = -\varepsilon\sum_{i=1}^{n}\left(\alpha_i\mathcal{P}_i\dot{x}\right), \tag{11}$$

while the infinitesimal time translation $\delta_\varepsilon x = -\varepsilon\dot{x}$ corresponds to the unbounded conservation law with $\alpha_i = 1$. This is manifestation of general no-go statement about unboundedness of energy in the theories with higher derivatives. Unless the higher-derivative theory is highly constrained, the usual Noether theorem cannot connect a positive conserved quantity to the time translation invariance, see for instance discussion in [8] and references therein.

2. The Lagrange anchor for the PU oscillator

The generalization of the Noether theorem suggests that the correspondence between the symmetries and characteristics (and hence conservation laws) is established by the linear differential operator[1]

$$V = \sum_{i=1}^{2n-1}\overset{(i)}{V}(t, x, \dot{x}, \dots, \overset{(2n-1)}{x})\frac{d^i}{dt^i}$$

that associates the characteristic Q with the symmetry

$$\delta_\varepsilon x = \varepsilon V(Q) = \sum_{i=1}^{2n-1}\overset{(i)}{V}\frac{d^iQ}{dt^i}\,, \qquad \delta_\varepsilon\Big(\frac{\delta S}{\delta x}\Big)\Big|_{\frac{\delta S}{\delta x}=0} = 0\,. \tag{12}$$

Invariance of the equation of motion under transformation (12) implies certain compatibility condition for the Lagrange anchor [12]. In the simplest case of linear equations $\delta S/\delta x = T(x)$ and the Lagrange anchor with no field and time dependence,

$$\overset{(i)}{V} = \text{const}, \qquad i = 0, \dots, 2n-1\,,$$

this compatibility condition takes the form [9]

$$V^*T^* - TV = 0\,. \tag{13}$$

[1] The notation of this section is adapted for the case of the PU oscillator. For general definitions of the Lagrange anchor and correspondence between symmetries and conservation laws see [12, 13].

For a self-adjoint wave operator $T = T^*$ (which is always true for Lagrangian theories) and a self-adjoint Lagrange anchor $V = V^*$, relation (13) takes even more simple form

$$[V, T] = 0\,. \tag{14}$$

The obvious solution $V = 1$ corresponds to the canonical Lagrange anchor which is always admissible for Lagrangian theory. It establishes the Noether correspondence between symmetries and conservation laws (10). The canonical Lagrange anchor cannot connect bounded from below integral of motion with translation in time, we have to be interested in the non-canonical Lagrange anchors. We have the following n-parameter family of the Lagrange anchors for the PU oscillator of order $2n$:

$$V = \sum_{i=1}^{n} \beta_i \mathcal{P}_i\,, \tag{15}$$

with β_i being arbitrary real constants. The details about derivation of this Lagrange anchor can be found in [9].

The Lagrange anchor (15) associates the general conservation law (8) with the symmetry

$$\delta_\varepsilon x = \varepsilon V(Q) = -\varepsilon \sum_{i=1}^{n} \Big(\alpha_i \beta_i \mathcal{P}_i \dot{x}\Big) + \varepsilon \Omega R(\alpha_i, \beta_j) T(x)\,, \tag{16}$$

where

$$R(\alpha_i, \beta_j) = \sum_{i,j=1}^{n} \Big(\alpha_i \beta_i - \alpha_i \beta_j\Big) \prod_{r \neq i} \frac{1}{\omega_i^2 - \omega_r^2} \prod_{s \neq j} \frac{1}{\omega_j^2 - \omega_s^2} \prod_{k \neq j, i} \Big(\frac{d^2}{dt^2} + \omega_k^2\Big)\,.$$

The second term in (16) is given by the linear combination of equations of motion and their differential consequences, and thus should be considered as trivial. Below, we will consider symmetries modulo trivial terms.

The crucial difference between Noether's correspondence (10) and (16) is that the symmetry (16) depends on n free parameters. We can use this ambiguity of the Lagrange anchor to connect the general integral of motion (8) with translations in time. Whenever $\alpha_i \neq 0$ the desired correspondence

$$V(Q) = -\dot{x} + \Omega R(\alpha_i, 1/\alpha_j) T(x) \tag{17}$$

is established for

$$\beta_i = \frac{1}{\alpha_i}\,. \tag{18}$$

In contrast to the Noether energy, the conservation law (8) can be bounded or unbounded from below depending on the value of α's. The bounded integrals of motion (8) are associated with time translations by differential Lagrange anchor

$$V = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathcal{P}_i, \qquad (-1)^i \alpha_i > 0\,. \tag{19}$$

From the classical viewpoint the relationship (12) is as good as Noether's one. In particular, it allows one to define the generalization of the Dickey bracket of conservation laws and admits BRST-description [14]. Thus, the correspondence between the bounded from below conservation law (8), the Lagrange anchor (19) and the time translation (17) ensures the stability of the PU oscillator theory even if the canonical energy is unbounded.

Let us give one more argument that makes analogy between the energy and conservation law associated with the time translation more explicit. It is well known that different Lagrange anchors result in different quantizations of one and the same classical system [13, 15]. In the first-order formalism, the integrable[2] Lagrange anchor always defines a Poisson bracket on the phase space of the system, while the corresponding integral of energy becomes the Hamiltonian [13, 16, 17]. The canonical Lagrange anchor corresponds to the canonical Poisson brackets and Hamiltonian that follows from the Ostrogradsky formalism [18, 19], while non-canonical Lagrange anchors correspond to non-canonical Poisson brackets and Hamiltonians.

3. The case of resonance

Let us consider the fourth order PU oscillator in the case of equal frequencies $\omega=\omega_1=\omega_2$. The equation of motion reads

$$T(x)=\Big(\frac{1}{\omega}\frac{d^2}{dt^2}+\omega\Big)^2x=0\,,\qquad T=\Big(\frac{1}{\omega}\frac{d^2}{dt^2}+\omega\Big)^2. \tag{20}$$

The solutions to the equation of motion demonstrate runaway behavior with linear time dependence of the oscillation amplitude

$$x(t)=A\sin(\omega t+\varphi_0)+Bt\sin(\omega t+\varphi_1)$$

with A, B, φ_0 and φ_1 being arbitrary real constants. The system (20) still has two impendent integrals of motion

$$I_1=\frac{1}{2}\Big(\frac{1}{\omega^2}\dddot{x}+\dot{x}\Big)^2+\frac{1}{2}\Big(\frac{1}{\omega}\ddot{x}+\omega x\Big)^2,\quad I_2=\frac{1}{2}\left(\frac{\ddot{x}^2-2\dddot{x}\dot{x}}{\omega^2}-2\dot{x}^2-\omega^2x^2\right).$$

The first integral is obtained from (7) by taking limit $\omega_1\to\omega_2$ with special renormalization of the α-constants. The second one is just the Noether energy. In contrast to the case of unequal frequencies, it is impossible to find two independent bounded from below quadratic integrals of motion. Only the integral of motion I_1 is bounded from below. However, an attempt to associate it with time translation fails.

The characteristic for I_1 reads

$$\frac{dI_1}{dt}=Q_1T(x),\qquad Q_1=\Big(\frac{1}{\omega^2}\dddot{x}+\dot{x}\Big).$$

[2]See the definition of integrability in [12]. The field-independent Lagrange anchor (15) is automatically integrable.

There are two-parameter family of Lagrange anchors for PU oscillator (20)

$$V = \frac{\beta_2}{\omega^2}\frac{d^2}{dt^2} + \beta_1\,. \tag{21}$$

The corresponding symmetry reads

$$V(Q_1) = \frac{\beta_1}{\omega^2}T(x) + \frac{\beta_2 - \beta_1}{\omega^2}\dddot{x} + (\beta_2 - \beta_1)\dot{x}\,. \tag{22}$$

In (22), the third derivative vanishes if and only if the symmetry (22) is trivial, i.e., $\beta_1 = \beta_2$. In view of above, there is no time-independent bounded from below conservation law that could be associated to time translation. This result demonstrates the fact that has been already observed in [7], where it was found that PU oscillator with resonance does not admit Hamiltonian formulation with any bounded Hamiltonian.

Conclusion

We observe that for higher-derivative theories, the stability does not necessarily require the Noether energy to be bounded from below. The classical stability can be ensured by a weaker condition that the model admits a bounded integral of motion. Once the equations of motion admit the Lagrange anchor such that maps the bounded integral to the time translation, the theory can retain stability at quantum level. Both the conserved quantity and the Lagrange anchor are not uniquely defined by the equations of motion and may exist even in non-singular models. This allows us to expand the stability analysis to the wide class of higher-derivative theories, including non-singular ones. The general idea is exemplified by the Pais–Uhlenbeck oscillator. Using the ambiguity of choice of the Lagrange anchor and bounded conserved quantity, we demonstrate the stability of Pais–Uhlenbeck oscillator when all the frequencies are different.

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Deformation Quantization with Separation of Variables and Gauge Theories

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Abstract. We construct a gauge theory on a noncommutative homogeneous Kähler manifold by using the deformation quantization with separation of variables for Kähler manifolds. A model of noncommutative gauge theory that is connected with an ordinary Yang–Mills theory in the commutative limit is given. As an examples, we review a noncommutative $\mathbb{C}P^N$ and construct a gauge theory on it. We also give details of the proof showing that the noncommutative $\mathbb{C}P^N$ constructed in this paper coincides with the one given by Bordemann, Brischle, Emmrich and Waldmann [1].

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Keywords. Noncommutative geometry, gauge theory.

1. Deformation quantization of Kähler manifolds

Let M be a Kähler manifold with Kähler form ω and associated Poisson structure $\{\cdot,\cdot\}$. We note that for the Kähler manifold (M,ω) we have a Kähler potential Φ on any open subset given by the following.

$$\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \qquad g_{i\bar{j}} = \frac{\partial^2 \Phi}{\partial z^i \partial \bar{z}^j},$$

where $g_{i\bar{j}}$ is a Hermitian metric. In this article, a noncommutative space is constructed by using deformation quantization which is defined as follows.

Definition 1. Let $\mathcal{F}$ be a set of formal power series of $\hbar$:

$$\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k,\ f_k \in C^\infty(M) \right\}.$$

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A star product of $f, g \in \mathcal{F}$ is defined as

$$f * g = \sum C_k(f, g)\hbar^k,$$

such that the product satisfies the following four conditions. (1). $*$ is an associative product. (2). C_k is a bi-differential operator. (3). C_0 and C_1 are defined as

$$C_0(f, g) = fg, \qquad C_1(f, g) - C_1(g, f) = i\{f, g\},$$

where $\{f, g\}$ is the Poisson bracket. (4). $f * 1 = 1 * f = f$.

For Kähler manifolds Karabegov introduced a star product with the additional property of separation of variables [3]. Recall that $*$ is called a star product with separation of variables when

$$a * f = af, \ \ f * b = fb$$

for any function f, any local holomorphic function a and any local anti-holomorphic function b. Karabegov showed that for an arbitrary ω, there exists a star product with separation variables [3]. The star product is constructed by a formal power series of differential operators L_f for any $f \in C^\infty(M)[[\hbar]]$ satisfying

$$L_f g := f * g.$$

Then L_f has the form $L_f = \sum_{n=0}^{\infty} \hbar^n A_n$, where $A_n = a_{n,\alpha}(f) \prod_i \left(D^{\bar{i}}\right)^{\alpha_i}$ with $D^{\bar{i}} := g^{\bar{i}j}\partial_j$. We also use $D^i := g^{i\bar{j}}\partial_{\bar{j}}$ in the following. It is required that L_f satisfies

$$L_f 1 = f * 1 = f, \ \ L_f\left(L_g h\right) = f * (g * h) = (f * g) * h = L_{L_f g} h.$$

L_f is determined by the following condition,

$$[L_f, \partial_{\bar{i}}\Phi + \hbar\partial_{\bar{i}}] = 0, \tag{1}$$

and $A_0 = f$.

2. Gauge theory

A differential calculus on noncommutative spaces can be constructed based on the derivations of the algebra $C^\infty(M)[[\hbar]]$ with its star product, where a derivation $\mathbf{d}$ is a linear operator satisfying the Leibniz rule, i.e., $\mathbf{d}(f * g) = \mathbf{d}f * g + f * \mathbf{d}g$. In a commutative space, vector fields are derivations. However first-order differential operators in noncommutative spaces do not satisfy the Leibniz rule in general. A derivation $\mathcal{L}$ is called an inner derivation when $P \in C^\infty(M)[[\hbar]]$ exists such that $\mathcal{L}(f) = [P, f]_* := P * f - f * P$, for any $f \in C^\infty(M)[[\hbar]]$. Inner derivations corresponding to vector fields play an important role, when we construct physical field theories. Note that $[P, f]_*$ includes higher derivative terms of f for a generic P. As for the characterization of inner derivation, we have the following.

Proposition 2. [Müller-Bahns and N. Neumaier] [5, 6] *In deformation quantization defined above, every inner derivations given as the vector fields are the Killing vector fields.*

We construct a gauge theory on a noncommutative deformed homogeneous Kähler manifold $M = \mathcal{G}/\mathcal{H}$ whose covariant derivatives are given as inner derivations corresponding to the Killing vector fields. In the following, we consider $U(n)$ gauge theories for simplicity. All following results can be applied for any matrix groups. Proofs of theorems in this section are given in [5].

We introduce noncommutative $U(n)$ transformations as deformations of unitary transformations. If $g \in U(n)$, then $g^\dagger g = I$, where $g^\dagger$ is the Hermitian conjugate of g and I is the identity matrix. As a natural extension, we define $G := M_n(C^\infty(M)[[\hbar]])$ such that for $U = \sum_{k=0}^{\infty} \hbar^k U^{(k)}$ and $U^\dagger = \sum_{k=0}^{\infty} \hbar^k U^{(k)\dagger} \in G$,

$$U^\dagger * U = \sum_{n=0}^{\infty} \hbar^n \sum_{m=0}^{n} U^{(m)\dagger} * U^{(n-m)} = I.$$

For an arbitrary $U^{(0)} : M \to U(n)$ there exist $U^{(k)}$ $(k = 1, 2, \dots)$ satisfying the above condition by solving it recursively at each order.

In a homogeneous Kähler manifold $M = \mathcal{G}/\mathcal{H}$, there are holomorphic Killing vector fields $\mathcal{L}_a = \zeta_a^i(z)\partial_i + \zeta_a^{\bar{i}}(\bar{z})\partial_{\bar{i}}$ corresponding to an isometry group, such that

$$[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc} \mathcal{L}_c,$$

where f_{abc} is a structure constant of $\mathfrak{g}$ and this $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$. As we saw in the previous section, $\mathcal{L}_a$ satisfies the Leibniz rule,

$$\mathcal{L}_a(f * g) = (\mathcal{L}_a f) * g + f * (\mathcal{L}_a g).$$

The Killing vectors are normalized here as

$$\eta^{ab}\zeta_a^i \zeta_b^{\bar{j}} = g^{i\bar{j}}, \quad \eta^{ab}\zeta_a^i \zeta_b^j = 0, \quad \eta^{ab}\zeta_a^{\bar{i}} \zeta_b^{\bar{j}} = 0,$$

where η^{ab} is the inverse of the Killing form of the Lie algebra $\mathfrak{g}$.

On a commutative homogeneous Kähler manifold $M = \mathcal{G}/\mathcal{H}$ we introduce a gauge connection $\mathcal{A}_a^{(0)}$ by

$$\mathcal{A}_a^{(0)} = \zeta_a^\mu A_\mu = \zeta_a^i A_i + \zeta_a^{\bar{i}} A_{\bar{i}},$$

where A_i and $A_{\bar{i}}$ are the gauge fields on M. The curvature is defined as $\mathcal{F}_{ab}^{(0)} := \mathcal{L}_a \mathcal{A}_b^{(0)} - \mathcal{L}_b \mathcal{A}_a^{(0)} - i[\mathcal{A}_a^{(0)}, \mathcal{A}_b^{(0)}] - i f_{abc}\mathcal{A}_c^{(0)}$, where $[A, B] = AB - BA$. $\mathcal{F}_{ab}^{(0)}$ is related to the curvature of A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu - i[A_\mu, A_\nu]$, as $\mathcal{F}_{ab}^{(0)} = \zeta_a^\mu \zeta_b^\nu F_{\mu\nu}$. It is shown that

$$\eta^{ac}\eta^{bd}\mathcal{F}_{ab}^{(0)}\mathcal{F}_{cd}^{(0)} = g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Following the above observation, we consider a gauge theory on a noncommutative space. For a gauge field $\mathcal{A}_a := \sum_{k=0}^{\infty} \hbar^k \mathcal{A}_a^{(k)}$, a gauge transformation is defined by

$$\mathcal{A}_a \to \mathcal{A}'_a = iU^{-1} * \mathcal{L}_a U + U^{-1} * \mathcal{A}_a * U.$$

Let us define the curvature of $\mathcal{A}_a$ by

$$\mathcal{F}_{ab} := \mathcal{L}_a \mathcal{A}_b - \mathcal{L}_b \mathcal{A}_a - i[\mathcal{A}_a, \mathcal{A}_b]_* - i f_{abc} \mathcal{A}_c.$$

Then $\mathcal{F}_{ab}$ transforms covariantly:

$$\mathcal{F}_{ab} \to \mathcal{F}'_{ab} = U^{-1} * \mathcal{F}_{ab} * U.$$

Using this, we obtain a gauge invariant action.

Theorem 3. *A gauge invariant action for the gauge field is given by*

$$S_g := \int_{\mathcal{G}/\mathcal{H}} \mu_g \ \mathrm{tr}\left(\eta^{ac}\eta^{bd}\mathcal{F}_{ab} * \mathcal{F}_{cd}\right),$$

where μ_g is the trace density on $\mathcal{G}/\mathcal{H}$.

The gauge invariance of the action is obtained by the cyclic symmetry of the trace density. The existence of the trace density, $\int_M f * g\mu_g = \int_M g * f\mu_g$, is guaranteed by [4]. When we consider a noncommutative deformation of a homogeneous Kähler manifold which does not break its isometry group, it is expected that the above action is the unique action on it which has a connection with the Yang–Mills theory in the commutative limit.

3. Non commutative deformation of $\mathbb{C}P^N$

In this section, we review a noncommutative $\mathbb{C}P^N$. The detailed derivations of the following results are found in [5, 7].

Let z^i $(i = 1, 2, \ldots, N)$ be the inhomogeneous coordinates of $\mathbb{C}P^N$. Then, the Kähler potential of $\mathbb{C}P^N$ is given by

$$\Phi = \ln\left(1 + |z|^2\right), \qquad (|z|^2 = \sum_i z^i \bar{z}^i).$$

The metric $(g_{i\bar{j}})$ and its inverse $(g^{\bar{i}j})$ are derived from Φ as

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 + |z|^2)\delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2}, \quad g^{\bar{i}j} = (1 + |z|^2)\left(\delta_{ij} + z^j \bar{z}^i\right).$$

Solving the recursion relation (1), we obtain an explicit star product for arbitrary functions f and g:

$$f * g = \sum_{n=0}^{\infty} \frac{\alpha_n(\hbar)}{n!} g_{j_1\bar{k}_1} \cdots g_{j_n\bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f\right) (D^{\bar{k}_1} \cdots D^{\bar{k}_n} g).$$

Here $\alpha_m(t)$ is defined as

$$\alpha_m(t) = \frac{\Gamma(1 - m + \frac{1}{t})}{\Gamma(1 + \frac{1}{t})}.$$

This $\alpha_m(t)$ is the generating function of the Stirling number of the second kind.

By using this, we get

$$z^i * z^j = z^i z^j, \qquad z^i * \bar{z}^j = z^i \bar{z}^j, \qquad \bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \tag{2}$$

$$\begin{aligned}\bar{z}^i * z^j &= \bar{z}^i z^j + \hbar \delta_{ij}(1+|z|^2)_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)\\ &\quad + \frac{\hbar}{1-\hbar}\bar{z}^i z^j (1+|z|^2)_2F_1\left(1,2;2-1/\hbar;-|z|^2\right).\end{aligned} \tag{3}$$

For $\mathbb{C}H^N$, we have obtained similar results. (See [7].)

3.1. Inner derivations and gauge theory on noncommutative $\mathbb{C}P^N$

As an example, we study differentials in a noncommutative $\mathbb{C}P^N$. Note that the isometry group of this $\mathbb{C}P^N$ is $SU(N+1)$. We give concrete expressions of the Killing vectors corresponding to the generators of $su(N+1)$, the Lie algebra of $SU(N+1)$. Generators $(T_a)_{AB}$ of $su(N+1)$ in the fundamental representation satisfy

$$[T_a, T_b] = i f_{abc} T_c, \qquad \mathrm{Tr}\ T_a = 0, \qquad \mathrm{Tr}\ T_a T_b = \delta_{ab},$$
$$(T_a)_{AB}(T_a)_{CD} = \delta_{AD}\delta_{BC} - \frac{1}{N+1}\delta_{AB}\delta_{CD},$$

where f_{abc} is the structure constant of $su(N+1)$, $a = 1, 2, \ldots, N^2+2N$, and $A, B = 0, 1, \ldots, N$. Set

$$\begin{aligned}\mathcal{L}_a = \zeta_a^i \partial_i + \zeta_a^{\bar{i}} \partial_{\bar{i}} &= (T_a)_{00}\left(z^i \partial_i - \bar{z}^i \partial_{\bar{i}}\right) + (T_a)_{0i}\left(z^i z^j \partial_j + \partial_{\bar{i}}\right)\\ &\quad + (T_a)_{i0}\left(-\partial_i - \bar{z}^i \bar{z}^j \partial_{\bar{j}}\right) + (T_a)_{ij}\left(-z^j \partial_i + \bar{z}^i \partial_{\bar{j}}\right).\end{aligned}$$

Then $\{\mathcal{L}_a\}$ forms the generators of the isometry $SU(N+1)$ which satisfies $[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc}\mathcal{L}_c$. For the noncommutative $\mathbb{C}P^N$ trace density μ_g is given by the Riemannian volume form. The Yang–Mills type action is constructed as

$$S_g := \int_{\mathbb{C}P^N} \sqrt{g} dz^1 \cdots dz^N d\bar{z}^1 \cdots d\bar{z}^N\ \mathrm{tr}\left(\mathcal{F}_{ab} * \mathcal{F}_{cd}\eta^{ac}\eta^{bd}\right), \tag{4}$$

and all classical calculation of this Yang–Mills theory can be done by using the Killing vectors given above. Similarly to the $\mathbb{C}P^N$, we can construct this type of the Yang–Mills theory on a noncommutative $\mathbb{C}H^N$ with concrete expressions [5, 7].

4. Proof for equality of star products

In this section, we show that our star product for $\mathbb{C}P^N$ coincides with the one given by Bordemann et al. [1], which is a detailed version of the discussions in [7]. Schlichenmaier also showed equivalences among various $*$ products by using rather different ways from our method [8, 9].

A non-trivial part of star products in this paper is given as (3). We denote by $*_B$ the star product in [1]. $\bar{z}^i *_B z^j$ is given as

$$\begin{aligned}
&\bar{z}^i z^j + \sum_{m=1}^{\infty} \hbar^m \sum_{s=1}^{m} \sum_{k=1}^{s} \frac{k^{m-1}(-1)^{m-k}}{s!(s-k)!(k-1)!} \left(|\zeta|^2\right)^s \\
&\qquad\qquad \times \frac{\partial^s}{\partial\bar{\zeta}^{A_1}\cdots\partial\bar{\zeta}^{A_s}} \left(\frac{\bar{\zeta}^i}{\bar{\zeta}^0}\right) \frac{\partial^s}{\partial\zeta^{A_1}\cdots\partial\zeta^{A_s}} \left(\frac{\zeta^j}{\zeta^0}\right) \\
&= \bar{z}^i z^j + \hbar\delta_{ij}(1+|z|^2)G_1(1+|z|^2) + \hbar\bar{z}^i z^j (1+|z|^2) G_2(1+|z|^2). \qquad (5)
\end{aligned}$$

Here, ζ^i and $\bar{\zeta}^i$ are the homogeneous coordinates and

$$G_1(1+|z|^2) = \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=1}^{s+1} \frac{\hbar^m s! k^m (-1)^{m+1-k}}{(s+1-k)!(k-1)!} \left(1+|z|^2\right)^s, \qquad (6)$$

$$G_2(1+|z|^2) = \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=1}^{s+1} \frac{\hbar^m (s+1)! k^m (-1)^{m+1-k}}{(s+1-k)!(k-1)!} \left(1+|z|^2\right)^s. \qquad (7)$$

Comparing with (3), we will show

$$G_1(1+|z|^2) = {}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right), \qquad (8)$$

$$G_2(1+|z|^2) = \frac{1}{1-\hbar} {}_2F_1\left(1,2;2-1/\hbar;-|z|^2\right). \qquad (9)$$

The proof for (8) and (9) contains the following two steps.

Step 1. ${}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)$ and ${}_2F_1\left(1,2;2-1/\hbar;-|z|^2\right)$ satisfy

$$\left[x(1-x)\partial_x^2 + (1-1/\hbar-3x)\partial_x - 1\right] {}_2F_1\left(1,1;1-1/\hbar;x\right) = 0, \qquad (10)$$

$$\left[x(1-x)\partial_x^2 + (2-1/\hbar-4x)\partial_x - 2\right] {}_2F_1\left(1,2;2-1/\hbar;x\right) = 0, \qquad (11)$$

where $x := -|z|^2$. Therefore, for $y := 1-x = 1+|z|^2$, we prove that $G_1(y)$ and $G_2(y)$ satisfy

$$\left[y(1-y)\partial_y^2 + (2+1/\hbar-3y)\partial_y - 1\right] G_1(y) = 0, \qquad (12)$$

$$\left[y(1-y)\partial_y^2 + (2+1/\hbar-4y)\partial_y - 2\right] G_2(y) = 0. \qquad (13)$$

Step 2. The Gauss hypergeometric function behaves as

$$F(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}x + O(x^2), \qquad (14)$$

and thus we prove that

$$G_1(1+|z|^2) = 1 + \frac{\hbar}{1-\hbar}|z|^2 + O\left((-|z|^2)^2\right), \qquad (15)$$

$$G_2(1+|z|^2) = \frac{1}{1-\hbar} + \frac{2\hbar}{(1-\hbar)(1-2\hbar)}|z|^2 + O\left((-|z|^2)^2\right). \qquad (16)$$

4.1. Proof

First we prove $G_1(1+|z|^2) = {}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)$.

Step 1. We list each terms of the left-hand side of (12):

$$-y^2\partial_y^2 G_1(y) = \sum_{s=2}^{\infty} y^s s! \left\{ \sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{\hbar^m s(s-1)k^m(-1)^{m-k}}{(s+1-k)!(k-1)!} + \sum_{k=1}^{s+1} \frac{\hbar^s s(s-1)k^s(-1)^{s-k}}{(s+1-k)!(k-1)!} \right\},$$

$$y\partial_y^2 G_1(y) = \sum_{s=1}^{\infty} y^s s! \left\{ \sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{\hbar^m (s+1)^2 s k^m(-1)^{m-k+1}}{(s+2-k)!(k-1)!} + \sum_{m=s+1}^{\infty} \frac{\hbar^m (s+1)^2 s(s+2)^m(-1)^{m-s+1}}{(s+1)!} \right\},$$

$$2\partial_y G_1(y) = \sum_{s=0}^{\infty} y^s s! \left\{ \sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{2\hbar^m (s+1)^2 k^m(-1)^{m-k+1}}{(s+2-k)!(k-1)!} + \sum_{m=s+1}^{\infty} \frac{2\hbar^m (s+1)^2 (s+2)^m(-1)^{m-s+1}}{(s+1)!} \right\},$$

$$\frac{1}{\hbar}\partial_y G_1(y) = \sum_{s=0}^{\infty} y^s s! \left\{ \sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{\hbar^m (s+1)^2 k^{m+1}(-1)^{m-k}}{(s+2-k)!(k-1)!} + \sum_{m=s+1}^{\infty} \frac{\hbar^m (s+1)^2 (s+2)^{m+1}(-1)^{m-s}}{(s+1)!} + \sum_{k=1}^{s+2} \frac{\hbar^s (s+1)^2 k^{s+1}(-1)^{s-k}}{(s+2-k)!(k-1)!} \right\},$$

$$-3y\partial_y G_1(y) = \sum_{s=1}^{\infty} y^s s! \left\{ \sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{3\hbar^m s k^m(-1)^{m-k}}{(s+1-k)!(k-1)!} + \sum_{k=1}^{s+1} \frac{3\hbar^s s k^s(-1)^{s-k}}{(s+1-k)!(k-1)!} \right\},$$

$$-G_1(y) = \sum_{s=0}^{\infty} y^s s! \left\{ \sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{\hbar^m k^m(-1)^{m-k}}{(s+1-k)!(k-1)!} + \sum_{k=1}^{s+1} \frac{\hbar^s k^s(-1)^{s-k}}{(s+1-k)!(k-1)!} \right\}.$$

It is easy to check that the y^0, y^1 terms are 0. The vanishing of linear terms of $y^s s!$ $(s \geq 2)$ can be shown as

$$\sum_{m=s+1}^{\infty}\sum_{k=1}^{s+1} \frac{\hbar^m k^m(-1)^{m-k}}{(s+2-k)!(k-1)!}[s(s-1)(s+2-k) - (s+1)^2 s - 2(s+1)^2 + (s+1)^2 k + 3s(s+2-k) + (s+2-k)]$$

$$+\sum_{m=s+1}^{\infty}\frac{\hbar^m(s+1)^2(s+2)^m(-1)^{m-s}}{(s+1)!}[-s-2+(s+2)]$$
$$+(-\hbar)^s\left\{\sum_{k=1}^{s+1}\frac{(-1)^k k^s}{(s+1-k)!(k-1)!}[s(s-1)+3s+1]\right.$$
$$\left.+(s+1)^2\sum_{k=1}^{s+2}\frac{(-1)^k k^{s+1}}{(s+2-k)!(k-1)!}\right\}=0.$$

Here, we used

$$\sum_{k=1}^{s+1}\frac{(-1)^k k^s}{(s+1-k)!(k-1)!}=-\frac{1}{s!}\sum_{k=0}^{s}{}_sC_k(k+1)^s=(-1)^{s+1}, \tag{17}$$
$$\sum_{k=1}^{s+2}\frac{(-1)^k k^{s+1}}{(s+2-k)!(k-1)!}=-\frac{1}{(s+1)!}\sum_{k=0}^{s+1}{}_{s+1}C_k(k+1)^{s+1}=(-1)^s \tag{18}$$

which are obtained from (A.3) in Appendix. Thus, $G_1(1+|z|^2)$ satisfies the same differential equation for ${}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)$.

Step 2. Next, we show that the boundary condition of $G_1(1+|z|^2)$ coincides with the one for ${}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)$. The constant terms of $G_1(1+|z|^2)$ are

$$\sum_{m=0}^{\infty}\sum_{s=0}^{m}\sum_{k=1}^{s+1}\frac{\hbar^m s!k^m(-1)^{m-k+1}}{(s+1-k)!(k-1)!}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{s=k}^{m}\frac{\hbar^m s!(k+1)^m(-1)^{m-k}}{(s-k)!k!}.$$

Using (A.5), we obtain

$$\sum_{s=k}^{m}\frac{s!}{(s-k)!}=\frac{(m+1)!}{(k+1)(m-k)!}.$$

Furthermore, (A.2) implies that

$$\sum_{k=0}^{m}\frac{\hbar^m(k+1)^m(m+1)!(-1)^{m-k}}{(k+1)!(m-k)!}=\delta_{m,0}.$$

Then, we find that the constant term of $G_1(1+|z|^2)$ is given as

$$\sum_{m=0}^{\infty}\delta_{m,0}=1. \tag{19}$$

Note that $|z|^2$ linear term in $G_1(1+|z|^2)$ is given by

$$\sum_{m=0}^{\infty}\sum_{s=0}^{m}\sum_{k=1}^{s+1}\frac{\hbar^m s!sk^m(-1)^{m-k+1}}{(s+1-k)!(k-1)!}=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{s=k}^{m}\frac{\hbar^m s!s(k+1)^m(-1)^{m-k}}{(s-k)!k!}.$$

Using (A.5), we obtain

$$\sum_{s=k}^{m}\frac{s!s}{(s-k)!}=\frac{(m+2)!}{(k+2)(m-k)!}-\frac{(m+1)!}{(k+1)(m-k)!}.$$

From (A.2) and (A.4), the following formula is given:

$$\sum_{k=0}^{m} \frac{(k+1)^m(-1)^k}{k!} \left[\frac{(m+2)!}{(k+2)(m-k)!} - \frac{(m+1)!}{(k+1)(m-k)!} \right] = (-1)^m - \delta_{m,0}.$$

Finally the $|z|^2$ linear term in $G_1(1+|z|^2)$ is given by

$$\sum_{m=0}^{\infty} \hbar^m (-1)^m \left[(-1)^m - \delta_{m,0} \right] = \sum_{m=1}^{\infty} \hbar^m = \frac{\hbar}{1-\hbar}. \tag{20}$$

(19) and (20) show that the behavior of $G_1(1+|z|^2)$ near the origin coincides with ${}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)$.

From Steps 1 and 2, we get $G_1(1+|z|^2) = {}_2F_1\left(1,1;1-1/\hbar;-|z|^2\right)$.

Similarly, we can prove that $G_2(1+|z|^2) = \dfrac{1}{1-\hbar}{}_2F_1\left(1,2;2-1/\hbar;-|z|^2\right)$. These facts mean that $\bar{z}^i * z^j = \bar{z}^i *_B z^j$. The relations of the other star products between coordinates are trivial because of the star products with separation of variables. Therefore, equality between the star product given in this paper and $*_B$ in [1] is proved.

5. Summary

In the first half of this paper, we review a construction of a gauge theory on a noncommutative homogeneous Kähler manifold. We use the deformation quantization with separation of variables for Kähler manifolds. In particular, a noncommutative $\mathbb{C}P^N$ and its noncommutative gauge theory are given with explicit expressions. In this paper, we omit the part about $\mathbb{C}H^N$, but a similar noncommutative deformation and a gauge theory can be constructed. In the latter part of this paper, we provided a detailed proof to show that the noncommutative $\mathbb{C}P^N$ given in this paper coincides with the one given by Bordemann et al. The proof is done by directly comparing the concrete expressions of these two star products.

Appendix

Following formulas are given in p. 3 and p. 4 in [2].

$$\sum_{k=0}^{m} (-1)^k {}_mC_k k^m = (-1)^m m!, \tag{A.1}$$

$$\sum_{k=0}^{m} (-1)^k {}_mC_k k^n = 0, \qquad (m > n), \tag{A.2}$$

$$\sum_{k=0}^{n} (-1)^k {}_nC_k (\alpha+k)^n = (-1)^n n!, \tag{A.3}$$

$$\sum_{k=0}^{N} (-1)^k {}_NC_k (\alpha+k)^{n-1} = 0, \qquad (N \geq n \geq 1) \tag{A.4}$$

$$\sum_{n=k}^{m} \frac{n!}{(n-k)!} = \frac{(m+1)!}{(k+1)(m-k)!}. \tag{A.5}$$

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Physics of Spectral Singularities

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Abstract. Spectral singularities are certain points of the continuous spectrum of generic complex scattering potentials. We review the recent developments leading to the discovery of their physical meaning, consequences, and generalizations. In particular, we give a simple definition of spectral singularities, provide a general introduction to spectral consequences of $\mathcal{PT}$-symmetry (clarifying some of the controversies surrounding this subject), outline the main ideas and constructions used in the pseudo-Hermitian representation of quantum mechanics, and discuss how spectral singularities entered in the physics literature as obstructions to these constructions. We then review the transfer matrix formulation of scattering theory and the application of complex scattering potentials in optics. These allow us to elucidate the physical content of spectral singularities and describe their optical realizations. Finally, we survey some of the most important results obtained in the subject, drawing special attention to the remarkable fact that the condition of the existence of linear and nonlinear optical spectral singularities yield simple mathematical derivations of some of the basic results of laser physics, namely the laser threshold condition and the linear dependence of the laser output intensity on the gain coefficient.

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1. Introduction

The term 'spectral singularity' entered mathematical literature through the work of Jack Schwartz [1] who coined this name for a mathematical object discovered by Mark Aronovich Naimark in 1954 [2]. Naimark had come across spectral singularities and worked out some of their consequences in his attempts at generalizing the well-known spectral theory of self-adjoint Schrödinger operators,

$$H = -\frac{d^2}{dx^2} + v(x), \tag{1}$$

defined on the half-line, i.e., for $x \in [0, \infty)$, to situations where $v(x)$ was a complex scattering potential. This marks the starting point of a comprehensive study of spectral singularities that has attracted the attention of mathematicians for over half a century [3–10]. For further references, see [11].

Following the pioneering work of Naimark, the notion of spectral singularity was generalized in various directions [3, 5, 6, 8–10]. In particular, Kemp [3] considered spectral singularities of the Schrödinger operators (1) defined on the full line, i.e., $x \in \mathbb{R}$. These admit a simple description in terms of certain solutions of the Schrödinger equation [11]

$$-\psi''(x) + v(x)\psi(x) = k^2\psi(x), \qquad x \in \mathbb{R}, \tag{2}$$

called the Jost solutions.

Let $v(x)$ be a real or complex scattering potential defined on $\mathbb{R}$, and suppose that $|v(x)| \to 0$ as $x \to \pm\infty$ in such a manner that [3]

$$\int_{-\infty}^{\infty} (1 + |x|)|v(x)|dx < \infty. \tag{3}$$

Then for each $k \in \mathbb{R}$, the Schrödinger equation (2) admits a pair of solutions $\psi_{k\pm}$ fulfilling the asymptotic boundary conditions [3]:

$$\lim_{x\to\pm\infty} e^{\mp ikx}\psi_{k\pm}(x) = 1, \qquad \lim_{x\to\pm\infty} e^{\mp ikx}\psi'_{k\pm}(x) = \pm ik. \tag{4}$$

These are the celebrated *Jost solutions* of (2).

Definition 1. Let $v : \mathbb{R} \to \mathbb{C}$ be a function satisfying (3) and H be the Schrödinger operator (1) that is defined by v on $\mathbb{R}$. A real and positive number $k_\star^2$ is called a *spectral singularity* of H or v, if the Jost solutions $\psi_{k_\star\pm}$ of (2) are linearly dependent.

It is not difficult to see that the Jost solutions correspond to the scattering states of the potential $v(x)$. But as we shall see below, for real scattering potentials they are always linearly independent. This is why physicists did not pay much attention to spectral singularities throughout the twentieth century.

The situation began to change in 1998 by the discovery of a class of complex potentials that possessed a real spectrum [12]. A well-known example is $v(x) = ix^3$, whose spectrum is discrete, real, and positive [13]. This unexpected result was initially associated with the fact that the corresponding Schrödinger operator (1) was invariant under the parity-time-reversal transformation, also known as spacetime reflection [12],

$$\psi(x) \longrightarrow \mathcal{PT}\psi(x) = \psi(-x)^*, \tag{5}$$

where ψ is an arbitrary square-integrable function, i.e., $\psi \in L^2(\mathbb{R})$, and $\mathcal{P}$ and $\mathcal{T}$ are respectively the parity and time-reversal operators defined by

$$\mathcal{P}\psi(x) := \psi(-x), \qquad \mathcal{T}\psi(x) := \psi(x)^*. \tag{6}$$

We postpone the discussion of the spectral implications of $\mathcal{PT}$-symmetry to Section 2. Here we suffice to mention that during the last 16 years there has been a great interest in the study of $\mathcal{PT}$-symmetric potentials. A substantial amount of the early work on this topic consisted of searching for other examples of complex potentials possessing a real spectrum. A rather straightforward method of constructing such potentials is to generate them from real potentials via non-unitary similarity transformations. The simplest example is a complex translation of the form $v(x) \to v(x + \mathfrak{z})$, where $\mathfrak{z}$ is a complex parameter [15]. To the best of our knowledge, it was in this context that spectral singularities entered in the physics literature.

In 2005 Boris Samsonov noted that the application of complex translations on a real potential could yield complex scattering potentials supporting spectral singularities [16]. He also proposed means of removing these spectral singularities by performing certain supersymmetry transformations. In Samsonov's words this meant "curing" a "disease", for he maintained that "Hamiltonians with spectral singularities are 'bad'." This is a typical reaction when one encounters a 'singularity'. However, the history of science teaches us that some of the greatest discoveries of mankind have their root at unwanted 'singularities'. One of the aims of the present article is to show that the same applies to spectral singularities, as they provide the mathematical basis for one of the most important discoveries of all times, namely lasers.

2. $\mathcal{PT}$-symmetry versus pseudo-hermiticity

Samsonov's article [16] failed to provide the necessary incentive for the study of the physical aspects of spectral singularities. But soon after, spectral singularities were to reveal their presence in the study of a delta-function potential with an imaginary coupling constant, i.e., $v(x) = i\alpha\,\delta(x)$ with $\alpha \in \mathbb{R}$, [17]. The motivation for this study was provided by attempts at finding a set of necessary and sufficient conditions for the reality of the spectrum of a non-Hermitian linear operator H. A well-advertised claim is that $\mathcal{PT}$-symmetry provides such a condition [18]. This is certainly not true if we take (6) as the definition of $\mathcal{P}$ and $\mathcal{T}$, for there are infinity of examples of real potentials, such as $v(x) = x^2 + \sin x$, that do not commute with $\mathcal{PT}$ but have a real spectrum.

In order to ensure the validity of the above claim, we need to reinterpret what we mean by $\mathcal{PT}$-symmetry or generalize it appropriately. First, we recall the following obvious consequences of (6).

$$[\mathcal{P}, \mathcal{T}] = 0, \qquad \mathcal{P}^2 = \mathcal{T}^2 = (\mathcal{PT})^2 = I, \tag{7}$$

where I stands for the identity operator. The following is a precise definition of $\mathcal{PT}$-symmetry.

Definition 2. Let $\mathcal{P}$ and $\mathcal{T}$ be the linear operators defined on $L^2(\mathbb{R})$ by (6). Then a linear operator H acting in $L^2(\mathbb{R})$ is said to be *$\mathcal{PT}$-symmetric*, if it commutes with $\mathcal{PT}$, i.e., $[H, \mathcal{PT}] = 0$. Moreover, suppose that H has a discrete and non-degenerate spectrum, and there is a complete set[1] of eigenvectors ψ_n of H that are also eigenvectors of $\mathcal{PT}$. Then H is said to have an *unbroken* or *exact $\mathcal{PT}$-symmetry.*

For the cases where H has an exact $\mathcal{PT}$-symmetry, there are $\epsilon_n \in \mathbb{C}$ such that $\mathcal{PT}\psi_n = \epsilon_n\psi_n$. Then, in view of (7), $\psi_n = (\mathcal{PT})^2\psi_n = \mathcal{PT}(\epsilon_n\psi_n) = |\epsilon_n|^2\psi_n$. This shows that $\epsilon_n = e^{i\alpha_n}$ for some real number α_n. Now, setting $\tilde{\psi}_n := e^{i\alpha_n/2}\psi_n$, we find $\mathcal{PT}\tilde{\psi}_n = e^{-i\alpha_n/2}\mathcal{PT}\psi_n = e^{i\alpha_n/2}\psi_n = \tilde{\psi}_n$. Therefore, exact $\mathcal{PT}$-symmetry of H means the existence of a complete set of $\mathcal{PT}$-invariant eigenvectors of H. This argument relies only on the fact that $\mathcal{PT}$ is an antilinear operator[2] squaring to I. Therefore, it applies to all such operators $\mathcal{X}$.[3] We use this observation to introduce the notion of exact antilinear symmetry.

Definition 3. Let H and $\mathcal{X}$ be respectively linear and antilinear operators acting in a Hilbert space $\mathscr{H}$, and I be the identity operator on $\mathscr{H}$. H is said to be $\mathcal{X}$-symmetric, if $[\mathcal{X}, H] = 0$. Furthermore, suppose that H has a discrete spectrum and $\mathcal{X}^2 = I$. Then H is said to have an exact $\mathcal{X}$-symmetry if there is a complete set of eigenvectors ψ_n of H that are invariant under $\mathcal{X}$, i.e., $\mathcal{X}\psi_n = \psi_n$.

The following is a useful property of exact antilinear symmetry. Its application to $\mathcal{PT}$-symmetry is the reason for the claim that exact $\mathcal{PT}$-symmetry implies the reality of the spectrum.

Theorem 1. *Eigenvalues of every linear operator H that has an exact antilinear symmetry are real.*

Proof. Let E_n be an eigenvalue of H and ψ_n be a corresponding $\mathcal{X}$-invariant eigenvector. Then, in view of the fact that $\mathcal{X}\psi_n = \psi_n$ and $[\mathcal{X}, H] = 0$,

$$\begin{aligned} E_n^* &= \frac{\langle\psi_n|E_n^*\mathcal{X}\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} = \frac{\langle\psi_n|\mathcal{X}E_n\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} = \frac{\langle\psi_n|\mathcal{X}H\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} \\ &= \frac{\langle\psi_n|H\mathcal{X}\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} = \frac{\langle\psi_n|H\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} = E_n. \end{aligned}$$ □

This theorem suggests generalizing $\mathcal{PT}$-symmetry to the presence of an antilinear symmetry. In order to avoid using the same symbols for different concepts, we use 'PT-symmetry' to refer to this generalization.

Definition 4. We say that H is *PT-symmetric* if it has an exact antilinear symmetry.

[1]This means that the span of this set is dense in $L^2(\mathbb{R})$.
[2]An antilinear operator $\mathcal{X}$ acting in a complex vector space V is one whose domain is a subspace of V and fulfills the antilinearity condition: $\mathcal{X}(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1^*\mathcal{X}v_1 + \alpha_2^*\mathcal{X}v_2$ for all $\alpha_1, \alpha_2 \in \mathbb{C}$ and $v_1, v_2 \in V$.
[3]Because every antilinear operator acting in $L^2(\mathbb{R})$ can be expressed in the form $\mathcal{QT}$ for some linear operator $\mathcal{Q}$, a natural choice of notation for the antilinear operators $\mathcal{X}$ is $\mathcal{QT}$ [14].

In view of this definition, Theorem 1 is equivalent to the statement that PT-symmetry is a sufficient condition for the reality of the spectrum of H. The converse can also be established if $\mathscr{H}$ is finite dimensional or if we impose a further technical condition[4] on H, [20]. Therefore, introducing the above notion of PT-symmetry secures the validity of the claim that it is a necessary and sufficient condition for the reality of the spectrum of a large class of linear operators. However the price one pays for doing so is a clear distinction between PT-symmetry and the parity-time-reversal (spacetime reflection) symmetry that we label by $\mathcal{PT}$-symmetry. Indeed unlike Hermiticity which is a sufficient condition for the reality of the spectrum, PT-symmetry is both necessary and sufficient. But this PT-symmetry does not mean spacetime reflection ($\mathcal{PT}$) symmetry. Similarly to Hermiticity exact $\mathcal{PT}$-symmetry is a sufficient but not necessary condition for the reality of the spectrum of a linear operator. Non-exact $\mathcal{PT}$-symmetry, which can be immediately checked for a given operator, is neither necessary nor sufficient.

A major difficulty with the above notion of PT-symmetry is that there does not exist a universal choice for the antilinear symmetry appearing in Definition 4; each PT-symmetric linear operator H has its own set of antilinear operators $\mathcal{X}$ which make H exactly $\mathcal{X}$-symmetric. If we denote this set by $\mathcal{S}_H$, then PT-symmetry of H is equivalent to the condition that $\mathcal{S}_H$ is nonempty. In practice, in order to check if this is the case, one must determine an appropriate set of eigenvectors ψ_n of H, make sure that they form a complete set, and try to construct an antilinear operator $\mathcal{X}$ that leaves ψ_n's invariant and squares to I. This is generally a difficult task.

The question of finding a necessary and sufficient condition for the reality of the spectrum of a non-Hermitian operator has a more illuminating answer.

Theorem 2. *Let H be a linear operator with a complete set of eigenvectors that acts in a finite-dimensional inner-product space. Then H has a real spectrum if and only if there is a positive-definite operator η_+ intertwining H and its adjoint $H^\dagger$, i.e.,*

$$H^\dagger \eta_+ = \eta_+ H, \tag{8}$$

alternatively $H^\dagger$ is related to H by the similarity transformation:

$$H^\dagger = \eta_+ H\, \eta_+^{-1}. \tag{9}$$

Conditions (8) and (9), that we call 'η_+-pseudo-Hermiticity' of H, was derived in Refs. [20, 22, 23] for a more general class of operators. These act in a possibly infinite-dimensional Hilbert space $\mathscr{H}$, have a discrete spectrum, and possess a complete biorthonormal eigensystem $\{(\psi_n, \phi_n)\}$, [19]. The latter means the existence of a sequence of complex numbers $\{E_n\}$ and a pair of sequences of vectors,

[4]This is the condition of the existence of a set of eigenvectors of H that forms a Riesz basis, i.e., it can be mapped to an orthonormal basis by an invertible bounded operator [19].

$\{\psi_n\}$ and $\{\phi_n\}$, which satisfy

$$H\psi_n = E_n\psi_n, \qquad\qquad H^\dagger\phi_n = E_n^*\phi_n, \tag{10}$$

$$\langle\psi_m|\phi_n\rangle = \delta_{mn}, \qquad\qquad \sum\nolimits_n |\psi_n\rangle\langle\phi_n| = I. \tag{11}$$

Here $\langle\cdot|\cdot\rangle$ stands for the inner product of $\mathscr{H}$. We use the term 'diagonalizable' to mean that H admits a complete biorthonormal eigensystem.

Theorem 2 admits an infinite-dimensional generalization provided that we impose further restrictions on H and η_+ [19]. For pedagogical reasons we postpone discussing these to the final three paragraphs of this section. For the moment, we follow the physicists' tradition of assuming that what we know about finite dimensions is essentially valid in infinite dimensions. For example, we take the following condition as the definition of a Hermitian or self-adjoint operator:

$$\langle\phi|H\psi\rangle = \langle H\phi|\psi\rangle,$$

where ϕ and ψ are arbitrary elements of $\mathscr{H}$ (hence neglecting domain issues).

A key observation made in [20, 22, 23] is that the reality of the spectrum of H is related to the fact that we can turn it into a Hermitian operator by modifying the inner product of $\mathscr{H}$ properly. Using the term 'Hermitizablility' for the latter property, we can say that *a diagonalizable operator with a real spectrum need not be Hermitian, but it is necessarily Hermitizable.* Conversely, *every Hermitizable operator is diagonalizable and has a real spectrum.* Therefore, *Hermitizability is a necessary and sufficient condition for the reality of the spectrum of H.*[5]

The modified inner product that achieves the Hermitization of H is determined by the operator η_+ according to

$$\langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle, \tag{12}$$

where $\psi,\phi\in\mathscr{H}$ are arbitrary. In other words, if $\mathscr{H}_{\eta_+}$ labels the Hilbert space obtained by endowing the set of vectors belonging to $\mathscr{H}$ with the inner product $\langle\cdot,\cdot\rangle_{\eta_+}$, then $H:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$ is Hermitian.

The operator η_+ that defines the modified inner product (12) is usually called a 'metric operator.' It is not difficult to show that its positive square root, $\rho := \sqrt{\eta_+}$, defines a unitary operator mapping $\mathscr{H}_{\eta_+}$ onto $\mathscr{H}$ and that $h := \rho H\rho^{-1}$ is a Hermitian operator acting in $\mathscr{H}$, [29]. This provides a direct evidence for the reality of the spectrum of H, for H and h are isospectral. It also makes a connection with an earlier work on quasi-Hermitian operators that is done in the context of nuclear physics [30].

Another useful result of Refs. [20, 22, 23] is the following spectral representation of the metric operator.

$$\eta_+ = \sum\nolimits_n |\phi_n\rangle\langle\phi_n|. \tag{13}$$

[5] A rigorous extension of this result to infinite dimensions requires special care. In mathematics literature, it is studied in the context of 'symmetrizable' [24] and 'quasi-Hermitian operators' [25]. For more recent developments, see [26–28].

Because the eigenvectors ϕ_n of $H^\dagger$ are not unique, this equation signifies the non-uniqueness of the metric operator [29, 31]. Different choices for $\{\phi_n\}$ determine different metric operators for H. Once such a choice is made, we can construct the corresponding Hilbert space $\mathscr{H}_{\eta_+}$ and view H as a Hamiltonian operator acting in $\mathscr{H}_{\eta_+}$. By construction $H : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$ is a Hermitian operator. Therefore $(\mathscr{H}_{\eta_+}, H)$ determines a unitary quantum system. The pure states of this system are represented by the rays in $\mathscr{H}_{\eta_+}$, the observables are given by Hermitian operators acting in $\mathscr{H}_{\eta_+}$, and the dynamics is governed by the time-dependent Schrödinger equation defined by H in $\mathscr{H}_{\eta_+}$.

Applications of the above method of constructing metric operators and the modified inner products for various toy models have been explored in the literature. A comprehensive list of references published prior to 2010 is given in the review article [19]. Here we confine our attention to a very simple example that was originally considered in [32].

