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Oktay Veliev

Multidimensional Periodic Schrödinger Operator

Perturbation Theory and Applications

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To the memory of my mother and father and to my family.

Preface

The book is devoted to the spectral theory of the multidimensional Schrödinger operator $L(q)$ generated in $L_2(\mathbb{R}^d)$ by the differential expression

$$-\Delta u(x) + q(x)u(x),$$

where $x \in \mathbb{R}^d,\ d \geq 2$ and q is a real periodic, relative to a lattice Ω, potential. This operator describes the motion of a particle in the bulk matter. To describe the brief synopsis of the book let us introduce some notations and recall some well-known definitions. It is well known that the spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$ for $t \in F^*$ generated in $L_2(F)$ by the same differential expression and the conditions

$$u(x+\omega) = e^{i\langle t,\omega\rangle}u(x),\ \forall \omega \in \Omega,$$

where $\langle \cdot,\cdot \rangle$ is the inner product in $\mathbb{R}^d$, t is a crystal momentum (quasimomentum), $F =: \mathbb{R}^d/\Omega$ and $F^* =: \mathbb{R}^d/\Gamma$ are the fundamental domains (primitive cells) of the lattices Ω and Γ respectively, and

$$\Gamma =: \{\delta \in \mathbb{R}^d : \langle \delta, \omega \rangle \in 2\pi\mathbb{Z}, \forall \omega \in \Omega\}$$

is the reciprocal lattice, i.e., is the lattice dual to Ω. The spectrum of $L_t(q)$ consists of the eigenvalues

$$\Lambda_1(t) \leq \Lambda_2(t) \leq \ldots$$

These eigenvalues are called the Bloch eigenvalues. They define functions $\Lambda_n : t \to \Lambda_n(t)$ for $n = 1, 2, \ldots$ of t that are called the band functions of $L(q)$.
The n-th band function Λ_n is continuous with respect to t and its range

$$\delta_n =: \{\Lambda_n(t) : t \in F^*\}$$

is the n-th band of the spectrum $\sigma(L(q))$ of $L(q)$:

$$\sigma(L(q)) = \cup_{n=1}^{\infty} \delta_n.$$

The eigenfunctions of $L_t(q)$ are known as the Bloch functions.

The book consists of five chapters. The first chapter presents preliminary definitions and statements to be used in the next chapters. Besides, we give a brief discussion of what is known from the literature and what is presented in the book about the perturbation theory of $L(q)$. In the second chapter, first, we obtain the asymptotic formulas of arbitrary order for the Bloch eigenvalue and Bloch function of the periodic Schrödinger operator $L(q)$ of arbitrary dimension, when the corresponding quasimomentum lies far from the diffraction hyperplanes

$$D_\delta =: \{x \in \mathbb{R}^d : |x|^2 = |x+\delta|^2\}$$

for small values of δ. Then we study the case, when the corresponding quasimomentum lies near a diffraction hyperplane and gets the complete perturbation theory for the multidimensional Schrödinger operator with a periodic potential. Moreover, we construct and estimate the measures of the isoenergetic surfaces in the high energy region which implies the validity of the Bethe-Sommerfeld conjecture for arbitrary dimension and arbitrary lattice. This conjecture was formulated in 1928 and claims that there exist only a finite number of gaps (the spaces between the bands δ_n and δ_{n+1} for $n = 1, 2, \ldots$) in the spectrum $\sigma(L(q))$ of $L(q)$. Note that the construction of the perturbation theory of $L(q)$ is connected with the investigation of the complicated picture of the crystal diffraction. The regular perturbation theory does not work in this case, since the Bloch eigenvalues of the free operator are situated very close to each other in the high energy region.

In the third chapter, using the asymptotic formulas obtained in the second chapter, we determine constructively a family of the spectral invariants of $L(q)$ from the given Bloch eigenvalues. Some of these invariants are explicitly expressed by the Fourier coefficients of the potential which present the possibility of determining the potential constructively by using the Bloch eigenvalues as the input data.

In the fourth chapter, we consider the inverse problems of the three-dimensional Schrödinger operator with a periodic potential q by the spectral invariants obtained in the third chapter. First, we construct a set of trigonometric polynomials which is dense in the Sobolev space $W_2^s(F)$, where $s > 3$, in the $\mathbb{C}^\infty$- topology and every element of this set can be determined constructively and uniquely, modulo inversion $x \to -x$ and translations $x \to x + \tau$ for $\tau \in \mathbb{R}^3$, from the given spectral invariants that were determined constructively from the given Bloch eigenvalues. Then a special class V of the periodic potentials is constructed, which can be easily and constructively determined from the spectral invariants and hence from the given Bloch eigenvalues. Moreover, we consider the stability of the algorithm for the unique determination of the potential $q \in V$ of the three-dimensional Schrödinger operator with respect to the spectral invariants and Bloch eigenvalues.

In the fifth chapter we summarize our results from the point of view of both physicists and mathematicians. I am thankful to Claus Ascheron and Peter Wölfle for their advices that help to improve the readability of the book.

Contents

Chapter 1
Preliminary Facts

Abstract In this chapter we present some definitions and statements from the points of view of both physicists and mathematicians to be used in the next chapters. We mean especially the definitions of the lattices, periodic functions, Brillouin zones, Schrödinger operator, Bloch eigenvalues, Bloch functions, diffraction planes, band structures and Fermi surfaces. Moreover, we try to explain the transition between these notions due to the understanding of the physicists and mathematicians. Besides, we give a brief discussion of what is known from the literature and what is presented in the book about the perturbation theory of the multidimensional Schrödinger operator with a periodic potential. For this aim we consider the large Bloch eigenvalues and the corresponding Bloch functions of the one-dimensional periodic Schrödinger operator by the approach of Chap. 2, since it helps to compare the well-known one-dimensional case with the multidimensional case and to see the complexity of the results obtained in this book.

1.1 Lattices, Brillouin Zones, and Periodic Functions in $\mathbb{R}^d$

The structure of the crystals can be described in terms of the lattice (called in geometry and crystallography, a Bravais lattice), with a group of atoms attached to every lattice point. The Bravais lattice in

$$\mathbb{R}^d =: \left\{(x_1, x_2, \ldots, x_d) : x_1 \in \mathbb{R},\ x_2 \in \mathbb{R}, \ldots,\ x_d \in \mathbb{R}\right\},$$

where $\mathbb{R}$ is the set of all real numbers, is defined by d linearly independent vectors $\omega_1, \omega_2, \ldots, \omega_d$. In the case $d = 3$ these vectors are known as fundamental translations vectors such that every atomic arrangement looks the same in every respect when viewed from the point $\mathbf{r}$ as when viewed from the point

$$\mathbf{r} + \sum_{k=1}^{3} n_k \omega_k,$$

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where n_1, n_2 and n_3 are integers. The lattice Ω generated by the vectors ω_1, $\omega_2, \dots, \omega_d$ is the set of all linear combinations of these vectors with the integer coefficients:

$$\Omega = \left\{ \omega = \sum_{k=1}^{d} n_k \omega_k : n_1 \in \mathbb{Z},\ n_2 \in \mathbb{Z}, \dots, n_d \in \mathbb{Z} \right\}, \tag{1.1.1}$$

where $\mathbb{Z}$ is the set of all integers. The vectors $\omega_1, \omega_2, \dots, \omega_d$ used for the generation of Ω are known as the primitive vectors or basis vectors for the lattice. The parallelotope (d-dimensional parallelogram)

$$F = \left\{ x = \sum_{k=1}^{d} y_k \omega_k : y_1 \in [0,1),\ y_2 \in [0,1), \dots, y_d \in [0,1) \right\} \tag{1.1.2}$$

is called the fundamental parallelotope or the primitive unit cell of the lattice. In the cases $d = 2$ and 3 the parallelotope F is the parallelogram and parallelepiped, respectively. It has the origin in $\mathbb{R}^d$ as one corner and the vectors $\omega_1, \omega_2, \dots, \omega_d$ form the sides which meet at that corner. Thus a crystal is characterized by its regular periodically repeated structure. The smallest unit of this structure is called the primitive unit cell. The primitive cells (parallelotopes) are joined together filling the entire volume and giving rise to the periodicity of the crystal lattice.

The measure $\mu(F)$ (generalized volume) of the parallelotope F is equal to the absolute value of the determinant of the $d \times d$ matrix $(\omega_{i,j})$ created from the d row vectors

$$\omega_1 = (\omega_{1,1}, \omega_{1,2}, \dots, \omega_{1,d}),\ \omega_2 = (\omega_{2,1}, \omega_{2,2}, \dots, \omega_{2,d}), \dots,\ \omega_d = (\omega_{d,1}, \omega_{d,2}, \dots, \omega_{d,d}).$$

Everywhere, for simplicity of notation and without loss of generality we assume that the generalized volume (measure $\mu(F)$) of the parallelotope F is equal to 1. Thus

$$\mu(F) = \left|\det(\omega_{i,j})\right| = 1. \tag{1.1.3}$$

There are infinitely many choices for the basis vectors and hence for the unit cells. In other words, the set of generators for a lattice is not uniquely determined. It is well-known that the vectors $b_1, b_2, \dots, b_d$ are the other generators of Ω if and only if there is a $d \times d$ matrix $A = (a_{i,j})$ with integer matrix elements and $|\det A| = 1$ such that

$$b_i = \sum_{j=1}^{d} a_{i,j} \omega_j$$

for $i = 1, 2, \dots, d$. Therefore, condition (1.1.3) is not a restriction for the choices for the basis vectors of the lattice Ω.

Note that when F is translated through all the vectors in the lattice Ω fills all of the space $\mathbb{R}^d$ without overlapping. Therefore the fundamental domain (unit cell) F of the lattice Ω can be identified with the factor space (quotient group) $\mathbb{R}^d/\Omega$ which is the set of equivalent classes, where the equivalence of two elements x and y of $\mathbb{R}^d$ is defined as follows: we say that x and y are equivalent if $x - y \in \Omega$. Thus any measurable set M that contains, for each $x \in \mathbb{R}^d$, exactly one representative of the set

$$x + \Omega =: \{x + y : y \in \Omega\}$$

is called a unit cell of the lattice Ω. It is also clear that $\mathbb{R}^d/\Omega$ is a d-dimensional torus (direct product of d circles).

We say that a function $f : \mathbb{R}^d \to \mathbb{C}$ is periodic with respect to the lattice Ω if

$$f(x + \omega) = f(x)$$

for all $\omega \in \Omega$, where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $\mathbb{C}$ is the set of all complex numbers. Note that the periodic function f can be regarded in this case as a function on the torus $\mathbb{R}^d/\Omega$. It is clear that the wave function $e^{i\langle\gamma,x\rangle}$ is periodic with respect to the lattice Ω if and only if

$$\langle\gamma, \omega\rangle \in 2\pi\mathbb{Z}, \tag{1.1.4}$$

for all $\omega \in \Omega$, where $\gamma \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^d$. The set of all vectors $\gamma \in \mathbb{R}^d$ satisfying (1.1.4), that is,

$$\Gamma =: \{\gamma \in \mathbb{R}^d : \langle\gamma, \omega\rangle \in 2\pi\mathbb{Z}, \forall\omega \in \Omega\} \tag{1.1.5}$$

is the lattice dual to Ω and is called the reciprocal lattice. The basis vectors of the reciprocal lattice Γ are the vectors $\gamma_1, \gamma_2, \ldots, \gamma_d$ satisfying

$$\langle\gamma_i, \omega_i\rangle = 2\pi \quad \& \quad \langle\gamma_i, \omega_j\rangle = 0 \tag{1.1.6}$$

for $i, j = 1, 2, \ldots, d$ and $j \neq i$. Thus the fundamental parallelotope of the lattice Γ is

$$F^* = \left\{\omega = \sum_{k=1}^{d} a_k\gamma_k : a_1 \in [0, 1),\ a_2 \in [0, 1), \ldots, a_d \in [0, 1)\right\}. \tag{1.1.7}$$

As we noted above, F^* can be identified with the fundamental domain $\mathbb{R}^d/\Gamma$ of the lattice Γ.

The other and famous fundamental domains (unit cells) of the reciprocal lattice Γ are the Brillouin zones. The first Brillouin zone (called Brillouin zone) of Γ is defined to be the set of points $x \in \mathbb{R}^d$ in reciprocal space which are nearer (not necessarily unique) to the origin than any point $x + \gamma$ with $\gamma \in \Gamma$ and $\gamma \neq 0$.

The nth Brillouin zone is the set of all points x in the reciprocal space which have the origin as their (not necessarily unique) nth nearest point of the set

$$x + \Gamma = \left\{ y \in \mathbb{R}^d : \ y = x + \gamma, \gamma \in \Gamma \right\}. \tag{1.1.8}$$

Note that any interior point of the nth Brillouin zone is the unique nth nearest point. If the several points of (1.1.8) are the nth nearest points (i.e. are equidistant from the origin) then these points belong to the boundaries of the Brillouin zones and only one of them belongs to the nth Brillouin zone.

It readily follows from this definition the following properties of the Brillouin zone:

(a) All zones have equal volumes,

(b) Each zone can be translated into the first zone so as to fill it exactly by translating different pieces of the zone by appropriate reciprocal lattice-vectors.

(c) For arbitrary fixed n the nth Brillouin zone contains unique element from any equivalent classes defined as follows: x and y are equivalent if $x - y \in \Gamma$. Therefore the Brillouin zones can be identified with the fundamental domain $\mathbb{R}^d / \Gamma$ of the lattice Γ.

The geometrical description of the Brillouin zones will be given in Sect. 1.3.

Now let us give the brief description of the problem discussed above. The reciprocal lattice vectors are the special wave vectors γ for which the free electron wave function $e^{i\langle \gamma, x\rangle}$ is periodic with respect to the direct lattice. The wave vectors having this property will be said to belong to the reciprocal lattice. The primitive vectors $\gamma_1, \gamma_2, \ldots, \gamma_d$ of the reciprocal lattice can be generated from the primitive vectors $\omega_1, \omega_2, \ldots, \omega_d$ of the direct lattice by the equalities (1.1.6). A crystal is made up of a periodic arrangement of one or more atoms (the basis) repeated at each Bravais lattice point. Consequently, the crystal looks the same when viewed from any equivalent lattice point, namely those separated by the translation of one unit cell. Every periodic function is associated with a Bravais lattice. You can think of the function as being defined in a primitive unit cell and then repeating the primitive unit cell at every point of the Bravais lattice.

As we noted above the wave function $e^{i\langle \gamma, x\rangle}$ is periodic, with respect to the lattice Ω, if and only if $\gamma \in \Gamma$. One can easily verify that the system

$$\left\{ e^{i\langle \gamma, x\rangle} : \ \gamma \in \Gamma \right\} \tag{1.1.9}$$

is an orthonormal basis in the Hilbert space $L_2(F)$ of square integrable functions with the inner product

$$(f, g) = \int_F f(x)\overline{g(x)}dx.$$

Indeed, by (1.1.3) we have

$$\left\| e^{i\langle \gamma, x\rangle} \right\|^2 = \int_F \left| e^{i\langle \gamma, x\rangle} \right|^2 dx = \int_F 1dx = \mu(F) = 1,$$

where $\|\cdot\|$ is the norm in the space $L_2(F)$ defined by

$$\|f\| = \left(\int_F |f(x)|^2 \, dx\right)^{1/2}.$$

The orthogonality of the system (1.1.9) means that

$$\left(e^{i\langle\gamma,x\rangle}, e^{i\langle\widetilde{\gamma},x\rangle}\right) = \int_F e^{i\langle\delta,x\rangle} dx = 0$$

for all $\widetilde{\gamma} \neq \gamma$, where $\delta = \gamma - \widetilde{\gamma} \in \Gamma$. The last integral can be calculated by using the substitution

$$(x_1, x_2, \ldots, x_d) \leftrightarrow (y_1, y_2, \ldots, y_d),$$

where $y_1, y_2, \ldots, y_d$ equal to the coefficients of the expansion x in the basis $\omega_1, \omega_2, \ldots, \omega_d$ [see (1.1.2)] and by (1.1.2) this substitution transforms the parallelotope F to the cube $[0, 1)^d$. Moreover the Jacobian J of this substitution is nonzero since the vectors $\omega_1, \omega_2, \ldots, \omega_d$ are linearly independent. Therefore using

$$x = \sum_{k=1}^{d} y_k \omega_k \tag{1.1.10}$$

and taking into account that $\delta \in \Gamma \backslash \{0\}$, that is,

$$\delta = \sum_{k=1}^{d} n_k \gamma_k$$

where $n_1, n_2, \ldots, n_d$ are integers and at least one of them is not zero we have

$$\int_F e^{i\langle\delta,x\rangle} dx = |J| \int_0^1 \int_0^1 \ldots \int_0^1 e^{i2\pi n_1 y_1} e^{i2\pi n_2 y_2} \ldots e^{i2\pi n_d y_d} dx_1 dx_2 \ldots dx_d = 0.$$

Since the system

$$\left\{e^{i2\pi n_1 y_1} e^{i2\pi n_2 y_2} \ldots e^{i2\pi n_d y_d} : n_1 \in \mathbb{Z}, n_2 \in \mathbb{Z}, \ldots, n_d \in \mathbb{Z}\right\}$$

is complete in $L_2\left([0, 1)^d\right)$, the above substitution shows that (1.1.9) is complete in the Hilbert space $L_2(F)$ and hence is an orthonormal basis. Therefore every function $q \in L_2(F)$ has the decomposition

$$q(x) = \sum_{\gamma \in \Gamma} q_\gamma e^{i\langle\gamma,x\rangle}, \tag{1.1.11}$$

where

$$q_\gamma = \left(q, e^{i\langle\gamma,x\rangle}\right) = \int_F q(x)e^{-i\langle\gamma,x\rangle}dx$$

for $\gamma \in \Gamma$ are the Fourier coefficients of q with respect to the orthonormal system (1.1.9) and the Fourier series (1.1.11) converges to q in the norm of $L_2(F)$. If

$$\sum_{\gamma\in\Gamma} |q_\gamma| < \infty,$$

then the series (1.1.11) converges uniformly to the periodic function q.

The smoothness of q depends on the Fourier coefficients. For simplicity, let us first consider the case $d = 1$. Let $\Omega = \mathbb{Z}$. Then $\Gamma = 2\pi\mathbb{Z}$ and the system

$$\{e^{i2\pi nx} : n \in \mathbb{Z}\} \tag{1.1.12}$$

is the orthonormal basis in $L_2[0, 1]$. Using the integrations by part, one can readily see that if the sth derivative of the periodic functions q of period 1 belongs to $L_2[0, 1]$ then the Fourier coefficient $q_n^{(s)}$ of $q^{(s)}$ with respect to (1.1.12) satisfies the equality

$$q_n^{(s)} =: (2\pi n)^{-s} q_n$$

where

$$q_n = \int_0^1 q(x)e^{-i2\pi nx}dx$$

is the Fourier coefficient of q. Therefore the periodic function q belongs to the Sobolev space

$$W_2^s[0, 1] =: \left\{f : f^{(s)} \in L_2[0, 1]\right\}$$

if and only if

$$\sum_{n\in\mathbb{Z}} |2\pi n|^{2s} |q_n|^2 < \infty$$

Similarly for arbitrary dimension d the relation $q \in W_2^s(F)$ for the periodic, with respect to the lattice Ω, function q means that

$$\sum_{\gamma\in\Gamma} |q_\gamma|^2 |\gamma|^{2s} < \infty$$

1.2 Schrödinger Operator and Bloch Functions

The energy operator is often referred to as the Hamiltonian and it is also called (in nonrelativistic quantum mechanics) the Schrödinger operator. The Schrödinger operator with a periodic potential arises in the quantum theory of crystals and describes the motion of a particle in the crystal. The ions forming a crystal lattice Ω actually generate a periodic field and one can examine the motion of a electron in this field. Thus if $V(x)$ is the potential seen an electron at x then $V(x+\omega) = V(x)$ for all $\omega \in \Omega$. The wave function $u(x)$ of the electron placed in the periodic potential V must satisfy the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta u(x) + V(x)u(x) = Eu(x),$$

where

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2},$$

$\hbar$ is Planck's constant, m and E are respectively the mass of the electron and its energy eigenvalue.

In the mathematical literature the Schrödinger equation is written in the form

$$-\Delta u(x) + q(x)u(x) = \Lambda u(x), \tag{1.2.1}$$

where

$$q(x) = \frac{2m}{\hbar^2}V(x),\ \Lambda = \frac{2m}{\hbar^2}E.$$

The Schrödinger operator $L(q)$ with a real periodic, relative to a lattice Ω, potential q is defined in space $L_2(\mathbb{R}^d)$ as follows, where $L_2(\mathbb{R}^d)$ is the Hilbert space of square integrable functions with the inner product

$$(f,g)_{\mathbb{R}^d} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx.$$

Let D be the set of all functions $u \in L_2(\mathbb{R}^d)$ such that

(i) u is compactly supported, that is, the set

$$\left\{x \in \mathbb{R}^d : f(x) \neq 0\right\}$$

is a bounded closed subset of $\mathbb{R}^d$,

(ii) $\dfrac{\partial u}{\partial x_j}$ exists and is an absolutely continuous function of x_j for $j = 1, 2, \ldots, d$,

(iii) $-\Delta u + qu \in L_2(\mathbb{R}^d)$.

Let $L^0(q)$ be an operator defined in D by

$$L^0(q)u = -\Delta u + qu.$$

One can readily verify that $L^0(q)$ is a symmetric operator, that is,

$$\left(L^0(q)f, g\right)_{\mathbb{R}^d} = \left(f, L^0(q)g\right)_{\mathbb{R}^d}$$

for all $f, g \in D$. The Schrödinger operator (Hamiltonian) $L(q)$ is the self-adjoint extension of $L^0(q)$. The existence and uniqueness of the extension are well-known (see [BeShu]).

Now we consider the connection of the Hamiltonian $L(q)$ with the Bloch Functions. Recall that Bloch wave or Bloch state, named after Felix Bloch, is the wave function of a particle (usually, an electron) placed in a periodic potential q. Bloch's theorem states that for a particle moving in the periodic potential, the wave functions $\Psi(x)$ are of the form

$$\Psi(x) = e^{i\langle t,x\rangle}p(x), \tag{1.2.2}$$

where $p(x)$ is a periodic function with the same periodicity that the potential q has and $t \in \mathbb{R}^d$ is a crystal momentum (quasimomentum). The exact form of $p(x)$ depends on the potential associated with atoms (ions) that form the solid. The motion of an electron in the free space, where the potential q is zero everywhere, is described by the simplest form of the Schrödinger equation

$$-\Delta u(x) = \lambda u(x)$$

and the wave function $e^{i\langle t,x\rangle}$ is the solution of this equation, since

$$-\Delta e^{i\langle t,x\rangle} = |t|^2 e^{i\langle t,x\rangle}.$$

Thus by Bloch's theorem the wave function $\Psi(x)$ of the electron in the periodic potential is the product of the wave function $e^{i\langle t,x\rangle}$ of the electron in the free space and the periodic function $p(x)$. The wave function expressed by Eq. (1.2.2) is called the Bloch wave or Bloch state.

The Bloch's theorem is very important, since by applying this theorem, the wave function in a macroscopic crystal containing as many atoms as the Avogadro number can be determined by solving the Schrödinger equation into which information from just one unit cell is inserted.

One of the often used (in mathematics) forms of Bloch's theorem is the following (see [Eas]):

Theorem (*Bloch*) *Let* S *consist of the real numbers* Λ *for which the* Eq. (1.2.1) *has a non-trivial bounded solution in* $\mathbb{R}^d$. *If* $\Lambda \in S$ *then* (1.2.1) *has a solution* $\Psi_t(x, \Lambda)$ *of the form*

$$\Psi_t(x, \Lambda) = e^{i\langle t,x\rangle}p(x), \tag{1.2.3}$$

where p is a periodic function having the same periodicity that the potential q has, *the vector $t \in \mathbb{R}^d$ in* (1.2.3) *is called a crystal momentum* (*quasimomentum*) *and S is said to be stability set of the* Eq. (1.2.1)

The solution of (1.2.1) of the form (1.2.3) is called the Bloch solution of (1.2.1) (see [Ku]). It readily follows from (1.2.3) that if $\omega \in \Omega$, where Ω is the period lattice of the potential q and hence of p, then

$$\Psi_t(x+\omega,\Lambda) = e^{i\langle t,x+\omega\rangle} p(x+\omega) = e^{i\langle t,x\rangle} e^{i\langle t,\omega\rangle} p(x) = e^{i\langle t,\omega\rangle}\Psi_t(x,\Lambda)$$

Therefore the Bloch solution $\Psi_t(x,\Lambda)$ of (1.2.1) can be considered as an eigenfunction of the eigenvalue problem (1.2.1) and

$$u(x+\omega) = e^{i\langle t,\omega\rangle} u(x) \tag{1.2.4}$$

for all $\omega \in \Omega$. Conversely, if $\Psi(x,\Lambda)$ is an eigenfunction of this eigenvalue problem then by (1.2.4) we have

$$|\Psi(x+\omega,\Lambda)| = |\Psi(x,\Lambda)|$$

for all $\omega \in \Omega$. It implies that $\Psi(x,\Lambda)$ is bounded in $\mathbb{R}^d$ and by Bloch's theorem has the form (1.2.3), that is, $\Psi(x,\Lambda)$ is the Bloch solution of (1.2.1). Thus $\Psi(x,\Lambda)$ is a Bloch solution of (1.2.1) if and only if it is an eigenfunction of the eigenvalue problem (1.2.1) and (1.2.4) for some values of the quasimomentum $t \in \mathbb{R}^d$. The corresponding eigenvalue $\Lambda(t)$ is called the Bloch eigenvalue for the crystal momentum t. In other words, the Bloch eigenvalue $\Lambda(t)$ and Bloch function $\Psi_t(x,\Lambda)$ for fixed crystal momentum t are the eigenvalue and eigenfunction of $-\Delta + q$ acting on the space

$$\left\{ u \in H^2_{loc}(\mathbb{R}^d) : u(x+\omega) = e^{i\langle t,\omega\rangle} u(x), \forall \omega \in \Omega \right\},$$

where $H^2_{loc}(\mathbb{R}^d)$ is the space of locally square integrable functions u such that $\partial^\alpha u$, for $|\alpha| \le 2$, is also locally square integrable.

In the language of the operator theory the Bloch eigenvalue $\Lambda(t)$ and Bloch function $\Psi_t(x,\Lambda)$ for fixed crystal momentum t are the eigenvalue and eigenfunction of the differential operator $L_t(q)$ generated in $L_2(F)$ by the differential expression

$$-\Delta u(x) + q(x)u(x) \tag{1.2.5}$$

and the boundary conditions (1.2.4), where in the writing the boundary conditions in the form (1.2.4) we take it that the eigenfunction u is extended to the whole $\mathbb{R}^d$ as continuously differentiable functions. More precisely, the operator $L_t(q)$ can be defined in $L_2(\overline{F})$ as the differential operator generated by (1.2.5) and the boundary conditions

$$u(x+\omega_j) = e^{i\langle t,\omega_j\rangle} u(x),\ u_{y_j}(x+\omega_j) = e^{i\langle t,\omega_j\rangle} u_{y_j}(x) \tag{1.2.6}$$

for $x \in \overline{F}(j)$ and $j = 1, 2, \ldots, d$, where $\overline{F}$ is the closure of the parallelotope (1.1.2), that is,

$$\overline{F} =: \left\{ x = \sum_{k=1}^{d} y_k \omega_k : y_1 \in [0,1], y_2 \in [0,1], \ldots, y_d \in [0,1] \right\} \tag{1.2.7}$$

is the closed parallelotope,

$$\overline{F}(j) =: \left\{ x = \sum_{k \in \{1,2,\ldots,d\} \backslash \{j\}} y_k \omega_k : y_1 \in [0,1], y_2 \in [0,1], \ldots, y_d \in [0,1] \right\} \tag{1.2.8}$$

is the face of the boundary $\partial\overline{F}$ of the parallelotope $\overline{F}$ generated by $\omega_1, \omega_2, \ldots, \omega_{j-1}, \omega_{j+1}, \omega_{j+2}, \ldots, \omega_d$ and $u_{y_j} =: \frac{\partial u}{\partial y_j}$ is the derivative of u with respect to the variable y_j defined by (1.1.10) [see also (1.2.7)].

Note that the boundary conditions (1.2.6) mean that the values of u and u_{y_j} on the face $\omega_j + \overline{F}(j)$ of $\partial\overline{F}$ are equal to $e^{i\langle t, \omega_j \rangle}$ times of their values on opposite face $\overline{F}(j)$. The boundary conditions (1.2.6) are equivalent to the conditions (1.2.4) if, as we noted above, in the writing the boundary conditions in the form (1.2.4) we take it that the eigenfunction u is extended to the whole $\mathbb{R}^d$ as continuously differentiable functions. Therefore in the next chapters for simplicity we say that the operator $L_t(q)$ is generated in $L_2(F)$ by the differential expression (1.2.5) and boundary conditions (1.2.4). Thus the operator $L_t(q)$ is defined as follows. Domain of definition $D(L_t(q))$ of $L_t(q)$ is the set of $u \in L_2(\overline{F})$ such that:

(a) $\dfrac{\partial u}{\partial x_j}$ exists and is an absolutely continuous function of x_j for $j = 1, 2, \ldots, d$,

(b) $-\Delta u + qu \in L_2(\overline{F})$,

(c) u satisfies the boundary conditions (1.2.6).

For $u \in D(L_t(q))$ the operator $L_t(q)$ is defined by

$$L_t(q)u = -\Delta u + qu$$

It is well-known the following statements about the spectral properties of $L_t(q)$ and $L(q)$:

Theorem (*On the spectra of the operators $L_t(q)$ and $L(q)$*).

(a) *The spectrum $\sigma(L_t(q))$ of the operator $L_t(q)$ is discrete and consists of the eigenvalues*

$$\Lambda_1(t) \leq \Lambda_2(t) \leq \cdots \tag{1.2.9}$$

such that $\Lambda_j(t) \to \infty$ as $j \to \infty$ which are the Bloch eigenvalues with the fixed quasimomentum t. The corresponding normalized eigenfunctions (*Bloch functions*)

$$\Psi_{1,t}(x), \Psi_{2,t}(x), \ldots$$

form an orthonormal basis in $L_2(F)$.

(b) *The function* Λ_n *is continuous with respect to* t *and its range*

$$\delta_n =: \left\{\Lambda_n(t) : t \in F^*\right\},$$

where F^* *is the fundamental parallelotope of the reciprocal lattice* Γ, *is a closed interval of* $\mathbb{R}$.

(c) *The operator* $L(q)$ *has no eigenvalue and has only the continuous spectrum. The spectrum* $\sigma(L(q))$ *of the operator* $L(q)$, *the stability set* S *defined in the above formulation of Bloch's theorem, and the union of the spectra of the operators* $L_t(q)$ *for* $t \in F^*$ *are the same, that is,*

$$\sigma(L(q)) = S = \bigcup_{t\in F^*} \sigma(L_t(q)) = \bigcup_{t\in F^*} \left(\bigcup_{n=1}^{\infty} \{\Lambda_n(t)\}\right) = \bigcup_{n=1}^{\infty} \delta_n. \tag{1.2.10}$$

Thus $\sigma(L(q))$ consists of the intervals δ_n for $n = 1, 2, \ldots$, that are called the band of the spectrum of $L(q)$. The spaces between neighboring bands are called the band gaps or the gaps in the spectrum of $L(q)$. In the physical literature these bands and gaps are named as energy bands (allowed regions of energy) and forbidden regions of energy respectively

Note that the rigorous proof of this theorem can be found in [Eas] (see also [BeShu, ReSi]). First the physicists observed that the spectrum of $L(q)$ has a band structure [SomBe, Ki, Mad]. The eigenfunctions $\Psi_{1,t}(x), \Psi_{1,t}(x), \ldots,$ of $L_t(q)$ for all values of the quasimomentum t are the Bloch waves [Bl]. For the multidimentional case Gelfand proved Parseval's relation for the Bloch waves in $L_2(\mathbb{R}^d)$ [Gel]. Oder and Keller [OdKe] proved that the spectrum of $L(q)$ is the union of all Bloch eigenvalues $\Lambda_1(t), \Lambda_2(t), \ldots,$ for all $t \in F^\star$. Thomas [Th] proved that the spectrum of $L(q)$ is absolutely continuous. Wilson [Wi] studied the analytic properties of $\Lambda_n(t)$ as a function of the quasimomentum t.

Now let us discuss this theorem from the point of view of the mathematicians and physicists. The statement (a) follows from the fact that $L_t(q)$ is a self-adjoint operator defined in a bounded region of $\mathbb{R}^d$.

Now we discuss (b). The function Λ_n is continuous with respect to t due to the following. Let $P_n(x)$ be a function defined by

$$P_n(x) = e^{-i\langle t,x\rangle}\Psi_{n,t}(x), \tag{1.2.11}$$

where $\Psi_{n,t}(x)$ is the eigenfunction of $L_t(q)$ corresponding to the eigenvalue $\Lambda_n(t)$, that is,

$$-\Delta\Psi_{n,t}(x) + q(x)\Psi_{n,t}(x) = \Lambda_n(t)\Psi_{n,t}(x), \tag{1.2.12}$$

$$\Psi_{n,t}(x+\omega) = e^{i\langle t,\omega\rangle}\Psi_{n,t}(x), \forall\omega \in \Omega. \tag{1.2.13}$$

Using (1.2.11), (1.2.12), and (1.2.13) one can easily verify that $P_n(x)$ satisfies the following equalities

$$-\triangle P_n(x) - \langle 2it, \nabla \rangle \, P_n(x) + \langle t, t \rangle \, P_n(x) + q(x) P_n(x) = \Lambda_n(t) P_n(x)$$

and

$$P_n(x + \omega) = P_n(x) \tag{1.2.14}$$

for all $\omega \in \Omega$. Hence $\Lambda_n(t)$ is the eigenvalue of the operator generated by the operation

$$-\triangle - \langle 2it, \nabla \rangle + \langle t, t \rangle + q \tag{1.2.15}$$

and the periodic boundary conditions. Since the periodic boundary conditions do not depend on t and the operation (1.2.15) continuously depends on t the eigenvalue $\Lambda_n(t)$ also continuously depends on t. Therefore its range

$$\delta_n =: \left\{ \Lambda_n(t) : t \in F^* \right\}, \tag{1.2.16}$$

where F^* is the fundamental parallelotope of the reciprocal lattice Γ, is an interval of $\mathbb{R}$. The closedness of δ_n will be discussed later.

Now let us discuss (c). The operator $L(q)$ is associated with the whole space $\mathbb{R}^d$ and by the Floquet theory (see [Ku]) the Schrö dinger equation (1.2.1) has no solution belonging to $L_2(\mathbb{R}^d)$. Therefore $L(q)$ has no eigenvalue. In fact, the numbers $\Lambda_n(t)$ are not the eigenvalues of the operator $L(q)$ since the corresponding Bloch solutions $\Psi_{n,t}(x)$ do not belong to $L_2(\mathbb{R}^d)$ and by definition, Λ is an eigenvalue of the operator $L(q)$ if there exists

$$\Psi \in D(L(q)) \subset L_2(\mathbb{R}^d)$$

such that

$$L(q)\Psi = \Lambda \Psi. \tag{1.2.17}$$

Therefore Bloch eigenvalues are called the generalized eigenvalues of the operator $L(q)$. However, in some literatures $\Lambda_n(t)$ is named as an eigenvalue of $L(q)$; that is natural, say, in the following sense. Instead of the operator $L(q)$) in whole space $\mathbb{R}^d$ one can consider an operator $L(q, \mathbf{n})$ in the very large parallelotope

$$\overline{F}_{\mathbf{n}} = \left\{ x = \sum_{k=1}^{d} y_k \omega_k : y_1 \in [-n_1, n_1], \, y_2 \in [-n_2, n_2], \ldots, y_d \in [-n_d, n_d] \right\}, \tag{1.2.18}$$

with the periodic boundary conditions, where $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ and n_1, $n_2, \ldots, n_d$ are large positive integers. Due to the fact that $\mathbb{R}^d$ is a limit of $\overline{F}_{\mathbf{n}}$ as $n_j \to \infty$ for $j = 1, 2, \ldots, d$, the eigenvalues of the operator $L(q, \mathbf{n})$ or the limit points of its eigenvalues can be named (in some sense) the eigenvalues of $L(q)$.

Moreover using this argument, it was proved that (see [Eas]) the set of limit points, of the eigenvalues of $L(q, \mathbf{n})$ as $n_j \to \infty$ for $j = 1, 2, \ldots, d$, coincides with $\sigma(L(q))$. On the other hand, using another argument one can see that the set of all eigenvalues of $L(q, \mathbf{n})$ and their limit points as $n_j \to \infty$ for $j = 1, 2, \ldots, d$, coincide with the set of Bloch eigenvalues

$$\left\{\Lambda_n(t) : t \in F^{\star}, n \in \mathbb{N}\right\}.$$

These arguments encourage to believe the validity of (1.2.10).

To be more precise let us define the operator $L(q, \mathbf{n})$ precisely. Moreover consideration the Schrödinger operator in the bounded and large parallelotope $\overline{F}_{\mathbf{n}}$ is interesting, since an electron in a metal must be confined in a bounded space. The effect of a finite size of a system on the motion of an electron must be taken into account. The electron wave function $u(x)$ is assumed along the parallelotope $\overline{F}_{\mathbf{n}}$. Since macroscopic crystal contains as many atoms as the Avogadro number it is interesting to consider the large parallelotope $\overline{F}_{\mathbf{n}}$ which means that $n_1, n_2, \ldots, n_d$ are large numbers. Let us impose the periodic boundary conditions

$$u(x + 2n_j\omega_j) = u(x), u_{y_j}(x + 2n_j\omega_j) = u_{y_j}(x) \tag{1.2.19}$$

on this parallelotope for $x \in \overline{F}_{\mathbf{n}}(j)$ and $j = 1, 2, \ldots, d$, where $\overline{F}_{\mathbf{n}}(j)$ is the face of the boundary $\partial\overline{F}_{\mathbf{n}}$ of the parallelotope $\overline{F}_{\mathbf{n}}$ which is parallel to $\overline{F}(j)$ [see (1.2.8)] and passes through the point $-n_j\omega_j$ and the variable y_j is defined by (1.1.10). Note that the boundary conditions (1.2.19) means that the values of u and u_{y_j} on the face $\overline{F}_{\mathbf{n}}(j)$ of the parallelotope $\overline{F}_{\mathbf{n}}$ are equal to their values on the opposite face.

Let $L(q, \mathbf{n})$ be an operator generated in $L_2(\overline{F}_{\mathbf{n}})$ by the differential expression (1.2.5) and the boundary conditions (1.2.19). Since $L(q, \mathbf{n})$ is associated with the bounded domain $\overline{F}_{\mathbf{n}}$ of $\mathbb{R}^d$ its spectrum is discrete and consists of the eigenvalues. One can readily verify that the set of the eigenvalues of $L(q, \mathbf{n})$ are the union of the Bloch eigenvalues $\Lambda_n(t)$ for $n \in \mathbb{N}$ and $t \in A(\mathbf{n})$, where

$$A(\mathbf{n}) = \left\{ t = \sum_{j=1}^{d} \frac{k_j}{2n_j}\gamma_j : k_j = 0, 1, \ldots, 2n_j;\ j = 1, 3, \ldots, d \right\}. \tag{1.2.20}$$

Indeed, if $u(x)$ satisfies the first condition of (1.2.6) for $t \in A(\mathbf{n})$, then applying it $2n_j$ times and using (1.1.6) we obtain that

$$u(x + 2n_j\omega_j) = \exp\left(\sum_{j=1}^{d} \frac{k_j}{2n_j}\gamma_j, 2n_j\omega_j\right) u(x) = e^{i2\pi k_j} u(x) = u(x),$$

that is, the first condition of (1.2.19) holds. In the same way one can show that the second condition of (1.2.6) implies the second condition of (1.2.19). The proof of the converse statements are similar (see [Eas]). Thus we have

$$\sigma(L(q,\mathbf{n})) = \bigcup_{t \in A(\mathbf{n})} \sigma(L_t(q)). \tag{1.2.21}$$

Denote by Σ the union of the spectrum $\sigma(L(q,\mathbf{n}))$ of the operators $L(q,\mathbf{n})$ for $\mathbf{n} \in \mathbb{N}^d$. It is clear that the closure $\overline{\Sigma}$ of Σ is the set of all limit points of the spectrum $\sigma(L(q,\mathbf{n}))$ as $n_j \to \infty$ for $j = 1, 2, \ldots, d$. Since, as we noted above, the set of these limit points is $\sigma(L(q))$, we have

$$\overline{\Sigma} = \sigma(L(q)). \tag{1.2.22}$$

On the other hand, taking into account that the set of all limit points of

$$\left\{ \frac{k}{2n} : k = 0, 1, \ldots, 2n \right\}$$

as $n \to \infty$ is $[0, 1]$ and using the equalities (1.2.20) and (1.1.7) one can readily see that the set of all limit points of $A(\mathbf{n})$ as $n_j \to \infty$ for $j = 1, 2, \ldots, d$ is F^*. Therefore (1.2.21) and the continuity of the function $\Lambda_n(t)$ on F^* show that

$$\overline{\Sigma} = \bigcup_{t \in F^*} \sigma(L_t(q)). \tag{1.2.23}$$

Thus we tried to explain the reason of the well-known equalities

$$S = \bigcup_{t \in F^*} \sigma(L_t(q)) = \overline{\Sigma} = \sigma(L(q)). \tag{1.2.24}$$

Now let us discuss the well-known mathematical statements described above and some properties of the Bloch eigenvalues $\Lambda_n(t)$ and the Bloch functions $\Psi_{n,t}(x)$ from the point in view of physicists. Considering $\Lambda_n(t)$ as an eigenvalue of the boundary value problem (1.2.1) and (1.2.4) and taking into account that for any $\gamma \in \Gamma$, where Γ is the reciprocal lattice, the equality

$$e^{i\langle t+\gamma,\omega\rangle} = e^{i\langle t,\omega\rangle} \tag{1.2.25}$$

holds, we obtain

$$\Lambda_n(t+\gamma) = \Lambda_n(t) \tag{1.2.26}$$

and

$$\Psi_{n,t+\gamma}(x) = \Psi_{n,t}(x) \tag{1.2.27}$$

for all $\gamma \in \Gamma$.

By (1.2.26) for given n the energy eigenvalue $\Lambda_n(t)$ is periodic with periodicity of a reciprocal lattice. The energies $\Lambda_n(t)$ associated with the index n vary continuously with the wave vector t and form an energy band δ_n identified by the band index n.

All distinct values of $\Lambda_n(t)$ occur for t-values within the fundamental domain $\mathbb{R}^d/\Gamma$ of the lattices Γ, say within the first Brillouin zone or the unit cell (fundamental parallelotope F^*) of the reciprocal lattice. In (1.2.26) replacing γ by γ_j for $j = 1, 2, \ldots, d$ and using (1.1.7) we see that $\Lambda_n(t)$ takes the same values in the opposite faces of the closed parallelotope $\overline{F^*}$. Therefore the bands δ_n for $n = 1, 2, \ldots,$ of the spectrum of $L(q)$ are closed intervals, since they are the images of the closed parallelotope $\overline{F^*}$ under the continuous function $\Lambda_n(t)$. These intervals are allowed zones of energy and the spaces between the neighboring intervals are forbidden zones.

The Bloch wave energy eigenstate $\Psi_{n,t}(x)$ is written with subscripts n and t, where n is a discrete index, called the band index, which is present because there are many different Bloch waves with the same quasimomentum t (each has a different periodic component p). Within a band (i.e., for fixed n), $\Psi_{n,t}(x)$ varies continuously with t, if its energy $\Lambda_n(t)$ is a simple eigenvalue. Since (1.2.27) holds for any reciprocal lattice vector γ, all distinct Bloch waves occur for t-values within the first Brillouin zone of the reciprocal lattice.

Suppose an electron is in a Bloch state $\Psi_{n,t}(x)$. It follows from (1.2.11), (1.2.14), and (1.2.25) that

$$\Psi_{n,t}(x) = e^{i\langle t,x\rangle} P_n(x) = e^{i\langle t+\gamma,x\rangle} P_{n,\gamma}(x), \tag{1.2.28}$$

where P_n and $P_{n,\gamma}$ for $\gamma \in \Gamma$ are periodic with the same periodicity as the crystal lattice Ω. Thus the actual quantum state of the electron is entirely determined by $\Psi_{n,t}(x)$, not t or $P_n(x)$ directly, since t or $P_n(x)$ are not unique. Indeed, if $\Psi_{n,t}(x)$ can be written as above using t, it can also be written using $t + \gamma$, where γ is any reciprocal lattice vector [see (1.2.27)] and this replacement changes the periodic component $P_n(x)$ in (1.2.28).

Equality (1.2.27) shows that the wave vectors (quasimomenta) that differ by a reciprocal lattice vector are equivalent, in the sense that they characterize the same set of Bloch states. The first Brillouin zone is a restricted set of wave vectors with the property that no two of them are equivalent, yet every possible wave vector is equivalent to one (and only one) vector in the first Brillouin zone. Hence, if we restrict to the first Brillouin zones, then every Bloch state has a unique t. Therefore the first Brillouin zone is often used to depict all of the Bloch states without redundancy, for example in a band structure, and it is used for the same reason in many calculations.

1.3 Band Structure, Fermi Surfaces and Perturbations

In Sect. 1.2 we discussed the description of the levels of an electron in a periodic potential in terms of a family of continuous functions $\Lambda_n(t)$ called as the band functions. For each n, the set of electronic levels specified by $\Lambda_n(t)$ is called an energy band. The information contained in these functions for different n and t is referred to as the band structure of the solid. The electron in the free space corresponds to the Schrödinger operator with zero potential. In the case $q = 0$ the eigenvalues and eigenfunctions of $L_t(q)$ are $|\gamma + t|^2$ and $e^{i\langle\gamma+t,x\rangle}$ for $\gamma \in \Gamma$, since

$$-\Delta e^{i\langle\gamma+t,x\rangle} = |\gamma + t|^2 e^{i\langle\gamma+t,x\rangle},$$

the function $e^{i\langle\gamma+t,x\rangle}$ satisfies (1.2.4) and the system

$$\left\{e^{i\langle\gamma+t,x\rangle} : \gamma \in \Gamma\right\}$$

is an orthonormal basis in $L_2(F)$.

(i) Diffraction hyperplanes and Brillouin zones. The eigenvalue $|\gamma + t|^2$ of $L_t(0)$ coincides with the other eigenvalue $|\gamma + t + \delta|^2$, that is, $|\gamma + t|^2$ is a multiple eigenvalue of $L_t(0)$ if and only if $\gamma + t$ belongs to the diffraction hyperplane

$$D_\delta =: \{x \in \mathbb{R}^d : |x|^2 = |x + \delta|^2\} \tag{1.3.1}$$

for some $\delta \in \Gamma$. By (1.3.1), $x \in D_\delta$ if and only if the points x and $x+\delta$ have the same distance from the origin. Therefore D_δ is the hyperplane normal to the reciprocal lattice vector $-\delta$ at their midpoint. Moreover by the same reason D_δ is the boundary of the Brillouin zones defined in Sect. 1.1. The diffraction hyperplanes play a crucial role in the perturbation theory. Let us have a look the diffraction hyperplanes and Brillouin zones in the following cases:

Case 1. $d = 1$. Consider the case of one-dimensional Schrödinger operator $L(q)$ with a periodic, with respect to the lattice $\mathbb{Z}$, potential q. Then the reciprocal lattice is $2\pi\mathbb{Z}$ and the solution of the equation

$$|x|^2 = |x + 2\pi n|^2$$

in $\mathbb{R}$ is the point πn. Thus in this case the diffraction hyperplanes are the points πn for $n = \pm 1, \pm 2, \ldots$ that are the boundaries of the Brillouin zones. The first Brillouin zone is $(-\pi, \pi]$. The second Brillouin zone is $(\pi, 2\pi] \cup (-2\pi, -\pi]$ and the nth Brillouin zone is $((n-1)\pi, n\pi] \cup (-n\pi, -(n-1)\pi]$.

Case 2. $d = 2$. Let the reciprocal lattice Γ be the two-dimensional lattice in $\mathbb{R}^2$ and δ be a vector of the lattice. Then $x \in D_\delta$ if and only if x lies in the line normal to the vector $-\delta$ at its midpoints. Thus in this case the diffraction hyperplanes are the lines normal to the reciprocal lattice vectors at their midpoints and the nth Brillouin zone is the union of the polygons bounded by the diffraction lines.

Similarly in the case $d = 3$ the diffraction hyperplanes are the planes normal to the reciprocal lattice vectors at their midpoints. Therefore the reciprocal space is partitioned into polyhedra bounded by the planes normal to the reciprocal lattice vectors at their midpoints. These planes are boundaries of the Brillouin zone. Hence the Brillouin zone appears in reciprocal space as an assembly of polyhedra bounded by the planes normal to the reciprocal lattice vectors at their midpoints.

(ii) Isoenergetic surface. The isoenergetic surface representing the momentum distribution of the electrons is also constructed in reciprocal space. Note that the isoenergetic surface $I_q(\lambda)$ corresponding to the energy λ refers to a constant energy surface and is defined by

$$I_q(\lambda) = \{t \in F^* : \exists N, \Lambda_N(t) = \lambda\},$$

that is, $I_q(\lambda)$ is the set of quasimomenta t in the primitive cell F^* of the reciprocal lattice for which there exists a Bloch eigenvalue $\Lambda_N(t)$ coinciding with the constant energy λ, where the band function $\Lambda_N(t)$ is defined in Sect. 1.2. This surface for some special and important (in physics) value of λ is called the Fermi surface. Since for the free electrons (in the case $q = 0$) the band functions are $|\gamma + t|^2$, the isoenergetic surface $I_0(\lambda)$ in this case is

$$I_0(\lambda) = \{t \in F^* : \exists \gamma \in \Gamma, |\gamma + t|^2 = \rho^2\}$$

which is the translation of the sphere $\{x \in \mathbb{R}^d : |x| = \rho\}$, where $\lambda = \rho^2$, to the primitive cell F^* by the vectors of the reciprocal lattice Γ. In fact this sphere can be illustrated as the isoenergetic surfaces of the free electron.

(iii) Perturbation of the free electron. Now we discuss how the free-electron is perturbed by the periodic potential and then demonstrate it in the one-dimensional case (see **iv**). The effect of the periodic potential on the electron can be treated in the reciprocal space in terms of the interaction of the isoenergetic surface with the diffraction hyperplanes, that is, with the boundaries of the Brillouin zones. The isoenergetic surface begins to be distorted from a sphere before making contacts with the Brillouin zone planes. The gaps in the spectrum emerges as a result of distortion of the isoenergetic surface in the diffraction planes. Recall that the spectrum of the Schrödinger operator $L(q)$ with a periodic potential consists of the energy bands δ_n for $n = 1, 2, \ldots,$ that are defined in (1.2.16) and named as the allowed bands. The gap in the spectrum is the region between the energy bands δ_n and δ_{n+1} and in the physical literature is named the forbidden band or the energy gap. This means that the electron is not allowed to take energies between the allowed bands δ_n and δ_{n+1} and, hence, there appears an energy discontinuity. Thus an energy gap appears across the Brillouin zone plane. The isoenergetic surface becomes discontinuous, being separated into pieces by the zone boundary. This means that a part of the isoenergetic surface appears in the $(n + 1)$th zone but the rest remains in the nth zone, leaving unoccupied states holes. It can be easily seen in the one-dimensional case [see the example below in (**iv**)].

The formation of the energy gap can also be discussed from the point of view of the diffraction phenomena of the Bloch wave. For this let us recall the Bragg reflection. The quasimomentum $\gamma + t$ is said to satisfy the Laue condition or the Bragg condition if it belongs to the diffraction plane D_δ for some δ, that is,

$$|\gamma + t|^2 = |\gamma + t + \delta|^2.$$

The Bloch wave changes its direction due to the Bragg reflection.

In the following one-dimensional example we demonstrate both how the interaction of the isoenergetic surface with the diffraction hyperplanes and how the diffraction phenomena of the Bloch waves result in the energy gap. Note that the band structure calculation of a real lattice is much more complicated and this example should be looked upon as a simple demonstration.

(iv) One-dimensional Model. Let $H(q)$ be the one-dimensional Schrödinger operator (named as Hill's operator) generated in $L_2(-\infty, \infty)$ by the expression

$$-y'' + q(x)y, \tag{1.3.2}$$

where q is a real-valued function satisfying $q(x) = q(x+1)$. Note that there are a lot of books and papers about the one-dimensional case (see [DuSch, Eas, Le, MaVi, Mar, Na, Ti] and the references on them), where the spectrum of $H(q)$ is investigated and the asymptotic formulas for the eigenvalues λ and the eigenfunctions Ψ when $\lambda \to \infty$ were obtained by different methods. Here we consider the large Bloch eigenvalues and the corresponding Bloch functions of $H(q)$ by the approach which is useful for understanding the results of Chap. 2. Moreover, it helps to compare the well-known one-dimensional case with the multidimensional case and to see the complexity of the results obtained in this book.

For simplicity assume that

$$\sup_{x\in[0,1]} |q(x)| = M < \infty \quad \& \int_0^1 q(x)dx = 0. \tag{1.3.3}$$

Note that the first condition in (1.3.3) can be replaced by $q \in L_1[0, 1]$ (see [VeDe, VeDu]) and the second condition is assumed without loss of generality. Thus the period lattice of the potential q is $\mathbb{Z}$ and the reciprocal lattice is $2\pi\mathbb{Z}$. As we stressed above if the reciprocal lattice is $2\pi\mathbb{Z}$ then the diffraction planes are the points πn for $n = \pm 1, \pm 2, \ldots,$ since the Bragg condition holds at them. We see below that this is indeed the wave number at which the energy gap appears. Moreover we see readily the cases when the plane wave $e^{i(2\pi n+t)x}$ is reflected and when it is not reflected by the crystals.

Let us recall some well-known results about $H(q)$ that we use for the discussion of this model. The spectrum $\sigma(H)$ of the operator $H(q)$ is the union of the spectra $\sigma(H_t)$ of the operators $H_t(q)$ for $t \in [0, 2\pi)$, which are generated in $L_2[0, 1]$ by the expression (1.3.2) and the t-periodic boundary conditions

$$y(1) = e^{it}y(0),\ y'(1) = e^{it}y'(0).$$

In the case $q = 0$ the eigenvalues and eigenfunctions of $H_t(0)$ are respectively $(2\pi n + t)^2$ and $e^{i(2\pi n+t)x}$ for $n \in \mathbb{Z}$. All eigenvalues of $H_t(0)$ for $t \neq 0, \pi$ are simple, while the eigenvalues of $H_0(0)$, except 0, and $H_\pi(0)$ are double. Since the eigenvalues of $H_{-t}(q)$ coincide with those of $H_t(q)$, we discuss only the case $t \in [0, \pi]$. For simplicity let us investigate the case $t \in [0, \frac{\pi}{2}]$. (The case $t \in (\frac{\pi}{2}, \pi]$ can be considered in the same way). By well-known perturbation theory the eigenvalues

$$\lambda_0(t) \leq \lambda_{-1}(t) \leq \lambda_1(t) \leq \lambda_{-2}(t) \leq \lambda_2(t) \leq \cdots \tag{1.3.4}$$

of $H_t(q)$ for $t \in [0, \frac{\pi}{2}]$ satisfy the inequalities

$$\left|\lambda_n(t) - (2\pi n + t)^2\right| \leq M \tag{1.3.5}$$

for all $n \in \mathbb{Z}$ due to (1.3.3).