Consider the case that $v(x) = 0$, i.e., H is the second derivative operator acting in $L^2(\mathbb{R})$. It is well known that H is Hermitian and has a nonnegative real continuous spectrum. We can easily check that it satisfies the η_+-pseudo-Hermiticity relation (9) for

$$\eta_+ := e^{-\kappa\mathcal{P}} = \cosh(\kappa)I - \sinh(\kappa)\mathcal{P}, \tag{14}$$

where κ is an arbitrary real number, and $\mathcal{P}$ is the parity operator (6). Equation (14) defines a genuine metric operator. Substituting it in (12) yields the following expression for the corresponding modified inner product.

$$\langle\phi,\psi\rangle_{\eta_+} = \cosh(\kappa)\int_{-\infty}^{\infty}\phi(x)^*\psi(x)dx - \sinh(\kappa)\int_{-\infty}^{\infty}\phi(x)^*\psi(-x)dx. \tag{15}$$

For the standard position operator X, that is given by $X\psi(x) := x\psi(x)$, we can use (15) to show that

$$\langle\phi, X\psi\rangle_{\eta_+} - \langle X\phi,\psi\rangle_{\eta_+} = 2\sinh(\kappa)\int_{-\infty}^{\infty} x\phi(x)^*\psi(-x)dx.$$

This quantity differs from zero for $\kappa \neq 0$ and $\psi(x) = x\phi(x) = x\,e^{-x^2}$. Therefore, as an operator acting in $\mathscr{H}_{\eta_+}$, X is not Hermitian, unless if $\kappa = 0$. The same holds for the standard momentum operator, $P := -i\frac{d}{dx}$.

Clearly the positive square root of the metric operator (14) has the form $\rho = e^{-\kappa\mathcal{P}/2}$. In view of this relation, it is easy to show that the operators X_{η_+} and P_{η_+} defined by

$$X_{\eta_+} := \rho^{-1}X\,\rho = \eta_+^{-1}X = [\cosh(\kappa)I + \sinh(\kappa)\mathcal{P}]X,$$
$$P_{\eta_+} := \rho^{-1}P\,\rho = \eta_+^{-1}P = [\cosh(\kappa)I + \sinh(\kappa)\mathcal{P}]P,$$

are Hermitian operators acting in $\mathscr{H}_{\eta+}$. Because $[X_{\eta_+}, P_{\eta_+}] = iI$, we can take X_{η_+} and P_{η_+} to represent the position and momentum observables of the system defined by $(\mathscr{H}_{\eta+}, H)$. For all real numbers κ, this is just a free particle moving on a straight line. But for different choices of κ, we have different operators representing the position and momentum of the particle. This has some peculiar consequences. For example, for $\kappa \neq 0$, the spatially localized states of the particle correspond to a linear combination of two Dirac delta-functions rather than a single delta-function [32]!

When $\mathscr{H}$ is an infinite-dimensional Hilbert space the above constructions are valid provided that we impose some additional technical conditions. Specifically, the complete biorthonormal eigensystem $\{(\psi_n, \phi_n)\}$ should be bounded [19]. This is equivalent to the condition that $\{\psi_n\}$ and $\{\phi_n\}$ are Riesz bases of $\mathscr{H}$, which means that they can be mapped to an orthonormal basis by a bounded invertible operator [19, 35]. The boundedness of $\{(\psi_n, \phi_n)\}$ implies that the metric operator η_+ must be a positive automorphism, i.e., a positive invertible operator that is defined everywhere in $\mathscr{H}$ (which makes it bounded) and has a bounded inverse [19].

It turns out that if we define the Schrödinger operator for the potential ix^3 as a linear operator (with maximal domain) acting in $\mathscr{H} := L^2(\mathbb{R})$, then we cannot satisfy (8) or (9) using a bounded positive-definite operator η_+ that is inversely bounded. Therefore, strictly speaking, an appropriate metric operator does not exist for this potential [33]. As we explain below this is a mathematical technicality that can be circumvented by paying due attention to the role of the linear operators representing physical observables in quantum mechanics.

Consider redefining the Hilbert space $\mathscr{H}$ and the operator H in such a way that the new Hilbert space $\mathscr{H}'$ includes the eigenvectors of H and the new operator H', which acts in $\mathscr{H}'$, shares both the spectrum and eigenvectors of H, [34]. Because we can only prepare state vectors which are superpositions of the eigenvectors of the relevant observables, as far as H is concerned both $\mathscr{H}$ and $\mathscr{H}'$ include all the prepareable state vectors, and H and H' are equivalent as representations of a quantum mechanical observable. As shown in Ref. [34], for a given metric operator η_+, which may violate the conditions of boundedness or inverse boundedness, it is possible to construct $\mathscr{H}'$ and H' in such a way that they have the above-mentioned properties and in addition H' be a Hermitian operator. Therefore although one cannot use $(\mathscr{H}, H)$ to define a unitary quantum system directly, one can construct $\mathscr{H}'$ and H' which contain the same physically relevant ingredients and use $(\mathscr{H}', H')$ to define a unitary quantum system.

3. Singularities of the metric operators

Consider the Hilbert space $\mathscr{H}$ obtained by endowing $\mathbb{C}^2$ with the Euclidean inner product. The elements of $\mathscr{H}$ and the linear operators acting in it can be respectively represented by 2×1 and 2×2 matrices in the standard basis of $\mathbb{C}^2$. Using the same symbol for the matrix representations and the corresponding vectors and operators, we consider constructing the most general metric operator for

$$H := \begin{bmatrix} 0 & 1 \\ x^2 & 0 \end{bmatrix}, \tag{16}$$

where $x \in \mathbb{R}$. It is easy to show that for this operator,

$$E_n = (-1)^n x, \qquad \psi_n = \frac{N_n}{\sqrt{2}} \begin{bmatrix} (-1)^n \\ x \end{bmatrix}, \qquad \phi_n = \frac{1}{\sqrt{2} N_n^*} \begin{bmatrix} (-1)^n \\ x^{-1} \end{bmatrix}, \tag{17}$$

where $n = 1, 2$ and N_n are arbitrary nonzero complex coefficients possibly depending on x.

Inserting the last of Equations (17) in (13) and introducing $a_\pm := (|N_2|^{-2} \pm |N_1|^{-2})/2$, we find

$$\eta_+ = \begin{bmatrix} a_+ & a_- x^{-1} \\ a_- x^{-1} & a_+ x^{-2} \end{bmatrix}.$$

This relation identifies $x = 0$ with a singularity of all possible metric operators for H. Note also that H loses the property of being diagonalizable precisely for this value of x. This is an example of what is called an exceptional point [36, 37] or a non-Hermitian degeneracy [38].

The term 'exceptional point' is introduced by Kato in his study of the effects of perturbations of a linear operator on its spectral properties [39]. The following is a widely used definition of this concept which differs slightly from Kato's.

Definition 5. Let V be a vector space, m be a positive integer, $H(x) : V \to V$ be a linear operator depending on m real parameters $x_1, x_2, \ldots, x_m$. We identify these with local coordinates of a point x of a parameter space (a smooth manifold) M. Suppose that for each $x \in M$ the eigenvalues of $H(x)$ have finite geometric multiplicity and form a countable set of isolated points of $\mathbb{C}$ that we denote by $E_n(x)$. Here n is a spectral label taking values in a discrete set $\mathcal{N}$. Let $\mu_n(x)$ be the geometric multiplicity of $E_n(x)$, i.e., the dimension of the span of eigenvectors of $H(x)$ that are associated with the eigenvalue $E_n(x)$. A point x_0 of M is called an exceptional point of $H(x)$ if there are $n \in \mathcal{N}$, $\epsilon \in \mathbb{R}^+$, and a parameterized curve in M, i.e., a continuos function, $\gamma : (-\epsilon, \epsilon) \to M$, such that $\gamma(0) = x_0$ and for all $t \neq 0$, $\mu_n(\gamma(t)) \neq \mu_n(x_0)$.

For the case that V is endowed with the structure of an inner-product space, we can speak of the adjoint of $H(x)$ and decide whether it is Hermitian. If for all $x \in M$, $H(x)$ is a Hermitian operator, the geometric multiplicity of the eigenvalues $E_n(x)$ do not undergo discontinuous changes and an exceptional point cannot exist. Therefore, non-Hermiticity is a necessary condition for the emergence of an exceptional point.

It turns out that exceptional points have a number of interesting physical realizations. See for example [36–38, 40–42] and references therein. In particular, they lead to certain geometric phases which have been the subject of intensive theoretical [36–38, 40, 41, 43] and experimental studies [44–46] since the early 1990's.

The two-dimensional model (16) can be easily generalized to higher-dimensional matrix Hamiltonians $H(x)$, [37]. If we choose the eigenvectors of $H(x)$ in such a way that they are nonsingular functions of x, then exceptional points appear as the singularities of the eigenvectors of $H(x)^\dagger$ and consequently the corresponding metric operator (13).

Definition 5 introduces exceptional points in terms of a condition on the eigenvalues of $H(x)$. If this operator acts in a Hilbert space, we can speak of its spectrum. This is a subset of $\mathbb{C}$ that in addition to the eigenvalues may contain other numbers. The latter constitute two disjoint sets called the continuous and the residual spectra of $H(x)$ [47]. Hermitian operators have an empty residual spectrum. The same is true for a large class of non-Hermitian operators.

A natural question that arises in the study of non-Hermitian operators with a real spectrum is how to generalize the notions of diagonalizability and the metric operator for operators whose spectrum includes a continuous part. The first step in this direction was taken in Ref. [48]. It involved a direct extension of the approach developed for operators with a discrete spectrum to the imaginary $\mathcal{PT}$-symmetric barrier potential,

$$v(x) = \begin{cases} -i\zeta & \text{for} \quad -1 \le x \le 0, \\ i\zeta & \text{for} \quad 0 < x \le 1, \\ 0 & \text{for} \quad |x| \le 1, \end{cases} \qquad \zeta \in \mathbb{R}. \tag{18}$$

To the best of our knowledge, this provided the first example of a $\mathcal{PT}$-symmetric potential which admitted an optical realization [49]. The next step was to carry out the same analysis for the delta-function potential with a complex coupling constant [17],

$$v(x) = \mathfrak{z}\,\delta(x), \qquad \mathfrak{z} \in \mathbb{C}. \tag{19}$$

The treatment of (18) and (19) that was offered in [48] and [17] is perturbative in nature. But there is an important difference; for imaginary values of $\mathfrak{z}$ regardless of how small $|\mathfrak{z}|$ is, the perturbative calculation of the metric operator for (19) is obstructed by the emergence of a singularity. In the remainder of this section, we provide a general description of this phenomenon and its relation to spectral singularities that was originally noticed in [17] and explored more thoroughly for the double-delta-function potential in [50]:

$$v(x) = \mathfrak{z}_-\,\delta(x+a) + \mathfrak{z}_+\,\delta(x-a), \qquad \mathfrak{z}_\pm \in \mathbb{C},\ a \in \mathbb{R}^+. \tag{20}$$

Let $v_z : \mathbb{R} \to \mathbb{C}$ be a scattering potential depending on complex parameters $z_1, z_2, \ldots, z_m$, that we collectively denote by z, i.e., $z := (z_1, z_2, \ldots, z_m)$. Suppose that the Schrödinger operator $H_z := -\frac{d^2}{dx^2} + v_z(x)$ acts in $L^2(\mathbb{R})$ and has a real and purely continuous spectrum given by $[0, \infty)$, i.e., its point and residual spectra are empty. Then the nonzero elements of the spectrum of H_z correspond to the numbers k^2 appearing on the right-hand side of the Schröndinger equation (2). These are associated with a linearly independent pair of solutions of this equation that we denote by $\psi^{(z)}_{k,a}$ with $a = 1, 2$;

$$H_z \psi^{(z)}_{k,a} = k^2 \psi^{(z)}_{k,a}. \tag{21}$$

Because $\psi_{k,a}^{(z)}$ do not belong to $L^2(\mathbb{R})$, they are not eigenvectors of H_z. We refer to them as 'generalized eigenfunctions' of H_z. Similarly, we can construct generalized eigenfunctions of $H_z^\dagger := -\frac{d^2}{dx^2} + v_z(x)^*$ that we denote by $\phi_{k,a}^{(z)}$. These satisfy

$$H_z^\dagger \phi_{k,a}^{(z)} = k^2 \phi_{k,a}^{(z)}. \tag{22}$$

We can generalize the notion of 'diagonalizability' for H_z, by demanding the existence of an eigensystem $\{(\psi_{k,a}^{(z)}, \phi_{k,a}^{(z)})\}$ which satisfy the following biorthonormality and completeness relations [48].

$$\langle \phi_{k,a}^{(z)} | \psi_{q,b}^{(z)} \rangle = \delta_{ab}\delta(k-q), \qquad \sum_{a=1}^{2} \int_0^\infty dk\ |\psi_{k,a}^{(z)}\rangle\langle\phi_{k,a}^{(z)}| = I. \tag{23}$$

Similarly we generalize the expression (13) for the metric operator:

$$\eta_+ = \sum_{a=1}^{2} \int_0^\infty dk\ |\phi_{k,a}^{(z)}\rangle\langle\phi_{k,a}^{(z)}|. \tag{24}$$

Now, we demand that $v_z(x)^* = v_{z^*}(x)$. Then it is easy to see that

$$H_z^\dagger \psi_{k,a}^{(z^*)} = \left[-\frac{d^2}{dx^2} + v_z(x)^*\right] \psi_{k,a}^{(z^*)} = H_{z^*} \psi_{k,a}^{(z^*)} = k^2 \psi_{k,a}^{(z^*)}. \tag{25}$$

Because for each $k \in \mathbb{R}^+$, the Schrödinger equation $H_z^\dagger \psi = k^2 \psi$ has two linearly independent solutions, Equations (22) and (25) imply that $\phi_{k,a}^{(z)}$ are linear combinations of $\psi_{k,a}^{(z^*)}$, i.e., there are $J_{ab}(k) \in \mathbb{C}$ such that

$$\phi_{k,a}^{(z)} = \sum_{b=1}^{2} J_{ab}^{(z)}(k) \psi_{k,b}^{(z^*)}. \tag{26}$$

It is also not difficult to show that $\langle \psi_{k,a}^{(z^*)} | \psi_{q,b}^{(z)} \rangle$ is proportional to $\delta(k-q)$, i.e., there are $K_{ab}^{(z)}(k) \in \mathbb{C}$ such that

$$\langle \psi_{k,a}^{(z^*)} | \psi_{q,b}^{(z)} \rangle = K_{ab}^{(z)}(k)\delta(k-q). \tag{27}$$

Inserting (26) in the first equation in (23) and making use of (27), we find [50]

$$\mathbf{J}(k)^{(z)*}\mathbf{K}^{(z)}(k) = \mathbf{I}, \tag{28}$$

where $\mathbf{J}^{(z)}(k)$ and $\mathbf{K}^{(z)}(k)$ are 2×2 matrices having $J_{ab}^{(z)}(k)$ and $K_{ab}^{(z)}(k)$ as their entries, and $\mathbf{I}$ is the 2×2 identity matrix. Similarly, using (24), (26), and (28), we obtain

$$\eta_+ = \sum_{a,b=1}^{2} \int_0^k dk\ \mathcal{E}_{ab}^{(z)}(k) |\psi_{k,a}^{(z^*)}\rangle\langle\psi_{k,b}^{(z^*)}|, \tag{29}$$

where $\mathcal{E}_{ab}^{(z)}(k)$ are entries of the matrix $[\mathbf{K}^{(z)}(k)\mathbf{K}^{(z)}(k)^\dagger]^{-1}$.

Equation (28) implies that $\det(\mathbf{K}^{(z)}(k)) \neq 0$. But in general there is no reason why this relation should hold for all k and z. Explicit calculations for the potentials (19) and (20) show that the values of k^2 for which $\det(\mathbf{K}^{(z)}(k)) = 0$ are precisely the spectral singularities of the Schrödinger operator H_z, [50]. We are not aware of a proof of this statement for a general complex scattering potential. The proof for the double-delta-function potential that was given in [50] revealed a useful connection between spectral singularities and the transfer matrix $\mathbf{M}(k)$ of scattering theory [51]. It turned out that $\det(\mathbf{K}^{(z)}(k))$ was proportional to the $M_{22}(k)$ entry of $\mathbf{M}(k)$ with a nonzero proportionality factor. This provided the key observation that immediately led to the explanation of the physical meaning of a spectral singularity [52]. We give a detailed discussion of these developments in the next section.

We close this section by noting that according to (29), spectral singularities are also singularities of the metric operator. In this sense they are generalizations of the phenomenon of exceptional points to the linear operators that possess a nonempty continuous spectrum.

4. Scattering theory and spectral singularities

Consider a possibly complex scattering potential $v(x)$ satisfying (3). The left- and right-incident scattering solutions of the Schrödinger equation, that we respectively denote by $\psi_k^l(x)$ and $\psi_k^r(x)$, satisfy the following asymptotic boundary conditions.

$$\psi_k^l(x) \to \begin{cases} \mathscr{A}^l(k)\left[e^{ikx} + R^l(k)e^{-ikx}\right] & \text{as} \quad x \to -\infty, \\ \mathscr{A}^l(k)T^l(k)e^{ikx} & \text{as} \quad x \to \infty, \end{cases} \tag{30}$$

$$\psi_k^r(x) \to \begin{cases} \mathscr{A}^r(k)T^r(k)e^{-ikx} & \text{as} \quad x \to -\infty, \\ \mathscr{A}^r(k)\left[e^{-ikx} + R^r(k)e^{ikx}\right] & \text{as} \quad x \to \infty, \end{cases} \tag{31}$$

where $\mathscr{A}^{l/r}$, $R^{l/r}$, and $T^{l/r}$ are in general complex-valued functions. Because the Schrödinger equation (2) is linear, the choice of $\mathscr{A}^{l/r}$ does not affect the physically measurable quantities. This is not the case for $R^{l/r}$ and $T^{l/r}$, which are known as the left/right reflection and transmission amplitudes. Their modulus squared, $|R^{l/r}|^2$ and $|T^{l/r}|^2$, determine the left/right reflection and transmission coefficients that can be measured in experiments.[6]

A well known consequence of the linearity of the Schrödinger equation (2) is that $T^l = T^r$, [52–54].[7] We therefore use T for $T^{l/r}$. It is also easy to see that $\psi_k^{l/r}$

[6]Notice that some authors use the symbols $R^{r/l}$ and $T^{r/l}$ for reflection and transmission coefficients.

[7]This arises from Wronskian identities [52, 54] and has nothing to do with the reality of the potential as claimed in [55].

coincide with the Jost solutions $\psi_{k\pm}$ for $\mathscr{A}^{l/r}(k)T(k)=1$;

$$\psi_{k+}(x) \to \begin{cases} T(k)^{-1}\left[e^{ikx}+R^l(k)e^{-ikx}\right] & \text{as} \quad x\to-\infty, \\ e^{ikx} & \text{as} \quad x\to\infty, \end{cases} \tag{32}$$

$$\psi_{k-}(x) \to \begin{cases} e^{-ikx} & \text{as} \quad x\to-\infty, \\ T(k)^{-1}\left[e^{-ikx}+R^r(k)e^{ikx}\right] & \text{as} \quad x\to\infty. \end{cases} \tag{33}$$

The existence of the Jost solutions implies that $T(k)\neq 0$, i.e., perfectly absorbing potentials [56] do not exist.

The coefficients of the $e^{\pm ikx}$ that appear on the right-hand side of (32) and (33) turn out to coincide with the entries of a 2×2 complex matrix known as the transfer matrix.

Because $v(x)\to 0$ as $x\pm\infty$, every solution $\psi(x)$ of the Schrödinger equation (2) satisfy

$$\psi(x)\to A_\pm(k)e^{ikx}+B_\pm(k)e^{-ikx} \qquad \text{as} \qquad x\to\pm\infty, \tag{34}$$

where $A_\pm(k)$ and $B_\pm(k)$ are complex coefficients. The transfer matrix $\mathbf{M}(k)$ is defined by the relation

$$\begin{bmatrix} A_+(k) \\ B_+(k) \end{bmatrix} = \mathbf{M}(k) \begin{bmatrix} A_-(k) \\ B_-(k) \end{bmatrix}.$$

In light of (32), (33), and (34), we can relate the entries $M_{ij}(k)$ of the transfer matrix $\mathbf{M}(k)$ with the reflection and transmission amplitudes. This results in [52]

$$M_{11}=T-\frac{R^lR^r}{T}, \qquad M_{12}=\frac{R^r}{T}, \qquad M_{21}=-\frac{R^l}{T}, \qquad M_{22}=\frac{1}{T}, \tag{35}$$

which, in particular, imply $\det \mathbf{M}(k)=1$. Furthermore, we can use these relations to express (32) and (33) in the form

$$\psi_{k+}(x) \to \begin{cases} M_{22}(k)e^{ikx}-M_{21}(k)e^{-ikx} & \text{as} \quad x\to-\infty, \\ e^{ikx} & \text{as} \quad x\to\infty, \end{cases} \tag{36}$$

$$\psi_{k-}(x) \to \begin{cases} e^{-ikx} & \text{as} \quad x\to-\infty, \\ M_{22}(k)e^{-ikx}+M_{12}(k)e^{ikx} & \text{as} \quad x\to\infty. \end{cases} \tag{37}$$

The following characterization of spectral singularities is a direct consequence of these equations.

Theorem 3. *Let $v:\mathbb{R}\to\mathbb{C}$ and H be as in Definition 1, $\mathbf{M}(k)$ be the transfer matrix of v, $M_{ij}(k)$ be the entries of $\mathbf{M}(k)$, and $k_\star$ be a positive real number. Then $k_\star^2$ is a spectral singularity of H (or v) if and only if $M_{22}(k_\star)=0$.*

Proof. $k_\star^2$ is a spectral singularity of H whenever $\psi_{k_\star-}$ and $\psi_{k_\star+}$ are linearly dependent. According to (36) and (37) and the fact that these equations determine $\psi_{k\pm}$ uniquely, this happens if and only if $M_{22}(k_\star)=0$. □

Combining the statement of Theorem 3 with Equation (35) yields the physical meaning of spectral singularities, namely that spectral singularities are the real and positive values of the energy $k_\star^2$ at which reflection and transmission amplitudes diverge [52]. The latter is a characteristic property of resonances, for they satisfy the outgoing boundary conditions [57]. As seen from (36) and (37), for the cases that $k_\star$ corresponds to a spectral singularity and $\psi_{k_\star\pm}$ become linearly dependent, they also satisfy the outgoing boundary conditions.

The main distinction between the wave function for a resonance and the Jost solutions $\psi_{k_\star\pm}$ at a spectral singularity $k_\star^2$ is that, unlike the latter, the former satisfies the Schrödinger equation for a non-real value of k^2. Because the imaginary part of k^2 for a resonance determines its width, we can identify spectral singularities with the energies of certain zero-width resonances. Note, however, that spectral singularities determine genuine non-decaying scattering states with real and positive energy [52]. This distinguishes them from the bound states in the continuum [58]. Although the latter are also associated with zero-width resonances, their wave function is a square-integrable solutions of the Schrödinger equation. For a discussion of other differences between spectral singularities and bound states in the continuum, see [59].

The fact that the reflection and transmission amplitudes and consequently the reflection and transmission coefficients $|R^{l/r}(k)|^2$ and $|T(k)|^2$ diverge for a resonance does not conflict with the well-known unitarity condition

$$|R^{l/r}(k)|^2 + |T(k)|^2 = 1, \tag{38}$$

because the k-value for a resonance is not real. For a spectral singularity, k is real and (38) is violated. This provides a simple proof of the following result.

Theorem 4. *Real potentials cannot support a spectral singularity.*

In the standard formulation of quantum mechanics, the Hamiltonian operator H is required to be Hermitian and the potential functions v are necessarily real-valued. Therefore, they do not display spectral singularities. The same applies to the pseudo-Hermitian representation of quantum mechanics [19] where H may not be Hermitian but Hermitizable. This is because the presence of a spectral singularities obstructs the existence of a metric operator that achieves the Hermitization process. However, complex scattering potentials have a number of applications in other areas of physics. The primary example is the optical potentials used in modeling optically active material. This is the arena in which the role and implications of spectral singularities have so far been studied. We devote the next section to a brief description of the optical realizations of spectral singularities.

5. Spectral singularities in optics

Consider an isotropic charge-free linear medium whose electromagnetic properties changes along one direction, that we take to be the x-axis in a Cartesian coordinate system. We can encode these properties in the definition of the refractive index of the medium $\mathbf{n}(x)$ which is a generally complex quantity. Suppose that we are interested in the propagation of a linearly polarized time-harmonic electromagnetic wave in this medium. If we choose our y-axis along the polarization direction, we can express the electric field in the form $\vec{E}(\vec{r},t) = e^{-i\omega t}\mathscr{E}(\vec{r})\hat{e}_y$, where $\vec{r} := (x,y,z)$, ω is the angular frequency of the wave, $\mathscr{E}(\vec{r})$ is a solution of

$$\left[\nabla^2 + k^2\mathbf{n}(x)^2\right]\mathscr{E}(\vec{r}) = 0, \tag{39}$$

$\hat{e}_y$ is the unit vector along the positive y-axis, $k := \omega/c$ is the wavenumber, and c is the speed of light in vacuum [60].

Equation (39) admits solutions depending only on x; $\mathscr{E}(\vec{r}) = \psi(x)$. In view of (39), $\psi(x)$ satisfies the Schrödinger equation (2) corresponding to the potential

$$v(x) := k^2[1 - \mathbf{n}(x)^2]. \tag{40}$$

If the medium is confined to a compact region in empty space, $\mathbf{n}(x) = 1$ for sufficiently large values of $|x|$. This together with the fact that $\mathbf{n}$ is a complex-valued function imply that $v(x)$ is a (finite-range) complex scattering potential. Therefore, optical potentials (40) provide a fertile ground for the investigation of the physical implications of spectral singularities. Ref. [52], which offers the first such investigation, explores spectral singularities in a medium described by an optical potential of the form (18).

The physical meaning of these spectral singularities is more easily understood for a simpler model that consists of a homogeneous optically active infinite planar slab of length L placed in vacuum [61, 62]. This corresponds to a complex barrier potential,

$$v(x) = \left\{\begin{array}{ccc} \mathfrak{z} & \text{for} & |x| \le L/2, \\ 0 & \text{for} & |x| > L/2, \end{array}\right. \qquad \mathfrak{z} := k^2[1 - \mathfrak{n}^2], \tag{41}$$

where $\mathfrak{n}$ stands for the refractive index of the slab.

Inside the slab, where $|x| \le L/2$, the Schrödinger equation (2) admits a solution of the form $\psi(x) = \mathscr{E}_0 e^{ik\,\mathfrak{n}(x+L/2)}$, where $\mathscr{E}_0$ is a constant. This corresponds to a right-going plane wave

$$\vec{E}(\vec{r},t) = \mathscr{E}_0 e^{i[k\,\mathfrak{n}(x+L/2)-\omega t]}\hat{e}_y, \qquad |x| \le L/2.$$

If we use η and κ to respectively denote the real and imaginary parts of $\mathfrak{n}$, so that $\mathfrak{n} = \eta + i\kappa$, we find

$$|\vec{E}(\vec{r},t)|^2 = |\mathscr{E}_0|^2 e^{-k\kappa(2x+L)}, \qquad |x| \le L/2.$$

In particular, as the wave travels through the slab, its intensity changes from $|\mathscr{E}_0|^2$ to $|\mathscr{E}_0|^2 e^{-2k\kappa L}$, i.e., it undergoes an exponential loss or gain of intensity by a factor of $e^{-2k\kappa L}$ depending on whether $\kappa > 0$ or $\kappa < 0$. Because of this a medium that has a positive (respectively negative) value for κ is called a lossy (respectively gain) medium. The factor $2k|\kappa|$ that determines the amount of the exponential loss (gain) per unit distance traversed by the wave is called the attenuation (respectively gain) coefficient. In terms of the wavelength, $\lambda := 2\pi/k$, this quantity takes the form $4\pi|\kappa|/\lambda$. In particular, the *gain coefficient* is given by [63]

$$g := -\frac{4\pi\kappa}{\lambda}. \tag{42}$$

Because the complex barrier potential (41) is exactly solvable, we can easily determine its transfer matrix and explore its spectral singularities. This is done in Refs. [61, 62]. Here we suffice to state that the relation $M_{22}(k_\star) = 0$, which determines the spectral singularities $k_\star^2$ whenever $k_\star \in \mathbb{R}^+$, reduces to the following complex transcendental equation [62].

$$e^{-2i\,\mathfrak{n}k_\star L} = \left(\frac{\mathfrak{n}-1}{\mathfrak{n}+1}\right)^2. \tag{43}$$

The right-hand side of this relation is a well-known quantity in optics called the *reflectivity* $\mathcal{R}$. If we compute the modulus (absolute-value) of both sides of (43) and use (42) in the resulting expression, we obtain [62]

$$g = \frac{1}{2L}\ln\frac{1}{|\mathcal{R}|^2}. \tag{44}$$

This equation that is a consequence of the existence of a spectral singularity is one of the basic relations of laser physics known as the laser threshold condition [63]. The right-hand side of (44) is the minimum amount of gain necessary for a (mirrorless) slab laser to begin emitting laser light. It is called the threshold gain coefficient.

Every laser amplifies the background noise to sizable intensities and emits it as coherent electromagnetic radiation. This is precisely what a spectral singularity does, because it leads to infinite reflection and transmission coefficients that are capable of amplifying extremely week background electromagnetic waves to considerable intensities. The fact that the waves emitted from both sides of a slab laser have the same intensity and phase (are coherent) also follows from (43). This is indeed a general property of spectral singularities, because they are invariant under the space reflection (parity) $\mathcal{P}$. Under $\mathcal{P}$ the transfer matrix $\mathbf{M}(k)$ of every scattering potential transforms as

$$\mathbf{M}(k) \stackrel{\mathcal{P}}{\longleftrightarrow} \sigma_1\mathbf{M}(k)^{-1}\sigma_1, \tag{45}$$

where σ_1 is the first Pauli matrix, i.e., the 2×2 matrix with zero diagonal and unit off-diagonal entries, [64, 65]. According to (45), $M_{22}(k)$ is $\mathcal{P}$-invariant. Therefore, the same holds for the spectral singularities that are given by the real and positive zeros of $M_{22}(k)$.

In Ref. [66], we develop a nonlinear generalization of spectral singularities that apply to nonlinearities that are confined in space (have compact support.) It turns out that the mathematical relation describing these nonlinear spectral singularities for the above simple slab model supplemented with a weak Kerr nonlinearity yields an equation relating the output intensity I of the slab laser to its gain coefficient [67]. For a typical optical gain medium [63], which satisfies $|\kappa| \ll 1 < \eta$, this equation takes the following form.

$$I = \frac{f(\eta)(g - g_{th})}{\sigma\, g_{th}}, \tag{46}$$

where f is a real-valued function taking strictly positive values, g is the gain coefficient (42), g_{th} is the threshold gain coefficient that is given by the right-hand side of (44), and σ is the Kerr coefficient which, for generic gain media, takes small but positive values.

Because $f(\eta) > 0$, $\sigma > 0$, and $I \geq 0$, Eq. (46) implies that there is no power emitted from a slab laser unless we have $g > g_{th}$, and for $g > g_{th}$ the intensity of emitted wave increases linearly as a function of $g - g_{th}$. Both of these statements are among the basic results of the physics of lasers. Here they follow as logical consequences of the purely mathematical condition of the existence of a nonlinear spectral singularity. Let us also mention that (46) has a more general domain of validity. In Ref. [68], we explore the consequences of the emergence of nonlinear spectral singularities for a weakly nonlinear $\mathcal{PT}$-symmetric bilayer slab. This consists of a pair of adjacent infinite homogeneous planar slabs with complex-conjugate refractive index, $\eta \pm i\kappa$, so that one's gain is balanced by other's loss [65, 69]. The laser output intensity computed using the condition of the appearance of a nonlinear spectral singularity is also given by (46), albeit with a different choice for the function f, [68].

Another interesting development having its root in optical spectral singularities is the discovery of perfect coherent absorbers (CPA) which are also called antilasers [64, 70–73]. These are optical devices that function as time-reversed lasers, i.e., they completely absorb coherent electromagnetic waves.

Under the time-reversal transformation (6), scattering potentials $v(x)$ and their transfer matrix $\mathbf{M}(k)$ transform according to

$$v(x) \xleftrightarrow{\mathcal{T}} v(x)^*, \qquad\qquad \mathbf{M}(k) \xleftrightarrow{\mathcal{T}} \sigma_1 \mathbf{M}(k)^* \sigma_1. \tag{47}$$

In light of these relations, the time-reversal transformation $\mathcal{T}$ converts an optical potential (40) describing a gain media into that of a lossy medium, and induces the transformation:

$$M_{11}(k) \xleftrightarrow{\mathcal{T}} M_{22}(k)^*.$$

This, in particular, means that the spectral singularities of $v(x)$ correspond to the real values of the wavenumber k at which the $M_{11}(k)$ entry of the transfer matrix of the time-reversed potential, $v(x)^*$, vanishes. At this wavenumber the optical system modeled by $v(x)^*$ serves as a CPA. In other words, CPA action is a realization of the spectral singularities of the time-reversed (complex-conjugate) optical potential [64, 65].

6. Concluding remarks

Spectral singularities were introduced by Naimark more than sixty years ago and has since become a subject of research in operator theory. Given their interesting mathematical implications, it is quite surprising that their relevance to scattering theory and their physical meaning could not be understood earlier than in 2009. It turns out that the optics of gain media offers various physical models in which this concept can be realized. The study of the optical realization of spectral singularities shows that they form a mathematical basis for lasers. This observation could be made much earlier, had the optical physicists knew about spectral singularities. Indeed, the solution of the wave equations with outgoing boundary conditions, which leads to spectral singularities for real wavenumbers, has been employed in laser theory previously [74].

The discovery of the physical aspects of spectral singularities has boosted interest in their study particularly among physicists. During the past five years there have appeared a number of research publications on the subject. The following is a list of those that we did not elude to above.

- Refs. [75–78] address some of the formal and conceptual aspects of the subject.
- Refs. [50, 65, 79–81] explore specific toy models supporting spectral singularities.
- Refs. [82, 83] study the application of semiclassical approximation and perturbation theory for determining spectral singularities of non-homogeneous gain media with planar symmetry.
- Refs. [84, 85] examine the optical spectral singularities in spherical and cylindrical geometries. In particular, [85] offers a detailed and careful treatment of spectral singularities in the whispering gallery modes. These correspond to the cylindrical and spherical lasers.
- Refs. [86, 87] discuss some of the applications of spectral singularities in condensed matter physics.
- Refs. [88, 89] consider spectral singularities in certain optically active waveguides and elaborate on their regularization due to the presence of nonlinearities.
- Ref.[90] offers an extension of the analysis of [62] to waves with a non-normal incidence angle.
- Refs. [91–93] are some other publications that discuss spectral singularities.

The recent development of a nonlinear generalization of spectral singularities [66] has opened the way towards applications of this concept in the vast territory of nonlinear waves. The fact that the simple applications in effectively one-dimensional optical systems yield a mathematical derivation of the known behavior of the laser output intensity provides ample motivation for further study of nonlinear spectral singularities in other areas of physics.

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On the Moduli Space of Yang–Mills Fields on $\mathbb{R}^4$

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Abstract. We consider the problem of description of the structure of the moduli space of Yang–Mills fields on $\mathbb{R}^4$ with gauge group G. According to harmonic spheres conjecture, this moduli space should be closely related to the space of harmonic spheres in the loop space ΩG. Since the structure of the latter space is much better understood, the proof of conjecture will help to clarify the structure of the moduli space of Yang–Mills fields. We propose an idea how to prove the harmonic spheres conjecture using the twistor methods.

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1. Yang–Mills fields and instantons

Let G be a compact Lie group (*gauge group*). A *gauge potential* on $\mathbb{R}^4$ is a connection A in a principal G-bundle over $\mathbb{R}^4$ identified with a 1-form on $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$ of G. In the case when G coincides with the group $\mathrm{U}(n)$ of unitary $(n \times n)$-matrices this form can be written as

$$A = \sum_{\mu=1}^{4} A_\mu(x) dx_\mu$$

where $x = (x_1, x_2, x_3, x_4)$ are coordinates on $\mathbb{R}^4$ and coefficients $A_\mu(x)$ are smooth functions on $\mathbb{R}^4$ with values in the algebra of skew-Hermitian $(n \times n)$-matrices.

A *gauge G-field F* is the curvature of the connection A given by the 2-form on $\mathbb{R}^4$ with values in the Lie algebra $\mathfrak{g}$

$$F = DA = dA + \frac{1}{2}[A, A]$$

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where $D : \Omega^1(\mathbb{R}^4, \mathfrak{g}) \to \Omega^2(\mathbb{R}^4, \mathfrak{g})$ is the exterior covariant derivative generated by the connection A. In the case $G = \mathrm{U}(n)$ this form is written as

$$F = \sum\nolimits_{\mu,\nu=1}^{4} F_{\mu\nu}(x) dx_\mu \wedge dx_\nu$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

with $\partial_\mu := \partial/\partial x_\mu$, $\mu = 1, 2, 3, 4$.

Introduce the *Yang–Mills action* functional given by the formula

$$S(A) = \frac{1}{2} \int_{\mathbb{R}^4} \|F\|^2 d^4x$$

where the norm $\|F\|$ is computed with the help of a given fixed invariant inner product on the Lie algebra $\mathfrak{g}$. In the case $G = \mathrm{U}(n)$ one can take for such a product $\langle X, Y\rangle := -\mathrm{tr}(XY)$. In this case the formula for the action $S(A)$ will be rewritten in the form

$$S(A) = -\frac{1}{2} \int_{\mathbb{R}^4} \mathrm{tr}(*F \wedge F)$$

where $*$ is the Hodge star-operator on $\mathbb{R}^4$.

The functional $S(A)$ is invariant under *gauge transformations* given by the smooth mappings $g : \mathbb{R}^4 \to G$, tending to the unit $e \in G$ at infinity. Under the action of these transformations gauge potentials and fields transform according to the following formulas

$$A \longmapsto A_g := g^{-1}dg + g^{-1}Ag, \quad g : F \longmapsto F_g := g^{-1}Fg$$

where the group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation. In the case $G = \mathrm{U}(1)$ the gauge transform is given by the multiplication by the gauge factor $g(x) = e^{i\theta(x)}$ so that the corresponding gauge potential transformation coincides with the gradient transform $A \mapsto A - id\theta$ while the gauge field F does not change.

A gauge field F is called the *Yang–Mills field* if it is extremal for the action functional $S(A)$ and has finite Yang–Mills action $S(A) < \infty$. The corresponding gauge potential A is called the *Yang–Mills connection.*

The Euler–Lagrange equations for the functional $S(A)$ have the form

$$D^*F = 0$$

where $D^* : \Omega^2(\mathbb{R}^4, \mathfrak{g}) \to \Omega^1(\mathbb{R}^4, \mathfrak{g})$ is the operator formally adjoint to D. In the case $\mathbb{R}^4$ it coincides with $D^* = - * D*$ where $*$ is the Hodge operator so that the Euler–Lagrange equations for $S(A)$ are rewritten in the form

$$D * F = 0.$$

The obtained equation is called the *Yang–Mills equation* and is often supplemented by the *Bianchi identity*

$$DF = 0$$

which is automatically satisfied for gauge fields F.

A gauge field F is called *selfdual* (resp. *anti-selfdual*) if

$$*F = F \quad (\text{resp. } *F = -F).$$

The Bianchi identity implies immediately that solutions of the *duality equations*

$$*F = \pm F$$

satisfy automatically the Yang–Mills equation.

If we write down the form F as the sum

$$F = F_+ + F_-,$$

where $F_\pm = \frac{1}{2}(*F \pm F)$, then the formula for the Yang–Mills action will rewrite as

$$S(A) = \frac{1}{2}\int_{\mathbb{R}^4} \left(\|F_+\|^2 + \|F_-\|^2\right) d^4x.$$

For the gauge fields F with finite Yang–Mills action the quantity

$$k(A) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \left(-\|F_+\|^2 + \|F_-\|^2\right) d^4x$$

turns out to be an integer-valued topological invariant called the *topological charge* of the field F. If we extend, using the Uhlenbeck compactness theorem, the connection A with finite Yang–Mills action to a connection in some associated vector bundle E over the compactification S^4 of $\mathbb{R}^4$ then this invariant will be expressed in terms of Chern classes of this bundle. For example, in the case of $G = \mathrm{SU}(2)$ it coincides with the 2-nd Chern class.

Comparing the above formulas for the action $S(A)$ and topological charge $k(A)$, we arrive at the estimate

$$S(A) \geq 4\pi^2 |k(A)|.$$

From the same formulas we see that the minimum of the action $S(A)$ on the topological class of gauge fields of gauge potentials with finite Yang–Mills action and fixed topological charge $k(A) = k$ is equal to $4\pi^2|k|$ and is attained for $k > 0$ on anti-selfdual fields while for $k < 0$ it is attained on selfdual fields.

An anti-selfdual field with finite Yang–Mills action is called the *instanton* and a selfdual field with finite Yang–Mills action is called the *anti-instanton.*

Instantons and anti-instantons realize the local minima of the action $S(A)$, however this functional has also non-minimal critical points (cf. [12–15]).

One of the main goals of the Yang–Mills theory is the investigation of the structure of the *moduli space* $\mathcal{M}_k$ of Yang–Mills fields with fixed topological charge k given by the quotient

$$\mathcal{M}_k = \frac{\{\text{Yang–Mills fields with fixed topological charge } k\}}{\{\text{gauge transforms}\}}.$$

We are still far from the complete understanding of the structure of this space, however an analogous problem for the instantons, i.e., the description of the moduli space of instantons on $\mathbb{R}^4$, was solved by Atiyah, Drinfeld, Hitchin and Manin with the help of the twistor approach presented in the next Section.

2. Twistor interpretation of instantons and Yang–Mills fields

Denote by S^4 the four-dimensional sphere identified with the one-point compactification of $\mathbb{R}^4$. In the same way as the two-dimensional sphere S^2 is identified with the Riemann sphere $\mathbb{CP}^1$ the four-dimensional sphere S^4 can be identified with the quaternion projective line $\mathbb{HP}^1$, consisting of pairs of quaternions $[q, q']$ (not simultaneously equal to zero), defined up to multiplication (from the right) by a non-zero quaternion. Quaternions $q \in \mathbb{H}$ may be written in the complex form as $q = z_1 + z_2 j$ where z_1, z_2 are complex numbers and j is an imaginary unit, anti-commuting with the usual imaginary unit $i \in \mathbb{C}$.

Define the *basic twistor bundle* over S^4

$$\pi : \mathbb{CP}^3 \xrightarrow{\mathbb{CP}^1} \mathbb{HP}^1$$

by the following tautological formula

$$[z_1, z_2, z_3, z_4] \longmapsto [z_1 + z_2 j, z_3 + z_4 j],$$

where the 4-tuple of complex numbers $[z_1, z_2, z_3, z_4] \in \mathbb{CP}^3$ (not simultaneously equal to zero) is defined up to multiplication by a non-zero complex number and the pair of quaternions $[z_1 + z_2 j, z_3 + z_4 j] \in \mathbb{HP}^1$ is defined up to multiplication (from the right) by a non-zero quaternion. The fibre of π coincides with the complex projective line $\mathbb{CP}^1$ invariant under the multiplication from the right by j, i.e., under the map

$$j : [z_1, z_2, z_3, z_4] \longmapsto [-z_2, z_1, -z_4, z_3].$$

The constructed bundle may be considered as a complex version of the Hopf bundle

$$S^7 \xrightarrow{S^3} S^4$$

generated by the projection $S^7 \subset \mathbb{C}^4 \to \mathbb{CP}^3$.

The restriction of π to the Euclidean space $\mathbb{R}^4 = S^4 \setminus \{\infty\}$ coincides with the bundle

$$\pi : \mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty \longrightarrow \mathbb{R}^4$$

where the omitted complex projective line $\mathbb{CP}^1_\infty$ is identified with the fibre $\pi^{-1}(\infty)$ at $\infty \in S^4$.

This bundle admits the following nice geometric interpretation due to Atiyah (cf. [2]). Namely, the space $\mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty$ is foliated by parallel projective planes $\mathbb{CP}^2$ intersecting in $\mathbb{CP}^3$ on the projective line $\mathbb{CP}^1_\infty$. Consider the fibre $\pi^{-1}(q)$ of our bundle at an arbitrary point $q \in \mathbb{R}^4$. With any point $z \in \pi^{-1}(q)$ of this fibre we can associate a complex structure J_z on the tangent space $T_q\mathbb{R}^4 \cong \mathbb{R}^4$ by identifying (with the help of the tangent map π_*) this space with the complex plane from our family, going through z. Thus the fibre $\pi^{-1}(q)$ of the twistor bundle at q is identified with the space of complex structures on the tangent space $T_q\mathbb{R}^4$ compatible with the metric.

The moduli space $\mathcal{N}_k$ of instantons with the fixed topological charge k, which is defined as

$$\mathcal{N}_k = \frac{\{G\text{-instantons on } \mathbb{R}^4 \text{ with topological charge } k\}}{\{\text{gauge transforms}\}},$$

admits the following interpretation in terms of the twistor bundle

$$\mathbb{CP}^3 \setminus \mathbb{CP}^1_\infty \to \mathbb{R}^4.$$

According to the theorem of Atiyah–Ward [5] there exists a one-to-one correspondence between:

$$\left\{\begin{matrix}\text{moduli space of } G\text{-} \\ \text{instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{equivalence classes of based} \\ \text{holomorphic } G^{\mathbb{C}}\text{-bundles over} \\ \mathbb{CP}^3\text{, trivial on } \pi\text{-fibers}\end{matrix}\right\}.$$

Here, $G^{\mathbb{C}}$ is the complexification of the group G and $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^3$ is called *based* if it is provided with a fixed trivialization on $\mathbb{CP}^1_\infty = \pi^{-1}(\infty)$.

Using this twistor interpretation of instantons, Atiyah, Drinfeld, Hitchin and Manin gave a full description of the moduli space of instantons known now under the name of ADHM-construction (cf. [3]).

In order to extend these results to arbitrary Yang–Mills fields we would like to have the twistor interpretation of these fields. Such interpretation was proposed in the papers by Manin [11], Witten [18] and Isenberg–Green–Yasskin [10]. To formulate their construction denote by $(\mathbb{CP}^3)^*$ the dual projective space identified with the space of complex projective planes $\mathbb{CP}^2$ in $\mathbb{CP}^3$. Consider the space F of flags in $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$, consisting of pairs: (point; plane, containing this point). In homogeneous coordinates $([z];[\xi])$ on $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$ this space is identified with the subspace

$$Q = \{([z];[\xi]) : (z,\xi) = 0\}$$

where $(\cdot,\cdot)$ denotes the natural pairing between $\mathbb{CP}^3$ and $(\mathbb{CP}^3)^*$. The twistor construction, mentioned above, gives a description of holomorphic Yang–Mills fields on the complexified space $\mathbb{C}S^4$ identified with the Grassmann manifold $G_1(\mathbb{CP}^3) \equiv G_2(\mathbb{C}^4)$. Namely, such fields correspond to the equivalence classes of holomorphic $G^{\mathbb{C}}$-bundles over Q, satisfying the following two conditions:

1) they are trivial on all quadrics of the form

$$Q(l) = \{(\text{point } z;\text{ projective plane } p) :\ z \in l \subset p\}$$

where l is a projective line in $\mathbb{CP}^3$;

2) they extend to holomorphic bundles on the 3rd infinitesimal neighborhood $Q^{(3)}$ of the subspace Q in $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$.

3. Atiyah–Donaldson construction and harmonic spheres conjecture

There exists another twistor description of the moduli space of instantons given by Atiyah [1] and Donaldson [7] which may be considered as a two-dimensional reduction of the Atiyah–Ward theorem. According to Atiyah–Donaldson, there exists a one-to-one correspondence between:

$$\left\{\begin{array}{l}\text{moduli space of}\\ G\text{-instantons on}\\ \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{equivalence classes of based holomorphic}\\ G^{\mathbb{C}}\text{-bundles over } \mathbb{CP}^1\times\mathbb{CP}^1,\text{ trivial on}\\ \mathbb{CP}^1_\infty\cup\mathbb{CP}^1_\infty\end{array}\right\}.$$

Here, $G^{\mathbb{C}}$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ is called *based* if it is provided with a fixed trivialization at $(\infty,\infty) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ and we denote by $\mathbb{CP}^1_\infty \cup \mathbb{CP}^1_\infty$ the union of two projective lines at infinity of the form

$$\mathbb{CP}^1_\infty \cup \mathbb{CP}^1_\infty = (\mathbb{CP}^1 \times \infty) \cup (\infty \times \mathbb{CP}^1).$$

Atiyah has proposed an interpretation of the right-hand side of the correspondence established by the Atiyah–Donaldson theorem in terms of the holomorphic spheres in the loop space ΩG.

Recall that the *loop space* of a compact Lie group G is the homogeneous space

$$\Omega G = LG/G$$

where $LG = C^\infty(S^1, G)$ is the *loop group* of G, i.e., the group of C^∞-smooth maps $S^1 \to G$ with the pointwise multiplication and G in the denominator is identified with the subgroup of constant maps $S^1 \to g_0 \in G$. The space ΩG is a Kähler Frechet manifold with the complex structure induced from the representation of ΩG as the homogeneous space of a complex Lie group:

$$\Omega G = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}}$$

where $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$ is the complex loop group of G and $L_+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is the subgroup of the loop group $LG^{\mathbb{C}}$, consisting of the maps extending to holomorphic maps of the unit disc $\Delta \to LG^{\mathbb{C}}$.

The theorem of Atiyah asserts that there exists a one-to-one correspondence between:

$$\left\{\begin{array}{l}\text{equivalence classes of based holomorphic}\\ G^{\mathbb{C}}\text{-bundles over } \mathbb{CP}^1\times\mathbb{CP}^1,\text{ trivial on the}\\ \text{union } \mathbb{CP}^1_\infty\cup\mathbb{CP}^1_\infty\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{based holomoprhic}\\ \text{spheres } f:\mathbb{CP}^1\to\\ \Omega G\end{array}\right\}.$$

Here, a map $\mathbb{CP}^1 \to \Omega G$ is called *based* if it sends the point $\infty \in \mathbb{CP}^1$ into the class $[G] \in \Omega G$.

The two given theorems of Atiyah and Donaldson imply that there exists a one-to-one correspondence between:

$$\left\{\begin{array}{l}\text{moduli space of } G\text{-}\\ \text{instantons on } \mathbb{R}^4\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{based holomorphic}\\ \text{spheres } f:\mathbb{CP}^1\to\Omega G\end{array}\right\}.$$

Define the *energy* of a smooth sphere $f : \mathbb{CP}^1 \to \Omega G$ in the loop space ΩG by the Dirichlet integral of the form

$$E(f) = \frac{1}{2}\int_{\mathbb{C}} \|df(z)\|^2 \frac{|dz \wedge d\bar{z}|}{(1+|z|^2)^2}$$

where the norm $\|df(z)\|$ is computed with respect to the Riemannian metric of the loop space ΩG (recall that this space is Kähler and so has a Riemannian metric compatible with its complex structure), and the integral over the complex plane $\mathbb{C}$ is taken with respect to the conformal metric on $\mathbb{C}$. The critical points of the energy functional are called the *harmonic spheres* in ΩG. The local minima of this functional are given by the holomorphic and anti-holomorphic spheres in ΩG.

So the Atiyah–Donaldson theorem establishes a one-to-one correspondence between the local minima of two functionals, namely:

$$\left\{\begin{matrix}\text{Yang–Mills action on gauge } G\text{-} \\ \text{fields on } \mathbb{R}^4\end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix}\text{energy of smooth} \\ \text{spheres in } \Omega G\end{matrix}\right\}$$

with local minima given respectively by

$$\left\{\begin{matrix}\text{instantons and} \\ \text{anti-instantons}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic and anti-} \\ \text{holomorphic spheres}\end{matrix}\right\}.$$

If we replace the local minima by the arbitrary critical points of the corresponding functionals then we arrive at the *harmonic spheres conjecture* asserting that it should exist a one-to-one correspondence between:

$$\left\{\begin{matrix}\text{moduli space of Yang–} \\ \text{Mills } G\text{-fields on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{based harmonic spheres} \\ f : \mathbb{P}^1 \to \Omega G\end{matrix}\right\}.$$

The described transition from the local minima to the critical points of the functionals may be also considered as a kind of the “realification” procedure. Indeed, if we replace the smooth spheres in the right-hand side of the correspondence by smooth functions $f : \mathbb{C} \to \mathbb{C}$ then the described procedure will reduce to the trivial transition from the holomorphic and anti-holomorphic functions to the arbitrary harmonic functions (being the sums of holomorphic and anti-holomorphic functions). In the case of smooth spheres in the loop space ΩG this transition from holomorphic and anti-holomorphic spheres to harmonic ones becomes non-trivial due to the non-linearity of the Euler–Lagrange equations for the energy functional on smooth spheres.

Apart from the Atiyah–Donaldson theorem and given heuristic considerations there is one more evidence in favor of the harmonic spheres conjecture. Namely, in the paper by Friedrich and Habermann [9] it is proved that there exists a one-to-one correspondence between the moduli space of Yang–Mills fields on the two-dimensional sphere S^2 and harmonic loops in ΩG being the critical points of the energy functional on loops. Recall that the energy of a smooth loop $\gamma \in \Omega G$ is given by the Dirichlet integral of the form

$$E(\gamma) = \frac{1}{2}\int_{S^1} \|\gamma^{-1}(\lambda)\gamma'(\lambda)\|^2 d\lambda$$

where $\|\cdot\|$ is the invariant norm on the Lie algebra $\mathfrak{g}$ of the group G. The critical points of this functional are called the *harmonic loops* in ΩG.

Representing the space $\mathbb{R}^4$ as the product $\mathbb{R}^2 \times \mathbb{R}^2$, we can consider the Friedrich–Habermann's result as a variant of the harmonic spheres conjecture "at a point" establishing a one-to-one correspondence between:

$$\left\{\begin{matrix}\text{moduli space of Yang–}\\ \text{Mills } G\text{-fields on } \mathbb{R}^2\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{harmonic loops}\\ \text{in } \Omega G\end{matrix}\right\}.$$

So it is not surprising that the Friedrich–Habermann construction uses, instead of the basic twistor bundle $\mathbb{CP}^3 \to S^4$, being the complex analogue of the Hopf bundle $S^7 \to S^4$, another Hopf bundle $S^3 \to S^2$.