First let us give the rigorous mathematical proof of the asymptotic formulas and then discuss the band structure from the point of view of the physicists. To obtain the asymptotic formula for the eigenvalues $\lambda_n(t)$ and corresponding normalized eigenfunctions $\Psi_{n,t}(x)$ of $H_t(q)$, let us use the following relation

$$(\lambda_n(t) - (2\pi k + t)^2)(\Psi_{n,t}, e^{i(2\pi k+t)x}) = (q\Psi_{n,t}, e^{i(2\pi k+t)x}) \tag{1.3.6}$$

which can be obtained from the equation

$$-\Psi''_{n,t}(x) + q(x)\Psi_{n,t}(x) = \lambda_n(t)\Psi_{n,t}(x)$$

by multiplying $e^{-i(2\pi k+t)x}$ and integrating the resulting expression over $[0, 1]$ by parts, where $(\cdot, \cdot)$ denotes the inner product in $L_2[0, 1]$. By (1.3.3) and Schward's inequality we have

$$\left|(q\Psi_{n,t}, e^{i(2\pi k+t)x})\right| \leq M. \tag{1.3.7}$$

If $t \in [0, \frac{\pi}{2}]$ then $\left|(2\pi n + t)^2 - (2\pi k + t)^2\right| \geq 2\pi(|n-k|)\,|(2\pi\,|n+k| - \pi)|$ for $k \neq \pm n$. This with (1.3.5) gives us

$$\left|\lambda_n(t) - (2\pi k + t)^2\right| > 3\pi^2\,|(n-k)(n+k)| \tag{1.3.8}$$

for $k \neq \pm n$ and for the large values of n.

It follows from (1.3.6)–(1.3.8) that

$$\sum_{k\in\mathbb{Z}, k\neq\pm n} \left|(\Psi_{n,t}(x), e^{i(2\pi k+t)x})\right|^2 = \sum_{k\in\mathbb{Z}, k\neq\pm n} \frac{M^2}{(3\pi^2(n-k)(n+k))^2} = O(\frac{1}{n^2}).$$

Hence

$$\left\|\sum_{k\in\mathbb{Z}, k\neq\pm n} (\Psi_{n,t}(x), e^{i(2\pi k+t)x})e^{i(2\pi k+t)x}\right\| = O(\frac{1}{n}).$$

Therefore the expansion of $\Psi_{n,t}(x)$ by the orthonormal basis $\{e^{i(2\pi n+t)x} : n \in \mathbb{Z}\}$ has the form

$$\Psi_{n,t}(x) = u_n(t)e^{i(2\pi n+t)x} + v_n(t)e^{i(-2\pi n+t)x} + O(n^{-1}), \tag{1.3.9}$$

where $u_n(t) = (\Psi_{n,t}, e^{i(2\pi n+t)x})$, $v_n(t) = (\Psi_{n,t}, e^{i(-2\pi n+t)x})$,

$$|u_n(t)|^2 + |v_n(t)|^2 = 1 + O(n^{-2}). \tag{1.3.10}$$

Now we consider the following two cases. First let us consider the case when the quasimomentum $2\pi n + t$ is far from the diffraction points πk, that is, there exists a positive constant $c \ll 1$ such that $t \in [c, \frac{\pi}{2}]$. Then

$$\left|(2\pi n + t)^2 - (-2\pi n + t)^2\right| \geq 8\pi\,|n|\,c.$$

Therefore using (1.3.5) and (1.3.6) for $k = -n$ we obtain

$$\left|\lambda_n(t) - (-2\pi n + t)^2\right| \geq 8\pi\,|n|\,c - M$$

and

$$(\Psi_{n,t}(x), e^{i(-2\pi n+t)x}) = O(n^{-1})$$

This with (1.3.9) and (1.3.10) implies that

$$\Psi_{n,t}(x) = e^{i(2\pi n+t)x} + O(\frac{1}{n}) \tag{1.3.11}$$

for $t \in [c, \frac{\pi}{2}]$.

Now using (1.3.11) in (1.3.6), letting $k = n$ and taking into account the second relation of (1.3.3) we obtain that

$$\lambda_n(t) = (2\pi n + t)^2 + O\left(\frac{1}{n}\right). \tag{1.3.12}$$

Now let us consider the case $t \in [0, c]$, that is, the case when the quasimomentum $2\pi n + t$ is close the diffraction point $2\pi n$. In the case $t = 0$ the eigenvalues $(2\pi n)^2$ for $n \neq 0$ of the unperturbed operator $H_0(0)$ are double and the corresponding eigenfunctions are the linear combinations of $e^{i2\pi nx}$ and $e^{-i2\pi nx}$. All eigenvalues of $H_t(0)$ for $t \neq 0, \pi$ are simple. However if t is very close to 0 then the eigenvalues $(2\pi n + t)^2$ and $(-2\pi n + t)^2$ are close to each other.

Let us consider the case $t = 0$. Since the eigenvalues $(2\pi n)^2$ for $n \neq 0$ of the unperturbed operator $H_0(0)$ are double, by (1.3.4) and (1.3.5) the perturbed operator $H_0(q)$ has two eigenvalues (counting multiplicity) denoted by $\lambda_n =: \lambda_n(0)$ and $\lambda_{-n} =: \lambda_{-n}(0)$ such that

$$\lambda_{-n} \le \lambda_n, \left|\lambda_{\pm n} - (2\pi n)^2\right| \le M.$$

First let us prove that the eigenvalues λ_n and λ_{-n} are simple if

$$|nq_{2n}|^{-1} = o(1), \tag{1.3.13}$$

that is, $|q_{2n}| \gg \frac{1}{n}$, where

$$q_{2n} = \left(q, e^{i4\pi nx}\right) = \int_0^1 q\,(x)\, e^{-i4\pi nx} dx.$$

Suppose to the contrary that λ_n is a double eigenvalue, that is, $\lambda_n = \lambda_{-n}$. Then by (1.3.9) and (1.3.10) the corresponding eigenspace is close to the span of the plane waves $e^{i2\pi nx}$ and $e^{-i2\pi nx}$, and there exists an eigenfunction of the form $e^{-2\pi nx} + O\left(n^{-1}\right)$. Using this eigenfunction instead of $\Psi_{n,0}(x)$ in the formula

$$(\lambda_n - (2\pi n)^2)(\Psi_{n,0}, e^{i2\pi nx}) = (q\Psi_{n,0}, e^{i2\pi nx}), \tag{1.3.14}$$

obtained from (1.3.6) by taking $t = 0$ and $k = n$, we get $O\left(n^{-1}\right) = q_{2n} + O\left(n^{-1}\right)$ which contradicts (1.3.13). Thus the eigenvalues λ_n and λ_{-n} are simple for large values of n if (1.3.13) holds.

Now, for simplicity, let us consider the case when q is an even function. Then

$$q_{2n} = \int_0^1 q(x) \cos 4\pi nx dx \in \mathbb{R} \tag{1.3.15}$$

and without loss of generality it can be assumed that $q_{2n} > 0$. Moreover in the case of even potential q, it is well-known that (see [Eas, MaVi]) the periodic solutions and hence the eigenfunction $\Psi_n(x) =: \Psi_{n,0}(x)$ is either even or odd function. Therefore, by (1.3.9) either $v_n = u_n + O(n^{-1})$ or $v_n = -u_n + O(n^{-1})$, where

$$u_n = (\Psi_n, e^{i2\pi nx}), v_n = (\Psi_n, e^{-i2\pi nx}).$$

In the first case from (1.3.9) and (1.3.10) one can easily obtain that

$$\Psi_n(x) = u_n e^{i2\pi nx} + u_n e^{-i2\pi nx} + O(n^{-1}) = \sqrt{2} \cos 2\pi nx + O(n^{-1}). \tag{1.3.16}$$

Using this and taking into account that $(\Psi_n, \Psi_{-n}) = 0$, where $\Psi_{-n} =: \Psi_{-n,0}(x)$, we obtain

$$\Psi_{-n}(x) = u_n e^{i2\pi nx} - u_n e^{-i2\pi nx} + O(n^{-1}) = \sqrt{2} \sin 2\pi nx + O(n^{-1}). \tag{1.3.17}$$

Now using (1.3.16) and (1.3.17) in (1.3.14) and taking into account that $\lambda_n - (2\pi n)^2$ is a real number,

$$(\cos 2\pi nx)^2 = \frac{1}{2}(1 + \cos 4\pi nx),\ (\sin 2\pi nx)^2 = \frac{1}{2}(1 - \cos 4\pi nx),$$

and then using (1.3.15) we get

$$\lambda_n = (2\pi n)^2 + q_{2n} + O\left(n^{-1}\right) \tag{1.3.18}$$

and

$$\lambda_{-n} = (2\pi n)^2 - q_{2n} + O\left(n^{-1}\right) \tag{1.3.19}$$

respectively. Note that the condition $\lambda_{-n} \leq \lambda_n$ and formulas (1.3.18) and (1.3.19) show that we have to take $v_n = u_n + O(n^{-1})$ if $q_{2n} > 0$ and therefore (1.3.16) and (1.3.17) hold.

In the same way one can show that the eigenvalues

$$\mu_{-1} \leq \mu_1 \leq \mu_{-2} \leq \mu_2 \leq \cdots$$

of $H_\pi(q)$ and the corresponding eigenfunction $\Phi_{-1}, \Phi_1, \Phi_{-2}, \Phi_2, \ldots$ satisfy the following asymptotic formulas

$$\mu_n = (2n\pi - \pi)^2 + q_{2n-1} + O\left(n^{-1}\right) \tag{1.3.20}$$

$$\mu_{-n} = (2\pi n - \pi)^2 - q_{2n-1} + O\left(n^{-1}\right) \tag{1.3.21}$$

and

$$\Phi_n(x) = \sqrt{2}\cos(2\pi n - \pi)x + O(\frac{1}{n}). \tag{1.3.22}$$

$$\Phi_{-n}(x) = \sqrt{2}\sin(2\pi n - \pi)x + O(\frac{1}{n}). \tag{1.3.23}$$

It is well-known that [Eas, MaVi, Ti] the spectrum of $H(q)$ consists of the intervals

$$[\lambda_0, \mu_{-1}], [\mu_1, \lambda_{-1}], [\lambda_1, \mu_{-2}], [\mu_2, \lambda_{-2}], \ldots, [\lambda_{j-1}, \mu_{-j}], [\mu_j, \lambda_{-j-1}], \tag{1.3.24}$$

where $j = 3, 4 \ldots$, that are the energy bands. Therefore the gaps in the spectrum (energy gaps) of the Hill's operator $H(q)$ consist of the intervals

$$\Delta_1 = (\mu_{-1}, \mu_1), \Delta_2 = (\lambda_{-1}, \lambda_1), \ldots, \Delta_{2j-1} = (\mu_{-j}, \mu_j), \Delta_{2j} = (\lambda_{-j}, \lambda_j), \tag{1.3.25}$$

where $j = 2, 3, \ldots$, that are the forbidden zones. Then (1.3.18)–(1.3.21) imply that the length $|\Delta_n|$ of the nth forbidden zone Δ_n (gap of the spectrum) satisfies the asymptotic formula

$$|\Delta_n| = 2\,|q_n| + O\left(\frac{1}{n}\right). \tag{1.3.26}$$

From the point of view of mathematicians the gaps arise as follows. For any real periodic potential q the spectrum of the Hill's operator $H(q)$ consists of the intervals (1.3.24). The ends of the intervals are periodic and antiperiodic eigenvalues. In the case of unperturbed operator $H(0)$ these intervals are

$$[0, \pi^2], [\pi^2, (2\pi)^2], \ldots, [(2n\pi)^2, ((2n+1)\pi)^2], [((2n+1)\pi)^2, ((2n+2)\pi)^2] \tag{1.3.27}$$

for $n = 1, 2, \ldots$ The right end of the nth band coincides with the left end of the $(n+1)$th band and these ends are the double eigenvalues $(n\pi)^2$ of periodic (if n is an even number) or antiperiodic (if n is an odd number). Under the perturbation q these double eigenvalues (double eigenvalue can be considered as two coinciding eigenvalues) are separated and one eigenvalue goes to the left and becomes the right end λ_{-j} of the nth band (if $n = 2j$) of the perturbed operator $H(q)$ and the other eigenvalue goes to the right and becomes the left end λ_j of the $(n+1)$th band of $H(q)$. The space Δ_{2j} between these ends λ_{-j} and λ_j can not be occupied by the Bloch eigenvalues $\lambda_{-j}(t)$ and $\lambda_j(t)$, since for $t \in [0, c]$, where $c \ll 1$, the eigenvalues $\lambda_{-j}(t)$ and $\lambda_j(t)$ together with $\lambda_{-j}(0) =: \lambda_{-j}$ and $\lambda_j(t) =: \lambda_j$ go to the left and right respectively and hence arise gaps in the spectrum.

Now we summarize the discussed statements about the one-dimensional Schr ödinger operator $H(q)$ with a periodic potential q in the language of physicists. In the above example, we rigorously constructed the Bloch waves in the high energy region by asymptotic method that is very similar to the two-wave approximation. As we noted above the Bragg condition is satisfied at $\pm\pi n$, since the reciprocal lattice is $2\pi\mathbb{Z}$. The isoenergetic surfaces $I_0((\pi n)^2)$ corresponding to the energy $(\pi n)^2$ consist of two points $-\pi n$ and πn and these points are the diffraction planes of the reciprocal lattice. Under the perturbations the isoenergetic surfaces are separated into pieces by the zone boundary and part of the isoenergetic surface appears in the $(n+1)$th zone but the rest remains in the nth zone, leaving unoccupied states holes.

Formula (1.3.11) means that the plane wave $e^{i(2\pi n+t)x}$ is almost not reflected by the crystals if the wave number $2\pi n + t$ is far from the diffraction planes πn. Formulas (1.3.16) and (1.3.17) show that under perturbation q the plane waves $e^{i2\pi nx}$ and $e^{-i2\pi nx}$ interface each other. The standing waves $\sqrt{2}\cos 2\pi nx$ and $\sqrt{2}\sin 2\pi nx$ are the results of the interference between two waves $e^{i2\pi nx}$ and $e^{-i2\pi nx}$ traveling in the opposite directions. On the other hand, it is well-known that the eigenvalues of $H_t(q)$ for $t \neq 0, \pi$ are simple. Therefore if $\lambda_n(0)$ is a simple eigenvalue, then $\Psi_{n,t}(x)$ continuously depend on $t \in [0, \pi)$. This situation with (1.3.16) and (1.3.17) shows that if t is close to zero then under perturbation q the plane waves $e^{i(2\pi n+t)x}$ and $e^{-i(2\pi n+t)x}$ interface each other. Moreover, these situations with (1.3.22) and (1.3.23) show the same result when t is close to π. Thus the electrons in the crystal are arranged in the energy bands separated by the forbidden regions, called energy gaps or band gaps, in the energy for which no wavelike electron orbitals exist. The band gap is a result of the interference between two waves traveling in the opposite directions. The plane wavefunction $e^{i(2\pi n+t)x}$ represents the running wave and carries the momentum $k = 2\pi n + t$. If $t \neq 0, \pi$ then this wave function is the travelling wave. However, the wave function at $t = 0$ is not wave $e^{i2\pi nx}$ or $e^{-i2\pi nx}$ travelling to the right or left, respectively. Namely when the Bragg reflection condition $t = 0$ is satisfied by the wave vector $2\pi n + t$ a wave travelling to the right is Bragg-reflected to travel to the left and vice versa. As a result the standing waves $\sqrt{2}\cos 2\pi x$ and $\sqrt{2}\sin 2\pi x$ are obtained from the travelling waves $e^{i2\pi nx}$ and $e^{-i2\pi nx}$. The two standing waves $\sqrt{2}\cos \pi x$ and $\sqrt{2}\sin \pi x$ pile up the electrons at the different regions. Therefore the two waves have different values of the potential energy which is the origin of the energy gap. It is well-known and we can see from the above example that the magnitude of the energy gap depends on the Fourier coefficients of the periodic potential. Thus the effect of the periodic potential is to produce an energy gap in the band structure of the one-dimensional case and the energy gap appears when the Bragg condition is satisfied at $\pm\pi n$. In other words, when the wave vector is near to these diffraction planes the Bloch wave is expressed by a linear combination of the unperturbed plane waves $e^{i(2\pi n+t)x}$ and $e^{-i(2\pi n+t)x}$ perturbed by the lattice planes. The running wave $-\pi n$ is reflected to the wave πn by receiving the crystal momentum $2\pi n$ from the lattice planes and the reflected wave $-\pi n$ is again reflected to the wave πn by receiving the crystal momentum $2\pi n$ from the lattice planes. This process is infinitely repeated, resulting in a cosine- or sine-type stationary wave. Under the above condition on the potential, the energy of the sine-type Bloch wave is lowered and the energy of the cosine-type Bloch wave is raised. Thus, the difference in the energy between these two stationary states must be responsible for the formation of the energy gap.

1.4 Some Discussions of the Perturbation Theory

In this section we discuss the perturbation theory and isoenergetic surfaces for the multidimensional Schrödinger operator $L(q)$ in the high energy region. This case, for the first time, was investigated in the papers [Ve1, Ve2, Ve3, Ve4, Ve5, Ve6]. In Chap. 2 we consider it in detail. Now we only describe briefly the crucial points and complexity of this theory. For this, first let us recall that, in general, the perturbation theory is easy if the potential q is smaller than the distance between the eigenvalues of the unperturbed operator $L(0)$. In other words, as well-known from the quantum mechanics, if the perturbation is small compared to the energy difference between the states, then we can use the regular perturbation theory to calculate the wave functions and energy levels. The perturbation theory breaks down, however, in those cases when the potential cannot be considered as a small perturbation. This happens when the magnitude of the potential becomes comparable with the energy separation. To be more precise let us define a constant h for the energy separation, named as the energy separation constant, as follows. One can readily see from the **One-dimensionel model** (see Sect. 1.3) that there are two cases:

Case 1. **Isolated eigenvalue**. An eigenvalue λ is isolated if all other eigenvalues are far from λ (see the case $t \in [c, \frac{\pi}{2}]$). Then the energy separation constant h is a distance from λ to the set of all other eigenvalues.

Case 2. **Isolated pair of eigenvalues**. If the two eigenvalues λ_1 and λ_2 are close to each other and the others are far from these eigenvalues (see the case $t \in [0, c)$), then the energy separation constant h is a distance from the set $\{\lambda_1, \lambda_2\}$ to the set of all other eigenvalues.

If $\|q\| \ll h$ then the perturbation theory is easy and well-known, since in Case 1 and Case 2 one can use the regular perturbation theory and two wave approximations, respectively, where the relation $h \gg 1$ means that h is a sufficiently large number. The inequality $\|q\| \ll h$ which easifies the perturbation theory occurs in the following two cases:

First case: The perturbation q is bounded or $\|q\| = O(1)$ and the energy separation constant h tends to infinity as the eigenvalues go to infinity. This case is the one-dimensional case in the high energy region and we demonstrated it in the **One-dimensionel model** (see Sect. 1.3) and noted that this case was investigated very well, there are a lot of books and papers about it.

Second case: The energy separation constant h is greater than some constant and the potential q is replaced by εq, where ε is a small parameter, that is, $\|\varepsilon q\| \ll h$. This case can be used for the small eigenvalues of the multidimensional operator $L(\varepsilon q)$ to obtain the formulas for $\varepsilon \to 0$. Indeed if the eigenvalue $|\gamma + t|^2$ has a distance greater than some constant from the other eigenvalues then the small perturbation εq can be investigated by the regular perturbation theory. Moreover if $|\gamma+t|^2$ coincides with (or it is near to) the eigenvalue $|\gamma + t + \delta|^2$ but has a distance greater than some constant from the other eigenvalues, that is, if $\gamma + t$ lies in (or it is near to) only one Bragg plane D_δ, then a weak periodic potential εq has its major effect on those free electron levels whose wave vectors are close to ones at which the Bragg reflection can occur. In this case, in order to find the energy levels and the wave functions one can use, for example, the two wave approximations. We will discuss this case in detail in Chap. 5.

Thus in the **first and second case**, we can use the regular perturbation theory to calculate the wave functions and energy levels.

Now we are ready to discuss the multidimensional operator $L(q)$ in the high energy region. In this case, in the big contrary of the **first and second case** (see above) we meet with the situation $h \ll \|q\|$ instead of $\|q\| \ll h$, since the denseness of the Bloch eigenvalues of the free operator increases infinitely with the increasing energy and hence the distance between the eigenvalues tends to zero or the multiplicity of the eigenvalues tend to infinity. To describe this case more precisely, let us introduce some notations. The relation $a(\rho) \sim b(\rho)$ as $\rho \to \infty$ means that $a(\rho) = O(b(\rho))$ and $b(\rho) = O(a(\rho))$, that is, there exist constants c_1 and c_2 such that

$$c_1 b(\rho) < a(\rho) < c_2 b(\rho).$$

In this case we say that $a(\rho)$ is of order $b(\rho)$. Let $E(\rho)$ be the number of the Bloch eigenvalues (counting multiplicity) of the unperturbed operator $L_t(0)$ lying in the interval $[\rho^2, \rho^2 + 1)$. The number $E(\rho)$ depends on $t \in F^*$, however, in average, $E(\rho) \sim \rho^{d-2}$, since $|\gamma + t|^2 \in [\rho^2, \rho^2 + 1)$ if and only if

$$\gamma + t \in \{x \in \mathbb{R}^d : \rho^2 \leq |x|^2 < \rho^2 + 1\} =: W(\rho). \tag{1.4.1}$$

On the other hand, the spherical washer $W(\rho)$ is filled with the translations of F^* by the vectors γ of the reciprocal lattice Γ, and

$$\mu(W(\rho)) \sim \rho^{d-2}\mu(F^*),$$

where $\mu(A)$ denotes the volume of the set A. Thus in the interval $[\rho^2, \rho^2+1)$ of length 1 there are, in average, $E(\rho)$ Bloch eigenvalues $|\gamma + t|^2$ of the free operator, where $E(\rho) \sim \rho^{d-2}$. It means that the eigenvalues are densely situated in the high energy region $[\rho^2, \rho^2 + 1)$ and for the energy separation constant $h(\rho)$ (now it depends on ρ) one can write the equality

$$h(\rho) = O(\rho^{2-d}). \tag{1.4.2}$$

Hence in the multidimensional case in the high energy region the bounded potential q cannot be considered as a small perturbation, since

$$\|q\| \sim \rho^{d-2}h(\rho) \gg h(\rho) \tag{1.4.3}$$

for $d > 2$ and $\rho \gg 1$. Therefore the regular perturbation theory is ineffective in this case. In Chap. 2 we consider this case in detail. Now we only describe briefly the following three problems **(a), (b) and (c)** which are the crucial and remarkable points of the perturbation theory of the multidimensional operator $L(q)$ in the high energy region.

(a) Simplicity problem. *Determine the set of quasimomenta $\gamma + t$ such that the corresponding Bloch eigenvalues $\Lambda(\gamma + t) \in [\rho^2, \rho^2 + 1)$ of $L_t(q)$ are simple.*

The complexity of this problem is the following. The eigenvalue $\Lambda(\gamma + t) \in [\rho^2, \rho^2 + 1)$ is a result of moving of the Bloch eigenvalues $|\gamma+t|^2$ of the free electron under the perturbation q. In the interval $[\rho^2, \rho^2 + 1)$ of length 1 there are, in average, $E(\rho)$ Bloch eigenvalues $|\widetilde{\gamma} + t|^2$ of $L_t(0)$, where $\widetilde{\gamma} \in \Gamma$ and $E(\rho) \sim \rho^{d-2}$. After the periodic perturbation q all these eigenvalues move and some of them move of order 1 and hence each of the resulting eigenvalues $\Lambda(\widetilde{\gamma} + t)$ of $L_t(q)$ may coincide with $\Lambda(\gamma+t)$. Thus we need to control the moving of all eigenvalues $|\widetilde{\gamma}+t|^2 \in [\rho^2, \rho^2+1)$ for some values of t in order that all resulting eigenvalues $\Lambda(\widetilde{\gamma} + t)$ do not coincide with $\Lambda(\gamma + t)$ and hence $\Lambda(\gamma + t)$ becomes a simple eigenvalue. Therefore it seems that it is impossible to find the values of the quasimomenta $\gamma + t$ for which the corresponding Bloch eigenvalues $\Lambda(\gamma + t)$ of $L_t(q)$ are simple. The importance of the simplicity of $\Lambda(\gamma + t)$ is the following. The simplicity of $\Lambda(\gamma + t)$ is necessary for the investigation of the corresponding Bloch wave $\Psi_{\gamma+t}(x)$ and for proving that it is close to the plane wave $e^{i\langle\gamma+t,x\rangle}$ that is, satisfies the formula

$$\Psi_{\gamma+t}(x) = e^{i\langle\gamma+t,x\rangle} + O(|\gamma+t|^{-\alpha}), \tag{1.4.4}$$

where $\alpha > 0$. The last equality means that the plane wave $e^{i\langle\gamma+t,x\rangle}$ goes through the crystal almost without the diffraction. On the other hand, it is well known that the plane wave $e^{i\langle\gamma+t,x\rangle}$ is reflected by the crystal if $\gamma + t$ belongs to (or it is near to) a diffraction hyperplane D_δ for some $\delta \in \Gamma$. Then the reflected wave $e^{i\langle\gamma+\delta+t,x\rangle}$ interferes with the initial wave $e^{i\langle\gamma+t,x\rangle}$ (see [BS], [Ki, Mad]) and (1.4.4) does not hold. As we noted above there are, in average, $E(\rho)$ eigenvalues

$$|\gamma+t|^2, |\gamma+t+\delta_1|^2, |\gamma+t+\delta_2|^2, \ldots, |\gamma+t+\delta_n|^2,$$

where $n = E(\rho) \sim \rho^{d-2}$, lying in the interval $[\rho^2, \rho^2 + 1)$. On the other hand, by choosing the coordinate axis so that the direction of δ coincides with the direction of $(1, 0, 0, \ldots, 0)$, we can easily verify that if

$$|\gamma+t|^2 - |\gamma+t+\delta|^2 = c$$

then the quasimomentum $\gamma + t$ lies on the distance $\frac{|c|}{|\delta|}$ from the diffraction plane D_δ. Therefore all the diffraction planes $D_{\delta_1}, D_{\delta_2}, \ldots, D_{\delta_n}$, may reflect the wave $e^{i\langle\gamma+t,x\rangle}$ with the fixed quasimomentum t. If we do not fix t, then all diffraction planes passing through the washer $W(\rho)$ may reflect the wave $e^{i\langle\gamma+t,x\rangle}$ if the corresponding eigenvalue $|\gamma+t|^2$ lies in the interval $[\rho^2, \rho^2 + 1)$. On the other hand, the number $D(\rho)$ of the diffraction planes having nonempty intersection with the sphere

$$S(\rho) = \{x \in \mathbb{R}^d : |x| = \rho\}$$

and hence with $W(\rho)$ is of order ρ^d, that is, $D(\rho) \sim \rho^d$. Thus the second problem is the following.

(b) Bragg Reflection Problem. *Determine the set of quasimomenta $\gamma+t \in W(\rho)$ for which the plane wave $e^{i\langle\gamma+t,x\rangle}$ under the periodic perturbationq goes through the crystal without the essential influence of the $D(\rho)$ diffraction hyperplanes, where $D(\rho) \sim \rho^d$.*

That is why the mathematical difficulties of the perturbation theory of the multidimensional operator $L(q)$ in the high energy region have a physical nature—a complicated picture of diffraction inside the crystal.

As we explained above in one-dimensional case it is very easy to explain the arising of the gaps in the spectrum. Briefly speaking, there are only two Bloch eigenvalues $(-n\pi)^2$ and $(n\pi)^2$ of the free operator lying at the point $\lambda = (n\pi)^2$ and the isoenergetic surface $I_0((n\pi)^2)$ consists only of the two points $-n\pi$ and $n\pi$ which are the diffraction planes. Under the perturbation q one eigenvalue goes to the left and one to the right and the gap in the neighborhood of $(n\pi)^2$ emerges as a result of these movings.

In the big contrary of the one-dimensional case, in the multidimensional case the set of all Bloch eigenvalues $|\gamma + t|^2$ of the unperturbed operator $L(0)$ lying at the same point ρ^2 as much as the points of the sphere $S(\rho)$, since $|\gamma+t|^2 = \rho^2$ if and only if $(\gamma+t) \in S(\rho)$. Some of these eigenvalues $|\gamma+t|^2$ are multiple. Recall that $|\gamma+t|^2$ is multiple if $\gamma+t$ lies in the intersection of the sphere $S(\rho)$ and diffraction planes and the all other eigenvalues are simple. If the sphere is large, then after the perturbation q the probability that all these eigenvalues go away from the point ρ^2 and the other Bloch eigenvalues do not come to this point and hence the isoenergetic surface $I_q(\rho^2)$ becomes an empty set is very small. (Hence the probability of the validity of the Bethe-Sommerfeld conjecture is close to 1). However as we noted above there are the $D(\rho)$ diffraction planes intersecting $S(\rho)$ for large ρ, where $D(\rho) \sim \rho^d$, and the isoenergetic surface begins to be distorted from a sphere before making contacts with the diffraction planes. Thus the isoenergetic surface is divided into a lot of pieces. Therefore the rigorous mathematical investigation of the perturbations of all these eigenvalues and to prove that the isoenergetic surface $I_q(\rho^2)$ can not become an empty set are extremely complicated. Thus the third problem is the following:

(c) Isoenergetic Surfaces Problem. *Determine the shape and measure of the isoenergetic surface $I_q(\rho^2)$ of $L(q)$ which emerges from the isoenergetic sphere $S(\rho)$ of $L(0)$ as a result of its distortion and separation into very small pieces by the $D(\rho)$ diffraction planes intersecting $S(\rho)$, where $D(\rho) \sim \rho^d$.*

To answer all these **three problems (a), (b) and (c),** in Chap. 2 we develop a new mathematical approach to this problem. The momentum space is divided into two domains: U (non-resonance domain) and V (resonance domain) and the eigenvalues $|\gamma + t|^2$, for large $\gamma \in \Gamma$, are divided into two groups: non-resonance ones if $\gamma + t \in U$ and resonance ones if $\gamma + t \in V$ and various asymptotic formulae are obtained for the perturbations of each groups. (The precise definitions of U and V are given in the introduction of Chap. 2). For the first time in the papers [Ve1, Ve2, Ve3, Ve4] we constructed the set $B \subset U$, called as a simple set, such that if $\gamma+t \in B$, then the corresponding Bloch eigenvalue $\Lambda(\gamma+t)$ is simple and satisfies

$$\Lambda(\gamma + t) = |\gamma + t|^2 + O(|\gamma + t|^{-\alpha}),$$

where $\alpha > 0$ and the Bloch function $\Psi_{\gamma+t}(x)$, corresponding to the eigenvalue $\Lambda(\gamma+t)$ satisfies (1.4.4). Moreover we proved that the simple set B has the asymptotically full measure on $\mathbb{R}^d$ and constructed a part of the isoenergetic surface $I_q(\rho^2) \subset B$ for large ρ which is a union of the smooth surfaces and has the measure asymptotically close to the measure of the sphere $S(\rho)$. Thus, we constructed the set $B \subset U$ that positively solves all the problems **(a), (b) and (c)** described above. Therefore the main difficulty and the crucial point of the investigations of the Bloch functions and isoenergetic surfaces and hence of the perturbation theory of $L(q)$ is the construction and estimation of the set B. We discuss it in detail in the introduction of Chap. 2. Note that, in Chap. 2, we construct the simple set in the non-resonance domain U so that it contains a big part of the isoenergetic surfaces of $L(q)$. However in the case of the resonance domain V we construct the simple set so that it can be easily used for the constructive determination (in Chap. 3) a family of the spectral invariants by the given Bloch eigenvalues. Then in Chap. 4, we constructively determine the potential q by these spectral invariants. We will continue these discussions at the end (in Chap. 5) of this book after the construction a perturbation theory (Chap. 2) and its applications (Chaps. 3 and 4).

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Chapter 2
Asymptotic Formulas for the Bloch Eigenvalues and Bloch Functions

Abstract In this chapter we construct a perturbation theory for the multidimensional Schrödinger operator with a periodic potential. This chapter consists of 6 sections. First section is the introduction, where we define the non-resonance and resonance domains U and V, describe briefly the scheme of this chapter and discuss the related papers. The asymptotic formulas of arbitrary order for the Bloch eigenvalues when the corresponding quasimomentum lies in the non-resonance and resonance domains are obtained in Sects. 2.2 and 2.3 respectively. In Sect. 2.4, we obtain asymptotic formulas for the Bloch functions when the quasimomentum lies in a set $B \subset U$ which has asymptotically full measure in the momentum (reciprocal) space. In Sect. 2.5, we construct and investigate the large part of the isoenergetic surfaces in the high energy region which implies the validity of the Bethe-Sommerfeld conjecture. Note that the method of this chapter is the first and unique by which the asymptotic formulas for the Bloch eigenvalues and Bloch functions and the validity of the conjecture for arbitrary lattice and arbitrary dimension were proved. In Sect. 2.6, we obtain the asymptotic formulas for the Bloch functions when the corresponding quasimomentum lies in a set $B_\delta \subset V$ which is near to the diffraction hyperplane D_δ and is constructed so that it can be easily used for the constructive determination (in Chap. 3) a family of the spectral invariants by the given Bloch eigenvalues.

2.1 Introduction

We consider the Schrödinger operator

$$L(q) = -\Delta + q$$

in $L_2(\mathbb{R}^d)$ for $d \geq 2$ with a periodic (relative to a lattice Ω) potential q, where

$$q \in W_2^s(F), \quad s \geq s_0 =: \frac{3d-1}{2}(3^d + d + 2) + \frac{d3^d}{4} + d + 6, \tag{2.1.1}$$

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$F =: \mathbb{R}^d/\Omega$ is a fundamental domain of Ω. Without loss of generality it can be assumed that the measure $\mu(F)$ of F is 1 and the mean value of the potential q on F is 0. As we noted in Sect. 1.2 of Chap. 1, the spectrum $\sigma(L(q))$ of $L(q)$ is the union of all Bloch eigenvalues $\Lambda_n(t)$ for $t \in F^*$ and $n \in \mathbb{N}$, that is, the union of all eigenvalues of $L_t(q)$ for all $t \in F^\star$:

$$\sigma(L(q)) = \bigcup_{t\in F^*} \sigma(L_t(q)) = \bigcup_{n=1}^{\infty} \left\{\Lambda_n(t) : t \in F^*\right\}, \tag{2.1.2}$$

where $F^\star =: \mathbb{R}^d/\Gamma$, Γ is the lattice dual to Ω and $L_t(q) = -\Delta + q$ is defined in $L_2(F)$ by the quasiperiodic boundary conditions [see (1.2.4) of Chap. 1]. The normalized eigenfunction $\Psi_{n,t}(x)$ of $L_t(q)$ corresponding to the eigenvalue $\Lambda_n(t)$ is known as Bloch function and satisfies

$$L_t(q)\Psi_{n,t}(x) = \Lambda_n(t)\Psi_{n,t}(x). \tag{2.1.3}$$

In the case $q = 0$ the eigenvalues and eigenfunctions of $L_t(q)$ are $|\gamma + t|^2$ and $e^{i\langle\gamma+t,x\rangle}$ for $\gamma \in \Gamma$:

$$L_t(0)e^{i\langle\gamma+t,x\rangle} = |\gamma + t|^2 e^{i\langle\gamma+t,x\rangle}. \tag{2.1.4}$$

In the papers [Ve1, VeMol, Ve2, Ve3, Ve4] for the first time the eigenvalues $|\gamma + t|^2$, for large $\gamma \in \Gamma$, were divided into two groups: non-resonance ones and resonance ones and various asymptotic formulae were obtained for the perturbations of each groups. To give the precise definitions of the non-resonance and resonance eigenvalue $|\gamma + t|^2$ of order ρ^2 (written as $|\gamma + t|^2 \sim \rho^2$, for definiteness suppose

$$\gamma + t \in R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)),$$

where $R(\rho) = \{x \in \mathbb{R}^d : |x| < \rho\})$ for large parameter ρ, we write the potential $q \in W_2^s(F)$ in the form

$$q(x) = \sum_{\gamma\in\Gamma} q_\gamma e^{i\langle\gamma,x\rangle} = P(x) + G(x), \tag{2.1.5}$$

where

$$q_\gamma = (q, e^{i\langle\gamma,x\rangle}) = \int_F q(x)e^{-i\langle\gamma,x\rangle}dx,$$

$$P(x) = \sum_{\gamma\in\Gamma(\rho^\alpha)} q_\gamma e^{i\langle\gamma,x\rangle}, \quad G(x) = \sum_{\gamma\notin\Gamma(\rho^\alpha)} q_\gamma e^{i\langle\gamma,x\rangle},$$

$$\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 < |\gamma| < \rho^\alpha)\},$$

and $\alpha = \frac{1}{\varkappa}$, $\varkappa = 3^d + d + 2$. The relation $|\gamma + t|^2 \sim \rho^2$ means that there exist constants c_1 and c_2 such that

$$c_1\rho < |\gamma + t| < c_2\rho.$$

Here and in the subsequent relations we denote by c_i for $i = 1, 2, \ldots$ the positive, independent of ρ, constants. Recall that the relation $q \in W_2^s(F)$ [see (1.1.13) of Chap. 1] means that

$$\sum_{\gamma\in\Gamma} |q_\gamma|^2|\gamma|^{2s} < \infty. \tag{2.1.6}$$

This implies that if $s \geq d$, then

$$\sup_{x\in F} |G(x)| = \sup_{x\in F} |\sum_{\gamma\notin\Gamma(\rho^\alpha)} q_\gamma e^{i\langle\gamma,x\rangle}| \leq \sum_{|\gamma|\geq\rho^\alpha} |q_\gamma| = O(\rho^{-p\alpha}), \tag{2.1.7}$$

where $p = s - d$. By the well-known perturbation theory [Kat] it follows from (2.1.7) that the influence of G to the eigenvalue $|\gamma + t|^2$ is $O(\rho^{-p\alpha})$. To observe the influence of the trigonometric polynomial P to the eigenvalue $|\gamma + t|^2$, we use the formula

$$(\Lambda_N - |\gamma + t|^2)b(N, \gamma) = (\Psi_{N,t}q, e^{i\langle\gamma+t,x\rangle}), \tag{2.1.8}$$

where $(\cdot, \cdot)$ is the inner product in $L_2(F)$ and

$$b(N, \gamma) = (\Psi_{N,t}, e^{i\langle\gamma+t,x\rangle}),$$

which is obtained by multiplying both sides of (2.1.3) by $e^{i\langle\gamma+t,x\rangle}$ and using (2.1.4). We say that (2.1.8) is the binding formula for $L_t(q)$ and $L_t(0)$, since it connects the eigenvalues and eigenfunctions of $L_t(q)$ and $L_t(0)$. Introducing expansion (2.1.5) of $q(x)$ into (2.1.8) and taking into account (2.1.7), we get

$$(\Lambda_N(t) - |\gamma + t|^2)b(N, \gamma) = \sum_{\gamma_1\in\Gamma(\rho^\alpha)} q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha}). \tag{2.1.9}$$

If Λ_N is close to $|\gamma + t|^2$ and $\gamma + t$ does not belong to any of the sets

$$V_{\gamma_1}(\rho^{\alpha_1}) =: \{x \in \mathbb{R}^d : ||x|^2 - |x + \gamma_1|^2| \leq \rho^{\alpha_1}\} \cap (R(\frac{3\rho}{2})\backslash R(\frac{\rho}{2})) \tag{2.1.10}$$

for $\gamma_1 \in \Gamma(\rho^\alpha)$, where $\alpha_1 = 3\alpha$, that is, $\gamma + t$ is far from the diffraction planes D_{γ_1} for $\gamma_1 \in \Gamma(\rho^\alpha)$, then

$$||\gamma + t|^2 - |\gamma - \gamma_1 + t|^2| > \rho^{\alpha_1}, |\Lambda_N(t) - |\gamma - \gamma_1 + t|^2| > \frac{1}{2}\rho^{\alpha_1} \tag{2.1.11}$$

for all $\gamma_1 \in \Gamma(\rho^{\alpha})$. Therefore, it follows from (2.1.8) that

$$b(N, \gamma - \gamma_1) = \frac{(\Psi_{N,t} q, e^{i\langle \gamma - \gamma_1 + t, x\rangle})}{\Lambda_N(t) - |\gamma - \gamma_1 + t|^2} = O(\rho^{-\alpha_1}). \tag{2.1.12}$$

This with the obvious inequality

$$\sum_{\gamma \in \Gamma} |q_\gamma| < c_3 \tag{2.1.6a}$$

[see (2.1.6) and take into account that $s > d$] implies that the right-hand side of (2.1.9) is $O(\rho^{-\alpha_1})$. Moreover, we prove that there exists an index N such that $\frac{1}{b(N, \gamma)}$ times the right-hand side of (2.1.9) is $O(\rho^{-\alpha_1})$, i.e.,

$$\Lambda_N(t) = |\gamma + t|^2 + O(\rho^{-\alpha_1}). \tag{2.1.13}$$

Thus we see that if $\gamma + t$ does not belong to any of the sets $V_{\gamma_1}(\rho^{\alpha_1})$ [see (2.1.10)] for $\gamma_1 \in \Gamma(\rho^{\alpha})$, then the influence of the trigonometric polynomial P and hence the influence of the potential q [see (2.1.5) and (2.1.7)] to the eigenvalue $|\gamma + t|^2$ is not significant and there exists an eigenvalue of the operator $L_t(q)$ satisfying (2.1.13). This case is called the non-resonance case. More precisely, we give the following definitions:

Definition 2.1.1 Let ρ be a large parameter, $\alpha_k = 3^k \alpha$ for $k = 1, 2, \ldots,$ and

$$V_{\gamma_1}(c_4\rho^{\alpha_1}) =: \{x \in \mathbb{R}^d : ||x|^2 - |x + \gamma_1|^2| \le c_4\rho^{\alpha_1}\} \cap (R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)),$$

$$E_1(c_4\rho^{\alpha_1}, p) =: \bigcup_{\gamma_1 \in \Gamma(p\rho^{\alpha})} V_{\gamma_1}(c_4\rho^{\alpha_1}),\ U(c_4\rho^{\alpha_1}, p) =: (R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))\backslash E_1(c_4\rho^{\alpha_1}, p),$$

$$E_k(c_4\rho^{\alpha_k}, p) =: \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^{\alpha})} (\cap_{i=1}^k V_{\gamma_i}(c_4\rho^{\alpha_k})),$$

where p is defined in (2.1.7), the intersection $\cap_{i=1}^k V_{\gamma_i}$ in the definition of E_k is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ that are linearly independent. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain and $|\gamma + t|^2$ is called a non-resonance eigenvalue if $\gamma + t \in U(\rho^{\alpha_1}, p)$. The domains $V_{\gamma_1}(\rho^{\alpha_1})$ for $\gamma_1 \in \Gamma(p\rho^{\alpha})$ are called the resonance domains and $|\gamma + t|^2$ is called a resonance eigenvalue if $\gamma + t \in V_{\gamma_1}(\rho^{\alpha_1})$. The domain

$$V'_{\gamma_1}(\rho^{\alpha_1}) =: V_{\gamma_1}(\rho^{\alpha_1})\backslash E_2,$$

i.e., the part of the resonance domain $V_{\gamma_1}(\rho^{\alpha_1})$, which does not contain the intersections of two resonance domains is called a single resonance domain.

It is clear that asymptotic formula (2.1.13) holds if we replace $V_{\gamma_1}(\rho^{\alpha_1})$ by $V_{\gamma_1}(c_4\rho^{\alpha_1})$. Note that changing the value of c_4 in the definition of $V_{\gamma_1}(c_4\rho^{\alpha_1})$, we obtain the different definitions of the non-resonance eigenvalues (for the simplicity of the notations we take $c_4 = 1$). However, in any case we obtain the same asymptotic formulas and the same perturbation theory, that is, this changing does not change anything for the asymptotic formulas. Therefore we can define the non-resonance eigenvalue in different way. Instead of the resonance domain $V_{\gamma_1}(c_4\rho^{\alpha_1})$ the set

$$W_{\gamma_1,\alpha_1} = \{x \in \mathbb{R}^d : ||x|^2 - |x+\gamma_1|^2| < |x|^{\alpha_1}\}$$

can be considered (see [Ve2, Ve3, Ve4]). Since

$$V_{\gamma_1}(\frac{1}{2}\rho^{\alpha_1}) \subset (R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)) \cap W_{\gamma_1,\alpha_1} \subset V_{\gamma_1}(\frac{3}{2}\rho^{\alpha_1}),$$

in all considerations the resonance domain $V_{\gamma_1}(\rho^{\alpha_1})$ can be replaced by

$$W_{\gamma_1,\alpha_1} \cap (R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)).$$

Moreover, instead of the domain $V_{\gamma_1}(\rho^{\alpha_1})$ the set

$$\{x \in \mathbb{R}^d : |\langle x, \gamma_1\rangle| < \varepsilon|x||\gamma_1|\},$$

where $\varepsilon \ll 1$, also can be considered (see [Ve1, VeMol]). In any case we use the same idea: breafly speaking, the eigenvalues $|\gamma + t|^2 \sim \rho^2$ are non-resonance if $\gamma + t$ far from the diffraction planes

$$D_\delta =: \{x \in \mathbb{R}^d : |x|^2 = |x+\delta|^2\}$$

for $\delta = O(\rho^\alpha)$. Nevertheless it is suitable to define the non-resonance eigenvalue in different way depending on the form of the potential. Namely, the domain W_{γ_1,α_1} is suitable, when the potential is the trigonometric polynomial. In case of smooth potential we need to introduce a large parameter ρ and consider $V_{\gamma_1}(\rho^{\alpha_1})$. Note that all considered eigenvalues $|\gamma + t|^2$ of $L_t(0)$ satisfy the relations

$$\frac{1}{2}\rho < |\gamma + t| < \frac{3}{2}\rho.$$

Therefore in the asymptotic formulas instead of $O(\rho^a)$ one can take $O(|\gamma + t|^a)$.

In Sect. 2.2 to investigate the perturbations of the non-resonance eigenvalues $|\gamma+t|^2$ we take the operator $L_t(0)$ for an unperturbed operator and q for a perturbation. Iterating binding formula (2.1.8) for $L_t(q)$ and $L_t(0)$, namely, using (2.1.12) in (2.1.9) and then using decomposition (2.1.5) and continuing this process, we prove that (2.1.13) and the asymptotic formulas of arbitrary order hold. More precisely, we obtain the following results. For each $\gamma + t \in U(\rho^{\alpha_1}, p)$ there exists an eigenvalue $\Lambda_N(t)$ of the operator $L_t(q)$ satisfying the formulae

$$\Lambda_N(t) = |\gamma + t|^2 + F_{k-1}(\gamma + t) + O(|\gamma + t|^{-k\alpha_1}) \tag{2.1.14}$$

for $k = 1, 2, \ldots, [\frac{1}{3}(p - \frac{1}{2}\varkappa(d-1))]$, where $[a]$ denotes the integer part of a, $F_0 = 0$, and F_{k-1} (for $k > 1$) is expressed by the potential q and the eigenvalues of $L_t(0)$. Besides, we prove that if the conditions

$$|\Lambda_N(t) - |\gamma + t|^2| < \frac{1}{2}\rho^{\alpha_1}, \tag{2.1.15}$$

$$|b(N, \gamma)| > c_5\rho^{-c\alpha}, \tag{2.1.16}$$

where $0 \le c < p - \frac{1}{4}d3^d$, hold then the following statements are valid:

(a) if $\gamma + t \in U(\rho^{\alpha_1}, p)$, then $\Lambda_N(t)$ satisfies (2.1.14) for $k = 1, 2, \ldots, [\frac{1}{3}(p-c)]$;

(b) if $\gamma + t \in E_s \backslash E_{s+1}$, where $s = 1, 2, \ldots, d-1$, then

$$\Lambda_N(t) = \lambda_j(\gamma + t) + O(|\gamma + t|^{-(p-c-\frac{1}{4}d3^d)\alpha}), \tag{2.1.17}$$

where λ_j is an eigenvalue of a matrix $C(\gamma + t)$ (see below for the explanation of C in the three-dimensional case). Moreover, we prove that every large eigenvalue of the operator $L_t(q)$ for all values of t satisfies one of these formulae (see Theorems 2.2.1 and 2.2.2).

The results of Sect. 2.2 were obtained in [Ve1, Ve2, Ve3, Ve4] and their enlarged forms were written in [Ve6, Ve9]. The non-resonance eigenvalues for the three-dimensional Schrödinger operator $L_t(q)$ were considered in [Ve3]. Moreover, in [Ve3] we observed that if $\gamma + t \in V_\delta(\rho^{\alpha_1}) \backslash E_2$ and $\gamma_1 \in \Gamma(\rho^\alpha) \backslash \{n\delta : n \in \mathbb{Z}\}$, where δ is the element of Γ of minimal norm in its direction, then it follows from the definition of E_2 that the inequalities obtained from (2.1.11) by replacing α_1 with α_2 hold. Hence

$$b(N, \gamma - \gamma_1) = O(\rho^{-\alpha_2})$$

[see (2.1.12)] and (2.1.9) has the form

$$(\Lambda_N(t) - |\gamma + t|^2) b(N, \gamma) = \sum_{n \in \mathbb{Z}, n\delta \in \Gamma(\rho^\alpha)} q_{n\delta} b(N, \gamma - n\delta) + O(\frac{1}{\rho^{\alpha_2}}). \tag{2.1.18}$$

This gives an idea that the influence of $q(x) - q^\delta(x)$, where

$$q^\delta(x) = \sum_{n \in \mathbb{Z}} q_{n\delta} e^{in\langle \delta, x \rangle}, \tag{2.1.19}$$

is not significant and there exist eigenvalues of $L_t(q)$ which are close to the eigenvalues of $L_t(q^\delta)$. Using this idea in [VeMol], to investigate the resonance eigenvalues we used the approximation of the Green functions of $L_t(q)$ by the Green functions of $L_t(q^\delta)$. Note that in [Ve3] (see Theorem 2 of [Ve3]) writing the equations obtained from (2.1.18) by replacing $|\gamma + t|^2$ with $|\gamma + t + n\delta|^2$ for $n \in \mathbb{Z}$, $n\delta \in \Gamma(\rho^\alpha)$, we got the system of equations from which one can conclude that the probable approximations of the eigenvalues of the three-dimensional Schrödinger operator $L_t(q)$, besides $|\gamma + t|^2$, are the eigenvalues of the matrix C, where C is a finite submatrix of the matrix corresponding to the operator $L_t(q^\delta)$. However, in the d-dimensional case, to investigate the perturbation of the eigenvalue $|\gamma+t|^2$ when the corresponding quasimomentum $\gamma + t$ lies in the intersection of k resonance domains we have to consider more complicated system and matrix (see (2.2.15) and [Ve2, Ve4]).

Here we write the non-resonance case so that it can easily be used in Sect. 2.3, where we consider in detail the single resonance domains $V_\delta(\rho^{\alpha_1})\backslash E_2$, since there are similarities between the investigations of the non-resonance and the single resonance cases. To see the similarities and differences between these cases, that is, between Sects. 2.2 and 2.3, let us give the following comparison. As we noted above in the non-resonance case the influence of the potential q is not significant, while in the single resonance case the influence of $q - q^\delta$ is not significant. Therefore, in Sect. 2.2 for the investigation of the non-resonance case we take the operator $L_t(0)$ for an unperturbed operator and q for a perturbation, while in Sect. 2.3 for the investigation of the single resonance case we take the operator $L_t(q^\delta)$ for an unperturbed operator and $q - q^\delta$ for a perturbation. In Sect. 2.2 to obtain the asymptotic formula for the non-resonance case we iterate the formula (2.1.8) [called binding formula for $L_t(q)$ and $L_t(0)$] connecting the eigenvalues and eigenfunctions of $L_t(q)$ and $L_t(0)$. Similarly, in Sect. 2.3 for the investigation of the eigenvalues corresponding to the quasimomentum lying in the single resonance domain $V_\delta(\rho^{\alpha_1})\backslash E_2$ (see Definition 2.1.1), we iterate a formula [called binding formula for $L_t(q)$ and $L_t(q^\delta)$] connecting the eigenvalues and eigenfunctions of $L_t(q)$ and $L_t(q^\delta)$. The binding formula for $L_t(q)$ and $L_t(q^\delta)$ can be obtained from the binding formula (2.1.8) for $L_t(q)$ and $L_t(0)$ by replacing the perturbation q and the eigenvalues $|\gamma+t|^2$ and eigenfunctions $e^{i\langle\gamma+t,x\rangle}$ of the unperturbed (for the non-resonance case) operator $L_t(0)$ with the perturbation $q - q^\delta$ and the eigenvalues and eigenfunctions of the unperturbed (for the single resonance case) operator $L_t(q^\delta)$ respectively. To write this formula first we consider the eigenvalues and eigenfunctions of $L_t(q^\delta)$. For this let us introduce the following notations which will be used during the book.

Notation 2.1.1 *Let δ be a visible element of Γ, that is, δ is the element of Γ of minimal norm in its direction. Denote by Ω_δ the sublattice $\{h \in \Omega : \langle h, \delta\rangle = 0\}$ of Ω in the hyperplane $H_\delta = \{x \in \mathbb{R}^d : \langle x, \delta\rangle = 0\}$ and denote by Γ_δ the lattice of H_δ which is dual to Ω_δ, that is, $\Gamma_\delta =: \{a \in H_\delta : \langle a, k\rangle \in 2\pi\mathbb{Z}, \forall k \in \Omega_\delta\}$. The function q^δ defined by (2.1.19) is called the directional potential. The eigenvalues and eigenfunctions of the Schrödinger operator $L_t(q^\delta)$ with the directional potential q^δ can be indexed by pair (j, β) of the Cartesian product $\mathbb{Z} \times \Gamma_\delta$ [see* Lemma 2.3.1(b)] *and we denote them by $\lambda_{j,\beta}$ and $\Phi_{j,\beta}(x)$ respectively.*

By this notation we have

$$L_t(q^\delta)\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x). \tag{2.1.20}$$

Thus the binding formula for $L_t(q)$ and $L_t(q^\delta)$ is

$$(\Lambda_N(t) - \lambda_{j,\beta})b(N, j, \beta) = (\Psi_{N,t}, (q - q^\delta)\Phi_{j,\beta}), \tag{2.1.21}$$

where

$$b(N, j, \beta) = (\Psi_{N,t}, \Phi_{j,\beta}),$$

which can be obtained by multiplying both sides of (2.1.3) by $\Phi_{j,\beta}(x)$ and using (2.1.20). To prove the asymptotic formulas in the single resonance case we iterate the formula (2.1.21). The iterations of the formulas (2.1.8) and (2.1.21) are similar. Therefore the simple iterations of (2.1.8) in Sect. 2.2 help to read the complicated iterations of (2.1.21) in Sect. 2.3.