4. Twistor interpretation

Unfortunately, a direct extension of the Atiyah–Donaldson proof to the harmonic case is impossible because this proof is based on the monad method and is purely holomorphic. One can however try to reduce the proof of the harmonic spheres conjecture to the holomorphic case by "pulling-up" the both parts of the correspondence in the conjecture to their twistor spaces.

The twistor description of the moduli space of Yang–Mills fields in terms of the space of flags F in $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$ was given above. There is also the twistor description of harmonic spheres in the loop spaces which was proposed in our papers (cf., e.g., [17]). It is based on the following considerations.

The loop space ΩG can be isometrically embedded into the infinite-dimensional Hilbert–Schmidt Grassmannian $\mathrm{Gr}(H)$, and the description of harmonic maps into $\mathrm{Gr}(H)$ can be obtained by the generalization of the corresponding construction for the finite-dimensional Grassmannian $\mathrm{Gr}(\mathbb{C}^n)$.

In the finite-dimensional case harmonic spheres in the Grassmannian $\mathrm{Gr}(\mathbb{C}^n)$ coincide with the projections to $\mathrm{Gr}(\mathbb{C}^n)$ of holomorphic spheres in flag bundles $\mathrm{Fl}(\mathbb{C}^n) \to \mathrm{Gr}(\mathbb{C}^n)$, provided with some special almost complex structure. It implies that harmonic spheres in $\mathrm{Gr}(\mathbb{C}^n)$ can be obtained from a trivial one with the help of a Bäcklund-type procedure by successive adding of holomorphic and anti-holomorphic spheres.

In the infinite-dimensional situation the role of flag bundles is played by the so-called *virtual flag bundles* $\mathrm{Fl}(H)$ while the other part of the harmonic spheres construction extends to the Grassmannian $\mathrm{Gr}(H)$ by analogy with the finite-dimensional case.

It follows that the twistor version of the harmonic spheres conjecture should establish a one-to-one correspondence between holomorphic bundles over $\mathbb{CP}^3 \times (\mathbb{CP}^3)^*$, satisfying conditions 1),2) from Section 2, and holomorphic spheres in the virtual flag bundles $\mathrm{Fl}(H)$. Unfortunately, the descriptions of these two objects are given in different terms (the first description uses the dual projective twistor space while the second one not). So in order to prove the twistor version of the harmonic spheres conjecture one should first establish a correspondence between these descriptions.

We note in conclusion that the harmonic spheres conjecture will imply the existence of a Bäcklund-type procedure allowing to construct arbitrary Yang–Mills fields from a trivial one by the successive adding of instantons and anti-instantons. In particular, it would mean that there exist many more non-minimal Yang–Mills fields than instantons and anti-instantons separately.

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On Covariant Poisson Brackets in Field Theory

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Abstract. A general approach is proposed to constructing covariant Poisson brackets in the space of histories of a classical field-theoretical model. The approach is based on the concept of Lagrange anchor, which was originally developed as a tool for path-integral quantization of Lagrangian and non-Lagrangian dynamics. The proposed covariant Poisson brackets generalize the Peierls' bracket construction known in the Lagrangian field theory.

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1. Introduction

The least action principle provides the foundation for classical mechanics and field theory. A distinguishing feature of the Lagrangian equations of motion among other differential equations is that their solution space carries a natural symplectic structure, making it into a phase space. The physical observables, being identified with the smooth function(al)s on the phase space, are then endowed with the structure of a Poisson algebra. There are at least two different ways for describing this Poisson algebra. The first one is the standard Hamiltonian formalism, which requires an explicit splitting of space-time into space and time and introduction of canonical momenta. The main drawback of this approach is the lack of manifest covariance, which causes some complications in applying it to relativistic field theory. An alternative approach was proposed by Peierls in his seminal 1952 paper [1]. In that paper he invented what is now known as the Peierls brackets on the covariant phase space. In contrast to the usual (non-covariant) Hamiltonian formalism, where the phase space is identified with the space of initial data, the covariant phase space is the functional space consisting of all the trajectories obeying the Lagrangian equations of motion. Peierls' paper opened up the way for constructing

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a fully relativistic theory of quantum fields [2]. For more recent discussions of the Peierls brackets, on different levels of rigor, we refer the reader to [3–5].

In this paper, we explain how to extend the concept of covariant phase-space to the most general (i.e., not necessarily Lagrangian) theories. Our approach is based on the notion of Lagrange anchor, which was originally proposed in [6] as a tool for path-integral quantization of Lagrangian and non-Lagrangian theories. In most cases the existence of a Lagrange anchor appears to be less restrictive condition for the classical dynamics than the existence of an action functional. Furthermore, one and the same system of equations may admit a variety of different Lagrange anchors leading to nonequivalent quantizations. In the next sections, we will show that any Lagrange anchor gives rise to a Poisson structure in the space of solutions to the classical equations of motion. The corresponding Poisson brackets are fully covariant and reduce to the Peierls brackets in the case of Lagrangian theories endowed with the canonical Lagrange anchor. It is pertinent to note that for the mechanical systems described by ordinary differential equations in normal form, a relationship between the Lagrange anchors and Poisson brackets has been already established in [7].

Our exposition is mostly focused on the algebraic and geometric aspects of the construction, while more subtle functional analytical details are either ignored or treated in a formal way. These details, however, are not specific to our problem and can be studied, in principle, along the same lines as in the case of conventional Peierls' brackets.

2. Classical gauge systems

2.1. Kinematics

In modern language the classical fields are just the sections of a locally trivial fiber bundle $B \to M$ over the space-time manifold M. The typical fiber F of B is called the target space of fields. In case the bundle is trivial, i.e., $B = M \times F$, the fields are merely the mappings from M to F. In each trivializing coordinate chart $U \subset M$ a field $\varphi : M \to B$ is described by a collection of functions $\varphi^i(x)$, where $x \in U$ and φ^i are local coordinates in F. These functions are often called the components of the field φ.

Formally, one can think of $\Gamma(B)$ – the space of all field configurations – as a smooth manifold $\mathcal{M}$ with the continuum infinity of dimensions and $\varphi^i(x)$ playing the role of local coordinates. In other words, the different local coordinates $\varphi^i(x)$ on $\mathcal{M}$ are labeled by the space-time point $x \in M$ and the discrete index i. To emphasize this interpretation of fields as coordinates on the infinite-dimensional manifold $\mathcal{M}$ we will include the space-time point x into the discrete index i and write φ^i for $\varphi^i(x)$; in so doing, the summation over the "superindex" i implies usual summation for its discrete part and integration over M for x. In the physical literature this convention is known as DeWitt's condensed notation [2].

Proceeding with the infinite-dimensional geometry above, we identify the "smooth functions" on the "manifold" $\mathcal{M}$ with the infinitely differentiable functionals of field φ. These functionals form a commutative algebra, which will be denoted by Φ. If $\delta\varphi^i$ is an infinitesimal variation of field, then, according to the condensed notation, the corresponding variation of a functional $S \in \Phi$ can be written in the form

$$\delta S = S_{,i}\, \delta\varphi^i \,, \tag{1}$$

where the comma denotes the functional derivative.

The concepts of vector fields, differential forms and exterior differentiation on $\mathcal{M}$ are naturally introduced through the functional derivatives, see, e.g., [13]. In particular, the variations $\delta\varphi^i$ span the space of 1-forms and the functional derivatives $\delta/\delta\varphi^i$ define a basis in the tangent space $T_\varphi\mathcal{M}$. So, we can speak of the tangent and cotangent bundles of $\mathcal{M}$.

The tangent and cotangent bundles are not the only vector bundles that can be defined over $\mathcal{M}$. Given a vector bundle $E \to M$ over the space-time manifold, we define the vector bundle $\mathcal{E} \to \mathcal{M}$ whose sections a smooth functionals of fields with values in $\Gamma(E)$. In other words, a section $\xi \in \Gamma(\mathcal{E})$ takes each field configuration $\varphi \in \mathcal{M}$ to a section $\xi[\varphi] \in \Gamma(E)$. Here we do not require the section $\xi[\varphi]$ to be smooth; discontinuous or even distributional sections are also allowed. We will refer to $\mathcal{E}$ as the vector bundle associated with E. The dual vector bundle $\mathcal{E}^*$ is defined to be the vector bundle associated with E^*.

In order to justify our subsequent constructions some restrictions are to be imposed on the structure of the underlying space-time manifold. Our basic assumption will be that M is a globally hyperbolic manifold endowed with a volume form. In the most of field-theoretical models both the structures come from a Lorentzian metric on M. The globally hyperbolic manifolds is a natural arena for the theory of hyperbolic differential equations with well-posed Cauchy problem. By definition, each globally hyperbolic manifold M admits a global time function whose level surfaces provide a foliation of M into space-like Cauchy surfaces N, so that $M \simeq \mathbb{R} \times N$. Using the direct product structure, one can cut M into the "past" and the "future" with respect to a given instant of time $t \in \mathbb{R}$:

$$M_t^- = (-\infty, t] \times N \,, \qquad M_t^+ = [t, \infty) \times N \,, \qquad M = M_t^- \cup M_t^+ \,.$$

Given a vector bundle $E \to M$, we define the following subspaces in the space of sections $\Gamma(E)$:

- $\Gamma_0(E) = \{\xi \in \Gamma(E) \,|\, \mathrm{supp}\ \xi \text{ is compact}\}$;
- $\Gamma_{\mathrm{sc}}(E) = \{\xi \in \Gamma(E) \,|\, \mathrm{supp}\ \xi \text{ is spatially compact}\}$;
- $\Gamma_-(E) = \{\xi \in \Gamma_{\mathrm{sc}}(E) \,|\, \mathrm{supp}\ \xi \subset M_t^- \text{ for some } t\}$;
- $\Gamma_+(E) = \{\xi \in \Gamma_{\mathrm{sc}}(E) \,|\, \mathrm{supp}\ \xi \subset M_t^+ \text{ for some } t\}$.

Here the spatially compact means that the intersection $M_t^- \cap \mathrm{supp}\,\xi \cap M_{t'}^+$ is compact for any $t \geq t'$. We will refer to the elements of $\Gamma_-(E)$ and $\Gamma_+(E)$ as the sections with retarded and advanced support, respectively.

A differentiable functional A is said to be compactly supported if $A_{,i} \in \Gamma_0(T^*\mathcal{M})$. For example, a local functional, like the action functional, is compactly supported if it is given by an integral over a compact domain. The formally smooth and compactly supported functionals form a subalgebra $\Phi_0 \subset \Phi$. Let now $\mathcal{E}$ be a vector bundle associated with E. We say that a section $\xi \in \Gamma(\mathcal{E})$ has retarded, advanced or compact support if $\xi[\varphi] \in \Gamma(E)$ does so for any field configuration $\varphi \in \mathcal{M}$. The sections with the mentioned support properties form subspaces in $\Gamma(\mathcal{E})$, which will be denoted by $\Gamma_-(\mathcal{E})$, $\Gamma_+(\mathcal{E})$, and $\Gamma_0(\mathcal{E})$, respectively.

When dealing with local field theories it is also useful to introduce the subspace of local sections $\Gamma_{\mathrm{loc}}(\mathcal{E}) \subset \Gamma(\mathcal{E})$. This consists of those sections of E whose components are given, in each coordinate chart, by smooth functions of the field φ and its partial derivatives up to some finite order. For instance, the components of the Euler–Lagrange equations $S_{,i} = 0$ constitute a section of $\Gamma_{\mathrm{loc}}(T^*\mathcal{M})$.

2.2. Dynamics

The dynamics of fields are specified by a set of differential equations

$$T_a[\varphi] = 0\,. \tag{2}$$

Here a is to be understood as including a space-time point. According to our definitions, the left-hand sides of the equations can be viewed as components of a local section of some vector bundle $\mathcal{E}$ over $\mathcal{M}$. We call $\mathcal{E}$ the *dynamics bundle*. Since we do not assume the field equations (2) to come from the least action principle, the discrete part of the condensed index a may have nothing to do with that of i labeling the field components. In the special case of Lagrangian systems, the dynamics bundle coincides with the cotangent bundle $T^*\mathcal{M}$ and the field equations are determined by the exact 1-form (1), with S being the action functional.

Let Σ denote the space of all solutions to the field equations (2). Geometrically, we can think of Σ as a smooth submanifold of $\mathcal{M}$ and refer to Σ as the *dynamical shell* or just the *shell*. For the Lagrangian systems the shell is just the set of all stationary points of the action S. By referring to Σ as a smooth submanifold we mean that the standard regularity conditions hold for the field equations [8].

Given the shell, a functional $A \in \Phi_0$ is said to be trivial iff $A|_\Sigma = 0$. Clearly, the trivial functionals form an ideal of the algebra Φ_0. Denoting this ideal by Φ_0^{triv}, we define the quotient-algebra $\Phi_0^\Sigma = \Phi_0/\Phi_0^{\mathrm{triv}}$. The regularity of the field equations imply that for each trivial functional $A \in \Phi_0^{\mathrm{triv}}$ there exists a (distributional) section $\xi \in \Gamma(\mathcal{E}^*)$ such that $A = \xi^a T_a$. In other words, the trivial functionals are precisely those that are proportional to the equations of motion and their differential consequences. By definition, the elements of the algebra Φ_0^Σ are given by the equivalence classes of functionals from Φ_0, where two functionals A and B are considered to be equivalent if $A - B \in \Phi_0^{\mathrm{triv}}$. In that case we will write $A \approx B$. Formally, one can think of Φ_0^Σ as the space of smooth, compactly supported functionals on Σ.

2.3. Gauge symmetries and physical observables

The first functional derivatives of the field equations (2) constitute the Jacobi operator J,

$$J_{ai} = T_{a,i} \,. \tag{3}$$

Geometrically, J defines a homomorphism from the tangent to the dynamics bundle.

The field equations (2) are said to be *gauge invariant* if there exist a vector bundle $\mathcal{F} \to \mathcal{M}$ together with a section $R = \{R^i_\alpha\}$ of $\mathcal{F}^* \otimes T\mathcal{M}$ such that

$$J_{ai} R^i_\alpha \approx 0 \quad \Leftrightarrow \quad R_\alpha T_a = C^b_{\alpha a} T_b \,. \tag{4}$$

In local field theory it is also assumed that $R^i_\alpha[\varphi]$ is the integral kernel of a differential operator $R[\varphi] : \Gamma(\mathcal{F}) \to \Gamma(TM)$ for each $\varphi \in \mathcal{M}$.

For the sake of simplicity we assume the bundle $\mathcal{F}$ to be trivial and consider $R_\alpha = \{R^i_\alpha\}$ as a collection of vector fields on $\mathcal{M}$. This vector fields are called the gauge symmetry generators. The terminology is justified by the fact that for any infinitesimal section $\varepsilon \in \Gamma_0(\mathcal{F})$ the infinitesimal change of field $\varphi^i \to \varphi^i + \delta_\varepsilon \varphi^i$, where

$$\delta_\varepsilon \varphi^i = R^i_\alpha \varepsilon^\alpha \,,$$

maps solutions of (2) to solutions. In other words, the vector fields R_α are tangent to the dynamical shell Σ. The gauge symmetry transformations are said to be trivial if $R \approx 0$. If the vector bundle $\mathcal{F}$ is big enough to accommodate all nontrivial gauge symmetries, then we call $\mathcal{F}$ the *gauge algebra bundle* and refer to R_α as a complete set of gauge symmetry generators. It follows from the definition (4) that the vector fields R_α define an on-shell involutive vector distribution on $\mathcal{M}$, i.e.,

$$[R_\alpha, R_\beta] \approx C^\gamma_{\alpha\beta} R_\gamma \,,$$

for some C's. This distribution will be denoted by $\mathcal{R}$ and called *gauge distribution.*

A functional $A \in \Phi_0$ is gauge invariant if

$$A_{,i} \, R^i_\alpha \approx 0 \,.$$

In that case we say that A represents a *physical observable.* The gauge invariant functionals form a subalgebra $\Phi_0^{\mathrm{inv}} \subset \Phi_0$. Two gauge invariant functionals A and A' are considered as equivalent or represent the same physical observable if $A \approx A'$. So, we identify the physical observables with the equivalence classes of gauge invariant functionals from Φ_0. This definition is consistent as the trivial functionals are automatically gauge invariant and the property of being gauge invariant passes through the quotient $\Phi_0^{\mathrm{inv}}/\Phi_0^{\mathrm{triv}}$. In what follows we will identify physical observables with their particular representatives in Φ_0^{inv}.

3. The Lagrange anchor

According to our definitions each classical field theory is completely specified by a pair $(\mathcal{E}, T)$, where $\mathcal{E} \to \mathcal{M}$ is a vector bundle over the configuration space of fields and T is a particular section of $\Gamma_{\mathrm{loc}}(\mathcal{E})$. The solution space Σ is then identified with the zero locus of the section T. Whereas the classical equations of motion $T_a[\varphi] = 0$ are enough to formulate the classical dynamics they are certainly insufficient for constructing a quantum-mechanical description of fields. Any quantization procedure has to involve one or another additional structure. Within the path-integral quantization, for instance, it is the action functional that plays the role of such an extra structure. The procedure of canonical quantization relies on the Hamiltonian form of dynamics, involving a non-degenerate Poisson bracket and a Hamiltonian. Either approach assumes the existence of a variational formulation for the classical equations of motion (the least action principle) and becomes inapplicable beyond the scope of variational dynamics. The extension of these quantization methods to non-variational dynamics was proposed in [6, 9]. In particular, the least action principle of the Lagrangian formalism was shown to admit a far-reaching generalization based on the concept of a *Lagrange anchor*.

Like many fundamental concepts, the notion of a Lagrange anchor can be introduced and motivated from various perspectives. Some of these motivations and interpretations can be found in Refs. [6, 12, 13]. For our present purposes it is convenient to define the Lagrange anchor V as a linear operator making the on-shell commutative diagram

$$\begin{tikzcd} \Gamma(T\mathcal{M}) \arrow[r, "J"] & \Gamma(\mathcal{E}) \\ \Gamma(\mathcal{E}^*) \arrow[u, "V"] \arrow[r, "J^*"] & \Gamma(T^*\mathcal{M}). \arrow[u, "V^*"'] \end{tikzcd} \tag{5}$$

Here J^* and V^* denote the formal dual of the operators J and V. The on-shell commutativity of the diagram means that

$$J \circ V \approx V^* \circ J^* \,. \tag{6}$$

Due to the regularity condition, the off-shell form of the last equality reads

$$J_{ai}V^i_b - V^i_a J_{bi} = C^d_{ab} T_d \tag{7}$$

for some C's.

In the case of Lagrangian theories $\mathcal{E} = T^*\mathcal{M}$ and we can take $V = 1$. Then (7) reduces to the commutativity of the second functional derivatives, $J_{ij} = S_{,ij} = J_{ji}$. The operator $V = 1$ is referred to as the *canonical Lagrange anchor* for Lagrangian equations of motion. It should be noted that even for Lagrangian equations $S_{,i} = 0$ there may exist non-canonical Lagrange anchors.

As with the generators of gauge symmetries, we can think of the Lagrange anchor as a collection of vector fields $V_a = \{V^i_a\}$ on $\mathcal{M}$. These generate a (singular) vector distribution $\mathcal{V}$, which we call the *anchor distribution*. From the physical standpoint, $\mathcal{V}$ defines the possible directions of quantum fluctuations on $\mathcal{M}$. For the Lagrangian theories endowed with the canonical Lagrange anchor $V = 1$ all directions are allowable and equivalent. At the other extreme we have zero Lagrange anchor, $V = 0$, for which the corresponding quantum system remains pure classical (no quantum fluctuations). In the intermediate situation only a part of physical degrees of freedom may fluctuate and/or the intensity of fluctuations around a particular field configuration $\varphi \in \mathcal{M}$ may vary with φ.

Unlike the gauge distribution $\mathcal{R}$, the anchor distribution $\mathcal{V}$ is not generally involutive even on shell. Nonetheless, using the regularity condition, one can derive the following commutation relations [10]:

$$\begin{aligned} [V_a, V_b]^i &\approx C_{ab}^d V_d^i + D_{ab}^\alpha R_\alpha^i + J_{aj} W_b^{ji} - J_{bj} W_a^{ji}\,, \\ [R_\alpha, V_a]^i &\approx C_{\alpha a}^b V_b^i + D_{\alpha a}^\beta R_\beta^i + J_{aj} W_\alpha^{ji}\,, \end{aligned} \tag{8}$$

where the coefficients W's are symmetric in ij and the C's are defined by Rels. (4) and (7). By definition, the coefficients C_{ab}^d and $C_{\alpha a}^b$ are given by the integral kernels of trilinear differential operators, while the coefficients D's and W's may well be non local. Locality of the latter coefficients will be our additional assumption. It is automatically satisfied for the so-called *integrable Lagrange anchors* as they were defined in [11]. We will not dwell here on the concept of integrability of the Lagrange anchors referring the interested reader to the cited paper.

4. Covariant Poisson brackets

The cornerstone of our construction is an *advanced/retarded fluctuation* $V_A^\pm$ caused by a physical observable A. By definition, $V_A^\pm$ is a vector field from $\Gamma_\pm(T\mathcal{M})$ satisfying the condition

$$V_A^\pm T_a \approx V_a A\,. \tag{9}$$

It is not hard to see that the last equation defines $V_A^\pm$ uniquely up to adding a vector field from $\mathcal{R}$ and on-shell vanishing terms [10].

Now we define the advanced/retarded brackets of two physical observables by the relation

$$\{A, B\}^\pm = V_A^\pm B - V_B^\pm A\,, \qquad \forall A, B \in \Phi_0^{\mathrm{inv}}\,. \tag{10}$$

These brackets are well defined on shell as the ambiguity related to the choice of the fluctuations,

$$V_A^\pm \quad \rightarrow \quad V_A^\pm + \xi^\alpha R_\alpha + T_a X^a\,, \qquad \xi \in \Gamma_\pm(\mathcal{F})\,, \quad X^a \in \Gamma_\pm(T\mathcal{M})\,, \tag{11}$$

results in on-shell vanishing terms. Using Rels. (8) one can prove the following main statement.

Proposition 1. *Brackets* (10) *map physical observables to physical observables and satisfy all the properties of Poisson brackets: antisymmetry, bi-linearity, the Leibnitz rule and Jacobi identity.*

Proof. See [10]. □

In [10], it was also shown that the advanced and retarded fluctuations are connected to each other by the following *reciprocity relation*:

$$V_A^- B \approx V_B^+ A\,. \tag{12}$$

It just says that the retarded effect of A on B is equal to the advanced effect of B on A, and vice versa. As an immediate consequence of (12) we obtain the on-shell equality

$$\{A,B\}^{\pm} \approx \pm\tilde{V}_A B\,, \tag{13}$$

where the difference $\tilde{V}_A = V_A^+ - V_A^-$ is called the *causal fluctuation.*

Let us now compare the covariant Poisson brackets (10) with the usual Peierls' brackets in Lagrangian field theory. In the latter case the dynamics of fields are governed by some action functional $S[\varphi]$. As was explained in Sec. 3, the corresponding equations of motion $S_{,i}[\varphi] = 0$ admit the canonical Lagrange anchor given by the unit operator $V = 1$ on $\Gamma(T^*\mathcal{M})$. The definition of the advanced/retarded fluctuation (9) takes the form

$$V_A^{\pm i} S_{,ij} \approx A_{,j}\ . \tag{14}$$

In the absence of gauge symmetries this equation can be solved for $V_A^{\pm}$ with the help of the advanced/retarded Green function $G^{\pm ij}$. By definition,

$$G^{\pm in} S_{,nj} = S_{,jn}\, G^{\pm ni} = \delta^i_j \quad \text{and} \quad G^{-ij} = 0 = G^{+ji} \quad \text{if} \quad j > i\,. \tag{15}$$

Here $j > i$ means that the time associated with the index i lies to the past of the time associated with the index j. Besides (15), the advanced and retarded Green functions satisfy the so-called reciprocity relation $G^{\pm ij} = G^{\mp ji}$. In terms of the Green functions the advanced/retared solution to (14) reads

$$V_A^{\pm i} = G^{\pm ij} A_{,j}\ . \tag{16}$$

and the causal fluctuation takes the form $\tilde{V}_A^i = V_A^{+i} - V_A^{-i} = \tilde{G}^{ij} A_{,j}$, where the difference $\tilde{G} = G^+ - G^-$ is known as the causal Green function. In view of the reciprocity relation, $\tilde{G}^{ij} = -\tilde{G}^{ji}$. Substituting (16) into (13), we get

$$\{A,B\}^{\pm} = \pm A_{,i}\, \tilde{G}^{ij} B_{,j}\ . \tag{17}$$

The antisymmetry of the brackets as well as the derivation property are obvious. The direct verification of the Jacobi identity for (17) can be found in [2].

5. An example: the Pais–Uhlenbeck oscillator

The PU oscillator is described by the fourth-order differential equation

$$\left(\frac{d^2}{dt^2} + \omega_1^2\right)\left(\frac{d^2}{dt^2} + \omega_2^2\right) x = 0\,, \tag{18}$$

where the constants ω_1 and ω_2 have the meaning of frequencies. The advanced/retarded Green function $G^{\pm}(t_2 - t_1)$ for (18) is given by

$$G^{\pm}(t) = \pm\frac{\theta(\mp t)}{\omega_2^2 - \omega_1^2}\left(\frac{\sin\omega_1 t}{\omega_1} - \frac{\sin\omega_2 t}{\omega_2}\right),$$

with $\theta(t)$ being the Heaviside step function.

Equation (18) admits the two-parameter family of the Lagrange anchors [14]

$$V=\alpha+\beta\frac{d^2}{dt^2}\,,\qquad \alpha,\beta\in\mathbb{R}\,. \tag{19}$$

In this particular case the defining condition for the Lagrange anchor (7) reduces to the commutativity of the operator V with the fourth-order differential operator defining the equation of motion (18). Notice that the equation of motion (18) is Lagrangian and the canonical Lagrange anchor corresponds to $\alpha=1$, $\beta=0$.

The advanced Poisson brackets are given by

$$\begin{aligned}\{x(t_1),x(t_2)\}^+ &= V\tilde{G}(t_1-t_2)\\ &= \left(\frac{\alpha-\beta\omega_1^2}{\omega_2^2-\omega_1^2}\right)\frac{\sin\omega_1(t_1-t_2)}{\omega_1}-\left(\frac{\alpha-\beta\omega_2^2}{\omega_2^2-\omega_1^2}\right)\frac{\sin\omega_2(t_1-t_2)}{\omega_2}\,.\end{aligned}$$

Differentiating with respect to t_1, t_2 and setting $t_1=t_2$, we obtain the following Poisson brackets of the phase-space variables $z=(x,\dot{x},\ddot{x},\dddot{x})$:

$$\begin{gathered}\{\dot{x},x\}^+=\beta\,,\qquad \{\dot{x},\ddot{x}\}^+=\{\dddot{x},x\}^+=\alpha-\beta(\omega_1^2+\omega_2^2)\,,\\ \{\ddot{x},\dddot{x}\}^+=\alpha(\omega_1^2+\omega_2^2)-\beta(\omega_1^4+\omega_1^2\omega_2^2+\omega_2^4)\,,\end{gathered} \tag{20}$$

and the other brackets vanish. For $\alpha=1$, $\beta=0$, this yields the standard Poisson brackets on the phase space of the PU oscillator.

With the Poisson brackets (20) the equations of motion (18) can be written in the Hamiltonian form

$$\dot{z}^i=\{H,z^i\}^+\,,\qquad i=1,2,3,4\,,$$

where the Hamiltonian is given by

$$H=\frac{1}{2}\frac{(\dddot{x}+\omega_1^2\dot{x})^2+\omega_2^2(\ddot{x}+\omega_1^2x)^2}{(\omega_1^2-\omega_2^2)(\alpha-\beta\omega_2^2)}-\frac{1}{2}\frac{(\dddot{x}+\omega_2^2\dot{x})^2+\omega_1^2(\ddot{x}+\omega_2^2x)^2}{(\omega_1^2-\omega_2^2)(\alpha-\beta\omega_1^2)}\,.$$

As was first noticed in [14], this Hamiltonian is positive definite whenever

$$\omega_1^2>\frac{\alpha}{\beta}>\omega_2^2\,. \tag{21}$$

Clearly, the canonical Lagrange anchor $(\alpha=1,\beta=0)$ does not satisfy these inequalities for any frequencies $\omega_{1,2}$. On the other hand, in the absence of resonance ($\omega_1\neq\omega_2$), one can always choose the non-canonical Lagrange anchor (19) to meet inequalities (21). Upon quantization the positive-definite Hamiltonian will have a positive energy spectrum and a well-defined ground state. The last property is crucial for the quantum stability of the system [14]. So, we see that non-canonical Lagrange anchors may offer certain advantages over the canonical one, when the issuers of quantum stability of higher-derivative systems are concerned.

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Part III: Groups and Algebras

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Weyl Group Orbit Functions in Image Processing

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Abstract. Generalized Fourier transform and related convolution based on Weyl group orbit functions transform can be used in spatial image filtering. In this paper we use the family of E-orbit functions of A_2 to provide an example of such a filtering.

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Keywords. Fourier type transform, image processing, Weyl groups, orbit functions.

1. Introduction

In [1] we described one of possible applications of Weyl group orbit functions and their discretization in image processing. In standard digital data processing a convolution of the image and a spatial filter is used to change a brightness value in each point. Fourier transform changes the convolution into a simple multiplication of Fourier images. We generalized this method using the orbit transform [2, 3]. In this paper we present an example of this technique applied to the family of E-functions [4] of the Weyl group of A_2.

The paper is organized as follows. In Section 2 we provide a brief description of spatial image filtering using the classical Fourier analysis and convolution. Sections 3 and 4 summarize some necessary terms from the theory of Weyl group orbit functions and their discretization. In Section 5 we present our technique of using E-orbit convolution in image processing. The paper is concluded with an example of image filtering and some remarks.

2. Spatial image filtering

The most simple technique of spatial filtering lies in the use of changing the intensity level of the image in each point. The neighboring points (8 or 24) are used to change the central point. Mathematically, this is reached by convolution of the image and 3×3 or 5×5 matrix, called the convolution kernel. Different values of the entries of the kernel provide different types of filters, e.g., blurring, sharpening and edge-detecting.

For defining the convolution in the case of the orbit convolution we proceed in a similar way as for the discrete Fourier transform. For two-dimensional functions f and g we have

$$\begin{aligned}(f * g)(x,y) &= \sum_{k=-\lfloor \frac{M}{2} \rfloor}^{\lfloor \frac{M-1}{2} \rfloor} \sum_{l=-\lfloor \frac{N}{2} \rfloor}^{\lfloor \frac{N-1}{2} \rfloor} f(x-k, y-l) g(k,l) \\ &= \frac{1}{M} \frac{1}{N} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} (Ff)(k,l)(Fg)(k,l) e^{2\pi i \left(\frac{kx}{M} + \frac{ly}{N}\right)},\end{aligned}$$

where $M, N \in \mathbb{Z}$, and F is the Fourier transform.

The standard convolution theorem then claims that the Fourier image of a convolution is a multiplication of Fourier images of the functions,

$$F(f * g)(x,y) = (Ff)(x,y)(Fg)(x,y).$$

3. Even Weyl groups of simple Lie algebras of rank 2

Simple Lie algebras of rank two A_2, C_2 and G_2 are described by simple roots $\Delta = \{\alpha_1, \alpha_2\} \in \mathbb{R}^2$. Coroots are defined as $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$; weights ω_i are dual to coroots, $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. We define the coroot lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee$ and the weight lattice $P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

Reflections r_1 and r_2 with respect to the hyperplanes orthogonal to the simple roots α_1 and α_2 generate the corresponding Weyl group W. The action of W on simple roots gives a root system $W(\Delta)$ in $\mathbb{R}^2$. The affine Weyl group W^{aff} is defined as $W^{\text{aff}} = Q^\vee \rtimes W$ and its fundamental domain is denoted by F.

The even Weyl group W^e is defined as $W^e = \{w \in W \mid \det(w) = 1\}$. Corresponding even affine Weyl group is denoted W_e^{aff} and its fundamental domain is given by $F^e = F \cup r_i(\operatorname{int} F)$, where r_i is a simple reflection and $\operatorname{int} F$ denotes the interior of F. Figure 1 shows root systems and fundamental domains of A_2, C_2 and G_2.

4. *E*-orbit functions and *E*-orbit transform

For the rest of the paper we fix the Weyl group W to be the Weyl group of A_2. Several families of Weyl group orbit functions can be defined in the context of the group W. E-orbit functions are defined for every $x \in \mathbb{R}^2$ and $\lambda \in P$ as

$$\Xi_\lambda(x) = \sum_{w \in W^e} e^{2\pi i \langle w(\lambda), x \rangle}.$$

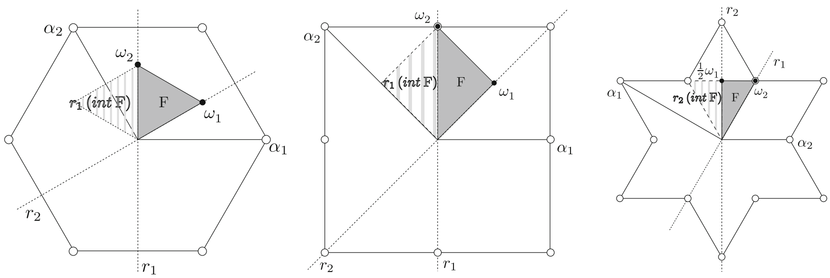

FIGURE 1. Root systems A_2, C_2 and G_2. The grey triangle denotes the fundamental domain of the corresponding affine Weyl group.

The E-orbit functions are described in detail in [4]. They are invariant with respect to the even affine Weyl group, therefore, we restrict them on the domain F^e.

The method of discretization of E-orbit functions is described in detail in [3], here we provide a brief summary for the case of A_2. We fix an integer M and define a finite grid of points F^e_M. We consider a scalar product of functions f, g sampled on the grid,

$$\langle f,\, g\rangle_{F^e_M} = \sum_{x\in F^e_M} \varepsilon^e(x) f(x)\overline{g(x)}$$

where the weight function $\varepsilon^e(x)$ is given by the order of the even Weyl orbit of x, $\varepsilon^e(x) = \frac{|W^e|}{|\,\mathrm{stab}_{W^e}(x)|}$. Finite family of orbit functions which are pairwise orthogonal with respect to this scalar product is given by a grid of parameters Λ^e_M labeling the functions.

For every $\lambda, \lambda' \in \Lambda^e_M$ it holds that

$$\langle \Xi_\lambda,\, \Xi_{\lambda'}\rangle_{F^e_M} = c|W^e|M^2 h^{e\vee}_\lambda \delta_{\lambda\lambda'},$$

where the coefficient $h^{e\vee}_\lambda$ is the order of the stabilizer of λ, c is the determinant of the corresponding Cartan matrix of W and $|W^e|$ is the order of the even Weyl group.

In the case of the Weyl group of A_2 the two grids are of the following form:

$$\begin{aligned} F^e_M &= \left\{\frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \,|\, s_0, s_1, s_2 \in \mathbb{Z}^{\geq 0}, s_0+s_1+s_2 = M\right\} \\ &\quad \cup \left\{\frac{-s_1}{M}\omega_1^\vee + \frac{s_1+s_2}{M}\omega_2^\vee \,|\, s_0, s_1, s_2 \in \mathbb{Z}^{>0}, s_0+s_1+s_2 = M\right\}, \\ \Lambda^e_M &= \left\{t_1\omega_1 + t_2\omega_2 \,|\, t_0, t_1, t_2 \in \mathbb{Z}^{\geq 0}, t_0+t_1+t_2 = M\right\} \\ &\quad \cup \left\{-t_1\omega_1 + (t_1+t_2)\omega_2 \,|\, t_0, t_1, t_2 \in \mathbb{Z}^{>0}, t_0+t_1+t_2 = M\right\}. \end{aligned} \tag{1}$$

The orders of the two grids are the same and equal to $\frac{(M+1)(M+2)}{2}$. The values of $\varepsilon^e(x)$ and $h^{e\vee}_\lambda$ are listed in Table 1.

$x \in F_M^e$	$\varepsilon^e(x)$	$\lambda \in \Lambda_M^e$	$h_\lambda^{e\vee}$
$[s_0, s_1, s_2]$	3	$[t_0, t_1, t_2]$	1
$[s_0, s_1, 0]$	3	$[t_0, t_1, 0]$	1
$[s_0, 0, s_2]$	3	$[t_0, 0, t_2]$	1
$[0, s_1, s_2]$	3	$[0, t_1, t_2]$	1
$[0, 0, s_2]$	1	$[0, 0, t_2]$	3
$[0, s_1, 0]$	1	$[0, t_1, 0]$	3
$[s_0, 0, 0]$	1	$[t_0, 0, 0]$	3

Table 1. The coefficients $\varepsilon^e(x)$ and $h_\lambda^{e\vee}$ of A_2. The variables s_i, t_i, $i = 0, 1, 2$, are non negative integers and have the same meaning as in (1).

The discrete orthogonality allows us to perform a Fourier like transform, called E-orbit transform. We consider a function f sampled on the points of F_M^e. We can interpolate it by a sum of E-functions

$$I_M(x) = \sum_{\lambda \in \Lambda_M^e} F_\lambda \Xi_\lambda(x),$$

where we require $f(x) = I_M(x)$ for every $x^e \in F_M$. Therefore, the coefficients F_λ equal to

$$F_\lambda = \frac{\langle f, \, \Xi_\lambda \rangle_M}{\langle \Xi_\lambda, \, \Phi_\lambda \rangle_M} = \frac{1}{c|W^e|M^2 h_\lambda^{e\vee}} \sum_{x \in F_M^e} \varepsilon^e(x) f(x) \overline{\Xi_\lambda(x)}. \tag{2}$$

5. *E*-orbit convolution and image filtering

The E-orbit convolution is defined for discrete functions f, g sampled on F_M^e and $u \in F_M^e$ as

$$(f * g)(u) := \sum_{x \in F_M^e} \varepsilon^e(x) \sum_{w \in W^e} f(x) g(u - w(x)).$$

Then, the orbit convolution theorem holds:

$$(f * g)(u) = \sum_{\lambda \in \Lambda_M^e} c|W^e|M^2 h_\lambda^{e\vee} F_\lambda G_\lambda \Xi_\lambda(u), \tag{3}$$

where F_λ and G_λ are the E-orbit transforms of f and g given by (2).

Spatial filtering using the orbit convolution is proceed in the following way: a digital image is sampled on a grid F_M^e for a suitable M. A filter is a discrete function sampled on much smaller grid, different values of the filters have different result on the image-edge detecting, blurring, edge enhancing, etc. Brightness values in the new image are computed using the orbit convolution, applying the convolution theorem (3).

6. Example

We choose the family E-functions of Weyl group of A_2. We define two filters, an edge detecting filter and a blurring filter:

$$g_{\text{edge}} = \frac{1}{3}\begin{pmatrix} & 0 & \\ 0 & & -1 \\ & 3 & \end{pmatrix}, \quad g_{\text{blur}} = \frac{1}{48}\begin{pmatrix} & & 1 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ & & 1 & & \end{pmatrix}.$$

We apply the filters to the image of Lena. The results are shown on Figures 2 and 3.

FIGURE 2. The original image on the left and the result of the edge detection on the right.

FIGURE 3. The image with added noise on the left and the result of the bluring filter on the right.

7. Conclusion

In this paper we used the family of E-orbit functions to provide processing of a digital image using theory described in [1]. Compared to the method using the family of C-functions [5], this one is more suitable to use in image processing, mainly because of two reasons. The shape of fundamental domain corresponds better to square images and the procedure is faster thanks to smaller number of summands in the expression of orbit functions. However, the method is still slow compared to the classical ones. In our future we are interested in modifying the fast Fourier transform for orbit transforms to improve the speed of our procedure.

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Poisson and Fourier Transforms for Tensor Products and an Overalgebra

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Abstract. For the group $G = \mathrm{SL}(2,\mathbb{R})$, we write explicitly operators intertwining irreducible finite-dimensional representations T_k with tensor products $T_l \otimes T_m$ (we call them Poisson and Fourier transforms), they are differential operators, and also we write explicit formulae for compositions of these transforms with Lie operators of the overgroup $G \times G$

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In this paper we present some results on tensor products of irreducible finite-dimensional representations of the group $G = \mathrm{SL}(2,\mathbb{R})$. It is well known that the tensor product $T_l \otimes T_m$ decomposes into the direct multiplicity free sum:

$$T_l \otimes T_m = T_{|l-m|} + T_{|l-m|+1} + \cdots + T_{l+m-1} + T_{l+m}\,.$$

We write explicitly operators intertwining $T_l \otimes T_m$ and direct summands T_k, we call them Poisson and Fourier transforms. They turn out to be *differential operators*. For the Poisson transform we give three variants. The Fourier transform coincides up to a factor with Rankin–Cohen brackets [1, 7].

Further, we write explicitly formulae for compositions of Poisson and Fourier transforms with Lie operators of the overgroup $\widetilde{G} = G \times G$. It continues works [2–6] on infinite-dimensional representations on homogeneous spaces. Now we deal with *finite-dimensional* representations. Our approach does not need Plancherel formulae.

In this paper we essentially use that the tensor product $T_l \otimes T_m$ is equivalent to a representation of the group G in functions on the hyperboloid G/H of one sheet in $\mathbb{R}^3$ induced by a character of the diagonal subgroup H.

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We use the following notation for "generalized powers":

$$a^{[s]} = a\,(a+1)\cdots(a+s-1), \quad a^{(s)} = a\,(a-1)\cdots(a-s+1),$$

here a is a number or an operator, $s \in \mathbb{N} = \{0,1,2,\ldots\}$.

1. The group $\mathrm{SL}(2,\mathbb{R})$ and its representations

The group $G = \mathrm{SL}(2,\mathbb{R})$ consists of real matrices of the second order with unit determinant:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Any irreducible finite-dimensional representation T_k of the group G is labeled by the number k (the *highest weight*) such that $k \in (1/2)\mathbb{N}$. It acts on the space V_k of polynomials $\varphi(x)$ in x of degree $\leqslant 2k$ (so that $\dim V_k = 2k+1$) by

$$\left(T_k(g)\,\varphi\right)(x) = \varphi(x\cdot g\,)\,(\beta x+\delta)^{2k}, \quad x\cdot g = \frac{\alpha x+\gamma}{\beta x+\delta}\,, \tag{1}$$

we consider that G acts from the right.

The Lie algebra $\mathfrak{g}$ of the group G consists of real matrices of the second order with zero trace. A basis in $\mathfrak{g}$ consists of matrices:

$$L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

The commutation relations are:

$$[L_+, L_-] = -2L_1\,, \quad [L_+, L_1] = -L_+\,, \quad [L_1, L_-] = -L_-\,. \tag{3}$$

Representations of $\mathfrak{g}$ generated by representations of G we denote by the same symbols.

Change in (1) $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$, we obtain the *contragredient* representation $\widehat{T}_k$ of the group G:

$$\left(\widehat{T}_k(g)\varphi\right)(x) = \varphi(x\cdot\widehat{g}\,)\,(\gamma x+\alpha)^{2k}\,, \quad \widehat{g} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}.$$

The representation $\widehat{T}_k$ is equivalent to the representation T_k by means of the map

$$\varphi \mapsto \widehat{\varphi}, \quad \widehat{\varphi}(x) = x^{2k}\varphi\left(\frac{1}{x}\right). \tag{4}$$

For basis elements (2) we have:

$$T_k(L_-) = \frac{d}{dx}\,, \quad T_k(L_1) = x\,\frac{d}{dx} - k\,, \quad T_k(L_+) = x^2\,\frac{d}{dx} - 2kx\,. \tag{5}$$

and $\widehat{T}_k(L_\pm) = -T_k(L_\mp)$, $\widehat{T}_k(L_1) = -T_k(L_1)$.

2. Tensor products

Let us call the group $\widetilde{G} = G \times G$ by the *overgroup*. It contains three subgroups isomorphic to G: the diagonal G^d consisting of pairs (g, g), and two component subgroups G_1 and G_2 consisting of pairs (g, E) and (E, g) respectively, here E is the identity matrix, $g \in G$.

The tensor product $V_l \otimes V_m$ consists of polynomials $f(x, y)$ of degree $\leqslant 2l$ in x and degree $\leqslant 2m$ in y. The representation $T_{lm} = T_l \times T_m$ of $\widetilde{G}$ acts on $V_l \otimes V_m$ by

$$(T_{lm}(g_1, g_2)f)\,(x, y) = f(x \cdot g_1\,,\, y \cdot g_2\,) \cdot (\beta_1 x + \delta_1)^{2l} \cdot (\beta_2 y + \delta_2)^{2m}.$$

The restriction of T_{lm} to G^d is the tensor product $T_l \otimes T_m$ of representations T_l and T_m of G:

$$((T_l \otimes T_m)(g)f)\,(x, y) = f(x \cdot g\,,\, y \cdot g\,) \cdot (\beta x + \delta)^{2l} \cdot (\beta y + \delta)^{2m}.$$

Denote

$$r = m - l. \tag{6}$$

It is well known that $T_l \otimes T_m$ decomposes into the direct multiplicity free sum

$$T_l \otimes T_m = \sum_k T_k\,, \tag{7}$$

where k ranges over the set

$$|r|,\ |r| + 1,\ \ldots, l + m - 1,\ l + m\,. \tag{8}$$

Respectively, $V_l \otimes V_m$ decomposes into the direct sum

$$V_l \otimes V_m = \sum_k W_k\,. \tag{9}$$

Subspaces W_k are invariant and irreducible with respect to $T_l \otimes T_m$. To simplify writing, we do not show dependence on l, m – both on the right side of (9) and later on. The restriction of $T_l \otimes T_m$ to W_k is equivalent to T_k.

3. Poisson and Fourier transforms

Let k belong to the set (8). Denote

$$j = l + m - k, \tag{10}$$

so that $2l - j = k - r$, $2m - j = k + r$. We consider operators $M_k : V_k \to V_l \otimes V_m$ and $F_k : V_l \otimes V_m \to V_k$ intertwining representations $T_l \otimes T_m$ and T_k, i.e.,

$$M_k\, T_k(g) = (T_l \otimes T_m)\,(g) M_k\,,$$
$$T_k(g) F_k = F_k\,(T_l \otimes T_m)\,(g)\,,$$

where $g \in G$. We call operators M_k and F_k the Poisson and Fourier transforms, respectively.

Since decomposition (7) is multiplicity free, the image of M_k is the subspace W_k, and the transform F_k vanishes on all W_s except W_k. Therefore, transforms

M_k and F_k are defined uniquely up to a factor. The composition $F_k M_k$ is a scalar operator on V_k (multiplication by a number). First we construct M_k and then we normalize F_k such that $F_k M_k = \mathrm{id}$, i.e., F_k on W_k is the inverse to M_k:

$$F_k = M_k^{-1} \quad \text{on} \quad W_k. \tag{11}$$

A polynomial $f \in V_l \otimes V_m$ is reconstructed from its Fourier components $F_k f$ as follows:

$$f = \sum M_k \varphi_k \,, \quad \varphi_k = F_k f \,. \tag{12}$$

Theorem 1. *The Poisson transform $M_k : V_k \to V_l \otimes V_m$ can be written in one of the following three forms. Let $\varphi \in V_k$. Then*

$$(M_k \, \varphi)\,(x, y) = \sum_{s=0}^{k+r} \binom{k+r}{s} (k - r + 1)^{[s]} \,(y - x)^{2m-s} \left(\frac{d}{dx}\right)^{k+r-s} \varphi(x)\,, \tag{13}$$

$$= (y - x)^j \left\{ (y - x)\frac{d}{dx} + k - r + 1 \right\}^{[k+r]} \varphi(x)\,, \tag{14}$$

$$= (y - x)^{l+m+k+1} \left(\frac{d}{dx}\right)^{k+r} (y - x)^{-k+r-1}\, \varphi(x)\,, \tag{15}$$

where r and j are given by (6) *and* (10).

Proof. First we prove (13). Now instead of the representation $T_l \otimes T_m$ it is convenient to take the equivalent representation $T_l \otimes \widehat{T}_m$. Instead of variables x, y we take

$$\xi = x, \quad \eta = \frac{1}{y}\,. \tag{16}$$

The representation $T_l \otimes \widehat{T}_m$ acts on $V_l \otimes V_m$ by

$$\Big((T_l \otimes \widehat{T}_m)(g) f\Big)\,(\xi, \eta) = f(\xi \cdot g\,,\, \eta \cdot \widehat{g}\,) \cdot (\beta\, \xi + \delta)^{2l} \cdot (\gamma\, \eta + \alpha)^{2m}.$$

Let us "seat" $T_l \otimes \widehat{T}_m$ and $V_l \otimes V_m$ on the hyperboloid of one sheet $\mathcal{X}$ in $\mathbb{R}^3$ defined by equation $-x_1^2 + x_2^2 + x_3^2 = 1$. Realize $\mathcal{X}$ as the set of matrices

$$x = \frac{1}{2} \begin{pmatrix} 1 - x_3 & x_2 - x_1 \\ x_2 + x_1 & 1 + x_3 \end{pmatrix} \tag{17}$$

with determinant equal to 0. The group G acts on it: $x \mapsto g^{-1} x g$, transitively. The stabilizer of the point $x^0 = (0, 0, 1)$ is the subgroup H of diagonal matrices

$$h = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

Let us introduce on $\mathcal{X}$ horospherical coordinates ξ, η:

$$x = \frac{1}{N} \begin{pmatrix} -\eta\xi & -\eta \\ \xi & 1 \end{pmatrix}, \quad N = 1 - \xi\eta. \tag{18}$$

The action of G in these coordinates splits: if a point x has coordinates ξ, η, then the point $g^{-1}xg$ has coordinates $\widetilde{\xi} = \xi \cdot g$, $\widehat{\eta} = \eta \cdot \widehat{g}$. The initial point x^0 has coordinates $\xi = 0$, $\eta = 0$. The element

$$g_x = \begin{pmatrix} 1/N & \eta/N \\ \xi & 1 \end{pmatrix} \tag{19}$$

moves x^0 to the point x with coordinates ξ, η. The map

$$D : f \mapsto F = N^{-2m} f, \quad f \in V_l \otimes V_m, \tag{20}$$

transfers $V_l \otimes V_m$ to a space A_{lm} of some rational functions F of ξ, η. In virtue of (17) and (18) these rational functions are *polynomials* in x_1, x_2, x_3 on $\mathcal{X}$. For example, the polynomial $f = 1$ in $V_l \otimes V_m$ goes to the polynomial $((x_3+1)/2)^{2m}$. Denote $B_k = D(W_k)$.

The map D intertwines $T_l \otimes \widehat{T}_m$ and the representation U_r (recall that $r = m-l$) of G induced by the character $h \mapsto \lambda^{-2r}$ of the subgroup H. In horospherical coordinates the representation U_r is:

$$(U_r(g)F)(\xi,\eta) = F\left(\widetilde{\xi}, \widehat{\eta}\right)(\beta\xi + \delta)^{-2r}.$$

Let Z be the space of distributions on $\mathbb{R}$ concentrated at the point $t = 0$. It consists of linear combinations of the Dirac delta function $\delta(t)$ and its derivatives $\delta^{(p)}(t)$. Formulae (5) define a representation of the Lie algebra $\mathfrak{g}$ on Z, denote it by T_k again. Moreover, such a representation is defined not only for $k \in (1/2)\mathbb{N}$ but for arbitrary $k \in \mathbb{R}$. We need $k \in (1/2)\mathbb{Z}$. In particular, the representation T_{-k-1}, $k \in (1/2)\mathbb{N}$, acts on $\delta^{(p)}(t)$ as follows:

$$\begin{aligned} T_{-k-1}(L_1)\,\delta^{(p)}(t) &= (k-p)\,\delta^{(p)}(t)\,, \\ T_{-k-1}(L_+)\,\delta^{(p)}(t) &= p\,(2k+1-p)\,\delta^{(p-1)}(t)\,, \\ T_{-k-1}(L_-)\,\delta^{(p)}(t) &= \delta^{(p+1)}(t)\;. \end{aligned}$$

It has an invariant subspace spanned by $\delta^{(p)}(t)$, $p \geqslant 2k+1$. The corresponding factor space will be denoted by Z_{-k-1}, it is spanned by $\delta^{(p)}(t)$, $p \leqslant 2k$. The factor representation T_{-k-1} on Z_{-k-1} is equivalent to T_k, hence it arises to the group G.

Let us consider a bilinear form on the product $Z \times V_k$:

$$\langle \zeta\,,\, \varphi \rangle = \int_{-\infty}^{\infty} \zeta(t)\,\varphi(t)\,dt\,. \tag{21}$$

It is invariant with respect to the pair (T_{-k-1}, T_k), i.e.,

$$\langle T_{-k-1}(g^{-1})\,\zeta\,,\,\varphi\rangle = \langle \zeta\,,\, T_k(g)\,\varphi\rangle\,, \quad g \in G. \tag{22}$$

The distribution $\delta^{(k+r)}(t)$ (notice that $k+r \in \mathbb{N}$) is an eigenfunction for the subgroup H with the eigenvalue λ^{2r}. Therefore, the map $\widehat{M}_k : V_k \to A_{lm}$ defined by

$$\left(\widehat{M}_k\varphi\right)(x) = (-1)^{k+r}\,\langle T_{-k-1}(g_x^{-1})\,\delta^{(k+r)}\,,\,\varphi\rangle,$$

where g_x is given by (19), intertwines T_k with U_r. Therefore, the image of this map is just B_k. Because of (22), (21), (19) and (1) we get:

$$\left(\widehat{M}_k\varphi\right)(x) = \left(\frac{d}{dt}\right)^{k+r}\bigg|_{t=0} \varphi\left(\frac{\dfrac{1}{N}t+\xi}{\dfrac{\eta}{N}t+1}\right)\left(\frac{\eta}{N}t+1\right)^{2k},$$

where x is a point of $\mathcal{X}$ with coordinates ξ, η. A computation (for example, by induction on $k+r$) gives

$$\left(\widehat{M}_k\varphi\right)(x) = \sum_{s=0}^{k+r}\binom{k+r}{s}(k-r+1)^{[s]}\left(\frac{\eta}{N}\right)^s \varphi^{(k+r-s)}(\xi). \tag{23}$$

Now we go back from the hyperboloid to the tensor product $T_l \otimes T_m$ and variables x, y, see (16), so that $N = 1 - x/y$. In virtue of (20) and (4) we multiply the right-hand side of (23) by $(Ny)^{2m}$. It gives (13).