The brief scheme of the iteration of (2.1.21) is following. Using (2.1.5), decomposing $(q - q^\delta)\Phi_{j,\beta}$ by the eigenfunctions of $L_t(q^\delta)$ and putting this decomposition into (2.1.21), we get

$$\begin{aligned}(\Lambda_N(t) - \lambda_{j,\beta})b(N, j, \beta) &= O(\rho^{-p\alpha}) \\ &+ \sum_{(j_1,\beta_1)\in Q} A(j, \beta, j + j_1, \beta + \beta_1)b(N, j + j_1, \beta + \beta_1),\end{aligned} \tag{2.1.22}$$

where Q is a subset of the Cartesian product $\mathbb{Z} \times \Gamma_\delta$. Now using

$$b(N, j + j_1, \beta + \beta_1) = \frac{(\Psi_{N,t}, (q - q^\delta)\Phi_{j,\beta})}{(\Lambda_N(t) - \lambda_{j+j_1,\beta+\beta_1})},$$

which is obtained from (2.1.21) by replacing j, β with $j + j_1, \beta + \beta_1$, in (2.1.22), we get the once iteration of (2.1.21):

$$\begin{aligned}(\Lambda_N(t) - \lambda_{j,\beta})b(N, j, \beta) &= O(\rho^{-p\alpha}) \\ &+ \sum_{(j_1,\beta_1)\in Q} A(j, \beta, j + j_1, \beta + \beta_1)\frac{(\Psi_{N,t}, (q - q^\delta)\Phi_{j,\beta})}{(\Lambda_N(t) - \lambda_{j+j_1,\beta+\beta_1})}.\end{aligned} \tag{2.1.23}$$

Continuing this process we get the iterations of (2.1.21). Then we prove the asymptotic formulas, by using the iterations of (2.1.21), as follows. First we investigate in detail, the multiplicand $A(j, \beta, j + j_1, \beta + \beta_1)$ of (2.1.23) and prove the estimation

$$\sum_{(j_1,\beta_1)\in Q} |A(j, \beta, j + j_1, \beta + \beta_1)| < c_6 \tag{2.1.24}$$

(see Lemmas 2.3.2, 2.3.3, 2.3.4). Then we investigate the distance between the eigenvalues $\lambda_{j,\beta}$ and $\lambda_{j+j_1,\beta+\beta_1}$ (see Lemma 2.3.5) and hence estimate the denominator of the fractions in (2.1.23), since $\Lambda_N(t)$ is close to $\lambda_{j,\beta}$. Using this and (2.1.24) we prove that there exists an index N such that $\frac{1}{b(N,j,\beta)}$ times the right-hand side of (2.1.23) is $O(\rho^{-\alpha_2})$, from which we get

$$\Lambda_N(t) = \lambda_{j,\beta} + O(\rho^{-\alpha_2}) \tag{2.1.25}$$

(see Lemma 2.3.6, Theorem 2.3.1). At last using this formula in the arbitrary times iterations of (2.1.21), we obtain the asymptotic formulas of arbitrary order (Theorem 2.3.2). The results of Sect. 2.3 were obtained in [VeMol, Ve6, Ve9].

In Sect. 2.4, we investigate the Bloch functions in the non-resonance domain. To investigate the Bloch functions we need to find the values of the quasimomenta $\gamma + t$ for which the corresponding eigenvalues of $L_t(q)$ are simple. In the interval $[\rho^2, \rho^2 + 1)$ of length 1 there are, in average, ρ^{d-2} eigenvalues $|\gamma + t|^2$ of the unperturbed operator $L_t(0)$. Under the perturbation, all these eigenvalues move and some of them move or order 1. Therefore, it seems that it is impossible to find the values of quasimomenta $\gamma + t$ for which the corresponding eigenvalues of $L_t(q)$ are simple. For the first time in papers [Ve2, Ve3, Ve4] (in [Ve3] for $d = 3$ and in [Ve2, Ve4] for the cases: $d = 2$, $q \in L_2(F)$ and $d > 2$, q is a smooth potential) we found the required values of quasimomenta, namely we constructed and estimated the subset B of $U(\rho^{\alpha_1}, p)$ with the following remarkable properties (the expanded explanations of these properties were done in [Ve5, Ve6, Ve9]):

Property 1 (Simplicity). If $\gamma + t \in B$, then there exists a unique eigenvalue $\Lambda_N(t)$, denoted by $\Lambda(\gamma + t)$, of the operator $L_t(q)$ satisfying (2.1.13), (2.1.14). This is a simple eigenvalue of $L_t(q)$ and therefore we call the set B as the simple set.

Construction of the set B consists of two steps.

Step 1. We prove that all eigenvalues $\Lambda_N(t)$ of the operator $L_t(q)$ satisfying $\Lambda_N(t) \sim \rho^2$ lie in the ε_1 neighborhood of the numbers $F(\gamma + t)$ and $\lambda_j(\gamma + t)$, where

$$F(\gamma + t) = |\gamma + t|^2 + F_{k_1 - 1}(\gamma + t), \quad \varepsilon_1 = \rho^{-d-2\alpha}, \quad k_1 = [\frac{d}{3\alpha}] + 2 \tag{2.1.26}$$

[see (2.1.14), (2.1.17)]. We call these numbers as the known parts of the eigenvalues of $L_t(q)$. Moreover, for $\gamma + t \in U(\rho^{\alpha_1}, p)$ there exists $\Lambda_N(t)$ satisfying

$$\Lambda_N(t) = F(\gamma + t) + o(\rho^{-d-2\alpha}) = F(\gamma + t) + o(\varepsilon_1). \tag{2.1.27}$$

Step 2. By eliminating the set of quasimomenta $\gamma + t$, for which the known parts $F(\gamma + t)$ of $\Lambda_N(t)$ are situated from the known parts $F(\gamma' + t)$, $\lambda_j(\gamma' + t)$ $(\gamma' \neq \gamma)$ of the other eigenvalues at a distance less than $2\varepsilon_1$, we construct the set B with the following properties: if $\gamma + t \in B$, then the following conditions [called simplicity conditions for the eigenvalue $\Lambda_N(t)$ satisfying (2.1.27)] hold:

$$|F(\gamma+t)-F(\gamma'+t)|\geq 2\varepsilon_1 \tag{2.1.28}$$

for $\gamma'\in K\backslash\{\gamma\}$, $\gamma'+t\in U(\rho^{\alpha_1},p)$ and

$$|F(\gamma+t)-\lambda_j(\gamma'+t)|\geq 2\varepsilon_1 \tag{2.1.29}$$

for $\gamma'\in K$, $\gamma'+t\in E_k\backslash E_{k+1}$, $j=1,2,\ldots,$ where K is the set of $\gamma'\in\Gamma$ satisfying

$$|F(\gamma+t)-|\gamma'+t|^2|<\frac{1}{3}\rho^{\alpha_1}. \tag{2.1.30}$$

Thus the simple set B is defined as follows:

Definition 2.1.2 The simple set B is the set of

$$x\in U(\rho^{\alpha_1},p)\cap(R(\frac{3}{2}\rho-\rho^{\alpha_1-1})\backslash R(\frac{1}{2}\rho+\rho^{\alpha_1-1}))$$

such that $x=\gamma+t$, where $\gamma\in\Gamma$, $t\in F^\star$, and the simplicity conditions (2.1.28) and (2.1.29) hold.

As a consequence of the conditions (2.1.28) and (2.1.29), the eigenvalue $\Lambda_N(t)$ satisfying (2.1.27) does not coincide with the other eigenvalues.

To check the simplicity of $\Lambda_N(t)=:\Lambda(\gamma+t)$ (see Property 1) we prove that for any normalized eigenfunction $\Psi_{N,t}$ corresponding to $\Lambda_N(t)$ the equality

$$\sum_{\gamma'\in\Gamma\backslash\gamma}|b(N,\gamma')|^2=O(\rho^{-2\alpha_1}), \tag{2.1.31}$$

which is equivalent to

$$|b(N,\gamma)|^2=1+O(\rho^{-2\alpha_1}), \tag{2.1.31a}$$

holds. The equality (2.1.31a) implies the simplicity of $\Lambda_N(t)$. Indeed, if $\Lambda_N(t)$ is a multiple eigenvalue, then there exist two orthogonal normalized eigenfunctions satisfying (2.1.31a), which is impossible. In fact to prove the simplicity of $\Lambda_N(t)$ it is enough to show that for any normalized eigenfunction $\Psi_{N,t}$ corresponding to $\Lambda_N(t)$ the inequality

$$|b(N,\gamma)|^2>\frac{1}{2} \tag{2.1.31b}$$

holds. We proved this inequality in [Ve2, Ve3, Ve4] and as noted in Theorem 3 of [Ve3] and in [Ve5, Ve6, Ve9] the proof of this inequality does not differ from the proof of (2.1.31a) which equivalent to the following property:

Property 2 (asymptotic formulas for the Bloch functions). If $\gamma+t\in B$, then the eigenfunction $\Psi_{N,t}(x)$, denoted by $\Psi_{\gamma+t}(x)$, corresponding to the eigenvalue $\Lambda_N(t)=:\Lambda(\gamma+t)$ (see Property 1) is close to $e^{i\langle\gamma+t,x\rangle}$, namely

$$\Psi_{N,t}(x) =: \Psi_{\gamma+t}(x) = e^{i\langle\gamma+t,x\rangle} + O(|\gamma+t|^{-\alpha_1}). \tag{2.1.32}$$

Iterating (2.1.32), we get

$$\Psi_{\gamma+t}(x) = F^*_{k-1}(\gamma+t) + O(|\gamma+t|^{-k\alpha_1}) \tag{2.1.33}$$

for $k = 1, 2, \ldots$, where $F^*_{k-1}(\gamma+t)$ is expressed by q and the eigenvalues and eigenfunctions of $L_t(0)$ [see Theorem 2.4.2, formula (2.4.20)].

Note that the main difficulty and the crucial point of the investigation of the Bloch functions and hence the main difficulty of the perturbation theory of $L(q)$ is the construction and estimation of the simple set B. This difficulty of the perturbation theory of $L(q)$ is of a physical nature and it is connected with the complicated picture of the crystal diffraction. In the multidimensional case this becomes extremely difficult since in the 1 neighborhood of ρ^2 there are, in average, ρ^{d-2} eigenvalues and hence the eigenvalues can be highly degenerated. To see that the main part of the perturbation theory is the construction and estimation of the set B let us briefly prove that (the precise proof is given in Theorem 2.4.1) from the construction of B it easily follows the simplicity of the eigenvalues and the asymptotic formula (2.1.32) for the Bloch functions. As we noted above to prove the simplicity of $\Lambda_N(t)$ and (2.1.32) it is enough to prove that (2.1.31) holds, that is, we need to prove that the term $b(N, \gamma')$ in (2.1.31) is very small. If

$$|b(N, \gamma')| > c_5\rho^{-c\alpha},$$

then replacing γ by γ' in (2.1.15), (2.1.16), (2.1.14), (2.1.17), and (2.1.27) we see that $\Lambda_N(t)$ lies in ε_1 neighborhood of one of the numbers $F(\gamma'+t)$ and $\lambda_j(\gamma'+t)$, which contradicts to the simplicity conditions (2.1.28) and (2.1.29), since (2.1.27) holds.

Since the main part of the perturbation theory is the construction and estimation of the set B let us discuss the construction and the history of the construction of the simple set. For the first time in [Ve2, Ve3, Ve4] we constructed and estimated the simple set B. In [Ve3] we constructed the simple set for the three dimensional Schrödinger operator $L(q)$. If $d = 2, 3$, then the simplicity conditions (2.1.28) and (2.1.29) are relatively simple, namely in this case

$$F(\gamma+t) = |\gamma+t|^2$$

and the matrix $C(\gamma'+t)$, when $\gamma'+t$ lies in the single resonance domain, corresponds to the Schrödinger operator with directional potential (2.1.19) (see Theorems 1 and 2 in [Ve3]). Therefore the simple set is constructed in such way that if $\gamma+t \in B$, then the inequality

$$||\gamma+t|^2 - |\gamma'+t|^2| \geq \rho^{-a} \tag{2.1.34}$$

for $\gamma' + t \in U(\rho^{\alpha_1}, p)$, the inequality

$$||\gamma + t|^2 - \lambda_j(\gamma' + t)| \geq \rho^{-a} \tag{2.1.35}$$

for $\gamma' + t$ lying in the single resonance domain, and the inequality

$$||\gamma + t|^2 - |\gamma' + t|^2| \geq c_3$$

for $\gamma' + t$ lying in the intersection of two resonance domains hold, where $a > 0$. Thus for the construction of the simple set B of quasimomenta in case $d = 3$ we eliminated the vicinities of the diffraction planes [see (2.1.34)], the sets connected with the directional potential [see (2.1.35)], and the intersection of two resonance domains.

As the dimension d increases, the geometrical structure of B becomes more complicated for the following reason. Since the denseness of the eigenvalues of the free operator increases as d increases we need to use the asymptotic formulas of high accuracy and investigate the intersections of the higher order of the resonance domains. Then the functions $F(\gamma+t)$, $\lambda_j(\gamma+t)$ [see (2.1.28), (2.1.29)] taking part in the construction of B (see Definition 2.1.2) becomes more complicated. Therefore surfaces and sets defined by these functions and hence the construction and investigation of B become more intricate. Besides of this construction in [Ve2] we gave the additional idea for the nonsmooth potential, namely for the construction of the simple set B when the nonsmooth potentials $q \in L_2(\mathbb{R}^2/\Omega)$, we eliminated additionally a set, which is described in terms of the number of the states (see [Ve2] p. 47, [Ve6] Sect. 3 of Chap. 3, and [Ve7]). More precisely, we eliminated the translations $A_k^{'}$ of the set A_k by the vectors $\gamma \in \Gamma$, where

$$A_1 = \{x : N_x(K_\rho(\frac{M_0}{\rho})) > b_1\}, A_k = \{x : N_x(K_\rho(\frac{2^{k-1}M_0}{\rho})\backslash K_\rho(\frac{2^{k-2}M_0}{\rho})) > b_k\},$$

$$M_0 \gg 1, b_1 = (M_0)^{\frac{3}{2}}, b_k = (2^k M_0)^{\frac{3}{2}}, k \geq 2,$$

$$K_\rho(a) = \{x : ||x| - \rho| < a\}$$

and $N_x(A)$ is the number of the vectors $\gamma + x$ lying in A. These eliminations imply that if $\gamma + t$ is in the simple set then the number of the vectors γ' in A_k is less than or equal to b_k. On the other hand using the formula (2.1.8) it can be proved that

$$|b(N, \gamma')|^2 = O((2^k M_0)^{-2}).$$

As a result, the left-hand side of (2.1.31) becomes $o(1)$, which implies the simplicity of $\Lambda(\gamma + t)$ and the closest of the functions $\Psi_{\gamma+t}(x)$ and $e^{i\langle\gamma+t,x\rangle}$. The simple set B of the quasimomenta is constructed and investigated for the first time (hence the main difficulty and the crucial point of perturbation theory of $L(q)$ are investigated) in [Ve3] for $d = 3$ and in [Ve2, Ve4] for the cases: 1. $d = 2$, $q \in L_2(F)$; 2. $d > 2$, q is a smooth potential. Thus for the first time in the papers [Ve2, Ve3, Ve4, Ve5] using the simple set B in the non-resnance domain U we constructed the perturbation theory (asymptotic formulas for Bloch eigenvalues and Bloch functions) for the Schrödinger operator $L(q)$ of arbitrary dimension d.

Then, Karpeshina proved (see [Ka1, Ka2, Ka3]) the convergence of the perturbation series of the two and three dimensional Schrödinger operator $L(q)$ with a wide class of the nonsmooth potential q for a set, that is similar to B. In [FeKnTr1] the asymptotic formulas for the eigenvalues and Bloch functions of the two and three dimensional operator $L_t(q)$ were obtained by the investigation of the corresponding infinity matrix.

In Sect. 2.5 we consider the geometrical aspects of the simple set of the Schrödinger operator of arbitrary dimension. We prove that the simple set B has asymptotically full measure on $\mathbb{R}^d$. Moreover, we construct a part of the isoenergetic surfaces

$$\{t \in F^* : \exists N, \Lambda_N(t) = \rho^2\}$$

corresponding to ρ^2, which is a smooth surface and has the measure asymptotically close to the measure of the isoenergetic surface

$$\{t \in F^* : \exists \gamma \in \Gamma, |\gamma + t|^2 = \rho^2\}$$

of the operator $L(0)$. For this we prove that the set B has the following third property:
Property 3 (Geometric property, containment the overlapping intervals). For any large ρ the set B contains the intervals $\{a + sb : s \in [-1, 1]\} =: T(\rho)$ such that

$$\Lambda(a - b) < \rho^2, \quad \Lambda(a + b) > \rho^2.$$

Since for $\gamma + t \in T(\rho) \subset B$ the eigenvalue $\Lambda(\gamma + t)$ is simple (see Property 1), the function $\Lambda(x)$ is continuous on $T(\rho)$ and hence there exists $\gamma + t$ such that $\Lambda(\gamma + t) = \rho^2$, that is, the interval $\{\Lambda(\gamma + t) : (\gamma + t) \in T(\rho)\}$ (consisting of Bloch eigenvalues) overlap ρ^2 which implies the validity of the Bethe-Sommerfeld conjecture for arbitrary dimension and arbitrary lattice. This conjecture claims that there exists only a finite number of gaps in the spectrum of $L(q)$.
Property 4 (Containment the large part of the isoenergetic surface). Using the geometric **Property 3**, we construct the part of the isoenergetic surfaces and proved that for large ρ the isoenergetic surfaces

$$I_\rho(q) = \{t \in F^* : \exists n, \Lambda_n(t) = \rho^2\}$$

of $L(q)$, contains a set which consists of the smooth surfaces and has the measure asymptotically equal to the measure of the sphere $\{x \in \mathbb{R}^d : |x| = \rho\}$. The nonempty of $I_\rho(q)$ for large ρ implies the validity of the Bethe-Sommerfeld conjecture for arbitrary dimension and arbitrary lattice.

There are several different approaches for solving the Bethe-Sommerfeld conjecture. First method is the method of Skriganov. The Skriganov's method is based on the detailed investigation of the arithmetic and geometric properties of the lattice. Skriganov [Sk1, Sk2, Sk3, Sk4, Sk5] proved the validity of the Bethe-Sommerfeld conjecture for the Schrödinger operator with the dimension $d = 2, 3$ and the arbitrary lattice, with the dimension $d > 3$ and the rational lattice. Dahlberg and Trubowits [DaTru] gave the simple proof of this conjecture for the two dimensional Schrödinger operator using an asymptotic of the Bessel functions.

In papers [Ve1, Ve2, Ve3, Ve4] (see also [Ve6, Ve8, Ve9]), for the first time, we proved the validity of the Bethe-Sommerfeld conjecture for the arbitrary lattice and arbitrary dimension by using the asymptotic formulas and by the construction of the simple set B, that is, by the method of the perturbation theory. Then Karpeshina (see [Ka1]) proved this conjecture for the two and three dimensional Schrödinger operator $L(q)$ for a wide class of nonsmooth potentials q by the method of the perturbation theory.

Helffer and Mohamed [HeMo] proved it for $d \leq 4$, by investigating the integrated density of states. Since this and the other investigations [Ka7, Moh, MoPaPc, PaSh, So] about the integrated density of states have no any connection with the main themes (asymptotic formulas for Bloch eigenvalues and Bloch functions, spectral invariants and inverse problem) of this book, we do not discuss those results.

Parnovski [Pa] proved the validity of this conjecture for the arbitrary lattice and arbitrary dimension by the methods of the perturbation theory. As he wrote in [Pa] (see introduction), there are certain parallels between the approach of the paper [Pa] and the approach used in the paper [Ve4] of Veliev. Briefly, he wrote the following similarities and differences. Similarities: Precise asymptotic formulae for Bloch eigenvalues in the non-resonance regions and some, although not very precise, formulae in the resonance regions and the geometrical combinatorics. Differences: Veliev makes a heavy use of the asymptotic formulae for the eigenfunctions, the isoenergetic surface, whereas we don't need it. One can readily see that the similarities are Property 1 and Property 3 (see above) which are enough to prove the validity of this conjecture for the arbitrary lattice and arbitrary dimension. Thus in [Pa] Parnovski proved the validity of this conjecture by using the similarities. The differences, that is, asymptotic formulas for the Bloch functions and investigations of the isoenergetic surface are more important than the conjecture and are my additional investigation whose expanded explanation were done in [Ve5, Ve6, Ve9]. Hence the method of the papers [Ve1, Ve2, Ve3, Ve4] is the first and unique (for present) by which the validity of the Bethe-Sommerfeld conjecture for the arbitrary lattice and arbitrary dimension was proved, since [Pa] was written after all my papers and arxiv papers about it. Note that in the recent literature [Ka4, Ka5, Ka6, PaBa, PaSo, Ve8] the generalizations of some results to periodic magnetic Schrödinger, polyharmonic, and psevdodifferential operator were investigated. In order to avoid the technical complexity and taking into account that the book is devoted to $L(q)$ I do not discuss the generalizations.

In 4-th and 5-th sections we construct and investigate the simple set B with the properties 1–4. Note that one can read Sects. 2.4 and 2.5 without reading Sect. 2.3.

In Sect. 2.6, we construct the simple set in the resonance domain and obtain the asymptotic formulas of arbitrary order for the Bloch functions of the multidimensional Schrödinger operator $L(q)$ of arbitrary dimension d, where

$$q \in W_2^s(F), \quad s \geq 6(3^d(d+1)^2)+d, \tag{2.1.36}$$

when the corresponding quasimomentum lies in this simple set, by using the ideas of the Sects. 2.4 and 2.5. For the first time the asymptotic formulas for the Bloch function in the resonance case were obtained in [FeKnTr2] for $d = 2$. Then in [Ka2], in the resonance case, for $d = 2, 3$ and for a wide class of singular potentials q, including Coulomb potential, the isoenergetic surfaces were constructed and the convergence of the perturbation series for the Bloch functions was proved. In the paper [Ve9] we investigated the resonance case for arbitrary dimension d. Note that we construct the simple set in the non-resonance domain so that it contains a big part of the isoenergetic surfaces of $L(q)$. However in the case of resonance domain we construct the simple set so that it can be easily used for the constructive determination a family of the spectral invariants by the given Bloch eigenvalues and then to study the inverse problem of $L(q)$ by these spectral invariants in the next chapters.

In this chapter for the different types of the measures of the subset A of $\mathbb{R}^d$ we use the same notation $\mu(A)$. By $|A|$ we denote the number of elements of the set A and use the following obvious fact. If $a \sim \rho$, then

$$|\{\gamma + t : \gamma \in \Gamma, ||\gamma + t| - a| < 1\}| = O(\rho^{d-1}). \tag{2.1.37}$$

Therefore for the number of the eigenvalues $\Lambda_N(t)$ of $L_t(q)$ lying in $(a^2-\rho, a^2+\rho)$ the equality

$$|\{N : \Lambda_N(t) \in (a^2-\rho, a^2+\rho)\}| = O(\rho^{d-1}) \tag{2.1.37a}$$

holds. Besides, we use the inequalities:

$$\alpha_1 + d\alpha < 1 - \alpha, \quad d\alpha < \frac{1}{2}\alpha_d, \tag{2.1.38}$$

$$\alpha_k + (k-1)\alpha < 1, \quad \alpha_{k+1} > 2(\alpha_k + (k-1))\alpha \tag{2.1.39}$$

$$k_1 \leq \frac{1}{3}(p - \frac{1}{2}(\varkappa(d-1)), \quad 3k_1\alpha > d + 2\alpha, \tag{2.1.40}$$

for $k = 1, 2, \ldots, d$, which follow from the definitions of the numbers p, $\varkappa$, α, α_k, k_1 [see (2.1.5), (2.1.1), (2.1.26), and the Definition 2.1.1].

2.2 Asymptotic Formulas for the Eigenvalues

First we obtain the asymptotic formulas for the non-resonance eigenvalues by iteration of (2.1.9). If (2.1.15) holds and

$$\gamma + t \in U(\rho^{\alpha_1}, p),$$

then (2.1.11) holds. Therefore using the decomposition (2.1.5) in (2.1.12), we obtain

$$b(N, \gamma - \gamma_1) = \sum_{\gamma_2 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{\Lambda_N(t) - |\gamma - \gamma_1 + t|^2} + O(\rho^{-p\alpha}). \tag{2.2.1}$$

Substituting this for $b(N, \gamma - \gamma_1)$ into the right-hand side of (2.1.9) and isolating the terms containing the multiplicand $b(N, \gamma)$, we get

$$(\Lambda_N(t) - |\gamma + t|^2) b(N, \gamma) = \sum_{\gamma_1, \gamma_2 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{\Lambda_N(t) - |\gamma - \gamma_1 + t|^2} + O(\rho^{-p\alpha})$$
$$= \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{|q_{\gamma_1}|^2 b(N, \gamma)}{\Lambda_N(t) - |\gamma - \gamma_1 + t|^2} + \sum_{\substack{\gamma_1, \gamma_2 \in \Gamma(\rho^\alpha), \\ \gamma_1 + \gamma_2 \neq 0}} \frac{q_{\gamma_1} q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{\Lambda_N(t) - |\gamma - \gamma_1 + t|^2} + O(\rho^{-p\alpha}), \tag{2.2.2}$$

since

$$q_{\gamma_1} q_{\gamma_2} = |q_{\gamma_1}|^2$$

for $\gamma_1 + \gamma_2 = 0$ and the last summation is taken under the condition $\gamma_1 + \gamma_2 \neq 0$. The formula (2.2.2) is the once iteration of (2.1.9). Let us iterate it several times. It follows from the definition of $U(\rho^{\alpha_1}, p)$ that (see Definition 2.1.1) if

$$\gamma + t \in U(\rho^{\alpha_1}, p), \gamma_1 \in \Gamma(\rho^\alpha), \gamma_2 \in \Gamma(\rho^\alpha), \ldots, \gamma_k \in \Gamma(\rho^\alpha), \gamma_1 + \gamma_2 + \cdots + \gamma_k \neq 0,$$

and (2.1.15) holds, then

$$||\gamma + t|^2 - |\gamma - \gamma_1 - \gamma_2 - \cdots - \gamma_k + t|^2| > \rho^{\alpha_1},$$
$$|\Lambda_N(t) - |\gamma - \gamma_1 - \gamma_2 - \cdots - \gamma_k + t|^2| > \frac{1}{2}\rho^{\alpha_1}, \quad \forall k \leq p. \tag{2.2.3}$$

Therefore arguing as in the proof of (2.2.1), we get

$$b(N, \gamma - \sum_{j=1}^{k} \gamma_j) = \sum_{\gamma_{k+1} \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_{k+1}} b(N, \gamma - \gamma_1 - \gamma_2 - \cdots - \gamma_{k+1})}{\Lambda_N(t) - |\gamma - \gamma_1 - \gamma_2 - \cdots - \gamma_k + t|^2} + O(\frac{1}{\rho^{p\alpha}}) \tag{2.2.4}$$

for $k \le p$, $\gamma_1+\gamma_2+\cdots+\gamma_k \ne 0$. Now we iterate (2.1.9), by using (2.2.4), as follows. In (2.2.2) replace $b(N,\gamma-\gamma_1-\gamma_2)$ by its expression from (2.2.4) [in (2.2.4) replace k by 2] and isolate the terms containing $b(N,\gamma)$, then replace $b(N,\gamma-\gamma_1-\gamma_2-\gamma_3)$ for $\gamma_1+\gamma_2+\gamma_3 \ne 0$ by its expression from (2.2.4) and isolate the terms containing $b(N,\gamma)$. Repeating this process p_1 times, we obtain

$$(\Lambda_N(t)-|\gamma+t|^2)b(N,\gamma) = A_{p_1-1}(\Lambda_N,\gamma+t)b(N,\gamma)+C_{p_1}+O(\rho^{-p\alpha}), \quad (2.2.5)$$

where $p_1 =: [\frac{p}{3}]+1$,

$$A_{p_1-1}(\Lambda_N,\gamma+t) = \sum_{k=1}^{p_1-1} S_k(\Lambda_N,\gamma+t),$$

$$S_k(\Lambda_N,\gamma+t) = \sum_{\gamma_1,\ldots,\gamma_k\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_k}q_{-\gamma_1-\gamma_2-\cdots-\gamma_k}}{\prod_{j=1}^{k}(\Lambda_N(t)-|\gamma+t-\sum_{i=1}^{j}\gamma_i|^2)},$$

$$C_{p_1} = \sum_{\gamma_1,\ldots,\gamma_{p_1+1}\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_{p_1+1}}b(N,\gamma-\gamma_1-\gamma_2-\cdots-\gamma_{p_1+1})}{\prod_{j=1}^{p_1}(\Lambda_N(t)-|\gamma+t-\sum_{i=1}^{j}\gamma_i|^2)}.$$

Here the summations for S_k and C_{p_1} are taken under the additional conditions $\gamma_1+\gamma_2+\cdots+\gamma_s \ne 0$ for $s=1,2,\ldots,k$ and $s=1,2,\ldots,p_1$ respectively. These conditions and (2.2.3) show that the absolute values of the denominators of the fractions in S_k and C_{p_1} are greater than $(\frac{1}{2}\rho^{\alpha_1})^k$ and $(\frac{1}{2}\rho^{\alpha_1})^{p_1}$ respectively. Now using inequality (2.1.6a), we get

$$\begin{aligned} S_k(\Lambda_N,\gamma+t) &= O(\rho^{-k\alpha_1}), \quad \forall k=1,2,\ldots,p_1-1, \\ C_{p_1} &= O(\rho^{-p_1\alpha_1}) = O(\rho^{-p\alpha}), \end{aligned} \quad (2.2.6)$$

since $p_1 \ge 3p$ [see (2.2.5)] and $\alpha_1 = 3\alpha$ (see Definition 2.1.1), and hence $p_1\alpha_1 \ge p\alpha$. In the proof of (2.2.6) we used only the condition (2.1.15) for Λ_N. Therefore

$$S_k(a,\gamma+t) = O(\rho^{-k\alpha_1}) \quad (2.2.7)$$

for all $a \in \mathbb{R}$ satisfying

$$|a-|\gamma+t|^2| < \frac{1}{2}\rho^{\alpha_1}.$$

Theorem 2.2.1 (a) *Suppose* $\gamma+t \in U(\rho^{\alpha_1},p)$. *If* (2.1.15) *and* (2.1.16) *hold, then* $\Lambda_N(t)$ *satisfies* (2.1.14) *for* $k=1,2,\ldots,[\frac{1}{3}(p-c)]$, *where*

$$F_0(\gamma+t)=0,\ F_k(\gamma+t)=O(\rho^{-\alpha_1}), \quad \forall k=0,1,\ldots, \quad (2.2.8)$$

$$F_1(\gamma+t)=\sum_{\gamma_1\in\Gamma(\rho^{\alpha})}\frac{|q_{\gamma_1}|^2}{|\gamma+t|^2-|\gamma-\gamma_1+t|^2}, \tag{2.2.9}$$

$$F_s=A_s(|\gamma+t|^2+F_{s-1},\gamma+t)=\sum_{k=1}^{s}S_k(|\gamma+t|^2+F_{s-1},\gamma+t)$$
$$=\sum_{k=1}^{s}(\sum_{\gamma_1,\ldots,\gamma_k\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_k}q_{-\gamma_1-\gamma_2-\cdots-\gamma_k}}{\prod_{j=1}^{k}(|\gamma+t|^2+F_{s-1}-|\gamma+t-\sum_{i=1}^{j}\gamma_i|^2)}) \tag{2.2.10}$$

for $s=1,2,\ldots.$ *and the last summations in* (2.2.10) *are taken under the additional conditions* $\gamma_1+\gamma_2+\cdots+\gamma_j\neq 0$ *for* $j=2,3,\ldots,k.$

(b) *For each vector* $\gamma+t$ *from* $U(\rho^{\alpha_1},p)$ *there exists an eigenvalue* $\Lambda_N(t)$ *of* $L_t(q)$ *satisfying* (2.1.14) *for* $k=1,2,\ldots,[\frac{1}{3}(p-\frac{1}{2}\varkappa(d-1))]$.

Proof (a) Dividing both side of (2.2.5) by $b(N,\gamma)$ and using (2.1.16) and (2.2.6), we get the proof of (2.1.13). Thus the formula (2.1.14) for $k=1$ holds and $F_0=0$. Hence (2.2.8) for $k=0$ is also proved. Moreover, from (2.2.7), we obtain

$$S_k(|\gamma+t|^2+O(\rho^{-\alpha_1}),\gamma+t)=O(\rho^{-k\alpha_1}) \tag{2.2.11}$$

for $k=1,2,\ldots.$ Therefore (2.2.8) for arbitrary k follows from the definition of F_k [see (2.2.10)] by induction. Now we prove (2.1.14) by induction on k. Suppose (2.1.14) holds for $k=j<[\frac{1}{3}(p-c)]\leq p_1$, that is,

$$\Lambda_N(t)=|\gamma+t|^2+F_{j-1}(\gamma+t)+O(\rho^{-j\alpha_1}).$$

Substituting this into $A_{p_1-1}(\Lambda_N,\gamma+t)$ in (2.2.5), dividing both sides of (2.2.5) by $b(N,\gamma)$, using (2.1.16), and taking into account that

$$A_{p_1-1}(\Lambda_N,\gamma+t)=A_j(\Lambda_N,\gamma+t)+O(\rho^{-(j+1)\alpha_1})$$

[see (2.2.6) and the definition of A_{p_1-1} in (2.2.5)], we get

$$\Lambda_N(t)=|\gamma+t|^2+A_j(|\gamma+t|^2+F_{j-1}+O(\rho^{-j\alpha_1}),\gamma+t)$$
$$+O(\rho^{-(j+1)\alpha_1})+O(\rho^{-(p-c)\alpha})$$

On the other hand

$$O(\rho^{-(p-c)\alpha})=O(\rho^{-(j+1)\alpha_1}),$$

since $j+1\leq\frac{1}{3}[p-c]$, and $\alpha_1=3\alpha$. Therefore to prove (2.1.14) for $k=j+1$ it remains to show that

$$A_j(|\gamma+t|^2+F_{j-1}+O(\rho^{-j\alpha_1}),\gamma+t)$$
$$=A_j(|\gamma+t|^2+F_{j-1},\gamma+t))+O(\rho^{-(j+1)\alpha_1}) \tag{2.2.12}$$

[see the definition of F_j in (2.2.10)]. It can be checked by using (2.1.6a), (2.2.8), (2.2.11) and the obvious relation

$$\frac{1}{\prod_{j=1}^{s}(|\gamma+t|^2+F_{j-1}+O(\rho^{-j\alpha_1})-|\gamma+t-\sum_{i=1}^{s}\gamma_i|^2)}$$
$$-\frac{1}{\prod_{j=1}^{s}(|\gamma+t|^2+F_{j-1}-|\gamma+t-\sum_{i=1}^{s}\gamma_i|^2)}$$
$$=\frac{1}{\prod_{j=1}^{s}(|\gamma+t|^2+F_{j-1}-|\gamma+t-\sum_{i=1}^{s}\gamma_i|^2)}\left(\frac{1}{1-O(\rho^{-(j+1)\alpha_1})}-1\right)$$
$$=O(\rho^{-(j+1)\alpha_1}),\quad \forall s=1,2,\ldots.$$

The formula (2.2.9) is also proved, since by (2.2.10) and (2.2.8) we have

$$F_1=A_1(|\gamma+t|^2,\gamma+t)=S_1(|\gamma+t|^2,\gamma+t)$$
$$=\sum_{\gamma_1\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_1}q_{-\gamma_1}}{(|\gamma+t|^2-|\gamma+t-\gamma_1|^2)}. \tag{2.2.13}$$

(b) Let A be the set of indices N satisfying (2.1.15). Using (2.1.8) and Bessel's inequality, we obtain

$$\sum_{N\notin A}|b(N,\gamma)|^2=\sum_{N\notin A}\left|\frac{(\Psi_N(x),q(x)e^{i\langle\gamma+t,x\rangle})}{\Lambda_N-|\gamma+t|^2}\right|^2=O(\rho^{-2\alpha_1})$$

Hence, by the Parseval equality, we have

$$\sum_{N\in A}|b(N,\gamma)|^2=1-O(\rho^{-2\alpha_1}).$$

This and the inequality

$$|A|=O(\rho^{d-1})=O(\rho^{(d-1)\varkappa\alpha})$$

[see (2.1.37a) and the definition of α in (2.1.5)] imply that there exists a number N satisfying (2.1.16) for $c=\frac{1}{2}\varkappa(d-1)$. Thus $\Lambda_N(t)$ satisfies (2.1.14) due to (a). □

Theorem 2.2.1 shows that in the non-resonance case the eigenvalue of the operator $L_t(q)$ is close to the eigenvalue of the unperturbed operator $L_t(0)$. However, in Theorem 2.2.2 we prove that if

$$\gamma+t\in\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k})\backslash E_{k+1}$$

for $k \geq 1$, where $\gamma_1, \gamma_2, \ldots, \gamma_k$ are linearly independent vectors of $\Gamma(p\rho^{\alpha})$, then the corresponding eigenvalue of $L_t(q)$ is close to the eigenvalue of the matrix $C(\gamma + t, \gamma_1, \gamma_2, \ldots, \gamma_k)$ constructed as follows. Introduce the sets:

$$B_k =: B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{b : b = \sum_{i=1}^{k} n_i \gamma_i, n_i \in \mathbb{Z}, |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\},$$

$$B_k(\gamma + t) = \gamma + t + B_k = \{\gamma + t + b : b \in B_k\}, \tag{2.2.14}$$

$$B_k(\gamma + t, p_1) = \{\gamma + t + b + a : b \in B_k, |a| < p_1\rho^{\alpha}, a \in \Gamma\} = \{h_i + t : i = 1, 2, \ldots, b_k\},$$

where p_1 is defined in (2.2.5), $h_1 + t, h_2 + t, \ldots, h_{b_k} + t$ are the vectors of $B_k(\gamma + t, p_1)$, and $b_k =: b_k(\gamma_1, \gamma_2, \ldots, \gamma_k)$ is the number of the vectors of $B_k(\gamma + t, p_1)$. Define the matrix $C(\gamma + t, \gamma_1, \gamma_2, \ldots, \gamma_k) =: (c_{i,j})$ by

$$c_{i,i} = |h_i + t|^2, \quad c_{i,j} = q_{h_i - h_j}, \ \forall i \neq j, \tag{2.2.15}$$

where $i, j = 1, 2, \ldots, b_k$.

To prove Theorem 2.2.2 we use the following lemma.

Lemma 2.2.1 *Suppose*

$$\gamma + t \in (\cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}))\backslash E_{k+1}$$

and $h + t \in B_k(\gamma + t, p_1)$. *If* $(h - \gamma' + t) \notin B_k(\gamma + t, p_1)$, *where* $\gamma' \in \Gamma(\rho^{\alpha})$, *then*

$$||\gamma + t|^2 - |h - \gamma' - \gamma_1' - \gamma_2' - \cdots - \gamma_s' + t|^2| > \frac{1}{5}\rho^{\alpha_{k+1}} \tag{2.2.16}$$

for $s = 0, 1, \ldots, p_1 - 1$, *where* $\gamma_1' \in \Gamma(\rho^{\alpha})$, $\gamma_2' \in \Gamma(\rho^{\alpha}), \ldots, \gamma_s' \in \Gamma(\rho^{\alpha})$.

Proof It follows from the definitions of p_1 [see (2.2.5)] and p [see (2.1.5), (2.1.1)] that $p > 2p_1$. Therefore the conditions of Lemma 2.2.1 imply that

$$h - \gamma' - \gamma_1' - \gamma_2' - \cdots - \gamma_s' + t \in B_k(\gamma + t, p)\backslash B_k(\gamma + t)$$

for $s = 0, 1, \ldots, p_1 - 1$. By the definitions of $B_k(\gamma + t, p)$ and B_k [see (2.2.14)] we have

$$h - \gamma' - \gamma_1' - \gamma_2' - \cdots - \gamma_s' + t = \gamma + t + b + a,$$

where

$$|b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}, |a| < p\rho^{\alpha}, \gamma + t + b + a \notin \gamma + t + B_k, b \in B_k \subset P, \tag{2.2.17}$$

and $P = Span\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. In this notation (2.2.16) has the form

$$||\gamma + t + a + b|^2 - |\gamma + t|^2| > \frac{1}{5}\rho^{\alpha_{k+1}}, \tag{2.2.18}$$

where (2.2.17) holds. To prove (2.2.18) we consider two cases:

Case 1. $a \in P$. Since $b \in B_k \subset P$ [see (2.2.17)] we have $a + b \in P$. This with the third relation in (2.2.17) implies that $a + b \in P \backslash B_k$, i.e.,

$$a + b \in P, \quad |a + b| \geq \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}} \tag{2.2.19}$$

[see the definition of B_k in (2.2.14)]. Now to prove (2.2.18) we consider the orthogonal decomposition $\gamma + t = y + v$ of $\gamma + t$, where $v \in P$ and $y \perp P$. First we prove that the projection v of any vector

$$x \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$$

on P satisfies

$$|v| = O(\rho^{(k-1)\alpha + \alpha_k}). \tag{2.2.20}$$

For this we turn the coordinate axis so that P coincides with the span of the vectors $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_k$. Since $\gamma_s \in P$ we have

$$\gamma_s = \sum_{i=1}^{k} \gamma_{s,i} e_i, \quad \forall s = 1, 2, \ldots, k$$

Therefore the relation $x \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ and (2.1.10) imply

$$\sum_{i=1}^{k} \gamma_{s,i} x_i = O(\rho^{\alpha_k}), \quad \forall s = 1, 2, \ldots, k,$$

where $x = (x_1, x_2, \ldots, x_d)$, $\gamma_j = (\gamma_{j,1}, \gamma_{j,2}, \ldots, \gamma_{j,k}, 0, 0, \ldots, 0)$. Solving this system of equations by Cramer's rule, we obtain

$$x_n = \frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})}, \quad \forall n = 1, 2, \ldots, k, \tag{2.2.21}$$

where $b_{j,i}^n = \gamma_{j,i}$ for $n \neq j$ and $b_{j,i}^n = O(\rho^{\alpha_k})$ for $n = j$. Since the absolute value of the determinant $\det(\gamma_{j,i})$ is the volume of the parallelotope generated by the vectors $\gamma_1, \gamma_2, \ldots, \gamma_k$ we have

$$\left|\det(\gamma_{j,i})\right| \geq \mu(F) = 1.$$

On the other hand the relation $\gamma_j \in \Gamma(p\rho^{\alpha})$ and the definition of $b_{j,i}^n$ imply that

$$|\gamma_{j,i}| < p\rho^{\alpha}, \quad \det(b^n_{j,i}) = O(\rho^{\alpha_k+(k-1)\alpha}).$$

Therefore using (2.2.21), we get

$$x_n = O(\rho^{\alpha_k+(k-1)\alpha}), \quad \forall n = 1, 2, \ldots, k; \ \forall x \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}). \tag{2.2.22}$$

Hence (2.2.20) holds. The conditions $a \in P$, $b \in P$ and the orthogonal decomposition $\gamma + t = y + v$ of $\gamma + t$, where $v \in P$ and $y \perp P$ imply that

$$\langle y, v\rangle = \langle y, a\rangle = \langle y, b\rangle = 0,$$

and

$$|\gamma + t + a + b|^2 - |\gamma + t|^2 = |a + b + v|^2 - |v|^2. \tag{2.2.23}$$

Therefore using (2.2.20), (2.2.19), and the inequality $\alpha_{k+1} > 2(\alpha_k + (k-1)\alpha)$ [see the second inequality in (2.1.39)], we obtain the estimation (2.2.18).

Case 2. $a \notin P$. First we show that

$$||\gamma + t + a|^2 - |\gamma + t|^2| \geq \rho^{\alpha_{k+1}}. \tag{2.2.24}$$

Suppose that (2.2.24) does not hold. Then $\gamma + t \in V_a(\rho^{\alpha_{k+1}})$. On the other hand

$$\gamma + t \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_{k+1}})$$

(see the conditions of Lemma 2.2.1). Therefore we have $\gamma + t \in E_{k+1}$ which contradicts to the conditions of the lemma. Thus (2.2.24) is proved. Now, to prove (2.2.18) we write the difference $|\gamma + t + a + b|^2 - |\gamma + t|^2$ as the sum of

$$d_1 =: |\gamma + t + a + b|^2 - |\gamma + t + b|^2 \text{ and } d_2 =: |\gamma + t + b|^2 - |\gamma + t|^2.$$

Since

$$d_1 = |\gamma + t + a|^2 - |\gamma + t|^2 + 2\langle a, b\rangle,$$

it follows from the inequalities (2.2.24) and (2.2.17) that $|d_1| > \frac{2}{3}\rho^{\alpha_{k+1}}$. On the other hand, taking $a = 0$ in (2.2.23), we have

$$d_2 = |b + v|^2 - |v|^2.$$

Therefore (2.2.20), the first inequality in (2.2.17) and the second inequality in (2.1.39) imply that

$$|d_2| < \frac{1}{3}\rho^{\alpha_{k+1}}, \quad |d_1| - |d_2| > \frac{1}{3}\rho^{\alpha_{k+1}},$$

that is, (2.2.18) holds □

Now we are ready to prove the following

Theorem 2.2.2 (a) *Suppose*

$$\gamma + t \in (\cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k}))\backslash E_{k+1},$$

where $1 \leq k \leq d-1$. *If* (2.1.15) *and* (2.1.16) *hold, then there exists an index* j *such that* (2.1.17) *holds, where*

$$\lambda_1(\gamma+t) \leq \lambda_2(\gamma+t) \leq \cdots \leq \lambda_{b_k}(\gamma+t)$$

are the eigenvalues of the matrix $C(\gamma+t, \gamma_1, \gamma_2, \ldots, \gamma_k)$ *defined in* (2.2.15).

(b) *Every eigenvalue* $\Lambda_N(t)$ *of the operator* $L_t(q)$ *satisfies one of the formulas* (2.1.14) *and* (2.1.17) *for* $k = [\frac{1}{3}(p - \frac{1}{2}\varkappa(d-1))]$ *and* $c = \frac{\varkappa(d-1)}{2}$ *respectively.*

Proof (a) Writing the Eq. (2.1.9) for all $h_i + t \in B_k(\gamma+t, p_1)$, we obtain

$$(\Lambda_N - |h_i+t|^2)b(N, h_i) = \sum_{\gamma' \in \Gamma(\rho^\alpha)} q_{\gamma'} b(N, h_i - \gamma') + O(\rho^{-p\alpha}) \tag{2.2.25}$$

for $i = 1, 2, \ldots, b_k$ [see (2.2.14) for the definition of $B_k(\gamma+t, p_1)$]. It follows from (2.1.15) and Lemma 2.2.1 that if

$$(h_i - \gamma' + t) \notin B_k(\gamma+t, p_1),$$

then

$$|\Lambda_N(t) - |h_i - \gamma' - \gamma_1 - \gamma_2 - \cdots - \gamma_s + t|^2| > \frac{1}{6}\rho^{\alpha_{k+1}}, \tag{2.2.26}$$

where $\gamma' \in \Gamma(\rho^\alpha)$, $\gamma_j \in \Gamma(\rho^\alpha)$, $j = 1, 2, \ldots, s$ and $s = 0, 1, \ldots, p_1 - 1$. Therefore, using the p_1 times iterations of (2.2.1) taking into account (2.2.26), (2.1.6a) and the obvious inequality $p_1\alpha_{k+1} > p\alpha$ [see (2.2.5) and Definition 2.1.1 for the definitions of p_1 and α_{k+1}], we see that if

$$(h_i - \gamma' + t) \notin B_k(\gamma+t, p_1),$$

then

$$b(N, h_i - \gamma') = \sum_{\gamma_1, \ldots, \gamma_{p_1-1} \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} q_{\gamma_2} \ldots q_{\gamma_{p_1}} b(N, h_i - \gamma' - \sum_{i=1}^{p_1} \gamma_i)}{\prod_{j=0}^{p_1-1} (\Lambda_N - |h_i - \gamma' + t - \sum_{i=1}^{j} \gamma_i|^2)}$$
$$+ O(\rho^{-p\alpha}) = O(\rho^{p_1 \alpha_{k+1}}) + O(\rho^{-p\alpha}) = O(\rho^{-p\alpha}). \tag{2.2.27}$$

Hence (2.2.25) has the form

$$(\Lambda_N - |h_i + t|^2) b(N, h_i) = \sum_{\substack{\gamma' : \gamma' \in \Gamma(\rho^\alpha), \\ h_i - \gamma' + t \in B_k(\gamma + t, p_1)}} q_{\gamma'} b(N, h_i - \gamma') + O(\rho^{-p\alpha})$$

for $i = 1, 2, \ldots, b_k$. This system can be written in the matrix form

$$(C - \Lambda_N I)(b(N, h_1), b(N, h_2), \ldots b(N, h_{b_k})) = O(\rho^{-p\alpha}),$$

where the right-hand side of this system is a vector having the norm

$$\| O(\rho^{-p\alpha}) \| = O(\sqrt{b_k} \rho^{-p\alpha}).$$

Using the last two equalities, taking into account that one of the vectors $h_1 + t$, $h_2 + t, \ldots, h_{b_k} + t$ is $\gamma + t$ [see the definition of $B_k(\gamma + t, p_1)$ in (2.2.14)] and (2.1.16) holds, we obtain

$$c_5 \rho^{-c\alpha} < (\sum_{i=1}^{b_k} |b(N, h_i)|^2)^{\frac{1}{2}} \leq \|(C - \Lambda_N I)^{-1}\| \sqrt{b_k} c_7 \rho^{-p\alpha}. \tag{2.2.28}$$

Since $(C - \Lambda_N I)^{-1}$ is the symmetric matrix having the eigenvalues $(\Lambda_N - \lambda_i)^{-1}$ for $i = 1, 2, \ldots, b_k$, we have

$$\max_{i=1,2,\ldots,b_k} |\Lambda_N - \lambda_i|^{-1} = \|(C - \Lambda_N I)^{-1}\| > c_5 c_7^{-1} b_k^{-\frac{1}{2}} \rho^{-c\alpha + p\alpha}, \tag{2.2.29}$$

where b_k is the number of the vectors of $B_k(\gamma + t, p_1)$. It follows from the definition of $B_k(\gamma + t, p_1)$ [see (2.2.14)] and the obvious relations

$$|B_k| = O(\rho^{\frac{k}{2}\alpha_{k+1}}), |\Gamma(p_1 \rho^\alpha)| = O(\rho^{d\alpha}), \quad d\alpha < \frac{1}{2} 3^d \alpha = \frac{1}{2} \alpha_d$$

that

$$b_k = O(\rho^{d\alpha + \frac{k}{2}\alpha_{k+1}}) = O(\rho^{\frac{d}{2}\alpha_d}) = O(\rho^{\frac{d}{2} 3^d \alpha}), \quad \forall k = 1, 2, \ldots, d-1. \tag{2.2.30}$$

Thus (2.1.17) follows from (2.2.29) and (2.2.30).

(b) Let $\Lambda_N(t)$ be an eigenvalue of $L_t(q)$ lying in $(\frac{3}{4}\rho^2, \frac{5}{4}\rho^2)$. Denote by D the set of all vectors $\gamma \in \Gamma$ satisfying (2.1.15). Using (2.1.8), (2.1.15), Bessel's inequality, and Parseval's equality, we obtain

$$\sum_{\gamma \notin D} |b(N,\gamma)|^2 = \sum_{\gamma \notin D} |\frac{(\Psi_{N,t}q, e^{i\langle \gamma+t, x\rangle})}{\Lambda_N - |\gamma+t|^2}|^2$$
$$= O(\rho^{-2\alpha_1})\|\Psi_{N,t}q\| = O(\rho^{-2\alpha_1})$$

and

$$\sum_{\gamma \in D} |b(N,\gamma)|^2 = 1 - O(\rho^{-2\alpha_1}).$$

Since $|D| = O(\rho^{d-1})$ [see (2.1.37)], there exists $\gamma \in D$ such that

$$|b(N,\gamma)| > c_8 \rho^{-\frac{(d-1)}{2}} = c_8 \rho^{-\frac{(d-1)\varkappa}{2}\alpha},$$

that is, (2.1.16) for $c = \frac{(d-1)\varkappa}{2}$ holds. Now the proof of (b) follows from Theorem 2.2.1(a) and Theorem 2.2.2(a), since either $\gamma + t \in U(\rho^{\alpha_1}, p)$ or $\gamma + t \in E_k \backslash E_{k+1}$ for $k = 1, 2, \ldots, d-1$ [see (2.2.33)] □

Remark 2.2.1 The obtained asymptotic formulas hold true, without any changes in their proofs, if we replace $V_{\gamma_1}(\rho^{\alpha_1})$ by $V_{\gamma_1}(c_4\rho^{\alpha_1})$. Here we note that the non-resonance domain

$$U =: U(c_4\rho^{\alpha_1}, p) =: (R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho))\backslash \bigcup_{\gamma_1 \in \Gamma(p\rho^{\alpha})} V_{\gamma_1}(c_4\rho^{\alpha_1})$$

(see Definition 2.1.1) has an asymptotically full measure on $\mathbb{R}^d$ in the sense that

$$\lim_{\rho\to\infty} \frac{\mu(U \cap S(\rho))}{\mu(S(\rho))} = 1,$$

where

$$S(\rho) = \{x \in \mathbb{R}^d : |x| = \rho\}$$

is the sphere. Clearly, $S(\rho) \cap V_b(c_4\rho^{\alpha_1})$ is the part of sphere $S(\rho)$, which is contained between two parallel hyperplanes

$$\{x : |x|^2 - |x+b|^2 = -c_4\rho^{\alpha_1}\} \ \& \ \{x : |x|^2 - |x+b|^2 = c_4\rho^{\alpha_1}\}.$$

The distances of these hyperplanes from the origin are $O(\frac{\rho^{\alpha_1}}{|b|})$. Therefore, the relations

$$|\Gamma(p\rho^{\alpha})| = O(\rho^{d\alpha})$$

and $\alpha_1 + d\alpha < 1 - \alpha$ [see (2.1.38)] imply

$$\mu(S(\rho)\cap V_b(c_4\rho^{\alpha_1})) = O(\frac{\rho^{\alpha_1+d-2}}{|b|}), \mu(E_1\cap S(\rho)) = O(\rho^{d-1-\alpha}), \quad (2.2.31)$$

$$\mu(U(c_4\rho^{\alpha_1}, p)\cap B(\rho)) = (1+O(\rho^{-\alpha}))\mu(B(\rho)). \quad (2.2.32)$$

If

$$x \in \cap_{i=1}^{d} V_{\gamma_i}(\rho^{\alpha_d}),$$

then (2.2.22) holds for $k = d$ and $n = 1, 2, \ldots, d$. Hence we have

$$|x| = O(\rho^{\alpha_d+(d-1)\alpha}).$$

It is impossible, since $\alpha_d + (d-1)\alpha < 1$ [see the first inequality in (2.1.39)] and $x \in S(\rho)$. It means that

$$(\cap_{i=1}^{d} V_{\gamma_i}(\rho^{\alpha_k}))\cap S(\rho) = \emptyset$$

for $\rho \gg 1$. Thus for $\rho \gg 1$ we have

$$R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho) = (U(\rho^{\alpha_1}, p)\cup(\cup_{s=1}^{d-1}(E_s\backslash E_{s+1}))). \quad (2.2.33)$$

Remark 2.2.2 Here we note some properties of the known parts

$$|\gamma+t|^2 + F_k(\gamma+t) \ \&\ \lambda_j(\gamma+t)$$

(see Theorem 2.2.1 and Theorem 2.2.2) of the eigenvalues of $L_t(q)$. Denoting $\gamma+t$ by x we consider the function

$$F(x) = |x|^2 + F_k(x).$$

It follows from the definition of $F_k(x)$ that (see 2.2.10) $F(x)$ is continuous on $U(c_4\rho^{\alpha_1}, p)$. Let us prove the equalities

$$\frac{\partial F_k(x)}{\partial x_i} = O(\rho^{-2\alpha_1+\alpha}), \quad \forall i = 1, 2, \ldots, d;\ \forall k = 1, 2, \ldots, \quad (2.2.34)$$

for $x \in U(\rho^{\alpha_1}, p)$, by induction on k. If $k = 1$ then (2.2.34) follows from (2.1.6a) and the obvious relation

$$\frac{\partial}{\partial x_i}(\frac{1}{|x|^2-|x-\gamma_1|^2}) = \frac{-2\gamma_1(i)}{(|x|^2-|x-\gamma_1|^2)^2} = O(\rho^{-2\alpha_1+\alpha}), \quad (2.2.35)$$

where $\gamma_1(i)$ is the ith component of the vector $\gamma_1 \in \Gamma(p\rho^{\alpha})$. Now suppose that (2.2.34) holds for $k = s$. Using this and (2.2.8), replacing $|x|^2$ by $|x|^2 + F_s(x)$ in (2.2.35) and evaluating as above we obtain

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x|^2+F_s-|x-\gamma_1|^2}\right)=\frac{-2\gamma_1(i)+\frac{\partial F_s(x)}{\partial x_i}}{(|x|^2+F_s-|x-\gamma_1|^2)^2}=O(\rho^{-2\alpha_1+\alpha}).$$

This formula together with the definition (2.2.10) of F_k gives (2.2.34) for $k=s+1$.