After that, formulae (14) and (15) can be checked immediately, for example, by induction. □

For $f(x,y)$ we use notation

$$f^{(a,b)} = \frac{\partial^{a+b} f}{\partial x^a\, \partial y^b}\,.$$

Theorem 2. *The Fourier transform $F_k : V_l \otimes V_m \to V_k$ (recall that (11) holds) is given by the following formula. Let $f(x,y)$ be a polynomial in $V_l \otimes V_m$. Then*

$$(F_k f)(t) = c_k \sum_{\alpha=0}^{j}(-1)^{j-\alpha}\binom{2l-j+\alpha}{\alpha}\binom{2m-\alpha}{j-\alpha} f^{(j-\alpha,\,\alpha)}(t,t), \tag{24}$$

where j is given by (10),

$$c_k^{-1} = (2k)^{(k+r)}(2k+2)^{[j]} = \frac{1}{2k+1}(l+m+k+1)^{(2m+1)}.$$

Proof. Let us take decomposition (12). Differentiate f successively with respect to y and put $x = y = t$, then from (13) we obtain that $f^{(0,j)}(t,t)$, $j = 0, 1, 2, \ldots$, is a linear combination of functions $\varphi_k(t)$, $\varphi'_{k+1}(t), \ldots, \varphi^{(j)}_{l+m}(t)$, where $k = l+m-j$. We obtain a triangular system of equations for functions $\varphi_k(t)$ and their derivatives. It shows that $\varphi_k(t)$ is a linear combination of functions

$$f^{(0,j)}(t,t),\ \frac{d}{dt}f^{(0,j-1)}(t,t), \ldots, \left(\frac{d}{dt}\right)^{j-1} f^{(0,1)}(t,t),\ \left(\frac{d}{dt}\right)^{j} f(t,t)$$

and, therefore, $\varphi_k(t)$ is a linear combination of functions $f^{(j-\alpha,\,\alpha)}(t,t)$ where $\alpha = 0, 1, \ldots, j$. Thus, formula (24) means that a function $\varphi(t)$ in V_k is a linear combination of $f^{(j-\alpha,\,\alpha)}(t,t)$, where $f = M_k\varphi$. We have to show that coefficients of this linear combination are just ones from (24). We see from (13) that only one term, namely the term with $s = k + r$, gives a non-zero contribution, the latter is equal to

$$(k - r + 1)^{[k+r]} \, (-1)^{j-\alpha} \, j! \, \varphi(t),$$

notice that the first factor here is $(2k)^{[k+r]}$. Substitute it on the right-hand side of (24). Then this right-hand side becomes

$$\begin{aligned} & c_k \, (2k)^{[k+r]} \sum_{\alpha=0}^{j} j! \binom{2l - j + \alpha}{\alpha} \binom{2m - \alpha}{j - \alpha} \cdot \varphi(t) \\ & \quad = \frac{1}{(2k+2)^{[j]}} \sum_{\alpha=0}^{j} \binom{j}{\alpha} (2l - j + 1)^{[\alpha]} \, (2m - j + 1)^{[j-\alpha]} \cdot \varphi(t) = \varphi(t). \end{aligned}$$

Here we apply the binomial formula for generalized powers. □

4. Interaction of Poisson and Fourier transforms with the overalgebra

The Lie algebra of the overgroup $\widetilde{G}$ ("overalgebra") is the direct sum $\widetilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{g}$. The Lie algebras $\mathfrak{g}^d$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ of the subgroups G^d, G_1, G_2 consist of pairs (X, X), $(X, 0)$, $(0, X)$ respectively, here $X \in \mathfrak{g}$.

We find compositions of Lie operators $T_{lm}(X, Y)$ and transforms M_k and F_k. For pairs $(X, X) \in \mathfrak{g}^d$ the answer is simple, since M_k and F_k are intertwining operators. Therefore, it is sufficient to take pairs (X, Y) in a complementary subspace for $\mathfrak{g}^d$. For such a subspace we take the subalgebra $\mathfrak{g}_2$.

Let us introduce the following operators $S_k(X)$, $k \in \mathbb{N}$, and $E(X)$ on the real line $\mathbb{R}$. They depend on $X \in \mathfrak{g}$ linearly, and on basis (2) they are given by

$$\begin{aligned} E(L_-) &= 1\,, \quad & S_k(L_-) &= \frac{d^2}{dt^2}\,, \\ E(L_1) &= t\,, \quad & S_k(L_1) &= t\frac{d^2}{dt^2} - (2k+1)\frac{d}{dt}\,, \\ E(L_+) &= t^2\,, \quad & S_k(L_+) &= t^2\frac{d^2}{dt^2} - 2(2k+1)t\frac{d}{dt} + (2k+1)(2k+2). \end{aligned}$$

Lemma 3. *The following commutation relations hold:*

$$\begin{aligned} S_k([X, Y]) &= T_k(X)\, S_k(Y) - S_k(Y)\, T_{k+1}(X)\,, \\ E([X, Y]) &= T_k(X)\, E(Y) - E(Y)\, T_{k-1}(X)\,. \end{aligned}$$

Lemma 4. *The operators $S_k(X)$ map V_{k+1} to V_k.*

Indeed, $S_k(X)$ act on monomials as follows:

$$\begin{aligned} S_k(L_-)t^s &= s(s-1) \cdot t^{s-2}\,, \\ S_k(L_1)t^s &= s[s - (2k+2)] \cdot t^{s-1}\,, \\ S_k(L_+)t^s &= [s - (2k+1)][s - (2k+2)] \cdot t^s\,. \end{aligned}$$

Further, introduce the following coefficients α_k, β_k, γ_k (as before, we do not show dependence on l, m):

$$\alpha_k = \frac{(2m-k-r)(k-r+1)}{2(k+1)(2k+1)}\,, \tag{25}$$

$$\beta_k = \frac{1}{2} + \frac{r(2m-r+1)}{2k(k+1)}\,, \tag{26}$$

$$\gamma_k = -\frac{(2m+k-r+1)(k+r)}{2k(2k+1)}\,. \tag{27}$$

Theorem 5. *The operator $T_{lm}(0,X)$, $X \in \mathfrak{g}$, interacts with the Poisson transform M_k as follows:*

$$T_{lm}(0,X)\, M_k = \alpha_k \cdot\, M_{k+1}\, E(X) + \beta_k \cdot M_k\, T_k(X) + \gamma_k \cdot M_{k-1}\, S_{k-1}(X)\,. \tag{28}$$

The last term is correct, see Lemma 4*.*

Proof. First we take $X = L_-$. Then (28) is

$$\frac{\partial}{\partial y}\, M_k\, \varphi = \alpha_k \cdot M_{k+1}\, \varphi + \beta_k \cdot M_k\, \varphi' + \gamma_k \cdot M_{k-1}\, \varphi''\,, \tag{29}$$

the prime means differentiation. We use (14), the commutation relation:

$$\left\{(y-x)\frac{d}{dx} + a\right\}^{[s]} \frac{d}{dx} = \frac{d}{dx}\left\{(y-x)\frac{d}{dx} + a + 1\right\}^{[s]}$$

and the equality

$$\frac{\partial}{\partial y}\, M_k\, \varphi + \frac{\partial}{\partial x}\, M_k \varphi = M_k\, \varphi'.$$

which means that M_k intertwines $T_k(L_-)$ and $T_{lm}(L_-, L_-)$. It allows to check (29) with coefficients (25), (26), (27).

For $X = L_1$ and $X = L_+$ we use equality (28) with $X = L_-$ already proved and the commutation relations (3) – successively the first and the second ones, and corresponding relations for operators $S_k(X)$ and $E(X)$ from Lemma 3. $\square$

Theorem 6. *The operator $T_{lm}(0,X)$, $X \in \mathfrak{g}$, interacts with the Fourier transform F_k as follows:*

$$F_k\, T_{lm}(0,X) = \gamma_{k+1} \cdot S_k(X)\, F_{k+1} + \beta_k \cdot T_k(X)\, F_k + \alpha_{k-1} \cdot E(X)\, F_{k-1}\,. \tag{30}$$

Proof. Let us apply $T_{lm}(0,X)$ to decomposition (12). By (28) we obtain

$$T_{lm}(0,X)\, f = \sum_k \big\{\alpha_k\, M_{k+1}\, E(X)\, \varphi_k + \beta_k\, M_k\, T_k(X)\, \varphi_k + \gamma_k\, M_{k-1}\, S_{k-1}(X)\, \varphi_k\big\}.$$

Now we apply to it the Fourier transform F_c and use (12), it gives

$$F_c\, T_{lm}(0,X)\, f = \alpha_{c-1}\, E(X)\, \varphi_{c-1} + \beta_c\, T_c(X)\, \varphi_c + \gamma_{c+1}\, S_c(X)\, \varphi_{c+1}\,,$$

which is just (30). $\square$

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Unipotent Flow on $\mathbf{SL_2}(\mathbb{Z}) \setminus \mathbf{SL_2}(\mathbb{R})$: From Dynamics to Elementary Number Theory

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Abstract. In this paper we give a simple and elementary proof of a quantitative density result for unipotent flow on $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$.

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1. M. Ratner's Orbit Closure Theorem and dynamics on $\mathbf{SL}_n(\mathbb{Z}) \setminus \mathbf{SL}_n(\mathbb{R})$

One of the most important results in the modern Ergodic Theory is M. Ratner's Orbit Closure Theorem. It has many application in the theory of Dynamical System itself, as well as in various problems related to Number Theory. Ratner's theorem states that given any Lie group G, any discrete subgroup $\Gamma \subset G$ with fundamental domain $\Gamma \setminus G$ of finite measure and any "flow" φ generated by unipotent elements, the closure of every φ-orbit is homogeneous. In particular the closure of φ-orbit is dense "modulo Γ" in a certain closed subgroup $S \subset G$. For thr precise definitions and formulations and the history we refer to a wonderful book [8] and the bibliography therein, as well as to the original paper [10] by M. Ratner.

In the simplest case $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ Ratner's theorem gives a corollary that if the ratio β/γ is irrational then the orbit

$$\underline{\Lambda} \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

of the lattice

$$\underline{\Lambda} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathbb{Z}^2 \tag{1}$$

is dense in the space $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$ of two-dimensional lattices.

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The main results of the present paper (Theorems 1,2 below) give a quantitative version of this density statement by means of an elementary approach related to continued fractions and asymptotics for the number of solutions of the congruence $xy \equiv 1 \pmod{p}$, modulo prime number p where x and y belong to certain intervals of consecutive integers. Our continued fractions consideration is connected to a paper by M. Laurent and A. Nogueira [3].

Certain results similar to our Theorem 1 below are due to A. Strömbergsson [12] and F. Maucourant and B. Weiss [7]. From their theorems a result of the same form as our Theorem 1 follows immediately, but in the right-hand side of (6) there will be $O(t_\nu^{-\delta})$ with an effective positive δ which is not calculated explicitly. We do not compare our exponent to those from [7, 12]. The approach from [7, 12] relies on the methods of Dynamical Systems.

It worth mentioning that in the case $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ Ratner's theorem provides a simple and clear proof for the famous theorem by Margulis [4–6] (see also [2]) which establish the density of values of indefinite non-degenerate quadratic Q forms in ≥ 3 variables, provided that Q is not a scalar multiple of a form with integer coefficients. Unfortunatly we are not able to extend our elementary approach to this situation.

2. Two-dimensional lattices and 2 × 2 matrices

Suppose that $\rho > 0$. For reals A_1, B_1 we define

$$A_2 = -\rho A_1, B_2 = -\rho B_1. \tag{2}$$

Let $r \in (0,1)$ be a positive constant such that for any $A_1, B_1 \asymp Q$ the equation

$$xy - zw = 1 \tag{3}$$

has a solution $x, y, z, w \in \mathbb{Z}$ with

$$A_1 - Q^r \leq x \leq A_1 + Q^r,\ B_1 - Q^r \leq z \leq B_1 + Q^r,$$
$$A_2 - Q^r \leq y \leq A_2 + Q^r,\ B_2 - Q^r \leq w \leq B_2 + Q^r. \tag{4}$$

Of course r does not depend on ρ. (We can take as r any number greater than 3/4, see Section 5).

Theorem 1. *Suppose that we have two unimodular matrices*

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{5}$$

Suppose that $\delta \neq 0$ and $\beta/\delta \notin \mathbb{Q}$. Then there exists a sequence of reals $t_\nu \to \infty$, $\nu \to \infty$ and a sequence of unimodular integer matrices

$$\begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} l_1^1(\nu) & l_1^2(\nu) \\ l_2^1(\nu) & l_2^2(\nu) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

such that

$$\begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = O(|t_\nu|^{\frac{r-1}{r+1}}), \tag{6}$$

where the constant in $O(\cdot)$ *may depend on the size of matrices* (5).

Corollary. *Consider* $\underline{\Lambda}$ *defined in* (1) *and suppose that* $\beta/\delta \notin \mathbb{Q}$. *Let* $\Gamma \in SL_2(\mathbb{R})$ *be another unimodular matrix. Then there exists a sequence of reals* $t_\nu \to \infty$, $\nu \to \infty$ *such that for the lattice* $\underline{\Lambda}_{t_\nu} = \underline{\Lambda} \cdot \begin{pmatrix} 1 & 0 \\ t_\nu & 1 \end{pmatrix}$ *from the orbit of the lattice* $\underline{\Lambda}$ *in the horocycle flow on* $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$ *one has*

$$\mathrm{dist}(\underline{\Lambda}_{t_\nu}, \Gamma) = O(|t_\nu|^{\frac{r-1}{r+1}})$$

(*here we consider the natural distance in the space of lattices* (*see* [1])).

Our next result deals with the situation when the ratio β/δ is not very well approximable by rationals. In this situation we establish a "uniform" bound.

Theorem 2. *Consider a function* $\psi(t)$ *decreasing to zero as* $t \to +\infty$ *and such that* $\psi(t) = O(t^{-1})$. *Define* $\rho(t)$ *to be the function inverse to the function* $t \mapsto 1/\psi(t)$. *Suppose that under the conditions of Theorem* 1 *one has*

$$\min_{p \in \mathbb{Z}} |(\beta/\delta) \cdot q - p| \geq \psi(q), \quad \forall q = 1, 2, 3, \ldots. \tag{7}$$

Then there exists a positive constant C *such that for any* $T \geq 1$ *there exists a solution*

$$\begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad t \in \mathbb{R}$$

of the system of inequalities

$$\left\| \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \right\| \leq CT^{\frac{r}{1+r}} / \rho\left(T^{\frac{1}{1+r}}\right), \quad 1 \leq |t| \leq T. \tag{8}$$

Here $||\cdot||$ *stands for the maximum of absolute values of elements of a matrix.*

We should note here that if the series

$$\sum_{q=1}^{\infty} \psi(q)$$

converges, then for almost all real numbers ω one has

$$\min_{p \in \mathbb{Z}} |\xi q - p| \geq c(\omega)\psi(q), \;\; \forall q \in \mathbb{Z}_+,$$

with a certain positive $c(\omega)$ depending on ω.

Example. If $\psi(t) = t^{-\omega}$, $\omega \geq 1$ then $\rho(t) = t^{1/\omega}$ and the right-hand side of (8) is equal to

$$O\left(T^{\frac{r}{1+r} - \frac{1}{\omega(1+r)}}\right).$$

This bound is non-trivial provided $\omega < r^{-1}$. Moreover in the case when β/δ is a badly approximable number (that is, $\psi(t) = \kappa t^{-1}$ with a positive κ) the right-hand side of (8) is equal to $T^{\frac{r-1}{r+1}}$; in this case we see that the result of Theorem 1 holds uniformly.

Note that when r is close to 3/4 then the exponent $\frac{r-1}{r+1}$ is close to $-1/7$. We give all the proofs in Sections 3–5.

3. Lemmata

We consider Euclidean plane $\mathbb{R}^2$ with coordinates (u, v). Suppose that $|\alpha\delta - \beta\gamma| = 1$ and put

$$M = 4\max(|\alpha|, |\beta|, |\gamma|, |\delta|, |\delta|^{-1})$$

Consider the lattice

$$\Lambda = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \mathbb{Z}^2.$$

The following lemma is a simple result from continued fractions' theory.

Lemma 1. *Suppose that $\beta/\delta \notin \mathbb{Q}$. Let p_ν/q_ν and $p_{\nu+1}/q_{\nu+1}$ be two consecutive convergent fractions to β/δ. Then*

(i) $\begin{vmatrix} q_\nu & q_{\nu+1} \\ p_\nu & p_{\nu+1} \end{vmatrix} = (-1)^\nu$;

(ii) *vectors*

$$\mathbf{e}_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} q_\nu - \begin{pmatrix} \gamma \\ \delta \end{pmatrix} p_\nu, \quad \mathbf{e}_2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} q_{\nu+1} - \begin{pmatrix} \gamma \\ \delta \end{pmatrix} p_{\nu+1}$$

form a basis of Λ;

(iii) *the set*

$$\Pi_\nu = \{(u, v) \in \mathbb{R}^2 : \ |u| \le M q_{\nu+1}, \ |v| \le M q_{\nu+1}^{-1}\} \subset \mathbb{R}^2$$

contains a fundamental domain with respect to Λ, and hence any its shift $\mathbf{e} + \Pi_\nu$, $\mathbf{e} \in \mathbb{R}^2$ contains at least one point of Λ.

Corollary 1. *For any η and $R \in \mathbb{Z}_+$ the set*

$$\Pi_\nu(\eta; R) = \{(u, v) \in \mathbb{R}^2 : \ |u| \le 2MRq_{\nu+1}, \ |v - \eta| \le 2MRq_{\nu+1}^{-1}\}$$

contains $(2R+1)^2$ points of the lattice Λ of the form

$$\mathbf{e}_0 + l_1\mathbf{e}_1 + l_2\mathbf{e}_2, \quad l_j \in \mathbb{Z}, \ |l_j| \le R, \tag{9}$$

with some $\mathbf{e}_0 \in \Lambda$.

Lemma 2. *There exist $\mathbf{f}_1, \mathbf{f}_2 \in \Lambda$ such that*

(i) *$\mathbf{f}_1, \mathbf{f}_2$ form a basis of Λ;*

(ii) *$\mathbf{f}_j \in \Pi_\nu(\eta_j, q_{\nu+1}^r)$, $j = 1, 2$.*

Proof. Consider lattice points of the form (9) from Corollary 1 for parameters η_1 and η_2. Let these lattice points be

$$\mathbf{e}_0^1 + l_1^1 \mathbf{e}_1 + l_2^1 \mathbf{e}_2 \tag{10}$$

for parameter η_1 and

$$\mathbf{e}_0^2 + l_2^1 \mathbf{e}_1 + l_2^2 \mathbf{e}_2 \tag{11}$$

for parameter η_2. We may suppose that $\eta_1 \neq 0$. Put $\rho = |\eta_2/\eta_1|$, $Q = q_{\nu+1}, R = Q^r = q_{\nu+1}^r$ and

$$\begin{pmatrix} A_1, & A_2 \\ B_1 & B_2 \end{pmatrix} = \begin{pmatrix} q_\nu & q_{\nu+1} \\ -p_\nu & -p_{\nu+1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \eta_1 & \eta_2 \end{pmatrix}.$$

Then (2) is valid and $A_1 \asymp Q$.

Under the multiplication by the matrix $\begin{pmatrix} q_\nu & q_{\nu+1} \\ -p_\nu & -p_{\nu+1} \end{pmatrix}^{-1}$ lattice points (10) and (11) turn into integer points $\begin{pmatrix} x \\ z \end{pmatrix}$ and $\begin{pmatrix} y \\ w \end{pmatrix}$ respectively, satisfying (4). By the definition of r there exist x, y, z, w satisfying (3). So there exist l_i^j, $i, j = 1, 2$ such that two points (10) and (11) form a basis of Λ. □

Lemma 3. *Suppose that*

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix} = \begin{vmatrix} \Xi_1 & \Xi_2 \\ H_1 & H_2 \end{vmatrix} = 1. \tag{12}$$

Suppose that

$$\max_{j=1,2} |H_j - \eta_j| \leq \varepsilon \tag{13}$$

and $H_1, H_2 \neq 0$. Then

$$\left| \frac{\Xi_1 - \xi_1}{H_1} - \frac{\Xi_2 - \xi_2}{H_2} \right| \leq \frac{\varepsilon \cdot (|\xi_1| + |\xi_2|)}{|H_1 H_2|}. \tag{14}$$

Proof. As

$$\xi_1 \eta_2 - \Xi_1 H_2 - (\xi_2 \eta_1 - \Xi_2 H_1) = 0,$$

from (12) we see that

$$|(\xi_1 - \Xi_1) H_2 - (\xi_2 - \Xi_2) H_1| \leq \varepsilon \cdot (|\xi_1| + |\xi_2|).$$

So (14) follows. □

4. Proofs of theorems

Proof of Theorem 1. We take large ν. Let vectors $\mathbf{f}_j$ from Lemma 2 be of the form $\mathbf{f}_j = \left(\begin{array}{c} \Xi_j \\ H_j \end{array} \right)$, $j = 1, 2$. Then $H_1, H_2 \neq 0$ and $\max_{j=1,2} |H_j - \eta_j| \leq 2Mq_{\nu+1}^{r-1}$ and $\max_{j=1,2} |\Xi_j| \leq 2Mq_{\nu+1}^{1+r}$. Put $t_\nu = \frac{\xi_1 - \Xi_1}{H_1}$. Then by (14) we have

$$\left| t_\nu - \frac{\xi_2 - \Xi_2}{H_2} \right| = O(q_{\nu+1}^{r-1}).$$

Of course we have $|t_\nu| = O(q_{\nu+1}^{1+r})$. Now

$$\begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Xi_1 & \Xi_2 \\ H_1 & H_2 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & t_\nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Xi_1 & \Xi_2 \\ H_1 & H_2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = O(q_{\nu+1}^{r-1}).$$

So Theorem 1 follows. □

Proof of Theorem 2. Given real $U \geq 1$ put $U_* = \rho(U)$. By Minkowski's convex body theorem and condition (7) one can take the primitive point $(q, p) \in \mathbb{Z}^2$ such that

$$U_* \leq q \leq U. \quad |(\beta/\delta) \cdot q - p| \leq U^{-1}. \tag{15}$$

Then this point may be completed to a basis of $\mathbb{Z}^2$ by a point $(q', p') \in \mathbb{Z}^2$ such that

$$U_* \leq q' \leq 2U. \quad |(\beta/\delta) \cdot q - p| \leq U_*^{-1}. \tag{16}$$

From (15),16 we see that the rectangle

$$\{(u, v) \in \mathbb{R}^2 : \ |u| \leq MU, \ |v| \leq MU_*^{-1}\} \subset \mathbb{R}^2$$

contains a fundamental domain for Λ. Now we follow the argument of the proof of Theorem 1. We see that t may be taken to be $\ll U^{1+r}$, and we establish the bound

$$\left\| \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} - \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \right\| \ll U^r U_*^{-1}.$$

By putting $U = T^{\frac{1}{1+r}}$ we have

$$U^r U_*^{-1} = T^{\frac{r}{1+r}} / \rho \left(T^{\frac{1}{1+r}} \right),$$

and Theorem 2 follows. □

5. About admissible value of r

Here we show that any value $r > 3/4$ is good for our purpose. First of all we may suppose that $w = p$ is a prime number (here we use a well-known fact that between Q and $Q + Q^{3/4}$ for large Q there exists a prime number, see [9]).

Then we apply the well-known fact that for any two intervals I_1, I_2 of lengths $\gg p^{3/4+\varepsilon}$ there exist $x \in I_1, y \in I_2$ such that $xy \equiv 1 \pmod{p}$ (see [11, 13]). This result easily follows from A. Weil's bounds for Kloosterman sums. Now $z = \frac{xy-1}{p}$ will be an integer. Easy calculation shows that x, y, z, w will satisfy (4).

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Lie Superalgebras of Krichever–Novikov Type

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Abstract. Classically, starting from the Witt and Virasoro algebra important examples of Lie superalgebras were constructed. In this write-up of a talk presented at the Białowieża meetings we report on results on Lie superalgebras of Krichever–Novikov type. These algebras are multi-point and higher genus equivalents of the classical algebras. The grading in the classical case is replaced by an almost-grading. It is induced by a splitting of the set of points, were poles are allowed, into two disjoint subsets. With respect to a fixed splitting, or equivalently with respect to a fixed almost-grading, it is shown that there is up to rescaling and equivalence a unique non-trivial central extension of the Lie superalgebra of Krichever–Novikov type. It is given explicitly.

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1. Introduction

In the context of conformal field theory (CFT) the Witt algebra and its universal central extension, the Virasoro algebra, play an important role. These algebras encode conformal symmetry. To incorporate superconformal symmetry one is forced to extend the algebras to Lie superalgebras. Examples of them are the Neveu–Schwarz and the Ramond type superalgebras.

These algebras we call the classical algebras. They correspond to the genus zero situation. Krichever–Novikov algebras are higher genus and multi-point analogs of them. For higher genus, but still only for two points where poles are allowed,

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some of the algebras were generalised in 1986 by Krichever and Novikov [8–10]. In 1990 the author [13–16] extended the approach further to the general multi-point case. These extensions were not straight-forward generalizations. The crucial point was to introduce a replacement of the graded algebra structure present in the "classical" case. Krichever and Novikov found that an almost-grading, see Definition 2, will be enough to allow constructions in representation theory, like triangular decompositions, highest weight modules, Verma modules and so on. In [15, 16] it was realized that a splitting of the set A of points where poles are allowed into two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading. For every such splitting the corresponding almost-grading was given. Essentially different splittings (not just corresponding to interchanging of I and O) will yield essentially different almost-gradings. For the general theory (including the classical case) see the recent monograph [20].

In the context of conformal field theory and string theory the Neveu-Schwarz and the Ramond type superalgebras appear as superextensions of the classical algebras. Some physicists also studied superanalogs of the algebra of Krichever–Novikov type, but still only with two points where poles are allowed, e.g., [1–4, 21]. The multi-point case was developed by the author some time ago and recently published in [19]. See also [20].

Starting from Krichever–Novikov type superalgebras interesting explicit infinite-dimensional examples of Jordan superalgebras and antialgebras can be constructed. In this respect, see the work of Leidwanger and Morier-Genoud [11, 12], and Kreusch [7].

In this write-up we will recall the construction of the Krichever–Novikov (KN) type algebras for the multi-point situation and for arbitrary genus. The classical situation will be a special case. In particular, the construction of the Lie superalgebra is recalled. Its almost-graded structure, induced by a fixed splitting $A = I \cup O$, is given.

The main result presented here is the fact that up to rescaling the central element and equivalence of extension, there is only one non-trivial almost-graded central extension of the Lie superalgebra of KN type with even central element. We stress the fact, that this does not mean that there is essentially only one central extension. For an essentially different splitting we get an essentially different central extension. For higher genus there are even central extensions not related to any splitting. In the classical situation in this way uniqueness of the non-trivial central extension is again obtained. Recall that "classical" means genus zero and two points where poles are allowed.

We will give a geometric description for the defining cocycle, see (43). For the two-point case the form of the cocycle was given by Bryant in [4], correcting some omission in [1].

In the case of odd central elements we obtained that the corresponding central extension of the Lie superalgebra will split. For the proofs we have to refer to the original article [19], respectively to [20].

2. The classical Lie superalgebra

For the convenience of the reader we start with recalling the definition of a Lie superalgebra. Let $\mathcal{S}$ be a vector space which is decomposed into even and odd elements $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, i.e., $\mathcal{S}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. Furthermore, let $[.,.]$ be a $\mathbb{Z}/2\mathbb{Z}$-graded bilinear map $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ such that for elements x, y of pure parity

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]. \tag{1}$$

This says that

$$[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{0}}] \subseteq \mathcal{S}_{\bar{0}}, \qquad [\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{1}}, \qquad [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{0}}, \tag{2}$$

and $[x, y]$ is symmetric for x and y odd, otherwise anti-symmetric. Furthermore, $\mathcal{S}$ is a Lie superalgebra if in addition the super-Jacobi identity (x, y, z of pure parity)

$$(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0 \tag{3}$$

is valid. As long as the type of the arguments is different from (*even, odd, odd*) all signs can be put to $+1$ and we obtain the form of the usual Jacobi identity. In the remaining case we get

$$[x, [y, z]] + [y, [z, x]] - [z, [x, y]] = 0. \tag{4}$$

By the very definitions $\mathcal{S}_0$ is a Lie algebra.

In purely algebraic terms the Virasoro algebra $\mathcal{V}$ is given by generators $\{e_n (n \in \mathbb{Z}), t\}$ with relations

$$[e_n, e_m] = (m - n)e_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m} \cdot t, \qquad [t, e_n] = 0\,. \tag{5}$$

Without the central term t we obtain the Witt algebra $\mathcal{W}$.

For the classical (Neveu–Schwarz) Lie superalgebra we add an additional set of generators $\{\varphi_m \mid m \in \mathbb{Z} + \frac{1}{2}\}$ and complete the relations to

$$\begin{aligned}
[e_n, e_m] &= (m - n)e_{m+n} + \frac{1}{12}(n^3 - n)\,\delta_n^{-m}\, t, \\
[e_n, \varphi_m] &= \left(m - \frac{n}{2}\right)\, \varphi_{m+n}, \\
[\varphi_n, \varphi_m] &= e_{n+m} - \frac{1}{6}\left(n^2 - \frac{1}{4}\right)\, \delta_n^{-m}\, t, \\
[t, e_n] &= [t, \varphi_m] = 0\,.
\end{aligned} \tag{6}$$

These algebras can be realized in a geometric manner by considering vector fields and forms of weight $-1/2$ (see below) on the Riemann sphere S^2 (i.e., the Riemann surface of genus zero). If we take the vector fields $e_n = z^{n+1}\frac{d}{dz}$, the forms of weight $-1/2$ given by $\varphi_m = z^{m+1/2}(dz)^{-1/2}$, let the vector field e_n act by taking the Lie derivative, and set $[\varphi_m, \varphi_n] = \varphi_m \cdot \varphi_n$ then we obtain the relations above without central terms. The element t is an additional element and the factors in front of it seem to be rather ad-hoc for the moment[1]. One verifies by direct calculations that by our prescription we obtain indeed a Lie superalgebra.

3. Higher genus generalization

Starting from the geometric realization above we can extend this to arbitrary genus. For the whole contribution let Σ be a compact Riemann surface (without boundary). We do not put any restriction on the genus $g = g(\Sigma)$. Furthermore, let A be a finite subset of Σ. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e., $A = I \cup O$. Set $N := \#A$, $K := \#I$, $M := \#O$, with $N = K + M$. More precisely, let

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_M) \tag{7}$$

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets.

Our objects, algebras, structures, ... will be meromorphic objects defined on Σ which are holomorphic outside the points in A. To introduce them let $\mathcal{K} = \mathcal{K}_\Sigma$ be the canonical line bundle of Σ, respectively, the locally free canonical sheaf. The local sections of the bundle are the local holomorphic differentials. If $P \in \Sigma$ is a point and z a local holomorphic coordinate at P then a local holomorphic differential can be written as $f(z)dz$ with a local holomorphic function f defined in a neighborhood of P. A global holomorphic section can be described locally for a covering by coordinate charts $(U_i, z_i)_{i \in J}$ by a system of local holomorphic functions $(f_i)_{i \in J}$, which are related by the transformation rule induced by the coordinate change map $z_j = z_j(z_i)$ and the condition $f_i dz_i = f_j dz_j$. This says

$$f_j = f_i \cdot \left(\frac{dz_j}{dz_i}\right)^{-1}. \tag{8}$$

With respect to a coordinate covering a meromorphic section of $\mathcal{K}$ is given as a collection of local meromorphic functions $(h_i)_{i \in J}$ for which the transformation law (8) is true.

In the following λ is either an integer or a half-integer. If λ is an integer then

(1) $\mathcal{K}^\lambda = \mathcal{K}^{\otimes \lambda}$ for $\lambda > 0$,
(2) $\mathcal{K}^0 = \mathcal{O}$, the trivial line bundle, and
(3) $\mathcal{K}^\lambda = (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$.

Here as usual $\mathcal{K}^*$ denotes the dual line bundle to the canonical line bundle. The dual line bundle is the holomorphic tangent line bundle, whose local sections are the holomorphic tangent vector fields $f(z)(d/dz)$. If λ is a half-integer, then we first have to fix a "square root" of the canonical line bundle, sometimes called a *theta-characteristics*. This means we fix a line bundle L for which $L^{\otimes 2} = \mathcal{K}$.

[1]Below we will give for the factors a geometric expression.

After such a choice of L is done we set $\mathcal{K}^\lambda = \mathcal{K}_L^\lambda = L^{\otimes 2\lambda}$. In most cases we will drop mentioning L, but we have to keep the choice in mind. Also the structure of the algebras we are about to define will depend on the choice. But the main properties will remain the same.

Remark. A Riemann surface of genus g has exactly 2^{2g} non-isomorphic square roots of $\mathcal{K}$. For $g = 0$ we have $\mathcal{K} = \mathcal{O}(-2)$ and $L = \mathcal{O}(-1)$, the tautological bundle which is the unique square root. Already for $g = 1$ we have 4 non-isomorphic ones. As in this case $\mathcal{K} = \mathcal{O}$ one solution is $L_0 = \mathcal{O}$. But we have also the other bundles L_i, $i = 1, 2, 3$.

As above we can talk about holomorphic and meromorphic sections of K^λ. In local coordinates z_i we can write such sections as $f_i dz_i^\lambda$, with f_i being a local holomorphic, respectively meromorphic function.

We set

$$\begin{aligned}\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } \mathcal{K}^\lambda \mid \\ \text{such that } f \text{ is holomorphic over } \Sigma \setminus A\}.\end{aligned} \tag{9}$$

Here the set of A is fixed and we drop it in the notation. Obviously, $\mathcal{F}^\lambda$ is an infinite-dimensional $\mathbb{C}$-vector space. Recall that in the case of half-integer λ everything depends on the theta characteristic L. We call the elements of the space $\mathcal{F}^\lambda$ *meromorphic forms of weight* λ (with respect to the theta characteristic L). Altogether we set

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{10}$$

4. Algebraic structure

4.1. Associative multiplication

The natural map of the locally free sheaves of rank one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s, t) \mapsto s \otimes t, \tag{11}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{12}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\, dz^\lambda, t\, dz^\nu) \mapsto s\, dz^\lambda \cdot t\, dz^\nu = s \cdot t\; dz^{\lambda+\nu}. \tag{13}$$

If there is no danger of confusion then we will mostly use the same symbol for the section and for the local representing function.

The following is obvious

Proposition 1. *The vector space $\mathcal{F}$ with operation $\cdot$ is an associative and commutative graded $\big($over $\frac{1}{2}\mathbb{Z}\big)$ algebra. Moreover, $\mathcal{F}^0$ is a subalgebra.*

We also use $\mathcal{A} := \mathcal{F}^0$. Of course, it is the algebra of meromorphic functions on Σ which are holomorphic outside of A. The spaces $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.

4.2. Lie algebra structure

Next we define a Lie algebra structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s,t) \mapsto [s,t], \tag{14}$$

which is defined in local representatives of the sections by

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda)s\frac{dt}{dz} + \nu\, t\frac{ds}{dz}\right) dz^{\lambda+\nu+1}, \tag{15}$$

and bilinearly extended to $\mathcal{F}$.

Proposition 2.

(a) *The bilinear map* $[.,.]$ *defines a Lie algebra structure on* $\mathcal{F}$.
(b) *The space* $\mathcal{F}$ *with respect to* $\cdot$ *and* $[.,.]$ *is a Poisson algebra.*

Proof. This is done by local calculations. For details see [17, 20]. □

Proposition 3. *The subspace* $\mathcal{L} := \mathcal{F}^{-1}$ *is a Lie subalgebra with respect to the operation* $[.,.]$, *and the* $\mathcal{F}^\lambda$*'s are Lie modules over* $\mathcal{L}$.

As forms of weight -1 are vector fields, $\mathcal{L}$ could also be defined directly as the Lie algebra of those meromorphic vector fields on the Riemann surface Σ which are holomorphic outside of A. The product (15) gives the usual Lie bracket of vector fields and the Lie derivative for their actions on forms:

$$[e,f]_|(z) = \left[e(z)\frac{d}{dz}, f(z)\frac{d}{dz}\right] = \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ , \tag{16}$$

$$\nabla_e(f)_|(z) = L_e(f)_| = e\,.\,f_| = \left(e(z)\frac{df}{dz}(z) + \lambda f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ . \tag{17}$$

4.3. Superalgebra of half-forms

Next we consider the associative product

$$\cdot\ \mathcal{F}^{-1/2} \times \mathcal{F}^{-1/2} \to \mathcal{F}^{-1} = \mathcal{L}, \tag{18}$$

and introduce the vector space and the product

$$\mathcal{S} := \mathcal{L} \oplus \mathcal{F}^{-1/2}, \quad [(e,\varphi),(f,\psi)] := ([e,f] + \varphi\cdot\psi, e\,.\,\varphi - f\,.\,\psi). \tag{19}$$

Usually we will denote the elements of $\mathcal{L}$ by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

Definition (19) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a "super-bracket" (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e,\varphi] := -[\varphi,e] := e\,.\,\varphi =_| \left(e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz}\right)(dz)^{-1/2}, \tag{20}$$

and

$$[\varphi,\psi] = \varphi\cdot\psi. \tag{21}$$

We call the elements of $\mathcal{L}$ elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ elements of odd parity. For such elements x we denote by $\bar{x} \in \{\bar{0}, \bar{1}\}$ their parity.

The sum (19) can be described as $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, where $\mathcal{S}_{\bar{i}}$ is the subspace of elements of parity $\bar{i}$.

Proposition 4 ([19, Prop. 2.5]). *The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Definition 1. The algebra $\mathcal{S}$ is the Lie superalgebra of Krichever–Novikov type.

Remark. The introduced Lie superalgebra corresponds classically to the Neveu-Schwarz superalgebra. In string theory physicists considered also the Ramond superalgebra as string algebra (in the two-point case). The elements of the Ramond superalgebra do not correspond to sections of the dual theta characteristics. They are only defined on a 2-sheeted branched covering of Σ, see, e.g., [1, 3]. Hence, the elements are only multi-valued sections. As here we only consider honest sections of half-integer powers of the canonical bundle, we do not deal with the Ramond algebra.

The choice of the theta characteristics corresponds to choosing a spin structure on Σ. Furthermore, this bundle is related to graded Riemann surfaces. See Bryant [4] for more details on this aspect.

5. Almost-graded structure

Recall the classical situation. This is the Riemann surface $\mathbb{P}^1(\mathbb{C}) = S^2$, i.e., the Riemann surface of genus zero, and the points where poles are allowed are $\{0, \infty\}$. In this case the algebras introduced in the last chapter are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [8] there is a weaker concept, called an almost-grading which to a large extent is a valuable replacement of an honest grading. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The (almost-)grading is fixed by exhibiting certain basis elements in the spaces $\mathcal{F}^\lambda$ as homogeneous.

Definition 2. Let $\mathcal{L}$ be a Lie or an associative algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,
(ii) There exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}. \tag{22}$$

Elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called homogeneous subspace of degree n.

In a similar manner almost-graded modules over almost-graded algebras are defined. Of course, we can extend in an obvious way the definition to superalgebras, respectively even to more general algebraic structures. This definition makes complete sense also for more general index sets $\mathbb{J}$. In fact we will consider the index set $\mathbb{J} = (1/2)\mathbb{Z}$ for our superalgebra. The even elements (with respect to the super-grading) will have integer degree, the odd elements half-integer degree.

As already mentioned above the almost-grading for $\mathcal{F}^\lambda$ is introduced by exhibiting certain elements $f^\lambda_{m,p} \in \mathcal{F}^\lambda$, $p = 1, \dots, K$ which constitute a basis of the subspace $\mathcal{F}^\lambda_m$ of homogeneous elements of degree m. Here $m \in \mathbb{J}_\lambda$ with $\mathbb{J}_\lambda = \mathbb{Z}$ (for λ integer) or $\mathbb{J}_\lambda = \mathbb{Z} + 1/2$ (for λ half-integer). The basis elements $f^\lambda_{m,p}$ of degree m are required to have order

$$\operatorname{ord}_{P_i}(f^\lambda_{m,p}) = (n + 1 - \lambda) - \delta^p_i$$

at the point $P_i \in I$, $i = 1, \dots, K$. The prescription at the points in O is made in such a way that the element $f^\lambda_{m,p}$ is essentially uniquely given. For more details on the prescription see [20, Chapter 4] or the original article [14]. In the classical case we have $\deg(e_n) = n$ and $\deg(\varphi_m) = m$. *Warning:* The spaces $\mathcal{F}^\lambda_m$ depend on the splitting of A.

For the property of being almost-graded the following result is crucial. It is obtained by calculating residues and estimating orders.

Proposition 5 ([20, Thm. 3.8]). *There exist constants R_1 and R_2 (depending on the number and splitting of the points in A and of the genus g) independent of $n, m \in \mathbb{J}$ and λ and ν such that for the basis elements*

$$\begin{aligned}
f^\lambda_{n,p} \cdot f^\nu_{m,r} &= f^{\lambda+\nu}_{n+m,r}\delta^r_p \\
&\qquad + \sum_{h=n+m+1}^{n+m+R_1} \sum_{s=1}^{K} a^{(h,s)}_{(n,p)(m,r)} f^{\lambda+\nu}_{h,s}, \quad a^{(h,s)}_{(n,p)(m,r)} \in \mathbb{C}, \\
[f^\lambda_{n,p}, f^\nu_{m,r}] &= (-\lambda m + \nu n)\, f^{\lambda+\nu+1}_{n+m,r}\delta^r_p \\
&\qquad + \sum_{h=n+m+1}^{n+m+R_2} \sum_{s=1}^{K} b^{(h,s)}_{(n,p)(m,r)} f^{\lambda+\nu+1}_{h,s}, \quad b^{(h,s)}_{(n,p)(m,r)} \in \mathbb{C}.
\end{aligned} \tag{23}$$

The constants R_i can be explicitly calculated (if needed).

As a direct consequence we obtain

Theorem 6. *The algebras $\mathcal{L}$ and $\mathcal{S}$ are almost-graded Lie, respectively Lie superalgebras. The almost-grading depends on the splitting of the set A into I and O. More precisely,*

$$\mathcal{F}^\lambda = \bigoplus_{m \in \mathbb{J}_\lambda} \mathcal{F}^\lambda_m, \qquad \textit{with} \quad \dim \mathcal{F}^\lambda_m = K. \tag{24}$$

and there exist R_2, R_3 (*independent of* n *and* m) *such that*

$$[\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h\ , \qquad [\mathcal{S}_n, \mathcal{S}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_3} \mathcal{S}_h.$$

Also from (23) we directly conclude

Proposition 7. *For all* $m, n \in \mathbb{J}_\lambda$ *and* $r, p = 1, \ldots, K$ *we have*

$$\begin{aligned} [e_{n,p}, e_{m,r}] &= (m-n) \cdot e_{n+m,r}\, \delta_r^p + h.d.t. \\ e_{n,p} \cdot \varphi_{m,r} &= (m - \frac{n}{2}) \cdot \varphi_{n+m,r}\, \delta_r^p + h.d.t. \\ \varphi_{n,p} \cdot \varphi_{m,r} &= e_{n+m,r}\, \delta_r^p + h.d.t. \end{aligned} \tag{25}$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ *and* $n+m+R_i$.

See (6) for an example in the classical case (by ignoring the central extension appearing there for the moment).

Remark. Leidwanger and Morier-Genoux introduced in [11] also a Jordan superalgebra based on the Krichever–Novikov objects given by

$$\mathcal{J} := \mathcal{F}^0 \oplus \mathcal{F}^{-1/2} = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}. \tag{26}$$

Recall that $\mathcal{F}^0$ is the associative algebra of meromorphic functions. The (Jordan) product is defined via the algebra structure for the spaces $\mathcal{F}^\lambda$ by

$$\begin{aligned} f \circ g &:= f \cdot g &&\in \mathcal{F}^0, \\ f \circ \varphi &:= f \cdot \varphi &&\in \mathcal{F}^{-1/2}, \\ \varphi \circ \psi &:= [\varphi, \psi] &&\in \mathcal{F}^0. \end{aligned} \tag{27}$$

By rescaling the second definition with the factor $1/2$ one obtains a Lie antialgebra. See [11] for more details and additional results on representations. Using the results presented here one obtains an almost-grading (depending on a splitting $A = I \cup O$) of the Jordan superalgebra

$$\mathcal{J} = \bigoplus_{m \in 1/2\mathbb{Z}} \mathcal{J}_m. \tag{28}$$

Hence, it makes sense to call it a Jordan superalgebra of KN type. Calculated for the introduced basis elements we get (using Proposition 5)

$$\begin{aligned} A_{n,p} \circ A_{m,r} &= A_{n+m,r}\, \delta_r^p + \text{h.d.t.} \\ A_{n,p} \circ \varphi_{m,r} &= \varphi_{n+m,r}\, \delta_r^p + \text{h.d.t.} \\ \varphi_{n,p} \circ \varphi_{m,r} &= \frac{1}{2}(m-n)\mathcal{A}_{n+m,r}\, \delta_r^p + \text{h.d.t.} \end{aligned} \tag{29}$$

6. Central extensions

A central extension of a Lie algebra W is defined on the vector space direct sum $\widehat{W} = \mathbb{C} \oplus W$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ its Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{W}] = 0, \quad x, y \in W. \tag{30}$$

Then $\widehat{W}$ will be a Lie algebra, e.g., fulfill the Jacobi identity, if and only if Φ is antisymmetric and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y). \tag{31}$$

There is the notion of equivalence of central extensions. It turns out that two central extensions are equivalent if and only if the difference of their defining 2-cocycles Φ and Φ' is a coboundary, i.e., there exists a $\phi : W \to \mathbb{C}$ such that

$$\Phi(x, y) - \Phi'(x, y) = d_1\phi(x, y) = \phi([x, y]). \tag{32}$$

In this way the second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ of W with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions. The class $[0]$ corresponds to the trivial (i.e., split) central extension. Hence, to construct central extensions of our Lie algebras we have to find such Lie algebra 2-cocycles.

For the superalgebra case central extension are obtained with the help of a bilinear map

$$\Phi : \mathcal{S} \times \mathcal{S} \to \mathbb{C} \tag{33}$$

via an expression completely analogous to (30). Additional conditions for Φ follow from the fact that the resulting extension should be again a superalgebra. This implies that for homogeneous elements $x, y, z \in \mathcal{S}$ ($\mathcal{S}$ might be an arbitrary Lie superalgebra) we need

$$\Phi(x, y) = -(-1)^{\bar{x}\bar{y}}\Phi(x, y). \tag{34}$$

If x and y are odd then the bilinear map Φ will be symmetric, otherwise it will be antisymmetric. The super-cocycle condition reads in complete analogy with the super-Jacobi relation as

$$(-1)^{\bar{x}\bar{z}}\Phi(x, [y, z]) + (-1)^{\bar{y}\bar{x}}\Phi(y, [z, x]) + (-1)^{\bar{z}\bar{y}}\Phi(z, [x, y]) \;=\; 0. \tag{35}$$

As we will need it anyway, I will write it out for the different type of arguments. For (*even, even, even*), (*even, even, odd*), and (*odd, odd, odd*) it will be of the "usual form" of the cocycle condition

$$\Phi(x, [y, z]) + \Phi(y, [z, x]) + \Phi(z, [x, y]) \;=\; 0. \tag{36}$$

For (*even, odd, odd*) we obtain

$$\Phi(x, [y, z]) + \Phi(y, [z, x]) - \Phi(z, [x, y]) \;=\; 0. \tag{37}$$

Now we have to decide which parity our central element should have. In our context, as we want to extend the central extension of the vector field algebra to the superalgebra, the natural choice is that the central element should be even. This implies that our bilinear form Φ has to be an even form. Consequently,

$$\Phi(x, y) = \Phi(y, x) = 0, \quad \text{for} \;\; \bar{x} = 0, \bar{y} = 1. \tag{38}$$

In this case only (37) for the (*even, odd, odd*) and (36) for the (*even, even, even*) case will give relations which are not trivially zero.

Given a linear form $\phi : \mathcal{S} \to \mathbb{C}$ we assign to it

$$\delta_1\phi(x,y) = \phi([x,y]). \tag{39}$$

As in the classical case $\delta_1\phi$ will be a super-cocycle. A super-cocycle Φ will be a coboundary if and only if there exists a linear form $\phi : \mathcal{S} \to \mathbb{C}$ such that $\Phi = \delta_1\phi$. As ϕ is a linear form it can be written as $\phi = \phi_{\bar{0}} \oplus \phi_{\bar{1}}$ where $\phi_{\bar{0}} : \mathcal{S}_{\bar{0}} \to \mathbb{C}$ and $\phi_{\bar{1}} : \mathcal{S}_{\bar{1}} \to \mathbb{C}$. Again we have the two cases of the parity of the central element. Let Φ be a coboundary $\delta_1\phi$. If the central element is even then Φ will also be a coboundary with respect to a ϕ with $\phi_{\bar{1}} = 0$. In other words this ϕ is even. In the odd case we can take $\phi_{\bar{0}} = 0$ and ϕ is odd.

After fixing a parity of the central element we consider the quotient spaces

$$\mathrm{H}^2_{\bar{0}}(\mathcal{S},\mathbb{C}) := \{\text{even cocycles}\}/\{\text{even coboundaries}\}, \tag{40}$$

$$\mathrm{H}^2_{\bar{1}}(\mathcal{S},\mathbb{C}) := \{\text{odd cocycles}\}/\{\text{odd coboundaries}\}. \tag{41}$$

These cohomology spaces classify central extensions of $\mathcal{S}$ with even (respectively odd) central elements up to equivalence. Equivalence is defined as in the non-super setting.

To define a super-cocycle we have to introduce the following objects.

Definition 3. Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$. A system of local holomorphic functions $R = (R_\alpha(z_\alpha))$ is called a holomorphic *projective connection* if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{42}$$

the Schwartzian derivative. Here $'$ means differentiation with respect to the coordinate z_α.

It is a classical result [5, 6] that every Riemann surface admits a holomorphic projective connection R. From the definition it follows that the difference of two projective connections is a quadratic differential. In fact starting from one projective connection we will obtain all of them by adding quadratic differentials to it.

If we have a cocycle Φ for the algebra $\mathcal{S}$ we obtain by restriction a cocycle for the algebra $\mathcal{L}$. For arguments with mixed parity we know that $\Phi(e,\psi) = 0$. A naive try to put just anything for $\Phi(\varphi,\psi)$ will not work as (37) relates the restriction of the cocycle on $\mathcal{L}$ with its values on $\mathcal{F}^{-1/2}$.

Proposition 8 ([19, Prop. 5.1]). *Let C be any closed* (*differentiable*) *curve on Σ not meeting the points in A, and let R be any* (*holomorphic*) *projective connection,* *then the bilinear extension of*

$$\begin{aligned}
\Phi_{C,R}(e,f) &:= \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \\
\Phi_{C,R}(\varphi,\psi) &:= -\frac{1}{24\pi \mathrm{i}} \int_C (\varphi'' \cdot \psi + \varphi \cdot \psi'' - R \cdot \varphi \cdot \psi)\, dz \\
\Phi_{C,R}(e,\varphi) &:= 0
\end{aligned} \tag{43}$$

gives a Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$. The cocycle class does not depend on the chosen connection R.

A similar formula was given by Bryant in [4]. By adding the projective connection in the second part of (43) he corrected some formula appearing in [1]. He only considered the two-point case and only the integration over a separating cycle. See also [7] for the multi-point case, where still only the integration over a separating cycle is considered.

The following remarks are in order. For the proof of the claims see [19].

1. Adding the projective connection is necessary to make the integrand a well-defined differential.
2. Different R will yield cohomologous cocycles.
3. With respect to the curve C only its homology class in $\mathrm{H}_1(\Sigma \setminus A, \mathbb{Z})$ is relevant for the cocycle.

But a different cycle class will change the cocycle in an essential manner. We cannot expect uniqueness if $g > 0$ or $N > 2$. We should not forget, that we want to extend our almost-grading of $\mathcal{S}$ to the centrally extended Lie superalgebra by assigning a degree to the central element. This only works if our cocycle is "local" [8] in the following sense: There exists $M_1, M_2 \in \mathbb{Z}$ such that

$$\forall n, m : \quad \psi(W_n, W_m) \neq 0 \implies M_1 \leq n + m \leq M_2.$$

What is local is defined in terms of the almost-grading and hence depends on the splitting $A = I \cup O$.

If the integration path C in (43) is a *separating cycle* C_S, i.e., a cycle which separates the points in I from the points in O, then the cocycle $\Phi_{C_S,R}$ is local. A special choice for C_S is the collection of circles around the points in I. This shows that in this case the cocycle can be calculated via residues.

How about the opposite direction? Given a local cocycle, can it be described as such a geometric cocycle? The answer is yes and this is the main result which we present.

Theorem 9 ([19, Thm. 5.5]). *Given a local* (*even*) *cocycle for the Lie superalgebra $\mathcal{S}$ of Krichever–Novikov type then up to coboundary it is a multiple of $\Phi_{C_S,R}$. Hence, up to rescaling and equivalence there is a unique non-trivial almost-graded central extension of $\mathcal{S}$.*

Recall that the almost-grading is fixed by the splitting. We will only give some general remarks on the proof.

1. We start with a local cocycle for $\mathcal{S}$ and restrict it to $\mathcal{L}$, the vector field subalgebra.
2. This gives a local cocycle for the vector field algebra $\mathcal{L}$.
3. For such local cocycles my earlier classification results [18] show that it is unique (in the above sense) and can by given by an expression of the type $\Phi_{C_S,R}$ for $\mathcal{L}$.
4. This expression we extend to $\mathcal{S}$ by using the full expression of $\Phi_{C_S,R}$.
5. The difference between the initial cocycle and the extended one vanishes if restricted to $\mathcal{L}$.
6. Next we show that each local cocycle of $\mathcal{S}$ which vanishes on $\mathcal{L}$ vanishes in total. This is done by some induction process using the almost-gradedness and the locality (more precisely, the boundedness from above is enough).

Theorem 10 ([19, Thm. 5.6]). *All local cocycles of odd type are coboundaries. Hence, there does not exist non-trivial almost-graded odd central extensions of $\mathcal{S}$.*

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On n-ary Lie Algebras of Type (r, l)

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Abstract. These notes are devoted to the multiple generalization of a Lie algebra introduced by A.M. Vinogradov and M.M. Vinogradov. We compare definitions of such algebras in the usual and invariant case. Furthermore, we show that there are no simple n-ary Lie algebras of type $(n-1, l)$ for $l > 0$.

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1. Introduction

This paper is devoted to the study of the multiple generalization of a Lie algebra introduced in [11]. More precisely in [11] A.M. Vinogradov and M.M. Vinogradov proposed a two-parameter family of n-ary algebras that generalized *Filippov n-algebras* and *Lie n-algebras*. V.T. Filippov [3] considered alternating n-ary algebras V satisfying the following Jacobi identity:

$$\{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_n\}\} = \sum_{i=1}^{n} \{b_1, \ldots, b_{i-1}\{a_1, \ldots, a_{n-1}, b_i\}, \ldots, b_n\}, \qquad (1)$$

where a_i, $b_j \in V$. In other words, the linear maps $\{a_1, \ldots, a_{n-1}, -\} : V \to V$ are derivations of the n-ary bracket $\{b_1, \ldots, b_n\}$. Another natural n-ary generalization of the standard Jacobi identity has the following form:

$$\sum_{(I,J)} (-1)^{(I,J)} \{\{a_{i_1}, \ldots, a_{i_n}\}, a_{j_1}, \ldots, a_{j_{n-1}}\} = 0, \qquad (2)$$

where the sum is taken over all ordered unshuffle multi-indexes $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_{n-1})$ such that (I, J) is a permutation of the multi-index $(1, \ldots, 2n-1)$. We call alternating n-ary algebras satisfying Jacobi identity (2) *Lie n-algebras.*

The n-ary algebras of type (1) appear naturally in Nambu mechanics [9] in the context of Nambu–Poisson manifolds, in supersymmetric gravity theory and

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in supersymmetric gauge theories, the Bagger–Lambert–Gustavsson Theory, see [1] for details. The n-ary algebras of type (2) were considered for instance by P. Michor and A.M. Vinogradov in [8] and by P. Hanlon and M.L. Wachs [4]. The homotopy case was studied in [10] in the context of the Schlesinger–Stasheff homotopy algebras and L_∞-algebras. Such algebras are related to the Batalin–Fradkin–Vilkovisky theory and to string field theory [6].

The paper is structured as follows. In Section 2 we remind the definition of the Nijenhuis–Richardson bracket on $\bigwedge V^* \otimes V$, where V is a finite-dimensional vector space. In Section 3 we give a definition of Vinogradovs' algebras. In Section 4 we compare definitions of usual Vinogradovs' algebras and Vinogradovs' algebras with a symmetric non-degenerate invariant form. In Section 4 we show that there are no simple n-ary Lie algebras of type $(n-1, l)$, $l > 0$. The case of n-ary Lie algebras of type $(n-1, 0)$ was studied in [7].

2. Nijenhuis–Richardson bracket

In this section we follow [11]. Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. We put $I^n := (1, \ldots, n)$ and we denote by I and J the ordered multi-indexes $I = (i_1, \ldots, i_l)$ and $J = (j_1, \ldots, j_k)$ such that $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_k$. Let $|I| := l$ and $|J| := k$. Assume that $i_p \neq j_q$ for all p, q. Then we have the non-ordered multi-index

$$(I, J) := (i_1, \ldots, i_l, j_1, \ldots, j_k).$$

We denote by $I + J$ the ordering of the multi-index (I, J) and by $(-1)^{(I,J)}$ the parity of the permutation that maps (I, J) into $I + J$. We set

$$a_{I^n} := (a_1, \ldots, a_n) \quad a_I := (a_{i_1}, \ldots, a_{i_l}), \quad a_J := (a_{j_1}, \ldots, a_{j_k}).$$

Let us take $L \in \bigwedge^l V^* \otimes V$ and $K \in \bigwedge^k V^* \otimes V$. We define:

$$L[K](a_{I^{l+k-1}}) = \sum_{I+J=I^{l+k-1}} (-1)^{(I,J)} L(K(a_I), a_J). \tag{3}$$

Here we assume that $|I| = k$, $|J| = l - 1$.

Definition 1. The *Nijenhuis–Richardson bracket*

$$[L, K]^{NR} \in \bigwedge^{l+k-1} V^* \otimes V,$$

of $L \in \bigwedge^l V^* \otimes V$ and $K \in \bigwedge^k V^* \otimes V$ is defined by the following formula:

$$[L, K]^{NR} := (-1)^{(l-1)(k-1)} L[K] - K[L].$$

For $L \in \bigwedge^l V^* \otimes V$, we denote by $L_{a_1,\ldots,a_p}$ the following $(l-p)$-linear map:

$$L_{a_1,\ldots,a_p}(b_1, \ldots, b_{l-p}) := L(a_1, \ldots, a_p, b_1, \ldots, b_{l-p}),$$

where $a_i, b_j \in V$.

3. Vinogradovs' algebras

In this subsection we recall the definition of n-ary algebras introduced by A.M. Vinogradov and M.M. Vinogradov in [11]. The graded versions of such algebras are studied in [12]. These n-ary algebras are a generalization of n-ary Lie algebras of type (1) and of type (2) for even n. Let V be a finite-dimensional complex or real vector space.

Definition 2. An *n-ary algebra structure* μ on V is an n-linear map:

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ times}} \to V.$$

An *n-ary algebra* is a pair (V, μ).

Sometimes we will use the bracket notation $\{a_1, \ldots, a_n\}$ instead of $\mu(a_1, \ldots, a_n)$.

Definition 3. An n-ary algebra is called *skew-symmetric* if

$$\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\} = -\{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\} \tag{4}$$

for any $a_p \in V$.

Definition 4 ([11]). Let (V, μ) be an n-ary skew-symmetric algebra. We say that (V, μ) is an *n-ary Lie algebra of type* (r, l) if the following holds:

$$[\mu_{a_1,\ldots,a_r}, \mu_{b_1,\ldots,b_l}]^{NR} = 0, \tag{5}$$

where $a_i, b_j \in V$ and $0 \leq l \leq r < n$.

Example 1. Filippov n-ary algebras or n-ary algebras satisfying (1) are exactly n-ary Lie algebras of type $(n - 1, 0)$. For even n, the n-ary Lie algebras satisfying (2) are exactly n-ary algebras of type $(1, 0)$. (See [11] for details.)

Remark 5. The theory of Filippov n-ary algebras is relatively well developed. For instance, there is a classification of simple real and complex Filippov n-ary algebras and an analog of the Levi decomposition [7]. W.X. Ling in [7] proved that there exists only one simple finite-dimensional n-ary Filippov algebra over an algebraically closed field of characteristic 0 for any $n > 2$. The simple Filippov n-ary superalgebras in the finite- and infinite-dimensional case were studied in [2]. It was shown there that there are no simple linearly compact n-ary Filippov superalgebras which are not n-ary Filippov algebras, if $n > 2$. A classification of linearly compact n-ary Filippov algebras was given in [2].

Remark 6. The n-ary algebras of type (2) were studied for instance in [8] and [4]. In [13] the n-ary algebras of type (2) endowed with a non-degenerate invariant form were considered.

4. Nijenhuis–Richardson and Poisson brackets on V

Let V be a finite-dimensional vector space that is supplied with a non-degenerate symmetric bilinear form $(\,,)$. Then $\bigwedge V$ possesses a natural structure of a Poisson superalgebra defined by the following formulas:

$$[x, y] := (x, y), \quad x, y \in W;$$
$$[v, w_1 \cdot w_2] := [v, w_1] \cdot w_2 + (-1)^{vw_1} w_1 \cdot [v, w_2],$$
$$[v, w] = -(-1)^{vw}[w, v],$$

where v, w, w_i are homogeneous elements in $\bigwedge W$. (See for example [5] or [13] for details.) Let us take any element $\tilde{\mu} \in \bigwedge^{n+1} V$. Then we can define an n-ary algebra structure on V in the following way:

$$\{a_1, \ldots, a_n\} := [a_1, [\ldots, [a_n, \tilde{\mu}] \ldots]], \ \ a_i \in V. \tag{6}$$

Definition 7. A skew-symmetric n-ary algebra structure is called *invariant with respect to the form* $(\,,)$ if the following holds:

$$(b, \{a_1, \ldots, a_n\}) = -(a_1, \{b, a_2, \ldots, a_n\}) \tag{7}$$

for any $b, a_i \in V$.

The following observation was noticed in [13]:

Proposition 8. *Assume that V is finite dimensional and $(\,,)$ is non-degenerate. Any skew-symmetric invariant n-ary algebra structures can be obtained by construction* (6)*. In other words, for any skew-symmetric invariant n-ary algebra structure $\mu \in \bigwedge^n V^* \otimes V$, there exists $\tilde{\mu} \in \bigwedge^{n+1} V$ such that the map μ coincides with the map defined by the formula* (6)*.*

Assume that (V, μ) is an n-ary Lie algebra of type (r, l) and the algebra structure μ is invariant with respect to $(\,,)$. It follows from Proposition 8 that the algebra (V, μ) has another n-ary algebra structure $\tilde{\mu} \in \bigwedge^{n+1}(V)$. In this section we study the relationship between these two n-ary algebra structures. Since μ and $\tilde{\mu}$ define the same algebra structure on V, we have:

$$\mu(a_1, \ldots, a_l) = [a_1, \ldots, [a_l, \tilde{\mu}] \ldots].$$

Let us take two skew-symmetric invariant n-ary algebra structures $L \in \bigwedge^l V^* \otimes V$ and $K \in \bigwedge^k V^* \otimes V$. It follows from (3) that

$$L[K](a_{I^{l+k-1}}) = \sum_{I+J=I^{l+k-1}} (-1)^{(I,J)}[[a_I, \tilde{K}], [a_J, \tilde{L}]], \tag{8}$$

where $|I| = k$, $|J| = l - 1$ and for simplicity we denote

$$[a_I, \tilde{L}] := [a_{i_1}, \ldots, [a_{i_l}, \tilde{L}] \ldots].$$

Proposition 9. *Let us take two skew-symmetric invariant n-ary algebra structures $L \in \bigwedge^l V^* \otimes V$ and $K \in \bigwedge^k V^* \otimes V$. Then we have:*

$$[a_{I^{l+k-1}}, [\tilde{L}, \tilde{K}]] = -(-1)^{(k+1)(l+1)}[L, K]^{NR}(a_{I^{l+k-1}}).$$

Proof. We set

$$a_{I^p} := (a_1, \dots, a_p); \quad [a_{I^p}, \tilde{\mu}] := [a_1, \dots [a_p, \tilde{\mu}]].$$

Furthermore, we have:

$$\begin{aligned}
[a_{I^{l+k-1}}, [\tilde{L}, \tilde{K}]] = & \sum_{I+J=I^{l+k-1}} (-1)^{(I,J)+(k+1)(l+1)} [[a_I, \tilde{L}], [a_J, \tilde{K}]] \\
& + \sum_{I'+J'=I^{l+k-1}} (-1)^{(I',J')+(l+1)k} [[a_{I'}, \tilde{L}], [a_{J'}, \tilde{K}]] \\
= & \sum_{I+J=I^{l+k-1}} (-1)^{(I,J)+(k+1)(l+1)} [[a_I, \tilde{L}], [a_J, \tilde{K}]] \\
& + \sum_{I'+J'=I^{l+k-1}} (-1)^{(J',I')} [[a_{I'}, \tilde{L}], [a_{J'}, \tilde{K}]] \\
= & \sum_{I+J=I^{l+k-1}} (-1)^{(I,J)+(k+1)(l+1)} [[a_I, \tilde{L}], [a_J, \tilde{K}]] \\
& + \sum_{I'+J'=I^{l+k-1}} -(-1)^{(J',I')} [[a_{J'}, \tilde{K}], [a_{I'}, \tilde{L}]],
\end{aligned}$$

where $|I| = l$, $|J| = k - 1$ and $|I'| = l - 1$, $|J;| = k$. By equation (8), we get

$$\begin{aligned}
[a_{I^{l+k-1}}, [\tilde{L}, \tilde{K}]] &= (-1)^{(k+1)(l+1)} K[L](a_{I^{l+k-1}}) - L[K](a_{I^{l+k-1}}) \\
&= -(-1)^{(k+1)(l+1)} [L, K]^{NR}(a_{I^{l+k-1}}).
\end{aligned}$$

The proof is complete. □

Let $L \in \bigwedge^t V^* \otimes V$ be a skew-symmetric invariant n-ary algebra structures. Clearly,

$$L_{a_1,\dots,a_p} = c[a_1, \dots, [a_p, \tilde{L}]],$$

where $c \neq 0$. Proposition 9 implies the following. If a skew-symmetric invariant algebra (V, μ) is given by an identity of type

$$[\mu_{a_1,\dots,a_r}, \mu_{a_1,\dots,a_l}]^{NR} = 0,$$

then the same algebra can be given by the following relation:

$$[[a_{I^r}\tilde{\mu}], [a_{I^l}\tilde{\mu}]] = 0.$$

The result of our study is the following.

Theorem 10. *Let V be a finite-dimensional vector space with a non-degenerate symmetric bilinear form $(,)$ and $\mu \in \bigwedge^n V^* \otimes V$ be an invariant algebra structure on V. Then (V, μ) is an n-ary Lie algebra of type (r, l) if and only if the following holds:*

$$[[a_{I^r}\tilde{\mu}], [a_{I^l}\tilde{\mu}]] = 0$$

for all $a_i, b_j \in V$.

5. Algebras of type $(n-1, l)$

In this section we consider n-ary algebras of type $(n-1, l)$, where $0 < l \leq n-1$. More precisely, we study n-ary skew-symmetric algebras satisfying (5) for $k = n-1$ and $l > 0$.

Lemma 1. Jacobi identity (5) of type $(n-1, l)$, where $0 \leq l \leq n-1$, is equivalent to the following identity:

$$\{a_1, \dots, a_{n-1}, \{b_1, \dots, b_n\}\} = \sum\nolimits_{i=1}^{n-l} \{b_1, \dots, \{a_1, \dots, a_{n-1}, b_i\}, \dots, b_n\}. \tag{9}$$

Proof. We will need the following formula, see [11]:

$$[L, K]_a^{NR} = [L_a, K]^{NR} + (-1)^{l-1}[L, K_a]^{NR},$$

where $L \in \bigwedge^{l} V^* \otimes V$ and $K \in \bigwedge^{k} V^* \otimes V$. Let (V, μ) is an n-ary Lie algebra of type $(n-1, l)$. Let us take $a_i, b_j, c_t \in V$ and denote $L := \mu_{a_1,\dots,a_n}$ and $K := \mu_{b_1,\dots,b_l}$. We have

$$[L, K]_{c_1,\dots,c_k}^{NR} = \sum\nolimits_i (-1)^{k-i}[L_{c_i}, K_{c_1,\dots,\hat{c}_i,\dots,c_k}]^{NR} + [L, K_{c_1,\dots,c_i,\dots,c_k}]^{NR}$$

where $k = n - l$. Further,

$$[L, K_{c_1,\dots,c_i,\dots,c_k}]^{NR} = \{a_1, \dots, a_{n-1}, \{b_1, \dots, b_l, c_1, \dots, c_k\}\};$$
$$[L_{c_i}, K_{c_1,\dots,\hat{c}_i,\dots,c_k}]^{NR} = -\{b_1, \dots, b_l, c_1, \dots, \hat{c}_i, \dots, c_k, \{a_1, \dots, a_{n-1}\}\}.$$

It is clear that $[L, K]^{NR} = 0$ if and only if $[L, K]_{c_1,\dots,c_k}^{NR} = 0$. The last equality is equivalent to:

$$\begin{aligned}
&\{a_1, \dots, a_{n-1}, \{b_1, \dots, b_l, c_1, \dots, c_k\}\}\\
&\qquad = \sum\nolimits_{i=1}^{k} (-1)^{k-i}\{b_1, \dots, b_l, c_1, \dots, \hat{c}_i, \dots, c_k, \{a_1, \dots, a_{n-1}\}\},\\
&\{a_1, \dots, a_{n-1}, \{b_1, \dots, b_l, c_1, \dots, c_k\}\}\\
&\qquad = \sum\nolimits_{i=1}^{k} \{b_1, \dots, b_l, c_1, \dots, \{a_1, \dots, a_{n-1}\}, \dots, c_k\},\\
&\{a_1, \dots, a_{n-1}, \{c_1, \dots, c_k, b_1, \dots, b_l\}\}\\
&\qquad = \sum\nolimits_{i=1}^{k} \{c_1, \dots, \{a_1, \dots, a_{n-1}\}, \dots, c_k, b_1, \dots, b_l\}.
\end{aligned}$$

The proof is complete. □

Example 2. For instance, 2-ary Lie algebras of type $(1, 1)$ are exactly the associative skew-symmetric algebras. Using the fact that these algebras are skew-symmetric, it is easy to see that all such algebras have the following property: $\{\{a, b\}, c\} = 0$. Therefore, this class is not reach.

Definition 11. A vector subspace $W \subset V$ is called an *ideal* of a skew-symmetric n-ary algebra (V, μ) if $\mu(V, \dots, V, W) \subset W$.

For example $\{0\}$ and V are always ideals of (V, μ). Such ideals are called *trivial*.

Definition 12. An n-ary Lie algebra of type $(n-1, l)$ is called *simple* if it is not one-dimensional and it does not have any non-trivial ideals.

Theorem 13. *Let V be a Lie algebra of type $(n-1, l)$, where $0 < l \leq n-1$, over $\mathbb{C}$. Assume that V does not possess any non-trivial ideals. Then $\dim V = 1$. In other words, there are no simple algebras of type $(n-1, l)$, where $0 < l \leq n-1$.*

Proof. Assume that $\dim V > 1$ and V does not possess any non-trivial ideals. Let us interchange b_{n-l} and b_{n-l+1} and use (9). We get:

$$\begin{aligned}&\{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_{n-l-1}, b_{n-l+1}, b_{n-l}, b_{n-l+2}, \ldots, b_n\}\}\\&\quad = \sum_{i=1}^{n-l-1} \{b_1, \ldots, \{a_1, \ldots, a_{n-1}, b_i\}, \ldots, b_n\}\\&\qquad + \{b_1, \ldots, b_{n-l-1}, \{a_1, \ldots, a_{n-1}, b_{n-l+1}\}, b_{n-l}, b_{n-l+2}, \ldots, b_n\}.\end{aligned}$$

From this equation and (9), we have:

$$\begin{aligned}&\{b_1, \ldots, b_{n-l-1}, \{a_1, \ldots, a_{n-1}, b_{n-l}\}, b_{n-l+1} \ldots, b_n\}\\&\quad = \{b_1, \ldots, b_{n-l}, \{a_1, \ldots, a_{n-1}, b_{n-l+1}\}, b_{n-l+2}, \ldots, b_n\}.\end{aligned} \tag{10}$$

Denote by $\mathcal{L}$ the vector subspace in $\mathrm{End}(V)$ generated by $\mu_{a_1,\ldots,a_{n-1}}$, where $a_i \in V$. Let $D = \mu_{a_1,\ldots,a_{n-1}}$ for some $a_i \in V$. We can rewrite (10) in the form:

$$\begin{aligned}&\{b_1, \ldots, b_{n-l-1}, D(b_{n-l}), b_{n-l+1}, \ldots, b_n\}\\&\quad = \{b_1, \ldots, b_{n-l}, D(b_{n-l+1}), b_{n-l+2}, \ldots, b_n\}.\end{aligned} \tag{11}$$

The n-ary algebra (V, μ) is skew-symmetric and equation (11) holds for any b_i. Hence, we can assume that the following equation holds

$$\{b_1, \ldots, D(b_i), b_{i+1}, \ldots, b_n\} = \{b_1, \ldots, b_i, D(b_{i+1}), \ldots, b_n\}$$

for any i. Applying this formula several times, we get:

$$\{b_1, \ldots, D(b_i), b_{i+1}, \ldots, b_n\} = \{b_1, \ldots, b_{i+j-1}, D(b_{i+j}), \ldots, b_n\}$$

for all i, j. Furthermore, let us take $D_1, D_2 \in \mathcal{L}$. First of all, we see:

$$\begin{aligned}\{b_1, \ldots, D_1(D_2(b_i)), b_{i+1}, \ldots, b_n\} &= \{b_1, \ldots, D_2(b_i), \ldots, D_1(b_{i+j}), \ldots, b_n\}\\&= \{b_1, \ldots, b_i, \ldots, D_2(D_1(b_{i+j})), \ldots, b_n\}.\end{aligned}$$

We put $B := D_1 \circ D_2 - D_2 \circ D_1$. Therefore,

$$\{b_1, \ldots, B(b_i), \ldots, b_{i+j}, \ldots, b_n\} = -\{b_1, \ldots, b_i, \ldots, B(b_{i+j}), \ldots, b_n\}.$$

Furthermore,

$$\begin{aligned}\{B(b_1), b_2, b_3, \ldots, b_{i+1}, \ldots, b_n\} &= -\{b_1, B(b_2), b_3, \ldots, b_{i+1}, \ldots, b_n\}\\&= \{b_1, b_2, B(b_3), \ldots, b_{i+1}, \ldots, b_n\};\\\{B(b_1), b_2, b_3, \ldots, b_{i+1}, \ldots, b_n\} &= -\{b_1, b_2, B(b_3), \ldots, b_{i+1}, \ldots, b_n\}.\end{aligned}$$

Therefore,

$$\{B(b_1), b_2, b_3, \ldots, b_n\} = 0$$

for all $b_i \in V$. If $B(b_1) \neq 0$, we see that $\langle B(b_1) \rangle$ is a non-trivial one-dimensional ideal in V. Hence, $B(b_1) = 0$ for all $b_1 \in V$. Therefore, $D_1 \circ D_2 - D_2 \circ D_1 = 0$ and $\mathcal{L}$ is a usual abelian Lie algebra. Since, V does not possess any non-trivial ideals, the $\mathcal{L}$-module V should be irreducible. By Schur's Lemma $\mathcal{L} \subset \{a \,\mathrm{id}\} \subset \mathrm{End}\, V$ and $\dim V = 1$. □

Example 3. The classification of simple complex and real Filippov n-ary algebras or Lie algebras of type $(n-1, 0)$ was done in [7]: there is one series of complex Filippov n-ary algebras A_k, where k is a natural number, and several real forms for each A_k. All these algebras have invariant forms and in terms of Poisson bracket they are given by the top form of V.

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Factorization Method for (q, h)-Hahn Orthogonal Polynomials

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Abstract. We investigate a version of factorization method on a sequence of Hilbert spaces related to (q, h)-ladder. As an example we present (q, h)-orthogonal polynomials.

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1. Introduction

In this paper we shall first recall the basics of the factorization method. We will consider the class of second order (q, h)-difference operators with factorization into first-order operators related to some generalization of the Hahn equation.

Recall that according to [10–12] ladders are the sequences of vector spaces V_k and operators $\mathbf{A}_k^{\pm}$ acting between them as follows

$$\ldots V_{k-1} \underset{\mathbf{A}_k^-}{\overset{\mathbf{A}_k^+}{\rightleftarrows}} V_k \underset{\mathbf{A}_{k+1}^-}{\overset{\mathbf{A}_{k+1}^+}{\rightleftarrows}} V_{k+1} \ldots$$

We will assume that the "commutator" of the ladder operators

$$q\mathbf{A}_{k+1}^-\mathbf{A}_{k+1}^+ - \mathbf{A}_k^+\mathbf{A}_k^- = a_k - qa_{k+1}, \qquad k \in \mathbb{Z}, \tag{1}$$

is a scalar, $a_k \in \mathbb{R}$. The operators

$$\mathbf{H}_k := \mathbf{A}_k^+\mathbf{A}_k^- + a_k = q\left(\mathbf{A}_{k+1}^-\mathbf{A}_{k+1}^+ + a_{k+1}\right) \tag{2}$$

are called loop operators (left and right loop operators). If the operators $\mathbf{H}_k$ admit the factorization given by (2) then the eigenproblem

$$\mathbf{H}_k\psi_k^n = \lambda_k^n\psi_k^n \tag{3}$$

can be rewritten in one of the following forms

$$\mathbf{A}_k^+\mathbf{A}_k^-\psi_k^n = (\lambda_k^n - a_k)\psi_k^n\ , \tag{4}$$

$$\mathbf{A}_{k+1}^-\mathbf{A}_{k+1}^+\psi_k^n = (q^{-1}\lambda_k^n - a_{k+1})\psi_k^n\ . \tag{5}$$

There exist some important classes of factorizations with the property $\lambda_k^0 = a_k$. In these cases any nonzero solution of the equation (ground state)

$$\mathbf{A}_k^-\psi_k^0 = 0 \tag{6}$$

is automatically a solution of equation (3). Moreover (4) and (5) can be used to construct new solutions

$$\psi_k^n = \mathbf{A}_k^+\cdots\mathbf{A}_{k-n+1}^+\psi_{k-n}^0 \tag{7}$$

with eigenvalues

$$\lambda_k^n = q^{-n}a_{k-n} \tag{8}$$

for any $n \in \mathbb{N}$.

Some results concerning the factorization method for the difference (q-difference in particular) equations were presented in papers [1–6, 8]. It is a generalization of the well-known method for differential operators developed for example in [13, 15–18].

Our investigation will be based on (q,h)-discretization of real line (Section 2). In place of standard h-discretization related to an arithmetic progression or q-discretization related to the geometric progression we will use both. This method leads to (q,h)-difference operators, which in the limit $h \to 0$, $q \to 1$ correspond to differential operators. Finally in Section 3, we shall present application of this method to (q,h)-Hahn orthogonal polynomials. In the limit $h \to 0$ we obtain the q-Hahn orthogonal polynomials [7, 9, 14], which were considered in Ref. [5]. In the limit $h \to 0$, $q \to 1$ this corresponds to the classical orthogonal polynomials. In Section 4 we consider an example, which leads to the discrete generalization of the Hermite II polynomials.

2. The (q,h)-discretization

Let us consider the spaces V_k consisting of continuous functions defined on the closure of (q,h)-interval

$$[a,b]_{q,h} := \{q^n a + [n]_q h :\ n \in \mathbb{N}\cup\{0\}\} \cup \{q^n b + [n]_q h :\ n \in \mathbb{N}\cup\{0\}\}, \tag{9}$$

where $0 < q < 1$, $h \in \mathbb{R}\setminus\{0\}$, $a, b \in \mathbb{R}$. We use the following notation for q-number

$$[n]_q = \frac{1-q^n}{1-q}. \tag{10}$$

We recall the definition of (q,h)-shift operators S and S^{-1} (see [11])

$$S\psi(x) := \psi(qx+h), \tag{11}$$

$$S^{-1}\psi(x) := \psi\left(q^{-1}x - q^{-1}h\right) \tag{12}$$

and the (q,h)-derivative operator [9]

$$D_{q,h}\psi(x) := \frac{\psi(x)-\psi(qx+h)}{(1-q)x-h} = \frac{1}{(1-q)x-h}\left(1-S\right)\psi(x). \tag{13}$$

Next, we define the (q,h)-integral on the (q,h)-interval $[0,b]_{q,h}$

$$\int_0^b \psi(x)d_{q,h}x = \sum_{n=0}^{\infty}\left((1-q)b-h\right)q^n\psi\left(q^n b+[n]_q h\right). \tag{14}$$

In the case $[a,b]_{q,h}$ we have

$$\int_a^b \psi(x)d_{q,h}x = \int_0^b \psi(x)d_{q,h}x - \int_0^a \psi(x)d_{q,h}x. \tag{15}$$

We also introduce the scalar products using (q,h)-integral on the spaces V_k in the following way

$$\langle\psi|\varphi\rangle_k := \int_a^b \overline{\psi(x)}\varphi(x)\varrho_k(x)d_{q,h}x, \tag{16}$$

where ϱ_k are weight functions. Then, we define the Hilbert spaces

$$\mathcal{H}_k = \{\psi_k \in V_k :\ \langle\psi_k|\psi_k\rangle_k < +\infty\}.$$

Let us present several basic properties which will be useful in the sequel

$$S\left((1-q)x-h\right) = q\left((1-q)x-h\right), \tag{17}$$
$$S^{-1}\left((1-q)x-h\right) = q^{-1}\left((1-q)x-h\right), \tag{18}$$
$$S^n\psi(x) = \psi\left(q^n x+[n]_q h\right), \tag{19}$$
$$S^\infty\psi(x) = \psi\left(\frac{h}{1-q}\right), \tag{20}$$

where $S^\infty\psi(x) = \lim_{n\to\infty} S^n\psi(x)$.

3. Factorization method in application to (q,h)-Hahn orthogonal polynomials

In this section we consider the class of (q,h)-Hahn polynomials $\{P_k^n\}_{n=0}^{\infty}$. We shown that they are orthogonal with respect to the scalar product (16) given by (q,h) analogues of Jackson's integral

$$\langle P_k^n, P_k^m\rangle_k = \int_a^b \overline{P_k^n(x)}P_k^m(x)\varrho_k(x)d_{q,h}x = \alpha_k^n\delta_{nm}, \tag{21}$$

where α_k^n is a constant.

At the beginning we introduce lowering $\mathbf{A}_k^- : \mathcal{H}_k \longrightarrow \mathcal{H}_{k-1}$ and raising operators $\mathbf{A}_k^+ : \mathcal{H}_{k-1} \longrightarrow \mathcal{H}_k$ (called also annihilation and creation operators)

$$\mathbf{A}_k^- = D_{q,h} = \frac{1}{(1-q)x-h}\left(1-S\right), \tag{22}$$

$$\begin{aligned}\mathbf{A}_k^+ &= -B(x)D_{q,h}S^{-1} - A_k(x)\\ &= -\frac{B(x)}{(1-q)x-h}\left(S^{-1}-1\right) - A_k(x),\end{aligned} \tag{23}$$

where the polynomials B and A_k are of the form

$$B(x) = b_2\left((1-q)x-h\right)^2 + b_1\left((1-q)x-h\right) + b_0, \tag{24}$$

$$\begin{aligned}A_k(x) = {}& q^{-2k}\left(c_1 - b_2\left(1-q^{2k}\right)\right)\left((1-q)x-h\right)\\ &+ q^{-k}\left(c_0 - b_1\left(1-q^k\right)\right),\end{aligned} \tag{25}$$

the parameters $b_2, b_1, b_0, c_0, c_1 \in \mathbb{R}$. Let us note that the parameters b_2, b_1, b_0, c_0, c_1 can depend on q and h. Then the (q,h)-Hahn ladder is given by

$$\ldots \mathcal{H}_{k-1} \underset{\mathbf{A}_k^- = D_{q,h}}{\overset{\mathbf{A}_k^+ = -BD_{q,h}S^{-1} - A_k}{\rightleftarrows}} \mathcal{H}_k \underset{\mathbf{A}_{k+1}^- = D_{q,h}}{\overset{\mathbf{A}_{k+1}^+ = -BD_{q,h}S^{-1} - A_{k+1}}{\rightleftarrows}} \mathcal{H}_{k+1} \ldots$$

We can easily check that the commutator condition (1) is fulfilled if the sequence $\{a_k\}$ is given by the recursion relation

$$a_k = q^{-1}a_{k-1} + (1-q)q^{-2k}\left(c_1 - b_2\left(1-q^{2k}\right)\right). \tag{26}$$

The explicit formula for a_k is written as

$$a_k = q^{-k}a_0 + (1-q^k)q^{-2k}c_1 + (1-q^k)\left(q^{-k+1} - q^{-2k}\right)b_2. \tag{27}$$

From equation (6) and $\mathbf{A}_k^-$ defined in (22), we find that the ground state is a constant function which we normalize to one

$$\psi_k^0 \equiv 1. \tag{28}$$

Moreover we observe that (7) implies that the functions

$$P_k^n = \mathbf{A}_k^+\mathbf{A}_{k-1}^+ \cdots \mathbf{A}_{k-n+1}^+ 1 \tag{29}$$

belong to the space V_k. The computation of the first three polynomials gives:

$$P_1^0(x) = 1, \tag{30}$$

$$P_k^1(x) = -A_k(x), \tag{31}$$

$$P_k^2(x) = A_k(x)A_{k-1}(x) + (1-q)q^{-2k+1}(c_1 - b_2(1-q^{2k-2}))B(x). \tag{32}$$

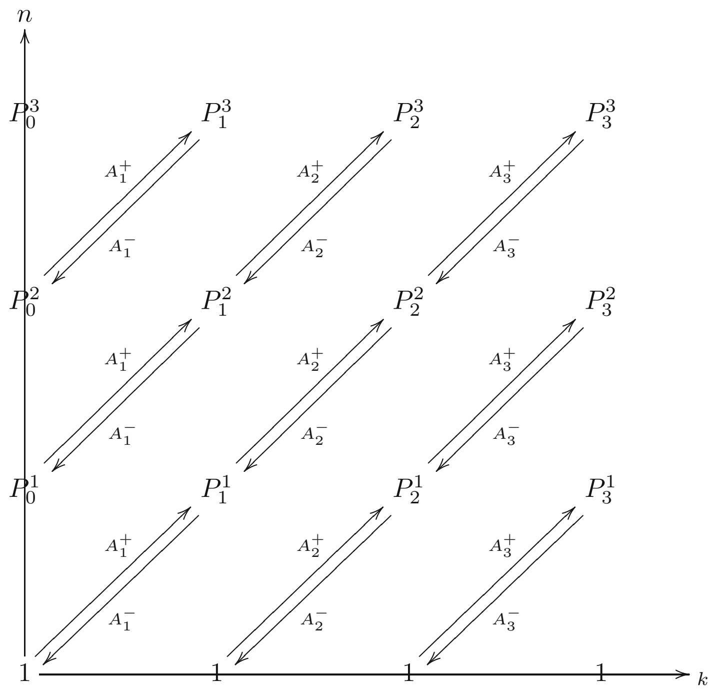
The (q,h)-Hahn orthogonal polynomials satisfy the following the (q,h)-Hahn equation

$$\mathbf{A}_k^+\mathbf{A}_k^- P_k^n = \lambda_k^n P_k^n, \tag{33}$$

where

$$\lambda_k^n = q^{-n} a_{k-n} - a_k = (q^n - 1)\, q^{-2k} c_1 + (1 - q^n) \left(q^{-2k} - q^{1-n}\right) b_2. \tag{34}$$

Finally, we present the action of the operators $\mathbf{A}_k^+$ and $\mathbf{A}_k^-$ in the diagram



Note that if $b_2 = \dfrac{\tilde{b_2}}{(1-q)^2}$, $b_1 = \dfrac{\tilde{b_1}}{1-q}$, $c_1 = \dfrac{\tilde{c_1}}{1-q}$ and we put $h = 0$, then we get q-Hahn orthogonal polynomials, see [5, 7, 9].

Now, we prove that the raising operators (23) are adjoint to lowering operators (22)

$$\mathbf{A}_k^+ = \left(\mathbf{A}_k^-\right)^* \tag{35}$$

with respect to the scalar product given by (16), where the weight function ϱ_k is defined by the (q, h)-Pearson equation

$$D_{q,h}\left(B(x)\varrho_k(x)\right) = A_k(x)\varrho_k(x) \tag{36}$$

and satisfies the recurrence relation

$$\varrho_k(x) = S\left(B(x)\varrho_{k+1}(x)\right). \tag{37}$$

Furthermore, from relations (36) and (37) we arrive at

$$\frac{\varrho_{k-1}(x)}{\varrho_k(x)} = B(x) - ((1-q)x-h)\, A_k(x), \tag{38}$$

$$\frac{\varrho_{k-1}(x)}{\varrho_k(qx+h)} = B(qx+h). \tag{39}$$

Moreover, we assume that the integration interval $[a,b]_{q,h}$ is obtained from the conditions

$$B(a)\varrho(a) = 0, \tag{40}$$

$$B(b)\varrho(b) = 0. \tag{41}$$

From the definition of the adjoint operator we calculate

$$\begin{aligned}
&\langle \left(\mathbf{A}_k^-\right)^* \psi_{k-1}|\varphi_k\rangle_k = \langle \psi_{k-1}|\mathbf{A}_k^- \varphi_k\rangle_{k-1} \\
&\quad = \int_a^b \psi_{k-1}(x)\left(D_{q,h}\varphi_k(x)\right)\varrho_{k-1}(x) d_{q,h}x \\
&\quad = \int_a^b \frac{\psi_{k-1}(x)}{((1-q)x-h)}\,(1-S)\,\varphi_k(x)\varrho_{k-1}(x) d_{q,h}x \\
&\quad = \int_a^b \frac{\psi_{k-1}(x)}{((1-q)x-h)}\frac{\varrho_{k-1}(x)}{\varrho_k(x)}\varphi_k(x)\varrho_k(x) d_{q,h}x \\
&\qquad - \sum_{n=0}^{\infty} \psi_{k-1}(q^n b + [n]_q h)\varphi_k(q^{n+1}b + [n+1]_q h)\varrho_{k-1}(q^n b + [n]_q h) \\
&\qquad + \sum_{n=0}^{\infty} \psi_{k-1}(q^n a + [n]_q h)\varphi_k(q^{n+1}a + [n+1]_q h)\varrho_{k-1}(q^n a + [n]_q h) \\
&\quad = \int_a^b \frac{\psi_{k-1}(x)}{((1-q)x-h)}\frac{\varrho_{k-1}(x)}{\varrho_k(x)}\varphi_k(x)\varrho_k(x) d_{q,h}x \\
&\qquad - \sum_{n=1}^{\infty} \psi_{k-1}(q^{n-1} b + [n-1]_q h)\varphi_k(q^{n}b + [n]_q h)\varrho_{k-1}(q^{n-1} b + [n-1]_q h) \\
&\qquad + \sum_{n=1}^{\infty} \psi_{k-1}(q^{n-1} a + [n-1]_q h)\varphi_k(q^{n}a + [n]_q h)\varrho_{k-1}(q^{n-1} a + [n-1]_q h) \\
&\quad = \int_a^b \frac{\psi_{k-1}(x)}{((1-q)x-h)}\frac{\varrho_{k-1}(x)}{\varrho_k(x)}\varphi_k(x)\varrho_k(x) d_{q,h}x \\
&\qquad - \int_a^b \frac{\varrho_{k-1}(q^{-1}x - q^{-1}h)}{\varrho_k(x)\,((1-q)x-h)}\left(S^{-1}\psi_{k-1}(x)\right)\varphi_k(x)\varrho_k(x) d_{q,h}x \\
&\quad = \left\langle \left(-\frac{B(x)}{(1-q)x-h}\left(S^{-1}-1\right) - A_k(x)\right)\psi_{k-1}|\varphi_k \right\rangle_k,
\end{aligned} \tag{42}$$

see also [8]. Note that we used above the properties (38), (39), (40) and (41). Formula (42) immediately implies that the operators are mutually adjoint $\mathbf{A}_k^+ = \left(\mathbf{A}_k^-\right)^*$.

Finally, we show that the polynomials (29) are orthogonal. If we assume $n \geq m$ then

$$\begin{aligned}
\langle P_k^n | P_k^m \rangle_k &= \left\langle \mathbf{A}_k^+ \mathbf{A}_{k-1}^+ \cdots \mathbf{A}_{k-n+1}^+ 1 | \mathbf{A}_k^+ \mathbf{A}_{k-1}^+ \cdots \mathbf{A}_{k-m+1}^+ 1 \right\rangle_k \\
&= \left(q^{-m} a_{k-m} + a_k\right) \left\langle \mathbf{A}_{k-1}^+ \mathbf{A}_{k-2}^+ \cdots \mathbf{A}_{k-n+1}^+ 1 | \mathbf{A}_{k-1}^+ \mathbf{A}_{k-2}^+ \cdots \mathbf{A}_{k-m+1}^+ 1 \right\rangle_{k-1} \\
&= \cdots = \sum_{i=1}^{m} \left(q^{-m+i-1} a_{k-m} + a_{k-i+1}\right) \delta_{nm},
\end{aligned} \tag{43}$$

where we used (1), (6), (33) and (34).

4. Example

In this section we consider the case when

$$b_2 = b_1 = c_0 = 0 \quad \text{and} \quad b_0 = 1. \tag{44}$$

The polynomials (24) and (25) become

$$B(x) = 1, \tag{45}$$

$$A_k(x) = q^{-2k} c_1 \left((1-q)x - h\right). \tag{46}$$

The lowering and raising operators have the form

$$\mathbf{A}_k^- = \frac{1}{(1-q)x - h} (1 - S), \tag{47}$$

$$\mathbf{A}_k^+ = -\frac{1}{(1-q)x - h} \left(S^{-1} - 1\right) - q^{-2k} c_1 \left((1-q)x - h\right). \tag{48}$$

The (q, h)-Hahn equation in this case is the equation for the discrete (q, h)-generalization of the Hermite II polynomials

$$\begin{aligned}
&\left(1 - ((1-q)x - h)^2 q^{-2k} c_1\right) P_k^n(qx + h) \\
&\quad - \left(1 + q - ((1-q)x - h)^2 q^{-2k} c_1\right) P_k^n(x) + q P_k^n(q^{-1}x - q^{-1}h) \\
&= (1 - q^n) \left((1-q)x - h\right)^2 q^{-2k} c_1 P_k^n(x),
\end{aligned} \tag{49}$$

because in the limit $q \to 1$ we obtain the q-Hermite II orthogonal polynomials [5, 7]. These polynomials are orthogonal with respect to the scalar product, where the weight function is specified by the expression

$$\varrho_k(x) = \prod_{n=0}^{\infty} \frac{1}{1 - ((1-q)x - h)^2 q^{2(n-k)} c_1}. \tag{50}$$

The computation of the first four polynomials gives:

$$P_k^0(x) = 1, \tag{51}$$

$$P_k^1(x) = -q^{-2k} c_1 \left((1-q)x - h\right), \tag{52}$$

$$P_k^2(x) = q^{-2k+1} c_1 \left(1 - q + q^{-2k+1} c_1 \left((1-q)x - h\right)^2\right), \tag{53}$$

$$\begin{aligned} P_k^3(x) = {} & -q^{-4k+1} c_1^2 \left((1-q)x - h\right) \\ & \times \left(1 - q + q^{-2k+3} c_1 \left((1-q)x - h\right)^2\right). \end{aligned} \tag{54}$$

The operators satisfy the following commutator relations

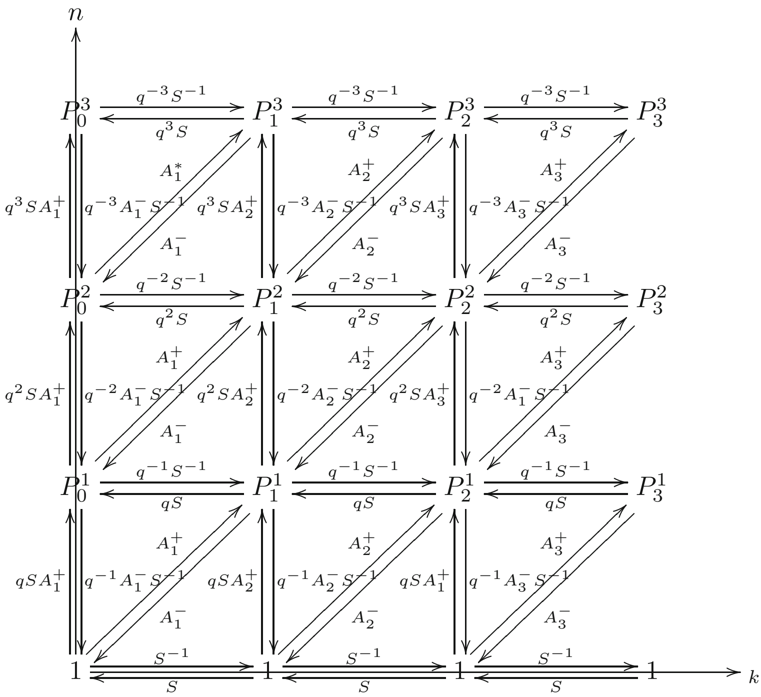
$$q\mathbf{A}_k^+ S^{-1} = S^{-1}\mathbf{A}_{k-1}^+, \tag{55}$$

$$\mathbf{A}_k^+ S = qS\mathbf{A}_{k+1}^+, \tag{56}$$

$$q\mathbf{A}_k^- S^{-1} = S^{-1}\mathbf{A}_{k-1}^-, \tag{57}$$

$$\mathbf{A}_k^- S = qS\mathbf{A}_{k+1}^-. \tag{58}$$

We present the above in the diagram



5. Conclusions

Let us note that if we introduce the operator

$$(\mathcal{S}\psi)(x) = \psi\left(x + \frac{h}{1-q}\right), \tag{59}$$

then we have the following relations between (q, h)-analysis and q-analysis

$$D_{q,h} = \mathcal{S}^{-1} \circ D_{q,0} \circ \mathcal{S}, \qquad \int d_{q,h}x = \mathcal{S}^{-1} \circ \int d_{q,0}x \circ \mathcal{S}. \tag{60}$$

These identities reduce (q, h)-calculations to q-calculations, see [19]. However, the motivation for the use of the (q, h)-calculations is the possibility of a simple transition between difference calculations and q-difference calculations.

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Examples of Hamiltonian Systems on the Space of Deformed Skew-symmetric Matrices

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Abstract. We consider a dual space to a Lie algebra of deformed skew-symmetric matrices equipped with two compatible Poisson brackets: the Lie–Poisson bracket and the frozen bracket. We obtain a bi-Hamiltonian integrable system. As an example we present a case of a system on a two-dimensional quadric which can be seen as a generalization of the Neumann system.

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1. Introduction

The aim of this paper is to present a particular example of the integrable system on the Lie–Poisson space related to the space of deformed skew-symmetric $n \times n$ matrices $\mathcal{A}_{a_1,\ldots,a_{n-1}}$.

We start by briefly recalling basic definitions and facts about spaces of deformed symmetric $n \times n$ matrices $\mathcal{S}_{a_1,\ldots,a_{n-1}}$ and skew-symmetric $n \times n$ matrices $\mathcal{A}_{a_1,\ldots,a_{n-1}}$. The deformation is described by a sequence of parameters $a_1, \ldots, a_{n-1}$. By a certain choice of parameters we can get Lie algebras $\mathfrak{so}(n)$, $\mathfrak{so}(p,q)$, $\mathfrak{e}(n-1)$ or the Lie algebra of the Galilean group.

These deformed sets were introduced in paper [11] where it was shown that $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ is a Lie algebra with respect to the standard matrix commutator. Various integrable systems in that case were studied in a series of papers [3–6]. In paper [2] another Lie structure on $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ was investigated, see (4), leading to generalizations of the Euler rigid body and the Clebsh system.

Interesting cases are related to situations when some parameters are put equal to zero or change signs. In this way we can consider Lie algebra contractions, see [7]. For example in [9, 14] the one-parameter family of Lie algebras containing

$\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(3)$ and $\mathfrak{e}(2)$ is presented. The Lie algebra $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ can be seen as a multi-parameter generalization of that situation.

The next section of the paper is also based on the paper [2] and is devoted to the investigation of the Lie–Poisson structure on the dual space L_+ to $\mathcal{A}_{a_1,\ldots,a_{n-1}}$. Moreover we consider a frozen Poisson bracket on L_+, see formula (21), which is compatible with the Lie–Poisson bracket. It allows us to use the Magri method to obtain a family of functions in involution, see [1, 8], – Casimirs for one of those brackets are in involution with respect to the other bracket. Thus we are able to obtain a Hamiltonian system with a rich family of integrals in motion.

In the last section we discuss an example for $n = 5$. We choose a certain fixed element ρ_0 and diagonal matrix $S \in \mathcal{S}_{a_1,\ldots,a_{n-1}}$. It turns out that a generic symplectic leaf for a frozen Poisson bracket can be identified with $\mathbb{R}^6$. We introduce convenient coordinates on L_+ and obtain an integrable system on the tangent bundle to a quadric surface. If this quadric is a sphere we can interpret this system as a version of the Neumann system, see [10, 12, 13].

2. Lie algebra $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ of deformed skew-symmetric matrices

Let us recall basic definitions and properties of objects considered throughout the paper. For more details, see [11] or [2].

Let us choose a sequence of real numbers $a_1, \ldots, a_{n-1}$ and define the following sets of $n \times n$ real matrices

$$\begin{aligned}\mathcal{A}_{a_1,\ldots,a_{n-1}} := \{X =&(x_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \\ & x_{ij} = -a_i \cdots a_{j-1} x_{ji} \text{ for } j > i, x_{ii} = 0\}\end{aligned} \tag{1}$$

and

$$\mathcal{S}_{a_1,\ldots,a_{n-1}} := \{S = (s_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid s_{ij} = a_i \cdots a_{j-1} s_{ji} \text{ for } j > i\}. \tag{2}$$

These sets consist of skew-symmetric and symmetric matrices deformed in such a way that elements above diagonal are multiplied by terms $a_i \cdots a_{j-1}$. For example an element of $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ looks like this:

$$\begin{pmatrix} 0 & -a_1x_{21} & -a_1a_2x_{31} & -a_1a_2a_3x_{41} & \cdots & -a_1a_2\cdots a_{n-1}x_{n1} \\ x_{21} & 0 & -a_2x_{32} & -a_2a_3x_{42} & \cdots & -a_2a_3\cdots a_{n-1}x_{n2} \\ x_{31} & x_{32} & 0 & -a_3x_{43} & \cdots & -a_3a_4\cdots a_{n-1}x_{n3} \\ x_{41} & x_{42} & x_{43} & 0 & \cdots & -a_4a_5\cdots a_{n-1}x_{n4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \cdots & 0 \end{pmatrix}. \tag{3}$$

These sets were introduced in Ref. [11] in the context of operators acting on a Hilbert space. The set $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ was endowed with the Lie algebra structure with the normal commutator. Subsequently in paper [2] it was shown that $\mathcal{A}_{a_1,\ldots,a_{n-1}}$

possesses another Lie algebra structure with respect to the Lie bracket given as a deformed commutator

$$[X, Y]_S := XSY - YSX \tag{4}$$

for $X, Y \in \mathcal{A}_{a_1,\ldots,a_{n-1}}$ and a fixed $S \in \mathcal{S}_{a_1,\ldots,a_{n-1}}$. In the generic case (i.e., for all parameters $a_i \neq 0$) we obtain a Lie algebra isomorphic to $\mathfrak{so}(n)$ or $\mathfrak{so}(p,q)$. However when we admit the case that one or more parameters assume value zero, we get different Lie algebras which may be considered as Wigner–Inönü contractions, see, e.g., [7].

To facilitate the work with elements of $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ and $\mathcal{S}_{a_1,\ldots,a_{n-1}}$ the following notation was introduced. Let P_l denote the projector onto $l+1,\ldots,n$ vectors of the canonical basis of $\mathbb{R}^n$:

$$P_l = \sum_{i=l+1}^{n} |i\rangle\langle i| \tag{5}$$

and by δ_l we will denote the matrix

$$\delta_l := |l+1\rangle\langle l+1| + \sum_{i=l+2}^{n} a_{l+1}a_{l+2}\cdots a_{i-1}|i\rangle\langle i|, \tag{6}$$

which has the following block form

$$\delta_l = \left(\begin{array}{c|ccccc} \mathbb{O} & & & \mathbb{O} & & \\ \hline & 1 & 0 & 0 & \ldots & 0 \\ & 0 & a_{l+1} & 0 & \ldots & 0 \\ \mathbb{O} & 0 & 0 & a_{l+1}a_{l+2} & \ldots & 0 \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & 0 & \ldots & a_{l+1}a_{l+2}\cdots a_{n-1} \end{array}\right). \tag{7}$$

Moreover let us denote by $k_1,\ldots,k_N$ the indices of all parameters a_i equal to zero, namely

$$a_{k_1} = \cdots = a_{k_N} = 0, \tag{8}$$

where the sequence $k_1,\ldots,k_N$ is chosen to be strictly increasing. In order to be able to write formulas in a consistent way we put additionally $k_0 = 0$ and $k_{N+1} = n$ (thus we have $P_0 = P_{k_0} = \mathbb{1}$ and $\delta_{k_{N+1}} = \delta_n = P_{k_{N+1}} = P_n = 0$).