Now denoting $\lambda_i(\gamma+t)-|\gamma+t|^2$ by $r_i(\gamma+t)$ we prove that

$$|r_i(x)-r_i(x')|\le 2\rho^{\frac{1}{2}\alpha_d}|x-x'|,\ \forall i. \tag{2.2.36}$$

It is clear that

$$r_1(x)\le r_2(x)\le\cdots\le r_{b_k}(x)$$

are the eigenvalues of the matrix

$$C(x)-|x|^2I=:\widetilde{C}(x),$$

where $C(x)$ is defined in (2.2.15). By definition, only the diagonal elements of the matrix

$$\widetilde{C}(x)=(\widetilde{c}_{i,j}(x))$$

depend on x and they are

$$\widetilde{c}_{i,j}(x)=|x+a_i|^2-|x|^2=2\langle x,a_i\rangle+|a_i|^2, \tag{2.2.37}$$

where $x=\gamma+t$, $a_i=h_i+t-x$ and $h_i+t\in B_k(\gamma+t,p_1)$. Using the equality $\alpha_d=3^d\alpha$ (see Definition 2.1.1) and the definition of $B_k(\gamma+t,p_1)$ [see (2.2.14)], we get

$$|a_i|<\frac{1}{2}\rho^{\frac{1}{2}\alpha_k}+p_1\rho^{\alpha}<\rho^{\frac{1}{2}\alpha_d}$$

for $k<d$. Therefore taking into account that $\widetilde{C}(x)-\widetilde{C}(x')$ is a diagonal matrix with diagonal entries

$$\widetilde{c}_{i,j}(x)-\widetilde{c}_{i,j}(x')=2\langle x-x',a_i\rangle$$

[see (2.2.37)], we have

$$\|\widetilde{C}(x)-\widetilde{C}(x')\|\le 2\rho^{\frac{1}{2}\alpha_d}|x-x'|$$

which yields (2.2.36).

2.3 Bloch Eigenvalues Near the Diffraction Planes

In this section we obtain the asymptotic formulae for the eigenvalues corresponding to the quasimomentum $\gamma+t$ lying near the diffraction hyperplane D_δ, namely lying in the single resonance domain

$$V_\delta^{'}(\rho^{\alpha_1}) =: V_\delta(\rho^{\alpha_1})\backslash E_2$$

defined in Definition 2.1.1, where δ is the element of Γ of minimal norm in its direction, that is, δ is the element of Γ such that

$$\{\langle\delta,\omega\rangle : \omega\in\Omega\} = 2\pi\mathbb{Z}.$$

In Sect. 2.2, to obtain the asymptotic formulas for the eigenvalues corresponding to the quasimomentum $\gamma+t$ lying far from the diffraction planes we considered the operator $L_t(q)$ as the perturbation of the operator $L_t(0)$ with q. As a result the asymptotic formulas for these eigenvalues of $L_t(q)$ were expressed in terms of the eigenvalues of $L_t(0)$. To obtain the asymptotic formulae for the eigenvalues corresponding to the quasimomentum $\gamma+t$ lying near the diffraction plane D_δ we consider the operator $L_t(q)$ as the perturbation of the operator $L_t(q^\delta)$, where the directional potential q^δ is defined in (2.1.19), with $q-q^\delta$. Hence it is natural that the asymptotic formulas, which will be obtained in this section, are expressed in terms of the eigenvalues of $L_t(q^\delta)$. Therefore first of all we need to investigate the eigenvalues and eigenfunctions of $L_t(q^\delta)$. Here we use Notation 2.1.1. Denote by F_δ the fundamental domain H_δ/Γ_δ of Γ_δ. Then $t\in F^* = \mathbb{R}^d/\Gamma$ has a unique decomposition

$$t = a+\tau+|\delta|^{-2}\langle t,\delta\rangle\,\delta, \tag{2.3.1}$$

where $a\in\Gamma_\delta$, $\tau\in F_\delta$. Define the sets Ω' and Γ' by

$$\Omega' = \{h+l\delta^* : h\in\Omega_\delta, l\in\mathbb{Z}\},$$

and

$$\Gamma' = \{b+(p-(2\pi)^{-1}\langle b,\delta^*\rangle)\delta : b\in\Gamma_\delta, p\in\mathbb{Z}\},$$

where δ^* is the element of Ω satisfying $\langle\delta^*,\delta\rangle = 2\pi$.

Lemma 2.3.1 (a) *The following relations hold:*

$$\Omega = \Omega', \Gamma = \Gamma'.$$

(b) *The eigenvalues and eigenfunctions of the operator $L_t(q^\delta)$ are*

$$\lambda_{j,\beta}(v,\tau) = |\beta+\tau|^2 + \mu_j(v(\beta,t)),\ \Phi_{j,\beta}(x) = e^{i\langle\beta+\tau,x\rangle}\varphi_{j,v(\beta,t))}(\zeta)$$

for $j \in \mathbb{Z}$, $\beta \in \Gamma_\delta$, *where* $v(\beta, t)$ *is the fractional part of*

$$|\delta|^{-2} \langle t, \delta \rangle - (2\pi)^{-1} \left\langle \beta - a, \delta^* \right\rangle,$$

τ *and* a *are uniquely determined from decomposition* (2.3.1)*. Here* $\mu_j(v(\beta, t))$ *and* $\varphi_{j,v(\beta,t)}(\zeta)$ *are the eigenvalues and normalized eigenfunctions of the operator* $T_{v(\beta,t)}(Q(\zeta))$ *generated by the boundary value problem*

$$-|\delta|^2 y''(\zeta) + Q(\zeta) y(\zeta) = \mu y(\zeta),$$

$$y(\zeta + 2\pi) = e^{i2\pi v} y(\zeta),$$

where, $\zeta = \langle \delta, x \rangle$, $Q(\zeta) = q^\delta(x)$ *and for simplicity of the notation, instead of* $v(\beta, t)$ *we write* $v(\beta)$ (*or* v) *if* t (*or* t *and* β), *for which we consider* $v(\beta, t)$, *is unambiguous.*

Proof (a) For each vector ω of the lattice Ω assign

$$h = \omega - (2\pi)^{-1} \langle \omega, \delta \rangle \delta^*.$$

Using the relations $\langle \omega, \delta \rangle =: 2\pi l \in 2\pi\mathbb{Z}$, and $\langle \delta^*, \delta \rangle = 2\pi$ we see that $h \in \Omega$ and $\langle h, \delta \rangle = 0$, i.e., $h \in \Omega_\delta$. Hence $\Omega \subset \Omega'$. Now for each vector γ of the lattice Γ assign $b = \gamma - |\delta|^{-2} \langle \gamma, \delta \rangle \delta$. It is not hard to verify that $b \in H_\delta$ and $\langle b, \omega \rangle = \langle \gamma, \omega \rangle \in 2\pi\mathbb{Z}$ for $\omega \in \Omega_\delta \subset \Omega$. Therefore $b \in \Gamma_\delta$. Moreover

$$\left\langle b, \delta^* \right\rangle = \left\langle \gamma, \delta^* \right\rangle - 2\pi \langle \gamma, \delta \rangle |\delta|^{-2}.$$

Since $\langle \gamma, \delta^* \rangle \in 2\pi\mathbb{Z}$, that is, $\langle \gamma, \delta^* \rangle = 2\pi n$, where $n \in \mathbb{Z}$, we have

$$\langle \gamma, \delta \rangle |\delta|^{-2} = n - (2\pi)^{-1} \left\langle b, \delta^* \right\rangle.$$

Therefore we obtain an orthogonal decomposition

$$\gamma = b + \langle \gamma, \frac{\delta}{|\delta|} \rangle \frac{\delta}{|\delta|} = b + (n - (2\pi)^{-1} \left\langle b, \delta^* \right\rangle) \delta \tag{2.3.2}$$

of $\gamma \in \Gamma$, where $b \in \Gamma_\delta$, and $n \in \mathbb{Z}$. Hence $\Gamma \subset \Gamma'$. On the other hand, if $b \in \Gamma_\delta$, $h \in \Omega_\delta$ and $n, l \in \mathbb{Z}$, then

$$\left\langle h + l\delta^*, b + (n - (2\pi)^{-1} \left\langle b, \delta^* \right\rangle) \delta \right\rangle = \langle h, b \rangle + 2\pi n l \in 2\pi\mathbb{Z}.$$

Thus we have the relations (see the definitions of the sets Ω' and Γ')

$$\Omega \subset \Omega', \Gamma \subset \Gamma', \left\langle \omega', \gamma' \right\rangle \in 2\pi\mathbb{Z}, \forall \omega' \in \Omega', \ \forall \gamma' \in \Gamma'. \tag{2.3.3}$$

Since Ω is the set of all vectors $\omega \in \mathbb{R}^d$ satisfying $\langle \omega, \gamma \rangle \in 2\pi\mathbb{Z}$ for all $\gamma \in \Gamma$ and Γ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $\langle \omega, \gamma \rangle \in 2\pi\mathbb{Z}$ for all $\omega \in \Omega$, the relations in (2.3.3) imply $\Omega' \subset \Omega$, $\Gamma' \subset \Gamma$ and hence $\Omega = \Omega'$, $\Gamma = \Gamma'$.

(b) Since $\beta + \tau$ is orthogonal to δ, turning the coordinate axis so that δ coincides with one of the coordinate axis and taking into account that the Laplace operator is invariant under rotation, one can easily verify that

$$(-\Delta + q^{\delta}(x))\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x)$$

Now using the relation $\langle \delta, \omega \rangle = 2\pi l$, where $\omega \in \Omega$, $l \in \mathbb{Z}$, and the definitions of $\Phi_{j,\beta}(x)$ and $\varphi_{j,v}(\langle \delta, x \rangle)$ we obtain

$$\Phi_{j,\beta}(x+\omega) = e^{i\langle \beta+\tau, x+\omega \rangle}\varphi_{j,v}(\langle \delta, x+\omega \rangle) = \Phi_{j,\beta}(x)e^{i\langle \beta+\tau,\omega \rangle + i2\pi l v(\beta,t)}.$$

Replacing τ and ω by $t - a - |\delta|^{-2}\langle t, \delta \rangle \delta$ and $h + l\delta^*$ respectively, where

$$h \in \Omega_\delta, l \in \mathbb{Z},$$

[see (2.3.1) and the first equality of (a)], and then using

$$\langle h, \delta \rangle = 0, \left\langle \delta^*, \delta \right\rangle = 2\pi$$

one can easily verify that

$$\langle \beta + \tau, \omega \rangle = \langle t, \omega \rangle + \langle \beta - a, h \rangle - 2\pi l[|\delta|^{-2}\langle t, \delta \rangle - (2\pi)^{-1}\left\langle \beta - a, \delta^* \right\rangle].$$

From this, using the relation

$$\langle \beta - a, h \rangle \in 2\pi\mathbb{Z},$$

(since $\beta - a \in \Gamma_\delta$, $h \in \Omega_\delta$), and taking into account that $v(\beta, t)$ is a fractional part of the expression in the last square bracket, we infer

$$\Phi_{j,\beta}(x+\omega) = e^{i\langle t,\omega \rangle}\Phi_{j,\beta}(x).$$

Thus $\Phi_{j,\beta}(x)$ is an eigenfunction of $L_t(q^\delta)$.

Now we prove that the system

$$\{\Phi_{j,\beta} : j \in \mathbb{Z}, \beta \in \Gamma_\delta\}$$

contains all eigenfunctions of $L_t(q^\delta)$. Assume the converse. Then there exists a nonzero function $f \in L_2(F)$, which is orthogonal to all elements of this system. Using (2.3.1), (2.3.2) and the definition of $v(\beta, t)$ [see Lemma 2.3.1(b)], we get

$$\gamma + t = \beta + \tau + (j + v)\delta, \tag{2.3.4}$$

where $\beta \in \Gamma_\delta$, $\tau \in F_\delta$, $j \in \mathbb{Z}$, and $v = v(\beta, t)$. Since $e^{i(j+v)\zeta}$ can be decomposed by the basis

$$\{\varphi_{j,v(\beta,t))}(\zeta) : j \in \mathbb{Z}\}$$

the function $e^{i\langle \gamma+t,x\rangle} = e^{i\langle \beta+\tau,x\rangle}e^{i(j+v)\zeta}$ [see (2.3.4)] can be decomposed by the system

$$\{\Phi_{j,\beta}(x) = e^{i\langle \beta+\tau,x\rangle}\varphi_{j,v(\beta,t))}(\zeta) : j \in \mathbb{Z}\}.$$

Then the above assumption

$$(\Phi_{j,\beta}, f) = 0$$

for $j \in \mathbb{Z}$, $\beta \in \Gamma_\delta$ implies that

$$(f, e^{i\langle \gamma+t,x\rangle}) = 0$$

for all $\gamma \in \Gamma$. This is impossible, since the system $\{e^{i\langle \gamma+t,x\rangle} : \gamma \in \Gamma\}$ is a basis of $L_2(F)$ □

Remark 2.3.1 It is clear that every vector x of $\mathbb{R}^d$ has the decompositions

$$x = \gamma + t$$

and

$$x = \beta + \tau + (j + v)\delta,$$

where $\gamma \in \Gamma$, $t \in F$ and $\beta \in \Gamma_\delta$, $\tau \in F_\delta$, $j \in \mathbb{Z}$, $v \in [0, 1)$. We say that the first and second decompositions are Γ and Γ_δ decompositions, respectively. Thus

$$\gamma + t = \beta + \tau + (j + v(\beta, t))\delta$$

[see (2.3.4)] is the Γ_δ decomposition of $\gamma + t$. As we noted in Lemma 2.3.1 instead of $v(\beta, t)$ we write $v(\beta)$ (or v) if t (or t and β), for which we consider $v(\beta, t)$, is unambiguous. The decomposition (2.3.4) of $\gamma + t$ is an orthogonal decomposition, since $\beta \in \Gamma_\delta$, $\tau \in F_\delta$, and δ is orthogonal to both Γ_δ and F_δ. Hence

$$|\gamma + t|^2 = |\beta + \tau|^2 + |(j + v)\delta|^2.$$

Therefore, one can easily verify that, if $\gamma + t \in V_\delta(\rho^{\alpha_1})$ (see Definition 2.1.1), then

$$||(j + v + 1)\delta|^2 - |(j + v)\delta|^2| < \rho^{\alpha_1}.$$

Using this and the equality $\alpha_1 = 3\alpha$, we get

$$|(j + v)\delta| < r_1, |j\delta| < r_1, r_1 > 2\rho^{\alpha}, \tag{2.3.5}$$

where $r_1 = \frac{\rho^{\alpha_1}}{|2\delta|} + |2\delta|$. To the eigenvalue

$$|\gamma + t|^2 = |\beta + \tau|^2 + |(j + v)\delta|^2$$

of $L_t(0)$ assign the eigenvalue

$$\lambda_{j,\beta}(v, \tau) = |\beta + \tau|^2 + \mu_j(v)$$

of $L_t(q^\delta)$, where $|(j + v)\delta|^2$ is the eigenvalue of $T_v(0)$ and $\mu_j(v)$ is the eigenvalue of $T_v(Q)$ [see Lemma 2.3.1(b)] satisfying

$$|\mu_j(v) - |(j + v)\delta|^2| \leq \sup |Q(\zeta)|, \quad \forall j \in \mathbb{Z}. \tag{2.3.6}$$

The eigenvalue $\lambda_{j,\beta}(v, \tau)$ of $L_t(q^\delta)$ can be considered as the perturbation of the eigenvalue

$$|\gamma + t|^2 = |\beta + \tau|^2 + |(j + v)\delta|^2$$

of $L_t(0)$ by q^δ. Then we see that the influence of q^δ is significant for $\beta+\tau+(j+v)\delta \in V_\delta(\rho^{\alpha_1})$, namely for the small values of j.

Now we prove that if

$$\beta + \tau + (j + v)\delta \in V_\delta(\rho^{\alpha_1}),$$

then there is an eigenvalue $\Lambda_N(t)$ of $L_t(q)$ which is close to the eigenvalue $\lambda_{j,\beta}(v, \tau)$ of $L_t(q^\delta)$, that is, we prove that the influence of $q - q^\delta$ is not significant if the quasimomentum lies in $V_\delta(\rho^{\alpha_1})\backslash E_2$. To prove this we consider the operator $L_t(q)$ as the perturbation of the operator $L_t(q^\delta)$ with $q-q^\delta$ and use (2.1.21) called the binding formula for $L_t(q)$ and $L_t(q^\delta)$. Recall that we obtained the asymptotic formulas for the perturbation of the non-resonance eigenvalue $|\gamma+t|^2$ by iterating the binding formula (2.1.8) for the unperturbed operator $L_t(0)$ and the perturbed operator $L_t(q)$ (see Sect. 2.2). Similarly, now to obtain the asymptotic formulas for the perturbation of the resonance eigenvalue we iterate the binding formula (2.1.21) for the unperturbed operator $L_t(q^\delta)$ and perturbed operator $L_t(q)$. For this (as in the non-resonance case) we decompose $(q - q^\delta)\Phi_{j,\beta}$ by the basis

$$\{\Phi_{j',\beta'} : j' \in \mathbb{Z}, \beta' \in \Gamma_\delta\}$$

and put this decomposition into (2.1.21). Let us find this decomposition. Using (2.3.2) for $\gamma_1 \in \Gamma(\rho^\alpha)$ and (2.1.5), we get

$$\gamma_1 = \beta_1 + (n_1 - (2\pi)^{-1} \langle \beta_1, \delta^* \rangle)\delta, \; e^{i\langle \gamma_1, x\rangle} = e^{i\langle \beta_1, x\rangle} e^{i(n_1-(2\pi)^{-1}\langle \beta_1,\delta^*\rangle)\zeta},$$

$$q(x) - Q(\zeta) = \sum_{(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} c(n_1,\beta_1) e^{i\langle\beta_1,x\rangle} e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta} + O(\rho^{-p\alpha}),$$

$$(q(x) - Q(\zeta))\Phi_{j,\beta}(x)$$
$$= \sum_{(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} c(n_1,\beta_1) e^{i\langle\beta_1+\beta+\tau,x\rangle} e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta} \varphi_{j,v(\beta)}(\zeta) + O(\rho^{-p\alpha}), \tag{2.3.7}$$

where $c(n_1,\beta_1) = q_{\gamma_1}$,

$\Gamma'(\rho^\alpha) = \{(n_1,\beta_1) : \beta_1 \in \Gamma_\delta\backslash\{0\}, n_1 \in \mathbb{Z}, \beta_1 + (n_1 - (2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\delta \in \Gamma(\rho^\alpha)\}$.

Note that if $(n_1,\beta_1) \in \Gamma'(\rho^\alpha)$, then

$$|\beta_1 + (n_1 - (2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\delta| < \rho^\alpha$$

and

$$|\beta_1| < \rho^\alpha, |(n_1 - (2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\delta| < \rho^\alpha < \frac{1}{2}r_1, \tag{2.3.8}$$

since β_1 is orthogonal to δ and $r_1 > 2\rho^\alpha$ [see (2.3.5)]. To decompose the right-hand side of (2.3.7) by the basis $\{\Phi_{j',\beta'}\}$ we use the following lemma

Lemma 2.3.2 (a) *If j and m satisfy the inequalities $|m| > 2|j|$, $|m\delta| \geq 2r$, then*

$$(\varphi_{j,v}, e^{i(m+v)\zeta}) = O(|m\delta|^{-s-1}) = O(\rho^{-(s+1)\alpha}), \tag{2.3.9}$$
$$(\varphi_{m,v}, e^{i(j+v)\zeta}) = O(|m\delta|^{-s-1}). \tag{2.3.10}$$

where $r \geq r_1 = \frac{\rho^{\alpha_1}}{|2\delta|} + |2\delta|$, $\varphi_{j,v}$ is the eigenfunction of the operator $T_v(Q)$, and $Q \in W_2^s[0,2\pi]$.

Proof (a) To prove (2.3.9) we iterate the formula

$$(\mu_j(v) - |(m+v)\delta|^2)(\varphi_{j,v}, e^{i(m+v)\zeta}) = (\varphi_{j,v}Q, e^{i(m+v)\zeta}), \tag{2.3.11}$$

by using the decomposition

$$Q(\zeta) = \sum_{|l_1|<\frac{|m|}{2s}} q_{l_1\delta} e^{il_1\zeta} + O(|m\delta|^{-(s-1)}) \tag{2.3.12}$$

Note that (2.3.11) and (2.3.12) are the one-dimensional cases of (2.1.8) and (2.1.5) and the iteration of (2.3.11) is simpler than the iteration of (2.1.8) [see (2.1.9) and (2.2.5)]. If $|j| < \frac{|m|}{2}$, and $|l_i| < \frac{|m|}{2s}$ for $i = 1, 2, \ldots, k$, where $k = [\frac{s}{2}]$, then the inequalities

$$|m+v-l_1-l_2-\cdots-l_q|-|j|>\frac{1}{5}|m|,$$
$$|m|-|j+v-l_1-l_2-\cdots-l_q|>\frac{1}{5}|m|$$

hold for $q=0,1,\ldots,k$. Therefore by (2.3.6), we have

$$(|\mu_j-|(m-l_1-l_2-\cdots-l_q+v)\delta|^2|)^{-1}=O(|m\delta|^{-2}), \tag{2.3.13}$$
$$(|\mu_m-|(j-l_1-l_2-\cdots-l_q+v)\delta|^2|)^{-1}=O(|m\delta|^{-2}), \tag{2.3.14}$$

for $q=0,1,\ldots,k$. Iterating (2.3.11) k times, by using (2.3.13), we get

$$(\varphi_j, e^{i(m+v)\zeta})=\sum_{|l_1\delta|,|l_2\delta|,\ldots,|l_{k+1}\delta|<\frac{|m\delta|}{2s}} q_{l_1\delta}q_{l_2\delta}\ldots q_{l_{k+1}\delta}$$
$$\times\frac{(\varphi_j, e^{i(m-l_1-l_2-\cdots-l_{k+1}+v)\zeta})}{\sqcap_{q=0}^{k}(\mu_j-|(m-l_1-l_2-\cdots-l_q+v)\delta|^2)}+O(|m\delta|^{-s-1}). \tag{2.3.15}$$

Now (2.3.9) follows from (2.3.13), (2.3.15) and from inequality (2.1.6a). Formula (2.3.10) can be proved in the same way by using (2.3.14) instead of (2.3.13). Note that in (2.3.9) and (2.3.10) instead of $O(|m\delta|^{-s-1})$ we can write $O(\rho^{-(s+1)\alpha})$, since $|m\delta|\geq r\geq r_1>2\rho^{\alpha}$ [see (2.3.5)] □

Lemma 2.3.3 *If* $|j\delta|<r$ *and* $(n_1,\beta_1)\in\Gamma'(\rho^{\alpha})$, *then*

$$e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)$$
$$=\sum_{|j_1\delta|<9r}a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)+O(\rho^{-(s-1)\alpha}), \tag{2.3.16}$$

where r *and* $\Gamma'(\rho^{\alpha})$ *are defined in* Lemma 2.3.2(a) *and in* (2.3.7) *respectively, and*

$$a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)=(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)).$$

Proof Since

$$e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)$$

is equal to its Fourier series with the orthonormal basis

$$\{\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta):j_1\in\mathbb{Z}\}$$

it suffices to show that

$$\sum_{j_1:|j_1\delta|\geq 9r}|a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)|=O(\rho^{-(s-1)\alpha}).$$

For this we prove

$$|a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)|=O(|j_1\delta|^{-s}) \tag{2.3.17}$$

for all j_1 satisfying $|j_1\delta|\geq 9r$ and take into account that $r\geq r_1>\rho^{\alpha}$ [see the last inequality in (2.3.5)]. Decomposing $\varphi_{j,v(\beta)}$ over $\{e^{i(m+v)\zeta}:m\in\mathbb{Z}\}$ and using the last inequality in (2.3.8), we obtain

$$e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_j(\zeta)=\sum_{m\in\mathbb{Z}}(\varphi_j,e^{i(m+v)\zeta})e^{i(m+n+v(\beta+\beta_1))\zeta}, \tag{2.3.18}$$

where $n\in\mathbb{Z}$ and $|n\delta|<r$. This and the decomposition

$$\varphi_{j+j_1}(\zeta)=\sum_{m\in\mathbb{Z}}(\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))\zeta})e^{i(m+v(\beta+\beta_1))v}$$

imply that

$$a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)=\sum_{m\in\mathbb{Z}}(\varphi_j,e^{i(m-n+v)\zeta})(\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))\zeta}) \tag{2.3.19}$$

where $j,\ j_{1,}\ n$ satisfy the conditions

$$|j\delta|<r,|j_1\delta|\geq 9r,|n\delta|<r$$

due to the conditions in Lemma 2.3.3, (2.3.17) and (2.3.18) respectively. Consider two cases:

Case 1: $|m\delta|>\frac{1}{3}|j_1\delta|\geq 3r$. In this case using the conditions of (2.3.19), we get $|(m-n)\delta|>2r$ and $|m-n|>|j|$. Therefore (2.3.9) implies that

$$(\varphi_j,e^{i(m-n+v)\zeta})=O(|m\delta|^{-s-1}),\quad\sum_{|m|>\frac{1}{3}|j_1|}|(\varphi_j,e^{i(m-n+v)\zeta})|=O(|j_1\delta|^{-s}).$$

Case 2: $|m|\leq\frac{1}{3}|j_1|$. Again using the conditions of (2.3.19) we obtain that $|j_1+j|>2|m|$. Therefore it follows from (2.3.10) that

$$(\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))\zeta})=O(|(j_1+j)\delta|^{-(s-1)})=O(|j_1\delta|^{-s-1}),$$
$$\sum_{|m|\leq\frac{1}{3}|j_1|}|(\varphi_{j+j_1}(\zeta),e^{i(m+v(\beta+\beta_1))\zeta})|=O(|j_1\delta|^{-s}).$$

These estimations for these two cases together with (2.3.19) yield (2.3.17) □

Now it follows from (2.3.7) and (2.3.16) that

$$(q(x)-Q(\zeta))\Phi_{j',\beta'}(x)=O(\rho^{-p\alpha})$$
$$+\sum_{(n_1,j_1,\beta_1)\in G(\rho^{\alpha},9r)} c(n_1,\beta_1)a(n_1,\beta_1,j,\beta',j'+j_1,\beta'+\beta_1)e^{i\langle\beta_1+\beta'+\tau,x\rangle}\varphi_{j'+j_1,v(\beta'+\beta_1)}(\zeta) \tag{2.3.20}$$

for all j' satisfying $|j'\delta|<r$, where

$$G(\rho^{\alpha},9r)=\{(n,j,\beta):|j\delta|<9r,(n,\beta)\in\Gamma'(\rho^{\alpha}),\beta\neq 0\}.$$

In (2.3.20) the multiplicand

$$e^{i\langle\beta_1+\beta'+\tau,x\rangle}\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta)=\Phi_{j'+j_1,\beta'+\beta_1}(x)$$

does not depend on n_1. Its coefficient is

$$\overline{A(j',\beta',j'+j_{1,}\beta'+\beta_1}=\sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^{\alpha})} c(n_1,\beta_1)a(n_1,\beta_1,j',\beta',j'+j_{1,}\beta'+\beta_1). \tag{2.3.21}$$

Lemma 2.3.4 *If* $|\beta'|\sim\rho$ *and* $|j'\delta|<r$, *where*

$$r\geq r_1=\frac{\rho^{\alpha_1}}{|2\delta|}+|2\delta|,$$

then

$$(q(x)-Q(\zeta))\Phi_{j',\beta'}(x)$$
$$=\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\overline{A(j',\beta',j'+j_{1,}\beta'+\beta_1})\Phi_{j'+j_{1,}\beta'+\beta_1}(x)+O(\rho^{-p\alpha}), \tag{2.3.22}$$

where

$$Q(\rho^{\alpha},9r)=\{(j,\beta):|j\delta|<9r,0<|\beta|<\rho^{\alpha}\}.$$

Moreover,

$$\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}|A(j',\beta',j'+j_{1,}\beta'+\beta_1)|<c_9, \tag{2.3.23}$$

where c_9 *does not depend on* (j',β').

Proof Formula (2.3.22) follows from (2.3.20) and (2.3.21). Now we prove (2.3.23). Since $c(n_1,\beta_1)=q_{\gamma_1}$ [see (2.3.7)], by the relations (2.1.6a) and (2.3.21) we need to prove the inequality

$$\sum_{j_1} |a(n_1, \beta_1, j', \beta', j' + j_1, \beta' + \beta_1)| < c_9 (c_3)^{-1}. \tag{2.3.24}$$

For this we use (2.3.19) and prove the inequalities:

$$\sum_{m\in\mathbb{Z}} |(\varphi_{j'}, e^{i(m-n+v(\beta')\zeta})| < c_{10}, \tag{2.3.25}$$

$$\sum_{j_1\in\mathbb{Z}} |(\varphi_{j'+j_1}, e^{i(m+v(\beta_1+\beta'))\zeta})| < c_{11}. \tag{2.3.26}$$

Since the distance between the numbers

$$|v\delta|^2, |(1+v)\delta|^2, \ldots,$$

and similarly the distance between the numbers

$$|(-1+v)\delta|^2, |(-2+v)\delta|^2, \ldots,$$

where $v \in [0, 1]$, are not less than c_{12}, it follows from (2.3.6) that the number of the elements of the sets

$$A = \{m : |(m - n + v(\beta'))\delta|^2 \in [\mu_{j'}(v(\beta')) - 1, \mu_{j'}(v(\beta')) + 1]\},$$
$$B = \{j_1 : \mu_{j'+j_1}(v(\beta_1 + \beta')) \in [|(m + v(\beta_1 + \beta'))\delta|^2 - 1, |(m + v)\delta|^2 + 1]\}$$

is less than c_{13}. Now in (2.3.25) and (2.3.26) isolating the terms with $m \in A$ and $j_1 \in B$ respectively, applying (2.3.11) to the other terms and then using

$$\sum_{m\notin A} \frac{1}{|\mu_{j'}(v') - |(m-n+v')\delta|^2|} < c_{14},$$

$$\sum_{j_1\notin B} \frac{1}{|\mu_{j'+j_1}(v_1') - |(m+v_1')\delta|^2|} < c_{14}$$

we get the proofs of (2.3.25) and (2.3.26). Thus (2.3.24) and hence (2.3.23) are proved. Clearly the constants $c_{14}, c_{13}, c_{12}, c_{11}, c_{10}$ can be chosen independently on (j', β'). Therefore c_9 does not depend on (j', β') □

Replacing (j, β) by (j', β') in (2.1.21) and using (2.3.22), we get

$$(\Lambda_N - \lambda_{j',\beta'})b(N, j', \beta') = (\Psi_N, (q - Q)\Phi_{j',\beta'}) = O(\rho^{-p\alpha})$$
$$+ \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} A(j', \beta', j' + j_1, \beta' + \beta_1)b(N, j' + j_1, \beta' + \beta_1) \tag{2.3.27}$$

for $|\beta'| \sim \rho$ and $|j'\delta| < r$, where $b(N, j, \beta) = (\Psi_N, \Phi_{j,\beta})$. Note that if $|j'\delta| < r$, then the summation in (2.3.27) is taken over $Q(\rho^{\alpha}, 9r)$. Therefore if $|j\delta| < r_1$, where r_1 is defined in (2.3.5), then we have

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha}) + \sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1)} A(j, \beta, j + j_1, \beta + \beta_1)b(N, j + j_1, \beta + \beta_1). \tag{2.3.28}$$

Thus (2.3.28) is obtained from (2.3.27) by interchanging j', β', r, and j, β, r_1. Now to find the eigenvalue $\Lambda_N(t)$, which is close to $\lambda_{j,\beta}$, where $|j\delta| < r_1$, we are going to iterate (2.3.28) as follows. Since $|j\delta| < r_1$ and $(j_1, \beta_1) \in Q(\rho^{\alpha}, 9r_1)$, we have

$$|(j + j_1)\delta| < 10r_1.$$

Therefore in (2.3.27) interchanging j', β', r, and $j + j_1, \beta + \beta_1, 10r_1$ and then introducing the notations

$$r_2 = 10r_1,\ j^2 = j + j_1 + j_2,\ \beta^2 = \beta + \beta_1 + \beta_2,$$

we obtain

$$(\Lambda_N - \lambda_{j+j_1,\beta_1+\beta})b(N, j + j_1, \beta + \beta_1) = O(\rho^{-p\alpha}) + \sum_{(j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)} b(N, j^2, \beta^2)A(j + j_1, \beta + \beta_1, j^2, \beta^2). \tag{2.3.29}$$

Clearly, there exists an eigenvalue $\Lambda_N(t)$ satisfying

$$|\lambda_{j,\beta} - \Lambda_N(t)| \le 2M,$$

where $M = \sup |q(x)|$. Moreover, in the next lemma (Lemma 2.3.5), we will prove that if $|\beta| \sim \rho$, and $(j_1, \beta_1) \in Q(\rho^{\alpha}, 9r_1)$, then

$$|\lambda_{j,\beta} - \lambda_{j+j_1,\beta+\beta_1}| > \frac{5}{9}\rho^{\alpha_2},\ |\Lambda_N(t) - \lambda_{j+j_1,\beta+\beta_1}| > \frac{1}{2}\rho^{\alpha_2}. \tag{2.3.30}$$

Therefore dividing both side of (2.3.29) by $\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}$, we get

$$b(N, j + j_1, \beta_1 + \beta) = O(\rho^{-p\alpha-\alpha_2}) + \sum_{(j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)} \frac{A(j + j_1, \beta + \beta_1, j^2, \beta^2)b(N, j^2, \beta^2)}{\Lambda_N - \lambda_{j+j_1,\beta_1+\beta}}. \tag{2.3.31}$$

Putting the obtained formula for $b(N, j + j_1, \beta_1 + \beta)$ into (2.3.28), we obtain

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha})$$
$$+ \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)\\(j_2,\beta_2)\in Q(\rho^\alpha,9r_2)}} \frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)b(N,j^2,\beta^2)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \tag{2.3.32}$$

Thus we got the once iteration of (2.3.28). It will give the first term of the asymptotic formula for Λ_N. For this we find the index N such that $b(N, j, \beta)$ is not very small (see Lemma 2.3.6) and (2.3.30) is satisfied, i.e., the denominator of the fraction in (2.3.32) is a big number. Then dividing both sides of (2.3.32) by $b(N, j, \beta)$, we get the asymptotic formula for $\Lambda_N(t)$ (see Theorem 2.3.1).

Lemma 2.3.5 *Let* $\gamma + t =: \beta + \tau + (j + v)\delta \in V'(\rho^{\alpha_1}) =: V_\delta(\rho^{\alpha_1})\backslash E_2$ [*see* (2.3.4), Remark 2.3.1 and Definition 2.1.1], *and* $(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)$, $(j_k, \beta_k) \in Q(\rho^\alpha, 9r_k)$, *where* r_1 *is defined in* (2.3.5) *and* $r_k = 10r_{k-1}$ *for* $k = 2, 3, \ldots, p-1$. *Then*

$$|j\delta| = O(\rho^{\alpha_1}),\ |j_k\delta| = O(\rho^{\alpha_1}),\quad |\beta_k| < \rho^\alpha \tag{2.3.33}$$

for $k = 1, 2, \ldots, p-1$. *Moreover if*

$$|j'\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2},\quad |\beta' - \beta| < (p-1)\rho^\alpha$$

and $\beta' \in \Gamma_\delta$, $j^k = j+j_1+\cdots+j_k$, $\beta^k = \beta+\beta_1+\cdots+\beta_k$, *where* $k = 1, 2, \ldots, p-1$, *then*

$$|\lambda_{j,\beta} - \lambda_{j',\beta'}| > \frac{5}{9}\rho^{\alpha_2},\quad \forall\beta' \neq \beta, \tag{2.3.34}$$

$$|\lambda_{j,\beta}(v,\tau) - \lambda_{j^k,\beta^k}| > \frac{5}{9}\rho^{\alpha_2},\quad \forall\beta^k \neq \beta. \tag{2.3.35}$$

Proof The relations in (2.3.33) follow from (2.3.5) and the definitions of r_1, r_k, $Q(\rho^\alpha, 9r_k)$ (see Lemma 2.3.4). Inequality (2.3.35) follows from (2.3.34) and (2.3.33). It remains to prove (2.3.34). Since

$$|\lambda_{j,\beta} - \lambda_{j',\beta'}| \geq ||\beta' + \tau|^2 - |\beta + \tau|^2| - |\mu_j - \mu_{j'}|, \tag{2.3.36}$$

it is enough to prove the following two inequalities

$$|\mu_j - \mu_{j'}| < \frac{1}{3}\rho^{\alpha_2},$$

and

$$||\beta+\tau|^2-|\beta'+\tau|^2|>\frac{8}{9}\rho^{\alpha_2}. \tag{2.3.37}$$

The first inequality follows from

$$|j'\delta|<\frac{1}{2}\rho^{\frac{1}{2}\alpha_2}, |j\delta|=O(\rho^{\alpha_1})$$

[see the conditions of this lemma and (2.3.33)] and (2.3.6), since $\alpha_2=3\alpha_1$. Now we prove (2.3.37). The conditions

$$|\beta'-\beta|<(p-1)\rho^{\alpha}, \quad |\delta|<\rho^{\alpha}$$

imply that there exist $n\in\mathbb{Z}$ and $\gamma'\in\Gamma$ such that

$$\gamma'=\beta'-\beta+(n+(2\pi)^{-1}\langle\beta'-\beta,\delta^*\rangle)\delta\in\Gamma(p\rho^{\alpha}). \tag{2.3.38}$$

Since $\beta'-\beta\neq 0$ [see (2.3.34)] and $\beta'-\beta\in\Gamma_\delta$, that is, $\langle\beta'-\beta,\delta\rangle=0$ the relation (2.3.38) implies that

$$\gamma'\in\Gamma(p\rho^{\alpha})\backslash\delta R.$$

This together with the condition

$$\gamma+t=\beta+\tau+(j+v)\delta\in V_\delta(\rho^{\alpha_1})\backslash E_2$$

(see the assumption of the lemma and the definition of E_2 in Definition 2.1.1) gives

$$\gamma+t\notin V_{\gamma'}(\rho^{\alpha_2}),$$

that is,

$$||\gamma+t|^2-|\gamma+t+\gamma'|^2|\geq\rho^{\alpha_2}.$$

From this using the orthogonal decompositions (2.3.4) and (2.3.38) of $\gamma+t$ and γ' respectively, taking into account that β,τ,β' are orthogonal to δ and then using the relations

$$|j\delta|=O(\rho^{\alpha_1}), |(n+(2\pi)^{-1}\langle\beta'-\beta,\delta^*\rangle)\delta|=O(\rho^{\alpha}), \alpha_2>2\alpha$$

[see (2.3.33), (2.3.38) and Definition 2.1.1], we obtain (2.3.37) □

Lemma 2.3.6 *Let $h_1,h_2,\ldots,h_m$ be the elements of $L_2(F)$, where $m=p_1-1$ and $p_1=[\frac{p}{3}]+1$. Then for every eigenvalue $\lambda_{j,\beta}\sim\rho^2$ of the operator $L_t(q^\delta)$ there exist an eigenvalue $\Lambda_N(t)$ and a corresponding normalized eigenfunction $\Psi_{N,t}(x)$ of the operator $L_t(q)$ such that:*

(i) $|\lambda_{j,\beta} - \Lambda_N(t)| \leq 2M$, *where* $M = \sup |q(x)|$,
(ii) $|b(N, j, \beta)| > c_{15}\rho^{-\frac{1}{2}(d-1)}$,
(iii) $|b(N, j, \beta)|^2 > \frac{1}{2m}\sum_{i=1}^{m} |(\Psi_N, \frac{h_i}{\|h_i\|})|^2 \geq \frac{1}{2m}|(\Psi_N, \frac{h_i}{\|h_i\|})|^2$ *for* $i = 1, 2, \ldots, m$.

Proof Let A, B and C be the sets of indexes N satisfying (i), (ii), (iii) respectively. Using (2.1.21), Bessel's inequality and Parseval's equality, we get

$$\sum_{N \notin A} |b(N, j, \beta)|^2 = \sum_{N \notin A} |\frac{(\Psi_N, (q - Q)\Phi_{j,\beta})}{\Lambda_N - \lambda_{j,\beta}}|^2$$
$$< (2M)^{-2} \left\|(q - Q)\Phi_{j,\beta}\right\|^2 \leq \frac{1}{4}$$

and

$$\sum_{N \in A} |b(N, j, \beta)|^2 \geq \frac{3}{4}.$$

On the other hand the inequality $|A| < c_{16}\rho^{(d-1)}$ [see (2.1.37a)] and the definition of B imply that if

$$c_{15}^2 < \frac{1}{4c_{16}},$$

then

$$\sum_{N \in A \backslash B} |b(N, j, \beta)|^2 < \frac{1}{4}.$$

Therefore using the relation $A = (A \backslash B) \cup (A \cap B)$, we obtain

$$\sum_{N \in A \cap B} |b(N, j, \beta)|^2 > \frac{1}{2}.$$

Now to prove the lemma we show that there exists $N \in A \cap B$ satisfying (iii). Assume that the assertion (iii) does not hold for all $N \in A \cap B$. Using the last inequality, the assumption that (iii) does not holds for $N \in A \cap B$ and then the Bessel inequality, we get

$$\frac{1}{2} < \sum_{N \in A \cap B} |b(N, j, \beta)|^2 < \frac{1}{2m} \sum_{i=1}^{m} \sum_{N \in A} |(\Psi_N, \frac{h_i}{\|h_i\|})|^2$$
$$\leq \frac{1}{2m} \sum_{i=1}^{m} ||\frac{h_i}{\|h_i\|}||^2 = \frac{1}{2}.$$

This contradiction completes the proof of the lemma □

Theorem 2.3.1 *For every eigenvalue $\lambda_{j,\beta}(v,\tau)$ of $L_t(q^{\delta})$ such that $\beta+\tau+(j+v)\delta \in V'_{\delta}(\rho^{\alpha_1})$ there exists an eigenvalue Λ_N of $L_t(q)$, denoted by $\Lambda_N(\lambda_{j,\beta}(v,\tau))$, satisfying*

$$\Lambda_N(\lambda_{j,\beta}(v,\tau)) = \lambda_{j,\beta}(v,\tau) + O(\rho^{-\alpha_2}). \tag{2.3.39}$$

Proof In the proof of this theorem we use the inequalities

$$p_1\alpha_2 > p\alpha, \quad p\alpha - \frac{1}{2}(d-1) > \alpha_2 \tag{2.3.40}$$

which follow from the definitions of p, α, α_2 and p_2 given in (2.1.5), Definition 2.1.1, and (2.2.5). By Lemma 2.3.6 there exists an eigenvalue $\Lambda_N(t)$ satisfying (i)–(iii) for

$$h_i(x) = \sum_{\substack{(j_1,\beta_1)\in Q(\rho^{\alpha},9r_1),\\ (j_2,\beta_2)\in Q(\rho^{\alpha},9r_2)}} \frac{\overline{A(j,\beta,j^1,\beta^1)}\,\overline{A(j^1,\beta^1,j^2,\beta^2)}\Phi_{j^2,\beta^2}(x)}{(\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1})^i},$$

where $i=1,2,\ldots,m$; $m=p_1-1$ and $Q(\rho^{\alpha},9r)$ is defined in Lemma 2.3.4. By the definition of $Q(\rho^{\alpha},9r_1)$ we have $\beta_1 \neq 0$. Therefore (2.3.34) and the assertion (i) of Lemma 2.3.6 yield (2.3.30). Hence, in the brief notations

$$a=\lambda_{j,\beta}, \quad z=\lambda_{j+j_1,\beta+\beta_1},$$

we have

$$|\Lambda_N - a| < 2M, \quad |z-a| > \frac{1}{2}\rho^{\alpha_2}.$$

Using the relations

$$\frac{1}{\Lambda_N - z} = -\sum_{i=1}^{\infty}\frac{(\Lambda_N-a)^{i-1}}{(z-a)^i} = -\sum_{i=1}^{m}\frac{(\Lambda_N-a)^{i-1}}{(z-a)^i} + O(\rho^{-p_1\alpha_2})$$

and the first inequality of (2.3.40), we see that (2.3.32) can be written in the form

$$(\Lambda_N - \lambda_{j,\beta})b(N,j,\beta,) = \sum_{i=1}^{m}(\Lambda_N-a)^{i-1}(\Psi_N,\frac{h_i}{\|h_i\|})\|h_i\| + O(\rho^{-p\alpha}).$$

Dividing both sides of the equality by $b(N,j,\beta)$, using assertions (ii), (iii) of Lemma 2.3.6, and the second inequality of (2.3.40), we get

$$|(\Lambda_N - \lambda_{j,\beta})| < (2m)^{\frac{1}{2}} \sum_{i=1}^{m} |\Lambda_N - a|^{i-1} \|h_i\| + O(\rho^{-\alpha_2}).$$

On the other hand, (2.3.23) and (2.3.35) imply that

$$\|h_i\| = O(\rho^{-\alpha_2}).$$

These relations and the above inequality $|\Lambda_N - a| < 2M$, yield the proof of the theorem □

Thus we iterated (2.3.28) once and got (2.3.32) from which (2.3.39) was obtained. Now to obtain the asymptotic formulas of arbitrary order, we repeat this iteration $2p_1$ times. For this we need to estimate the distance between $\lambda_{j,\beta}(v,\tau)$ and $\lambda_{j',\beta}(v,\tau)$ for $j' \neq j$, namely we use the following lemma.

Lemma 2.3.7 *There exists a positive function $\varepsilon(\rho)$ such that $\varepsilon(\rho) \to 0$ as $\rho \to \infty$ and the set*

$$A(\varepsilon(\rho)) =: (\varepsilon(\rho), \frac{1}{2} - \varepsilon(\rho)) \cup (\frac{1}{2} + \varepsilon(\rho), 1 - \varepsilon(\rho))$$

is a subset of

$$W(\rho) =: \{v \in (0,1) : |\mu_j(v) - \mu_{j'}(v)| > \frac{2}{\ln \rho}, \quad \forall j', j \in \mathbb{Z}, j' \neq j\}.$$

If $v(\beta) \in W(\rho)$, then

$$|\lambda_{j,\beta}(v,\tau) - \lambda_{j',\beta}(v,\tau)| > 2(\ln \rho)^{-1}, \quad \forall j' \neq j. \tag{2.3.41}$$

Proof Denote by $\widetilde{\mu}_1(v), \widetilde{\mu}_2(v), \ldots,$ the eigenvalues of $T_v(Q)$ numbered in nondecreasing order:

$$\widetilde{\mu}_1(v) \leq \widetilde{\mu}_2(v) \leq \cdots$$

It is well-known that the spectrum of Hill's operator $T(Q)$ consists of the intervals

$$\Delta_{2j-1} =: [\widetilde{\mu}_{2j-1}(0), \widetilde{\mu}_{2j-1}(\frac{1}{2})], \quad \Delta_{2j} =: [\widetilde{\mu}_{2j}(\frac{1}{2}), \widetilde{\mu}_{2j}(1)]$$

for $j = 1, 2, \ldots$. The length of the jth interval Δ_j of the spectrum tends to infinity as j tends to infinity. The distance between neighboring intervals, that is, the length of the gaps in the spectrum tends to zero. The eigenvalues $\widetilde{\mu}_{2j-1}(v)$ and $\widetilde{\mu}_{2j}(v)$ are the increasing continuous functions in the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively and

$$\widetilde{\mu}_j(1+v) = \widetilde{\mu}_j(v) = \widetilde{\mu}_j(1-v).$$

Since

$$\lim_{\rho\to\infty}(\ln\rho)^{-1}\to 0,$$

the length of the interval Δ_j is sufficiently greater than $(\ln\rho)^{-1}$ for $\rho\gg 1$ and there are numbers $\varepsilon'_j(\rho)$, $\varepsilon''_j(\rho)$ in $(0,\frac{1}{2})$ such that

$$\begin{aligned}
\widetilde{\mu}_{2j-1}(\varepsilon'_{2j-1}(\rho)) &= \widetilde{\mu}_{2j-1}(0)+(\ln\rho)^{-1},\\
\widetilde{\mu}_{2j-1}(\frac{1}{2}-\varepsilon''_j(\rho)) &= \widetilde{\mu}_{2j-1}(\frac{1}{2})-(\ln\rho)^{-1},\\
\widetilde{\mu}_{2j}(\frac{1}{2}+\varepsilon'_{2j}(\rho)) &= \widetilde{\mu}_{2j}(\frac{1}{2})+(\ln\rho)^{-1},\\
\widetilde{\mu}_{2j}(1-\varepsilon''_j(\rho)) &= \widetilde{\mu}_{2j}(1)-(\ln\rho)^{-1}.
\end{aligned}\tag{2.3.42}$$

Denote

$$\varepsilon'(\rho)=\sup_j\varepsilon'_j(\rho),\quad \varepsilon''(\rho)=\sup_j\varepsilon''_j(\rho),\quad \varepsilon(\rho)=\max\{\varepsilon'(\rho),\varepsilon''(\rho)\}.$$

To prove that $\varepsilon(\rho)\to 0$ as $\rho\to\infty$ we show that both $\varepsilon'(\rho)$ and $\varepsilon''(\rho)$ tend to zero as $\rho\to\infty$. If $\rho_1<\rho_2$ then

$$\varepsilon'_j(\rho_2)<\varepsilon'_j(\rho_1),\quad \varepsilon'(\rho_2)<\varepsilon'(\rho_1),$$

since $\widetilde{\mu}_{2j-1}(v)$ and $\widetilde{\mu}_{2j}(v)$ are the increasing functions in the intervals $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$ respectively. Hence

$$\varepsilon'(\rho)\to a\in[0,\frac{1}{2}]$$

as $\rho\to\infty$. Suppose that $a>0$. Then there is a sequence $\rho_k\to\infty$ as $k\to\infty$ such that $\varepsilon'(\rho_k)>\dfrac{a}{2}$ for all k. This implies that there is a sequence $\{i_k\}$ and without loss of generality it can be assumed that there is a sequence $\{2j_k-1\}$ of odd numbers such that

$$\varepsilon'_{2j_k-1}(\rho_k)>\frac{a}{4}$$

for all k. Since $\widetilde{\mu}_{2j-1}(v)$ increases in $(0,\frac{1}{2})$ and

$$\widetilde{\mu}_{2j_k-1}(\varepsilon'_{2j_k-1}(\rho_k))-\widetilde{\mu}_{2j_k-1}(0)=(\ln\rho_k)^{-1}$$

we have

$$|\widetilde{\mu}_{2j_k-1}(\frac{a}{4})-\widetilde{\mu}_{2j_k-1}(0)|\le(\ln\rho_k)^{-1}\to 0$$

as $k \to \infty$, which contradicts to the well-known asymptotic formulas for the eigenvalues $\widetilde{\mu}_j(v)$, for $v = 0$ and $v = \frac{a}{4}$, where $a \in (0, \frac{1}{2}]$. Thus we proved that $\varepsilon'(\rho) \to 0$ as $\rho \to \infty$. In the same way we prove this for $\varepsilon''(\rho)$, and hence for $\varepsilon(\rho)$. Now suppose $v \in A(\varepsilon(\rho))$. Using (2.3.42) and the definition of $\varepsilon(\rho)$, and taking into account that $\widetilde{\mu}_{2j-1}(v)$ and $\widetilde{\mu}_{2j}(v)$ increase in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively, we obtain that the eigenvalues $\widetilde{\mu}_1(v), \widetilde{\mu}_2(v), \ldots$ are contained in the intervals

$$[\widetilde{\mu}_{2j-1}(0) + (\ln\rho)^{-1}, \widetilde{\mu}_{2j-1}(\frac{1}{2}) - (\ln\rho)^{-1}], [\widetilde{\mu}_{2j}(\frac{1}{2}) + (\ln\rho)^{-1}, \widetilde{\mu}_{2j}(1) - (\ln\rho)^{-1}]$$

for $j = 1, 2, \ldots,$ and in each interval there exists a unique eigenvalue of T_v. Therefore the distance between the neighboring eigenvalues of T_v for $v \in A(\varepsilon(\rho))$ is not less than the distance between these intervals, which is not less than $2(\ln\rho)^{-1}$. Hence the inequality in the definition of $W(\rho)$ holds, that is, $A(\varepsilon(\rho)) \subset W(\rho)$. Inequality (2.3.41) is a consequence of the definition of $W(\rho)$ □

It follow from (2.3.35), (2.3.41) and (2.3.39) that

$$|\Lambda_N(\lambda_{j,\beta}) - \lambda_{j^k,\beta^k}(v,\tau)| > c(\beta^k, \rho), \quad \forall v(\beta) \in W(\rho), \tag{2.3.43}$$

where $(j_k, \beta_k) \in Q(\rho^\alpha, 9r_k)$, $k = 1, 2, \ldots, p-1$; $c(\beta^k, \rho) = (\ln\rho)^{-1}$ when $\beta^k = \beta$, $j^k \neq j$ and $c(\beta^k, \rho) = \frac{1}{2}\rho^{\alpha_2}$ when $\beta^k \neq \beta$.