The following obvious relations hold:

$$\delta_{k_i}P_{k_j} = 0, \qquad P_{k_i}P_{k_j} = P_{k_j} \text{ for } j > i \tag{9}$$

and

$$\delta_{k_i}\delta_{k_j} = 0 \tag{10}$$

for $i \neq j$. Moreover all P_{k_i} and δ_{k_j} commute and the matrices δ_{k_i} are invertible when restricted to the range of the projectors $P_{k_i} - P_{k_{i+1}}$. We will denote that pseudoinverse elements by $\iota(\delta_{k_i})$:

$$\iota(\delta_{k_i})\delta_{k_i} = \delta_{k_i}\iota(\delta_{k_i}) = P_{k_i} - P_{k_{i+1}}. \tag{11}$$

It can be shown that a matrix X is an element of $\mathcal{A}_{a_1,\dots,a_{n-1}}$ iff the following system of equations holds:

$$\delta_{k_i} X P_{k_i} + P_{k_i} X^T \delta_{k_i} = 0, \qquad i = 0,1,2,\dots,N \tag{12}$$

and a similar condition can be given for a matrix S to belong to $\mathcal{S}_{a_1,\dots,a_{n-1}}$:

$$\delta_{k_i} S P_{k_i} - P_{k_i} S^T \delta_{k_i} = 0, \qquad i = 0,1,2,\dots,N. \tag{13}$$

These conditions imply that the elements of $\mathcal{A}_{a_1,\dots,a_{n-1}}$ and $\mathcal{S}_{a_1,\dots,a_{n-1}}$ have the following "stairs-diagonal" block form:

$$X = \left(\begin{array}{c|c|c|c} X_0 & \mathbb{O} & \cdots & \mathbb{O} \\ \hline * & X_1 & \cdots & \mathbb{O} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline * & * & \cdots & X_N \end{array}\right), \tag{14}$$

$$S = \left(\begin{array}{c|c|c|c} S_0 & \mathbb{O} & \cdots & \mathbb{O} \\ \hline * & S_1 & \cdots & \mathbb{O} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline * & * & \cdots & S_N \end{array}\right), \tag{15}$$

where $X_i \in \mathcal{A}_{a_{k_i+1},\dots,a_{k_{i+1}-1}}$, $S_i \in \mathcal{S}_{a_{k_i+1},\dots,a_{k_{i+1}-1}}$ for $i = 0,\dots,N$ and $*$ denotes arbitrary matrices of suitable sizes. Formally it can be formulated as the following proposition:

Proposition 1. *For $X \in \mathcal{A}_{a_1,\dots,a_{n-1}}$ and $S \in \mathcal{S}_{a_1,\dots,a_{n-1}}$ the following equalities are valid:*

$$(\mathbb{1} - P_{k_i}) X P_{k_i} = (\mathbb{1} - P_{k_i}) S P_{k_i} = 0, \tag{16}$$

$$P_{k_i} X P_{k_i} = X P_{k_i}, \quad P_{k_i} S P_{k_i} = S P_{k_i} \tag{17}$$

for $i = 0,\dots,N$.

For the proofs of aforementioned facts we refer the reader to [2]. Using these propositions it is straightforward to prove that $\mathcal{A}_{a_1,\dots,a_{n-1}}$ is indeed a Lie algebra with respect to the bracket (4).

3. Integrable systems on $(\mathcal{A}_{a_1,\dots,a_{n-1}})^*$

We intend to investigate the Lie–Poisson bracket on the dual space to $\mathcal{A}_{a_1,\dots,a_{n-1}}$. In order to do it let us identify $(\mathcal{A}_{a_1,\dots,a_{n-1}})^*$ with a space of upper-triangular matrices L_+

$$L_+ := \{\rho = (\rho_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid \rho_{ij} = 0 \text{ for } i \geq j\}. \tag{18}$$

This identification is given by the following natural pairing

$$\langle \rho\,;\, X\rangle = \mathrm{Tr}(\rho X), \tag{19}$$

where $\rho \in L_+$ and $X \in \mathcal{A}_{a_1,\dots,a_{n-1}}$, see [11].

Canonical Lie–Poisson brackets on L_+ are defined by the well-known formula

$$\{f,g\}_S(\rho) = \langle \rho\ ;\ [Df(\rho), Dg(\rho)]_S\rangle, \tag{20}$$

where $f,g \in C^\infty(L_+)$, $\rho \in L_+$. The derivative $Df(\rho)$ here is considered as an element of $\mathcal{A}_{a_1,\dots,a_{n-1}}$.

One may consider $\{\cdot\,,\,\cdot\}_S$ as a family of compatible Poisson brackets on L_+ parametrized by $S \in \mathcal{S}_{a_1,\dots,a_{n-1}}$, see [2]. However here we will consider a different situation. In order to get a bi-Hamiltonian structure we will introduce a frozen Poisson bracket

$$\{f,g\}_{\rho_0}(\rho) = \langle \rho_0\ ;\ [Df(\rho), Dg(\rho)]_S\rangle, \tag{21}$$

where ρ_0 is a fixed element of L_+. It is a known fact that frozen bracket is compatible with Lie–Poisson bracket in the sense that their arbitrary linear combination is again a linear bracket.

For the purpose of this paper we will choose ρ_0 to be

$$\rho_0 = |1\rangle\langle 2| = \begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{22}$$

In this case we can write down a formula for $\{\cdot\,,\,\cdot\}_{\rho_0}$ explicitly in coordinates:

$$\begin{aligned}\{f,g\}_{\rho_0}(\rho) = &\sum_{k=3}^{n} a_2\cdots a_{k-1}s_{k2}\left(\frac{\partial f}{\partial\rho_{12}}\frac{\partial g}{\partial\rho_{2k}} - \frac{\partial g}{\partial\rho_{12}}\frac{\partial f}{\partial\rho_{2k}}\right)\\ &+\sum_{k=3}^{n} a_1\cdots a_{k-1}s_{k1}\left(\frac{\partial f}{\partial\rho_{12}}\frac{\partial g}{\partial\rho_{1k}} - \frac{\partial g}{\partial\rho_{12}}\frac{\partial f}{\partial\rho_{1k}}\right)\\ &+\sum_{k=3}^{n} a_2\cdots a_{k-1}s_{kk}\left(\frac{\partial f}{\partial\rho_{1k}}\frac{\partial g}{\partial\rho_{2k}} - \frac{\partial g}{\partial\rho_{2k}}\frac{\partial f}{\partial\rho_{1k}}\right)\\ &+\sum_{p=3}^{n-1}\sum_{k=p+1}^{n} a_2\cdots a_{k-1}s_{kp}\left(\frac{\partial f}{\partial\rho_{1k}}\frac{\partial g}{\partial\rho_{2k}} - \frac{\partial g}{\partial\rho_{2k}}\frac{\partial f}{\partial\rho_{1k}}\right.\\ &\qquad\left.+\frac{\partial f}{\partial\rho_{1p}}\frac{\partial g}{\partial\rho_{2p}} - \frac{\partial g}{\partial\rho_{2p}}\frac{\partial f}{\partial\rho_{1p}}\right).\end{aligned} \tag{23}$$

Moreover if we choose matrix S to be diagonal (see [2] for the description of isomorphisms which allow to do it without loss of generality if all parameters a_i are non-zero), the preceding formula (21) simplifies to

$$\{f,g\}_{\rho_0}(\rho) = \sum_{k=3}^{n} a_2\cdots a_{k-1}s_{kk}\left(\frac{\partial f}{\partial\rho_{1k}}\frac{\partial g}{\partial\rho_{2k}} - \frac{\partial g}{\partial\rho_{2k}}\frac{\partial f}{\partial\rho_{1k}}\right). \tag{24}$$

Note that this Poisson bracket depends only on coordinates ρ_{13}, ρ_{14}, $\dots$, ρ_{1n}, ρ_{23}, ρ_{24}, $\dots$, ρ_{2n} so the dynamics happens in fact in a $2(n-2)$-dimensional subspace. We may thus consider a Poisson inclusion

$$\mathrm{Mat}_{2\times(n-2)}(\mathbb{R}) \longrightarrow L_+. \tag{25}$$

Note that if $S = \mathbb{1}$ and $a_2 = \cdots a_{n-1} = 1$ we get a canonical Poisson bracket on $T^*\mathbb{R}^{n-2}$.

Casimirs for the Lie–Poisson bracket $\{\cdot,\cdot\}_S$ were computed in paper [2]:

$$C^l(\rho) = \frac{1}{2l}\operatorname{Tr}\left((\rho - \delta^{-1}\rho^T\delta)S^{-1}\right)^{2l}, \tag{26}$$

in the case when all parameters a_i are non-zero. Since for any choice of an element ρ_0, the Lie–Poisson bracket $\{\cdot,\cdot\}_S$ and the frozen bracket $\{\cdot,\cdot\}_{\rho_0}$ are compatible, these Casimirs are in involution with respect to the frozen bracket $\{\cdot,\cdot\}_{\rho_0}$, namely:

$$\{C^l, C^k\}_{\rho_0} = 0. \tag{27}$$

The situation $a_1 = \cdots = a_{n-1} = 1$, $S = \mathbb{1}$ was investigated in [6] and the situation $a_1 = \cdots = a_{n-1} = 1$ with arbitrary S – in [5].

4. Examples

Let us consider a case $n = 5$ with a matrix $S = \operatorname{diag}(s_1^{-1}, s_2^{-1}, s_3^{-1}, s_4^{-1}, s_5^{-1})$. In this case the frozen Poisson bracket $\{\cdot,\cdot\}_{\rho_0}$ (see (21)) can be expressed as

$$\begin{aligned}\{f,g\}_{\rho_0}(\rho) = \frac{a_2}{s_3}\left(\frac{\partial f}{\partial\rho_{13}}\frac{\partial g}{\partial\rho_{23}} - \frac{\partial f}{\partial\rho_{23}}\frac{\partial g}{\partial\rho_{13}}\right) + \frac{a_2a_3}{s_4}\left(\frac{\partial f}{\partial\rho_{14}}\frac{\partial g}{\partial\rho_{24}} - \frac{\partial f}{\partial\rho_{24}}\frac{\partial g}{\partial\rho_{14}}\right)\\ + \frac{a_2a_3a_4}{s_5}\left(\frac{\partial f}{\partial\rho_{15}}\frac{\partial g}{\partial\rho_{25}} - \frac{\partial f}{\partial\rho_{25}}\frac{\partial g}{\partial\rho_{15}}\right).\end{aligned} \tag{28}$$

Explicit formulas for the Casimirs for the Lie–Poisson bracket $\{\cdot,\cdot\}_S$ are the following:

$$\begin{aligned}C_1(\rho) = &-\frac{1}{a_1a_2a_3a_4}\big(a_2a_3a_4s_1s_2\rho_{12}^2 + a_3a_4s_1s_3\rho_{13}^2 + a_4s_1s_4\rho_{14}^2\\ &+ s_1s_5\rho_{15}^2 + a_1a_3a_4s_2s_3\rho_{23}^2 + a_1a_4s_2s_4\rho_{24}^2 + a_1s_2s_5\rho_{25}^2\\ &+ a_1a_2a_4s_3s_4\rho_{34}^2 + a_1a_2s_3s_5\rho_{35}^2 + a_1a_2a_3s_4s_5\rho_{45}^2\big),\end{aligned} \tag{29}$$

$$\begin{aligned}C_2(\rho) = \frac{1}{2}C_1^2(\rho) - \frac{1}{a_1a_2^2a_3^2a_4}\big(&s_1s_2s_4s_5(\rho_{15}\rho_{24} - \rho_{14}\rho_{25} + a_2a_3\rho_{12}\rho_{45})^2\\ &+ a_2s_1s_3s_4s_5(\rho_{15}\rho_{34} - \rho_{14}\rho_{35} + a_3\rho_{13}\rho_{45})^2\\ &+ a_3a_4s_1s_2s_3s_4(\rho_{14}\rho_{23} - \rho_{13}\rho_{24} + a_2\rho_{12}\rho_{34})^2\\ &+ a_3s_1s_2s_3s_5(\rho_{23}\rho_{15} - \rho_{13}\rho_{25} + a_2\rho_{12}\rho_{35})^2\\ &+ a_1a_2s_2s_3s_4s_5(\rho_{34}\rho_{25} - \rho_{24}\rho_{35} + a_3\rho_{23}\rho_{45})^2\big).\end{aligned} \tag{30}$$

Let us introduce the following vector notation to simplify the formulas for Casimirs:

$$\vec{p} = (\rho_{13}, \rho_{14}, \rho_{15}), \tag{31}$$

$$\vec{q} = (\rho_{23}, \rho_{24}, \rho_{25}), \tag{32}$$

$$\vec{c} = (a_3\rho_{45}, -\rho_{35}, \rho_{34}), \tag{33}$$

$$V = \mathrm{diag}(s_4 s_5, a_3 s_3 s_5, a_3 a_4 s_3 s_4), \tag{34}$$

$$W = \mathrm{diag}(a_3 a_4 s_3, a_4 s_4, s_5) = a_3 a_4 s_3 s_4 s_5 V^{-1}. \tag{35}$$

From the formula for the frozen Poisson bracket (28) it is obvious that components of $\vec{c}$ and ρ_{12} are Casimirs. Thus we can now consider the frozen Poisson bracket $\{\cdot,\cdot\}_{\rho_0}$ as a Poisson bracket on $\mathbb{R}^6 \ni (\vec{p},\vec{q})$. In a generic case it is a symplectic leaf for that Poisson structure.

In the new variables the Casimirs C_1 and C_2 assume the form (up to trivial transformations)

$$\tilde{C}_1(\vec{p},\vec{q}) = a_2 a_3 a_4 s_1 s_2 \rho_{12}^2 + \frac{a_1 a_2}{a_3}(V\vec{c})\cdot\vec{c} + a_1 s_2 (W\vec{q})\cdot\vec{q} + s_1(W\vec{p})\cdot\vec{p}, \tag{36}$$

$$\begin{aligned}\tilde{C}_2(\vec{p},\vec{q}) &= a_2 s_1 s_3 s_4 s_5 (\vec{p}\cdot\vec{c})^2 + a_1 a_2 s_2 s_3 s_4 s_5 (\vec{q}\cdot\vec{c})^2 \\ &\quad + s_1 s_2 \big(V(\vec{q}\times\vec{p}) + a_2\rho_{12} V\vec{c}\big)\cdot\big(\vec{q}\times\vec{p} + a_2\rho_{12}\vec{c}\big).\end{aligned} \tag{37}$$

Note that for a Hamiltonian system with respect to the frozen bracket $\{\cdot,\cdot\}_{\rho_0}$ with a Hamiltonian either $\tilde{C}_1$ or $\tilde{C}_2$ we need one additional functionally independent integral of motion to integrate it. This additional integral can be chosen as

$$D_1(\vec{p},\vec{q}) = (V\vec{c})\cdot(\vec{q}\times\vec{p}). \tag{38}$$

It can be seen as a generalization of integral of motion considered in papers [6] and [5].

Combining Casimirs $\tilde{C}_1$, $\tilde{C}_2$ and the integral of motion D_1 we can obtain a simpler family of functionally independent functions in involution on $\mathbb{R}^6$:

$$h_1(\vec{p},\vec{q}) = \frac{1}{2}\big(a_1 s_2 (W\vec{q})\cdot\vec{q} + s_1 (W\vec{p})\cdot\vec{p}\big), \tag{39}$$

$$\begin{aligned}h_2(\vec{p},\vec{q}) = \frac{1}{2}\big(&a_2 s_1 s_3 s_4 (\vec{p}\cdot\vec{c})^2 + a_1 a_2 s_2 s_3 s_4 s_5 (\vec{q}\cdot\vec{c})^2 \\ &+ s_1 s_2 (V(\vec{q}\times\vec{p}))\cdot(\vec{q}\times\vec{p})\big)\end{aligned} \tag{40}$$

and D_1.

Let us put $a_1 = 0$ and take as a Hamiltonian the function h_2

$$h_2(\vec{p},\vec{q}) = \frac{1}{2}\left(a_2 s_1 s_3 s_4 (\vec{p}\cdot\vec{c})^2 + s_1 s_2 (V(\vec{q}\times\vec{p}))\cdot(\vec{q}\times\vec{p})\right). \tag{41}$$

In this case Hamilton's equations with respect to the frozen Poisson bracket $\{\cdot,\cdot\}_{\rho_0}$ assume the following form

$$\begin{cases}\dot{\vec{p}} = -a_2 a_3 s_1 s_2 (W\vec{p})\times(\vec{q}\times\vec{p}) \\ \dot{\vec{q}} = a_2^2 s_1 (\vec{p}\cdot\vec{c}) V\vec{c} - a_2 a_3 s_1 s_2 (W\vec{q})\times(\vec{q}\times\vec{p})\end{cases}. \tag{42}$$

We can reduce this system of equations by restricting to the level set $h_1^{-1}(s_1/2)$. By direct calculation we get $\vec{p}\cdot W\vec{p} = 1$. By introducing new coordinates we can consider this system as a system on the tangent bundle to a quadric surface $TQ \ni (\vec{x},\vec{y})$

$$\vec{x}\cdot W\vec{x} = 1, \qquad \vec{x}\cdot W\vec{y} = 0, \tag{43}$$

where the new coordinates are defined as follows

$$\vec{x} = \vec{p}, \tag{44}$$

$$\vec{y} = -a_2a_3s_1s_2(\vec{q} - (\vec{p}\cdot W\vec{q})\vec{p}). \tag{45}$$

In the new coordinates using triple vector product expansion

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a} \tag{46}$$

Hamilton's equations (42) can be rewritten as

$$\begin{cases} \dot{\vec{x}} = \vec{y} \\ \dot{\vec{y}} = -a_2^3a_3s_1^2s_2(\vec{x}\cdot\vec{c})V\vec{c} - (\vec{y}\cdot W\vec{y})\vec{x} + a_2^3a_3s_1^2s_2(\vec{x}\cdot\vec{c})(\vec{x}\cdot V\vec{c})\vec{x} \end{cases}. \tag{47}$$

In the case when $W = \mathbb{1}$ (or at least it is positive definite) we obtain a system on the sphere with a quadratic Hamiltonian. It can be viewed as a version of a Neumann system, see [10, 12, 13]. In the general case we get a system on a general two-dimensional quadric, e.g., on a one- or two-sheeted hyperboloid.

5. Conclusions

We have shown that the Magri method applied to the Lie algebra $\mathcal{A}_{a_1,\ldots,a_{n-1}}$ leads to interesting integrable systems. In this paper using the frozen bracket approach we were able to obtain a version of a Neumann system, and in paper [2] we have shown how to get Clebsh and Euler equations in a similar setting. We have restricted our attention to the low-dimensional $n = 5$ case. It is possible also to obtain similar results in higher dimensions using methods analogous to the ones used in the paper [6].

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Matrix Beta-integrals: An Overview

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Abstract. First examples of matrix beta-integrals were discovered in 1930–50s by Siegel and Hua and in the 60s Gindikin obtained multi-parametric series of such integrals. We discuss beta-integrals related to the symmetric spaces, their interpolation with respect to the dimension of a ground field, and adelic analogs; also we discuss beta-integrals related to flag spaces.

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Table of Contents

1. Introduction. The Euler and Selberg integrals

1.1. Euler beta-function

Recall the standard formulas for the Euler beta-function:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{(Euler)} \tag{1}$$

$$\int_{\mathbb{R}} \frac{dx}{(1+ix)^{\mu}(1-ix)^{\nu}} = \frac{2^{2-\mu-\nu}\pi\Gamma(\mu+\nu-1)}{\Gamma(\mu)\Gamma(\nu)} \qquad \text{(Cauchy)} \tag{2}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\sigma}} = \frac{\Gamma(\alpha)\Gamma(\sigma-\alpha)}{\Gamma(\sigma)} \tag{3}$$

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$$\int_0^\pi (\sin t)^\mu e^{i\nu t} dt = \frac{\pi}{2^\mu} \frac{\Gamma(1+\mu)}{\Gamma(1+\frac{\mu+\nu}{2})\Gamma(1+\frac{\mu-\nu}{2})} e^{i\pi\nu/2} \qquad \text{(Lobachevsky)} \qquad (4)$$

The integral (3) is obtained from (1) by the substitution $x = t/(1+t)$. Replacing the segment $[0,1]$ in (1) by the circle $|x| = 1$, after simple manipulations we get (4). Considering the stereographic projection of the circle to the line, we come to (2).

1.2. Beta-integrals

'Beta-integral' is an informal term for integrals of the type

$$\int \Bigl(\text{Product}\Bigr) = \text{Product of Gamma-functions.} \qquad (5)$$

First, we present two nice examples. The De Branges [1]–Wilson integral (1972, 1980) is given by

$$\frac{1}{2\pi} \int_{\mathbb{R}} \Bigl| \frac{\prod_{j=1}^4 \Gamma(a_j + ix)}{\Gamma(2ix)} \Bigr|^2 dx = \frac{\prod_{1\leqslant k<l\leqslant 4} \Gamma(a_k + a_l)}{\Gamma(a_1 + a_2 + a_3 + a_4)}.$$

Recall that the integrand is a weight function for the Wilson orthogonal polynomials, which occupy the highest level of the Askey hierarchy [2] of hypergeometric orthogonal polynomials.

The second example is the Selberg integral, [3] (1944),

$$\int_0^1 \cdots \int_0^1 \prod_{j=1}^n t_j^{\alpha-1}(1-t_j)^{\beta-1} \prod_{1\leqslant k<l\leqslant n} |t_k - t_l|^{2\gamma}\, dt_1 \cdots dt_n$$
$$= \prod_{j=1}^n \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)\,\Gamma\bigl(1+j\gamma\bigr)}{\Gamma\bigl(\alpha+\beta+(n+j-2)\gamma\bigr)\,\Gamma(1+\gamma)}. \qquad (6)$$

As the Euler integral, the Selberg integral has several versions, for instance

$$\int_0^\infty \cdots \int_0^\infty \prod_{j=1}^n x_j^{\alpha-1}(1+x_j)^{-\alpha-\beta-2\gamma(n-1)} \prod_{1\leqslant k<l\leqslant n} |x_k - x_l|^{2\gamma}\, dx_1 \cdots dx_n$$
$$= \prod_{j=1}^n \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)\,\Gamma\bigl(1+j\gamma\bigr)}{\Gamma\bigl(\alpha+\beta+(n+j-2)\gamma\bigr)\,\Gamma(1+\gamma)} \qquad (7)$$

$$\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^n (1-ix_k)^{-\alpha}(1+ix_k)^{-\beta} \prod_{1\leqslant k<l\leqslant n} |x_k - x_l|^{2\gamma}\, dx_1 \cdots dx_n$$
$$= 2^{-(\alpha+\beta)n+\gamma n(n-1)+n} \prod_{j=1}^n \frac{\Gamma\bigl(\alpha+\beta-(n+j-2)\gamma-1\bigr)\Gamma(1+j\gamma)}{\Gamma\bigl(\alpha-(j-1)\gamma)\bigr)\Gamma\bigl(\beta-(j-1)\gamma)\bigr)\Gamma(1+\gamma)} \qquad (8)$$

There exists a large family of beta-integrals (5), including one-dimensional integrals (see an overview of Askey [4]), multi-dimensional integrals, q-analogs, elliptic analogs; some collection of references is [2, 5–8, 10].

Below we discuss matrix analogs of integrals (1)–(4), (6)–(8).

1.3. Notation

- $\mathbb{K}$ denotes $\mathbb{R}$, $\mathbb{C}$, or quaternions $\mathbb{H}$, $\mathfrak{d} := \dim \mathbb{K}$.
- $[X]_p$ is the left upper corner of a matrix X of size $p \times p$;
- X^*, X^t are the adjoint matrix and the transposed matrix;
- $X > 0$ means that a matrix X is self-adjoint and *strictly* positive definite, $X > Y$ means that $X - Y > 0$;
- $\|X\|$ denotes the *norm of a matrix*, precisely the norm of the corresponding linear operator in the standard Euclidean space.
- $\mathrm{Mat}_{p,q}(\mathbb{K})$ is the space of all matrices of size $p \times q$ over $\mathbb{K}$;
- $\mathrm{Herm}_n(\mathbb{K})$ is the space of all Hermitian matrices ($X = X^*$) of size n;
- $\mathrm{Symm}_n(\mathbb{K})$ is the space of all symmetric matrices ($X = X^t$) of size n.

The Lebesgue measure on such spaces is normalized in the most simple way. For instance, for $\mathrm{Mat}_{p,q}(\mathbb{C})$, we write

$$dZ := \prod\nolimits_{1\leqslant k\leqslant p,\, 1\leqslant l\leqslant q} d\Re z_{kl}\, d\Im z_{kl}.$$

2. The Hua integrals

2.1. The Hua integrals

The famous book [11] *Harmonic analysis of functions of several complex variables in classical domains* by Hua, 1958, contains calculations of a family of matrix integrals. We present two examples.

Consider the space $\mathrm{B}_{m,n}$ of complex $m \times n$ matrices Z with $\|Z\| < 1$. The following identity holds

$$\int_{ZZ^*<1} \det(1 - ZZ^*)^\lambda\, dZ = \frac{\prod_{j=1}^n \Gamma(\lambda + j) \prod_{j=1}^m \Gamma(\lambda + j)}{\prod_{j=1}^{n+m} \Gamma(\lambda + j)} \pi^{nm}. \tag{9}$$

Next, consider the space $\mathrm{Symm}_n(\mathbb{R})$ of all real symmetric matrices of size n. The following identity holds

$$\int_{\mathrm{Symm}_n(\mathbb{R})} \frac{dT}{\det(1 + T^2)^\alpha} = \pi^{\frac{n(n+1)}{4}} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \prod_{j=1}^{n-1} \frac{\Gamma\big(2\alpha - (n+j)/2\big)}{\Gamma(2\alpha - j)}. \tag{10}$$

2.2. Comments: spaces and integrands

We can consider the following 10 series of *matrix spaces*

- $p \times q$ matrices over $\mathbb{R}$;
- symmetric $n \times n$ matrices ($X = X^t$) over $\mathbb{R}$;
- skew-symmetric $n \times n$ matrices ($X = -X^t$) over $\mathbb{R}$;
- $p \times q$ matrices over $\mathbb{C}$;
- symmetric $n \times n$ matrices over $\mathbb{C}$;
- skew-symmetric $n \times n$ matrices over $\mathbb{C}$;
- Hermitian $n \times n$ matrices ($X = X^*$) over $\mathbb{C}$;
- $p \times q$ matrices over $\mathbb{H}$;

- Hermitian $n \times n$ matrices ($X = X^*$) over $\mathbb{H}$;
- anti-Hermitian $n \times n$ matrices ($X = -X^*$) over $\mathbb{H}$.

For any space in this list, we consider a *matrix ball* $XX^* < 1$.

Integrals for all 'matrix spaces' and all 'matrix balls' [1]

$$\int \det(1 + XX^*)^{-\alpha} dX; \tag{11}$$

$$\int_{XX^*<1} \det(1 - XX^*)^{\gamma} dX \tag{12}$$

are long products of gamma-functions as (9)–(10). Actually, Hua evaluated 1/3 of these 20 integrals. Apparently, there is no text where all these integrals are evaluated (and a reason, which does not justify this, is explained in the next subsection).

The domain of integration $\mathrm{B}_{m,n} \subset \mathbb{C}^{nm}$ in (9), i.e., the matrix ball $\|Z\| < 1$, is a well-known object in differential geometry, representation theory, and complex analysis, since it is an Hermitian symmetric space[2], $\mathrm{B}_{p,q} = \mathrm{U}_{p,q} \big/ (\mathrm{U}_p \times \mathrm{U}_q)$. The pseudounitary group $\mathrm{U}_{p,q}$ acts on this domain by linear-fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : Z \mapsto U := (a + Zc)^{-1}(b + Zd). \tag{13}$$

The remaining 9 series of 'matrix balls' $XX^* < 1$ are also Riemannian symmetric spaces[3]. Up to a minor inaccuracy, all Riemannian noncompact symmetric spaces admit 'matrix ball' models. The group of isometries consists of certain linear-fractional transformations (see tables of symmetric spaces in [12, Addendum D]).

The meaning of the integrand $\det(1 - ZZ^*)^{\alpha}$ is less obvious[4]. However, any mathematician who has dealt with the unit circle $|z| < 1$ could observe that the expression $(1 - z\overline{z})^{\alpha}$ quite often appears in the formulas. The same holds for $\det(1 - ZZ^*)^{\alpha}$ in the case of the matrix balls. We only point out a nice behavior of the expression under linear-fractional transformation (13):

$$\det(1 - UU^*)^{\alpha} = \det(1 - ZZ^*)^{\alpha} |\det(a + zc)|^{-2\alpha}.$$

Thus integrals (12) are integrals of some reasonable expressions over non-compact symmetric spaces.

Integrals (11) are integrals over compact symmetric spaces written in coordinates. For instance, in (10) we integrate over the space $\mathrm{Symm}_n(\mathbb{R})$. But $\mathrm{Symm}_n(\mathbb{R})$ is a chart on the real Lagrangian Grassmannian (recall that if an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric, then its graph is a Lagrangian subspace in $\mathbb{R}^n \oplus \mathbb{R}^n$, see,

---

[1] Recall the definition of the *determinant* $\det(X) = \det_{\mathbb{H}}(X)$ *of a quaternionic matrix* X. Such matrix determines a transformation $\mathbb{H}^n \to \mathbb{H}^n$ and therefore an $\mathbb{R}$-linear transformation $X_{\mathbb{R}} : \mathbb{R}^4 \to \mathbb{R}^4$. We set $\det_{\mathbb{H}}(X) := \sqrt[4]{\det(X_{\mathbb{R}})}$.

[2] The spaces $\mathrm{B}_{p,q}$ also are known as *Cartan domains* of type I.

[3] below a '*symmetric space*' means a semisimple (reductive) symmetric space.

[4] Hua Loo Keng evaluated the volumes of Cartan domains and some compact symmetric spaces and observed that the calculations survive in a greater generality.

e.g., [12, Sect. 3.1]). The Lagrangian Grassmannian is a homogeneous (symmetric) space $\mathrm{U}_n/\mathrm{O}_n$, see, e.g., [12, Sect. 3.3]. All other 'matrix spaces' defined above are open dense charts on certain compact Riemannian symmetric spaces. Up to a minor inaccuracy, all compact symmetric spaces admit such charts (see tables of symmetric spaces in [12, Addendum D]).

2.3. Integration over eigenvalues

Consider the space $\mathrm{Herm}_n(\mathbb{K})$ of all Hermitian matrices[5] over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$; equip this space with the standard Lebesgue measure. To a matrix $X \in \mathrm{Herm}_n(\mathbb{K})$, we assign the collection of its eigenvalues

$$\Lambda : \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n. \tag{14}$$

Thus we get a map $X \mapsto \Lambda$ from $\mathrm{Herm}_n(\mathbb{K})$ to the wedge (14). The distribution of eigenvalues is given by the formula

$$C_n(\mathbb{K}) \prod\nolimits_{1 \leqslant k < l \leqslant n} |\lambda_k - \lambda_l|^{\mathfrak{d}} \, d\lambda_1 \cdots d\lambda_n,$$

where $C_n(\mathbb{K})$ is a certain (explicit) constant, $\mathfrak{d} = \dim K$. This can be reformulated as follows. Let F be a function on $\mathrm{Herm}_n(\mathbb{K})$ invariant with respect to the unitary group $\mathrm{U}(n, \mathbb{K})$[6],

$$F(uXu^{-1}) = F(X), \qquad u \in \mathrm{U}(n, \mathbb{K}).$$

Such F is a function of eigenvalues, $F(X) = f(\lambda_1, \dots, \lambda_n)$. Then the following integration formula holds

$$\begin{aligned} &\int_{\mathrm{Herm}_n(\mathbb{K})} F(X)\, dX \\ &\quad = C_n(\mathbb{K}) \int_{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n} f(\lambda_1, \dots, \lambda_n) \prod_{1 \leqslant k < l \leqslant n} |\lambda_k - \lambda_l|^{\mathfrak{d}} \, d\lambda_1 \cdots d\lambda_n. \end{aligned} \tag{15}$$

The formula is a relative of the Weyl integration formula, see derivations of several formulas of this kind in [11].

In the Hua integral (10), the integrand is

$$\det(1 + T^2)^{-\alpha} = \prod\nolimits_{j=1}^{n} (1 + \lambda_j^2)^{-\alpha} = \prod\nolimits_{j=1}^{n} (1 + i\lambda_j)^{-\alpha} (1 - i\lambda_j)^{-\alpha}.$$

Applying the integration formula (15) we reduce the Hua integral (10) to a special case of the Selberg integral (8). Moreover, we get also an explicit evaluation of a more general integral

$$\int \det(1 + iT)^{\alpha} \det(1 - iT)^{\beta} \, dT.$$

---

[5] $\mathrm{Herm}_n(\mathbb{R})$ is $\mathrm{Symm}_n(\mathbb{R})$.

[6] $\mathrm{U}(n, \mathbb{R})$ is the orthogonal group $\mathrm{O}(n)$, $\mathrm{U}(n, \mathbb{C})$ is the usual unitary group $\mathrm{U}(n)$, $\mathrm{U}(n, \mathbb{H})$ is the compact symplectic group $\mathrm{Sp}(2n)$.

Next, consider the space of all complex matrices of size $m\times n$, where $m\leqslant n$. To each matrix we assign a collection of its singular values[7]

$$\mu_1\geqslant\mu_2\geqslant\dots\geqslant\mu_m\geqslant 0.$$

The distribution of singular values is given by

$$\prod\nolimits_{1\leqslant k\leqslant n}\mu_k^{2(n-m)+1}\prod\nolimits_{1\leqslant k<l\leqslant m}(\mu_k^2-\mu_l^2)^2\prod\nolimits_{1\leqslant k\leqslant n}d\mu_k.$$

The integrand in the Hua integral (9) is $\prod(1-\mu_k^2)^2$. After the substitution $x_k=\mu_k^2$, this integral also is reduced to the Selberg integral (8).

All 20 integrals (11)–(12) are reduced to the Selberg integrals in this way[8].

2.4. An application of Hua calculations: projective systems of measures

Let us return to integral (10). Represent a matrix T as a block matrix of size $(n-1)+1$,

$$T=\begin{pmatrix}S&p\\p^t&q\end{pmatrix}.$$

Consider a function f on $\mathrm{Symm}_n(\mathbb{R})$ depending only on $S=[T]_{n-1}$. Then the following identity holds

$$\begin{gathered}\int_{\mathrm{Symm}_n(\mathbb{R})}f(S)\det\left(1+\begin{pmatrix}S&p\\p^t&q\end{pmatrix}^2\right)^{-\alpha}dS\,dp\,dq\\
=2^{\frac{n-1}{2}}\pi^{\frac{n}{2}}\frac{\Gamma(2\alpha+\frac{n+1}{2})\Gamma(\alpha-\frac12)}{\Gamma(\alpha)\Gamma(2\alpha-1)}\int_{\mathrm{Symm}_{n-1}(\mathbb{R})}f(S)\det(1+S^2)^{1/2-\alpha}dS.\end{gathered}\tag{16}$$

This formula can be extracted from the original Hua calculation (the formula (16) also implies (10)).

Now fix $\alpha>-1/2$ and consider a measure $\nu_{\alpha,n}$ on Symm_n given by

$$\nu_{\alpha,n}=s_{\alpha,n}\det(1+T^2)^{-\alpha-(n+1)/2}dT,$$

where the normalizing constant $s_{\alpha,n}$ is chosen to make the total measure $=1$. Consider the chain of projections

$$\dots\longleftarrow\mathrm{Symm}_{n-1}(\mathbb{R})\longleftarrow\mathrm{Symm}_n(\mathbb{R})\longleftarrow\cdots,$$

where each map sends a matrix $X\in\mathrm{Symm}_n(\mathbb{R})$ to its left upper corner $[X]_{n-1}(\mathbb{R})$. In accordance with (16), this map sends the measure $s_{\alpha,n}$ to the measure $s_{\alpha,n-1}$. By the Kolmogorov consistency theorem (see, e.g., [13, §2.9]) there is a measure ν_α on the space $\mathrm{Symm}_\infty(\mathbb{R})$ of infinite symmetric matrices whose image under each map $X\mapsto[X]_n$ is $\nu_{\alpha,n}$.

Next, consider the group of finitary[9] orthogonal $(\infty+\infty)$ block matrices having the structure $\left(\begin{smallmatrix}a&b\\-b&a\end{smallmatrix}\right)$. This group is isomorphic to the group U_∞ of finitary

[7] *Singular values* of a matrix Z are eigenvalues of $\sqrt{ZZ^*}$.

[8] In all these cases the parameter γ in the Selberg integrals is 1/2, 1, 2. For some exceptional symmetric spaces distributions of invariants give $\gamma=4$.

[9] We say that a matrix g is *finitary*, if $g-1$ has finite number of nonzero matrix elements

unitary matrices. It acts on $\mathrm{Symm}_\infty(\mathbb{R})$ by linear-fractional transformations (13), note that this formula makes sense. It is easy to show that the measure ν_α is quasi-invariant with respect to such transformations, and there arises a problem of decomposition of the space L^2. We also can regard our limit space as the inverse limit of the chain of Lagrangian Grassmannians,

$$\cdots \longleftarrow \mathrm{U}_{n-1}/\mathrm{O}_{n-1} \longleftarrow \mathrm{U}_n/\mathrm{O}_n \longleftarrow \cdots$$

Such construction exists for any series of compact symmetric spaces and leads to an interesting harmonic analysis on the limit objects, see [14–17].

2.5. Remarks

1. The construction of inverse limits does not admit an extension to non-compact symmetric spaces (i.e., to matrix balls). Of course, the chain of projections of *sets*
$$\cdots \longleftarrow \mathrm{B}_{p,q} \longleftarrow \mathrm{B}_{p+1,q+1} \longleftarrow \mathrm{B}_{p+2,q+2} \longleftarrow \cdots$$
is well defined. We can consider normalized probabilistic measures
$$s'_{\alpha,p,q,k} \det(1 - ZZ^*)^{\alpha-2k}$$
on $\mathrm{B}_{p+k,q+k}$. However, for sufficiently large k the integral
$$\int_{\mathrm{B}_{p+k,q+k}} \det(1 - ZZ^*)^{\alpha-2k}\, dZ$$
is divergent.
2. Projective limits exist for p-adic Grassmannians, see [18].

3. Beta-functions of symmetric spaces

3.1. The Gindikin beta-function of symmetric cones

Consider the space $\mathrm{Pos}_n(\mathbb{K})$ of positive definite $n \times n$ matrices over $\mathbb{K}$. The cone $\mathrm{Pos}_n(\mathbb{K})$ is a model of the symmetric space $\mathrm{GL}_n(\mathbb{K})/\mathrm{U}_n(\mathbb{K})$, the group $\mathrm{GL}_n(\mathbb{K})$ acts on $\mathrm{Pos}_n(\mathbb{K})$ by transformations $g: X \mapsto g^*Xg$.

Gindikin [19], 1965, considered a matrix Γ-function given by

$$\begin{aligned}\mathbf{\Gamma}[\mathbf{s}] &:= \int_{\mathrm{Pos}_n(\mathbb{K})} e^{-\operatorname{tr} X} \prod_{j=1}^{n} \det[X]_j^{s_j - s_{j+1}} \cdot \det X^{\mathfrak{d}n/2 - \mathfrak{d}/2+1}\, dX \\ &= (2\pi)^{n(n-1)\mathfrak{d}/4} \prod_{k=1}^{n} \Gamma\Bigl(s_k - (k-1)\frac{\mathfrak{d}}{2}\Bigr). \end{aligned} \tag{17}$$

Here $s_j \in \mathbb{C}$, $s_{n+1} := 0$; $[X]_p$ denotes upper left corners of size p of a matrix X. The expressions $s_j - s_{j+1}$ are written by aesthetic reasons, we can write

$$\prod\nolimits_{j=1}^{n} \det[X]_j^{\lambda_j}$$

with arbitrary λ_j. The factor $\det X^{\eth n/2-\eth/2+1}$ can be included to the latter product, but it is the density of the $\mathrm{GL}_n(\mathbb{K})$-invariant measure on $\mathrm{Pos}_n(\mathbb{K})$ and it is reasonable to split it from the product.

To evaluate the integral, Gindikin considers[10] the substitution $X = S^*S$, where S is an upper triangular matrix with positive elements on the diagonal. After this the integral splits into a product of one-dimensional integrals.

Also the following imitation of the beta-function take place:

$$\mathbf{B}[\mathbf{s},\mathbf{t}] := \int_{0<X<1} \prod_{j=1}^{n} \Bigl(\det[X]_j^{s_j-s_{j+1}} \cdot \det[1-X]_j^{t_j-t_{j+1}}\Bigr) \\ \times \det X^{\eth n/2-\eth/2+1} \det(1-X)^{\eth n/2-\eth/2+1}\, dX = \frac{\mathbf{\Gamma}[\mathbf{s}]\,\mathbf{\Gamma}[\mathbf{t}]}{\mathbf{\Gamma}[\mathbf{s}+\mathbf{t}]}. \tag{18}$$

A proof in [19] is an one-to-one imitation of the standard evaluation of the Euler beta-integral.

These integrals extend some results of 1920–30s (Whishart, Ingham, Siegel, see [21]).

3.2. Beta functions of Riemannian non-compact symmetric spaces

The domain of integration $0 < X < 1$ in (18) is itself the symmetric space $\mathrm{GL}_n(\mathbb{K})/\mathrm{U}_n(\mathbb{K})$. Indeed, the matrix ball $ZZ^* < 1$ in the space of Hermitian matrices is a model of the symmetric space $\mathrm{GL}_n(\mathbb{K})/\mathrm{U}_n(\mathbb{K})$). The inequality $ZZ^* < 1$ is equivalent to $-1 < Z < 1$, and we substitute $Z = -1 + 2X$.

Analogs of integrals (18) for 7 remaining series of Riemannian non-compact symmetric spaces were obtained in [22][11]. We give two well-representative examples.

In the first example we consider a symmetric space, which can be realized as a matrix wedge. Let W_n be the domain (*Siegel upper half-plane*) of $n \times n$ complex symmetric matrices Z with $\Re Z > 0$. This is a model of a symmetric space $\mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n$. We write $Z = T + iS$, where T, S are real symmetric matrices. Then

$$\int_{T=T^t>0,S=S^t} \prod_{j=1}^{n} \frac{\det[T]_j^{\lambda_j-\lambda_{j+1}}}{\det[1+T+iS]_j^{\sigma_j-\sigma_{j+1}} \det[1+T-iS]_j^{\tau_j-\tau_{j+1}}} \\ \times \det T^{-(n+1)} dT\, dS \\ = \prod_{k=1}^{n} \frac{2^{2-\sigma_k-\tau_k+n-k}\pi^k \Gamma(\lambda_k-(n+k)/2)\Gamma(\sigma_k+\tau_k-\lambda_k-(n-k)/2)}{\Gamma(\sigma_k-(n-k)/2)\Gamma(\tau_k-(n-k)/2)} \tag{19}$$

(we set $\lambda_{j+1} = \sigma_{j+1} = \tau_{j+1} = 0$).

There are also noncompact symmetric spaces, which do not admit realizations as convex matrix cones and convex matrix wedges. As an example, we consider the

[10] See also, [20].

[11] For the case of tubes $\mathrm{SO}_0(n,2)/\mathrm{SO}(n) \times \mathrm{SO}(2)$, which is slightly exceptional, see [23].

space $\mathrm{O}_{p,q}/\mathrm{O}_p \times \mathrm{O}_q$. Let $q \geqslant p$. We realize this space (for details, see [22], Sect.3) as the space of real block matrices of size $(q-p)+p$ having the form

$$R = \begin{pmatrix} 1 & 0 \\ 2L & K \end{pmatrix}, \qquad R + R^t > 0.$$

We represent K as $K = M + N$, where M is symmetric and K is skew-symmetric. Then the dissipativity condition $R + R^t > 0$ reduces to the form

$$\begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix} > 0$$

or equivalently $M - LL^t > 0$. We have the following integrals in coordinates L, M, K:

$$\int\limits_{\substack{M=M^t>0, N=-N^t \\ M-LL^t>0}} \prod_{j=1}^{p} \frac{\det[M - LL^t]_j^{\lambda_j - \lambda_{j+1}}}{\det[1 + M + N]_j^{\sigma_j - \sigma_{j+1}}} \times \det(M - LL^t)^{-(p+q)/2} dM\, dN\, dL =$$
$$= \prod_{k=1}^{p} \pi^{k-(q-p)/2-1} \frac{\Gamma(\lambda_k - (q+k)/2 + 1)\Gamma(\sigma_k - \lambda_k - (p-k)/2)}{\Gamma(\sigma_k - p + k)}. \tag{20}$$

3.3. Remarks

1. Integrals (19)–(20) were written to obtain Plancherel measure for Berezin representations of classical groups, see [22, 24].
2. I do not know perfect counterparts of the integrals (19)–(20) for compact symmetric spaces. Some beta-integrals over classical groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ were considered in [15], extensions to over compact symmetric spaces are more-or-less automatic. However, they depend on a smaller number of parameters.
3. *On analogs of the* Γ*-function.* To be definite, consider the space $\mathrm{Mat}_{n,n}(\mathbb{C})$. Consider a distribution

$$\varphi(Z) = \prod\nolimits_{j=1}^{n} |\det[Z]_j|^{\lambda_j} \det[Z]_j^{p_j},$$

where $p_j \in \mathbb{Z}$, $\lambda_j \in \mathbb{C}$. This expression is homogeneous in the following sense: for an upper triangular matrix A and a lower triangular matrix B,

$$\varphi(BZA) = \prod |a_{jj} b_{jj}|^{\sum_{k \leqslant j} \lambda_j} (a_{jj} b_{jj})^{\sum_{k \leqslant j} p_j} \varphi(Z).$$

The Fourier transform $\widehat{\varphi}$ of φ must be homogeneous. For λ_j in a general position this remark allows to write $\widehat{\varphi}$ up to a constant factor. This factor (it is a product of Gamma-functions and sines) can be regarded as a matrix analog of Gamma-function. See Stein [25], 1967, Sato, Shintani [26], 1974. I do not know an exhausting text on this topic.

4. Zeta-functions of spaces of lattices

Non compact symmetric spaces have p-adic counterparts, namely Bruhat–Tits buildings (see, e.g., [12, Chapter 10]). Since this topic is not inside common knowledge, we will discuss an adelic variant of matrix beta-integrals.

4.1. Space of lattices

A *lattice* in $\mathbb{Q}^n$ is a subgroup isomorphic to $\mathbb{Z}^n$. Denote by Lat_n the space of lattices in $\mathbb{Q}^n$. The group $\mathrm{GL}_n(\mathbb{Q})$ acts on the space Lat_n, the stabilizer of the standard lattice $\mathbb{Z}^n$ is $\mathrm{GL}_n(\mathbb{Z})$. Thus Lat_n is a homogeneous space

$$\mathrm{Lat}_n \simeq \mathrm{GL}_n(\mathbb{Q})/\mathrm{GL}_n(\mathbb{Z}).$$

4.2. Analog of beta-integrals

We consider two coordinate flags

$$0 \subset \mathbb{Z} \subset \mathbb{Z}^2 \subset \cdots \subset \mathbb{Z}^n; \qquad 0 \subset \mathbb{Q} \subset \mathbb{Q}^2 \subset \cdots \subset \mathbb{Q}^n.$$

Consider intersections of a lattice S with these flags, i.e.,

$$S \cap \mathbb{Z}^k \ \subset\ S \cap \mathbb{Q}^k \ \subset\ \mathbb{R}^k.$$

For a lattice $S \subset \mathbb{R}^k$ we denote by $\upsilon_k(S)$ the volume of the quotient $\mathbb{R}^k/S$. The following identity holds [27]:

$$\sum_{S\in\mathrm{Lat}_n(\mathbb{Q})} \prod_{j=1}^{n} \upsilon_k(S\cap\mathbb{Q}^k)^{-\beta_k+\beta_{k+1}}\ \upsilon_k(S\cap\mathbb{Z}^k)^{-\alpha_k+\alpha_{k+1}}$$
$$= \prod_{j=1}^{n} \frac{\zeta(-(\beta_j+j-1))\ \zeta(\alpha_j+\beta_j-n+j)}{\zeta(\alpha_j-n+j)}, \tag{21}$$

where ζ is the Riemann ζ-function,

$$\zeta(s) = \sum_{k=0}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \Bigl(1-\frac{1}{p^s}\Bigr)^{-1}.$$

4.3. On Berezin kernels

It seems that holomorphic discrete series representations of semisimple Lie groups have no p-adic analogs. However, in [27] there were obtained p-adic analogs of the Berezin kernels and of the Berezin–Wallach set. Let us explain this in our minimal language. We define a Berezin kernel on Lat_n by

$$K_\alpha(S,T) := \frac{\bigl(\upsilon_n(R)\,\upsilon_n(S)\bigr)^{\alpha/2}}{\bigl(\upsilon_n(R\cap S)\bigr)^{\alpha}}.$$

This kernel is positive definite if and only if $\alpha = 0,1,\dots,n-1,$ or $\alpha > n-1$. Positive definiteness of the kernel means that there exists a Hilbert space H_α and a total system of vectors $\delta_S \in H_\alpha$, where S ranges in Lat_n, such that

$$\langle \delta_S, \delta_T\rangle_{H_\alpha} = K_\alpha(S,T).$$

The group $\mathrm{GL}_n(\mathbb{Q}_p)$ acts in the spaces H_α. Further picture is parallel to the theory of Berezin kernels over $\mathbb{R}$ (see [24]). Formula (21) allows to obtain the Plancherel formula for this representation.

4.4. Remarks

1. An analog of Γ-function is the Tamagawa zeta-function [28], see also [29]. It is the sum
$$\sum \prod\nolimits_{k=1}^{n} v_k(S \cap \mathbb{Z}^k)^{-\alpha_k+\alpha_{k+1}}$$
over sublattices is $\mathbb{Z}^n$. It can be obtained from (21) by a degeneration.
2. Certainly, analogs of (21) for symplectic and orthogonal groups must exist. As far as I know they are not yet obtained.

5. Non-radial interpolation of matrix beta-integrals

5.1. Rayleigh tables

Again, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or quaternions $\mathbb{H}$, $\mathfrak{d} = \dim \mathbb{K}$. Consider Hermitian matrices of order n over $\mathbb{K}$.

Consider eigenvalues of $[X]_p$ for each p,
$$\lambda_{p1} \leqslant \lambda_{p2} \leqslant \cdots \leqslant \lambda_{pp}.$$
We get a table $\mathcal{L}$
$$\begin{array}{ccccccccccccc}
 & & & & & & \lambda_{11} & & & & & & \\
 & & & & & \lambda_{21} & & \lambda_{22} & & & & & \\
 & & & & \lambda_{31} & & \lambda_{32} & & \lambda_{33} & & & & \\
 & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\lambda_{n1} & & \lambda_{n2} & & \lambda_{n3} & \ldots & \lambda_{n(n-2)} & & \lambda_{n(n-1)} & & \lambda_{nn} & &
\end{array} \tag{22}$$
with the *Rayleigh interlacing condition*[12]
$$\cdots \leqslant \lambda_{(j+1)k} \leqslant \lambda_{jk} \leqslant \lambda_{(j+1)(k+1)} \leqslant \cdots$$
This means that the numbers λ_{kl} increase in 'north-east' and 'south-east' directions.

Denote by $\mathcal{R}_n$ the space of all *Rayleigh tables* (22).

Note that for $\mathbb{K} = \mathbb{R}$ the number of variables λ_{kl} coincides with $\dim \mathrm{Herm}_n(\mathbb{R})$ (but generally there are $2^{n(n-1)/2}$ matrices X with a given $\mathcal{L}$).

Now consider the image of the Lebesgue measure on $\mathrm{Herm}_n(\mathbb{K})$ under the map $\mathrm{Herm}_n(\mathbb{K}) \to \mathcal{R}_n$. In other words, consider the joint distribution of eigenvalues of

[12] This statement also is called the Rayleigh–Courant-Fisher theorem.

all $[X]_p$. It is given by the formula

$$d\rho_{\mathfrak{d}}(\mathcal{L}) = C_n(\mathfrak{d}) \frac{\prod\limits_{2\leqslant j\leqslant n} \prod\limits_{1\leqslant\alpha\leqslant j-1,\ 1\leqslant p\leqslant j} |\lambda_{(j-1)\alpha} - \lambda_{jp}|^{\mathfrak{d}/2-1}}{\prod\limits_{2\leqslant j\leqslant n-1} \prod\limits_{1\leqslant\alpha<\beta\leqslant j} (\lambda_{j\beta} - \lambda_{j\alpha})^{\mathfrak{d}-2}} \times \prod_{1\leqslant p<q\leqslant n} (\lambda_{nq} - \lambda_{np}) \prod_{1\leqslant j\leqslant n} \prod_{1\leqslant\alpha\leqslant j} d\lambda_{j\alpha}, \tag{23}$$

where

$$C_n(\mathfrak{d}) = \frac{\pi^{n(n-1)\mathfrak{d}/4}}{\Gamma^{n(n-1)/2}(\mathfrak{d}/2)}.$$

Notice that for $\mathbb{K} = \mathbb{C}$ we get a total cancellation in the expression (23). History of this formula is not quite clear. It seems that ideologically it is contained in book [30] by Gel'fand, Naimark (see evaluation of spherical functions of $\mathrm{GL}(n,\mathbb{C})$). The measure (23) is used in integral representation of Jack polynomials in paper [31] by Olshanski and Okounkov. A formal proof is contained in [32], see also [33] and [34].

5.2. Interpolation

Now we can assume that $\mathfrak{d}$ is an arbitrary complex number and interpolate matrix beta-integrals

$$\int_{\mathrm{Herm}_n(\mathbb{K})} \prod_{k=1}^{n-1} (1 + i[X]_k)^{-\sigma_k+\sigma_{k+1}-\mathfrak{d}/2} (1 - i[X]_k)^{-\tau_k+\tau_{k+1}-\mathfrak{d}/2} \times \det(1+iX)^{-\sigma_n} \det(1-iX)^{-\tau_n}\, dX = \frac{\prod \Gamma(\dots)}{\prod \Gamma(\dots)}$$

with respect to $\mathfrak{d} = \dim \mathbb{K}$:

$$\int_{\mathcal{R}_n} \prod_{j=1}^{n-1} \prod_{\alpha=1}^{j} (1 + i\lambda_{j\alpha})^{-\sigma_j+\sigma_{j+1}-\mathfrak{d}/2} (1 - i\lambda_{j\alpha})^{-\tau_j+\tau_{j+1}-\mathfrak{d}/2} \times \prod_{p=1}^{n} (1 + i\lambda_{np})^{-\sigma_n} (1 - i\lambda_{np})^{-\tau_n}\, d\rho_{\mathfrak{d}}(\Lambda) = \pi^{n(n-1)\mathfrak{d}/4+n} \cdot \prod_{j=1}^{n} \frac{\Gamma(\sigma_j + \tau_j - 1 - (j-1)\mathfrak{d}/2)}{\Gamma(\sigma_j)\Gamma(\tau_j)}.$$

Here integration is taken over the space of all Rayleigh tables and the measure $d\rho_{\mathfrak{d}}(\Lambda)$ is given by (23).

However, the proof [32] of the latter formula remains to be valid for a wider family of integrals,

$$\int \prod_{j=1}^{n-1} \prod_{\alpha=1}^{j} (1+i\lambda_{j\alpha})^{-\sigma_j+\sigma_{j+1}-\theta_{j\alpha}} (1-i\lambda_{j\alpha})^{-\tau_j+\tau_{j+1}-\theta_{j\alpha}}$$

$$\times \prod_{p=1}^{n} (1+i\lambda_{np})^{-\sigma_n}(1-i\lambda_{np})^{-\tau_n}$$

$$\times \prod_{j=1}^{n-1} \frac{\prod_{1\leqslant\alpha\leqslant j,\, 1\leqslant p\leqslant j+1} |\lambda_{j\alpha}-\lambda_{(j+1)p}|^{\theta_{j\alpha}-1}}{\prod_{1\leqslant\alpha<\beta\leqslant j} (\lambda_{j\beta}-\lambda_{j\alpha})^{\theta_{j\alpha}+\theta_{j\beta}-2}} \prod_{1\leqslant p<q\leqslant n} (\lambda_{nq}-\lambda_{np})\, d\Lambda$$

$$= \pi^n 2^{2n-\sum\limits_{j=1}^{n}(\sigma_j+\tau_j)} \prod_{1\leqslant\alpha\leqslant j\leqslant n-1} \Gamma(\theta_{j\alpha}) \cdot \prod_{j=1}^{n} \frac{\Gamma(\sigma_j+\tau_j-1-\sum_{\alpha=1}^{j-1}\theta_{(j-1)\alpha})}{\Gamma(\sigma_j)\Gamma(\tau_j)}.$$

Now the parameter $\mathfrak{d}$ is replaced by $(n-1)n/2$ parameters $\theta_{j\alpha}$

5.3. Remark

The Gindikin beta-integrals admit an interpolation in the same spirit [32]. For beta-integrals (19)–(20) over wedges and more general domains an interpolation is unknown.

6. Beta-integrals over flag spaces

6.1. Beta-integrals

Now we consider upper-triangular matrices $Z=\{z_{ij}\}$ over $\mathbb{K}$, $z_{ii}=1$, $z_{ij}=0$ for $i>j$. Denote the space of all upper-triangular matrices by $\mathrm{Triang}_n(\mathbb{K})$. Recall that the space of upper-triangular matrices is a chart on a flag space.

Let $[Z]_{pq}$ be left upper corners of Z of size $p\times q$, denote

$$s_{pq}(Z) := \det([Z]_{pq}[Z]^*_{pq}).$$

The following identity [35] holds

$$\int_{\mathrm{Triang}_n(\mathbb{K})} \prod_{1\leqslant p<q\leqslant n} s_{pq}(Z)^{-\lambda_{pq}}\, dZ = \pi^{n(n-1)/4} \prod_{1\leqslant p<q\leqslant n} \frac{\Gamma(\nu_{pq}-\mathfrak{d}/2)}{\Gamma(\nu_{pq})},$$

where the integration is taken over the space of upper-triangular matrices, and

$$\nu_{pq} := -\frac{1}{2}(q-p-1)\mathfrak{d} + \sum_{k,m:\, p\leqslant k<q,\, q\leqslant m\leqslant n} \lambda_{mk}.$$

6.2. Projectivity

Consider the map $Z \mapsto [Z]_{n-1}$ from $\mathrm{Triang}_n(\mathbb{K})$ to $\mathrm{Triang}_{n-1}(\mathbb{K})$. Consider a measure

$$\prod_{p=1}^{n-1} s_{pn}(z)^{-\lambda_p}\, dZ^{\{n\}}$$

on $\mathrm{Triang}_n(\mathbb{K})$. Assume

$$\lambda_p + \lambda_{p+1} + \dots + \lambda_{n-1} > \frac{1}{2}(n-p)\mathfrak{d} \quad \text{for all } p.$$

Then the pushforward of this measure under the forgetting map is

$$\pi^{\frac{(n-1)\mathfrak{d}}{2}} \prod_{1\leqslant p\leqslant n-1} \frac{\Gamma(\lambda_p + \dots + \lambda_n - (n-p)\mathfrak{d}/2)}{\Gamma(\lambda_p + \dots + \lambda_n - (n-p+1)\mathfrak{d}/2)} \times \prod_{p=1}^{n-2} s_{p(n-1)}([Z]_{n-1})^{-\lambda_p}\; d[Z]_{n-1}.$$

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Differential Equations on Complex Manifolds

Anton Savin and Boris Sternin

Abstract. We discuss linear partial differential equations with constant coefficients on complex manifold $\mathbb{C}^n$. Using the Sternin–Shatalov integral transform we solve complex Cauchy problem and consider two applications: describe the singularities of the solution of the Cauchy problem and solve a physical problem of sweeping the charge inside the domain (balayage inwards problem).

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Keywords. complex theory of differential equations, Cauchy problem, balayage inwards, propagation of singularities, Landau manifold, integral transform.

1. Complex Cauchy problem. Examples

In the complex space $\mathbb{C}^n$ with coordinates $x = (x^1, \dots, x^n)$ consider the partial differential operator of order m with constant coefficients

$$H\left(-\frac{\partial}{\partial x}\right) = \sum_{|\alpha|\le m} a_\alpha \left(-\frac{\partial}{\partial x}\right)^\alpha, \text{ where } \alpha = (i_1, i_2, \dots, i_n) \text{ is a multiindex,} \quad (1)$$

while $|\alpha| = \sum_k i_k$. The characteristic polynomial

$$H(p) = \sum_{|\alpha|\le m} a_\alpha p^\alpha$$

is called the Hamiltonian, while its top degree component is denoted by $H_m(p)$.

Let us pose the Cauchy problem for the operator (1) with data on a hypersurface $X \subset \mathbb{C}^n$, which we assume to be irreducible analytic manifold of codimension one.

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Definition 1. The *Cauchy problem* for operator (1) is a system of the form

$$\begin{cases} H\left(-\dfrac{\partial}{\partial x}\right) u(x) = f(x), \\ u(x) \quad \text{vanishes to order } m \text{ on } X, \end{cases} \tag{2}$$

where the right-hand side $f(x)$ and the unknown $u(x)$ are *analytic* functions. (Recall that a function is analytic if it has no singularities except ramifications and poles.)

Here we consider for simplicity the Cauchy problem with zero initial data on X.

Let us make several remarks concerning Cauchy problem (2).

Simple examples show that the *solution of the Cauchy problem can have singularities even if X and the right-hand side $f(x)$ have no singularities.*

Example. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 1, \\ u(x,t)\Big|_{x=t^2} = 0. \end{cases} \tag{3}$$

The solution $u(x,t) = t - \sqrt{x}$ is a ramifying function.

Example. In $\mathbb{C}^2$ consider the Cauchy problem for the Laplacian

$$\begin{cases} \dfrac{\partial^2 u}{\partial (x^1)^2} + \dfrac{\partial^2 u}{\partial (x^2)^2} = 1, \\ u(x) \quad \text{vanishes to order 2 on } X. \end{cases} \tag{4}$$

The solution of this Cauchy problem is equal to

$$u(x) = -\frac{1}{2}\ln r + \frac{r^2}{4} - \frac{1}{4}, \quad \text{where } r = \sqrt{(x^1)^2 + (x^2)^2},$$

and is a ramifying function with ramification along the lines $x^1 \pm ix^2 = 0$.

These examples show that in the complex case the solution of a differential equation has singularities and is a ramifying function.

Let us now formulate the Cauchy–Kovalevskaya theorem for the system (2).

Theorem 2 (Cauchy–Kovalevskaya). *In a neighborhood of a point $x_0 \in X$ there exists a unique holomorphic solution of the Cauchy problem* (2), *provided that $f(x)$ is holomorphic at x_0 and x_0 is not a characteristic point.*

Recall the definition of characteristic point, which appears here.

Definition 3. A point $x_0 \in X$ is *characteristic* (with respect to the Hamiltonian H), if

$$H_m\left(\frac{\partial s}{\partial x}(x_0)\right) = 0,$$

where $s(x) = 0$ is a local equation of X. Geometrically, this condition means that the normal vector of the surface lies on the characteristic, which is defined by the equation $H_m(p) = 0$.

Note one essential difference between the complex and real theories: in complex theory any(!) differential equation has characteristics, since the equation $H_m(p) = 0$ always has a solution by the main theorem of algebra. Hence, any Cauchy problem has characteristic points!

We should note that the Cauchy–Kovalevskaya theorem gives only the *existence* and does not give an explicit formula for the solution. Moreover, this theorem does not describe singularities of the solution on X (at the characteristic points), and does not describe the solution far from X. To solve these problems, a more precise and fine apparatus is necessary.

In real theory, such apparatus is the Fourier transform, which plays a fundamental role in modern theory of differential equations (in real domain!). In the first place, this ows to the fact that it algebraizes equations with constant coefficients. More precisely, in Fourier coordinates operators of differentiation are written as operators of multiplication by independent variables and one has the commutation relation

$$\mathcal{F}_{x\to p} \circ \left(-i\frac{\partial}{\partial x}\right) = p \circ \mathcal{F}_{x\to p}, \tag{5}$$

where $\mathcal{F}_{x\to p}$ is the Fourier transform.

Unfortunately, the Fourier transform does not extend to complex theory for the following reasons:

1) in complex theory functions can have arbitrary growth rate at infinity, so that the integral over a noncompact cycle of the form $\mathbb{R}^n$ is always divergent;
2) in complex theory the functions are as a rule ramified, so that in the transform we cannot use a fixed cycle of integration, but rather have to choose a cycle, which avoids singularities of the function and lies on the corresponding Riemannian surface.

An integral transform, which satisfies the commutation relation (5) and is defined using integration over compact cycles, was obtained by Sternin and Shatalov in 1985 [1, 2], see also [3].

Below we recall the definition of the Sternin–Shatalov integral transform, solve complex Cauchy problem for equations with constant coefficients, and then consider two applications: describe the singularities of the solution of the Cauchy problem and solve one problem from physics – problem of sweeping the charge inside the domain (balayage inwards problem).

2. Sternin–Shatalov integral transform

In this section, we recall the definition of the Sternin–Shatalov transform and describe its properties. First, the transform is defined for homogeneous functions. At the end of the section, we show how to define the transform for arbitrary functions.

Naturally, for applications to differential equations one uses the transform for non-homogeneous functions. However, we start the exposition with the transform of homogeneous functions, since in this case the formulas are more natural.

2.1. Transform of homogeneous functions ($\boldsymbol{F}$-transform)

1. Definition of the transform. Consider the complex projective space $\mathbb{CP}^n$.

Let $f(x)$ be a homogeneous function of order $k \in \mathbb{Z}$ on $\mathbb{C}^{n+1}_*$, i.e., we have

$$f(\lambda x) = \lambda^k f(x), \qquad \text{for all } \lambda \neq 0.$$

Definition 4. The *Sternin–Shatalov transform* of a function $f(x)$ is the function $(F_{x\to p} f)(p) = \widetilde{f}(p)$, which is defined by the formula

$$\widetilde{f}(p) = \begin{cases} (-1)^{n+k}(n+k)! \displaystyle\int_{h(p)} \operatorname*{Res}_{L_p} \frac{f(x)\omega(x)}{(xp)^{n+k+1}}, & \text{if } n+k+1 \geq 1, \\ \dfrac{1}{[-(n+k+1)]!} \displaystyle\int_{h_1(p)} f(x)\omega(x)(xp)^{-(n+k+1)}, & \text{if } n+k+1 \leq 0. \end{cases} \tag{6}$$

Here

- $xp = x^0 p_0 + x^1 p_1 + \cdots + x^n p_n$ is the phase function;
- $\omega(x) = \sum_j (-1)^j x^j dx^0 \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$ is the Leray form on $\mathbb{C}^{n+1}$;
- $L_p = \{x \mid xp = 0\} \subset \mathbb{CP}^n$ is the zero set of the phase function; this is a hyperplane depending on the parameter p;
- Res_{L_p} stands for the Leray residue of a differential form on the hyperplane L_p[1].

The integration in (6) is over special homology classes $h(p)$ and $h_1(p)$, which will be defined below. Let us show that the integrand in (6) is defined essentially canonically:

- first, any differential form of degree n on $\mathbb{CP}^n$ is proportional to the Leray form, hence, the integrand naturally contains the Leray form;
- second, similar to the Fourier transform, the integrand depends on p through the phase function xp only;
- third, the fraction of the form
$$\frac{f(x)\omega(x)}{(xp)^l}$$
is homogeneous of order zero in x (i.e., it defines a form on $\mathbb{CP}^n$) if and only if the order of homogeneity of the numerator $(k+n+1)$ is equal to the order of homogeneity l of the denominator, i.e., the power in the denominator in (6) is chosen canonically;

[1] We recall that the Leray residue of a closed form with first-order pole can be computed explicitly:

$$\operatorname*{Res}_{z=0} \left(\frac{dz}{z} \wedge \varphi_{\mathrm{reg}} + \psi_{\mathrm{reg}} \right) = \varphi_{\mathrm{reg}}|_{z=0},$$

where $\varphi_{\mathrm{reg}}, \psi_{\mathrm{reg}}$ are regular forms in a neighborhood of the hyperplane $z=0$. For computation of the Leray residue in the general case see [4, 5].

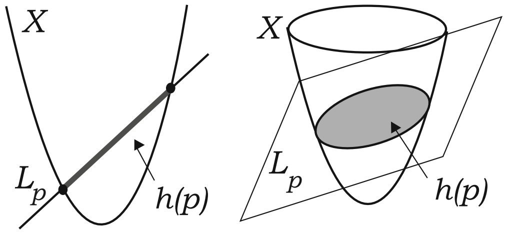

FIGURE 1. Vanishing cycle $h(p)$ in spaces of dimension two and three.

- finally, if $n+k+1 \geq 1$ we have to take residue, since the form has a pole at the plane L_p.

2. Construction of the homology classes $h(p)$ and $h_1(p)$. The transformation (6) is a *relative* transformation, i.e., to define it, we have to choose a codimension one submanifold $X \subset \mathbb{CP}^n$, which is assumed to be analytic and irreducible. This is quite natural from the point of view of the theory of differential equations, where X is just the space on which we pose Cauchy data.

Let $x_0 \in X$ be a regular point on X, which we assume to be nondegenerate in the sense that X and the tangent plane $L_{p_0} \subset \mathbb{CP}^n$ at this point have tangency of order two. Then for p close to p_0 we have the so-called *vanishing cycle* (see Figure 1)

$$h(p) \in H_{n-1}(L_p, X), \tag{7}$$

which lies on the secant plane L_p, while its boundary lies in X. This cycle is called vanishing since this cycle contracts to x_0 as $p \to p_0$. Similarly, one defines the vanishing cycle

$$h_1(p) \in H_n(\mathbb{CP}^n, L_p \cup X), \tag{8}$$

which lies in the ambient space $\mathbb{CP}^n$, while its boundary lies in the union $L_p \cup X$ (see Figure 2).

This defines classes $h(p)$ and $h_1(p)$ only for p close to p_0 (in other words, for the secant planes L_p close to the tangent plane L_{p_0}). However, using the Thom triviality theorem [6] one can show that these classes uniquely extend to all $p \in \mathbb{CP}^n$ as *ramifying* homology classes.

3. Spaces of ramified analytic functions. Consider the following question: what are the natural classes of functions for the transform (6)? Before we answer this question, let us make two remarks.

First, since the transform is defined by a ramifying integral, the result of applying this transform to an analytic (and even holomorphic!) function is a *ramifying analytic function*. Second, since the transform is defined by integrals over cycles with boundary in X, for the integrals to converge, it is necessary that the function $f(x)$ satisfies some conditions on this set.

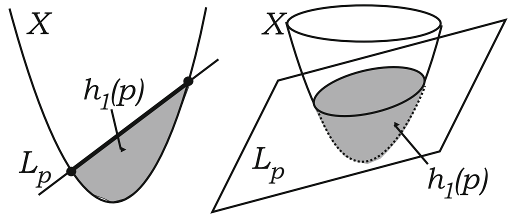

FIGURE 2. Vanishing cycle $h_1(p)$ in spaces of dimension two and three.

So, the transform (6) should be considered in spaces of ramifying analytic functions, which satisfy some conditions on the submanifold X. Let us give the corresponding definition.

Definition 5. The *Weighted space* $A_q^k(X)$ consists of ramifying analytic functions $f(x)$ homogeneous of order k such that:

1) they are holomorphic off $X \cup Y \subset \mathbb{CP}_x^n$, where $Y = Y_f$ denotes some analytic set, which depends on f;
2) given a regular point $x \in X \backslash Y$, near this point we have
$$|\, f(x')\,| \le C\,|s(x')|^q, \text{ with some constant } C, \tag{9}$$
here X is locally defined by the equation $s(x') = 0$.

It follows from this definition that the functions in $A_q^k(X)$ are zero on X for all $q > 0$.

One can show that the mapping (6) acts as
$$F_{x\to p}^k : A_q^k(X) \longrightarrow A_{q-k-(n+1)/2}^{-(n+k+1)}(\mathcal{L}X) \text{ for all } q > -1,$$
where $\mathcal{L}X \subset \mathbb{CP}_p^n$ is the *Legendre transform* of manifold X:
$$\mathcal{L}X = \overline{\{p \mid L_p \text{ is tangent to } X \text{ at a regular point } x \in X\}}$$
the closure of tangent planes to X at regular points. Let us note that the Legendre transform is easy to compute in examples. For instance, the Legendre transform of the quadric $X = \{x^2 = (x^1)^2\}$ is the quadric $\mathcal{L}X = \{p_1^2 = 4p_0p_2\}$.

4. Properties of the transform.

Theorem 6. *The transform* (6) *has the properties*

1. (*invertibility*) *The mapping* $F_{x\to p}^k : A_q^k(X) \longrightarrow A_{q-k-(n+1)/2}^{-(n+k+1)}(\mathcal{L}X)$ *is an isomorphism for all* $q > \max\left(-1, k + \frac{n+1}{2} - 1\right)$. *The inverse transform corresponds to the set* $\mathcal{L}X$ *and is equal to*
$$(-1)^{-k-1-\varepsilon}\left(\frac{i}{2\pi}\right)^{n-1} F_{p\to x}^{-(n+k+1)}, \qquad \text{where } \varepsilon = \begin{cases} 0, & \text{if } -k-2 \ge 1 \\ 1, & \text{if } -k-2 \le 0. \end{cases}$$

2. (*commutation relations*) *Given a function* $f \in A_q^k(X)$ *and* $q > 0$ *we have*

$$\begin{aligned} p_j \cdot F_{x\to p}^k(f) &= F_{x\to p}^{k-1}\left(-\frac{\partial f}{\partial x^j}\right), \\ \frac{\partial}{\partial p_j}(F_{x\to p}^k f) &= F_{x\to p}^{k+1}(x^j f). \end{aligned} \tag{10}$$

2.2. Transform of arbitrary functions ($\boldsymbol{R}$-transform)

For applications to differential equations we need arbitrary, i.e., *nonhomogeneous*, functions, defined in $\mathbb{C}^n$. One can extend the transform (6) to such functions, if we consider $\mathbb{C}^n$ as an affine chart in $\mathbb{CP}^n$. Let us describe the obtained transform.

1. Definition. To a function $f(x^1, \ldots, x^n)$ on $\mathbb{C}^n$, we assign the homogeneous function in $\mathbb{C}_*^{n+1}$:

$$(x^0)^{-n} f\left(\frac{x^1}{x^0}, \frac{x^2}{x^0}, \ldots, \frac{x^n}{x^0}\right). \tag{11}$$

Somewhat surprising (at first glance) is the factor $(x^0)^{-n}$ in this formula, which is responsible for the order of homogeneity. Actually, the choice of order equal to $-n$ gives the simplest expression for the transform R (see formula (12) below).

Definition 7. *R-transform* of function $f(x^1, \ldots, x^n)$ is the result of applying the transform $F_{x\to p}$ to the function (11). A direct computation shows that

$$(Rf)(p) = \int_{h(p)} \operatorname*{Res}_{L_p} \frac{f(x)dx^1 \wedge \cdots \wedge dx^n}{p_0 + p_1 x^1 + \cdots + p_n x^n}. \tag{12}$$

2. Properties of the transform $\boldsymbol{R}$. All the properties of the R-transform follow from those of the F-transform. We briefly formulate these properties.

The R-transform acts in the spaces

$$R:\ A_q(X) \longrightarrow A^{-1}_{q+\frac{n-1}{2}}(\mathcal{L}X),$$

where $A_q(X)$ stands for the space of ramifying analytic functions on $\mathbb{C}^n$, which satisfy near X estimates (9). The R-transform is invertible. Finally, we note that the commutation relations in this case are

$$R\left(-\frac{\partial f}{\partial x^j}\right) = p_j \frac{\partial}{\partial p_0} R(f), \qquad \text{for } f \in A_q(X), \quad q > 0$$

and differ from those in (10) in that they contain the derivative $\partial/\partial p_0$. Hence, unlike the real theory, in which the Fourier transform converts differential equations with constant coefficients into algebraic equations, in complex theory R-transform converts partial differential equations with constant coefficients into ordinary differential equations. Meanwhile, the corresponding ordinary differential equations are easy to solve. So, there are no difficulties here.

3. Example. Let us consider an example of the R-transform for a function of two variables.

Let $f(x) \equiv 1$ and X be the complex quadric $\{(x^1)^2 + (x^2)^2 = 1\}$. Let us compute the function $\widetilde{f} = R(1)$. This function is homogeneous of order -1, hence, it suffices to compute its values say for $p_1 = 1$. We have

$$\widetilde{f}(p_0, 1, p_2) = \int_{h(p_0,1,p_2)} \operatorname*{Res}_{L_p} \frac{dx^1 \wedge dx^2}{p_0 + x^1 + x^2 p_2}.$$

The residue is equal to

$$\operatorname*{Res}_{L_p} \frac{dx^1 \wedge dx^2}{p_0 + x^1 + x^2 p_2} = dx^2.$$

Note that it follows from the equation $L_p = \{p_0 + x^1 + x^2 p_2 = 0\}$ that x^2 can be considered as a coordinate on L_p. Hence, we have

$$\widetilde{f}(p_0, 1, p_2) = \int_{h(p_0,1,p_2)} dx^2.$$

Let us compute the homology class $h(p_0, 1, p_2)$. To this end, we compute the intersection $L_p \cap X$. It consists of two points

$$x^2 = \frac{p_0 p_2 \pm \sqrt{1 + p_2^2 - p_0^2}}{p_2^2 + 1}$$

and therefore the vanishing cycle of the corresponding quadric is just the segment, which joins these two points. Hence

$$\widetilde{f}(p_0, 1, p_2) = \frac{2\sqrt{1 + p_2^2 - p_0^2}}{1 + p_2^2}.$$

Taking into account the homogeneity of $\widetilde{f}$, we obtain finally

$$\boxed{\widetilde{f}(p) = \frac{2\sqrt{p_1^2 + p_2^2 - p_0^2}}{p_1^2 + p_2^2}}.$$

The latter function has singularities of two types: ramification in the nominator and polar singularities due to the denominator.

3. Solution of the Cauchy problem

Consider the Cauchy problem for an equation of order m

$$\begin{cases} H\left(-\dfrac{\partial}{\partial x}\right) u(x) = f(x), \\ u(x) \quad \text{vanishes to order } m \text{ on } X \end{cases} \tag{13}$$

with zero data on the submanifold X of codimension one.

Below we apply the Sternin–Shatalov transform and show that the Cauchy problem is uniquely solvable in spaces of ramifying functions; we also give an explicit formula for the solution of the Cauchy problem as a ramifying integral; finally, we describe the singularity set of the solution. Note that these results are *global*, i.e., valid in the entire space.

3.1. Theorem on the solvability of the Cauchy problem

We saw in the first section that even for a holomorphic right-hand side the solution of the Cauchy problem should be sought in the space of ramifying functions. Further, the condition that $u(x)$ vanishes to order m on the submanifold X can be restated as $u \in A_m(X)$, where $A_m(X)$ is the weighted space of ramifying functions, which was defined earlier.

So, to the problem (13) we assign the operator

$$H\left(-\frac{\partial}{\partial x}\right) : A_m(X) \to A_0(X), \tag{14}$$

or, more generally, operator

$$H\left(-\frac{\partial}{\partial x}\right) : A_{q+m}(X) \to A_q(X), \quad \text{defined for } q > -1 \tag{15}$$

acting in the scale of spaces $A_q(X)$.

To study the operator (15), we apply the Sternin–Shatalov transform.

Theorem 8 ([5]). *The operator* (15) *is invertible provided that* X *is not totally characteristic.*

Proof. 1. Consider the commutative diagram

$$\begin{tikzcd} A_{q+m}(X) \arrow[r, "H(-\partial/\partial x)"] \arrow[d, "R"'] & A_q(X) \arrow[d, "R"'] \\ A^{-1}_{q+m+(n-1)/2}(\mathcal{L}X) \arrow[r, "H(pd/dp_0)"'] & A^{-1}_{q+(n-1)/2}(\mathcal{L}X), \end{tikzcd} \tag{16}$$

where $\mathcal{L}X$ is the Legendre transform of X.

Since the R-transform is invertible, it follows from (16) that the invertibility problem for the initial operator reduces to the same problem for the family of ordinary differential equations with constant coefficients

$$H\left(p\frac{d}{dp_0}\right), \quad \text{where } p = (p_1, p_2, \dots, p_n), \tag{17}$$

in the spaces A^{-1}_q.

2. It follows from the properties of the function spaces in (16) that we should seek the solutions of the equation

$$H\left(p\frac{d}{dp_0}\right)\widetilde{u}(p_0, p) = \widetilde{f}(p_0, p), \tag{18}$$

that have a zero of order m on $\mathcal{L}X$. Suppose that the equation of $\mathcal{L}X$ is $p_0 = p_0(p)$, then it follows that the function $\widetilde{u}(p_0, p)$ should satisfy the conditions

$$\left(\frac{d}{dp_0}\right)^j \widetilde{u}\Big|_{p_0=p_0(p)} = 0, \text{ for all } j \le m-1, \tag{19}$$

i.e., it has zero Cauchy data for the ordinary differential equation (18). In other words, the operator $H(pd/dp_0)^{-1}$ is the inverse of the operator in the lower row in (16) and coincides with the resolving operator of the Cauchy problem for the equation (18) with zero Cauchy data (19) on X. The solution of this Cauchy problem can be constructed using the Green function. Straightforward estimates show that this operator acts in the spaces in question.

3. Now we can write out the inverse operator for $H(-\partial/\partial x)$ as the composition

$$H\left(-\frac{\partial}{\partial x}\right)^{-1} = R^{-1} \circ H\left(p\frac{d}{dp_0}\right)^{-1} \circ R, \tag{20}$$

i.e., we obtain an explicit solution of Cauchy problem in the form

$$\boxed{u = R^{-1} H\left(p\frac{d}{dp_0}\right)^{-1} (Rf)}\,. \tag{21}$$

□

3.2. Sternin–Shatalov formula for the solution of the Cauchy problem

If we substitute explicit expressions for the integral transforms R, R^{-1} and the integral operator $H(pd/dp_0)^{-1}$ in (21), we obtain an explicit expression for the solution of Cauchy problem (13) in terms of an iterated integral. It turns out, however, that this integral can be simplified, namely, it can be reduced to a double integral and one obtains a simple formula for the solution of the Cauchy problem, which was obtained by Sternin and Shatalov[2]. Let us describe this formula.

We assume for simplicity that the Hamiltonian H is a homogeneous function and consider the submanifolds

$$\Sigma_x = \{(p,y) \mid p(x-y) = 0\} \subset \mathbb{CP}^{n-1}_p \times \mathbb{CP}^n_y,$$
$$\operatorname{char} H = \{(p,y) \mid H(p) = 0\} \subset \mathbb{CP}^{n-1}_p \times \mathbb{CP}^n_y.$$

Theorem 9 ([5]). *If X is not totally characteristic, then the solution of the Cauchy problem is equal to*

$$u(x) = (n-m-1)!\left(\frac{i}{2\pi}\right)^{n-1} \int_{h(x)} \operatorname*{Res}_{\Sigma_x} \frac{f(y)dy \wedge \omega(p)}{H(p)(p(y-x))^{n-m}}, \qquad \text{if } n > m, \tag{22}$$

[2]In real theory, reductions of an iterated integral to a double integral are based on the Fubini theorem. However, in complex analysis the Fubini theorem does not work! More precisely, the relation between the iterated and double integral is described in terms of a certain spectral sequence (this is studied in detail in the paper [7]). Hence, the described reduction is a nontrivial step forward.

$$u(x) = \frac{(-1)^{m-n}}{(m-n)!}\left(\frac{i}{2\pi}\right)^{n-1}\int_{h_1(x)}\frac{f(y)(p(y-x))^{m-n}dy\wedge\omega(p)}{H(p)}, \qquad \text{if } n \le m, \tag{23}$$

where $\omega(p) = \sum_{j=1}^{n}(-1)^j p_j dp_1 \wedge\cdots\wedge \widehat{dp_j} \wedge\cdots\wedge dp_n$ *is the modified Leray form, while*

$$h(x) \in H_{2n-2}(\Sigma_x \setminus \operatorname{char} H, X) \tag{24}$$

$$h_1(x) \in H_{2n-1}(\mathbb{CP}_p^{n-1} \times \mathbb{CP}_y^n \setminus \operatorname{char} H, \Sigma_x \cup X) \tag{25}$$

are special ramified homology classes.

Remark 10. For completeness, we give the definitions of the ramified homology classes in Theorem 9. The class (24) is defined as a vanishing cycle of the family of quadrics $\Sigma_x \cap X \subset \Sigma_x$. The class (25) is defined by the equality $\partial h_1(x) = h(x)$, where ∂ is a boundary map in homology.

3.3. Localization of singularities

We can use formulas (22) and (23) for the exact solution of the Cauchy problem, to study the *localization problem* for the singularities of the solution. Namely, (22) expresses the solution as a ramified integral depending on a parameter. Integrals of this form originally appeared in physics as Feynman integrals (e.g., see [8]). Moreover, singularities of such integrals lie on special submanifolds called *Landau manifolds*. The Landau manifold for the integral (22) can be explicitly calculated and we therefore obtain localization of singularities of the integral and, hence, of the solution of the Cauchy problem. Let us formulate the resulting formula under the following assumptions:

- the right-hand side $f(x)$ is an entire function;
- the manifold X and the characteristic $\operatorname{char} H = \{p \mid H_m(p) = 0\} \subset \mathbb{CP}_p^{n-1}$ have no singularities;
- X is transverse to the set of points at infinity in $\mathbb{CP}^n$.

To localize the singularity set, we denote the set of characteristic points on X by $\operatorname{char} X$. Then we consider solutions of the Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial H_m}{\partial p}, \\ \dot{p} = -\dfrac{\partial H_m}{\partial x} = 0. \end{cases}$$

The projections of the solutions to the x-space are called *bicharacteristics*. Obviously, they are straight lines

$$x(t) = x(0) + t\frac{\partial H_m}{\partial p}, \quad t \in \mathbb{C}. \tag{26}$$

Theorem 11 ([5]). *The singularity set* $\operatorname{sing} u$ *of the solution of the Cauchy problem is contained in the union of bicharacteristics of the Hamiltonian* H *originating* *from the characteristic points of X, including characteristic points at infinity:*

$$\operatorname{sing} u \subset \bigcup_{x \in \operatorname{char} X} \left\{ x + t\frac{\partial H_m}{\partial p} \right\}. \tag{27}$$

Remark 12. Since the bicharacteristics of equation with constant coefficients are straight lines (26), it follows that the set on the right-hand side in (27) is a union of straight lines. This set is called the *characteristic conoid* of the Cauchy problem.

Example. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial (x^1)^2} + 5\dfrac{\partial^2 u}{\partial (x^2)^2} = 1, \\ u \text{ has zero of order } 2 \ \text{ on } X, \end{cases} \tag{28}$$

where $X = \{(x^1)^2 + (x^2)^2 = 1\}$ and f is holomorphic in $\mathbb{C}^2$.

Let us localize the singularities of the solution of this Cauchy problem. The Hamiltonian is $H(p) = p_1^2 + 5p_2^2$, the characteristic is

$$\operatorname{char} H = \{p_1^2 + 5p_2^2 = 0\}.$$

Characteristic points are obtained as the solutions of the system of equations

$$\begin{cases} (x^1)^2 + (x^2)^2 = 1, & (x^1, x^2) \in X, \\ p_1^2 + 5p_2^2 = 0, & (p_1, p_2) \in \operatorname{char} H, \\ p_1 = 2x^1, p_2 = 2x^2 & p = \partial s/\partial x. \end{cases}$$

This system has 4 solutions $(x^1, x^2) = \frac{1}{2}(\pm\sqrt{5}, \pm i)$. Projections of bicharacteristics, which pass through these 4 points, are equal to $(x^1, x^2) = \frac{1}{2}(\pm\sqrt{5}, \pm i)$ $+t(\pm 1, \pm i\sqrt{5})$. Hence, the singularities of the solution lie on the union of 4 lines

$$\sqrt{5}x^1 \pm ix^2 = \pm 2.$$

So, the singularities of the Cauchy problem (28) lie on these 4 lines.

Example. Let us now get back to Example 1 in the first section. A computation shows that there are no characteristic points in the affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$. But the solution has singularities. The question is: where do these singularities come from? Theorem 11 gives an answer to this question: the singularities are generated by characteristic points at infinity. A computation, which we omit here, shows that this is indeed the case.

4. Application. Balayage inwards

4.1. Statement of the problem

The classical balayage problem was introduced by Poincaré and was studied in several settings since then.

Here we consider the following statement (see monograph by H.S. Shapiro [9]) known as “balayage inwards” problem:

Given a density of charges f supported in a domain Ω, find a density of charges w with smaller support $\operatorname{supp} w \Subset \Omega$, *whose potential coincides with the potential of the original distribution f outside $\overline{\Omega}$.*

Of course, it is natural to try to make the support $\operatorname{supp} w$ of the new density of charges in a certain sense minimal.

Here we recall that the potential, induced by the charges with density f, is equal to

$$U^f(x) = \int\limits_{\Omega} f(y) E_n(x-y)\, dy. \tag{29}$$

Here $E_n(\xi)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$:

$$E_n(\xi) = \begin{cases} -\dfrac{1}{\operatorname{Vol}(\mathbb{S}^{n-1})} \cdot \dfrac{1}{|\xi|^{n-2}}, n > 2, \\ \dfrac{1}{2\pi} \ln \dfrac{1}{|\xi|}, n = 2 \end{cases}$$

($\operatorname{Vol}(\mathbb{S}^{n-1})$ is the volume of the unit sphere in $\mathbb{R}^n$.)

Example. Let Ω be a ball with center at the origin with uniform distribution of charges: $f|_{\Omega} \equiv 1$. It is well known (I. Newton), that in this case the same potential is obtained, if we place the uniform charge on an arbitrary smaller ball with the same center. Moreover, one can even place the charge at the center of the ball. This means that the solution of balayage inwards problem in this case is given by the distribution $w(x) = C\delta(x)$ proportional to the Dirac delta-function at the origin.

This example shows that it is natural to search for the desired distribution of charges w in the class of distributions supported in Ω. The corresponding potential in this case is defined by convolution (cf. (29)).

Remark 13. There are other problems in physics similar to balayage problem. Let us mention two such problems.

1. In geophysics, there is a problem of describing gravitationally equivalent bodies (motherbody problem): *Given a body D with known mass distribution, find a smaller body D_1, which produces the same gravitational field outside D.* Obviously, this problem can be reformulated as a balayage problem. However, there is one difference: a distribution of masses has to be a nonnegative function. For more details see [10].

2. In radiophysics, there is a problem of optimization of antenna sizes. We can consider antenna as a given distribution of currents in a domain D (domain occupied by the antenna). To construct antenna of a smaller size, means to find a distribution of currents which is supported in a smaller domain D_1 and produces the same electro-magnetic field outside D. Similar to the previous examples, this problem reduces to the study of properties of solutions of Helmholz equations (or, more generally, of Maxwell equations). For more details see [11].

Method of investigation of the problem. Note that in all three problems the minimal size of the object under construction is determined by the *singularities* of the solution of the corresponding equation inside the domain. Hence, study of such problems reduces to finding singularities of solutions of the corresponding equations. To study singularities of these equations (recall that the equations are defined in the real domain!) we complexify the problem, i.e., consider these equations in complex domain. The reason, why it is necessary to complexify the equation to study singularities of its solution lies in the very nature of elliptic equations. Indeed, we would like to study singularities of the solution. However, as is well known, singularities of solutions of differential equations propagate along characteristics. Meanwhile, elliptic equations have no real characteristics. This shows that singularities of the solution originated somewhere in the complex domain and propagate along (complex) characteristics into the real domain. Figuratively speaking, the singularities of the solution are obtained as a "complex rain", which falls down on the *real* space from some points in the *complex* space. Thus, to study singularities of the solution of real problems, one should follow the following scheme:

- pose the complex problem, which corresponds to the real problem;
- study the singularities of the solution of the complex problem;
- intersect the singularity set with the real space.

Below we realize this approach and solve the balayage inwards problem.

4.2. Solution of the problem

Main assumptions. In this section we give (following [12]) the solution of the balayage inwards problem using the methods of complex theory of differential equations under the assumption that the boundary $\partial\Omega$ and the original charge distribution f extend to the complex domain, namely:

1. The boundary $\partial\Omega$ of Ω is an irreducible algebraic hypersurface without singularities. The domain Ω is assumed to be bounded. Denote by X the complexification of $\partial\Omega$. This is an algebraic set in $\mathbb{C}^n$.
2. The function $f(x)$ is real analytic in Ω and admits analytic continuation to the compactification $\mathbb{CP}^n$ of $\mathbb{C}^n$ as a (possibly ramifying) function with singularities over analytic sets in $\mathbb{CP}^n$.

Reduction to a complex Cauchy problem. Let w be a distribution, which gives a solution of balayage problem. Consider the difference

$$u(x) = U^f(x) - U^w(x). \tag{30}$$

Then using properties of the potentials, one can show that the function (30) is a solution of the following (real!) Cauchy problem:

$$\begin{cases} \Delta u(x) = f(x) \text{ for } x \in \Omega \setminus \operatorname{supp} w, \\ u(x) \text{ vanishes to order 2 on } \partial\Omega. \end{cases} \tag{31}$$

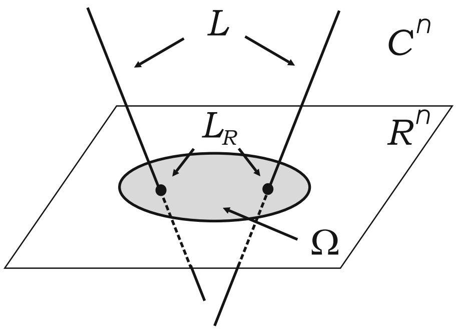

FIGURE 3. Complex rain gives real singularities.

Consider the complexification of (31), i.e., the problem

$$\begin{cases} \Delta u(x) = f(x), \ \text{for } x \in \mathbb{C}^n, \\ u(x) \text{ vanishes to order 2 on } X. \end{cases} \tag{32}$$

Under our assumptions the solution of the real problem (31) is obtained from the solution of the complex problem (32) as the restriction to the real space $\mathbb{R}^n \subset \mathbb{C}^n$ and is, as a rule, ramifying analytic function. Restricting this function to real $x \in \Omega$ and cutting it off (to obtain a univalued function) one can obtain the solution of (31) and, correspondingly, the solution of the balayage problem. Let us formulate this result.

Solution of balayage inwards problem. The solution of the problem can be obtained as follows:

1. The solution $u(x)$ of the complex Cauchy problem can be written out explicitly as a ramifying integral. We also know that the singularities of the solution lie on Landau manifold, which we denote by L (see Figure 3).
2. ("complex rain") Let $L_{\mathbb{R}} = L \cap \Omega$ be the real part of the singularity set, which lies inside Ω (see Figure 3). Note that this set lies compactly inside Ω, since the solution $u(x)$ has no singularities near $\partial\Omega$.
3. By item. 2 the function $u(x)$ has no singularities on the complement $\overline{\Omega} \setminus L_{\mathbb{R}}$. However, it might have ramifications. To obtain a univalued function, we choose a system of cuts $Z \subset \Omega$ with boundary in $L_{\mathbb{R}}$ (see Figure 4) such that $u(x)$ admits a univalued branch over $\overline{\Omega} \setminus Z$. Denote by $u(x)$ this branch and extend $u(x)$ as zero outside Ω. By construction, we have
$$u \in C^1(\mathbb{R}^n \setminus Z), \qquad \Delta u = f \text{ in } \Omega \setminus Z. \tag{33}$$
4. Suppose that u, which is defined in $\Omega \setminus Z$, admits a regularization to a distribution in Ω. We claim that the solution of the balayage problem is given by the distribution
$$w = \Delta(U^f - u), \quad \operatorname{supp} w \subset Z$$

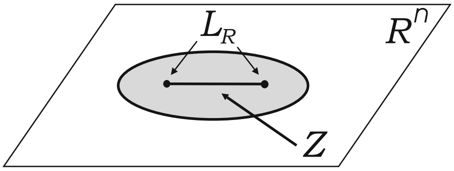

FIGURE 4. Cuts Z are necessary to obtain univalued function.

(we differentiate here in the sense of distributions). Indeed: 1) (33) implies that $\operatorname{supp} w \subset Z$; 2) outside Ω the potential U^w coincides with the original potential U^f:

$$U^w(x) = U^f(x) - u(x) = U^f(x), \qquad \text{for } x \in \mathbb{R}^n \setminus \Omega.$$

The balayage inwards problem is now solved.

4.3. Example

1. Ellipse. Consider the balayage problem for the domain bounded by the ellipse

$$\Omega = \left\{ \frac{(x^1)^2}{a^2} + \frac{(x^2)^2}{b^2} \le 1 \right\}, \quad a > b > 0.$$

Let us assume throughout the following that the initial distribution of charges is defined by an entire function.

Complex Cauchy problem:

$$\begin{cases} \Delta u = f \text{ in } \mathbb{C}^2, \\ u \text{ vanishes to order 2 on } X, \end{cases} \qquad X = \left\{ \frac{(x^1)^2}{a^2} + \frac{(x^2)^2}{b^2} = 1 \right\} \subset \mathbb{C}^2.$$

Let us compute the singularity set.
Hamiltonian: $H(p_1, p_2) = p_1^2 + p_2^2$,
Characteristic: $\operatorname{char} H = \{p_1 = \pm i p_2\} = \{(i,1)\} \cup \{(-i,1)\} \subset \mathbb{CP}_{1,p}$.
Conormal bundle:

$$N^*X = \left\{ \frac{(x^1)^2}{a^2} + \frac{(x^2)^2}{b^2} = 1, \quad \frac{x^1}{a^2} p_2 = \frac{x^2}{b^2} p_1 \right\}.$$

There are four characteristic points:

$$\left\{ \left(\mp \frac{a^2}{\sqrt{a^2 - b^2}}, \frac{ib^2}{\sqrt{a^2 - b^2}}, \pm i, 1 \right) \right\} \cup \left\{ \left(\pm \frac{a^2}{\sqrt{a^2 - b^2}}, -\frac{ib^2}{\sqrt{a^2 - b^2}}, \pm i, 1 \right) \right\}.$$

Four bicharacteristics passing through these points

$$x^1 \pm i x^2 = \pm \sqrt{a^2 - b^2}.$$

Hence, the characteristic conoid is just the union of these four lines

$$L = \{x^1 \pm i x^2 = \pm \sqrt{a^2 - b^2}\}.$$

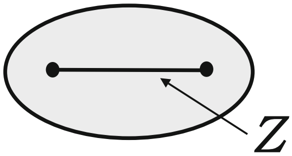

FIGURE 5. Singularities for ellipse.

Complex rain gives the real singularity set

$$L_{\mathbb{R}} = L \cap \mathbb{R}^2 = \{x^1 = \pm\sqrt{a^2 - b^2}, x^2 = 0\}.$$

which coincides with the focal points of the ellipse (see Figure 5).

One cut is necessary: for example, one can cut along the interfocal segment. It can be shown that the solution is indeed ramifying at the focal points and therefore the cut is necessary.

2. Paraboloid of revolution. Consider the balayage problem for the domain bounded by the paraboloid of rotation

$$\Omega = \left\{x^3 \geq (x^1)^2 + (x^2)^2\right\}.$$

A direct computation gives the localization of singularities of the complex Cauchy problem.

Lemma 14. *The singularities of the solution of the complex Cauchy problem with zero data on the surface*

$$X = \left\{x^3 = (x^1)^2 + (x^2)^2\right\} \subset \mathbb{C}^3$$

and entire right-hand side are contained in the set

$$L = \left\{(x^1)^2 + (x^2)^2 + \left(x^3 - \frac{1}{4}\right)^2 = 0\right\} \cup \{(x^1)^2 + (x^2)^2 = 0\}. \tag{34}$$

Complex rain gives localization of the real singularity set:

$$L_{\mathbb{R}} = L \cap \mathbb{R}^3 = \{x^1 = x^2 = 0\} \cup \{x^1 = x^2 = 0, x^3 = 1/4\},$$

which is just the vertical line divided by the point $(0, 0, 1/4)$ (the focal point of the paraboloid) into two rays.

In addition, since u has no singularities on $\partial\Omega$, it follows that the singularity set of the solution actually lies on the ray

$$\{x^1 = x^2 = 0, x^3 \geq 1/4\}.$$

The complement of this ray is simply-connected, hence, the solution is single-valued in this complement and, hence, no cuts are necessary (see Figure 6).

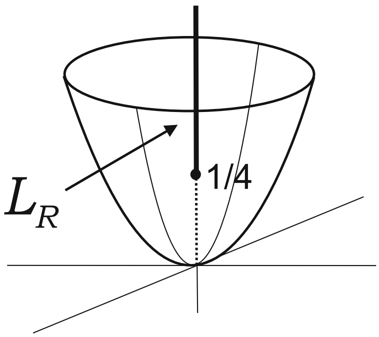

FIGURE 6. Singularities for paraboloid.

3. Ellipsoid of revolution. Consider the balayage problem for the domain bounded by the ellipsoid of rotation

$$\Omega = \left\{ (x^1)^2 + (x^2)^2 + \frac{(x^3)^2}{a^2} \le 1 \right\}, \qquad a \ne 1.$$

Here, as before, we assume that the original distribution of charges is defined by an entire function.

Lemma 15. *The singularities of the solution of the complex Cauchy problem with zero data on the surface*

$$X = \{(x^1)^2 + (x^2)^2 + (x^3)^2/a^2 = 1\}$$

and entire right-hand side are contained in the set

$$L = \left\{ (x^1)^2 + (x^2)^2 + \left(x_3 \pm \sqrt{a^2-1} \right)^2 = 0 \right\} \cup \{(x^1)^2 + (x^2)^2 = 0\}. \qquad (35)$$

In this case there are two possibilities.

1. Let $a > 1$ (prolate ellipsoid, see Figure 7). In this case complex rain gives the real singularity set

$$L_{\mathbb{R}} = L \cap \mathbb{R}^3 = \{x^1 = x^2 = 0\} \cup \{x^1 = x^2 = 0, x^3 = \pm\sqrt{a^2-1}\}.$$

This set is a union of the vertical line, which is divided by two points into two rays and a segment. Since the solution has no singularities on $\partial\Omega$, the singularity set is actually contained in the segment

$$\{x^1 = x^2 = 0, |x^3| \le \sqrt{a^2-1}\}.$$

The complement of this segment is a simply-connected domain and no cuts are needed.

2. Let $0 < a < 1$ (oblate ellipsoid, see Figure 8). In this case complex rain gives the real singularity set

$$L_{\mathbb{R}} = L \cap \mathbb{R}^3 = \{x^1 = x^2 = 0\} \cup \{(x^1)^2 + (x^2)^2 = 1 - a^2, x^3 = 0\}.$$

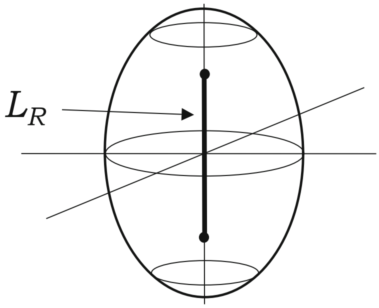

FIGURE 7. Singularities for prolate ellipsoid.

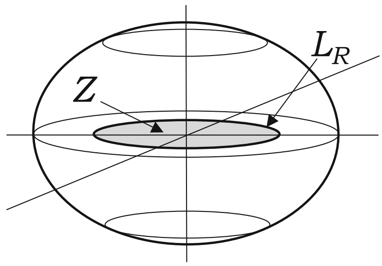

FIGURE 8. Singularities for oblate ellipsoid.

In addition, since u has no singularities on $\partial\Omega$, the singularities lie on the circle

$$\{(x^1)^2 + (x^2)^2 = 1 - a^2, x^3 = 0\}.$$

Making a cut along the disc, which bounds this circle, we obtain a simply-connected domain. The solution is a single-valued function in this domain.

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Asymptotic Properties of Solutions of Neutral Type Difference System with Delays

Ewa Schmeidel, Joanna Zonenberg and Barbara Łupińska

Abstract. We consider a three-dimensional nonlinear difference system with deviating arguments of the following form

$$\begin{cases} \Delta(x_n + p_n x_{n-\tau}) = a_n f(y_{n-l}) \\ \Delta y_n = b_n g(w_{n-m}), \\ \Delta w_n = \delta c_n h(x_{n-k}) \end{cases}$$

where the first equation of the system is a neutral type difference equation. First, the classification of nonoscillatory solutions of the considered system is presented. Next, we present the sufficient conditions for boundedness of a nonoscillatory solution. The obtained results are illustrated by examples.

Mathematics Subject Classification (2010). 39A10, 39A11, 39A12.

Keywords. Difference equation, neutral type, nonlinear system, nonoscillatory, bounded, unbounded solution.

1. Introduction

We consider a nonlinear three-dimensional difference system of the form

$$\begin{cases} \Delta(x_n + p_n x_{n-\tau}) = a_n f(y_{n-l}) \\ \Delta y_n = b_n g(w_{n-m}), \\ \Delta w_n = \delta c_n h(x_{n-k}) \end{cases} \tag{1}$$

where Δ denotes the forward difference operator $\Delta z_n = z_{n+1} - z_n$ for any real sequence (z_n), $n \in \mathbb{N}_{n_0} = \{n_0, n_0+1, \dots\}$, $n_0 = \max\{l, m, k, \tau\}$, l, m, k, τ are non negative integers, (p_n) is a real sequence and $\delta = \pm 1$. Here sequences (a_n), $(b_n)\colon \mathbb{N} \to \mathbb{R}_+ \cup \{0\}$, $(c_n)\colon \mathbb{N} \to \mathbb{R}_+$, where $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ denote the set of positive integers, real numbers and the set of positive real numbers, respectively. Moreover

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \infty. \tag{2}$$

Assume that $f, g, h\colon \mathbb{R} \to \mathbb{R}$ are functions such that there exist positive, different constants M_1, M_2, M_3 such that

$$\frac{f(u)}{u} \geq M_1, \ \frac{g(u)}{u} \geq M_2 \text{ and } \frac{h(u)}{u} \geq M_3 \text{ for } u \neq 0. \tag{3}$$

Set $M = \min\{M_1, M_2, M_3\}$.

By a solution of the system (1) we mean a real sequence

$$(x, y, w) = ((x_n), (y_n), (w_n))$$

which is defined for all $n \in \mathbb{N}$ and satisfies (1) for all $n \in \mathbb{N}_{n_0}$. A solution (x, y, w) of the system (1) is called nonoscillatory if all its components are nonoscillatory (that is either eventually positive or eventually negative). Otherwise it is called oscillatory. A solution (x, y, w) of the system (1) is called bounded if all its components are bounded. Otherwise it is called unbounded.

If the sequences $(a_n), (b_n)$ are positive, f, g are linear functions and $l = 0,\ m = 0$, the system (1) reduces to the third-order neutral type difference equation

$$\Delta\left(\frac{1}{b_n}\left(\Delta\frac{1}{a_n}\Delta(x_n + p_n x_{n-\tau}\right)\right) = \delta c_n h(x_{n-k}).$$

Such equations and their special cases have been studied by many authors, see for example, [1–4] and the references cited therein.

The background for difference systems can be found in the well-known monographs, see, for example, Agarwal [5], Agarwal, Bohner, Grace and O'Regan [6], Kocić and Ladas [7]. Usually, they consider two-dimensional difference systems (see, for example, [8–12]). Oscillatory results for three-dimensional system are investigated by Schmeidel in [13–15], Thandapani and Ponnammal in [16]. Results which are presented in this paper partially answered the open problem stated in [16].