Now to obtain the asymptotic formulas of the arbitrary order for $\Lambda_N(t)$ we iterate (2.3.28) $2p_1$ times by using (2.3.43), as follows. Since $|j\delta| < r_1$ [see (2.3.5)],

$$(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1), \quad (j_2, \beta_2) \in Q(\rho^\alpha, 9r_2)$$

[see (2.3.32)], and $j^2 = j + j_1 + j_2$ [see (2.3.29) for this notation], we have $|j^2\delta| < 10r_2$. Therefore in (2.3.27) interchanging j', β', r, and $j^2, \beta^2, 10r_2$ and using the notations $r_3 = 10r_2$, $j^3 = j^2 + j_3$, $\beta^3 = \beta^2 + \beta_3$ (see Lemma 2.3.5), we obtain

$$\begin{aligned}(\Lambda_N - \lambda_{j^2,\beta^2})b(N, j^2, \beta^2) &= O(\rho^{-p\alpha}) \\ &+ \sum_{(j_3,\beta_3)\in Q(\rho^\alpha, 9r_3)} b(N, j^3, \beta^3)A(j^2, \beta^2, j^3, \beta^3).\end{aligned} \tag{2.3.44}$$

Dividing both side of (2.3.44) by $\Lambda_N - \lambda_{j^2,\beta^2}$ and using (2.3.43), we get

$$\begin{aligned}b(N, j^2, \beta^2) &= O(\rho^{-p\alpha}(c(\beta^2, \rho))^{-1}) \\ &+ \sum_{(j_3,\beta_3)\in Q(\rho^\alpha, 9r_3)} \frac{b(N, j^3, \beta^3)A(j^2, \beta^2, j^3, \beta^3)}{\Lambda_N - \lambda_{j^2,\beta^2}}\end{aligned} \tag{2.3.45}$$

for $(j^2, \beta^2) \neq (j, \beta)$. In the same way we obtain

$$b(N, j^k, \beta^k) = O(\rho^{-p\alpha}(c(\beta^k, \rho))^{-1}) + \sum_{(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha, 9r_{k+1})} \frac{b(N, j^{k+1}, \beta^{k+1})A(j^k, \beta^k, j^{k+1}, \beta^{k+1})}{\Lambda_N - \lambda_{j^k,\beta^k}} \tag{2.3.46}$$

for $(j^k, \beta^k) \neq (j, \beta)$, $k = 3, 4, \ldots$. Now we isolate the terms in the right-hand side of (2.3.32) with the multiplicand $b(N, j, \beta)$, i.e., the case $(j^2, \beta^2) = (j, \beta)$, and replace $b(N, j^2, \beta^2)$ in (2.3.32) by the right-hand side of (2.3.45) when $(j^2, \beta^2) \neq (j, \beta)$ and use (2.3.30) and (2.3.43) to get

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = S_1'(\Lambda_N, \lambda_{j,\beta}))b(N, j, \beta) + O(\rho^{-p\alpha}) + \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha, 9r_1), \\ (j_2,\beta_2)\in Q(\rho^\alpha,9r_2),(j^2,\beta^2)\neq(j,\beta)}} \frac{A(j, \beta, j^1, \beta^1)A(j^1, \beta^1, j^2, \beta^2)b(N, j^3, \beta^3)}{(\Lambda_N - \lambda_{j+j_1,\beta+\beta_1})(\Lambda_N - \lambda_{j^2,\beta^2})}, \tag{2.3.47}$$

where

$$S_1'(\Lambda_N, \lambda_{j,\beta}) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)} \frac{A(j, \beta, j + j_1, \beta + \beta_1)A(j + j_1, \beta + \beta_1, j, \beta)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \tag{2.3.48}$$

The formula (2.3.47) is the twice iteration of (2.3.29). Repeating these processes $2p_1$ times, i.e., in (2.3.47) isolating the terms with the multiplicand $b(N, j, \beta)$ (i.e., the case $(j^3, \beta^3) = (j, \beta)$) and replacing $b(N, j^3, \beta^3)$ by the right-hand side of (2.3.46) (for $k = 3$) when $(j^3, \beta^3) \neq (j, \beta)$ etc., we obtain

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = (\sum_{k=1}^{2p_1} S_k'(\Lambda_N, \lambda_{j,\beta}))b(N, j, \beta) + C_{2p_1}' + O(\rho^{-p\alpha}), \tag{2.3.49}$$

where

$$S_k'(\Lambda_N, \lambda_{j,\beta}) = \sum(\prod_{i=1}^{k} \frac{A(j^{i-1}, \beta^{i-1}, j^i, \beta^i)}{(\Lambda_N - \lambda_{j^i,\beta^i})})A(j^k, \beta^k, j, \beta),$$

$$C_k' = \sum(\prod_{i=1}^{k} \frac{A(j^{i-1}, \beta^{i-1}, j^i, \beta^i)}{(\Lambda_N - \lambda_{j^i,\beta^i})})A(j^k, \beta^k, j^{k+1}, \beta^{k+1})b(N, j^{k+1}, \beta^{k+1}).$$

Here $j^0 = j$, $\beta^0 = \beta$, $j^i = j + j_1 + j_2 + \cdots + j_i$, $\beta^i = \beta + \beta_1 + \beta_2 + \cdots + \beta_i$ and the summation for S_k', and C_k' are taken under the conditions

$$(j_i, \beta_i) \in Q(\rho^\alpha, 9r_i),\ (j^i, \beta^i) \neq (j, \beta)$$

for $i = 2, 3, \ldots, k$ and for $i = 2, 3, \ldots, k+1$, respectively. Besides by the definition of $Q(\rho^{\alpha}, 9r_i)$ we have $\beta_k \neq 0$ for $k = 1, 2, \ldots$. Therefore $\beta^1 \neq \beta$ and the equality $\beta^i = \beta$ implies that $\beta^{i\pm1} \neq \beta$. Hence the numbers of the multiplicands $\Lambda_N - \lambda_{j^i,\beta^i}$ in the denominators of S'_k and C'_{2p_1} satisfying

$$|\Lambda_N(\lambda_{j,\beta}) - \lambda_{j^i,\beta^i}| > \frac{1}{2}\rho^{\alpha_2}$$

[see (2.3.43)] are not less than $\frac{k}{2}$ and p_1, respectively. Now using (2.3.23) and the first inequality of (2.3.40), we obtain

$$C'_{2p_1} = O((\rho^{-\alpha_2}\ln\rho)^{p_1}) = O(\rho^{-p\alpha}),\ S'_1(\Lambda_N, \lambda_{j,\beta}) = O(\rho^{-\alpha_2}), \tag{2.3.50}$$

$$S'_k(\Lambda_N, \lambda_{j,\beta}) = O((\rho^{-\alpha_2}\ln\rho)^{\frac{k}{2}}), \quad \forall k = 2, 3, \ldots, 2p_1.$$

To prove this estimation we use (2.3.43). Moreover, if a real number a satisfies

$$|a - \lambda_{j,\beta}| < (\ln\rho)^{-1}$$

then, by (2.3.35) and (2.3.37) we have

$$|a - \lambda_{j^k,\beta^k}(v, \tau)| > c(\beta^k, \rho).$$

Therefore using this instead of (2.3.43) and repeating the proof of (2.3.50) we obtain

$$S'_1(a, \lambda_{j,\beta}) = O(\rho^{-\alpha_2}),\ S'_k(a, \lambda_{j,\beta}) = O(\rho^{-\alpha_2}\ln\rho)^{\frac{k}{2}}), \quad \forall k = 2, 3, \ldots, 2p_1. \tag{2.3.51}$$

Theorem 2.3.2 *For every eigenvalue $\lambda_{j,\beta}(v, \tau)$ of the operator $L_t(q^{\delta})$ such that*

$$\beta + \tau + (j + v)\delta \in V'_{\delta}(\rho^{\alpha_1}), \quad v(\beta) \in W(\rho),$$

there exists an eigenvalue Λ_N, denoted by $\Lambda_N(\lambda_{j,\beta}(v, \tau))$, of $L_t(q)$ satisfying the formulas

$$\Lambda_N(\lambda_{j,\beta}(v, \tau)) = \lambda_{j,\beta}(v, \tau) + E_{k-1}(\lambda_{j,\beta}) + O(\rho^{-k\alpha_2}(\ln\rho)^{2k}), \tag{2.3.52}$$

where

$$E_0 = 0,\ E_s = \sum_{k=1}^{2p_2} S'_k(\lambda_{j,\beta} + E_{s-1}, \lambda_{j,\beta}),$$

for $s = 1, 2, \ldots,$ and

$$E_{k-1}(\lambda_{j,\beta}) = O(\rho^{-\alpha_2}(\ln\rho)), \tag{2.3.53}$$

for $k = 1, 2, \ldots, [\frac{1}{9}(p - \frac{1}{2}\varkappa(d-1)]$.

Proof The proof of this Theorem is similar to the proof of Theorem 2.2.1(a). By Theorem 2.3.1, formula (2.3.52) for the case $k = 1$ is proved and $E_0 = 0$. Hence (2.3.53) for $k = 1$ is also proved. The proof of (2.3.53), for arbitrary k, follows from (2.3.51) and the definition of E_s by induction. Now we prove (2.3.52) by induction. Assume that (2.3.52) is true for $k = s < [\frac{1}{9}(p - \frac{1}{2}\varkappa(d-1)]$, i.e.,

$$\Lambda_N = \lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}(\ln\rho)^{2s})).$$

Putting this expression for Λ_N into

$$\sum_{k=1}^{2p_1} S_k'(\Lambda_N, \lambda_{j,\beta}),$$

dividing both sides of (2.3.49) by $b(N, j, \beta)$, using (2.3.50), (2.3.51) and the assertion (ii) of Lemma 2.3.6 and the equality $\alpha_2 = 9\alpha$, we get

$$\begin{aligned}\Lambda_N &= \lambda_{j,\beta} + \sum_{k=1}^{2p_1} S_k'(\lambda_{j,\beta} + E_{s-1} + O(\frac{(\ln\rho)^{2s}}{\rho^{s\alpha_2}}), \lambda_{j,\beta}) + O(\rho^{-\frac{1}{9}(p-\frac{1}{2}\tau(d-1))\alpha_2})\\ &= O(\rho^{-\frac{1}{9}(p-\frac{1}{2}\tau(d-1))\alpha_2}) + \lambda_{j,\beta} + \sum_{k=1}^{2p_1} S_k^{'}(\lambda_{j,\beta} + E_{s-1}, \lambda_{j,\beta})\\ &\quad + \{\sum_{k=1}^{2p_1} S_k^{'}(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}(\ln\rho)^{2s}), \lambda_{j,\beta}) - \sum_{k=1}^{2p_1} S_k^{'}(\lambda_{j,\beta} + E_{s-1}, \lambda_{j,\beta})\}.\end{aligned}$$

To prove (2.3.52) for $k = s + 1$ we need to show that the expression in the curly brackets is equal to

$$O((\rho^{-(s+1)\alpha_2}(\ln\rho)^{2s+1}).$$

This can be checked by using the estimations (2.3.24), (2.3.53), (2.3.35), (2.3.37) and the obvious relation

$$\begin{aligned}&\frac{1}{\prod_{i=1}^{n}(\lambda_{j,\beta} + E_{s-1} + O(\rho^{-s\alpha_2}(\ln\rho)^{2s}) - \lambda_{j^i,\beta^i})} - \frac{1}{\prod_{i=1}^{n}(\lambda_{j,\beta} + E_{s-1} - \lambda_{j^i,\beta^i})}\\ &= \frac{1}{\prod_{i=1}^{n}(\lambda_{j,\beta} + E_{s-1} - \lambda_{j^i,\beta^i})}(\frac{1}{1 + O(\rho^{-s\alpha_2}(\ln\rho)^{2s}\ln\rho)} - 1)\\ &= O(\rho^{-(s+1)\alpha_2}(\ln\rho)^{2(s+1)}), \quad \forall n = 1, 2, \ldots, 2p_1. \qquad \square\end{aligned}$$

Remark 2.3.2 Here we note some properties of the known parts $\lambda_{j,\beta} + E_k$ [see (2.3.52)], where

$$\lambda_{j,\beta} = \mu_j(v) + |\beta + \tau|^2$$

[see Lemma 2.3.1(b)], of the eigenvalues of $L_t(q)$. We prove the equality

$$\frac{\partial(E_k(\mu_j(v)+|\beta+\tau|^2))}{\partial\tau_i}=O(\rho^{-2\alpha_2+\alpha}\ln\rho) \tag{2.3.54}$$

for $i=1,2,\ldots,d-1$, where $\tau=(\tau_1,\tau_2,\ldots,\tau_{d-1})$, $k<[\frac{1}{9}(p-\frac{1}{2}\tau(d-1)]$ and $v(\beta)\in W(\rho)$. To prove (2.3.54) for $k=1$ we evaluate the derivatives of

$$H(\beta^k,j^k,\tau,v)=:(\mu_j(v)+|\beta+\tau|^2-\mu_{j^k}(v)-|\beta^k+\tau|^2)^{-1}.$$

Since $\mu_j(v)$, and $\mu_{j'}(v)$ do not depend on τ_i, the function $H(\beta^k,j^k,\tau,v)$ for $\beta^k=\beta$ does not depend on τ_i. Besides it follows from the definition of $W(\rho)$ (see Lemma 2.3.7) that $H(\beta,j^k,\tau,v)=O(\ln\rho)$. For $\beta^k\neq\beta$ using (2.3.35), and the equality

$$|\beta^k-\beta|=|\beta_1+\beta_2+\cdots+\beta_i|=O(\rho^\alpha)$$

[see the last inequality in (2.3.33)], we obtain that the derivatives of $H(\beta^k,j^k,\tau,v)$ are equal to $O(\rho^{-2\alpha_2+\alpha})$. Therefore using (2.3.23) and the definition of $E_1(\lambda_{j,\beta})$ [see (2.3.52) and (2.3.49)], by the direct calculation, we get (2.3.54) for $k=1$. Now suppose that (2.3.54) holds for $k=s-1$. Using this, replacing $\mu_j+|\beta+\tau|^2$ by $\mu_j+|\beta+\tau|^2+E_{s-1}$ in $H(\beta^k,j^k,\tau,v)$ and arguing as above we get (2.3.54) for $k=s$.

2.4 Asymptotic Formulas for the Bloch Functions

In this section using the asymptotic formulas for the eigenvalues and the simplicity conditions (2.1.28) and (2.1.29), we obtain the asymptotic formulas for the Bloch functions with a quasimomentum of the simple set B defined in Definition 2.1.2. Note that the simple set B is investigated in the next section.

Theorem 2.4.1 *If $\gamma+t\in B$, then there exists a unique eigenvalue $\Lambda_N(t)$ satisfying (2.1.14) for $k=1,2,\ldots,[\frac{p}{3}]$, where p is defined in (2.1.7). This eigenvalue is a simple eigenvalue of $L_t(q)$ and the corresponding eigenfunction $\Psi_{N,t}(x)$, denoted by $\Psi_{\gamma+t}(x)$, satisfies (2.1.32) if $q\in W_2^{s_0}(F)$, where s_0 is defined in (2.1.1).*

Proof By Theorem 2.2.1(b) if $\gamma+t\in B\subset U(\rho^{\alpha_1},p)$, then there exists an eigenvalue $\Lambda_N(t)$ satisfying (2.1.14) for $k=1,2,\ldots,[\frac{1}{3}(p-\frac{1}{2}\varkappa(d-1))]$ and by the first inequality of (2.1.40), formula (2.1.14) holds for $k=k_1$. Therefore using (2.1.14) for $k=k_1$, the relation $3k_1\alpha>d+2\alpha$ [see the second inequality of (2.1.40)], and the notations of (2.1.26), we obtain that the eigenvalue $\Lambda_N(t)$ satisfies the asymptotic formula (2.1.27). Let $\Psi_{N,t}$ be an arbitrary normalized eigenfunction corresponding to $\Lambda_N(t)$. Since the normalized eigenfunction is defined up to the constant of modulus 1, without loss of generality it can be assumed that $\arg b(N,\gamma)=0$,

where

$$b(N,\gamma)=(\Psi_{N,t},e^{i\langle\gamma+t,x\rangle}).$$

Therefore to prove (2.1.32) it suffices to show that (2.1.31) holds. To prove (2.1.31) we estimate the following summations

$$\sum_{\gamma'\notin K}|b(N,\gamma')|^2,\qquad \sum_{\gamma'\in K\backslash\{\gamma\}}|b(N,\gamma')|^2 \tag{2.4.1}$$

separately, where K is defined by (2.1.30). Using (2.1.27) and (2.1.30), we get

$$|\Lambda_N(t)-|\gamma'+t|^2|>\frac{1}{4}\rho^{\alpha_1},\quad \forall\gamma'\notin K, \tag{2.4.2}$$

$$|\Lambda_N(t)-|\gamma'+t|^2|<\frac{1}{2}\rho^{\alpha_1},\quad \forall\gamma'\in K. \tag{2.4.3}$$

It follows from (2.1.8) and (2.4.2) that

$$\sum_{\gamma'\notin K}|b(N,\gamma')|^2=\|q\Psi_{N,t}\|^2O(\rho^{-2\alpha_1})=O(\rho^{-2\alpha_1}). \tag{2.4.4}$$

Now let us estimate the second summation in (2.4.1). For this, we prove that simplicity conditions (2.1.28) and (2.1.29) imply

$$|b(N,\gamma')|\le c_5\rho^{-c\alpha},\quad \forall\gamma'\in K\backslash\{\gamma\}, \tag{2.4.5}$$

where $c=p-d\varkappa-\frac{1}{4}d3^d-3$. The conditions $\gamma'\in K$, $\gamma+t\in B$ [see (2.1.30) and Definition 2.1.2], the notation (2.1.26) and the equality (2.2.8) yield the inclusion

$$\gamma'+t\in R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho).$$

By (2.2.33) there are two cases.

Case 1: $\gamma'+t\in U(\rho^{\alpha_1},p)$. Case 2: $\gamma'+t\in(E_s\backslash E_{s+1})$, where $s=1,2,\ldots,d-1$. To prove (2.4.5) in Case 1 and Case 2, we suppose that (2.4.5) does not hold, use Theorem 2.2.1(a) and Theorem 2.2.2(a) respectively to get a contradiction.

Case 1. If the inequality in (2.4.5) is not true, then by (2.4.3) the conditions of Theorem 2.2.1(a) hold and hence we have

$$\Lambda_N(t)=|\gamma'+t|^2+F_{k-1}(\gamma'+t)+O(\rho^{-3k\alpha}) \tag{2.4.6}$$

for $k\le[\frac{1}{3}(p-c)]=[\frac{1}{3}(d\varkappa+\frac{1}{4}d3^d+3)]$. On the other hand, it follows from the definitions $k_1=:[\frac{d}{3\alpha}]+2$ [see (2.1.26)], $\alpha=:\frac{1}{\varkappa}$ [see (2.1.5)] of k_1 and α that

$$k_1 \leq \frac{1}{3}d\varkappa + 2 < \frac{1}{3}(d\varkappa + \frac{1}{4}d3^d + 3),$$

that is, formula (2.4.6) holds for $k = k_1$. Therefore arguing as in the proof of (2.1.27) (see the beginning of the proof of this theorem), we get

$$\Lambda_N(t) - F(\gamma' + t) = o(\varepsilon_1).$$

This with (2.1.27) contradicts to (2.1.28). Thus (2.4.5) in Case 1 is proved. Similarly, if the inequality in (2.4.5) does not hold in Case 2, that is, for $\gamma' + t \in (E_s \backslash E_{s+1})$ and $\gamma' \in K$, then by (2.4.3) the conditions of Theorem 2.2.2(a) hold and

$$\Lambda_N(t) = \lambda_j(\gamma' + t) + O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}), \tag{2.4.7}$$

where $(p - c - \frac{1}{4}d3^d)\alpha = (d\varkappa + 3)\alpha > d + 2\alpha$. Hence we have

$$\Lambda_N(t) - \lambda_j(\gamma' + t) = o(\varepsilon_1).$$

This with (2.1.27) contradicts (2.1.29). Thus the inequality in (2.4.5) holds. Therefore, using $|K| = O(\rho^{d-1})$ [see (2.1.37)], $\varkappa\alpha = 1$ [see (2.1.5)], we get

$$\sum_{\gamma' \in K \backslash \{\gamma\}} |b(N, \gamma')|^2 = O(\rho^{-(2c-\varkappa(d-1))\alpha}) = O(\rho^{-(2p-(3d-1)\varkappa-\frac{1}{2}d3^d-6)\alpha}). \tag{2.4.8}$$

If $s = s_0$, that is, $p = s_0 - d$, then $2p - (3d - 1)\varkappa - \frac{1}{2}d3^d - 6 = 6$. Since $\alpha_1 = 3\alpha$, the equalities (2.4.4) and (2.4.8) imply (2.1.31). Thus we proved that the equality (2.1.32) holds for any normalized eigenfunction $\Psi_{N,t}(x)$ corresponding to any eigenvalue $\Lambda_N(t)$ satisfying (2.1.14). If there exist two different eigenvalues or multiple eigenvalue satisfying (2.1.14), then there exist two orthogonal normalized eigenfunctions satisfying (2.1.32), which is impossible. Therefore $\Lambda_N(t)$ is a simple eigenvalue. It follows from Theorem 2.2.1(a) that $\Lambda_N(t)$ satisfies (2.1.14) for $k = 1, 2, \ldots, [\frac{p}{3}]$, since (2.1.32) holds and hence (2.1.16) holds for $c = 0$ □

Remark 2.4.1 Since for $\gamma + t \in B$ there exists a unique eigenvalue satisfying (2.1.14) and (2.1.27), we denote this eigenvalue by $\Lambda(\gamma + t)$. Since this eigenvalue is simple, we denote the corresponding eigenfunction by $\Psi_{\gamma+t}(x)$. By Theorem 2.4.1 this eigenfunction satisfies (2.1.32). Clearly, for $\gamma + t \in B$ there exists a unique index $N =: N(\gamma + t)$ such that $\Lambda(\gamma + t) = \Lambda_{N(\gamma+t)}(t)$ and $\Psi_{\gamma+t}(x) = \Psi_{N(\gamma+t),t}(x)$.

Now we prove the asymptotic formulas of arbitrary order for $\Psi_{\gamma+t}(x)$.

Theorem 2.4.2 *If $\gamma + t \in B$, then the eigenfunction $\Psi_{\gamma+t}(x) =: \Psi_{N,t}(x)$ corresponding to the eigenvalue $\Lambda(\gamma+t) =: \Lambda_N(t)$ satisfies (2.1.33), for $k = 1, 2, \ldots, n$, where $n = [\frac{1}{6}(2p - (3d - 1)\varkappa - \frac{1}{2}d3^d - 6)]$,*

$$F_0^* = e^{i\langle\gamma+t,x\rangle},\ F_1^* = e^{i\langle\gamma+t,x\rangle} + \sum_{\gamma_1\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1} e^{i\langle\gamma+t+\gamma_1,x\rangle}}{|\gamma+t|^2 - |\gamma+\gamma_1+t|^2},$$

$$F_k^*(\gamma+t) = (1+\|\widetilde{F}_k\|)^{-1}(e^{i\langle\gamma+t,x\rangle} + \widetilde{F}_k(\gamma+t)),$$

$\widetilde{F}_k$ is obtained from F_k by replacing q_{γ_1} with $e^{i\langle\gamma-\gamma_1+t,x\rangle}$, and F_k is defined by (2.2.10).

Proof By Theorem 2.4.1, (2.1.33) for $k = 1$ is proved. To prove (2.1.33) for $2 \le k \le n$, first we prove the following equivalent relations

$$\sum_{\gamma'\in\Gamma^c(k-1)} |b(N,\gamma+\gamma')|^2 = O(\rho^{-2k\alpha_1}), \tag{2.4.9}$$

$$\Psi_{N,t}(x) = b(N,\gamma)e^{i\langle\gamma+t,x\rangle} + \sum_{\gamma'\in\Gamma(\frac{k-1}{n}\rho^\alpha)} b(N,\gamma+\gamma')e^{i\langle\gamma+t+\gamma',x\rangle} + H_k(x), \tag{2.4.10}$$

where

$$\Gamma^c(k-1) =: \Gamma\backslash(\Gamma(\frac{k-1}{n}\rho^\alpha) \cup \{0\})$$

and

$$\|H_k\| = O(\rho^{-k\alpha_1}).$$

The case $k = 1$ is proved due to (2.1.31). Assume that (2.4.9) is true for $k = m < n$. Then using (2.4.10) for $k = m$, and the obvious decomposition

$$q(x) = \sum_{\gamma_1\in\Gamma(\frac{1}{n}\rho^\alpha)} q_{\gamma_1} e^{i\langle\gamma_1,x\rangle} + O(\rho^{-p\alpha})$$

[see (2.1.5)], we obtain

$$\Psi_{N,t}(x)q(x) = H(x) + O(\rho^{-m\alpha_1}),$$

where $H(x)$ is a linear combination of $e^{i\langle\gamma+t+\gamma',x\rangle}$ for $\gamma' \in \Gamma(\frac{m}{n}\rho^\alpha) \cup \{0\}$. Hence

$$(H, e^{i\langle\gamma+t+\gamma',x\rangle}) = 0$$

for $\gamma' \in \Gamma^c(m)$. Thus, using (2.1.8), (2.4.2) and Bessel's inequality, we get

$$\sum_{\gamma':\gamma'\in\Gamma^c(m),\gamma+\gamma'\notin K}|b(N,\gamma+\gamma')|^2$$
$$=\sum_{\gamma':\gamma'\in\Gamma^c(m),\gamma+\gamma'\notin K}|\frac{(H(x)+O(\rho^{-m\alpha_1}),e^{i\langle\gamma+t+\gamma',x\rangle})}{\Lambda_N-|\gamma+\gamma'+t|^2}|^2$$
$$=\sum_{\gamma':\gamma'\in\Gamma^c(m),\gamma+\gamma'\notin K}|\frac{(O(\rho^{-m\alpha_1}),e^{i\langle\gamma+t+\gamma',x\rangle})}{\Lambda_N-|\gamma+\gamma'+t|^2}|^2=O(\rho^{-2(m+1)\alpha_1}). \quad (2.4.11)$$

On the other hand, using $\alpha_1=3\alpha$, (2.4.8), and the definition of n, we obtain

$$\sum_{\gamma':\gamma'\in\Gamma^c(m),\gamma+\gamma'\in K}|b(N,\gamma+\gamma')|^2\le\sum_{\gamma'\in K\backslash\{\gamma\}}|b(N,\gamma')|^2=O(\rho^{-2n\alpha_1}).$$

This with (2.4.11) implies (2.4.9) for $k=m+1$. Thus (2.4.10) is also proved. It follows from (2.4.9) that

$$\|\sum_{\gamma'\in(\Gamma(\rho^\alpha)\backslash\Gamma(\frac{k-1}{n}\rho^\alpha))}b(N,\gamma+\gamma')e^{i\langle\gamma+t+\gamma',x\rangle}\|=O(\rho^{-k\alpha_1}).$$

Therefore the formula (2.4.10) for $k\le n$ can be written in the form

$$\Psi_{N,t}-b(N,\gamma)e^{i\langle\gamma+t,x\rangle}-\widetilde{H}_k=\sum_{\gamma_1\in\Gamma(\rho^\alpha)}b(N,\gamma-\gamma_1)e^{i\langle\gamma-\gamma_1+t,x\rangle}, \quad (2.4.12)$$

where
$$\|\widetilde{H}_k\|=O(\rho^{-k\alpha_1}).$$

It is clear that the right-hand side of (2.4.12) can be obtained from the right-hand side of the equality

$$(\Lambda_N-|\gamma+t|^2)b(N,\gamma)+O(\rho^{-p\alpha})=\sum_{\gamma_1\in\Gamma(\rho^\alpha)}q_{\gamma_1}b(N,\gamma-\gamma_1),$$

which is (2.1.9), by replacing q_{γ_1} with $e^{i\langle\gamma-\gamma_1+t,x\rangle}$. Therefore in (2.4.12) doing the iteration which was done in order to obtain (2.2.5) from (2.1.9), we get

$$\Psi_{N,t}(x)-b(N,\gamma)e^{i\langle\gamma+t,x\rangle}-\widetilde{H}_k(x) \quad (2.4.13)$$
$$=\widetilde{A}_{k-1}(\Lambda_N,\gamma+t)b(N,\gamma)+\widetilde{C}_k+O(\rho^{-p\alpha}),$$

where $\widetilde{A}_k(\Lambda_N,\gamma+t)$ and $\widetilde{C}_k$ are obtained from $A_k(\Lambda_N,\gamma+t)$ and C_k by replacing q_{γ_1} with $e^{i\langle\gamma-\gamma_1+t,x\rangle}$, respectively and the term $O(\rho^{-p\alpha})$ in the right-hand side of (2.4.13) is a function whose norm is $O(\rho^{-p\alpha})$. Note that if follows from the definitions of the functions $\widetilde{F}_k$, $\widetilde{A}_k$, $\widetilde{C}_k$ that the estimations similar to the estimations of F_k, A_k, C_k hold for these functions and the proof of these estimations are the same. Namely, repeating the proof of (2.2.6) and (2.2.8) we see that

$$\|\widetilde{A}_{k-1}\| = O(\rho^{-\alpha_1}), \quad \|\widetilde{C}_k\| = O(\rho^{-k\alpha_1}), \quad \|\widetilde{F}_{k-1}(\gamma+t)\| = O(\rho^{-\alpha_1}). \quad (2.4.14)$$

Now using the equalities

$$b(N,\gamma) = 1 + O(\rho^{-2\alpha_1}), \quad (2.4.15)$$

$$\begin{aligned}\widetilde{A}_{k-1}(\Lambda_N, \gamma+t) &= \widetilde{A}_{k-1}(F_{k-2}(\gamma+t), \gamma+t) + O(\rho^{-k\alpha_1}) \\ &= \widetilde{F}_{k-1}(\gamma+t) + O(\rho^{-k\alpha_1})\end{aligned}$$

[see (2.1.31a), (2.1.14), (2.2.12) and the definition of $\widetilde{F}_k$] and dividing both side of (2.4.13) by $b(N,\gamma)$ we get

$$\begin{aligned}\frac{1}{b(N,\gamma)}\Psi_{N,t}(x) = e^{i\langle\gamma+t,x\rangle} + \widetilde{F}_{k-1}(\gamma+t) + O(\rho^{-k\alpha_1}) \\ + \frac{1}{b(N,\gamma)}(\widetilde{H}_k(x) + \widetilde{C}_k + O(\rho^{-p\alpha})). \quad (2.4.16)\end{aligned}$$

Moreover the relation

$$\|\widetilde{H}_k\| = O(\rho^{-k\alpha_1})$$

[see (2.4.12)], formulas (2.4.14), (2.4.15), and the inequality $p\alpha \geq n\alpha_1 \geq k\alpha_1$ (see definition of n) imply that

$$\|O(\rho^{-k\alpha_1}) + \frac{1}{b(N,\gamma)}(\widetilde{H}_k + \widetilde{C}_k + O(\rho^{-p\alpha}))\| = O(\rho^{-k\alpha_1}). \quad (2.4.17)$$

Therefore using the equality $\|\Psi_{N,t}\| = 1$, the assumption $\arg b(N,\gamma) = 0$, the last equality of (2.4.14) and taking into account that $\widetilde{F}_{k-1}(\gamma+t)$ is a linear combination of $e^{i\langle\gamma+t-\gamma_1,x\rangle}$ for $\gamma_1 \in \Gamma(\rho^\alpha)$ [since $\widetilde{F}_{k-1}(\gamma+t)$ is obtained from the right-hand side of (2.4.12)] and hence the functions $e^{i\langle\gamma+t,x\rangle}$ and $\widetilde{F}_{k-1}(\gamma+t)$ are orthogonal, from (2.4.16), we obtain

$$\frac{1}{b(N,\gamma)} = (1 + \|\widetilde{F}_{k-1}(\gamma+t)\|)) + O(\rho^{-k\alpha_1})), \quad (2.4.18)$$

$$\Psi_{N,t}(x) = (1 + \|\widetilde{F}_{k-1}\|)^{-1}(e^{i\langle\gamma+t,x\rangle} + \widetilde{F}_{k-1}(\gamma+t) + O(\rho^{-k\alpha_1})). \quad (2.4.19)$$

Thus (2.1.33) is proved. Let us consider the case $k=2$. Using (2.4.15) and (2.4.17) in (2.4.16) for $k=2$ and recalling the definitions of $\widetilde{F}_1$ and F_1 [see (2.2.13)], we get

$$\Psi_{N,t}(x) = e^{i\langle\gamma+t,x\rangle} + \sum_{\gamma_1\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1} e^{i\langle\gamma+t+\gamma_1,x\rangle}}{|\gamma+t|^2 - |\gamma+\gamma_1+t|^2} + O(\rho^{-2\alpha_1}), \quad (2.4.20)$$

that is, we obtain the proof of the equality for $F_1^*(\gamma+t)$ □

2.5 Simple Sets and Isoenergetic Surfaces

In this section we consider the simple set B defined in Definition 2.1.2 and construct a large part of the isoenergetic surfaces

$$I_\rho(q) = \{t \in F^* : \exists N, \Lambda_N(t) = \rho^2\}$$

of $L(q)$ corresponding to ρ^2 for large ρ. In the case $q = 0$ the isoenergetic surface

$$I_\rho(0) = \{t \in F^* : \exists\gamma \in \Gamma, |\gamma+t|^2 = \rho^2\}$$

is the translation of the sphere

$$B(\rho) = \{\gamma + t : t \in F^*, \gamma \in \Gamma, |\gamma+t|^2 = \rho^2\}$$

by the vectors $\gamma \in \Gamma$. For simplicity of formulation of the main result of this section we start with a conversation about it and introduce the needed notations.

Notation 2.5.1 *We construct a part of isoenergetic surfaces by using Property 3 (see the Introduction) of the simple set B, that is, by the investigation of the function $\Lambda(\gamma+t)$ in the set B, where $\Lambda(\gamma+t)$ is defined in* Remark 2.4.1. *In other words, we consider the part*

$$PI_\rho(q) =: \{t \in F^* : \exists\gamma \in \Gamma, \Lambda(\gamma+t) = \rho^2\},$$

of the isoenergetic surfaces $I_\rho(q)$. The set $PI_\rho(q)$ is the translation of

$$TPI_\rho(q) =: \{\gamma + t : \Lambda(\gamma+t) = \rho^2\}.$$

We say that $TPI_\rho(q)$ is the part of the translated (on the simple set B) isoenergetic surfaces. In this section we construct the subsets I'_ρ and I''_ρ of $TPI_\rho(q)$ and $PI_\rho(q)$ respectively and prove that the measures of these subsets are asymptotically equal to the measure of the isoenergetic surfaces $I_\rho(0)$ of $L(0)$. In other words, we construct a large (in some sense) part I''_ρ of isoenergetic surfaces $I_\rho(q)$ of $L(q)$. Since $\Lambda(\gamma+t)$ approximately equal to $F(\gamma+t)$ [see (2.1.27) and Remark 2.4.1*] it is natural to call*

$$S_\rho = \{x \in U(2\rho^{\alpha_1}, p) : F(x) = \rho^2\},$$

where U and $F(x)$ are defined in Definition 2.1.1 *and in* (2.1.26)*, as approximated isoenergetic surfaces in the non-resonance domain.*

Now we construct a part of the simple set B in the neighborhood of S_ρ that contains I'_ρ. For this we consider the surface S_ρ. As we noted in the Introduction [*see Step 2 and* (2.1.28)] *the eigenvalue $\Lambda(\gamma+t)$ does not coincide with the eigenvalues $\Lambda(\gamma+t+b)$ if*

$$|F(\gamma+t)-F(\gamma+t+b)|>2\varepsilon_1$$

for $\gamma+t+b\in U(\rho^{\alpha_1},p)$ and $b\in\Gamma\backslash\{0\}$. Therefore we eliminate

$$P_b=\{x:x\in S_\rho, x+b\in U(\frac{1}{2}\rho^{\alpha_1},p), |F(x)-F(x+b)|<3\varepsilon_1\} \tag{2.5.1}$$

for $b\in\Gamma$ from S_ρ, denote the remaining part of S_ρ by S'_ρ, and consider the ε neighborhood of S'_ρ. Thus

$$S'_\rho=:S_\rho\backslash(\cup_{b\in\Gamma}P_b), U_\varepsilon(S'_\rho)=\cup_{a\in S'_\rho}U_\varepsilon(a)\},$$

where

$$\varepsilon=\frac{\varepsilon_1}{7\rho}, U_\varepsilon(a)=\{x\in R^d:|x-a|<\varepsilon\}, \varepsilon_1=\rho^{-d-2\alpha}.$$

In Theorem 2.5.1 *we prove that the simplicity condition* (2.1.28) *holds in $U_\varepsilon(S'_\rho)$. Denote by*

$$Tr(E)=\{\gamma+x\in U_\varepsilon(S'_\rho):\gamma\in\Gamma, x\in E\}$$

and

$$Tr_{F^\star}(E)=:\{\gamma+x\in F^\star:\gamma\in\Gamma, x\in E\}$$

the translations of $E\subset R^d$ into $U_\varepsilon(S'_\rho)$ and $F^\star$ respectively. In order that the simplicity condition (2.1.29) *holds, we discard from $U_\varepsilon(S'_\rho)$ the translation $Tr(A(\rho))$ of the set $A(\rho)$ defined as follows*

$$A(\rho)=:\bigcup_{k=1}^{d-1}\left(\bigcup_{\gamma_1,\gamma_2,\ldots,\gamma_k\in\Gamma(p\rho^\alpha)}\left(\bigcup_{i=1}^{b_k}A_{k,i}(\gamma_1,\gamma_2,\ldots,\gamma_k)\right)\right), \tag{2.5.2}$$

where

$$A_{k,i}(\gamma_1,\ldots,\gamma_k)=\{x\in\left(\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k})\backslash E_{k+1}\right)\cap K_\rho:\lambda_i(x)\in(\rho^2-3\varepsilon_1,\rho^2+3\varepsilon_1)\},$$

$\lambda_i(x)$ and b_k are defined in Theorem 2.2.2, *and*

$$K_\rho=\{x\in\mathbb{R}^d:||x|^2-\rho^2|<\rho^{\alpha_1}\}.$$

As a result, we construct the part $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$ *of the simple set* B [*see* Theorem 2.5.1(a)] *which contains the set* I'_ρ [*see* Theorem 2.5.1(c)].

To prove the main result (Theorem 2.5.1) of this section we use the following properties, namely (2.5.3) and Lemma 2.5.1, of the set constructed in Notation 2.5.1:

$$\begin{aligned} \rho - \rho^{\alpha_1 - 1} < |x| < \rho + \rho^{\alpha_1 - 1}, \quad \forall x \in U_\varepsilon(K_\rho), \\ |\frac{\partial F}{\partial x_i}| < 3\rho, \quad \forall x \in U(\rho^{\alpha_1}, p) \cap U_\varepsilon(K_\rho), \\ U_\varepsilon(S'_\rho) \subset U(\rho^{\alpha_1}, p) \cap K_\rho. \end{aligned} \tag{2.5.3}$$

To prove (2.5.3) recall that

$$F(x) = |x|^2 + F_{k_1-1}(x), \quad \forall x \in U(c_4\rho^{\alpha_1}, p) \tag{2.5.4}$$

$$F_{k_1-1}(x) = O(\rho^{-\alpha_1}), \quad \forall x \in U(c_4\rho^{\alpha_1}, p) \tag{2.5.4a}$$

$$\frac{\partial F_{k_1-1}(x)}{\partial x_i} = O(\rho^{-2\alpha_1+\alpha}) = O(\rho^{-5\alpha}), \quad \forall x \in U(c_4\rho^{\alpha_1}, p) \tag{2.5.4b}$$

$$F(x) = \rho^2, \ |x| = \rho + O(\rho^{-\alpha_1 - 1}), \quad \forall x \in S_\rho \tag{2.5.4c}$$

[see (2.1.26), (2.2.8), (2.2.34) and the definition of S_ρ]. One can readily see that the inequalities in (2.5.3) follows from the definitions of K_ρ and (2.5.4), (2.5.4a), (2.5.4b). Since $S'_\rho \subset S_\rho$, using (2.5.4c), we obtain the inclusion $U_\varepsilon(S'_\rho) \subset K_\rho$. This inclusion with $S'_\rho \subset U(2\rho^{\alpha_1}, p)$ (see the definition of S'_ρ and S_ρ) implies the inclusion in (2.5.3).

Lemma 2.5.1 (a) *If* $x \in U_\varepsilon(S'_\rho)$ *and* $x + b \in U(\rho^{\alpha_1}, p) \cap K_\rho$, *where* $b \in \Gamma$, *then*

$$|F(x) - F(x+b)| > 2\varepsilon_1,$$

where

$$\varepsilon = \frac{\varepsilon_1}{7\rho}, \varepsilon_1 = \rho^{-d-2\alpha}.$$

(b) *If* $x \in U_\varepsilon(S'_\rho)$, *then* $x + b \notin U_\varepsilon(S'_\rho)$ *for all* $b \in \Gamma$.

(c) *If* E *is a bounded subset of* $\mathbb{R}^d$, *then* $\mu(Tr(E)) \le \mu(E)$.

(d) *If* $E \subset U_\varepsilon(S'_\rho)$, *then* $\mu(Tr_{F^\star}(E)) = \mu(E)$.

Proof (a) If $x \in U_\varepsilon(S'_\rho)$, then there exists a point a such that $a \in S'_\rho$ and $x \in U_\varepsilon(a)$. Since $a + b$ lies in ε neighborhood of $x + b$, where

$$x + b \in U(\rho^{\alpha_1}, p) \cap K_\rho,$$

we have

$$a+b\in U(\frac{1}{2}\rho^{\alpha_1},p).$$

Therefore using the definitions of S'_ρ and P_b [see (2.5.1)], we obtain $a\notin P_b$ and

$$|F(a)-F(a+b)|\geq 3\varepsilon_1. \tag{2.5.5}$$

On the other hand, using the last inequality of (2.5.3) and the obvious relations

$$|x-a|<\varepsilon, |x+b-a-b|<\varepsilon,$$

we obtain

$$|F(x)-F(a)|<3\rho\varepsilon, |F(x+b)-F(a+b)|<3\rho\varepsilon. \tag{2.5.6}$$

These inequalities with (2.5.5) give the proof of Lemma 2.5.1(a), since $6\rho\varepsilon<\varepsilon_1$.

(b) If x and $x+b$ lie in $U_\varepsilon(S'_\rho)$, then there exist the points a and c in S'_ρ such that $x\in U_\varepsilon(a)$ and $x+b\in U_\varepsilon(c)$. Repeating the proof of (2.5.6), we get

$$|F(c)-F(x+b)|<3\rho\varepsilon.$$

This, the first inequality in (2.5.6) and the relations

$$F(a)=\rho^2, F(c)=\rho^2$$

for $a\in S_\rho$, $c\in S_\rho$ give

$$|F(x)-F(x+b)|<\varepsilon_1,$$

where $x\in U_\varepsilon(S'_\rho)$ and $x+b\in U_\varepsilon(S'_\rho)\subset U(\rho^{\alpha_1},p)\cap K_\rho$ [see (2.5.3)] which contradicts the Lemma 2.5.1(a).

(c) Clearly, for any bounded set E there exist only a finite number of vectors $\gamma_1,\gamma_2,\ldots,\gamma_s$ such that

$$E(k)=:(E+\gamma_k)\cap U_\varepsilon(S'_\rho)\neq\emptyset$$

for $k=1,2,\ldots,s$ and $Tr(E)$ is the union of the sets $E(k)$. By the definition of $E(k)$ we have $E(k)-\gamma_k\subset E$,

$$\mu(E(k)-\gamma_k)=\mu(E(k)).$$

Moreover, by (b),

$$(E(k)-\gamma_k)\cap(E(j)-\gamma_j)=\emptyset$$

for $k\neq j$. Therefore (c) is true.

(d) Now let $E \subset U_\varepsilon(S'_\rho)$. Then by (b) the set E can be divided into a finite number of the pairwise disjoint sets $E_1, E_2, \ldots, E_n$ such that there exist the vectors $\gamma_1, \gamma_2, \ldots, \gamma_n$ satisfying

$$(E_k + \gamma_k) \subset F^\star, (E_k + \gamma_k) \cap (E_j + \gamma_j) \neq \emptyset$$

for $k, j = 1, 2, \ldots, n$ and $k \neq j$. Using $\mu(E_k + \gamma_k) = \mu(E_k)$, we get the proof of (d), since $Tr_{F^\star}(E)$ and E are the union of the pairwise disjoint sets $E_k + \gamma_k$ and E_k for $k = 1, 2, \ldots, n$ respectively □

In the following Theorem we use the sets defined in Notation 2.5.1.

Theorem 2.5.1 (a) *The set $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$ is a subset of the simple set B defined in* Definition 2.1.2. *For every connected open subset E of $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho)$ there exists a unique index N such that $\Lambda_N(t) = \Lambda(\gamma + t)$ for $\gamma + t \in E$, where $\Lambda(\gamma + t)$ is defined in* Remark 2.4.1. *Moreover,*

$$\frac{\partial}{\partial t_j}\Lambda(\gamma + t) = \frac{\partial}{\partial t_j}|\gamma + t|^2 + O(\rho^{1-2\alpha_1}), \quad \forall j = 1, 2, \ldots, d. \tag{2.5.7}$$

(b) *For the part $V_\rho =: S'_\rho\backslash U_\varepsilon(Tr(A(\rho)))$ of the approximated isoenergetic surface S_ρ, the following holds:*

$$\mu(V_\rho) > (1 - c_{17}\rho^{-\alpha}))\mu(B(\rho)). \tag{2.5.8}$$

Moreover, $U_\varepsilon(V_\rho)$ lies in the subset $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$ of the simple set B.

(c) *The isoenergetic surface $I(\rho)$ contains the set I''_ρ, which consists of the smooth surfaces and has the measure*

$$\mu(I''_\rho) = \mu(I'_\rho) > (1 - c_{18}\rho^{-\alpha})\mu(B(\rho)), \tag{2.5.9}$$

where I'_ρ is a part of the translated isoenergetic surfaces $TPI_\rho(q)$ of $L(q)$ which is contained in the subset $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$ of the simple set B. In particular the number ρ^2 for $\rho \gg 1$ lies in the spectrum of $L(q)$, that is, the number of the gaps in the spectrum of $L(q)$ is finite, where $q \in W_2^{s_0}(\mathbb{R}^d/\Omega)$, $d \geq 2$, $s_0 = \frac{3d-1}{2}(3^d + d + 2) + \frac{1}{4}d3^d + d + 6$ and Ω is an arbitrary lattice.

Proof (a) To prove that

$$U_\varepsilon(S'_\rho)\backslash Tr(A(\rho)) \subset B$$

we need to show that for each point $\gamma+t$ of $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$ the following assertions are true:

(1) $\gamma + t \in U(\rho^{\alpha_1}, p) \cap (R(\frac{3}{2}\rho - \rho^{\alpha_1 - 1})\backslash R(\frac{1}{2}\rho + \rho^{\alpha_1 - 1}))$.

(2) If $\gamma' \in K$, where K is defined by (2.1.30), and $\gamma' + t \in U(\rho^{\alpha_1}, p)$ then (2.1.28) holds.

(3) If $\gamma' \in K$ and $\gamma' + t \in E_k \backslash E_{k+1}$ then (2.1.29) holds.

The proof of **(1)** follows from the inclusion in (2.5.3).

The proof of **(2)**. If $\gamma' \in K$, then (2.1.30) holds. Since $\gamma + t \in U_\varepsilon(S'_\rho)$, there exists $a \in S'_\rho \subset S_\rho$ such that $\gamma + t \in U_\varepsilon(a)$. Then (2.5.6), the equalities $F(a) = \rho^2$ (see the definition of S_ρ in Notation 2.5.1) and $\varepsilon_1 = 7\rho\varepsilon$ [see Lemma 2.5.1(a)] give

$$F(\gamma + t) \in (\rho^2 - \varepsilon_1, \rho^2 + \varepsilon_1). \tag{2.5.10}$$

This with (2.1.30) implies that

$$\gamma' + t \in U(\rho^{\alpha_1}, p) \cap K_\rho.$$

Now in Lemma 2.5.1(a) considering x and $x + b$ as $\gamma + t$ and $\gamma' + t$ we get (2.1.28).

The proof of **(3)**. As in the proof of **(2)** the inclusion $\gamma' \in K$ yields

$$\gamma' + t \in (E_k \backslash E_{k+1}) \cap K_\rho.$$

On the other hand, $\gamma + t \notin Tr(A(\rho))$ which means that $\gamma' + t \notin A(\rho)$. Therefore it follows from the definition of $A(\rho)$ [see (2.5.2)] that

$$\lambda_i(\gamma' + t) \notin (\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1).$$

This with (2.5.10) implies (2.1.29).

Now let E be a connected open subset of $U_\varepsilon(S'_\rho)\backslash Tr(A(\rho) \subset B$. By Theorem 2.4.1 and Remark 2.4.1 for $a \in E \subset U_\varepsilon(S'_\rho)\backslash Tr(A(\rho)$ there exists a unique index $N(a)$ such that

$$\Lambda(a) = \Lambda_{N(a)}(a),\ \Psi_a(x) = \Psi_{N(a),a}(x),\ |(\Psi_{N(a),a}, e^{i\langle a,x\rangle})|^2 > \frac{1}{2}$$

and $\Lambda(a)$ is a simple eigenvalue. On the other hand, for fixed N the functions $\Lambda_N(t)$ and $(\Psi_{N,t}, e^{i\langle t,x\rangle})$ are continuous in a neighborhood of a if $\Lambda_N(a)$ is a simple eigenvalue. Therefore for each $a \in E$ there exists a neighborhood $U(a) \subset E$ of a such that

$$|(\Psi_{N(a),y}, e^{i\langle y,x\rangle})|^2 > \frac{1}{2}$$

for $y \in U(a)$. Since for $y \in E$ there is a unique integer $N(y)$ satisfying

$$|(\Psi_{N(y),y}, e^{(\langle y,x\rangle})|^2 > \frac{1}{2},$$

we have $N(y) = N(a)$ for $y \in U(a)$. Hence we proved that

$$\forall a \in E, \exists U(a) \subset E : N(y) = N(a), \quad \forall y \in U(a). \tag{2.5.11}$$

Now let a_1 and a_2 be two points of E , and let $C \subset E$ be an arc that joins these points. Let $U(y_1), U(y_2), \ldots, U(y_k)$ be a finite subcover of the open cover $\{U(a) : a \in C\}$ of the compact C, where $U(a)$ is the neighborhood of a satisfying (2.5.11). By (2.5.11), we have $N(y) = N(y_i) = N_i$ for $y \in U(y_i)$. Clearly, if $U(y_i) \cap U(y_j) \neq \emptyset$, then $N_i = N(z) = N_j$, where $z \in U(y_i) \cap U(y_j)$. Thus $N_1 = N_2 = \cdots = N_k$ and $N(a_1) = N(a_2)$.

To calculate the partial derivatives of the function $\Lambda(\gamma + t) = \Lambda_N(t)$ we write the operator L_t in the form

$$-\triangle - \langle 2it, \nabla \rangle + \langle t, t \rangle .$$

Then, it is clear that

$$\frac{\partial}{\partial t_j} \Lambda_N(t) = 2t_j(\Phi_{N,t}, \Phi_{N,t}) - 2i(\frac{\partial}{\partial x_j} \Phi_{N,t}, \Phi_{N,t}), \tag{2.5.12}$$

$$\Phi_{N,t}(x) = \sum_{\gamma' \in \Gamma} b(N, \gamma') e^{i\langle \gamma', x \rangle}, \tag{2.5.13}$$

where

$$\Phi_{N,t}(x) = e^{-i\langle t,x \rangle} \Psi_{N,t}(x).$$

If $|\gamma'| \geq 2\rho$, then using

$$\Lambda_N =: \Lambda(\gamma + t) = \rho^2 + O(\rho^{-\alpha}),$$

[see (2.1.27), (2.5.10)], and the obvious inequality

$$|\Lambda_N - |\gamma' - \gamma_1 - \gamma_2 - \cdots - \gamma_k + t|^2| > c_{19}|\gamma'|^2$$

for $k = 0, 1, \ldots, p$, where $|\gamma_1| < \frac{1}{4p}|\gamma'|$, and iterating (2.1.8) p times by using the decomposition

$$q(x) = \sum_{|\gamma_1| < \frac{1}{4p}|\gamma'|} q_{\gamma_1} e^{i\langle \gamma_1, x \rangle} + O(|\gamma'|^{-p})$$

we get

$$b(N, \gamma') = \sum_{\gamma_1, \gamma_2, \ldots} \frac{q_{\gamma_1} q_{\gamma_2} \ldots q_{\gamma_p} b(N, \gamma' - \sum_{i=1}^{p} \gamma_i)}{\prod_{j=0}^{p-1} (\Lambda_N - |\gamma' - \sum_{i=1}^{j} \gamma_i + t|^2)} + O(|\gamma'|^{-p}), \tag{2.5.14}$$

$$b(N, \gamma') = O(|\gamma'|^{-p}), \quad \forall |\gamma'| \geq 2\rho \tag{2.5.15}$$

By (2.5.15) the series in (2.5.13) can be differentiated term by term. Hence

$$-i(\frac{\partial}{\partial x_j}\Phi_{N,t},\Phi_{N,t})=\sum_{\gamma'\in\Gamma}\gamma'(j)|b(N,\gamma')|^2=\gamma(j)|b(N,\gamma)|^2+a_1+a_2, \tag{2.5.16}$$

where

$$a_1=\sum_{|\gamma'|\geq 2\rho}\gamma'(j)|b(N,\gamma')|^2, a_2=\sum_{|\gamma'|<2\rho,\gamma'\neq\gamma}\gamma'(j)|b(N,\gamma')|^2.$$

By (2.1.31) and (2.1.31a)

$$a_2=O(\rho^{-2\alpha_1+1}), \gamma(j)|b(N,\gamma)|^2=\gamma(j)(1+O(\rho^{-2\alpha_1}),$$

and by (2.5.15), $a_1=O(\rho^{-2\alpha_1})$. Therefore (2.5.12) and (2.5.16) imply (2.5.7).

(b) To prove the inclusion

$$U_\varepsilon(V_\rho)\subset U_\varepsilon(S'_\rho)\backslash Tr(A(\rho))$$

we need to show that if $a\in V_\rho$, then

$$U_\varepsilon(a)\subset U_\varepsilon(S'_\rho)\backslash Tr(A(\rho)).$$

This is clear, since the relations $a\in V_\rho\subset S'_\rho$ imply that $U_\varepsilon(a)\subset U_\varepsilon(S'_\rho)$ and the relation

$$a\notin U_\varepsilon(Tr(A(\rho)))$$

implies that

$$U_\varepsilon(a)\cap Tr(A(\rho))=\emptyset.$$

To prove (2.5.8) first we estimate the measures of S_ρ, S'_ρ and $U_{2\varepsilon}(A(\rho))$, namely we prove that

$$\mu(S_\rho)>(1-c_{20}\rho^{-\alpha})\mu(B(\rho)), \tag{2.5.17}$$

$$\mu(S'_\rho)>(1-c_{21}\rho^{-\alpha})\mu(B(\rho)), \tag{2.5.18}$$

$$\mu(U_{2\varepsilon}(A(\rho)))=O(\rho^{-\alpha})\mu(B(\rho))\varepsilon \tag{2.5.19}$$

(see below, Estimations 1, 2, 3). The estimation (2.5.8) of the measure of the set V_ρ is done in Estimation 4 by using Estimations 1, 2, 3.