In 2011, Schmeidel considered system (1) under less restrictive assumptions on (p_n), namely $\lim\limits_{n\to\infty} p_n = p \in \mathbb{R}$, $|p| \neq 1$. In Ref. [15] classification of the companion sequence of (x_n), and sequences (w_n), (y_n) was obtained. Here, we present a classification of components of solution of system (1). In [15], the author considered two cases: that sequences (x_n) and its companion sequence have the same sign and that they have opposite sign. In the present paper only the first case is considered. The presented theorems are obtained under more restrictive assumption on (p_n) but less restrictive assumptions on sequence (x_n) then in [15].

2. Some Basic Lemmas

Set

$$z_n = x_n + p_n x_{n-\tau}. \tag{4}$$

The sequence (z_n) is called companion sequence of the sequence (x_n) relative to (p_n).

We begin with some lemmas which will be useful for proving the main result of this paper. In 2005, Migda and Migda presented the following result (see [17, Lemma 1]).

Lemma 1. *Let* (x_n), (p_n) *be real sequences and* (z_n) *be a sequence defined by* (4), *for* $n \geq \tau$. *Assume that* (x_n) *is bounded,* $\lim_{n\to\infty} z_n = l \in \mathbb{R}$, $\lim_{n\to\infty} p_n = \bar{p} \in \mathbb{R}$. *If* $|\bar{p}| \neq 1$, *then* (x_n) *is convergent and* $\lim_{n\to\infty} x_n = \frac{l}{1+\bar{p}}$.

In [18] Jankowski, Schmeidel and Zonenberg presented the following two Lemmas:

Lemma 2. *Assume that* $x\colon \mathbb{N} \to \mathbb{R}$ *and*

$$\lim_{n\to\infty} p_n = p \quad \textit{where} \quad |p| < 1.$$

If sequence (z_n) *defined by* (4) *is bounded, then sequence* (x_n) *is bounded too.*

Lemma 3. *Assume that*

$$\lim_{n\to\infty} p_n = p \quad \textit{and} \quad p \geq 0$$

in (4). *If* $\lim_{n\to\infty} z_n = \infty$ *then* $\lim_{n\to\infty} x_n = \infty$.

The following Lemmas will be used.

Lemma 4. *Assume that condition* (3) *is satisfied. Let* (x, y, w) *be a solution of the system* (1) *such that sequence* (x_n) *is nonoscillatory. Then* (x, y, w) *is nonoscillatory and sequences* (y_n) *and* (w_n) *are monotonic for sufficiently large* n.

Lemma 5. *Assume that*

$$\lim_{n\to\infty} p_n = p \textit{ and } p \geq 0 \tag{5}$$

and condition (3) *is satisfied. Let* (x, y, w) *be a solution of the system* (1) *and let sequence* (y_n) (*or* (w_n)) *be nonoscillatory. Then there exists limit of sequence* (x_n) *and exactly one of the following two cases holds*

1. (x, y, w) *is nonoscillatory and sequences* (z_n), (y_n) *and* (w_n) *are monotonic for sufficiently large* n
2. $\lim_{n\to\infty} x_n = 0$.

Lemma 6. *Assume that conditions* (2), (3) *and* (5) *are satisfied. Let* (x, y, w) *be a solution of the system* (1). *If* (x_n) *is nonoscillatory and*

$$\lim_{n\to\infty} x_n \in \mathbb{R},$$

then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} w_n = 0.$$

Lemma 7. *Assume that conditions* (2), (3) *and* (5) *are satisfied, and* (x, y, w) *is a nonoscillatory solution of the system* (1)*. Then exactly one of the following three cases holds*

(I) $\operatorname{sgn} x_n = \operatorname{sgn} y_n = \operatorname{sgn} w_n$,

(II) $\operatorname{sgn} x_n = \operatorname{sgn} w_n \neq \operatorname{sgn} y_n$,

(III) $\operatorname{sgn} x_n = \operatorname{sgn} y_n \neq \operatorname{sgn} w_n$,

for large n*. Moreover, if* $\delta = -1$ *in the system* (1)*, then every nonoscillatory solution of* (1) *fulfills condition* (I) *or* (II)*, if* $\delta = 1$*, then every nonoscillatory solution of* (1) *fulfills condition* (I) *or* (III)*.*

For the proofs of Lemmas 4–7 see [15].

3. Main Results

Theorem 8. *Assume that conditions* (2), (3) *and* (5) *are satisfied. Then every nonoscillatory solution* (x, y, w) *of the system* (1) *fulfilling condition* (I) *is unbounded.*

Proof. Let (x, y, w) be a nonoscillatory solution of the system (1) for which condition (I) is satisfied. Without loss of generality, assume that $x_n > 0$, $y_n > 0$ and $w_n > 0$ for large n, say $n \geq n_1$. From these and from the second equation of the system (1), we see that sequence (y_n) is eventually increasing. Summing the first equation of the system (1) from $n_2 = n_1 + l$ to $n - 1$ we have

$$z_n = z_{n_2} + \sum_{i=n_2}^{n-1} a_i f(y_{i-l}), \text{ for } n \geq n_2.$$

Since $y_n > 0$ for $n \geq n_1$, by the first equation of the system (1) and condition (3), we get that sequence (z_n) is monotonic. Then there exists $\lim_{n\to\infty} z_n$. Since $x_n > 0$ for sufficiently large n and by condition (3), we get $\lim_{n\to\infty} z_n = L^* \geq 0$. Therefore, there exists an integer $n_2 \geq n_0$ such that $z_n \geq \frac{L^*}{2}$, for $n \geq n_2$. From that, by positivity of y_n and by (3), we obtain

$$z_n \geq \frac{L^*}{2} + M_1 \sum_{i=n_2}^{n-1} a_i y_{i-l}.$$

Since (y_n) is nondecreasing then

$$z_n \geq \frac{L^*}{2} + M_1 y_{n_2 - l} \sum_{i=n_2}^{n-1} a_i.$$

Taking n to infinity, using (2), we obtain that $\lim_{n\to\infty} z_n = \infty$. From the above and by Lemma 3, we see that $\lim_{n\to\infty} x_n = \infty$. Hence, every solution of the system (1) which fulfills (I) is unbounded. $\square$

Example 1. Let us consider the following system of difference equations

$$\begin{cases} \Delta\left(x_n + \dfrac{1}{2^n}x_{n-3}\right) = 8y_{n-2} \\ \qquad\qquad \Delta y_n = 16w_{n-3}, \qquad n \in \mathbb{N}_3. \\ \qquad\qquad \Delta w_n = x_{n-2} \end{cases} \tag{6}$$

Here $p_n = \frac{1}{2^n} \to 0$. All assumptions of Theorem 8 are satisfied. Then every unbounded solution of this system satisfies condition (I). It is easy to see that $(2^n, 2^{n+2}, 2^n)$ is one of such solutions.

Example 2. Let us consider the following system of difference equations

$$\begin{cases} \Delta(x_n + \left(\dfrac{3}{2}\right)^n x_{n-4}) = \left(2 + \dfrac{3^n}{2^{n+2}}\right) y_{n-1} \\ \qquad\qquad \Delta y_n = 2w_{n-2}, \qquad\qquad n \in \mathbb{N}_4. \\ \qquad\qquad \Delta w_n = 4x_{n-1} \end{cases} \tag{7}$$

Here $p_n = \left(\frac{3}{2}\right)^n \to \infty$. All assumptions of Theorem 8 are satisfied. Then every unbounded solution of this system satisfies condition (I). It is easy to see that $(2^n, 2^n, 2^{n+1})$ is one of such solutions.

Theorem 9. *Assume that conditions* (2), (3) *and* (5) *are satisfied. Then every nonoscillatory solution* (x, y, w) *of the system* (1) *fulfilling condition* (II) *is bounded.*

Proof. Assume that (x, y, w) is a nonoscillatory solution of the system (1) which satisfies condition (II). (Notice that, by Lemma 7, this system has such solution if and only if $\delta = -1$.) Without loss of generality, we assume that $x_n > 0$, $y_n < 0$ and $w_n > 0$ for large n. Hence, from the first equation of the system (1), sequence (z_n) is decreasing. Taking it into account, by (5), we get that (z_n) is positive too. We conclude that sequence (z_n) is bounded. In virtue of Lemma 1, we obtain that sequence (x_n) has also finite limit. By Lemma 6, the sequence (w_n) is bounded too. So, also the thesis holds. □

Example 3. Let us consider the following system of difference equations

$$\begin{cases} \Delta\left(x_n + \dfrac{1}{2}x_{n-1}\right) = \dfrac{2^{2n}}{1+2^{2n}}\left((y_{n-1})^3 + y_{n-1}\right) \\ \qquad\qquad \Delta y_n = w_n, \qquad\qquad n \in \mathbb{N}. \\ \qquad\qquad \Delta w_n = -2^{-3}x_n \end{cases} \tag{8}$$

Here $\delta = -1$. We notice that all assumptions of Theorem 9 are satisfied. Thus every unbounded solution of this system satisfies condition (I). It is easy to see that $(2^{-n}, -2^{-n-1}, 2^{-n-2})$ is one of such solutions.

Theorem 10. *Assume that conditions* (2), (3) *and* (5) *are satisfied, and*

$$\sum_{n=1}^{\infty} c_n = \infty. \tag{9}$$

Then system (1) *has no nonoscillatory solution* (x, y, w) *which fulfilled condition* (III)*.*

Proof. Assume to the contrary that (x, y, w) is a nonoscillatory solution of system (1) which satisfied condition (III). Notice that, by Lemma 7, this system has such solution if and only if that $\delta = 1$ in the third equation of the system. Without loss of generality $x_n > 0$ for large n, say $n \geq n_3$. Hence by Lemma 7 we have $y_n > 0$ and $w_n < 0$ for large n. Therefore by $x_n > 0$ and (5) we have that (z_n) is eventually positive. With the first equation of system (1) we obtain, that sequence (z_n) is nondecreasing and with the third equation of system (1) we obtain that (w_n) is increasing. Since $w_n < 0$ we have that $\lim_{n\to\infty} w_n = L^{**} \leq 0$. Suppose that $\lim_{n\to\infty} w_n = L^{**} < 0$. Then there exists $n_4 \in \mathbb{N}$ such that $w_n \leq L^{**}$ for $n \geq n_4$. Summing the second equation of system (1) from $n_5 = \max\{n_3 + k, n_4 + m\}$ to $n - 1$ we obtain

$$y_n = y_{n_5} + \sum_{i=n_5}^{n-1} b_i g(w_{i-m}) \text{ for } n \geq n_5.$$

Hence, by negativity of sequences (w_n), (3), we get

$$y_n \leq y_{n_5} + M_2 \sum_{i=n_5}^{n-1} b_i w_{i-m} < y_{n_5} + M_2 L^{**} \sum_{i=n_5}^{n-1} b_i,$$

for $n \geq n_5$. Letting n to infinity and using (2) we obtain $\lim_{n\to\infty} y_n = -\infty$. This contradicts positivity of sequence (y_n). So, $\lim_{n\to\infty} w_n = 0$.

Summing the third equation of system (1) from $n_6 = n_5 + k$ to $n - 1$, we have

$$w_n = w_{n_6} + \sum_{i=n_6}^{n-1} c_i h(x_{i-k}) \text{ for } n \geq n_6.$$

Then, by (3), we obtain

$$w_n \geq w_{n_6} + M_3 \sum_{i=n_6}^{n-1} c_i x_{i-k}.$$

Since (z_n) is nondecreasing and $z_n > 0$, there exists $\lim_{n\to\infty} z_n \geq 0$. Hence, by (5) and Lemma 5 obtain that there exists $\lim_{n\to\infty} x_n = L^{***} \geq 0$ too. Then there exists sufficiently large n, say $n \geq n_7$ such that $x_n \geq \frac{L^{***}}{2}$ for $n \geq n_7$. Hence

$$w_n \geq w_{n_6} + \frac{M_3 L^{***}}{2} \sum_{i=n_6}^{n-1} c_i \quad \text{for} \quad n > \max\{n_6, n_7\}.$$

The left-hand side of the above inequality tends to zero whereas the right-hand side, by (9), tends to infinity. This contradiction ends the proof. □

As an immediate consequence of Lemma 7, Theorem 8 and Theorem 10 we get the following result.

Corollary 11. *Let $\delta = 1$. Assume that conditions* (2), (3), (5) *and* (9) *are satisfied. Then every nonoscillatory solution of system* (1) *is unbounded.*

Remark 12. Corollary 11 may be formulated as follows. Let $\delta = 1$. Assume that conditions (2), (3), (5) and (9) are satisfied. Then every bounded solution of system (1) is oscillatory.

Example 4. Let us consider the following system of difference equations

$$\begin{cases} \Delta\left(x_n + \dfrac{1}{4}x_{n-1}\right) = \dfrac{3}{8}y_{n-1} \\ \Delta y_n = \dfrac{3}{2}\dfrac{2^{3n-3}}{2^n + 2^{3n-2}}\left((w_{n-1})^3 + w_{n-1}\right), \qquad n \in \mathbb{N}, \\ \Delta w_n = \dfrac{3}{4}x_{n-1} \end{cases} \tag{10}$$

with $\delta = 1$. All assumptions of Remark 12 are satisfied. Then every bounded solution of this system is oscillatory. It is easy to see that $\left(\frac{(-1)^n}{2^n}, \frac{(-1)^n}{2^n}, \frac{(-1)^n}{2^n}\right)$ is one of such solutions.

Example 5. Let us consider the following system of difference equations

$$\begin{cases} \Delta\left(x_n + \dfrac{1}{n}x_n\right) = \dfrac{2n^2+4n+1}{n^2+n}y_{n-1} \\ \Delta y_n = 2w_{n-1}, \qquad n \in \mathbb{N}, \\ \Delta w_n = 2x_{n-1} \end{cases} \tag{11}$$

where $\delta = 1$. We notice that all assumptions of Remark 12 are satisfied. Then every bounded solution of this system is oscillatory. So,$((-1)^n, (-1)^n, (-1)^n)$ is one of such solutions.

The next theorem shows that if $\delta = -1$ system (1) can have oscillatory bounded solutions, too.

Theorem 13. *Let $\delta = -1$. Assume that conditions* (2), (3) *and* (5) *are satisfied. Let (x, y, w) be a bounded solution of* (1) *with (y_n) (or (w_n)) bounded away from zero. Then the solution (x, y, w) is oscillatory.*

Proof. For the sake of contradiction, let (x, y, w) be a nonoscillatory, bounded solution of (1). Then (x_n), (y_n) and (w_n) are nonoscillatory and bounded. By Lemma 7 and Theorem 8, such solution is of type (II). Let $x_n < 0$, $y_n > 0$ and $w_n < 0$ for $n \geq n_8$. Summing the first equation of system (1) from $n = n_8$ to $n-1$ and using (4), we get

$$z_n = z_{n_8} + \sum_{i=n_8}^{n-1} a_i f(y_{i-l}).$$

By condition (3), we obtain

$$z_n \geq z_{n_8} + M_1 \sum_{i=n_8}^{n-1} a_i y_{i-l}. \tag{12}$$

Since (y_n) is positive and bounded away from zero, there exists a constant $\varepsilon > 0$ such that $y_n \geq \varepsilon$ for all n. Then, by (12), we get

$$z_n \geq z_{n_8} + M_1 \varepsilon \sum_{i=n_8}^{n-1} a_i.$$

Letting n to infinity and using (2) we obtain $\lim\limits_{n\to\infty} z_n = \infty$. Hence, by Lemma 3, we get $\lim\limits_{n\to\infty} x_n = \infty$. This contradicts the boundedness of (x_n). If (w_n) is bounded away from zero, then summing the second equation of the system (1) and using similar arguments we get a contradiction. This completes the proof. □

Similarly, we can prove the next theorem.

Theorem 14. *Let $\delta = -1$. Assume that conditions (2), (3), (5) and (9) are satisfied. Let (x, y, w) be a bounded solution of (1) with (x_n) bounded away from zero. Then the solution (x, y, w) is oscillatory.*

Example 6. Let us consider the following system of difference equations

$$\begin{cases} \Delta\left(x_n + \dfrac{n}{2n+1} x_{n-1}\right) = \dfrac{2(4n^2+10n+5)}{2n+1} y_n \\[2ex] \Delta y_n = \dfrac{4(n+2)}{2n+3} w_n, \qquad\qquad n \in \mathbb{N}, \\[2ex] \Delta w_n = -\dfrac{2n+6}{(2n+5)(2n+7)} x_n \end{cases} \tag{13}$$

where $\delta = -1$. We notice that all assumptions of Theorem 14 are satisfied. Hence, every bounded solution of system (13), with first component bounded away from zero, is oscillatory. One such solution is $\left(2(-1)^n, \frac{(-1)^{n+1}}{2n+3}, \frac{(-1)^n}{2n+5}\right)$.

Note that if one component of the solution of system (1) is oscillatory, then other components of this solution are oscillatory, too.

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Orbits of Darboux Groupoid for Hyperbolic Operators of Order Three

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Abstract. Darboux transformations are viewed as morphisms in a Darboux category. Darboux transformations of type I which we defined previously, make an important subgroupoid. We describe the orbits of this subgroupoid for hyperbolic operators of order three.

We consider the algebras of differential invariants for our operators. In particular, we show that the Darboux transformations of this class can be lifted to transformations of differential invariants (which we calculate explicitly).

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1. Introduction

Darboux transformations are discrete symmetries of linear differential equations, ordinary or in partial derivatives, and are also the basis of Bäcklund transformations of the non-linear theory. A description of all Darboux transformations for a given equation is an open problem. For the current state of the art see [3–5, 7, 14, 17, 18].

The best studied Darboux transformations are those that are obtained from a number of linearly independent solutions through the Wronskian formulas. In this way one can construct Darboux transformations for equations of a very general form [6]. For practical purposes one would want to have other types of Darboux transformations too, since 1) the transformations of Wronskian type are not invertible; 2) the other popular special case of Darboux transformations – two Laplace transformations – is not in this class (in particular they are invertible).

Recall that the Laplace transformations are two transformations defined for a very special type of equations, namely, second-order hyperbolic equations in two independent variables. They are related to factorizations of the corresponding operator. The known Laplace transformation method is based on the fact that Laplace transformations can be described by transformations of the corresponding gauge invariants. After a long history [1, 2, 8, 10, 11, 13, 19, 20], the hypothesis that can be traced back to Darboux was proved [14] that the Darboux transformations of any order for such equations are either of Wronskian type, or can be decomposed into a product of Laplace transformations.

In [12] it was first noted that 1) there are a number of examples of operators of third order of two independent variables where the "Wronskian" method fails; 2) there are a number of examples of operators of third order of two independent variables which admit invertible Darboux transformations. In this case we cannot pose the question whether these invertible transformations are a product of Laplace transformations or not, since Laplace transformations are only defined for the equation mentioned above.

In [11] we suggested an algebraic formalism for Darboux transformations. In [15] for operators of arbitrary orders and depending in arbitrary number of independent variables we singled out a class of invertible Darboux transformations – the *Darboux transformations of type I.* The inverse of Darboux transformations of type I can be described by compact formulas at the operators level. Following an analogy with Laplace transformations, in this paper we describe the orbits of Darboux transformations of type I in terms of differential invariants.

We consider Darboux transformations of type I for hyperbolic operators L of order three in two independent variables. We obtain the following results:

1. If we fix the principal symbol of auxiliary operator M (for example, $\sigma(M) = p_x$) and consider all possible Darboux transformations of the type I with $M = \partial_x + m$, where m is a parameter, then all of them just transform L into operators that differ by a gauge transformations. We obtained the exact formulas in terms of differential invariants using the generating set found in [9, 16].
2. We observed that Darboux transformations of the type I have their own invariants that can be given in terms of the gauge invariants of L.
3. We discovered the first example of a family of operators of order three for which there is an analogue of the Laplace infinite chain, and which we can solve using our method, while the major Computer Algebra Systems cannot.
4. We obtained a number of examples of orbit with non-trivial topological structure.

Overall, this work announces the first interesting progress on the orbits of Darboux groupoid.

2. Groupoid of Darboux transformations

Our approach is algebraic: we let K be a differential field of characteristic zero with commuting derivations ∂_x, ∂_y, and $K[D] = K[D_x, D_y]$ be the corresponding ring of linear partial differential operators over K, where D_x, D_y correspond to derivations ∂_x, ∂_y.

Definition 1. Consider category $\mathrm{Dar}(S)$:

1. The objects of $\mathrm{Dar}(S)$ are operators from $K[D]$ with the same arbitrary but fixed principal symbol S.
2. Pair (M, N) is a morphism with source L and target L_1 if
$$NL = L_1 M \quad \text{up to equivalence} \quad (M, N) \sim (M + AL\ , N + L_1 A)\,,$$
where $A \in K[D]$ is arbitrary. Another notation for this morphism indicating the source and target, is $L \xrightarrow{(M,N)} L_1$.
3. Consecutive morphisms can be composed:
$$(M, N) \cdot (M_1, N_1) = (M_1 M, N_1 N).$$
4. For any object L *the identity morphism* is $1_L = (1, 1)$.

The morphisms are called *Darboux transformations.*

In [11] it was proved that the composition of morphisms is well defined. It follows that if a Darboux transformation (M, N) with source L and target L_1 has an inverse (M', N'), then for some $A, G \in K[D]$ the following equalities hold:

$$M'M = 1 + AL\,, \tag{1}$$
$$MM' = 1 + GL_1\,, \tag{2}$$
$$N'N = 1 + LA\,, \tag{3}$$
$$NN' = 1 + L_1 G\,. \tag{4}$$

In [15] we showed that auxiliary operators A and G are related:

$$GN = MA\,,$$

and that two of the four conditions above are redundant: (3) and (4) follow from (1) and (2).

In general case, the automorphisms in this category are not trivial.

Example. For a given $L \in K[D]$, most of $M \in K[D]$ does not correspond to any Darboux transformation. Here are some known examples of Darboux transformations:

1. For any $L \in K[D]$ there is always a trivial Darboux transformation: $N = L_1$ and $M = L$, that is the intertwining relation becomes $L_1 L = L_1 L$. Those are clearly non-invertible by, e.g., failing condition (1).

2. The famous two Laplace transformations are defined for operators of the form $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$, $a, b, c \in K$ only. They are defined by setting $M = \partial_x + b$ or $M = \partial_y + a$ (then the corresponding N, L_1 can be found easily). They can be applied if quantities known as Laplace invariants, h and k are not zero, respectively. This condition implies that the transformations are invertible.
3. For different types of $L \in K$, one takes $\psi \in K[D_x, D_y]$, $L(\psi) = 0$ and constructs $M = \partial_x - \psi_x/\psi$. This M often leads to a Darboux transformation.
4. The fact that $M = \partial_x - \psi_x/\psi$ not always lead to a Darboux transformations was first discovered in [12].

3. Darboux transformations of type I

In [15] we singled out a class of Darboux transformations which can be completely described on the level of operators: it consists of Darboux transformations (M, N) with source L such that

$$L = CM + f\,, \;\; f \in K\,.$$

Here $L, M \in K[D]$. These transformations are always invertible, and the explicit formulas for the target L_1, N and its inverse (M', N') are

$$L_1 = M^{1/f}C + f\,, \quad N = M^{1/f}\,, \; M' = -\frac{1}{f}C\,, \quad N' = -C\frac{1}{f}\,,$$

and the auxiliary operators from (1) and (2) are $A = G = 1/f$. We shall refer to such (M, N) as to *Darboux transformations of type I.*

This class is large enough. Indeed, Laplace transformations, and every invertible Darboux transformation of first order (where the order of a Darboux transformation (M, N) is the order of M and N) is of type I.

4. Darboux transformations for gauge differential invariants

For operators of the form

$$L = \partial_x\partial_y(\partial_x + \partial_y) + a_{ij}\partial_x^i\partial_y^j\,, \tag{5}$$

where we use multi-index notation, $a_{ij} \in K$, $i, j = 0, 1, 2$ under gauge transformations $L \to e^{-g}Le^g$, $g \in K \setminus \{0\}$, a generating set for differential invariants is known [9, 16]:

$$\begin{aligned}
I_1 &= \, -2a_{20} + a_{11} - 2a_{02}\,,\\
I_2 &= \partial_x(a_{20}) - \partial_y(a_{02})\,,\\
I_3 &= a_{10} + a_{20}(a_{20} - a_{11}) + \partial_y(a_{20} - a_{11})\,,\\
I_4 &= a_{01} + a_{02}(a_{02} - a_{11}) + \partial_x(a_{02} - a_{11})\,,\\
I_5 &= a_{00} - a_{01}a_{20} - a_{10}a_{02} + a_{02}a_{20}a_{11} + \\
&\qquad + (2a_{02} - a_{11} + 2a_{20})\partial_x(a_{20}) + \partial_{xy}(a_{20} - a_{11} + a_{02})\,.
\end{aligned}$$

Lemma 2. *Operator of form* (5) *has an invertible Darboux transformation*

1. *With some* $M = \partial_x + m$ *if and only if* $I_4 + 2I_2 = I_{1x}$*;*
2. *With some* $M = \partial_y + m$ *if and only if* $I_{1y} + 2I_2 = I_3$*;*
3. *With some* $M = \partial_x + \partial_y + m$ *if and only if* $I_3 - I_4 = I_2$*.*

Here $m \in K$*.*

Lemma 3. *Invertible Darboux transformations for* (5) *with* M *of the form* $M = \partial_x + m$ *can be lifted to gauge invariants:* $(I_1, I_2, I_4, I_4, I_5) \mapsto (J_1, J_2, J_3, J_4, J_5)$*, where*

$$\begin{aligned}
J_1 &= I_1 - T_x\,,\\
J_2 &= I_2 + T_{xy}\,,\\
J_3 &= I_3 + I_2 + T_{xy}\,,\\
J_4 &= 0\,,\\
J_5 &= f - I_{4y}/2 + I_{3x} - I_{2y} + T_{xxy}/2\,,
\end{aligned}$$

where $f = I_5 - I_1 I_2 - I_{4y}/2$*,* $T = \ln(f)$*. Here* $f \neq 0$ *and* $L = CM + f$ *for some* $C \in K[D]$*.*

With M *of the form* $M = \partial_y + m$*:*

$$\begin{aligned}
J_1 &= I_1 - T_y\,,\\
J_2 &= I_2 - T_{xy}\,,\\
J_3 &= 0\,,\\
J_4 &= I_4 - I_2 + T_{xy}\,,\\
J_5 &= I_1 I_2 - I_1 T_{xy} - I_2 T_y - I_{3x}/2 + I_{2x} + I_{4y} + T_{xyy}/2 + T_{xy} T_y + f\,,
\end{aligned}$$

where $f = I_5 - I_{3x}$*,* $T = \ln(f)$*. Here* $f \neq 0$ *and* $L = CM + f$ *for some* $C \in K[D]$*.*

With M *of the form* $M = \partial_x + \partial_y + m$*:*

$$\begin{aligned}
J_1 &= I_1 + T_x + T_y\,,\\
J_2 &= I_2\,,\\
J_3 &= -I_4 + 2I_3 + I_{1y} + T_{xy} + T_{yy}\,,\\
J_4 &= I_{1x} + 2I_4 - I_3 + T_{xx} + T_{xy}\,,\\
J_5 &= I_{4y} + I_{3x} + I_{1xy}/2 - I_1 I_4 - T_x I_4 - T_y I_4 + T_{xxy}/2 + T_{xyy}/2 + f\,,
\end{aligned}$$

where $f = I_5 + I_1 I_4 + I_{1xy}/2$*,* $T = \ln(f)$*. Here* $f \neq 0$ *and* $L = CM + f$ *for some* $C \in K[D]$*.*

Lemma 4. *The following are invariants of invertible Darboux transformations:*

$$\begin{aligned}
A_x = I_2 + I_{1y}\ , &\quad \text{in case} \quad \sigma(M) = p_x\,,\\
A_y = I_2 - I_{1y}\ , &\quad \text{in case} \quad \sigma(M) = p_y\,,\\
I_2\ , &\quad \text{in case} \quad \sigma(M) = p_x + p_y\,.
\end{aligned}$$

5. New integrable PDE(s) of third order

From conditions above, it follows that if an infinite sequence of Darboux transformations of type I with $\sigma(M) = p_x$ exists, then starting from at least the second term of the sequence $I_4 = 0$, and therefore, $2I_2 = I_{1x}$.

For example, we set $I_4 = 0$, $2I_2 = I_{1x}$, and $I_1 = 2x - y$, $I_2 = I_3 = 1$, $I_5 = 0$. In particular, $L_0 = \partial_x\partial_y(\partial_x + \partial_y) - y\partial_x\partial_y + (x-y)\partial_y^2 - x^2 + xy - 1$ belongs to this gauge equivalence class. This operator has two different factorizations into three factors each. The factors have symbols $p_y, p_x, p_x + p_y$ and $p_x, p_x + p_y, p_y$, reading from the left to the right. Thus, the partial differential equation corresponding to L_0 is easy to solve provided we know how to solve first-order partial differential equations in our setting.

On the other hand, there is an infinite chain of invertible Darboux transformations with $\sigma(M) = p_x$ starting at this operator. The nth term, $n \geq 0$, of this chain has invariants

$$\begin{aligned} I_1^n &= 2x - y - 2n/(2x-y)\,, \\ I_2^n &= 1 + 2n/(2x-y)^2\,, \\ I_3^n &= n + 1 + n(n+1)/(2x-y)^2\,, \\ I_4^n &= 0\,, \\ I_5^n &= -2nx + ny - 4n^2/(2x-y)^3\,, \end{aligned}$$

where n is an index.

1. Any partial differential equation having the corresponding values of five invariants can be solved in quadratures using our method.
2. Starting from the second term in the chain none of the corresponding operators are factorizable, so we are getting equations of a new, more complicated type.
3. For a specific equations, let us consider, for example, one of the simplest that correspond to the values of invariants obtained on step two (corresponds to $n = 1$ in the formulas above):

$$\begin{aligned} u_{xxy} + u_{xyy} + \left(x - y - \frac{2}{2x-y}\right) u_{yy} - \left(y + \frac{2}{2x-y}\right) u_{xy} \\ + u_x + \left(xy - x^2 + \frac{y}{2x-y}\right) u_y - x - \frac{2}{2x-y} = 0\,. \end{aligned}$$

The latest version of one of the leading Computer Algebra System MAPLE [18] cannot solve this equation.

6. Orbits structures examples

The structure of the orbit of the famous Laplace transformations is just one chain, infinite or finite. Their analogues for operators of higher order – Darboux transformations of type I appear to generate much more interesting structures.

Below we show a number of examples. In the vertices of the graphs below are the values of the five gauge invariants. Over the arrows we indicate the principal symbol of the auxiliary operator M. If there is no arrow then there is no corresponding Darboux transformation. In other words, all invertible Darboux transformations of order one are shown here.

I. Orbits of finite length:

$$\begin{tikzcd}
(g_x,0,0,g_{xx},e^{g(x)}) \arrow[loop above, "p_y"] \arrow[r, "p_x"] & (0,0,0,0,e^{g(x)}) \arrow[loop above, "p_y"] \arrow[r, "p_x"] \arrow[l, bend left, "p_x+p_y"] & (-g_x,0,0,0,e^{g(x)}) \arrow[loop above, "p_y"] \arrow[l, bend left, "p_x+p_y"]
\end{tikzcd}$$

$$\begin{tikzcd}
(1/x,0,0,-1/x^2,x) \arrow[loop above, "p_y"] \arrow[r, leftrightarrow, "p_x"'] & (0,0,0,0,x) \arrow[loop above, "p_y"] \arrow[r, "p_x"] & (-1/x,0,0,0,x) \arrow[loop above, "p_y"] \arrow[l, bend left, "p_x+p_y"]
\end{tikzcd}$$

$$\begin{tikzcd}
(-1/y+1/x,0,0,-1/x^2,xy) \arrow[r, "p_x"] & (-1/y,0,0,0,xy) \arrow[r, "p_x"] \arrow[dl, "p_x+p_y"] & (-1/x-1/y,0,0,0,xy) \arrow[dl, "p_x+p_y"'] \\
(1/x,0,0,-1/x^2,xy) \arrow[r, "p_x"] \arrow[u, "p_y"] & (0,0,0,0,xy) \arrow[r, "p_x"] \arrow[u, "p_y"] \arrow[dl, "p_x+p_y"] & (-1/x,0,0,0,xy) \arrow[u, "p_y"] \arrow[dl, "p_x+p_y"] \\
(1/x+1/y,0,-1/y^2,-1/x^2,xy) \arrow[r, "p_x"] \arrow[u, "p_y"] & (1/y,0,-1/y^2,0,xy) \arrow[r, "p_x"] \arrow[u, "p_y"] & (1/y-1/x,0,-1/y^2,0,xy) \arrow[u, "p_y"]
\end{tikzcd}$$

II. Infinite orbits:

$$
\begin{tikzcd}
\cdots \arrow[r, "p_y", "p_x"'] & (1,0,0,0,e^{x+y}) \arrow[r, "p_y", "p_x"'] & (0,0,0,0,e^{x+y}) \arrow[r, "p_y", "p_x"'] \arrow[ll, bend left, "p_x+p_y"] & (-1,0,0,0,e^{x+y}) \arrow[r, "p_y", "p_x"'] \arrow[ll, bend left, "p_x+p_y"] & \cdots \arrow[ll, bend left, "p_x+p_y"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,0,x,0,1) \arrow[r, "p_x"] & (0,0,x,0,2) \arrow[r, "p_x"] & (0,0,x,0,3) \arrow[r, "p_x"] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & (-d,2d^2,0,-2d^2,x+y) \arrow[r, "p_x"] \arrow[dl, "p_x+p_y"] & (-2d,0,0,0,x+y+2d^3) \arrow[dl, "p_x+p_y"] \\
 & & & (d,d^2,0,-3d^2,x+y+4d^3) \arrow[r, "p_x"] & (0,0,0,0,x+y) \arrow[u, "p_y"] \arrow[r, "p_x"] \arrow[dl, "p_x+p_y"] & (-d,-d^2,-d^2,0,x+y+d^3) \arrow[u, "p_y"] \arrow[dl, "p_x+p_y"] \\
 & & & (2d,0,-2d^2,-2d^2,x+y+2d^3) \arrow[u, "p_y"] \arrow[r, "p_x"] \arrow[dl, "p_x+p_y"] & (d,-d^2,-3d^2,0,x+y+3d^3) \arrow[u, "p_y"] & \\
 & & (4d,0,-6d^2,-6d^2,x+y+20d^3) \arrow[dl, "p_x+p_y"] & & & \\
 & (6d,0,-12d^2,-12d^2,x+y+66d^3) \arrow[dl, "p_x+p_y"] & & & & \\
\cdots & & & & &
\end{tikzcd}
$$

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Part V: Special Economic Session

Geometric Methods in Physics. XXXIII Workshop 2014
Trends in Mathematics, 315–317

From Nicolaus Copernicus' Economic Law up to the Present Day Economic Disasters (Report of a Dilettante)

Bogdan Mielnik

Abstract. The commercial phenomena described by an almost unknown N. Copernicus treatise on money prepared in the XVIth century on request of the Polish king Sigismund I. the Old turn out relevant for the present day economic situation.

Why dilettante? Since the usual economy publications in scientific journals are dedicated to some narrow problems or just their narrow sections and are so saturated by specialist data that are practically unreadable even for a highly interested reader. Worse: they are deadly boring!

Why Copernicus? Almost everybody knows about Nicolaus Copernicus contribution to astronomy. Less known were the details of his frustrated life, described, e.g., in "Sleepwalkers" by Arthur Koestler [1] who considered Copernicus as a kind of hard working but intimidated clergyman, always careful not to antagonize the church, afraid also about his personal reputation (he had a concubine). The extensive volume of his work was finally prepared only thanks to the firm persistence of his younger follower, a protestant who had not the best opinion in catholic clergy circles. Copernicus himself had a strong support of two Polish bishops (one was his relative), yet he was terribly afraid of possible errors in his theory. His undecidability delayed so much the publication of the book, that he saw it, perhaps without understanding, some hours before his death due to the brain hemorrhage. Almost unknown for the first 300 years, then commented, criticized but widely accepted, his "De revolutionibus orbium coelestium" turned out one of key steps of our cosmology. Though, in fact, for a long time unwelcome by the church. Enough to mention that it was in the "Index Librorum Prohibitorum" until 1828.

The law of two Nicolai. Much less known is the fact that at the beginning of his career, still as a young man, Copernicus was the author of an important economic treaty. The first draft of "De estimacione monetae", written already in 1517 contained an observation that 'the bad coins' push out of the market the good ones.

The law was again formulated in the famous "De monetae cudendae ratione" in 1526 on the request of the Polish king Sigismund I the Old and it was presented personally, by the young canon to the town council of Grudziadz. Some concepts introduced in his work, such as the difference between the use value and exchange value were applied by Adam Smith, though they were known already to Aristotle. Some years after, an equivalent law was discovered by Sir Thomas Gresham and since then it is referred to as the Gresham-Copernicus law (why Gresham first? After all, he was Sir!).

Curiously, the law was noticed much earlier by another Nicolaus, Nicole Oresme a medieval bishop, interested as well in cosmology! [1] Yet, the bishop's work on cosmology was dedicated to a mathematical proof that the celestial spheres rotate around the Earth which is the fixed center of the universe, so I have more confidence to both laws of the second Nicolaus. However, why was the Copernicus law so important? Basically, his thesis described the metal coins, their contents of gold or silver. Yet, it seems that its validity obeys no such limits...

The great crisis. To illustrate this, let us make a jump from 1517 up to 1928, when the money were not necessarily coins. The great crisis affected even the most powerful people. Some of them, including bankers, were suiciding themselves jumping from the high floors of their banks. There seemed to be no remedy, certainly not the humorous but cruel advice of the famous Argentinian writer, Ramón Gómez de la Serna:

> If you decide to commit a suicide jumping from a skyscraper, don't choose a too high one, so that you won't change your mind in the middle of your flight

Curiously, a more helpful idea belonged to the known British economist John Maynard Keynes. Aware of the role of unemployment in the crisis, he advised his government; "Just make great investments to assure that people will have jobs, invest and invest, spend your money, even if you don't have it". Paradoxically, what seemed a joke, helped to restore an equilibrium before the World War II. The Keynesian economy centers appeared in many American and European universities. Can the 1928 crisis contribute to the present day economical knowledge?...

The following paper "Economics – Physics of Social Sciences or Art?" by L. Hardt introduces the present day view upon economics, but we must remember that the roots of economy lay far away in the past...

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[1] According to some historians, the parallel discoveries of important laws are not an exception but the rule in science; cf. Robert K. Merton [2]

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Geometric Methods in Physics. XXXIII Workshop 2014
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Economics – Physics of Social Sciences or Art?

Economic talk at the XXXIII Workshop on Geometric Methods in Physics

Lukasz Hardt

Abstract. Some questions about the nature of economics are raised, e.g., the one whether economics can give us universal laws describing the workings of the market. The discussion here presented refers also to the debate on the state of the economic theory in the wake of the recent global financial crises. The paper concludes that economics is unable to give us such explanations of real economic processes that do not need further investigations. Also, the author claims that due to the extensive use of metaphors and economists' creativity in modeling economic phenomena economics is close to art, and thus one can even talk about the beauty of the science of Adam Smith.

Keywords. Methodology of economics, economic laws and models, metaphors, economics and physics.

Introduction

In the wake of the post 2008 financial crisis many ask about the nature of economics – is it a real science or just a rhetorical device used by its practitioners to persuade policy makers and their fellow citizens? The debate on the state of economic theory was so intense in the first months of the recession that many leading economists proclaimed that the science of Adam Smith as such is in crisis. Take, for instance, the following opinion by Paul Krugman (2008 Nobel Prize winner):

> [...] the economics profession went astray because economists, as a group, mistook beauty, clad in impressive-looking mathematics, for truth [1].

Or, the one by David Colander and his co-authors:

> In our hour of greatest need, societies around the world are left to grope in the dark without a theory. That, to us, is a systemic failure of the economics profession [2, p. 250].

On the other hand, many popular jokes about economics have been circulating since the collapse of the Lehman Brothers and the resulting panic in the global financial market. JokEc web-page gives, for instance, such a joke:

> "Economics is the only field in which two people can get a Nobel Prize for saying the opposite thing", or even stronger: "Economics is the only field in which two people can share a Nobel Prize for saying opposing things".

The rationale behind the above joke could be the question of what is the nature of economic laws or is it possible to compare conclusively different economic theories or, in other words, whether we are condemned to constantly hear the battles such as the one between Keynesians and monetarists, and hence having Nobel Prize winners sharing the same prize for saying opposing things. These questions matter since economic theory is often used to legitimize economic policy. So, referring now metaphorically to physics one can conclude that economic theory is in entanglement state with economic policy (*my fellow physicist readers, please excuse me if I am using this metaphor incorrectly*). This brings our attention to the problem of reflexivity in social sciences, including economics, where an observer is also a participant in a system and there is a two-way feedback between the observer (here: economist) and the system (here: economy) [3, p. 331]. This is an old problem whether an agent deeply rooted in the system can access the information an agent outside the system would have. Many philosophers argue for impossibility of *God's-eye view* (e.g., W. Quine) while others, more metaphysically oriented, do not deny such a possibility (e.g., T. Lawson in economics). Although these issues are of great importance, I do not want to investigate them in details, but what I would like to do is to come back to the opening question whether we can know for sure the workings of the economy (economics as physics of social sciences) and thus instruct politicians how they should regulate markets, or we are just in a position of artists who contemplate the world (here: economy) and try to describe metaphorically its beauty and immense complexity with little hope to understand its internal logic.

Why physics?

One would ask me – why your *economics as physics of social sciences* stands for economics giving you the truth about the workings of the market, and why not comparing economics to biology or chemistry? Even on a very brief inspection of the language of modern neoclassical economics each physicist is to immediately notice that economics is dominated by *mechanistic* and not *organismic* thinking. If you do not believe me, please run a quick check on JSTOR database and ask how many papers in *American Economic Review* contain references to mechanisms in its titles and how many to organisms. The answer: *mechanistic titles* – 51; *organismic titles* – zero[1]. Such a mechanistic world-view of economics is due to its emergence in the seventeenth and eighteenth century European thought which

[1] *AER* is one of the most prestigious journals in economics. I ran my experiment on papers from this journal since its establishment in 1911 till 2009.

was under the influence of *Descartes-the-Mechanician*. For him, the human body is "just a statue or a machine made of earth" (in: *The Treaty on Man*) and its functioning is due to mechanical principles giving rise to "clocks, artificial fountains, mills, and other machines" (ibid.). Next, Adam Smith – *Adam* and *Smith* of modern economics, influenced by Cartesian thought and Humean empiricism, started to conceptualize the market as a machine. Please let me cite some of his remarks:

> "[...] a machine [educated worker] which abridges labour, and which, though it costs a certain expence, repays that expence with a profit" (*Wealth of Nations*, Book II, Chapter I, no 17);
>
> "[...] the immense machine of the universe" (*Theory of Moral Sentiments*, Part IV, Chapter II, no 48);
>
> "[...] the whole machine of the world" (*Theory of Moral Sentiments*, Part VII, Chapter II, no 41).

In his History of *Ancient Physics* he is even more explicit about his mechanistic philosophy:

> [...] the universe was regarded as a complete machine, as a coherent system, governed by general laws, and directed to general ends, viz. its own preservation and prosperity, and that of all the species that are in it.

After nearly one hundred years since A. Smith, the economics of the late 19th century was even closer to Newtonian physics. L. Walras, one of the founding fathers of neoclassical economics, made the following point in 1874: "[...] economics is a science which resembles the physico-mathematical sciences in every respect" [4, p. 71]. This is a triumph of I. Newton and realization of his dream of "[...] deriving all the phenomena of nature by the same kind of reasoning from mechanical principles" (in: Introduction to *Principles*). So, please do not be surprised by the words of Edgeworth: "the maximum of pleasure in psychics being the effect or a concomitant of a maximum physical energy" [5, p. 89]. The inflow of concepts taken from physics into economics was eased by the fact that at the leading universities in England in 19th century economic students took the same courses in mathematics as the physic students did. Moreover, in order to obtain the final degree in economics students were obliged to write the same exam in mathematics as their fellow colleagues from physics departments were required to pass. This exam was called the Tripos and lasted usually for eight days. Alfred Marshall – the author of *The Principles of Economics* (1890), the fundamental work in neoclassical economics, was obliged to solve, for instance, the problem of the following kind while studying at Cambridge:

> Determine the initial motion of a rigid body which receives a given impulse; and find the screw round which it will begin to twist. A rough inelastic heavy ring rolls, with its plane vertical, down an incline plane, on which lie a series of pointed obstacles which are equal and at equal distance from each other, and which are sufficiently high to prevent

> the ring from touching the plane. If the rings start from rest from a position in which it is in contact with two obstacles, prove that its angular velocity as it leaves the $(n+1)$th obstacle is given by
>
> $$w^2 = 2\frac{g}{a}\sin i \sin\gamma \cos^4\gamma \, \frac{1-\cos^{4n}\gamma}{1-\cos^4\gamma},$$
>
> where a is the radius of the ring, i is the inclination of the plane to the horizon, and $2g/a$ is the angle which two adjacent obstacles subtend at the center of the ring when it is in contact with both [6, p. 78].

After being exposed to such problems no one should be surprised why economists educated at Cambridge put economics into conceptual framework of Newtonian physics. They also started to believe that economic models can give precise insights into the workings of the market in the same vein as experiments in physics reveal fundamental laws of nature. Such an approach was reinforced by popularization of distinctively mathematical explanations in economics (I am to comment on it more extensively later in this essay). So, as Mirowski [7] rightly stressed it, economics in the late 19th century started its way towards becoming a cyborg science without place for moral reasoning and uncertainties in formulating economic laws. Therefore, economics as physics of social sciences follows, but is it really the case?

Economics is producing beliefs not laws

Many economists still treat the claims they are producing as universal regularities in the form of *if X, then* (*always*) *Y*, or, more often, they add the ceteris paribus clause and claim that in unchanged circumstances *X is to* (*always*) *produce Y*. I do not believe in such an interpretation of economic laws. Let me explain. Economists use models to produce theoretical claims. So, they construct artificial worlds by isolating given phenomena and processes. Such worlds usually take the form of mathematically structured entities. For instance, while modelling the price behaviour of firms, economists assume that the price of a given good is determined by the cost of production, precisely p equals the first derivative of cost function (i.e., marginal cost $/MC/$). They are conscious that other factors are also at play but the cost is the most important one and hence they strip off the target of its irrelevant features, thus obtaining a model. We can have multiple models for one target, the models of completely different character and operating in distinct conceptual spaces. For instance, the workings of the economy can be represented by a real machine, e.g., the Phillips-Newlyn hydraulic analogue of U.S. money flow, which can be depicted using diagrams (e.g., Morgan [8, p. 35]), and the diagrams as such can be explained in verbal terms as a fable. Therefore, the model builder acts as an artist, since they both use creativity and imagination in building the artificial worlds. As Frigg [9, p. 251] puts it clear "models share important aspects in common with literary fiction" or in Cartwright's words "a model is a work of fiction" [10, p. 153] and an "intellectual construction" (ibid., p. 144). So, one can model price behaviour of consumers using the artificial world of Tolkien's

Middle-earth, however, Hobbits in such a model should make economic decisions in accordance with neoclassical principle of maximizing utility. So, economic models are neither pure isolations, nor pure constructions, but believable worlds depicting structures that enable the workings of mechanisms that refer to the ones operating in the real world. But such structures can be represented using models of different ingredients, forms, contents, and proprieties.

Theoretical insights produced by models are not directly applicable to the real world. Such insights are always true within models but have the status of beliefs if applied to describe the real economic phenomena, e.g., in economic models p is always to equal MC, however, one cannot anticipate that in the real world this will be always the case, since other factors can be at play, so one can only believe that p is to tend to MC. Thus, we do not test the model as such vis-à-vis the real world, but just "an application of a model, a hypothesis stating that certain elements of a model are approximately accurate or good enough representations of what goes on in a given empirical situation" [11, p. 219]. Next, he adds: "The fact that a model turns out not to work under certain circumstances does not count as a refutation of the model but only as a failed test of its applicability in a given domain"' [11, p. 220]. Therefore, the closer a given empirical domain to the model's structure is, the higher probability that the model's insights are to correctly explain the workings of such a domain. For instance, in neoclassical models we assume that shops can freely set prices thus such models' insights will be more appropriate in free markets (e.g., in the US) than in highly regulated market environments – it is for sure unreasonable to expect North Korean shops to behave in a manner described by models taken from neoclassical economics.

Let me now comment on another factor moving economics closer to art, namely the similarities between modelling and metaphorizing economic phenomena. First, as in the case of isolating a given set of explaining items while building a model the goal is to isolate the most important ones, the same is with metaphors, since the choice of the metaphor (e.g., Aristotelian saying that *Achilles is a lion*) is such as to capture the crucial characteristic of the primary subject (Achilles) by equalizing it to the secondary one (lion). As in the case of isolation, the choice of a given metaphor depends on the researcher's needs. For instance, in describing Southeast Asian economies, one can say 'South Korea is a tiger', if one wants to underline the dynamism, courage, and risk-loving culture of Korean entrepreneurs, or one can just proclaim that 'South Korea is a hidden dragon', if the goal is to focus on the role of consciously built state interventionist economic policy. As in the case of isolation where we do not have a complete isomorphism between the target and the model, the same is with metaphors, since, for instance, saying that 'South Korea is a tiger' means that it is like and is not a tiger. So, models, in this respect, have some similarities with metaphors.

Therefore, modelling economic phenomena is a creative process requiring researchers to build artificial credible worlds capable of representing mechanisms and processes operating in the real markets. Models do not make themselves. They are made by economists. They are made in order to explain, but "forming models is not driven by purely logical process but rather involves the scientist's intuitive, imaginative, and creative qualities" (Morgan [8, p. 25]). So, model builder acts as an artist. The belief produced by a given model is similar to a moral of a fable. Therefore, economics cannot be accused of producing imprecise descriptions of the real world, since this imprecision is given by the very nature of the way economists explain. However, some of them should be criticized for believing in the universal laws of economics. Such laws do not exist. Paradoxically, Krugman is right that economists do not search for truth, however, they try to construct credible beliefs about the real world. On the other hand, Colander and his co-authors are not right in claiming that economists do not offer credible theories – they do it, but their theoretical claims are valid only vis-à-vis these domains that have the structures similar to the ones of the models producing them. For instance, many theoretical insights of labour market models are valid in countries with elastic labour law (e.g., U.S.) and not valid in many overregulated European markets.

Economics is beautiful

In the above section of my paper I stated that economists while building models act as artists. Now, I would like to go even further and claim that economics as such is beautiful not only because it describes the beautiful world but also because its method and language are beautiful. Here I agree with Samuelson and Nordhaus that "Economics is part of both these cultures, a subject that combines the rigors of science with the poetry of humanities" [12, p. 5]. It is worth to mention that also many physicists have an impression that both the world and its mathematical descriptions are beautiful. Heisenberg's words are worth to be mentioned here (*I hope my physicists readers will forgive me such a lengthy quotation*):

> In fact, the last few weeks were full of excitement for me. And perhaps I can best illustrate what I have experienced through the analogy that I have attempted, an as yet unknown ascent to the fundamental peak of atomic theory, with great efforts during the past five years. And now, with the peak directly ahead of me, the whole terrain of interrelationships in atomic theory is suddenly and clearly spread out before my eyes. That these interrelationships display, in all their mathematical abstraction, an incredible degree of simplicity, is a gift we can only accept humbly. Not even Plato could have believed them to be so beautiful. For these interrelationships cannot be invented; they have been there since the creation of the world [13, p. 22].

For sure, the status of economic laws is distinct from the laws of physics, and one cannot claim that *economic laws have been there since the creation of the world*, however, the presence of mathematics in economics make some economic explanations universal and robust to every possible changes in the context. Let me give a very simple example. As L. Robbins defined the economics as "the science which studies human behaviour as a relationship between ends and scarce means which have alternative uses", economists have invested a lot of energy in analysing human decision making. So, the story is the following:

> Imagine that mother has three children and twenty-three apples. She would like to give the same amount of apples to each child without cutting any[2].

She cannot do it since twenty-three cannot be divided evenly by three. So, what explains here is the appeal to essential mathematical proprieties. Moreover, what explicates is the mathematics as such and not its use. In other words, the explanation consists here of mathematically necessary facts (Lange [14, p. 506]). Such distinctively mathematical explanations are simple and thus beautiful, since "mathematics is the archetype of the beautiful" (Kepler). *Simplex sigillum veri. Pulchritudo splendor veritatis.*

So, let me recapitulate the above arguments for the beauty of economics. First, economists use metaphors in constructing economic models, so they act as poets, however, they are constraint by the existence of the external reality which they try to comprehend. Here, I disagree with a well-known Medawar's thesis that when science arrives, it expels literature. Second, the use of mathematics enables economists to dig deeply into the internal (and maybe *eternal*) structure of the reality. Third, as artists do their works after having been fascinated by certain proprieties of the world, the same holds for scientists, including economists. Thus the resulting vagueness of economics cannot be treated as its drawback, but rather as a result of the immense complexity of the world. In many cases economics deals with this complexity quite well, however, its explanations will never be complete, since "There can be no explanation which is not in need of further explanation" [15, p. 195]. This holds for both economics and physics. So, criticizing a given science for its incompleteness is deeply unscientific. However, what links particular sciences and art is a fundamental question – why such an infinite complexity of the world exists and why science works and why art successfully gives us beautiful accounts of the world? So, at the very fundamental level the questions of science, including economics and physics, as well as art converge towards essentially philosophical dilemmas. In such a perspective economics is both physics of social sciences and art. Last but not least, even being solely physics of social sciences economics would not give us final and conclusive explanations about the workings of the market, since each physicist is probably deeply conscious that such all-encompassing explanations do not exist. But this nonexistence is per se beautiful and fascinating, since "As long as a branch of science offers an abundance of problems, so long is it alive" (Hilbert [16, p. 438]). So, science will live forever!

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[2]This example is adapted from the one presented by Lange [14].

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