(c) Relation (2.5.9) is proved in Estimation 5. The theorem is proved □

Remark 2.5.1 Since

$$\Psi_{N,t}(x)=e^{i\langle t,x\rangle}\Phi_{N,t}(x)$$

and the series (2.5.13) can be differentiated term by term, arguing as in the proof of (2.5.16) and using the notation of Remark 2.4.1 we obtain

$$\operatorname{grad}\left(\Psi_{\gamma+t}(x)\right) = i(\gamma+t)e^{i\langle\gamma+t,x\rangle} + O(|\gamma|^{1-2\alpha_1}),$$

for $(\gamma + t) \in B$.

In Estimations 1–5 we use the notations:

$$G(+i,a) = \{x \in G : x_i > a\},\ G(-i,a) = \{x \in G : x_i < -a\},$$

where $x = (x_1, x_2, \ldots, x_d)$, $a > 0$. Recalling the definitions of the sets S'_ρ, $A(\rho)$, and using (2.5.3), it is not hard to verify that for any subset G of $U_\varepsilon(S'_\rho) \cup U_{2\varepsilon}(A(\rho))$, that is, for all considered sets G in these estimations and for any $x \in G$ the followings hold

$$\rho - 1 < |x| < \rho + 1,\ G \subset (\cup_{i=1}^{d}(G(+i,\rho d^{-1}) \cup G(-i,\rho d^{-1}))). \tag{2.5.20}$$

Indeed, (2.5.3) implies the inequalities in (2.5.20), and the inclusion in (2.5.20) follows from these inequalities.

If $G \subset S_\rho$, then by (2.5.4) and (2.5.4b) we have

$$\frac{\partial F(x)}{\partial x_k} > 0$$

for $x \in G(+k, \rho^{-\alpha})$. Therefore to calculate the measure of $G(+k, a)$ for $a \geq \rho^{-\alpha}$, we use the formula

$$\mu(G(+k,a)) = \int\limits_{\Pr_k(G(+k,a))} \left(\frac{\partial F}{\partial x_k}\right)^{-1} |grad(F)| dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_d, \tag{2.5.21}$$

where

$$\Pr_k(G) =: \{(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, x_{k+2}, \ldots, x_d) : x \in G\}$$

is the projection of G on the hyperplane $x_k = 0$. Instead of $\Pr_k(G)$ we write $\Pr(G)$ if k is unambiguous. If D is $m-$dimensional subset of $\mathbb{R}^m$, then to estimate $\mu(D)$, we use the formula

$$\mu(D) = \int\limits_{\Pr_k(D)} \mu(D(x_1, \ldots x_{k-1}, x_{k+1}, \ldots, x_m)) dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_m, \tag{2.5.22}$$

where

$$D(x_1, \ldots x_{k-1}, x_{k+1}, \ldots, x_m) = \{x_k : (x_1, x_2, \ldots, x_m) \in D\}.$$

Estimation 1 Here we prove (2.5.17) by using (2.5.21). During this estimation the set S_ρ is redenoted by G. First we estimate $\mu(G(+1, a))$ for $a = \rho^{1-\alpha}$ by using (2.5.21) for $k = 1$ and the relations

$$\frac{\partial F}{\partial x_1} > \rho^{1-\alpha}, (\frac{\partial F}{\partial x_1})^{-1}|grad(F)| = \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - \cdots - x_d^2}} + O(\rho^{-\alpha}), \tag{2.5.23}$$

$$\Pr(G(+1, a)) \supset \Pr(A(+1, 2a)), \tag{2.5.24}$$

where

$$x \in G(+1, a), A = B(\rho) \cap U(3\rho^{\alpha_1}, p),$$

and

$$B(\rho) = \{x \in \mathbb{R}^d : |x| = \rho\}.$$

Here (2.5.23) follows from (2.5.4), (2.5.4b) and (2.5.4c). Now we prove (2.5.24). If

$$(x_2, \ldots, x_d) \in \Pr_1(A(+1, 2a)),$$

then by definition of $A(+1, 2a)$ there exists x_1 such that

$$x_1 > 2a = 2\rho^{1-\alpha}, x_1^2 + x_2^2 + \cdots + x_d^2 = \rho^2, |\sum_{i \geq 1}(2x_i b_i - b_i^2)| \geq 3\rho^{\alpha_1} \tag{2.5.25}$$

for all $(b_1, b_2, \ldots, b_d) \in \Gamma(p\rho^\alpha)$. Therefore for $h = \rho^{-\alpha}$ we have

$$(x_1 + h)^2 + x_2^2 + \cdots + x_q^2 > \rho^2 + \rho^{-\alpha}, (x_1 - h)^2 + x_2^2 + \cdots + x_q^2 < \rho^2 - \rho^{-\alpha}.$$

This, (2.5.4) and (2.5.4a) give

$$F(x_1 + h, x_2, \ldots, x_d) > \rho^2, F(x_1 - h, x_2, \ldots, x_d) < \rho^2.$$

Since F is a continuous function (see Remark 2.2.2) on $U(c_4\rho^{\alpha_1}, p)$ there exists $y_1 \in (x_1 - h, x_1 + h)$ such that [see (2.5.25)]

$$y_1 > a, F(y_1, x_2, \ldots, x_d) = \rho^2.$$

Moreover

$$|2y_1 b_1 - b_1^2 + \sum_{i \geq 2}(2x_i b_i - b_i^2)| > \rho^{\alpha_1}, \tag{2.5.26}$$

because the expression under the absolute value in (2.5.26) differs from the expression under the absolute value in (2.5.25) by $2(y_1 - x_1)b_1$, where

$$|y_1 - x_1| < h = \rho^{-\alpha}, \ |b_1| < p\rho^{\alpha}, \ |2(y_1 - x_1)b_1| < 2p < \rho^{\alpha_1}.$$

Now recalling the definitions of $G(+1, a)$ and S_ρ we see that these relations imply the inclusion

$$(x_2, \dots, x_d) \in \mathrm{Pr}_1 G(+1, a).$$

Hence (2.5.24) is proved. Now (2.5.23), (2.5.24), and the obvious relation

$$\mu(\mathrm{Pr}_1 G(+1, a)) = O(\rho^{d-1})$$

[see (2.5.20)] give

$$\mu(G(+1, a)) = \int\limits_{\mathrm{Pr}(G(+1,a))} \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - \dots - x_d^2}} dx_2 dx_3 \dots dx_d + O(\frac{1}{\rho^\alpha})\mu(B(\rho))$$
$$\geq \int\limits_{\mathrm{Pr}(A(+1,2a))} \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - \dots - x_d^2}} dx_2 dx_3 \dots dx_d - c_{22}\rho^{-\alpha}\mu(B(\rho))$$
$$= \mu(A(+1, 2a)) - c_{22}\rho^{-\alpha}\mu(B(\rho)).$$

Similarly,

$$\mu(G(-1, a)) \geq \mu(A(-1, 2a)) - c_{22}\rho^{-\alpha}\mu(B(\rho)).$$

Now using the inequality

$$\mu(G) \geq \mu(G(+1, a)) + \mu(G(-1, a))$$

we get

$$\mu(G) \geq \mu(A(-1, 2a)) + \mu(A(+1, 2a)) - 2c_{22}\rho^{-\alpha}\mu(B(\rho)).$$

On the other hand, it follows from the obvious relation

$$\mu(\{x \in B(\rho) : -2a \leq x_1 \leq 2a\}) = O(\rho^{-\alpha})\mu(B(\rho))$$

that

$$\mu(A(-1, 2a)) + \mu(A(+1, 2a)) \geq \mu(A) - c_{22}\rho^{-\alpha}\mu(B(\rho)).$$

Therefore

$$\mu(G) > \mu(A) - 3c_{22}\rho^{-\alpha}\mu(B(\rho)).$$

This implies (2.5.17), since

$$\mu(A)) = (1 + O(\rho^{-\alpha}))\mu(B(\rho))$$

[see (2.2.32)].

Estimation 2 Here we prove (2.5.18). For this we estimate the measure of the set $S_\rho \cap P_b$ by using (2.5.21). During this estimation the set $S_\rho \cap P_b$ is redenoted by G. We choose the coordinate axis so that the direction of b coincides with the direction of $(1, 0, 0, \ldots, 0)$, i.e., $b = (b_1, 0, 0, \ldots, 0)$ and $b_1 > 0$. It follows from the definition of P_b [see (2.5.1)], (2.5.4), (2.5.4c) that if $(x_1, x_2, \ldots, x_d) \in G$, then

$$x_1^2 + x_2^2 + \cdots + x_d^2 + F_{k_1-1}(x) = \rho^2, \tag{2.5.27}$$

$$(x_1 + b_1)^2 + x_2^2 + x_3^2 + \cdots + x_d^2 + F_{k_1-1}(x + b) = \rho^2 + h, \tag{2.5.28}$$

where

$$h \in (-3\varepsilon_1, 3\varepsilon_1), \varepsilon_1 = \rho^{-d-2\alpha}.$$

Therefore subtracting (2.5.27) from (2.5.28) and using (2.5.4a), we get

$$(2x_1 + b_1)b_1 = O(\rho^{-\alpha_1}). \tag{2.5.29}$$

This and the inequalities in (2.5.20) imply

$$|b_1| < 2\rho + 3,\ x_1 = \frac{-b_1}{2} + O(\rho^{-\alpha_1} b_1^{-1}),\ |x_1^2 - (\frac{b_1}{2})^2| = O(\rho^{-\alpha_1}). \tag{2.5.30}$$

Consider two cases. Case 1: $b \in \Gamma_1$, where

$$\Gamma_1 = \{b \in \Gamma : |\rho^2 - |\frac{b}{2}|^2| < 3d\rho^{-2\alpha}\}.$$

In this case using the last equality in (2.5.30), (2.5.27), (2.5.4a), and taking into account that $b = (b_1, 0, 0, \ldots, 0)$ and $\alpha_1 = 3\alpha$, we obtain

$$x_1^2 = \rho^2 + O(\rho^{-2\alpha}), |x_1| = \rho + O(\rho^{-2\alpha-1}), x_2^2 + x_3^2 + \cdots + x_d^2 = O(\rho^{-2\alpha}). \tag{2.5.31}$$

Therefore

$$G \subset G(+1, a) \cup G(-1, a),$$

where $a = \rho - \rho^{-1}$. Using (2.5.21) and the obvious relation

$$\mu(\mathrm{Pr}_1(G(\pm 1, a)) = O(\rho^{-(d-1)\alpha})$$

[see (2.5.31)] and taking into account that the expression under the integral in (2.5.21) for $k = 1$ is equal to $1 + O(\rho^{-\alpha})$ [see (2.5.4b) and (2.5.31)], we get

$$\mu(G(\pm 1, a)) = O(\rho^{-(d-1)\alpha}).$$

Thus

$$\mu(G) = O(\rho^{-(d-1)\alpha}).$$

Since

$$|\Gamma_1| = O(\rho^{d-1})$$

[see (2.1.37)], we have

$$\mu(\cup_{b\in\Gamma_1}(S_\rho \cap P_b) = O(\rho^{-(d-1)\alpha+d-1}) = O(\rho^{-\alpha})\mu(B(\rho)). \tag{2.5.32}$$

Case 2: $|\rho^2 - |\frac{b}{2}|^2| \geq 3d\rho^{-2\alpha}$. Repeating the proof of (2.5.31), we get

$$|x_1^2 - \rho^2| > 2d\rho^{-2\alpha}, \ \sum_{k=2}^{d} x_k^2 > d\rho^{-2\alpha}, \ \max_{k\geq 2} |x_k| > \rho^{-\alpha}. \tag{2.5.33}$$

Therefore

$$G \subset \cup_{k\geq 2}(G(+k, \rho^{-\alpha}) \cup G(-k, \rho^{-\alpha})).$$

Now we estimate $\mu(G(+d, \rho^{-\alpha}))$ by using (2.5.21). If $x \in G(+d, \rho^{-\alpha})$, then according to (2.5.27) and (2.5.4b) the expression under the integral in (2.5.21) for $k = d$ is $O(\rho^{1+\alpha})$. Therefore the first equality in

$$\mu(D) = O(\varepsilon_1|b|^{-1}\rho^{d-2}), \ \mu(G(+d, \rho^{-\alpha})) = O(\rho^{d-1+\alpha}\varepsilon_1|b|^{-1}), \tag{2.5.34}$$

where the set $\mathrm{Pr}_d G(+d, \rho^{-\alpha})$ is redenoted by D, implies the second equality in (2.5.34). To prove the first equality in (2.5.34) we use (2.5.22) for $m = d-1$ and $k = 1$ and prove the relations

$$\mu(\mathrm{Pr}_1 D) = O(\rho^{d-2}),$$

$$\mu(D(x_2, x_3, \ldots, x_{d-1})) < 6\varepsilon_1|b|^{-1} \tag{2.5.35}$$

for $(x_2, x_3, \ldots, x_{d-1}) \in \mathrm{Pr}_1 D$. First relation follows from the inequalities in (2.5.20). Thus we need to prove (2.5.35). If $x_1 \in D(x_2, x_3, \ldots, x_{d-1})$, then by the definitions of $D(x_2, x_3, \ldots, x_{d-1})$ and D we have $(x_1 x_2, , \ldots, x_{d-1}) \in D$ and

$$(x_1, x_2, \ldots, x_d) \in G(+d, \rho^{-\alpha}) \subset G =: S_\rho \cap P_b.$$

Therefore (2.5.27) and (2.5.28) hold. Subtracting (2.5.27) from (2.5.28), we get

$$2x_1 b_1 + (b_1)^2 + F_{k_1-1}(x+b) - F_{k_1-1}(x) = h, \tag{2.5.36}$$

where $x_2, x_3, \ldots, x_{d-1}$ are fixed. Hence we have two equations (2.5.27) and (2.5.36) with respect to the unknown x_1 and x_d. Using (2.5.4b), the implicit function theorem, and the inequalities $|x_d| > \rho^{-\alpha}$, $\alpha_1 > 2\alpha$ from (2.5.27), we obtain

$$x_d = f(x_1), \ \frac{df}{dx_1} = -\frac{2x_1 + O(\rho^{-2\alpha_1+\alpha})}{2x_d + O(\rho^{-2\alpha_1+\alpha})}. \tag{2.5.37}$$

Substituting this in (2.5.36), we get

$$2x_1 b_1 + b_1^2 + F_{k_1-1}(x_1 + b_1, x_2, \ldots, x_{d-1}, \ f(x_1)) - F_{k_1-1}(x_1, \ldots, x_{d-1}, f) = h. \tag{2.5.38}$$

Using (2.5.4b), (2.5.37), the first equality in (2.5.30), and $x_d > \rho^{-\alpha}$ we see that the absolute value of the derivative (w.r.t. x_1) of the left-hand side of (2.5.38) satisfies the inequality

$$|2b_1 + O(\rho^{-2\alpha_1+\alpha}) + O(\rho^{-2\alpha_1+\alpha})\frac{x_1 + O(\rho^{-2\alpha_1+\alpha})}{x_d + O(\rho^{-2\alpha_1+\alpha})})| > b_1$$

for

$$x_1 = \frac{-b_1}{2} + O(\rho^{-\alpha_1})$$

[see (2.5.30)]. Therefore from (2.5.38), by the implicit function theorem, we get

$$|\frac{dx_1}{dh}| < \frac{1}{|b|}, \quad \forall h \in (-3\varepsilon_1, 3\varepsilon_1).$$

This inequality implies that the image $\{x_1(h) : h \in (-3\varepsilon_1, 3\varepsilon_1)\}$ of the interval $(-3\varepsilon_1, 3\varepsilon_1)$ [see (2.5.28)] under the differentiable function $x_1(h)$ is an interval I with the length less than $6\varepsilon_1|b|^{-1}$. Since $D(x_2, x_3, \ldots, x_{d-1})$ is a measurable subset of I, (2.5.35) holds. Thus (2.5.34) is proved. In the same way we get the same estimation for the sets $G(-d, \rho^{-\alpha})$, $G(+k, \rho^{-\alpha})$ and $G(-k, \rho^{-\alpha})$, where $k \geq 2$. Hence

$$\mu(S_\rho \cap P_b) = O(\rho^{d-1+\alpha}\varepsilon_1|b|^{-1})$$

for $b \notin \Gamma_1$. Since $|b| < 2\rho + 3$ [see (2.5.30)] and $\varepsilon_1 = \rho^{-d-2\alpha}$, taking into account that the number of the vectors of Γ satisfying $|b| < 2\rho + 3$ is $O(\rho^d)$, we obtain

$$\mu(\cup_{b\notin\Gamma_1}(S_\rho \cap P_b)) = O(\rho^{2d-1+\alpha}\varepsilon_1) = O(\rho^{-\alpha})\mu(B(\rho)).$$

This, (2.5.32) and (2.5.17) give the proof of (2.5.18).
Estimation 3 Here we prove (2.5.19). Denote $U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, \ldots, \gamma_k))$ by G, where

$$\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^\alpha), \quad k \leq d-1,$$

and $A_{k,j}$ is defined in (2.5.2). To estimate $\mu(G)$ we turn the coordinate axis so that

$$Span\{\gamma_{1,}\gamma_2, \ldots, \gamma_k\} = \{x = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) : x_1, x_2, \ldots, x_k \in \mathbb{R}\}.$$

Then by (2.2.22), we have

$$x_i = O(\rho^{\alpha_k+(k-1)\alpha})$$

for $i \le k$, $x \in G$. This, (2.5.20), and $\alpha_k + (k-1)\alpha < 1$ [see the first inequality in (2.1.39)] give

$$G \subset (\cup_{i>k}(G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1})),$$

$$\mu(\Pr(G(+i, \rho d^{-1}))) = O(\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)}) \tag{2.5.39}$$

for $i > k$. Now using these and (2.5.22) for $m = d$, we prove that

$$\mu(G(+i, \rho d^{-1})) = O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)}), \quad \forall i > k. \tag{2.5.40}$$

For this we redenote by D the set $G(+i, \rho d^{-1})$ and prove that

$$\mu((D(x_1, x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_d)) \le (42d^2 + 4)\varepsilon \tag{2.5.41}$$

for

$$(x_1, x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_d) \in \Pr_i(D)$$

and $i > k$, since using (2.5.41) and (2.5.39) in (2.5.22) one can easily get the proof of (2.5.40). Hence we need to prove (2.5.41). To prove (2.5.41) it is sufficient to show that if both $x = (x_1, x_2, \ldots, x_i, \ldots x_d)$ and $x' = (x_1, x_2, \ldots, x_i', \ldots, x_d)$ are in D, then

$$|x_i - x_i'| \le (42d^2 + 4)\varepsilon.$$

Assume the converse. Then

$$|x_i - x_i'| > (42d^2 + 4)\varepsilon.$$

Without loss of generality it can be assumed that $x_i' > x_i$. Then we have the inequalities

$$x_i' > x_i + (42d^2 + 4)\varepsilon, \;\; x_i > \rho d^{-1} \tag{2.5.42}$$

since

$$x = (x_1, x_2, \ldots, x_i, \ldots x_d) \in D =: G(+i, \rho d^{-1}).$$

By the definition of G the points x and x' lie in the 2ε neighborhood of $A_{k,j}$. Therefore there exist the points a and a' in $A_{k,j}$ such that $|x - a| < 2\varepsilon$ and $|x' - a'| < 2\varepsilon$. These inequalities with (2.5.42) imply that

$$\rho d^{-1} - 2\varepsilon < a_i < a_i', \; a_i' - a_i > 42d^2\varepsilon, \tag{2.5.43}$$

$$(a_i')^2 - (a_i)^2 > 2(\rho d^{-1} - 2\varepsilon)(a_i' - a_i), \tag{2.5.43a}$$

$$a_s, a_s' \in (x_s - 2\varepsilon, x_s + 2\varepsilon,), \; ||a_s| - |a_s'|| < 4\varepsilon \tag{2.5.44}$$

for $s \neq i$, since $x_s' = x_s$ for $s \neq i$. On the other hand, the inequalities in (2.5.20) hold for the points of $A_{k,j}$, that is, we have

$$|a_s| < \rho + 1, |a_s'| < \rho + 1.$$

These inequalities and (2.5.44) imply

$$||a_s|^2 - |a_s'|^2| < 12\rho\varepsilon$$

for $s \neq i$, and by (2.5.43)

$$\sum_{s \neq i} ||a_s|^2 - |a_s'|^2| < 12d\rho\varepsilon < \frac{2}{7}\rho d^{-1}(a_i' - a_i). \tag{2.5.45}$$

Using this and (2.5.43a), we get

$$||a|^2 - |a'|^2| > \frac{3}{2}\rho d^{-1}|a_i' - a_i|. \tag{2.5.46}$$

At last, the inequalities

$$a_i' - a_i > 42d^2\varepsilon, \; |a_s - a_s'| < 4\varepsilon \quad \text{for } s \neq i$$

for $s \neq i$ [see (2.5.43) and the inclusion in (2.5.44)] show that

$$|a - a'| < 2|a_i' - a_i|. \tag{2.5.46a}$$

Now we prove that (2.5.46) and (2.5.46a) contradict the inclusions $a \in A_{k,j}$ and $a' \in A_{k,j}$. Using (2.2.36), the obvious relation $\frac{1}{2}\alpha_d < 1$ [see definitions of α and α_d in (2.1.5) and in Definition 2.1.1] and (2.5.46a), we get

$$|r_j(a) - r_j(a')| < \rho^{\frac{1}{2}\alpha_d}|a - a'| < \frac{1}{2}\rho d^{-1}|a_i' - a_i|,$$

where

$$r_j(x) = \lambda_j(x) - |x|^2$$

(see Remark 2.2.2). This inequality, (2.5.46), the inequality

$$a_i' - a_i > 42d^2\varepsilon$$

[see (2.5.43)], and the relation $\varepsilon_1 = 7\rho\varepsilon$ [see Lemma 2.5.1(a)] imply

$$|\lambda_j(a) - \lambda_j(a')| \geq ||a|^2 - |a'|^2| - |r_j(a) - r_j(a')|$$
$$> \rho d^{-1}|a_i' - a_i| > 42d\rho\varepsilon > 6\varepsilon_1.$$

The obtained inequality

$$|\lambda_j(a) - \lambda_j(a')| > 6\varepsilon_1$$

contradicts with the inclusions $a \in A_{k,j}$, $a' \in A_{k,j}$, since by definition of $A_{k,j}$ [see (2.5.2)] both $\lambda_j(a)$ and $\lambda_j(a')$ lie in $(\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1)$. Thus (2.5.41) and hence (2.5.40) are proved. In the same way we get the same estimation for $G(-i, \frac{\rho}{d})$. Thus

$$\mu(U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, \ldots, \gamma_k))) = O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+d-1-k}).$$

Now taking into account that $U_{2\varepsilon}(A(\rho))$ is the union of $U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, \ldots, \gamma_k)$ for $k = 1, 2, .., d-1;\ j = 1, 2, \ldots, b_k(\gamma_1, \gamma_2, \ldots, \gamma_k)$, and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^{\alpha})$ [see (2.5.2)] and using

$$b_k = O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}})$$

[see (2.2.30)], and that the number of the vectors $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ for $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^{\alpha})$ is $O(\rho^{dk\alpha})$, we obtain

$$\mu(U_{2\varepsilon}(A(\rho))) = O(\varepsilon\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}+dk\alpha+k(\alpha_k+(k-1)\alpha)+d-1-k}).$$

Therefore to prove (2.5.19), it remains to show that

$$d\alpha + \frac{k}{2}\alpha_{k+1} + dk\alpha + k(\alpha_k + (k-1)\alpha) + d - 1 - k \leq d - 1 - \alpha$$

or

$$(d+1)\alpha + \frac{k}{2}\alpha_{k+1} + dk\alpha + k(\alpha_k + (k-1)\alpha) \leq k$$

for $1 \leq k \leq d-1$. Dividing both sides of the last inequality by $k\alpha$ and using

$$\alpha_k = 3^k\alpha, \alpha = \frac{1}{\varkappa}, \varkappa = 3^d + d + 2$$

[see (2.1.5) and Definition 2.1.1], we get

$$\frac{d+1}{k} + \frac{3^{k+1}}{2} + 3^k + k - 1 \leq 3^d + 2.$$

The left-hand side of this inequality gets its maximum at $k = d - 1$. Therefore we need to show that

$$\frac{d+1}{d-1} + \frac{5}{6}3^d + d \leq 3^d + 4$$

which follows from the inequalities

$$\frac{d+1}{d-1} \leq 3,\ d < \frac{1}{6}3^d + 1$$

for $d \geq 2$.

Estimation 4 Here we prove (2.5.8). During this estimation we denote by G the set $S'_\rho \cap U_\varepsilon(Tr(A(\rho))$. Since $V_\rho = S'_\rho \backslash G$ and (2.5.18) holds, it is enough to prove that

$$\mu(G) = O(\rho^{-\alpha})\mu(B(\rho)).$$

For this we use (2.5.20) and prove

$$\mu(G(+i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho)),\ \mu(G(-i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho)) \tag{2.5.47}$$

for $i = 1, 2, \ldots, d$ by using (2.5.21). By (2.5.4b), if $x \in G(+i, \rho d^{-1})$, then the expression under the integral in (2.5.21) for $k = i$ is less than $2(d + 1)^2$. Therefore to prove the first equality of (2.5.47) it is sufficient to prove

$$\mu(\Pr(G(+i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho)) \tag{2.5.48}$$

Clearly, if

$$(x_1, x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_d) \in \Pr_i(G(+i, \rho d^{-1})),$$

then

$$\mu(U_\varepsilon(G)(x_1, x_2, \ldots x_{i-1}, x_{i+1}, \ldots x_d)) \geq 2\varepsilon$$

and by (2.5.22), it follows that

$$\mu(U_\varepsilon(G)) \geq 2\varepsilon\mu(\Pr(G(+i, \rho d^{-1})). \tag{2.5.49}$$

Hence to prove (2.5.48) we need to estimate $\mu(U_\varepsilon(G))$. For this we prove that

$$U_\varepsilon(G) \subset U_\varepsilon(S'_\rho),\ U_\varepsilon(G) \subset U_{2\varepsilon}(Tr(A(\rho))),\ U_\varepsilon(G) \subset Tr(U_{2\varepsilon}(A(\rho))). \tag{2.5.50}$$

The first and second inclusions follow from $G \subset S'_\rho$ and $G \subset U_\varepsilon(Tr(A(\rho)))$ respectively (see the definition of G). Now we prove the third inclusion in (2.5.50). If $x \in U_\varepsilon(G)$, then by the second inclusion of (2.5.50) there exists b such that

$$b \in Tr(A(\rho)), \quad |x - b| < 2\varepsilon.$$

Then by the definition of $Tr(A(\rho))$ there exist $\gamma \in \Gamma$ and $c \in A(\rho)$ such that $b = \gamma + c$. Therefore

$$|x - \gamma - c| = |x - b| < 2\varepsilon, \quad x - \gamma \in U_{2\varepsilon}(c) \subset U_{2\varepsilon}(A(\rho)).$$

This together with

$$x \in U_\varepsilon(G) \subset U_\varepsilon(S'_\rho)$$

[see the first inclusion in (2.5.50)] gives $x \in Tr(U_{2\varepsilon}(A(\rho)))$ (see the definition of $Tr(E)$ in Notation 2.5.1), i.e., the third inclusion in (2.5.50) is proved. This inclusion, Lemma 2.5.1(c), and (2.5.19) imply that

$$\mu(U_\varepsilon(G)) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon.$$

Now using (2.5.49), we get the proof of (2.5.48) and hence the proof of the first equality of (2.5.47). The second equality of (2.5.47) can be proved in the same way ◇

Estimation 5 Here we prove (2.5.9). Divide the set $V_\rho =: V$, defined in Theorem 2.5.1(b), into pairwise disjoint subsets

$$V'(\pm 1, \rho d^{-1}) =: V(\pm 1, \rho d^{-1}), V'(\pm i, \rho d^{-1}) =: V(\pm i, \rho d^{-1}) \backslash (\cup_{j=1}^{i-1}(V(\pm j, \rho d^{-1})))$$

for $i = 2, 3, \ldots, d$. Take any point $a \in V'(+i, \rho d^{-1}) \subset S_\rho$ and consider the function $F(x)$ [see (2.5.4)] on the interval $(a - \varepsilon e_i, a + \varepsilon e_i)$, where $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots$. By (2.5.4c), we have

$$F(a) = \rho^2.$$

It follows from (2.5.4b) and the definition of $V'(+i, \rho d^{-1})$ that

$$\frac{\partial F(x)}{\partial x_i} > \rho d^{-1}$$

for $x \in (a - \varepsilon e_i, a + \varepsilon e_i)$. Therefore

$$F(a - \delta e_i) < \rho^2 - c_{23}\varepsilon_1, \quad F(a + \delta e_i) > \rho^2 + c_{23}\varepsilon_1, \tag{2.5.51}$$

where $\delta = \frac{\varepsilon}{2}$, $\varepsilon_1 = 7\rho\varepsilon$. Since

$$[a - \delta e_i, a + \delta e_i] \in U_\varepsilon(a) \subset U_\varepsilon(V_\rho) \subset U_\varepsilon(S'_\rho) \backslash Tr(A(\rho))$$

(see Theorem 2.5.1(b)), it follows from Theorem 2.5.1(a) that there exists an index N such that $\Lambda(y) = \Lambda_N(y)$ for $y \in U_\varepsilon(a)$ and $\Lambda(y)$ satisfies (2.1.27) (see

Remark 2.4.1). Hence (2.5.51) implies that

$$\Lambda(a - \delta e_i) < \rho^2, \quad \Lambda(a + \delta e_i) > \rho^2. \tag{2.5.52}$$

Moreover it follows from (2.5.7) that the derivative of $\Lambda(y)$ with respect to y_i is positive for $y \in [a - \delta e_i, a + \delta e_i]$. Hence $\Lambda(y)$ is a continuous and increasing function in $[a - \delta e_i, a + \delta e_i]$. Thus (2.5.52) implies that there exists a unique point $y(a, i) \in [a - \delta e_i, a + \delta e_i]$ such that

$$\Lambda(y(a, i)) = \rho^2.$$

Define $I'_\rho(+i)$ by

$$I'_\rho(+i) = \{y(a, i) : a \in V'(+i, \rho d^{-1})\}).$$

In the same way we define

$$I'_\rho(-i) = \{y(a, i) : a \in V'(-i, \rho d^{-1})\}$$

and put

$$I'_\rho = \cup_{i=1}^{d} (I'_\rho(+i) \cup I'_\rho(-i)).$$

To estimate the measure of I'_ρ we compare the measure of $V'(\pm i, \rho d^{-1})$ with the measure of $I'_\rho(\pm i)$ by using the formula (2.5.21) and the relations

$$\Pr(V'(\pm i, \rho d^{-1})) = \Pr(I'_\rho(\pm i)), \ \mu(\Pr(I'_\rho(\pm i))) = O(\rho^{d-1}), \tag{2.5.53}$$

$$\left(\frac{\partial F}{\partial x_i}\right)^{-1} |grad(F)| - \left(\frac{\partial \Lambda}{\partial x_i}\right)^{-1} |grad(\Lambda)| = O(\rho^{-2\alpha_1}), \tag{2.5.54}$$

where the first equality in (2.5.53) follows from the definition of $I'_\rho(\pm i)$, the second equality in (2.5.53) follows from the inequalities in (2.5.20), since $I'_\rho \subset U_\varepsilon(S'_\rho)$, and (2.5.54) follows from (2.5.4b) and (2.5.7). Using (2.5.53), (2.5.54), and (2.5.21), we get

$$\mu(V'(\pm i, \rho d^{-1})) - \mu(I'_\rho(\pm i)) = O(\rho^{d-1-2\alpha_1}). \tag{2.5.55}$$

On the other hand, if

$$y =: (y_1, y_2, \ldots, y_d) \in I'_\rho(+i) \cap I'_\rho(+j)$$

for $i < j$ then there are $a \in V'(+i, \rho d^{-1})$ and $a' \in V'(+j, \rho d^{-1})$ such that $y = y(a, i) = y(a', j)$ and $y \in [a - \delta e_i, a + \delta e_i]$, $y \in [a' - \delta e_j, a' + \delta e_j]$. These inclusions and definitions of $V'(+i, \rho d^{-1})$, $V'(+j, \rho d^{-1})$ imply that

$$\rho d^{-1} - \delta \leq y_i \leq \rho d^{-1}.$$

Therefore using the inequalities in (2.5.20), we get

$$\mu(\mathrm{Pr}_j(I'_\rho(+i) \cap I'_\rho(+j))) = O(\varepsilon\rho^{d-2}).$$

This equality, (2.5.21) for $k = j$ and (2.5.7) give

$$\mu((I'_\rho(+i) \cap I'_\rho(+j))) = O(\varepsilon\rho^{d-2}) \tag{2.5.56}$$

for all i and j. Similarly

$$\mu((I'_\rho(+i) \cap I'_\rho(-j))) = O(\varepsilon\rho^{d-2})$$

for all i and j. Now using (2.5.56) and (2.5.55), we obtain

$$\begin{aligned}\mu(I'_\rho) = \sum_i \mu(I'_\rho(+i)) + \sum_i \mu(I'_\rho(-i)) + O(\varepsilon\rho^{d-2}) = \sum_i \mu(V'(+i, \rho d^{-1}))\\ + \sum_i \mu(V'(-i, \rho d^{-1})) + O(\rho^{d-1-2\alpha_1}) = \mu(V_\rho) + O(\rho^{-2\alpha_1})\mu(B(\rho)).\end{aligned} \tag{2.5.57}$$

This and (2.5.8) yield the inequality (2.5.9) for I'_ρ. Now we define I''_ρ as follows. If $\gamma + t \in I'_\rho$ then

$$\Lambda(\gamma + t) = \rho^2,$$

where $\Lambda(\gamma + t)$ is a unique eigenvalue satisfying (2.1.27) (see Remark 2.4.1). Since

$$\Lambda(\gamma + t) = |\gamma + t|^2 + O(\rho^{-\alpha_1})$$

[see (2.1.27), (2.5.4), and (2.5.4a)], for fixed t there exist only a finite number of vectors $\gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma$ satisfying

$$\Lambda(\gamma_k + t) = \rho^2.$$

Hence I'_ρ is the union of the pairwise disjoint sets

$$I'_{\rho,k} =: \{\gamma_k + t \in I'_\rho : \Lambda(\gamma_k + t) = \rho^2\} \ (k = 1, 2, \ldots s).$$

The translation

$$I''_{\rho,k} = I'_{\rho,k} - \gamma_k = \{t \in F^* : \gamma_k + t \in I'_{\rho,k}\}$$

of $I'_{\rho,k}$ is a part of the isoenergetic surface I_ρ of $L(q)$. Put

$$I''_\rho = \cup_{k=1}^s I''_{\rho,k}.$$

If
$$t \in I''_{\rho,k} \cap I''_{\rho,m}$$
for $k \neq m$, then
$$\gamma_k + t \in I'_\rho \subset U_\varepsilon(S'_\rho)$$
and
$$\gamma_m + t \in U_\varepsilon(S'_\rho),$$
which contradict Lemma 2.5.1(b). Therefore I''_ρ is the union of the pairwise disjoint subsets $I''_{\rho,k}$ for $k = 1, 2, \ldots s$. Thus
$$\mu(I''_\rho) = \sum_k \mu(I''_{\rho,k}) = \sum_k \mu(I'_{\rho,k}) = \mu(I'_\rho).$$
This implies (2.5.9) for I''_ρ, since (2.5.9) is proved for I'_ρ [see (2.5.57)] $\square$

2.6 Bloch Functions Near the Diffraction Hyperplanes

In this section we obtain the asymptotic formulas for the Bloch function corresponding to the quasimomentum lying near the diffraction hyperplanes. Here we assume that (2.1.36) holds instead of (2.1.1). Besides, in this section, we define the number $\varkappa$ by $\varkappa = 4(3^d(d+1))$ instead of the definition $\varkappa = 3^d + d + 2$ of $\varkappa$ given in (2.1.5). The other numbers $p, \alpha_k, \alpha, k_1, p_1$ are defined as in the introduction. Clearly these numbers satisfy all inequalities of (2.1.38)–(2.1.40). Therefore the formulas obtained in the previous sections hold in these notations too. Moreover the following relations hold
$$k_2 < \frac{1}{9}(p - \frac{1}{2}\varkappa(d-1)),\ k_2\alpha_2 > d + 2\alpha,\ 4(d+1)\alpha_d = 1, \tag{2.6.1}$$
where $k_2 = [\frac{d}{9\alpha}] + 2$. In this section we construct a subset B_δ of $V'_\delta(\rho^{\alpha_1})$ such that if
$$\gamma + t =: \beta + \tau + (j+v)\delta \in B_\delta$$
(see Remark 2.3.1 for the notations), then there exists a unique eigenvalue $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ satisfying (2.3.52). Moreover we prove that $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ is a simple eigenvalue if $\beta + \tau + (j+v)\delta$ belongs to the set B_δ. Therefore we call the set B_δ the simple set in the resonance domain $V_\delta(\rho^{\alpha_1})$. Then we obtain the asymptotic formulas of arbitrary order for the eigenfunction $\Psi_N(x)$ corresponding to the eigenvalue $\Lambda_N(\lambda_{j,\beta}(v,\tau))$. At the end of this section we prove that B_δ has asymptotically full measure on $V_\delta(\rho^{\alpha_1})$. The construction of the simple set B_δ in the resonance domain $V_\delta(\rho^{\alpha_1})$ is similar to the construction of the simple set B in the non-resonance domain

(see Step 1 and Step 2 in the introduction). As in Step 2 we need to find the simplicity conditions for the eigenvalue $\Lambda_N(\lambda_{j,\beta})$. Since the first inequality in (2.6.1) holds, $\Lambda_N(\lambda_{j,\beta})$ satisfies (2.3.52) for $k = k_2$. Therefore it follows from the second inequality of (2.6.1) that

$$\Lambda_N(\lambda_{j,\beta}(v,\tau)) = E(\lambda_{j,\beta}(v,\tau)) + o(\rho^{-d-2\alpha}) = o(\varepsilon_1), \tag{2.6.2}$$

where

$$E(\lambda_{j,\beta}(v,\tau)) = \lambda_{j,\beta}(v,\tau) + E_{k_2-1}(\lambda_{j,\beta}(v,\tau)),\ \varepsilon_1 = \rho^{-d-2\alpha},$$

$$\lambda_{j,\beta}(v,\tau) \sim \rho^2,\ E_{k_2-1}(\lambda_{j,\beta}) = O(\rho^{-\alpha_2}(\ln\rho)), \tag{2.6.3}$$

$$\lambda_{j,\beta}(v,\tau) = |\beta+\tau|^2 + \mu_j(v) = |\beta+\tau|^2 + O(\rho^{2\alpha_1}), \tag{2.6.4}$$

$$E(\lambda_{j,\beta}(v,\tau)) = |\beta+\tau|^2 + O(\rho^{2\alpha_1}) \tag{2.6.5}$$

[see (2.3.53), Lemma 2.3.1(b), (2.3.6), (2.3.5), and the definition of $E(\lambda_{j,\beta}(v,\tau))$]. Due to (2.6.2) we call $E(\lambda_{j,\beta}(v,\tau))$ as the known part of $\Lambda_N(\lambda_{j,\beta}(v,\tau))$. Since the other eigenvalues lie in the ε_1-neighborhood of $\lambda_i(\gamma'+t)$, $F(\gamma'+t)$ (see Step 1 in the introduction), in order that $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ does not coincide with any other eigenvalue we use the following two simplicity conditions:

$$|E(\lambda_{j,\beta}(v,\tau)) - F(\gamma'+t)| \ge 2\varepsilon_1, \quad \forall \gamma' \in M_1, \tag{2.6.6}$$

$$|E(\lambda_{j,\beta}) - \lambda_i(\gamma'+t)| \ge 2\varepsilon_1, \quad \forall \gamma' \in M_2;\ \forall i = 1, 2, \ldots, b_k, \tag{2.6.7}$$

where M is the set of $\gamma' \in \Gamma$ satisfying

$$|E(\lambda_{j,\beta}(v,\tau)) - |\gamma'+t|^2| < \frac{1}{3}\rho^{\alpha_1},$$

M_1 is the set of $\gamma' \in M$ satisfying $\gamma'+t \in U(\rho^{\alpha_1}, p)$, M_2 is the set of $\gamma' \in M$ such that $\gamma'+t \notin U(\rho^{\alpha_1}, p)$ and $\gamma'+t$ has the Γ_δ decomposition

$$\gamma'+t = \beta' + \tau + (j' + v(\beta'))\delta$$

(see Remark 2.3.1) with $\beta' \neq \beta$.

Definition 2.6.1 The simple set B_δ in the resonance domain $V_\delta(\rho^{\alpha_1})$ is the set of

$$x \in V'_\delta(\rho^{\alpha_1}) \cap (R(\frac{3}{2}\rho - \rho^{\alpha_1-1}) \backslash R(\frac{1}{2}\rho + \rho^{\alpha_1-1}))$$

such that $x = \gamma + t$, where $\gamma \in \Gamma$, $t \in F^\star$ [it is Γ decomposition of x (see Remark 2.3.1)] and $x = \beta + \tau + (j + v(\beta))\delta$, where $\beta \in \Gamma_\delta$, $\tau \in F_\delta$, $j \in \mathbb{Z}$, $v(\beta) \in W(\rho)$ (it is Γ_δ decomposition of x and $W(\rho)$ is defined in Lemma 2.3.7), and the simplicity conditions (2.6.6) and (2.6.7) hold.

Using the simplicity conditions (2.6.6) and (2.6.7) we prove that $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ does not coincide with the other eigenvalues if

$$\beta + \tau + (j + v)\delta \in B_\delta.$$

The existence and properties of the sets B_δ will be considered at the end of this section. Recall that in Sect. 2.4 the simplicity conditions (2.1.28) and (2.1.29) implied the asymptotic formulas for the Bloch functions in the non-resonance domain. Similarly, here the simplicity conditions (2.6.6), (2.6.7) imply the asymptotic formulas for the Bloch functions in the resonance domain $V_\delta^{'}(\rho^{\alpha_1})$. For this we use the following lemma.

Lemma 2.6.1 *Let $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ be the eigenvalue of the operator $L_t(q)$ satisfying* (2.3.52)*, where*

$$\beta + \tau + (j + v)\delta =: \gamma + t \in B_\delta.$$

If for $\gamma' + t =: \beta' + \tau + (j' + v(\beta'))\delta$ at least one of the following conditions:

$$\gamma' \in M, \quad \beta' \neq \beta, \tag{2.6.8}$$

$$|\beta - \beta'| > (p-1)\rho^{\alpha}, \tag{2.6.9}$$

$$|\beta - \beta'| \le (p-1)\rho^{\alpha}, \quad |j'\delta| \ge h \tag{2.6.10}$$

holds, then

$$|b(N,\gamma')| \le c_5\rho^{-c\alpha}, \tag{2.6.11}$$

where

$$h =: 10^{-p}\rho^{\frac{1}{2}\alpha_2}, c = p - d\varkappa - \frac{1}{4}d3^d - 3, \ b(N,\gamma') = (\Psi_{N,t}, e^{i(\gamma'+t,x)}),$$

and $\Psi_{N,t}$ is any normalized eigenfunction of $L_t(q)$ corresponding to $\Lambda_N(\lambda_{j,\beta}(v,\tau))$.

Proof Repeating the proof of the inequality in (2.4.5) and instead of the simplicity conditions (2.1.28), (2.1.29) and the set K, using the simplicity conditions (2.6.6), (2.6.7), and the set M, we obtain the proof of (2.6.11) under the condition (2.6.8).

Suppose that the condition (2.6.9) holds. Consider two cases:

Case 1: $\gamma' \in M$. It follows from (2.6.9) that $\beta' \neq \beta$. Thus, in Case 1, condition (2.6.8) holds and hence (2.6.11) is true.

Case 2: $\gamma' \notin M$. The definition of M [see (2.6.7)] and (2.6.2) imply that

$$|\Lambda_N - |\gamma' + t|^2| > \frac{1}{4}\rho^{\alpha_1}, \quad \forall \gamma' \notin M. \tag{2.6.12}$$

Therefore using (2.1.9) and the definition of c [see (2.6.11)], we get

$$b(N,\gamma') = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} b(N,\gamma'-\gamma_1)}{\Lambda_N - |\gamma'+t|^2} + o(\rho^{-c\alpha}). \tag{2.6.13}$$

Since $\gamma_1 \in \Gamma(\rho^\alpha)$ we have $\gamma_1 = \beta_1 + a_1\delta$ [see (2.3.2)], where $\beta_1 \in \Gamma_\delta$, $a_1 \in \mathbb{R}$, $|\beta_1| < \rho^\alpha$ and

$$\gamma' - \gamma_1 + t =: (\beta' - \beta_1) + \tau + (j' + v(\beta') - a)\delta.$$

Moreover , it follows from (2.6.9) that $(\beta' - \beta_1) \neq \beta$. Therefore, if $\gamma' - \gamma_1 \in M$, then repeating the proof of (2.6.11) for Case 1, we obtain

$$|b(N,\gamma'-\gamma_1)| \le c_5\rho^{-c\alpha} \tag{2.6.14}$$

for $\gamma' - \gamma_1 \in M$. Now in (2.6.13) instead of $b(N,\gamma'-\gamma_1)$ for $\gamma'-\gamma_1 \in M$ writing $O(\rho^{-c\alpha})$, and using (2.1.9) for $b(N,\gamma'-\gamma_1)$ when $\gamma'-\gamma_1 \notin M$, we get

$$b(N,\gamma') = \sum_{\gamma_1,\gamma_2} \frac{q_{\gamma_1}q_{\gamma_2} b(N,\gamma'-\gamma_1-\gamma_2)}{(\Lambda_N - |\gamma'+t|^2)(\Lambda_N - |\gamma'-\gamma_1+t|^2)} + o(\rho^{-c\alpha}), \tag{2.6.15}$$

where the summation is taken under the conditions

$$\gamma_1 \in \Gamma(\rho^\alpha),\ \gamma_2 \in \Gamma(\rho^\alpha),\ \gamma'-\gamma_1 \notin M.$$

Moreover, it follows from (2.6.12) that the multiplicands in the denominators of (2.6.15) are the large numbers, namely

$$|\Lambda_N - |\gamma' - \sum_{i=1}^{j}\gamma_i + t|^2| > \frac{1}{4}\rho^{\alpha_1}, \tag{2.6.16}$$

for

$$\gamma' - \sum_{i=1}^{j}\gamma_i \notin M,$$

where $\gamma_i \in \Gamma(\rho^\alpha),\ j = 0,1,\ldots$. Arguing as in the proof of (2.6.14), we obtain

$$|b(N,\gamma'-\gamma_1-\gamma_2)| \le c_5\rho^{-c\alpha} \tag{2.6.17}$$

for $(\gamma'-\gamma_1-\gamma_2) \in M$. Repeating this process $p-1$ times, that is, in (2.6.15) instead of $b(N,\gamma'-\gamma_1-\gamma_2)$ for $\gamma'-\gamma_1-\gamma_2 \in M$ writing $O(\rho^{-c\alpha})$ [see (2.6.17)], and using (2.1.9) for $b(N,\gamma'-\gamma_1-\gamma_2)$ when $\gamma'-\gamma_1-\gamma_2 \notin M$ etc., we obtain

$$b(N,\gamma')=\sum_{\gamma_1,\gamma_2,\ldots,\gamma_{p-1}}\frac{q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_{p-1}}b(N,\gamma'-\sum_{i=1}^{p-1}\gamma_i)}{\prod_{j=0}^{p-2}(\Lambda_N-|\gamma'-\sum_{i=1}^{j}\gamma_i+t|^2)}+o(\rho^{-c\alpha}),\quad (2.6.18)$$

where the summation is taken under the conditions

$$\gamma'-\sum_{i=1}^{j}\gamma_i\notin M$$

for $j=0,1,\ldots,p-2$. Therefore (2.6.16) and (2.1.6a) imply (2.6.11) for Case 2.

Now assume that (2.6.10) holds. First we prove that the following implication

$$\gamma'-\sum_{i=1}^{s}\gamma_i\in M\Longrightarrow\beta'-\sum_{i=1}^{s}\beta_i\neq\beta,\quad (2.6.19)$$

where $s=0,1,\ldots,p-1$ and

$$\gamma_i\in\Gamma(\rho^{\alpha}),\ \gamma_i=\beta_i+a_i\delta,\ (\beta_i,\delta)=0,\ \beta_i\in\Gamma_\delta,\ a_i\in\mathbb{R}\quad (2.6.20)$$

[see (2.3.2) for this orthogonal decomposition of γ_i] is true. Assume the converse, i.e.,

$$\beta'-\sum_{i=1}^{s}\beta_i=\beta.$$

Then (2.6.20) and the equality $\gamma'+t=\beta'+\tau+(j'+v(\beta'))\delta$ (see Lemma 2.6.1) yield

$$\gamma'+t-\sum_{i=1}^{s}\gamma_i=\beta+\tau+(j'+v(\beta'))\delta-\sum_{i=1}^{s}a_i\delta.\quad (2.6.21)$$

Since $\gamma_i\in\Gamma(\rho^{\alpha}),\ \delta\in\Gamma(\rho^{\alpha})$, $v(\beta')\in[0,1]$ (see Lemma 2.3.1), and (2.6.20) is the orthogonal decomposition of γ_i we have

$$|a_i\delta|<\rho^{\alpha},\ |v(\beta')\delta|<\rho^{\alpha}.$$

On the other hand, by (2.6.10), $|j'\delta|\geq h$. Therefore the orthogonal decomposition (2.6.21) and the relations

$$h=10^{-p}\rho^{\frac{1}{2}\alpha_2},\ h^2\sim\rho^{\alpha_2},\ \alpha_2=3\alpha_1=9\alpha\quad (2.6.22)$$

imply that

$$|\gamma'+t-\sum_{i=1}^{s}\gamma_i|^2\geq|\beta+\tau|^2+\frac{1}{2}h^2.$$

Using this, (2.6.5), and (2.6.22) we obtain

$$|E(\lambda_{j,\beta}(v,\tau)) - |\gamma' + t - \sum_{i=1}^{s} \gamma_i|^2| > \rho^{\alpha_1}$$

which contradicts

$$\gamma' - \sum_{i=1}^{s} \gamma_i \in M.$$

Thus (2.6.19) is proved. This implication for $s = 0$ means that if $\gamma' \in M$ then $\beta' \neq \beta$. Therefore if (2.6.10) holds and $\gamma' \in M$, then (2.6.8) holds too and hence (2.6.11) holds. To prove (2.6.11) under condition (2.6.10) in case $\gamma' \notin M$ we repeat the proof of (2.6.11) in Case 2, that is, use (2.6.18), (2.6.12), and etc. □

Theorem 2.6.1 *If $\gamma + t = \beta + \tau + (j + v(\beta))\delta \in B_\delta$, then there exists a unique eigenvalue $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ satisfying (2.3.52). This is a simple eigenvalue and the corresponding eigenfunction $\Psi_{N,t}(x)$ satisfies the asymptotic formula*

$$\Psi_{N,t}(x) = \Phi_{j,\beta}(x) + O(\rho^{-\alpha_2} \ln \rho). \tag{2.6.23}$$

Proof The proof is similar to the proof of Theorem 2.4.1. Arguing as in the proof of the Theorem 2.4.1 we see that to prove this theorem it is enough to show that for any normalized eigenfunction Ψ_N corresponding to any eigenvalueΛ_N satisfying (2.3.52) the following equality holds

$$\sum_{(j',\beta')\in K_0} |b(N,j',\beta')|^2 = O(\rho^{-2\alpha_2}(\ln \rho)^2), \tag{2.6.24}$$

where

$$K_0 = \{(j',\beta') : j' \in \mathbb{Z}, \beta' \in \Gamma_\delta, (j',\beta') \neq (j,\beta)\},$$
$$b(N,j',\beta') = (\Psi_N, \Phi_{j',\beta'}).$$

Divide K_0 into subsets: K_1^c, $K_1 \cap S(p-1)$, $K_1 \cap S^c(p-1)$, where

$$K_1^c = K_0 \backslash K_1,\ \ S^c(n) = K_0 \backslash S(n),$$

$$K_1 = \{(j',\beta') \in K_0 : |\Lambda_N(t) - \lambda_{j',\beta'}| < h^2\},$$

$$S(n) = \{(j',\beta') \in K_0 : |\beta - \beta'| \le n\rho^{\alpha}, |j'\delta| < 10^n h\}$$

and h is defined in (2.6.22). If $(j',\beta') \in K_1^c$, then using (2.1.21), the definitions of K_1^c and h, we have

$$\sum_{(j',\beta')\in K_1^c} |b(N,j',\beta')|^2 = \sup_{x\in F} |q(x)-q^{\delta}(x)|^2 O(\frac{1}{\rho^{2\alpha_2}}) = O(\frac{1}{\rho^{2\alpha_2}}). \quad (2.6.25)$$

To consider the set $K_1 \cap S(p-1)$ we prove that

$$K_1 \cap S(n) = K_1 \cap \{(j',\beta) : j' \in \mathbb{Z}\} \subset \{(j',\beta) : |j'\delta| < 2h\} \quad (2.6.26)$$

for $n = 1, 2, \ldots, p-1$. Take any element (j', β) from $K_1 \cap \{(j',\beta) : j' \in \mathbb{Z}\}$. Since

$$\lambda_{j',\beta}(v,\tau) = |\beta+\tau|^2 + \mu_{j'}(v) = |\beta+\tau|^2 + |(j'+v)\delta|^2 + O(1),$$

where $v \in [0,1]$ [see Lemma 2.3.1(b) and (2.3.6)], using the definition of K_1, (2.6.2), (2.6.5) and (2.6.22), we obtain

$$|O(\rho^{2\alpha_1}) - |(j'+v)\delta|^2| < 2h^2, \quad |j'\delta| < 2h.$$

Hence the inclusion in (2.6.26) is proved and

$$K_1 \cap \{(j',\beta) : j' \in \mathbb{Z}\} \subset K_1 \cap S(n)$$

for $n = 1, 2, \ldots, p-1$. If the inclusion

$$K_1 \cap S(n) \subset K_1 \cap \{(j',\beta) : j' \in \mathbb{Z}\}$$

does not hold, then there is an element $(j^{'}, \beta^{'})$ of $K_1 \cap S(n)$ such that

$$0 < |\beta - \beta^{'}| \le n\rho^{\alpha} \le (p-1)\rho^{\alpha}, \ |j'\delta| < 10^n h < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$$

[see (2.6.22)]. Hence the pairs (j', β') and (j, β) satisfy the conditions of (2.3.34). Therefore using (2.3.34), (2.3.39) and (2.6.22) we get

$$|\Lambda_N - \lambda_{j',\beta'}| > \frac{1}{2}\rho^{\alpha_2} > h^2 \quad (2.6.27)$$

which contradicts the inclusion $(j', \beta') \in K_1$. Thus (2.6.26) is proved. Therefore

$$\sum_{(j',\beta')\in K_1\cap S(p-1)} |b(N,j',\beta')|^2 \le \sum_{j'\neq j,\ |j'\delta|<2h} |b(N,j',\beta)|^2 \quad (2.6.28)$$

For the estimation of $b(N, j', \beta)$ when $|j'\delta| < 2h$, we use (2.3.27) as follows. In (2.3.27) replacing β' and r by β and $2h$, we obtain

$$(\Lambda_N - \lambda_{j',\beta})b(N, j', \beta) = O(\rho^{-p\alpha})$$
$$+ \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,18h)} A(j', \beta, j' + j_1, \beta + \beta_1)b(N, j' + j_1, \beta + \beta_1). \quad (2.6.29)$$

By the definition of $Q(\rho^\alpha, 18h)$ we have $|\beta_1| < \rho^\alpha$, $|j_1\delta| < 18h$, and hence

$$|(j' + j_1)\delta| < 20h < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}.$$

Therefore in the right-hand side of (2.6.29) the multiplicand $b(N, j' + j_1, \beta + \beta_1)$ for $(j' + j_1, \beta + \beta_1) \in D(\beta)$, where

$$D(\beta) = \{(j, \beta + \beta_1) : |j\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}, \quad 0 < |\beta_1| < \rho^\alpha\},$$

takes part. Put

$$|b(N, j_0, \beta + \beta_0)| = \max_{(j,\beta+\beta_1)\in D(\beta)} |b(N, j, \beta + \beta_1)|.$$

By definition of $D(\beta)$ and by (2.6.22) we have

$$|\Lambda_N - \lambda_{j_0,\beta+\beta_0}| > \frac{1}{2}\rho^{\alpha_2}.$$

This together with (2.1.21) gives

$$|b(N, j_0, \beta + \beta_0)| = O(\rho^{-\alpha_2}).$$

Using this, (2.6.29) and (2.3.23), we get

$$|b(N, j', \beta)| < c_{24}|\Lambda_N - \lambda_{j',\beta}|^{-1}\rho^{-\alpha_2} \quad (2.6.30)$$

for $j' \neq j$, $|j'\delta| < 2h$, where

$$\Lambda_N - \lambda_{j',\beta} = \lambda_{j,\beta} - \lambda_{j',\beta} + O(\rho^{-\alpha_2}) = \mu_j(v) - \mu_{j'}(v) + O(\rho^{-\alpha_2})$$

[see (2.3.39) and Lemma 2.3.1(b)] and $v \in W(\rho)$ (see the definition of B_δ). Now using the definition of $W(\rho)$ (see Lemma 2.3.7) and (2.3.6) we obtain

$$\sum_{j'\neq j} |\Lambda_N - \lambda_{j',\beta}|^{-2} = O(\ln \rho).$$

This with (2.6.30) and (2.6.28) yield

$$\sum_{(j',\beta')\in K_1\cap S(p-1)} |b(N,j',\beta)|^2 = O(\rho^{-2\alpha_2}(\ln\rho)^2). \tag{2.6.31}$$

It remains to consider $K_1 \cap S^c(p-1)$. Let us prove that

$$b(N,j',\beta') = O(\rho^{-c\alpha}) \tag{2.6.32}$$

for $(j',\beta') \in K_1 \cap S^c(p-1)$, where the number c is defined in Lemma 2.6.1. For this using the decomposition of $\varphi_{j',v(\beta')}(s)$ by

$$\{e^{i(m+v(\beta'))s} : m \in \mathbb{Z}\},$$

we get

$$b(N,j',\beta') = \sum_m (\varphi_{j',v}(s), e^{i(m+v)s})(\Psi_{N,t}(x), e^{i(\beta'+\tau+(m+v)\delta,x)}). \tag{2.6.33}$$

If $|\beta-\beta'| > (p-1)\rho^{\alpha}$ then Lemma 2.6.1 [see (2.6.9)], (2.3.25), and (2.6.33) give the proof of (2.6.32). Thus we need to consider the case $|\beta-\beta'| \le (p-1)\rho^{\alpha}$. Then by the definition of $S^c(p-1)$ we have $|j'\delta| \ge 10^{p-1}h$. Write the right-hand side of (2.6.33) as $T_1 + T_2$, where

$$T_1 = \sum_{m:|m\delta|\ge h} T(m),\ T_2 \sum_{m:|m\delta|<h} T(m),$$
$$T(m) = (\varphi_{j',v}(s), e^{i(m+v)s})(\Psi_{N,t}(x), e^{i(\beta'+\tau+(m+v)\delta,x)}).$$

By (2.3.25) and Lemma 2.6.1 [see (2.6.10)] we have

$$T_1 = O(\rho^{-c\alpha}).$$

If $|m\delta| < h$, then the inequality $|j'| > 2|m|$ holds. Therefore using (2.3.10), taking into account that $|j'\delta| \sim \rho^{\alpha_2}$ [(see (2.6.22)] and that the number of summands in T_2 is less than ρ^{α_2}, we get $T_2 = O(\rho^{-c\alpha})$. The estimations for T_1, T_2 give (2.6.32). Now using

$$|K_1| = O(\rho^{(d-1)\varkappa\alpha}),$$

we get

$$\sum_{(j',\beta')\in K_1\cap S^c(p-1)} |b(N,j',\beta')|^2 = O(\rho^{-(2c-(d-1)\varkappa)\alpha}). \tag{2.6.34}$$

This, (2.6.25) and (2.6.31) give the proof of (2.6.24), since $(2c-(d-1)\varkappa)\alpha > \alpha_2$. $\square$

Now using Theorem 2.6.1, we obtain the asymptotic formulas of arbitrary order.

Theorem 2.6.2 *The eigenfunction* $\Psi_{N,t}(x)$, *defined in* Theorem 2.6.1*, satisfies the following asymptotic formulas*

$$\Psi_{N,t}(x) = E^*_{k-1}(x) + O(\rho^{-k\alpha_2} \ln \rho) \tag{2.6.35}$$

for $k = 1, 2, \ldots, n_1$, *where* $n_1 = [\frac{1}{9}(p - \varkappa(\frac{3d-1}{2}) - \frac{1}{4}d3^d - 3)]$,

$$E^*_0(x) = \Phi_{j,\beta}(x),\ \ E^*_k(x) = (1 + \|\widetilde{E}_k\|)^{-1}(\Phi_{j,\beta}(x) + \widetilde{E}_k(x)),$$

$\widetilde{E}_k$ *is obtained from* E_k *by replacing* $A(j, \beta, j + j_1, \beta + \beta_1)$ *with* $\Phi_{j+j_1,\beta+\beta_1}(x)$, *and* E_k *is defined in* Theorem 2.3.2*.*

Proof The proof of this theorem is very similar to the proof of Theorem 2.4.2. By Theorem 2.6.1, (2.6.35) for $k = 1$ was proved. To prove it for arbitrary k ($k \le n_1$) we prove the following equivalent formulas

$$\sum_{(j',\beta')\in S^c(k-1)} |b(N, j', \beta')|^2 = O(\rho^{-2k\alpha_2}(\ln \rho)^2), \tag{2.6.36}$$

$$\Psi_{N,t}(x) = \sum_{(j',\beta')\in S(k-1)\cup(j,\beta)} b(N, j', \beta')\Psi_{j',\beta'} + O(\rho^{-k\alpha_2} \ln \rho). \tag{2.6.37}$$

First consider the set $S^c(k-1) \cap K_1$. It follows from the relations

$$S(k-1) \cap K_1 = S(p-1) \cap K_1 \ \&\ S(k-1) \subset S(p-1)$$

for $0 < k < p$ [(see (2.6.26) and the definition of $S(k-1)$] that

$$(S(p-1))\backslash S(k-1)) \cap K_1 = \emptyset$$

and

$$S^c(k-1) = S^c(p-1) \cup (S(p-1)\backslash S(k-1)),\ S^c(k-1) \cap K_1 = S^c(p-1) \cap K_1.$$

Therefore using (2.6.34), the equalities $c = p - d\varkappa - \frac{1}{4}d3^d - 3$ (see Lemma 2.6.1), $\alpha_2 = 9\alpha$, and $n_1 = [\frac{1}{9}(p - \varkappa(\frac{3d-1}{2}) - \frac{1}{4}d3^d - 3)]$ (see Theorem 2.6.2), we obtain

$$\sum_{(j',\beta')\in S^c(k-1)\cap K_1} |b(N, j', \beta')|^2 = O(\rho^{-2n_1\alpha_2}).$$

Thus it remains to prove

$$\sum_{(j',\beta')\in S^c(k-1)\cap K_1^c} |b(N, j', \beta')|^2 = O(\rho^{-2k\alpha_2}(\ln\rho)^2) \tag{2.6.38}$$

for $k = 2, 3, \ldots, n_1$. By (2.3.22) and (2.6.35) we have

$$\Psi_N(x)(q(x) - Q(s)) = H(x) + O(\rho^{-\alpha_2}\ln\rho),$$

where $H(x)$ is a linear combination of $\Phi_{j,\beta}(x)$ and $\Phi_{j',\beta'}(x)$ for $(j', \beta') \in S(1)$, since $|j\delta| < r_1 < h$ [(see (2.3.5)]. Hence $H(x)$ is orthogonal to $\Phi_{j',\beta'}(x)$ for $(j', \beta') \in S^c(1)$. Therefore using (2.3.27) and the definition of K_1^c we have

$$\sum_{(j',\beta')\in S^c(1)\cap K_1^c} |b(N, j', \beta')|^2 = \sum \left|\frac{(O(\rho^{-\alpha_2}\ln\rho), \Phi_{j',\beta'})}{\Lambda_N - \lambda_{j',\beta'}}\right|^2$$
$$= O(\rho^{-4\alpha_2}(\ln\rho)^2).$$

Hence (2.6.38) for $k = 2$ is proved. Assume that this is true for $k = m$. Then (2.6.37) for $k = m$ holds too. This and (2.3.22) for $r = 10^{m-1}h$ give

$$\Psi_N(x)(q(x) - Q(s)) = G(x) + O(\rho^{-m\alpha_2}\ln\rho),$$

where G is a linear combination of $\Phi_{j,\beta}$ and $\Psi_{j',\beta'}$ for $(j', \beta') \in S(m)$. Thus G is orthogonal to $\Psi_{j',\beta'}$ for $(j', \beta') \in S^c(m)$. Using this and repeating the proof of (2.6.38) for $k = 2$ we obtain the proof of (2.6.38) for $k = m + 1$. Thus (2.6.36) and (2.6.37) are proved. Arguing as in the proof of Theorem 2.4.2 one can easily see that the formula (2.6.37) can be written in the form

$$\Psi_{N,t}(x) - b(N, j, \beta)\Psi_{j,\beta}(x) - \widetilde{G}_k(x)$$
$$= \sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r_1)} A(j, \beta, j + j_1, \beta + \beta_1)b(N, j + j_1, \beta + \beta_1), \tag{2.6.39}$$

where

$$\|\widetilde{G}_k\| = O(\rho^{-k\alpha_1}).$$

It is clear that the right-hand side of (2.6.39) can be obtained from the right-hand side of the equality

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) - O(\rho^{-p\alpha})$$
$$= \sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r_1)} A(j, \beta, j + j_1, \beta + \beta_1)b(N, j + j_1, \beta + \beta_1) \tag{2.6.40}$$

which is (2.3.28), by replacing $A(j, \beta, j + j_1, \beta + \beta_1)$ with $\Phi_{j,\beta,j+j_1,\beta+\beta_1}(x)$. Therefore in (2.6.39), doing the iteration which was done in order to obtain (2.3.49) from (2.3.28), we get

$$\Psi_{N,t}(x) - b(N, j, \beta)\Psi_{j,\beta}(x) - \widetilde{G}_k(x) = \sum_{k=1}^{2p_1} \widetilde{S'}_k(\Lambda_N, \lambda_{j,\beta}))b(N, j, \beta) + \widetilde{C'}_{2p_1} + O(\rho^{-p\alpha}), \tag{2.6.41}$$

where $\widetilde{S'}_k(\Lambda_N, \gamma+t)$ and $\widetilde{C'}_k$ are obtained from $S'(\Lambda_N, \gamma+t)$ and $C_k^{'}$ by replacing $A(j, \beta, j+j_1, \beta+\beta_1)$ with $\Phi_{j,\beta,j+j_1,\beta+\beta_1}(x)$ respectively and the term $O(\rho^{-p\alpha})$ in the right-hand side of (2.6.41) is a function whose norm is $O(\rho^{-p\alpha})$. The remaining part of the proof of this theorem is similar to the proof of Theorem 2.4.2 □

Now we consider the simple set B_δ in the resonance domain $V_\delta(\rho^{\alpha_1})$. As we noted in Remark 2.3.1 every vector w of $\mathbb{R}^d$ has the decomposition

$$w = \beta + \tau + (j + v)\delta,\ (\beta + \tau, \delta) = 0, \tag{2.6.42}$$

where $\beta \in \Gamma_\delta,\ \tau \in F_\delta,\ j \in \mathbb{Z},\ v \in [0, 1)$. Hence the space $\mathbb{R}^d$ is the union of the pairwise disjoint sets

$$P(\beta, j) =: \{\beta + \tau + (j + v)\delta : \tau \in F_\delta, v \in [0, 1)\}$$

for $\beta \in \Gamma_\delta,\ j \in \mathbb{Z}$. To prove that B_δ has an asymptotically full measure on $V_\delta(\rho^{\alpha_1})$, that is,

$$\lim_{\rho\to\infty} \frac{\mu(B_\delta)}{\mu(V_\delta(\rho^{\alpha_1}))} = 1 \tag{2.6.43}$$

we define the following sets:

$$R_1(\rho) = \{j \in \mathbb{Z} : |j| < \frac{\rho^{\alpha_1}}{2|\delta|^2} + \frac{3}{2}\},$$

$$S_1(\rho) = \{j \in \mathbb{Z} : |j| < \frac{\rho^{\alpha_1}}{2|\delta|^2} - \frac{3}{2}\},$$

$$R_2(\rho) = \{\beta \in \Gamma_\delta : \beta \in R_\delta(\frac{3}{2}\rho + d_\delta + 1)\backslash R_\delta(\frac{1}{2}\rho - d_\delta - 1))\},$$

$$S_2(\rho) = \{\beta \in \Gamma_\delta : \beta \in (R_\delta(\frac{3}{2}\rho - d_\delta - 1)\backslash R_\delta(\frac{1}{2}\rho + d_\delta + 1))\backslash(\bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} V_b^\delta(\rho^{\frac{1}{2}}))\},$$

where

$$R_\delta(\rho) = \{x \in H_\delta : |x| < \rho\}, \Gamma_\delta(\rho^{\alpha_d}) = \{b \in \Gamma_\delta : |b| < \rho^{\alpha_d}\},$$

$$V_b^\delta(\rho^{\frac{1}{2}}) = \{x \in H_\delta : ||x + b|^2 - |x|^2| < \rho^{\frac{1}{2}}\},$$

and
$$d_\delta = \sup_{x,y\in F_\delta} |x-y|$$

is the diameter of F_δ.

Moreover we define a subset $P'(\beta, j)$ of $P(\beta, j)$ as follows. Introduce the sets

$$A(\beta, b, \rho) = \{v \in [0,1) : \exists j \in \mathbb{Z}, |2(\beta, b) + |b|^2 + |(j+v)\delta|^2| < 4d_\delta \rho^{\alpha_d}\},$$
$$A(\beta, \rho) = \bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} A(\beta, b, \rho),\ S_3(\beta, \rho) = W(\rho)\backslash A(\beta, \rho)$$

and put
$$S_4(\beta, j, v, \rho) = \{\tau \in F_\delta : \beta + \tau + (j+v)\delta \in B_\delta\}$$

for $j \in S_1, \beta \in S_2, v \in S_3(\beta, j, \rho)$. Then define $P'(\beta, j)$ by

$$P'(\beta, j) = \{\beta + \tau + (j+v)\delta : v \in S_3(\beta, \rho), \tau \in S_4(\beta, j, v, \rho)\}.$$

It is not hard to see that (2.6.43) follows from the following relations:

$$\lim_{\rho\to\infty} \frac{|S_i(\rho)|}{|R_i(\rho)|} = 1, \quad \forall i = 1, 2, \tag{2.6.44}$$
$$B_\delta \supset \cup_{j\in S_1,\beta\in S_2} P'(\beta, j), \tag{2.6.45}$$
$$V_\delta(\rho^{\alpha_1}) \subset \cup_{j\in R_1,\beta\in R_2} P(\beta, j), \tag{2.6.46}$$
$$\lim_{\beta\to\infty} \frac{\mu(P'(\beta, j))}{\mu(P(\beta, j))} = 1. \tag{2.6.47}$$

To prove these relations we use the following lemma.

Lemma 2.6.2 *Let $w =: \beta + \tau + (j+v)\delta$. Then the following implications are true:*
(a) $w \in V_\delta(\rho^{\alpha_1}) \Rightarrow j \in R_1, \beta \in R_2$,
(b) $j \in S_1, \beta \in S_2 \Rightarrow w \in V_\delta(\rho^{\alpha_1}) \cap (R(\frac{3}{2}\rho - \rho^{\alpha_1 - 1})\backslash R(\frac{1}{2}\rho + \rho^{\alpha_1-1}))$,
(c) $j \in S_1, \beta \in S_2 \Rightarrow w \in V'_\delta(\rho^{\alpha_1}) \cap (R(\frac{3}{2}\rho - \rho^{\alpha_1 - 1})\backslash R(\frac{1}{2}\rho + \rho^{\alpha_1-1}))$.
Moreover (2.6.46), (2.6.45), *and* (2.6.44) *hold.*

Proof Since $(\beta + \tau, \delta) = 0$ [see (2.6.42)] the inclusion $\omega \in V_\delta(\rho^{\alpha_1})$ means that

$$||(j+v+1)\delta|^2 - |(j+v)\delta|^2| < \rho^{\alpha_1}$$

and
$$(\frac{1}{2}\rho)^2 < |\beta+\tau|^2 + |(j+v)\delta|^2 < (\frac{3}{2}\rho)^2$$

[see (2.1.10)], where $|v| < 1$, $|\tau| \le d_\delta = O(1)$ [see (2.6.42)]. Therefore by direct calculation we get the proofs of the implications (a) and (b).

Now we prove (c). Since (b) holds and

$$V_{\delta}^{'}(\rho^{\alpha_1}) = V_{\delta}^{'}(\rho^{\alpha_1})\backslash(\cup_{a\in\Gamma(p\rho^{\alpha})\backslash\delta\mathbb{R}} V_a(\rho^{\alpha_2})$$

(see Definition 2.1.1), it is enough to show that $w \notin V_a(\rho^{\alpha_2})$ for $a \in \Gamma(p\rho^{\alpha})\backslash\delta\mathbb{R}$. Using the orthogonal decomposition $a_1 + a_2\delta$ of $a \in \Gamma(p\rho^{\alpha})$ [see (2.3.2)], where $a_1 \in \Gamma_\delta$, $a_2 \in \mathbb{R}$, $\langle a_1, \delta\rangle = 0$ and $|a_1| < p\rho^{\alpha}$, $|a_2\delta| < p\rho^{\alpha}$, we obtain

$$|w + a|^2 - |w|^2 = d_1 + d_2,$$

where

$$d_1 = |\beta + a_1|^2 - |\beta|^2,$$

$$d_2 = |(j + a_2 + v)\delta|^2 - |(j + v)\delta|^2 + 2\langle a_1, \tau\rangle .$$

The requirements on j, a_1, and a_2 imply that

$$d_2 = O(\rho^{2\alpha_1}).$$

On the other hand the condition $\beta \in S_2$ gives $\beta \notin V_a^{\delta}(\rho^{\frac{1}{2}})$, i.e., $|d_1| \geq \rho^{\frac{1}{2}}$. Since $2\alpha_k < \frac{1}{2}$ for $k = 1, 2$ [see the equality in (2.6.1)], we have

$$||w + a|^2 - |w|^2| > \frac{1}{2}\rho^{\frac{1}{2}}, w \notin V_a(\rho^{\alpha_2}).$$

Thus (c) is proved.

The inclusion (2.6.46) follows from the implication (a).

If $w = \beta + \tau + (j + v)\delta$ belongs to the right-hand side of (2.6.45) then using the implication (c) we obtain $w \in V_{\delta}^{'}(\rho^{\alpha_1})$. Therefore (2.6.45) follows from the definitions of $P'(\beta, j)$ and $S_4(\beta, j, v, \rho)$. It remains to prove (2.6.44). Using the definitions of R_1, S_1 and inequalities $|\delta| < \rho^{\alpha}$, $\alpha_1 > 2\alpha$ we obtain that (2.6.44) for $i = 1$ holds.

Now we prove (2.6.44) for $i = 2$. If $\beta \in R_2$ then

$$\beta + F_\delta \subset R_\delta(\frac{3}{2}\rho + 2d_\delta + 1)\backslash R_\delta(\frac{1}{2}\rho - 2d_\delta - 1).$$

This implies that,

$$|R_2| < (\mu(F_\delta))^{-1}\mu(R_\delta(\frac{3}{2}\rho + 2d_\delta + 1)\backslash R_\delta(\frac{1}{2}\rho - 2d_\delta - 1)),$$

since the translations $\beta + F_\delta$ of F_δ for $\beta \in \Gamma_\delta$, are the pairwise disjoint sets having measure $\mu(F_\delta)$. Suppose $\beta + \tau \in D(\rho)$, where

$$D(\rho) = (R_\delta(\frac{3}{2}\rho - 1)\backslash R_\delta(\frac{1}{2}\rho + 1))\backslash(\bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} V_b^\delta(2\rho^{\frac{1}{2}})).$$

Then

$$\frac{3}{2}\rho - 1 < |\beta + \tau| < \frac{1}{2}\rho + 1$$

and

$$||\beta + \tau + b|^2 - |\beta + \tau|^2| \geq 2\rho^{\frac{1}{2}}$$

for $b \in \Gamma_\delta(\rho^{\alpha_d})$. Therefore using $|\tau| \leq d_\delta$ it is not hard to verify that $\beta \in S_2$. Hence the sets $\beta + F_\delta$ for $\beta \in S_2$ is a cover of $D(\rho)$. Thus

$$|S_2| \geq (\mu(F_\delta))^{-1}\mu(D(\rho).$$

This, the estimation for $|R_2|$, and the obvious relations

$$|\Gamma_\delta(\rho^{\alpha_d})| = O(\rho^{(d-1)\alpha_d}),$$
$$\mu((R_\delta(\frac{3}{2}\rho - 1)\backslash R_\delta(\frac{1}{2}\rho + 1))) = O(\rho^{d-1}),$$
$$\mu((R_\delta(\frac{3}{2}\rho - 1)\backslash R_\delta(\frac{1}{2}\rho + 1)) \cap V_b^\delta(2\rho^{\frac{1}{2}})) = O(\rho^{d-2}\rho^{\frac{1}{2}}),$$

$(d-1)\alpha_d < \frac{1}{2}$ [see the equality in (2.6.1)],

$$\lim_{\rho\to\infty} \frac{\mu((R_\delta(\frac{3}{2}\rho - 1)\backslash R_\delta(\frac{1}{2}\rho + 1)))}{\mu(R(\frac{3}{2}\rho + 2d_\delta + 1)\backslash R_\delta(\frac{1}{2}\rho - 2d_\delta - 1))} = 1,$$

and

$$S_2(\rho) \subset R_2(\rho)$$

imply (2.6.44) for $i = 2$ □

Theorem 2.6.3 *The simple set B_δ has an asymptotically full measure in the resonance set $V_\delta(\rho^{\alpha_1})$ in the sense that (2.6.43) holds.*

Proof The proof of the theorem follows from (2.6.44)–(2.6.47). By Lemma 2.6.2 we need to prove (2.6.47). Since the translations $P(\beta, j) - \beta - j\delta$ and $P'(\beta, j) - \beta - j\delta$ of $P(\beta, j)$ and $P'(\beta, j)$ are

$$\{\tau + v\delta : v \in [0, 1), \tau \in F_\delta\}$$

and

$$\{\tau + v\delta : v \in S_3(\beta, \rho), \tau \in S_4(\beta, j, v, \rho)\}$$

respectively, it is enough to prove

$$\lim_{\rho\to\infty} \mu(S_3(\beta,\rho)) = 1,\ \mu(S_4(\beta, j, v, \rho)) = \mu(F_\delta)(1 + O(\rho^{-\alpha})), \tag{2.6.48}$$

where $j \in S_1, \beta \in S_2, v \in S_3(\beta, \rho)$, and $O(\rho^{-\alpha})$ does not depend on v. To prove the first equality in (2.6.48) it is enough to show that

$$\mu(A(\beta, \rho)) = O(\rho^{-\alpha}), \tag{2.6.49}$$

since $W(\rho) \supset A(\varepsilon(\rho))$ and

$$\lim_{\rho\to\infty} \mu(A(\varepsilon(\rho)) = 1$$

(see Lemma 2.3.7). Using the definition of $A(\beta, \rho)$ and the obvious relation

$$|\Gamma_\delta(\rho^{\alpha_d})| = O(\rho^{(d-1)\alpha_d})$$

we see that (2.6.49) holds if

$$\mu(A(\beta, b, \rho)) = O(\rho^{-d\alpha_d}).$$

In other words, we need to prove that

$$\mu\{s \in \mathbb{R} : |f(s)| < 4d_\delta \rho^{\alpha_d}\} = O(\rho^{-d\alpha_d}), \tag{2.6.50}$$

where

$$f(s) = 2\langle \beta, b\rangle + |b|^2 + s^2|\delta|^2, \quad \beta \in S_2,\ b \in \Gamma_\delta(\rho^{\alpha_d}).$$

The last inclusions yield

$$|2\langle \beta, b\rangle + |b|^2| \geq \rho^{\frac{1}{2}}$$

for $|b| < \rho^{\alpha_d}$. This and the inequalities

$$|f(s)| < 4d_\delta \rho^{\alpha_d}$$

[see (2.6.50)] and $\alpha_d < \frac{1}{2}$ [see the equality in (2.6.1)] imply that

$$s^2|\delta|^2 > \frac{1}{2}\rho^{\frac{1}{2}}$$

from which we obtain

$$|f'(s)| > |\delta|\rho^{\frac{1}{4}}.$$

Therefore (2.6.50) follows from the equality in (2.6.1). Thus (2.6.49) and hence the first equality in (2.6.48) are proved.

Now we prove the second equality in (2.6.48). For this we consider the set $S_4(\beta, j, v, \rho)$ for $j \in S_1,\ \beta \in S_2,\ v \in S_3(\beta, \rho)$. By the definitions of S_4 and

B_δ the set $S_4(\beta, j, v, \rho)$ is the set of $\tau \in F_\delta$ such that $E(\lambda_{j,\beta}(v,\tau))$ satisfies the conditions (2.6.6) and (2.6.7). Thus, we need to consider these conditions. For this we use the decompositions

$$\gamma + t = \beta + \tau + (j+v)\delta, \ \gamma' + t = \beta' + \tau + (j' + v(\beta', t))\delta,$$

(see Remark 2.3.1) and the notations

$$\lambda_{j,\beta}(v,\tau) = \mu_j(v) + |\beta+\tau|^2, \ \lambda_i(\gamma'+t) = |\gamma'+t|^2 + r_i(\gamma'+t)$$

(see Lemma 2.3.1(b) and Remark 2.2.2). Denoting by b the vector $\beta' - \beta$, we write the decomposition of $\gamma' + t$ in the form

$$\gamma' + t = \beta + b + \tau + (j' + v(\beta+b, t))\delta.$$

Then to every $\gamma' \in \Gamma$ there corresponds $b = b(\gamma') \in \Gamma_\delta$. For $\gamma' \in M_1$ denote by $B^1(\beta, b(\gamma'), j, v)$ the set of all τ not satisfying (2.6.6). For $\gamma' \in M_2$ denote by $B^2(\beta, b(\gamma'), j, v)$ the set of all τ not satisfying (2.6.7), where M_1 and M_2 are defined in (2.6.6) and (2.6.7). Clearly, if

$$\tau \in F_\delta \backslash (\cup_{s=1,2}(\cup_{\gamma' \in M_s}(B^s(\beta, b(\gamma'), j, v))$$

then the inequalities (2.6.6) and (2.6.7) hold, that is, $\tau \in S_4(\beta, j, v, \rho)$. Therefore using $\mu(F_\delta) \sim 1$ and proving that

$$\mu(\cup_{\gamma' \in M_s} B^s(\beta, b(\gamma'), j, v)) = O(\rho^{-\alpha}), \ \forall s = 1, 2, \tag{2.6.51}$$

we get the proof of the second equality in (2.6.48). Now we prove (2.6.51). Using the above notations and the notations of (2.6.6) and (2.6.7) it is not hard to verify that if $\tau \in B^s(\beta, b(\gamma'), j, v)$, then

$$|2\langle \beta, b\rangle + |b|^2 + |(j' + v(\beta+b))\delta|^2 + 2\langle b, \tau\rangle - \mu_j(v) + h_s(\gamma'+t)| < 2\varepsilon_1, \tag{2.6.52}$$

where

$$h_1 = F_{k_1 - 1} - E_{k_2 - 1}, h_2 = r_i - E_{k_2 - 1}, \gamma' \in M_s,$$

and $s = 1, 2$. First we prove that if $b =: b(\gamma') \in \Gamma_\delta(\rho^{\alpha_d})$, then (2.6.52) does not hold. The assumption $v \in S_3(\beta, \rho)$ implies that $v \notin A(\beta, \rho)$. This means that

$$|2\langle \beta, b\rangle + |b|^2 + |(j' + v(\beta+b))\delta|^2| \ge 4 d_\delta \rho^{\alpha_d}.$$

Therefore if

$$|2\langle b, \tau\rangle - \mu_j(v) + h_s(\gamma'+t)| < 3 d_\delta \rho^{\alpha_d}, \tag{2.6.53}$$

then (2.6.52) does not hold. Thus to prove that (2.6.52) does not hold it is enough to show that (2.6.53) holds. Now we prove (2.6.53). The relations

$$b \in \Gamma_\delta(\rho^{\alpha_d}) \ \& \ \tau \in F_\delta$$

imply that

$$|2\langle b, \tau\rangle| < 2d_\delta \rho^{\alpha_d}.$$

The inclusion $j \in S_1$ and (2.3.6) imply that

$$\mu_j(v) = O(\rho^{2\alpha_1}).$$

By (2.2.8) and (2.3.53),

$$h_1 = O(\rho^{\alpha_1}).$$

Since $\alpha_d = 3^d\alpha = 3^{d-1}\alpha_1$, (2.6.53) for $s = 1$ is proved. Now we prove the equality

$$r_i = O(\rho^{\alpha_1})$$

which implies that

$$|h_2| = O(\rho^{\alpha_1})$$

and hence ends the proof of (2.6.53). The inclusion

$$\tau \in B^2(\beta, b(\gamma'), j, v)$$

means that (2.6.7) does not hold, that is,

$$|E(\lambda_{j,\beta}(v, \tau)) - \lambda_i(\gamma' + t)| < 2\varepsilon_1.$$

On the other hand, the inclusion $\gamma' \in M_2$ implies that $\gamma' \in M$ (see the definitions of M_2, and M) and hence

$$|E(\lambda_{j,\beta}(v, \tau)) - |\gamma' + t|^2| \leq \frac{1}{3}\rho^{\alpha_1}$$

The last two inequalities imply that

$$r_i(\gamma' + t) = O(\rho^{\alpha_1}).$$

Thus (2.6.53) is proved. Hence (2.6.52) for $b \in \Gamma_\delta(\rho^{\alpha_d})$ does not hold. It means that the sets $B^1(\beta, b, j, v)$ and $B^2(\beta, b, j, v)$ for $|b| < \rho^{\alpha_d}$ are empty.

To estimate the measure of the set

$$B^s(\beta, b(\gamma'), j, v)$$

for

$$\gamma' \in M_s, \quad |b(\gamma')| \geq \rho^{\alpha_d}, \ b \in \Gamma_\delta$$

we choose the coordinate axis so that the direction of b coincides with the direction of $(1, 0, 0, \ldots, 0)$, i.e., $b = (b_1, 0, 0, \ldots, 0)$, $b_1 > 0$ and the direction of δ coincides with the direction of $(0, 0, \ldots, 0, 1)$. Then H_δ and $B^s(\beta, b, j, v)$ can be considered as $\mathbb{R}^{d-1}$ and as a subset of F_δ respectively, where $F_\delta \subset \mathbb{R}^{d-1}$. Now let us estimate the measure of $B^s(\beta, b, j, v)$ by using (2.5.22) for

$D = B^s(\beta, b, j, v)$, $m = d - 1$, and $k = 1$. For this we prove that

$$\mu((B^s(\beta, b, j, v))(\tau_2, \tau_3, \ldots, \tau_{d-1})) < 4\varepsilon_1 |b|^{-1}, \tag{2.6.54}$$

for all fixed $(\tau_2, \tau_3, \ldots, \tau_{d-1})$. Assume the converse. Then there are two points

$$\tau = (\tau_1, \tau_2, \tau_3, \ldots, \tau_{d-1}) \in F_\delta \ \& \ \tau' = (\tau_1', \tau_2, \tau_3, \ldots, \tau_{d-1}) \in F_\delta$$

of $B^s(\beta, b, j, v)$, such that

$$|\tau_1 - \tau_1'| \geq 4\varepsilon_1 |b|^{-1}. \tag{2.6.55}$$

Since (2.6.52) holds for τ' and τ we have

$$|2b_1(\tau_1 - \tau_1') + g_s(\tau) - g_s(\tau')| < 4\varepsilon_1, \tag{2.6.56}$$

where

$$g_s(\tau) = h_s(\beta' + \tau + (j' + v(\beta + b))\delta).$$

Using (2.2.34), (2.2.36), (2.3.54), and the inequality $|b| \geq \rho^{\alpha_d}$, we obtain

$$|g_1(\tau) - g_1(\tau')| < \rho^{-\alpha_1} |\tau_1 - \tau_1'| < b_1 |\tau_1 - \tau_1'|, \tag{2.6.57}$$

$$|g_2(\tau) - g_2(\tau')| < 3\rho^{\frac{1}{2}\alpha_d} |\tau_1 - \tau_1'| < b_1 |\tau_1 - \tau_1'|. \tag{2.6.58}$$

These inequalities and (2.6.56) imply that

$$b_1 |\tau_1 - \tau_1'| < 4\varepsilon_1$$

which contradicts (2.6.55). Hence (2.6.54) is proved. Since $B^s(\beta, b, j, v) \subset F_\delta$ and $d_\delta = O(1)$, we have

$$\mu(\mathrm{Pr}_1 B^s(\beta, b, j, v)) = O(1).$$

Therefore formula (2.5.22), the inequalities (2.6.54) and $|b| \geq \rho^{\alpha_d}$ yield

$$\mu((B^s(\beta, b(\gamma'), j, v) = O(\varepsilon_1 |b(\gamma')|^{-1}) = O(\rho^{-\alpha_d}\varepsilon_1)$$

for $\gamma' \in M_s \subset M$ and $s = 1, 2$. This implies (2.6.51), since

$$|M| = O(\rho^{d-1}), \varepsilon_1 = \rho^{-d-2\alpha}, O(\rho^{d-1-\alpha_d}\varepsilon_1) = O(\rho^{-\alpha}).$$

□

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Chapter 3
Constructive Determination of the Spectral Invariants

Abstract This chapter describes the constructive determination of the spectral invariants explicitly expressed with respect to the Fourier coefficients of the potential by using the Bloch eigenvalues as input data. At the same time, it gives a rich set of invariants that is enough to determine the potential q. This chapter consists of five sections. First section is the introduction and preliminary facts where we discuss the related papers, describe briefly the scheme of this chapter and recall the results of Chap. 2 which are used essentially in this chapter. In Sect. 3.2, we develop the asymptotic formulas obtained in Chap. 2 and write the first and second term of the asymptotic formulas for the the Bloch eigenvalues in the explicit form. In Sect. 3.3, we investigate the derivatives of the band functions Λ_n with respect to the quasimomentum. In Sect. 3.4, using the results of the previous sections, we determine constructively a family of spectral invariants of this operator from the given Bloch eigenvalues. Some of these invariants generalize the well-known invariants and others are entirely new. The new invariants are explicitly expressed by Fourier coefficients of the potential which present the possibility of determining the potential constructively by using the Bloch eigenvalues as input data in the next chapter. Final section of this chapter is the Appendix, where we give some estimations and calculations of previous sections.

3.1 Introduction and Preliminary Facts

The main purpose of this chapter is the constructive determination of a family of spectral invariants of the Schrödinger operator $L(q) = -\Delta + q$ in $L_2(\mathbb{R}^d)$, $d \geq 2$, with a real periodic, relative to a lattice Ω in $\mathbb{R}^d$, potential q satisfying the smoothness condition

$$q \in W_2^s(F) \quad \& \quad s \geq 6(3^d(d+1)^2) + d$$

from the given Bloch eigenvalues $\Lambda_n(t)$ for large values of n and for the values of quasimomentum t lying near the diffraction hyperplanes.

To list the main results, we use Notation 2.1.1 of Chap. 2 (see introduction of Chap. 2). Denote by $M(\Gamma)$ and $M(\Gamma_\delta)$, the set of all visible points of the lattices Γ and

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Γ_δ, respectively. The spectral invariants are expressed by the Bloch eigenvalues and the Bloch functions of the Schrödinger operator $L(q^\delta)$ in $L_2(\mathbb{R}^d)$ with the directional potential $q^\delta(x)$ defined in (2.1.19) of Chap. 2 which is the restriction of the original potential q to the linear span of $\{e^{in\langle\delta,x\rangle} : n \in \mathbb{Z}\}$. The function q^δ depends only on one variable $\zeta = \langle\delta, x\rangle$ and can be written as

$$q^\delta(x) = Q(\langle\delta, x\rangle),\ Q(\zeta) = \sum_{n\in\mathbb{Z}} q_{n\delta} e^{in\zeta}.$$

The Bloch eigenvalues and the Bloch functions of the operator $L(q^\delta)$ are expressed by eigenvalues $\mu_j(v)$ and eigenfunctions $\varphi_{j,v}(\zeta)$ of the Sturm-Liouville operator $T_v(Q)$ defined in Lemma 2.3.1 of Chap. 2.

In the pioneering paper [EsRaTr1] about isospectral potentials, it was proven that if $q \in C^6(F)$, $\omega \in \Omega\backslash 0$, and δ is the visible point of Γ satisfying $\langle\delta, \omega\rangle = 0$, then given Bloch eigenvalues one may recover the eigenvalues of $T_v(Q)$ for $v = 0, \frac{1}{2}$ and the invariants $I(\omega, \delta, j, v)$ for $j \in \mathbb{Z}$, $v = 0, \frac{1}{2}$, where

$$I(\omega, \delta, j, v) = \int_F |Q_\omega(x)\varphi_{j,v}(\langle x, \delta\rangle)|^2 dx \tag{3.1.1}$$

if $\mu_j(v)$ is a simple eigenvalue,

$$I(\omega, \delta, j, v) = \int_F |Q_\omega(x)|^2((\varphi_{j+1,v}(\langle x, \delta\rangle))^2 + (\varphi_{j,v}(\langle x, \delta\rangle))^2)dx \tag{3.1.2}$$

if $\mu_j(v)$ is not a simple eigenvalue, namely if $\mu_j(v) = \mu_{j+1}(v)$, and $Q_\omega(x)$ is defined by

$$Q_\omega(x) = \sum_{\gamma:\gamma\in\Gamma, \langle\gamma,\omega\rangle\neq 0} \frac{\gamma}{\langle\omega, \gamma\rangle} q_\gamma e^{i\langle\gamma,x\rangle}. \tag{3.1.3}$$

The proofs given in [EsRaTr1] were nonconstructive. In [FeKnTr2], it was given a constructive way of determining the spectrum of $L_t(q^\delta)$ from the spectrum of $L_t(q)$ for the case $d = 2$.

In this chapter, we consider the Schrödinger operator $L(q)$ for arbitrary dimension d and using the given Bloch eigenvalues as input date, we constructively determine all eigenvalues of $T_v(Q)$ for all values of $v \in [0, 1)$ and a family of new spectral invariants

$$J(\delta, b, j, v) = \int_F |q_{\delta,b}(x)\varphi_{j,v}(\langle\delta, x\rangle)|^2 dx \tag{3.1.4}$$

for $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $j \in \mathbb{Z}$, and for all visible elements b and δ of Γ_δ and Γ respectively, where

$$q_{\delta,b}(x) = \sum_{\gamma\in S(\delta,b)\backslash\delta\mathbb{R}} \frac{\gamma}{\langle b, \gamma\rangle} q_\gamma e^{i\langle\gamma,x\rangle}, \tag{3.1.5}$$

$S(\delta, b) = P(\delta, b) \cap \Gamma$, and $P(\delta, b)$ is the plane containing δ, b and 0. The results of this chapter were published in [Ve10]. The formula (3.1.3) contains all Fourier coefficients q_γ of q except the Fourier coefficients corresponding to the vectors of a hyperplane. However, (3.1.5) contains only the Fourier coefficients corresponding to the vectors of the plane $P(\delta, b)$ except the vectors of $\delta\mathbb{R}$. If the potential q is a trigonometric polynomial and $d > 2$, then most of the polynomials (3.1.5) contain either 2 nonzero Fourier coefficients q_γ and $q_{-\gamma}$, where

$$q_{-\gamma} = \overline{q_\gamma},$$

or 4 nonzero Fourier coefficients q_{γ_1}, q_{γ_2}, $q_{-\gamma_1}$, $q_{-\gamma_2}$, or 6 nonzero Fourier coefficients q_{γ_1}, q_{γ_2}, q_{γ_3}, $q_{-\gamma_1}$, $q_{-\gamma_2}$, $q_{-\gamma_3}$. Moreover $\mu_n(v)$, for $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $j \in \mathbb{Z}$, is a simple eigenvalue and the corresponding eigenfunction $\varphi_{n,v}(\zeta)$ has a simple asymptotic decomposition. Therefore, substituting the asymptotic decomposition

$$\left|\varphi_{n,v}(\zeta)\right|^2 = A_0 + \frac{A_1(\zeta)}{n} + \frac{A_2(\zeta)}{n^2} + \cdots, \tag{3.1.6}$$

where $A_k(\zeta)$ is expressed via $Q(\zeta)$, into (3.1.4) we find the new invariants

$$J_k(\delta, b) = \int_F |q_{\delta,b}(x)|^2 A_k(\langle \delta, x \rangle) dx \tag{3.1.7}$$

for $k = 0, 1, 2, \ldots, \delta \in M(\Gamma), b \in M(\Gamma_\delta)$. Note that $J_k(\delta, b)$ is explicitly expressed by the Fourier coefficients of q. Moreover, if $d > 2$ and q is a trigonometric polynomial, then, in general, the number of the nonzero invariants (3.1.7) is greater than the number of nonzero Fourier coefficients of q and most of these invariants are explicitly expressed by m Fourier coefficients of q, where $m \leq 3$. This situation allows us to give (it will be given in the next chapter) an algorithm for finding the potential q from these spectral invariants.

Let us describe the brief scheme of the constructive determination of these invariants. We use the asymptotic formulas for the Bloch eigenvalues and Bloch function obtained in Chap. 2. First, by improving the asymptotic formulas for the Bloch eigenvalues and Bloch functions, in the high energy region and near diffraction hyperplanes, obtained in Chap. 2, we get the asymptotic formulas, where there are sharp estimations for the first and second terms of the asymptotic decomposition. To describe this improvement, let us introduce the following notations. The eigenvalues of the operator $L_t(0)$ with zero potential are $|\gamma + t|^2$ for $\gamma \in \Gamma$. If the quasimomentum $\gamma + t$ lies near the diffraction hyperplane D_δ, then the corresponding eigenvalue of $L_t(q)$ is close to the eigenvalue of the operator $L_t(q^\delta)$ with directional potential. To describe the eigenvalue of $L_t(q^\delta)$, we consider the lattice Γ_δ defined in Notation 2.1.1 (see introduction of Chap. 2). Let $F_\delta =: H_\delta / \Gamma_\delta$ be the fundamental domain of Γ_δ. In this notation, the quasimomentum $\gamma + t$ has an orthogonal decomposition

$$\gamma + t = \beta + \tau + (j + v)\delta, \tag{3.1.8}$$

where

$$\beta \in \Gamma_\delta \subset H_\delta, \tau \in F_\delta \subset H_\delta, j \in \mathbb{Z}, v \in [0,1),$$

and v depends on β and t. The eigenvalues and eigenfunctions of the operator $L_t(q^\delta)$ are

$$\lambda_{j,\beta}(v,\tau) = |\beta+\tau|^2 + \mu_j(v) \text{ and } \Phi_{j,\beta}(x) = e^{i\langle \beta+\tau, x\rangle}\varphi_{j,v}(\zeta),$$

respectively for $j \in \mathbb{Z}$ and $\beta \in \Gamma_\delta$. We say that the large quasimomentum (3.1.8) lies near the diffraction hyperplane $D_\delta = \{x \in \mathbb{R}^d : |x|^2 = |x+\delta|^2\}$ if

$$\frac{1}{2}\rho < |\beta| < \frac{3}{2}\rho,\ j = O(\rho^{\alpha_1}),\ \alpha_k = 3^k\alpha,\ \alpha = \frac{1}{4(3^d(d+1))}, \tag{3.1.9}$$

where ρ is a large parameter and $k = 1, 2, \ldots, d$. In this chapter we construct a set of the quasimomentum near the diffraction plane D_δ such that if $\beta + \tau + (j+v)\delta$ belongs to this set, then there exists a unique eigenvalue, denoted by $\Lambda_{j,\beta}(v,\tau)$, of $L_t(q)$ satisfying

$$\Lambda_{j,\beta}(v,\tau) = \lambda_{j,\beta}(v,\tau) + O(\rho^{-a}), \tag{3.1.10}$$

$$\Lambda_{j,\beta}(v,\tau) = \lambda_{j,\beta}(v,\tau) + \frac{1}{4}\int_F |f^2_{\delta,\beta+\tau}|\left|\varphi_{j,v}\right|^2 dx + O(\rho^{-3a+2\alpha_1}\ln\rho), \tag{3.1.11}$$

where $a = 1 - \alpha_d + \alpha$ and

$$f_{\delta,\beta+\tau}(x) = \sum_{\gamma:\gamma\in\Gamma\backslash\delta\mathbb{R},|\gamma|<\rho^\alpha} \frac{\gamma}{\langle \beta+\tau,\gamma\rangle} q_\gamma e^{i\langle\gamma,x\rangle}. \tag{3.1.12}$$

This is a simple eigenvalue and the corresponding eigenfunction $\Psi_{j,\beta}(x)$ satisfies

$$\Psi_{j,\beta}(x) = \Phi_{j,\beta}(x) + O(\rho^{-a}). \tag{3.1.13}$$

The remainders of the formulas (3.1.10), (3.1.11), (3.1.13) are $O(\rho^{-a})$, $O(\rho^{-3a+2\alpha_1}\ln\rho)$, $O(\rho^{-a})$ respectively, while the remainders of the corresponding formulas, obtained in Chap. 2, are $O(\rho^{-\alpha_2})$, $O(\rho^{-2\alpha_2}(\ln\rho)^4)$, $O(\rho^{-\alpha_2}\ln\rho)$ [see (2.3.39), (2.3.52), (2.6.23) of Chap. 2], where

$$a > 1 - \frac{1}{4(d+1)},\ -3a + 2\alpha_1 < -2,$$

but α_2 is a small number [see (3.1.9)]. Moreover, the second term of (3.1.11) has an explicit and a suitable form for the constructive determination of new invariants. Besides, we prove that the derivative of $\Lambda_{j,\beta}(v,\tau)$ in the direction of $h = \frac{\beta+\tau}{|\beta+\tau|}$ satisfies

$$|\beta+\tau|\frac{\partial\Lambda_{j,\beta}(v,\tau)}{\partial h} = |\beta+\tau|^2 + O(\rho^{2-2a}) \tag{3.1.14}$$

and the derivatives of the other simple eigenvalues, neighboring with $\Lambda_{j,\beta}(v,\tau)$, do not satisfy (3.1.14). Using these formulas, we constructively determine the eigenvalues of T_v for $v \in [0,1)$ and the invariants (3.1.4), (3.1.7). Then, using the asymptotic formulas for the eigenvalues and the eigenfunctions of T_v, we find $A_0(\zeta)$, $A_1(\zeta)$, $A_2(\zeta)$ [see (3.1.7)] and the invariants

$$\int_F \left|q_{\delta,b}(x)\right|^2 q^{\delta}(x)dx, \tag{3.1.15}$$

$$\int_F \left|q^{\delta}(x)\right|^2 dx \tag{3.1.16}$$

(see Appendix 4). If the potential q is a trigonometric polynomial, then most of the directional potentials have the form

$$q^{\delta}(x) = q_{\delta}e^{i\langle\delta,x\rangle} + q_{-\delta}e^{-i\langle\delta,x\rangle}. \tag{3.1.17}$$

In this case, by direct calculations, we show that

$$\begin{aligned} &A_0 = 1, A_1 = 0, A_2 = \frac{q^{\delta}(x)}{2} + a_1\left|q_{\delta}\right|^2, A_3 = a_2 q^{\delta}(x) + a_3\left|q_{\delta}\right|^2, \\ &A_4 = a_4 q^{\delta}(x) + a_5(q_{\delta}^2 e^{i2\langle\delta,x\rangle} + q_{-\delta}^2 e^{-i2\langle\delta,x\rangle}) + a_6, \end{aligned} \tag{3.1.18}$$

where $a_1, a_2, \ldots, a_6$ are the known constants (see Appendix 4). Moreover using (3.1.18), (3.1.16), and (3.1.7) for $k = 2, 4$, we find the invariant

$$\int_F |q_{\delta,b}(x)|^2(q_{\delta}^2 e^{i2\langle\delta,x\rangle} + q_{-\delta}^2 e^{-i2\langle\delta,x\rangle})dx \tag{3.1.19}$$

in the case (3.1.17). In the next chapter, we give an algorithm for finding the potential q by the invariants (3.1.15), (3.1.16), and (3.1.19).

3.2 First and Second Terms of the Asymptotics

First let us describe some results of Chap. 2 that we use in this chapter. In Chap. 2 (see Sect. 2.6) we constructed a set B_{δ}, which is called a simple set near the diffraction plane D_{δ}, such that if the quasimomentum $\gamma + t = \beta + \tau + (j+v)\delta$ [see (3.1.8)] belongs to the simple set B_{δ}, then there exists a unique eigenvalue Λ_N of $L_t(q)$ which is simple and satisfies

$$|\Lambda_N - E(\lambda_{j,\beta}(v,\tau))| < \varepsilon_1 \tag{3.2.1}$$

(see Theorems 2.6.1 and 2.6.2 of Chap. 2), where $\varepsilon_1 = \rho^{-d-2\alpha}$ and $E(\lambda_{j,\beta}(v,\tau))$ is called the known part of Λ_N. Besides, we proved that all other eigenvalues of the operator $L_t(q)$ lie in the ε_1 neighborhood of the numbers $F(\gamma'+t)$ and $\lambda_j(\gamma'+t)$, where $\gamma' \in \Gamma$, which are called as the known parts of the other eigenvalues. In order that Λ_N does not coincide with the other eigenvalues, we use the following two simplicity conditions

$$|E(\lambda_{j,\beta}(v,\tau)) - F(\gamma'+t)| \geq 2\varepsilon_1, |E(\lambda_{j,\beta}(v,\tau)) - \lambda_i(\gamma'+t)| \geq 2\varepsilon_1. \quad (3.2.2)$$

Briefly, B_δ is the set of $\beta+\tau+(j+v)\delta$ satisfying (3.2.2). Thus we constructed the set B_δ by eliminating the set of quasimomenta $\gamma+t \equiv \beta+\tau+(j+v)\delta$ for which the known part $E(\lambda_{j,\beta}(v,\tau))$ of the corresponding eigenvalue is situated from the known parts of the other eigenvalues at a distance less than $2\varepsilon_1$. Then we investigated the set B_δ. It is clear that every vector w of $\mathbb{R}^d$ can be written as $w=\gamma+t$, where $\gamma \in \Gamma$, $t \in F^*$, and hence has decomposition (3.1.8). In Chap. 2 [see the formula (2.6.45), Theorems 2.3.1 and 2.6.1 of Chap. 2] we proved that if

$$j \in S_1(\rho), \beta \in S_2(\rho), v \in S_3(\beta,\rho), \tau \in S_4(\beta, j, v, \rho), \quad (3.2.3)$$

then

$$\beta+\tau+(j+v)\delta \in B_\delta$$

and hence there exists a unique eigenvalue Λ_N of $L_t(q)$ which is simple and satisfies

$$\Lambda_N = \lambda_{j,\beta}(v,\tau) + O(\rho^{-\alpha_2}) \quad (3.2.4)$$

and the corresponding eigenfunction $\Psi_{N,t}(x)$ satisfies

$$\Psi_{N,t}(x) = \Phi_{j,\beta}(x) + O(\rho^{-\alpha_2}\ln\rho), \quad (3.2.5)$$

where α_2 is defined in (3.1.9) and the set S_1, S_2, S_3 , S_4 are defined as follows:

$$S_1(\rho) = \{j \in \mathbb{Z} : |j| < \frac{\rho^{\alpha_1}}{2|\delta|^2} - \frac{3}{2}\}, \quad (3.2.6)$$

$$S_2(\rho) = \{\beta \in \Gamma_\delta : \beta \in (R_\delta(\frac{3}{2}\rho - d_\delta - 1)\backslash R_\delta(\frac{1}{2}\rho + d_\delta + 1))\backslash(\bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} V_b^\delta(\rho^{\frac{1}{2}}))\},$$

where

$$d_\delta = \sup_{x,y\in F_\delta} |x-y|, R_\delta(c) = \{x \in H_\delta : |x| < c\},$$
$$\Gamma_\delta(c) = \{b \in \Gamma_\delta : 0 < |b| < c\},$$
$$V_b^\delta(c) = \{x \in H_\delta : ||x+b|^2 - |x|^2| < c\}.$$

For $\beta \in S_2(\rho)$ the set $S_3(\beta, \rho)$ is defined by

$$S_3(\beta, \rho) = W(\rho)\backslash A(\beta, \rho), \tag{3.2.7}$$

where

$$W(\rho) \equiv \{v \in (0,1) : |\mu_j(v) - \mu_{j'}(v)| > \frac{2}{\ln \rho}, \ \forall j', j \in \mathbb{Z}, j' \neq j\},$$
$$A(\beta, \rho) = \bigcup_{b \in \Gamma_\delta(\rho^{\alpha_d})} A(\beta, b, \rho),$$

and

$$A(\beta, b, \rho) = \{v \in [0,1) : \exists j \in \mathbb{Z}, |2\langle \beta, b\rangle + |b|^2 + |(j+v)\delta|^2| < 4d_\delta \rho^{\alpha_d}\}.$$

For

$$j \in S_1(\rho), \beta \in S_2(\rho), v \in S_3(\beta, \rho)$$

the set $S_4(\beta, j, v, \rho)$ is the set of $\tau \in F_\delta$ for which

$$\beta + \tau + (j+v)\delta \in B_\delta.$$

In other words, $S_4(\beta, j, v, \rho)$ is the set of $\tau \in F_\delta$ for which $E(\lambda_{j,\beta}(v, \tau))$ satisfies the simplicity conditions (3.2.2). Since the functions taking part in (3.2.2) are measurable, $S_4(\beta, j, v, \rho)$ is a measurable set. In Chap. 2 (see 2.6.48 of Chap. 2), we proved that

$$\mu(S_4(\beta, j, v, \rho)) = \mu(F_\delta)(1 + O(\rho^{-\alpha})). \tag{3.2.8}$$

Remark 3.2.1 If (3.2.3) holds, then there exists unique index $N(j, \beta, v, \tau)$, depending on j, β, v, τ, for which the eigenvalue $\Lambda_{N(j,\beta,v,\tau)}(t)$ satisfies (3.2.4). Instead of $N(j, \beta, v, \tau)$, we write $N(j, \beta)$ (or N) if v, τ (or j, β, v, τ) are unambiguous. In the asymptotic formulas (3.1.11)–(3.1.14), instead of $\Lambda_{N(j,\beta,v,\tau)}$ and $\Psi_{N(j,\beta,v,\tau),t}(x)$ we write $\Lambda_{j,\beta}(v, \tau)$ and $\Psi_{j,\beta}(x)$ respectively, in order to underline that $\Lambda_{j,\beta}(v, \tau)$ and $\Psi_{j,\beta}(x)$ are close to $\lambda_{j,\beta}(v, \tau)$ and $\Phi_{j,\beta}(x)$, where $\lambda_{j,\beta}(v, \tau)$ and $\Phi_{j,\beta}(x)$ are the eigenvalues and eigenfunction of the operator $L_t(q^\delta)$ with directional potential q^δ.

To prove the asymptotic formulas (3.1.10)–(3.1.14), which are suitable for the constructive determination of the spectral invariants, we put an additional conditions on β. Namely, we suppose that

$$\beta \notin \bigcup_{b \in \Gamma_\delta(p\rho^\alpha)} V_b^\delta(\rho^a), \tag{3.2.9}$$

where $V_b^{\delta}(\rho^a)$ and $\Gamma_\delta(p\rho^\alpha)$ are defined in (3.2.6). By definition of $V_b^{\delta}(\rho^a)$, the relation (3.2.9) yields

$$||\beta|^2 - |\beta+\beta_1|^2| \geq \rho^a,\ \forall \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{3.2.10}$$

Using the inequalities

$$|\beta_1| < p\rho^\alpha, |\tau| < d_\delta, a > 2\alpha$$

[see (3.1.11)], we obtain

$$||\beta+\tau|^2 - |\beta+\beta_1+\tau|^2| > \frac{8}{9}\rho^a,\ \forall \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{3.2.11}$$

Now we prove (3.1.10) by using (3.2.3), (3.2.11), and the following relation

$$(\Lambda_N(t) - \lambda_{j,\beta})b(N, j, \beta) = (\Psi_{N,t}, (q - q^\delta)\Phi_{j,\beta}), \tag{3.2.12}$$

where

$$b(N, j, \beta) = (\Psi_{N,t}, \Phi_{j,\beta}).$$

In Chap. 2, using (2.3.12), we proved that [see (2.3.22) and (2.3.23) of Chap. 2] if

$$|j\delta| < r, |\beta| > \frac{1}{2}\rho, \tag{3.2.13}$$

where

$$r \geq r_1 = \frac{\rho^{\alpha_1}}{2|\delta|} + 2|\delta|,$$

then the following decomposition

$$(q(x)-q^\delta(x))\Phi_{j,\beta}(x) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} A(j,\beta, j+j_1, \beta+\beta_1)\Phi_{j+j_1,\beta+\beta_1}(x) + O(\rho^{-p\alpha}) \tag{3.2.14}$$

of $(q(x) - q^\delta(x))\Phi_{j,\beta}(x)$ by eigenfunction of $L_t(q^\delta)$ holds, where

$$Q(\rho^\alpha, 9r) = \{(j, \beta) : |j\delta| < 9r, 0 < |\beta| < \rho^\alpha\}$$

and

$$\sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} |A(j,\beta, j+j_1, \beta+\beta_1)| < c_2. \tag{3.2.15}$$

Using this decomposition in (3.2.12), we get

$$(\Lambda_N(t) - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha}) + \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} A(j, \beta, j + j_1, \beta + \beta_1)b(N, j + j_1, \beta + \beta_1). \tag{3.2.16}$$

Remark 3.2.2 If $|j'\delta| < r$, $|\beta'| > \frac{1}{2}\rho$ and

$$|\Lambda_N - \lambda_{j',\beta'}| > c(\rho),$$

then by (3.2.16) we have

$$b(N, j', \beta') = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} \frac{A(j', \beta', j' + j_1, \beta' + \beta_1)b(N, j' + j_1, \beta' + \beta_1)}{\Lambda_N - \lambda_{j',\beta'}} + O(\frac{1}{\rho^{p\alpha}c(\rho)}).$$

If $j \in S_1(\rho)$, then $|j\delta| < r_1 = O(\rho^{\alpha_1})$ and in (3.2.16) instead of r we take r_1.

Theorem 3.2.1 *If* (3.2.3) *and* (3.2.9) *hold, then there exists a unique eigenvalue* $\Lambda_{j,\beta}(v, \tau)$ *of* $L_t(q)$ *which is simple and satisfies* (3.1.10)*.*

Proof Since there exists a unique eigenvalue $\Lambda_N(t)$ satisfying (3.2.4) and the corresponding eigenfunction satisfies (3.2.5) (see Remark 3.2.1), we have

$$b(N, j, \beta) = 1 + O(\rho^{-\alpha_2} \ln \rho).$$

Therefore, we need to prove that the right-hand side of (3.2.16) is $O(\rho^{-a})$. First we show that

$$b(N, j + j_1, \beta + \beta_1) = O(\rho^{-a}) \tag{3.2.17}$$

for

$$\beta_1 \in \Gamma_\delta(p\rho^\alpha),\ j = o(\rho^{\frac{a}{2}}),\ j_1 = o(\rho^{\frac{a}{2}}).$$

For this we prove the inequality

$$|\Lambda_N(t) - \lambda_{j+j_1,\beta+\beta_1}| > \frac{1}{2}\rho^a,\ \forall\beta_1 \in \Gamma_\delta(p\rho^\alpha),\ \forall j = o(\rho^{\frac{a}{2}}),\ \forall j_1 = o(\rho^{\frac{a}{2}}), \tag{3.2.18}$$

and use the formula

$$b(N, j + j_1, \beta_1 + \beta) = \frac{(\Psi_{N,t}, (q - q^\delta)\Phi_{j+j_1,\beta_1+\beta})}{\Lambda_N - \lambda_{j+j_1,\beta_1+\beta}} \tag{3.2.19}$$

which can be obtained from (3.2.12) by replacing the indices j, β with $j+j_1, \beta+\beta_1$. By (3.2.4), the inequality (3.2.18) holds if

$$|\mu_j(v) + |\beta+\tau|^2 - \mu_{j+j_1}(v) - |\beta+\beta_1+\tau|^2| > \frac{5}{9}\rho^a.$$

This inequality can be easily obtained by using (3.2.11), the equalities $j = o(\rho^{\frac{a}{2}})$, $j + j_1 = o(\rho^{\frac{a}{2}})$ [see the conditions on $j,\ j_1$ in (3.2.17), (3.2.18)], and the formula

$$\mu_n(v) = |(n+v)\delta|^2 + O(n^{-1}) \tag{3.2.20}$$

(see [Mar]). Note that the set of the eigenvalues of $T_v(0)$ with zero potential is a sequence

$$\{|(n+v)\delta|^2 : \ n \in \mathbb{Z}\}$$

and it is not hard to see that the set of the eigenvalues of T_v can be written as a sequence

$$\{\mu_n(v) : \ n \in \mathbb{Z}\}$$

satisfying (3.2.20). Thus (3.2.17) is proved. Using (3.2.17), the definition of $Q(\rho^\alpha, 9r_1)$, and the relations $r_1 = O(\rho^{\alpha_1})$ (see Remark 3.2.2), $\alpha_1 < \frac{a}{2}$, we obtain that all multiplicands $b(N, j+j_1, \beta+\beta_1)$ in the right-hand side of (3.2.16), in the case $r = r_1$, is $O(\rho^{-a})$. Hence (3.2.15) implies that the right-hand side of (3.2.16) is $O(\rho^{-a})$. □

To prove the asymptotic formula (3.1.11), we iterate (3.2.16), in the case $r = r_1$, as follows. If $|j\delta| < r_1$, then the summation in (3.2.16) is taken under condition

$$(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)$$

(see Remark 3.2.2). By the definition of $Q(\rho^\alpha, 9r_1)$ we have $|j_1\delta| < 9r_1$. Hence

$$|(j+j_1)\delta| < r_2,$$

where $r_2 = 10r_1$. Therefore, using (3.2.18) and Remark 3.2.2, we get

$$b(N, j+j_1, \beta_1+\beta) = \sum_{(j_2,\beta_2)\in Q(\rho^\alpha, 9r_2)} \frac{A(j(1),\beta(1), j(2),\beta(2)) b(N, j(2),\beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}} + O(\rho^{-p\alpha}),$$

where

$$j(k) = j + j_1 + j_2 + \cdots + j_k, \beta(k) = \beta + \beta_1 + \beta_2 + \cdots + \beta_k$$

for $k = 0, 1, 2, \ldots$ Using this in (3.2.16), we obtain

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-p\alpha}) + \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)(j_2,\beta_2)\in Q(\rho^\alpha,9r_2)} \frac{A(j, \beta, j(1), \beta(1))A(j(1), \beta(1), j(2), \beta(2))b(N, j(2), \beta(2))}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \tag{3.2.21}$$

To prove (3.1.11), we use this formula and the following lemma.

Lemma 3.2.1 *Suppose* (3.2.3) *and* (3.2.9) *hold. If* $j' \neq j$, $|j'\delta| < r$, *where*

$$r = O(\rho^{\frac{1}{2}\alpha_2}), r \geq r_1,$$

and

$$r_1 = \frac{\rho^{\alpha_1}}{2|\delta|} + 2|\delta|,$$

then

$$b(N(j, \beta), j', \beta) = O(\rho^{-2a} r^2 \ln \rho).$$

Proof To prove this lemma, we use the following formula obtained from (3.2.21) by replacing j and r_1 with j' and r respectively

$$(\Lambda_{N(j,\beta)} - \lambda_{j',\beta})b(N, j', \beta) = O(\rho^{-p\alpha}) + \sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r)\\(j_2,\beta_2)\in Q(\rho^\alpha,90r)}} \frac{A(j, \beta, j'(1), \beta(1))A(j'(1), \beta(1), j'(2), \beta(2))b(N, j'(2), \beta(2))}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}}, \tag{3.2.22}$$

where $j'(k) = j' + j_1 + j_2 + \cdots + j_k$ for $k = 0, 1, 2, \ldots$. By (3.2.17) we have

$$b(N, j'(2), \beta(2)) = O(\rho^{-a}) \tag{3.2.23}$$

for $\beta(2) \neq \beta$. If $j'(2) \neq j$, then using (3.1.10) and taking into account that

$$v \in S_3(\beta, \rho) \subset W(\rho)$$

[see the definition of $W(\rho)$ in (3.2.7)], we obtain

$$|\Lambda_{N(j,\beta)} - \lambda_{j',\beta}| > \frac{1}{\ln \rho}. \tag{3.2.24}$$

Therefore using, Remark 3.2.2, and (3.2.17), we see that

$$b(N, j'(2), \beta) = O(\rho^{-a} \ln \rho)$$

for $j'(2) \neq j$. Using this, (3.2.15), and the estimations (3.2.18), (3.2.23), we see that the sum of the terms of the right side of (3.2.22) with multiplicand $b(N, j'(2), \beta(2))$ for $(j'(2), \beta(2)) \neq (j, \beta)$ is $O(\rho^{-2a} \ln \rho)$. It means that the formula (3.2.22) can be written in the form

$$(\Lambda_N - \lambda_{j',\beta})b(N, j', \beta) = O(\rho^{-2a} \ln \rho) + C_1(j', \Lambda_N)b(N, j, \beta), \tag{3.2.25}$$

where

$$C_1(j', \Lambda_N) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} \frac{A(j', \beta, j' + j_1, \beta + \beta_1)A(j' + j_1, \beta + \beta_1, j, \beta)}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}}. \tag{3.2.26}$$

By (3.1.10), (3.2.18), (3.2.15) we have

$$\frac{1}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}} = \frac{1}{\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1}} = O(\rho^{-3a}),$$

$$C_1(j', \Lambda_N) = C_1(j', \lambda_{j,\beta}) + O(\rho^{-3a}), \tag{3.2.27}$$

where $C_1(j', \lambda_{j,\beta})$ is obtained from $C_1(j', \Lambda_N)$ by replacing Λ_N with $\lambda_{j,\beta}$ in the denominator of the fractions in (3.2.26). In Appendix 1 we prove that

$$C_1(j', \lambda_{j,\beta}) = O(\rho^{-2a}r^2) \tag{3.2.28}$$

for

$$|j'\delta| < r, (j_1, \beta_1) \in Q(\rho^\alpha, 9r), j \in S_1.$$

Therefore dividing both sides of (3.2.25) by $\Lambda_N - \lambda_{j',\beta}$ and using (3.2.24), (3.2.27), (3.2.28), we get the proof of the lemma. □

Theorem 3.2.2 *If* (3.2.3) *and* (3.2.9) *hold, then there exists a unique eigenvalue* $\Lambda_{j,\beta}(v, \tau)$ *of* $L_t(q)$ *which is simple and satisfies* (3.1.11).

Proof We prove this by using (3.2.21). To estimate the summation in the right side of (3.2.21), we divide the terms in this summation into three groups. The terms of the first, second, and third groups are the terms with multiplicands $b(N, j, \beta)$, $b(N, j(2), \beta)$ with $j(2) \neq j$, and $b(N, j(2), \beta(2))$ with $\beta(2) \neq \beta$ respectively. The sum of the terms of the first group is $C_1(\Lambda_N)b(N, j, \beta)$, where

$$C_1(\Lambda_N) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)} \frac{A(j, \beta, j + j_1, \beta + \beta_1)A(j + j_1, \beta + \beta_1, j, \beta)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}}. \tag{3.2.29}$$

The sum of the terms of the second group is

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)\\ (j_2,\beta_2)\in Q(\rho^\alpha,9r_2)}} \frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j(2),\beta)}{\Lambda_N-\lambda_{j+j_1,\beta+\beta_1}} b(N,j(2),\beta),$$

where $j(2)\neq j$. Since

$$r_2=10r_1=O(\rho^{\alpha_1})$$

(see Remark 3.2.2) the conditions on $j,\ j_1,\ j_2$ and Lemma 3.2.1 imply that

$$j(2)=O(\rho^{\alpha_1})$$

and

$$b(N,j(2),\beta)=O(\rho^{-2a+2\alpha_1}\ln\rho).$$

Using this, (3.2.15) and (3.2.18), we obtain that the sum of the terms of the second group is $O(\rho^{-3a+2\alpha_1}\ln\rho)$. The sum of the terms of the third group is

$$\sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)\\ (j_2,\beta_2)\in Q(\rho^\alpha,9r_2)}} \frac{A(j,\beta,j(1),\beta(1))A(j(1),\beta(1),j(2),\beta(2))}{\Lambda_N-\lambda_{j+j_1,\beta+\beta_1}} b(N,j(2),\beta(2)), \tag{3.2.30}$$

where $\beta(2)\neq\beta$. Using (3.2.18) and Remark 3.2.2, we get

$$b(N,j(2),\beta(2))=\sum_{(j_3,\beta_3)\in Q(\rho^\alpha,9r_3)} \frac{A(j(2),\beta(2),j(3),\beta(3))b(N,j(3),\beta(3))}{\Lambda_N-\lambda_{j(2),\beta(2)}}+O(\rho^{-p\alpha}),$$

where $r_3=10r_2$. Substituting it into (3.2.30) and isolating the terms with multiplicands $b(N,j,\beta)$, we see that the sum of the terms of the third group is

$$C_2(\Lambda_N)b(N,j,\beta)+C_3(\Lambda_N)+O(\rho^{-p\alpha}),$$

where

$$C_2(\Lambda_N)=\sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1),\\ (j_2,\beta_2)\in Q(\rho^\alpha,90r_1)}} \frac{A(j,\beta,j(1),\beta(1))A(j(1),\beta(1),j(2),\beta(2))A(j(2),\beta(2),j,\beta)}{(\Lambda_N-\lambda_{j+j_1,\beta+\beta_1})(\Lambda_N-\lambda_{j(2),\beta(2)})}, \tag{3.2.31}$$

$$C_3(\Lambda_N)=\sum_{\substack{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)\\ (j_2,\beta_2)\in Q(\rho^\alpha,9r_2),\\ (j_3,\beta_3)\in Q(\rho^\alpha,9r_3)}} \frac{(\prod_{k=1,2,3}A(j(k-1),\beta(k-1),j(k),\beta(k)))b(N,j(3),\beta(3))}{(\Lambda_N-\lambda_{j(1),\beta(1)})(\Lambda_N-\lambda_{j(2),\beta(2)})},$$

and $(j(3), \beta(3)) \neq (j, \beta)$. By (3.2.17) and Lemma 3.2.1 we have

$$b(N, j(3), \beta(3)) = O(\rho^{-a})$$

for $(j(3), \beta(3)) \neq (j, \beta)$. Using this, (3.2.15), and taking into account that

$$|\Lambda_N(t) - \lambda_{j(1),\beta(1)}| > \frac{1}{3}\rho^a, \; |\Lambda_N(t) - \lambda_{j(2),\beta(2)}| > \frac{1}{3}\rho^a$$

for $\beta(1) \neq \beta$, $\beta(2) \neq \beta$ [see (3.2.18)], we obtain

$$C_3(\Lambda_N) = O(\rho^{-3a}).$$

The estimations of the terms of the first, second and third groups imply that the formula (3.2.21) can be written in the form

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = (C_1(\Lambda_N) + C_2(\Lambda_N))b(N, j, \beta) + O(\rho^{-3a+2\alpha_1} \ln \rho), \tag{3.2.32}$$

where

$$N = N(j, \beta, v, \tau), \; \Lambda_{N(j,\beta,v,\tau)} = \Lambda_{j,\beta}(v, \tau)$$

(see Remark 3.2.1). Therefore, dividing both part of (3.2.32) by $b(N, j, \beta)$, where

$$b(N, j, \beta) = 1 + o(1)$$

[see (3.2.5)], we get

$$\Lambda_{j,\beta} = \lambda_{j,\beta} + C_1(\Lambda_{j,\beta})) + C_2(\Lambda_{j,\beta})) + O(\rho^{-3a+2\alpha_1} \ln \rho). \tag{3.2.33}$$

The calculations in Appendix 3 and in Appendix 2 show that

$$C_1(\Lambda_{j,\beta}(v, \tau)) = \frac{1}{4}\int_F \left|f_{\delta,\beta+\tau}(x)\right|^2 \left|\varphi_{j,v}(\langle\delta, x\rangle)\right|^2 dx + O(\rho^{-3a+2\alpha_1}), \tag{3.2.34}$$

$$C_2(\Lambda_{j,\beta}(v, \tau)) = O(\rho^{-3a+2\alpha_1}). \tag{3.2.35}$$

Therefore (3.1.11) follows from (3.2.33). □

Theorem 3.2.3 *If* (3.2.3) *and* (3.2.9) *hold, then the eigenfunction* $\Psi_{j,\beta}(x)$ *corresponding to the eigenvalue* $\Lambda_{j,\beta}(v, \tau)$, *where* $\Lambda_{j,\beta}(v, \tau)$ *is defined in* Theorem 3.2.1, *satisfies* (3.1.13).

Proof To prove (3.1.13) we need to show that

$$\sum_{(j',\beta'):(j',\beta')\neq(j,\beta)} |b(N(j, \beta), j', \beta')|^2 = O(\rho^{-2a}). \tag{3.2.36}$$

In Chap. 2 [see (2.6.36) of Chap. 2] we proved that

$$\sum_{(j',\beta')\in S^c(k-1)} |b(N,j',\beta')|^2 = O(\rho^{-2k\alpha_2}(\ln\rho)^2), \tag{3.2.37}$$

where

$$S^c(n) = K_0\backslash S(n),\ K_0 = \{(j',\beta') : j' \in \mathbb{Z}, \beta' \in \Gamma_\delta, (j',\beta') \neq (j,\beta)\},$$
$$S(n) = \{(j',\beta') \in K_0 : |\beta - \beta'| \leq n\rho^\alpha, |j'\delta| < 10^n h\},\ h = O(\rho^{\frac{1}{2}\alpha_2})$$

and k can be chosen such that $k\alpha_2 > a$, $k < p$. Therefore, it is enough to prove that

$$\sum_{(j',\beta')\in S(k-1)} |b(N,j',\beta')|^2 = O(\rho^{-2a}). \tag{3.2.38}$$

Using (3.2.18), (3.2.19), definition of $S(k-1)$ and the Bessel inequality for the basis

$$\{\Phi_{j',\beta'}(x) : j' \in \mathbb{Z}, \beta' \in \Gamma_\delta\},$$

we have

$$\sum_{(j',\beta'):(j',\beta')\in S(k-1),\beta'\neq\beta} |b(N,j',\beta')|^2$$
$$= \sum_{(j',\beta')} \frac{|(\Psi_N(q-q^\delta), \Phi_{j',\beta'})|^2}{|\Lambda_N - \lambda_{j',\beta'}|^2} = O(\rho^{-2a}). \tag{3.2.39}$$

In the case $\beta' = \beta$ and $j' \neq j$ using Lemma 3.2.1, we obtain

$$\sum_{(j',\beta)\in S(k-1),\, j'\neq j} |b(N,j',\beta)|^2 = O(\rho^{-4a+2\alpha_2}(\ln\rho)^2)K, \tag{3.2.40}$$

where K is the number of j' satisfying $(j',\beta) \in S(k-1)$. Note that we can use Lemma 3.2.1, since

$$|j'\delta| = O(\rho^{\frac{1}{2}\alpha_2}),\ \forall (j',\beta') \in S(k-1)).$$

It is clear that

$$K = O(\rho^{\frac{1}{2}\alpha_2}).$$

Since $\alpha_2 < \frac{a}{2}$ [see (3.1.9), (3.1.11)], the right side of (3.2.40) is $O(\rho^{-2a})$. Thus (3.2.40) and (3.2.39) give (3.2.38) $\square$

3.3 On the Derivatives of the Band Functions

Now we estimate the derivative of $\Lambda_N(t)$ by using the following lemma.

Lemma 3.3.1 *Let $\Lambda_N(\beta+\tau+v\delta)$, be a simple eigenvalue of L_t satisfying*

$$|\Lambda_N(\beta+\tau+v\delta)-|\beta+\tau|^2|<|\delta|^{-2}\rho^{\alpha_1} \tag{3.3.1}$$

where α_1 is defined in (3.1.9), β satisfies (3.2.3), and $\beta+\tau+v\delta-t\in\Gamma$. Then

$$|\beta+\tau|\frac{\partial\Lambda_N(t)}{\partial h}=\sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta}\langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2, \tag{3.3.2}$$

where $\frac{\partial\Lambda_N(t)}{\partial h}$ is the derivative of $\Lambda_N(t)$ in the direction of $h=\frac{\beta+\tau}{|\beta+\tau|}$. Moreover,

$$|b(N,j',\beta')|\le\frac{c_3}{(|\beta'+\tau|^2+|(j'+v)\delta|^2)|\beta'+\tau|^{2d+6}} \tag{3.3.3}$$

for all β' satisfying $|\beta'+\tau|\ge 4\rho$ and for all $j'\in\mathbb{Z}$.

Proof We find the derivative of $\Lambda_N(t)$ by using

$$\frac{\partial\Lambda_N(t)}{\partial t_j}=2t_j-2i(\frac{\partial}{\partial x_j}\Phi_{N,t},\Phi_{N,t}),$$

where

$$\Phi_{N,t}(x)=e^{-i\langle t,x\rangle}\Psi_{N,t}(x),$$

$t=(t_1,t_2,\ldots,t_d)$ [see (2.5.12) of Chap. 2]. Then

$$\frac{\partial\Lambda_N(t)}{\partial h}=\sum_{j=1}^{d}h_j\frac{\partial\Lambda_N(t)}{\partial t_j}=2\langle h,t\rangle-2i(\frac{\partial}{\partial h}\Phi_{N,t},\Phi_{N,t}). \tag{3.3.4}$$

To compute $\frac{\partial}{\partial h}\Phi_{N,t}(x)$, we prove that the decomposition

$$\Phi_{N,t}(x)=\sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta}b(N,j',\beta')e^{i\langle\beta'+\tau-t,x\rangle}\varphi_{j'}(\langle\delta,x\rangle) \tag{3.3.5}$$

of $\Psi_{N,t}$ by basis

$$\{\Psi_{j,\beta}:j\in\mathbb{Z},\beta\in\Gamma_\delta\}$$

can be differentiated term by term. Since $\langle \delta, h\rangle = 0$ and

$$\frac{\partial}{\partial h} e^{i\langle \beta'+\tau-t,x\rangle}\varphi_{j'}(\langle \delta, x\rangle) = i\langle \beta'+\tau-t, h\rangle e^{i\langle \beta'+\tau-t,x\rangle}\varphi_{j'}(\langle \delta, x\rangle),$$

we need to prove that

$$\frac{\partial}{\partial h}\Phi_{N,t}(x) = \sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta} i\langle \beta'+\tau-t, h\rangle b(N, j', \beta') e^{i\langle \beta'+\tau-t,x\rangle}\varphi_{j'}(\langle \delta, x\rangle). \tag{3.3.6}$$

Therefore, we consider the convergence of these series by estimating the multiplicand $b(N, j', \beta')$. First we estimate this multiplicand for $(j', \beta') \in E$, where

$$E = \{(j', \beta') : |(j'+v)\delta|^2 + |\beta'+\tau|^2 \geq 9\rho^2\},$$

by using the formula

$$b(N, j', \beta') = \frac{(\Psi_{N,t}, (q-q^\delta)\Phi_{j',\beta'})}{\Lambda_N - \lambda_{j',\beta'}} \tag{3.3.7}$$

which can be obtained from (3.2.19) by replacing $j + j_1, \beta + \beta_1$ with j', β'. By (3.2.3) and (3.3.1) we have

$$|\Lambda_N| < 3\rho^2. \tag{3.3.8}$$

This inequality, the condition $(j', \beta') \in E$, definition of $\lambda_{j',\beta'}$, and (3.2.20) give

$$\lambda_{j',\beta'} - \Lambda_N > \frac{1}{2}(|(j'+v)\delta|^2 + |\beta'+\tau|^2) > \rho^2 \tag{3.3.9}$$

for $(j', \beta') \in E$. Therefore, (3.3.7) implies that

$$|b(N, j', \beta')| \leq \frac{c_5}{|(j'+v)\delta|^2 + |\beta'+\tau|^2}, \ \forall (j', \beta') \in E. \tag{3.3.10}$$

Now we obtain the high order estimation for $b(N, j', \beta')$ when $|\beta'+\tau| \geq 4\rho$. In this case to estimate $b(N, j', \beta')$ we use the iterations of the formula in Remark 3.2.2. To iterate this formula, we use the relation

$$|\beta'+\tau-\beta_1-\beta_2-\cdots-\beta_k|^2 > \frac{3}{4}|\beta'+\tau|^2$$

for $k = 1, 2, \ldots, d+3$, where $|\beta_i| < \rho^\alpha$ for $i = 0, 1, \ldots, k$. This and (3.3.8) give

$$\lambda_{j'(k),\beta'(k)} - \Lambda_N > \frac{1}{5}|\beta' + \tau|^2, \ \forall|\beta' + \tau| \geq 4\rho, \tag{3.3.11}$$

where $\beta'(k) = \beta' + \beta_1 + \beta_2 + \cdots + \beta_k$. Moreover if $|j'\delta| < c$, where c is a positive number, then

$$(j_k, \beta_k) \in Q(\rho^\alpha, 10^{k-1}9c).$$

These conditions on j' and j_1 imply that

$$|j'(1)\delta| < 10c.$$

Therefore, in the formula in Remark 3.2.2 replacing j', β', r by $j'(1), \beta'(1), 10c$, we get

$$b(N, j'(1), \beta'(1)) = O(\rho^{-p\alpha}) + \sum_{(j_2,\beta_2)\in Q(\rho^\alpha,90c)} \frac{A(j'(1), \beta'(1), j'(2), \beta'(2))b(N, j'(2), \beta'(2))}{\Lambda_N - \lambda_{j'(1),\beta'(1)}}.$$

In the same way, we obtain

$$b(N, j'(k), \beta'(k)) = O(\rho^{-p\alpha}) + \sum_{(j_{k+1},\beta_{k+1})\in Q(\rho^\alpha,(10^k)9c)} \frac{A(j'(k), \beta'(k), j'(k+1), \beta'(k+1))b(N, j'(k+1), \beta'(k+1))}{\Lambda_N - \lambda_{j'(k),\beta'(k)}} \tag{3.3.12}$$

for $k = 1, 2, \ldots$ In the formula in Remark 3.2.2 for $r = c$ using (3.3.12) for $k = 1, 2, \ldots d+3$ successively, we get

$$b(N, j', \beta') = \sum \left(\prod_{i=0}^{d+3} \frac{A(j'(i), \beta'(i), j'(i+1), \beta'(i+1))}{\Lambda_N - \lambda_{j'(i),\beta'(i)}}\right) b(N, j'(d+4), \beta'(d+4)), \tag{3.3.13}$$

where sum is taken under conditions

$$(j_1, \beta_1) \in Q(\rho^\alpha, 9c), (j_2, \beta_2) \in Q(\rho^\alpha, 90c), \ldots, (j_{d+4}, \beta_{d+4}) \in Q(\rho^\alpha, (10^{d+3})9c).$$

Now using (3.2.14), (3.3.9), and (3.3.11), we obtain the proof of (3.3.3). It follows from (3.3.12) and (3.3.3) that the series in (3.3.5) can be differentiated term by term and (3.3.6) holds. Substituting (3.3.6) into (3.3.4) and using the Parseval equality, by direct calculation, we obtain the proof of the lemma. □

Theorem 3.3.1 *If* (3.2.3) *and* (3.2.9) *hold, then the eigenvalue* $\Lambda_{j,\beta}(v,\tau)$, *defined in* Theorem 3.2.1, *satisfies* (3.1.14).

Proof It follows from (3.3.3), (3.2.36), and (3.1.13) that

$$\sum_{j'\in\mathbb{Z},|\beta'+\tau|\geq 4\rho} \langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2 = O(\rho^{2-2a}),$$

$$\sum_{j'\in\mathbb{Z},|\beta'+\tau|<4\rho,(j',\beta')\neq(j,\beta)} \langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2 = O(\rho^{2-2a}),$$

$$\langle\beta+\tau,\beta+\tau\rangle|b(N,j,\beta)|^2 = |\beta+\tau|^2 + O(\rho^{2-2a}),$$

where $N = N(j,\beta,v,\tau)$, $\Lambda_{N(j,\beta,v,\tau)} = \Lambda_{j,\beta}(v,\tau)$ (see Remark 3.2.1). Therefore (3.1.14) follows from (3.3.2) □

To prove the main results of this paper we need the following lemmas.

Lemma 3.3.2 *If* $\Lambda_N(\beta+\tau+v\delta)$ *is a simple eigenvalue of* $L_t(q)$ *satisfying*

$$|\Lambda_N(\beta+\tau+v\delta) - |\beta+\tau|^2| < 2\rho^{\alpha}, N \neq N(j,\beta,v,\tau),$$

where $\beta+\tau+v\delta-t\in\Gamma$, α *is defined in* (3.1.9), *and* j,β,v,τ *satisfy* (3.2.3), (3.2.9), *then*

$$|\beta+\tau|\frac{\partial\Lambda_N(t)}{\partial h} < |\beta+\tau|^2 - \frac{1}{4}\rho^{2\alpha_d}.$$

Proof Here we note some reasons of the proof. It follows from (3.2.36) that

$$|b(N,j,\beta)|^2 = 1 + O(\rho^{-2a}) \text{ for } N = N(j,\beta). \tag{3.3.14}$$

Since $||\Phi_{j,\beta}(x)|| = 1$, using the Parseval's equality for the orthonormal basis

$$\{\Psi_N(x) : N = 1,2,\ldots\}$$

and (3.3.14), we get

$$|b(N,j,\beta)|^2 = O(\rho^{-2a}),\ \forall N \neq N(j,\beta). \tag{3.3.15}$$

This with the following long estimations of the other terms of the series of the right side of (3.3.2) implies the proof of this lemma. By Lemma 3.3.1, we have

$$|\beta+\tau|\frac{\partial\Lambda_N(t)}{\partial h} = \sum_{j'\in\mathbb{Z},\beta'\in\Gamma_\delta} \langle\beta+\tau,\beta'+\tau\rangle|b(N,j',\beta')|^2 = \sum_{i=1}^{7} C_i, \tag{3.3.16}$$

where

$$C_i = \sum_{\beta' \in A_i} \sum_{j' \in \mathbb{Z}} \langle \beta + \tau, \beta' + \tau \rangle |b(N, j', \beta')|^2 \tag{3.3.17}$$

and A_i is defined as follows:

$$A_1 = \{\beta' \in \Gamma_\delta : \beta' + \tau \notin R_\delta(4\rho)\},$$

where

$$\begin{aligned}
&R_\delta(c) = \{x \in H_\delta : |x| < c\}, \\
&\quad A_2 = \{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(4\rho) \backslash R_\delta(H + \frac{1}{9}\rho^{a-1})\}, \\
&\quad A_3 = \{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(H + \frac{1}{9}\rho^{a-1}) \backslash R_\delta(H + \rho^{\alpha_d - 1}), |\beta - \beta'| \geq \rho^{a-2\alpha}\}, \\
&\quad A_4 = \{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(H + \frac{1}{9}\rho^{a-1}) \backslash R_\delta(H + \rho^{\alpha_d - 1}), |\beta - \beta'| < \rho^{a-2\alpha}\}, \\
&\quad A_5 = \{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(H + \rho^{\alpha_d - 1}) \backslash R_\delta(H - \rho^{2\alpha_d - 1}), |\beta - \beta'| \geq \rho^{\alpha_d}\}, \\
&\quad A_6 = \{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(H + \rho^{\alpha_d - 1}) \backslash R_\delta(H - \rho^{2\alpha_d - 1}), |\beta - \beta'| < \rho^{\alpha_d}\}, \\
&\quad A_7 = \{\beta' \in \Gamma_\delta : \beta' + \tau \in R_\delta(H - \rho^{2\alpha_d - 1})\},
\end{aligned}$$

where $H = |\beta + \tau|$, $\beta \in S_2(\rho)$, and hence by the definition of $S_2(\rho)$ [see (3.2.6)] H satisfies the inequalities

$$\frac{1}{2}\rho < H < \frac{3}{2}\rho. \tag{3.3.18}$$

First we prove that

$$C_i = O(\rho^{2-2a}), \ \forall i = 1, 2, 4, 6. \tag{3.3.19}$$

It follows from (3.3.3) that (3.3.19) holds for $i = 1$. To prove (3.3.19) for $i = 2$ we use (3.3.7) and show that

$$\lambda_{j',\beta'} - \Lambda_N(t) > c_6 \rho^a. \tag{3.3.20}$$

First let us prove (3.3.20). By the condition

$$|\Lambda_N(\beta + \tau + v\delta) - |\beta + \tau|^2| < 2\rho^\alpha$$

of the lemma we have

$$\Lambda_N = H^2 + O(\rho^\alpha). \tag{3.3.21}$$

If $\beta' \in A_2$, then using (3.3.18), definition of $\lambda_{j',\beta'}$, and (3.2.20), we have

$$\lambda_{j',\beta'} > H^2 + c_7 \rho^a. \tag{3.3.22}$$

This, (3.3.21), and the inequality $a > \alpha$ imply (3.3.20). Now using (3.3.20), (3.3.7), the inequalities

$$|\beta + \tau| < \frac{3}{2}\rho, |\beta' + \tau| < 4\rho$$

and the Bessel inequality, we get the proof of (3.3.19) for $i = 2$.

To prove (3.3.19) for $i = 4$ we use the inequality

$$C_4 < c_8\rho^2(C_{4,1} + C_{4,2}),$$

where

$$C_{4,1} = \sum_{\beta' \in A_4} \sum_{j':|j'\delta| \ge \frac{1}{30}\rho^{\frac{a}{2}}} |b(N, j', \beta')|^2, \ C_{4,2} = \sum_{\beta' \in A_4} \sum_{j':|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} |b(N, j', \beta')|^2,$$

and prove that

$$C_{4,i} = O(\rho^{-2a}), \ \forall i = 1, 2. \tag{3.3.23}$$

It is clear that if $\beta' \in A_4$ and $|j'\delta| \ge \frac{1}{30}\rho^{\frac{a}{2}}$, then (3.3.22) holds. Therefore, repeating the proof of (3.3.19) for $i = 2$, we get the proof of (3.3.23) for $i = 1$.

Now we prove (3.3.23) for $i = 2$. It follows from (3.3.7) that

$$C_{4,2} = \sum_{\beta' \in A_4} \sum_{j':|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \frac{|(\Psi_N, (q - q^\delta)\Phi_{j',\beta'})|^2}{|\Lambda_N(t) - \lambda_{j',\beta'}|^2}. \tag{3.3.24}$$

Since $\alpha_d > \alpha$, it follows from (3.3.21) that the inequality

$$\lambda_{j',\beta'} - \Lambda_N(t) > c_9\rho^{\alpha_d}$$

holds for $\beta' \in A_4$ and $|j'\delta| < \rho^{\frac{a}{2}}$. Therefore, using (3.2.20), we obtain

$$\sum_{j':|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \frac{1}{|\Lambda_N(t) - \lambda_{j',\beta'}|^2} < c_{10}, \ \forall \beta' \in A_4, \tag{3.3.25}$$

where c_{10} does not depend on β'. Using this in (3.3.24) and denoting

$$|(\Psi_N, (q - q^\delta)\Phi_{n(\beta'),\beta'})| = \max_{j':|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} |(\Psi_N, (q - q^\delta)\Phi_{j',\beta'})|$$

(if max is gotten for several index $n(\beta')$, then we take one of them), we get

$$C_{4,2} < c_{11} \sum_{\beta' \in A_4} |(\Psi_N, (q - q^\delta)\Phi_{n(\beta'),\beta'})|^2.$$

Now using (3.2.14), (3.2.15) and then (3.3.7), we obtain

$$\begin{aligned} C_{4,2} &< c_{12}\rho^{-p\alpha} + c_{12} \sum_{\beta' \in A_4} |b(N, n(\beta') + j_1(\beta'), \beta' + \beta_1(\beta'))|^2 \\ &= c_{12}\rho^{-p\alpha} + c_{12} \sum_{\beta' \in A_4} \frac{|(\Psi_N, (q - q^\delta)\Phi_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')})|^2}{|\Lambda_N - \lambda_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')}|^2}, \end{aligned} \tag{3.3.26}$$

where

$$|b(N, n(\beta') + j_1(\beta'), \beta' + \beta_1(\beta'))| = \max_{(j_1,\beta_1) \in Q(\rho^\alpha, 9\frac{1}{30}\rho^{\frac{a}{2}})} |b(N, n(\beta') + j_1, \beta' + \beta_1)|.$$

To estimate $C_{4,2}$ let us prove that

$$|\Lambda_N - \lambda_{n(\beta')+j_1(\beta'),\beta'+\beta_1(\beta')}| > \frac{1}{8}\rho^a. \tag{3.3.27}$$

The inclusion

$$(j_1, \beta_1) \in Q(\rho^\alpha, 9\frac{1}{30}\rho^{\frac{a}{2}})$$

and the condition

$$|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}$$

imply that

$$|n(\beta')\delta + j_1(\beta')\delta| < \frac{1}{3}\rho^{\frac{a}{2}}$$

and by (3.2.20)

$$|\mu_{n(\beta')+j_1(\beta')}| < \frac{1}{8}\rho^a.$$

Therefore, by (3.3.21), to prove (3.3.27) it is enough to show that

$$|H^2 - |\beta' + \beta_1 + \tau|^2| > \frac{3}{8}\rho^a, \ \forall \beta' \in A_4, \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{3.3.28}$$

Since

$$||\beta' + \tau|^2 - H^2| < \frac{1}{2}\rho^a$$

[see definition of A_4 and use (3.3.18)], we need to prove that

$$||\beta' + \tau|^2 - |\beta' + \beta_1 + \tau|^2| > \frac{7}{8}\rho^a, \ \forall \beta' \in A_4, \beta_1 \in \Gamma_\delta(p\rho^\alpha). \tag{3.3.29}$$

Using

$$|\beta - \beta'| < \rho^{a-2\alpha}$$

(see definition of A_4), by calculations, we get

$$|\beta' + \tau|^2 - |\beta' + \beta_1 + \tau|^2 = -2\langle \beta' + \tau, \beta_1 \rangle - |\beta_1|^2$$
$$= -2\langle \beta + \tau, \beta_1 \rangle - |\beta_1|^2 - 2\langle \beta' - \beta, \beta_1 \rangle = -(|\beta + \beta_1 + \tau|^2 - |\beta + \tau|^2) + o(\rho^a).$$

This and (3.2.11) imply that (3.3.29) and hence (3.3.27) holds. Now to estimate the right-hand side of (3.3.26) we prove that if $\beta' \in A_4$, $\beta'' \in A_4$ and $\beta' \neq \beta''$, then

$$\beta' + \beta_1(\beta') \neq \beta'' + \beta_1(\beta''). \tag{3.3.30}$$

Assume that they are equal. Then we have $\beta'' = \beta' + b$, where $b \in \Gamma_\delta(2\rho^\alpha)$, since

$$\beta_1(\beta') \in \Gamma_\delta(\rho^\alpha), \beta_1(\beta'') \in \Gamma_\delta(\rho^\alpha).$$

It easily follows from the inclusions $\beta' \in A_4$ and $\beta' + b \in A_4$ that

$$||\beta' + \tau|^2 - |\beta' + \tau + b|^2| < \frac{1}{2}\rho^a$$

which contradicts (3.3.29). Thus (3.3.30) is proved. Therefore, using (3.3.26), (3.3.27) and the Bessel inequality, we obtain the proof of (3.3.23) for $i = 2$. Hence (3.3.19) is proved for $i = 4$.

Now we prove (3.3.19) for $i = 6$. First we note that $A_6 = \{\beta\}$. Indeed if $\beta' \neq \beta$ and $\beta' \in A_6$, then we have $\beta' = \beta + b$, where $b \in \Gamma_\delta(\rho^{\alpha_d})$, and from the relations $\beta \notin V_b^\delta(\rho^{\frac{1}{2}})$ [see (3.2.3) and the definition of S_2], $|\beta + \tau| = H$, we obtain that

$$||\beta' + \tau|^2 - H^2| > \frac{1}{2}\rho^{\frac{1}{2}}$$

which contradicts the inclusion

$$\beta' + \tau \in R_\delta(H + \rho^{\alpha_d - 1}).$$

Hence

$$C_6 = \sum_{j' \in \mathbb{Z}} \langle \beta + \tau, \beta + \tau \rangle |b(N, j', \beta)|^2 = H^2 \sum_{j' \in \mathbb{Z}} |b(N, j', \beta)|^2 = H^2 \sum_{i=1}^{3} C_{6,i},$$

where

$$C_{6,1} = |b(N, j, \beta)|^2, C_{6,2} = \sum_{|j'\delta| \geq \frac{1}{30}\rho^{\frac{a}{2}}} |b(N, j', \beta)|^2, C_{6,3} = \sum_{|j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}, j' \neq j} |b(N, j', \beta)|^2.$$

To prove (3.3.19) for $i = 6$ we show that

$$C_{6,i} = O(\rho^{-2a}), \ \forall i = 1, 2, 3. \tag{3.3.31}$$

By (3.3.15) this equality holds for $i = 1$. For

$$|j'\delta| \geq \frac{1}{30}\rho^{\frac{a}{2}}$$

the inequality (3.3.20) holds. Therefore, repeating the proof of (3.3.19) for $i = 2$, we get the proof of (3.3.31) for $i = 2$. Arguing as in the proof of (3.3.23) for $i = 2$, we obtain the proof of (3.3.31) for $i = 3$. Thus (3.3.19) is proved for $i = 6$.

Now we prove that

$$C_i \leq \sum_{\beta' \in A_i} \sum_{j' \in Z} |b(N, j', \beta')|^2 (H^2 - \frac{1}{3}\rho^{2\alpha_d}) \tag{3.3.32}$$

for $i = 3, 5, 7$. Consider the triangle generated by vectors $\beta + \tau$, $\beta' + \tau$, $\beta - \beta'$. For $\beta' \in A_3$ we have

$$H + \rho^{\alpha_d - 1} \leq |\beta' + \tau| \leq H + \frac{1}{9}\rho^{a-1}, |\beta - \beta'| \geq \rho^{a-2\alpha}.$$

Let θ be the angle between the vectors $\beta + \tau$, and $\beta' + \tau$. If $|\theta| \leq \frac{\pi}{2}$, then using the cosine theorem, we get

$$|\langle \beta + \tau, \beta' + \tau \rangle| = \frac{1}{2}(|\beta + \tau|^2 + |\beta' + \tau|^2 - |\beta - \beta'|^2) < H^2 - \frac{1}{3}\rho^{2\alpha_d},$$

since $a - 2\alpha > \alpha_d$. Using this and taking into account that

$$\langle \beta + \tau, \beta' + \tau \rangle < 0$$

for $\frac{\pi}{2} < |\theta| \leq \pi$, we get the proof of (3.3.32) for $i = 3$. If $\beta' \in A_5$ and $|\theta| \leq \frac{\pi}{2}$, then

$$|\langle \beta + \tau, \beta' + \tau \rangle| \leq H^2 - \frac{1}{3}\rho^{2\alpha_d}$$

and hence (3.3.32) holds for $i = 5$. If $\beta' \in A_7$, then

$$|\beta' + \tau| \leq H - \rho^{2\alpha_d - 1}$$

and by (3.3.18) we have

$$|\langle \beta + \tau, \beta' + \tau \rangle| \leq H^2 - \frac{1}{3}\rho^{2\alpha_d},$$

that is, (3.3.32) holds for $i = 7$ too. Now (3.3.32) and the Bessel inequality imply that

$$C_3 + C_5 + C_7 \leq H^2 - \frac{1}{3}\rho^{2\alpha_d} = |\beta + \tau|^2 - \frac{1}{3}\rho^{2\alpha_d}.$$

This, (3.3.19) and (3.3.2) give the proof of the lemma, since $2 - 2a < 2\alpha_d$ [see (3.1.11)]. □

3.4 The Construction of the Spectral Invariants

In this section we determine constructively a family of spectral invariants of this operator from the given Bloch eigenvalues. For this we use the following lemma.

Lemma 3.4.1 *Let b be a visible element of Γ_δ and $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then there exists $\rho(v)$ such that if $\rho \geq \rho(v)$, then there exists $\beta \in S_2(\rho)$ satisfying (3.2.9), the relation $v \notin A(\beta, \rho)$, and the inequalities*

$$\frac{1}{3}|\rho|^a < |\langle \beta + \tau, b \rangle| < 3|\rho|^a, \tag{3.4.1}$$

$$|\langle \beta + \tau, \gamma \rangle| > \frac{1}{3}|\rho|^a, \ \forall \gamma \in S(\delta, b) \backslash \delta\mathbb{R}, \tag{3.4.2}$$

$$|\langle \beta + \tau, \gamma \rangle| > \frac{1}{3}|\rho|^{a+2\alpha}, \ \forall \gamma \notin S(\delta, b), |\gamma| < |\rho|^\alpha, \tag{3.4.3}$$

$$\int_F \left| f_{\delta,\beta+\tau}(x) \right|^2 \left| \varphi_{n,v}(\langle \delta, x \rangle) \right|^2 dx < c_4 \rho^{-2a} \tag{3.4.4}$$

for $\tau \in F_\delta$, where S_2, $A(\beta, \rho)$, $f_{\delta,\beta+\tau}$, $S(\delta, b)$ are defined in (3.2.6), (3.2.7), (3.1.12), (3.1.5).

Proof Let n_1 be a positive integer satisfying the inequality

$$|(n_1 + v)\delta|^2 \leq 4\rho^{1+\alpha_d} < |(n_1 + 1 + v)\delta|^2.$$

Introduce the following sets

$$D_{b',j}(\rho,v,4)=\{x\in H_\delta : |2\langle x,b'\rangle+|b'|^2+|(j+v)\delta|^2|<4d_\delta\rho^{\alpha_d}\},$$
$$D(\rho,v,4)=\bigcup_{j=-n_1-3}^{n_1}\ \bigcup_{b'\in\Gamma_\delta(\rho^{\alpha_d})} D_{b',j}(\rho,v,4), \tag{3.4.5}$$

$$S_2'(\rho,b,v)=((V_b^\delta(4\rho^a)\backslash V_b^\delta(\rho^a))\backslash(D(\rho,v,4)\cup D_1(\rho^{\frac12})\cup D_2(\rho^{a+2\alpha})))\cap D_3, \tag{3.4.6}$$

where

$$D_1(\rho^{\frac12})=\bigcup_{b'\in\Gamma_\delta(\rho^{\alpha_d})} V_{b'}^\delta(\rho^{\frac12}),\ D_2(\rho^{a+2\alpha})=\bigcup_{b'\in\Gamma_\delta(p\rho^{\alpha})\backslash b\mathbb{R}} V_{b'}^\delta(\rho^{a+2\alpha}),$$

$$D_3=(R(\frac{3}{2}\rho-d_\delta-1)\backslash R(\frac{1}{2}\rho+d_\delta+1)).$$

Now we prove that the set $S_2'(\rho,b,v)$ contains an element $\beta\in\Gamma_\delta$ satisfying all assertions of Lemma 3.4.1. First let us prove that $S_2'(\rho,b,v)\cap\Gamma_\delta$ is nonempty subset of $S_2(\rho)$, that is,

$$S_2'(\rho,b,v)\cap\Gamma_\delta\subset S_2(\rho),\ S_2'(\rho,b,v)\cap\Gamma_\delta\neq\emptyset. \tag{3.4.7}$$

It follows from the definitions of $S_2'(\rho,b,v)$ and $S_2(\rho)$ [see (3.2.6)] that the first relation of (3.4.7) holds. To prove the second relation we consider the set

$$D'(\rho)=(V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a))\backslash(D(\rho,v,6)\cup D_1(2\rho^{\frac12})\cup D_2(2\rho^{a+2\alpha})))\cap D_4,$$

where

$$D_4=R(\frac{3}{2}\rho-1)\backslash R(\frac{1}{2}\rho+1).$$

If $\beta+\tau\in D'(\rho)$, where $\beta\in\Gamma_\delta,\tau\in F_\delta$, then $\beta\in S_2'(\rho,b,v)$. Therefore

$$\{\beta+F_\delta : \beta\in S_2'(\rho,b,v)\cap\Gamma_\delta\}$$

is a cover of $D'(\rho)$. Hence

$$|S_2'(\rho,b,v)\cap\Gamma_\delta|\geq(\mu(F_\delta))^{-1}\mu(D'(\rho)), \tag{3.4.8}$$

where $|S_2'(\rho,b,v)\cap\Gamma_\delta|$ is the number of elements of $S_2'(\rho,b,v)\cap\Gamma_\delta$. Thus, to prove the second relation of (3.4.7), we need to estimate $\mu(D'(\rho))$. It is not hard to verify that (see Remark 2.2.1 of Chap. 2)

$$\mu((V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a))\cap D_4)>c_{13}\rho^{d-2+a}. \tag{3.4.9}$$

Now we estimate

$$\mu((V_b^\delta(3\rho^a)\backslash V_b^\delta(2\rho^a))\cap D_1(2\rho^{\frac{1}{2}})\cap D_4).$$

If $b'\in(b\mathbb{R})\cap\Gamma_\delta(\rho^{\alpha_d})$, then one can easily verify that

$$V_{b'}^\delta(2\rho^{\frac{1}{2}})\cap D_4\subset V_b^\delta(2\rho^a)\cap D_4.$$

Therefore, we need to estimate the measure of

$$V_b^\delta(3\rho^a)\cap V_{b'}^\delta(2\rho^{\frac{1}{2}})\cap D_4$$

for $b'\in\Gamma_\delta(\rho^{\alpha_d})\backslash b\mathbb{R}$. For this we turn the coordinate axes so that the direction of $(1,0,0,\ldots,0)$ coincides with the direction of b', and the plane generated by b, b' coincides with the plane $\{(x_1,x_2,0,\ldots,0):x_1\in\mathbb{R},x_2\in\mathbb{R}\}$, that is, $b'=(|b'|,0,0,\ldots,0)$, $b=(b_1,b_2,0,\ldots,0)$. Then the condition

$$x\in V_b^\delta(3\rho^a)\cap V_{b'}^\delta(2\rho^{\frac{1}{2}})\cap D_4$$

implies that

$$x_1|b'|=O(\rho^{\frac{1}{2}}),\ x_1b_1+x_2b_2=O(\rho^a),\ x_1^2+x_2^2+\cdots+x_{d-1}^2=O(\rho^2). \tag{3.4.10}$$

First equality of (3.4.10) shows that

$$x_1=O(\rho^{\frac{1}{2}}).$$

Since b' and b are linearly independent vectors of Γ_δ, we have

$$|b'||b_2|\ge\mu(F_\delta),$$

where $|b'|<\rho^{\alpha_d}$. Therefore,

$$|b_2|\ge\mu(F_\delta)\rho^{-\alpha_d}$$

and the second equality of (3.4.10) implies that

$$x_2=O(\rho^{a+\alpha_d}).$$

The third equality of (3.4.10) shows that the set

$$V_b^{\delta}(3\rho^a) \cap V_{b'}^{\delta}(2\rho^{\frac{1}{2}}) \cap D_4$$

is a subset of

$$[-c_{14}\rho^{\frac{1}{2}}, c_{14}\rho^{\frac{1}{2}}] \times [-c_{14}\rho^{a+\alpha_d}, c_{14}\rho^{a+\alpha_d}] \times ([-c_{14}\rho, c_{14}\rho])^{d-3}$$

which has the measure $O(\rho^{d-3+\frac{1}{2}+a+\alpha_d})$. This with

$$|\Gamma_\delta(\rho^{\alpha_d})| = O(\rho^{(d-1)\alpha_d})$$

gives

$$\mu((V_b^{\delta}(3\rho^a) \cap D_1(2\rho^{\frac{1}{2}}) \cap D_4) = O(\rho^{d-3+\frac{1}{2}+a+d\alpha_d}) = o(\rho^{d-2+a}), \qquad (3.4.11)$$

since $d\alpha_d < \frac{1}{2}$ [see the definition of α_d in (3.1.9)]. In the same way, we get

$$\mu(V_b^{\delta}(3\rho^a) \cap D_2(2\rho^{a+2\alpha}) \cap D_4) = O(\rho^{d-3+2a+(d+4)\alpha}) = o(\rho^{d-2+a}), \quad (3.4.12)$$

since $a + (d+4)\alpha < 1$ [see (3.1.9) and (3.1.11)]. To estimate $\mu(D_{b',j}(\rho, v, 6))$ we turn the coordinate axes so that the direction of $(1, 0, 0, \ldots, 0)$ coincides with the direction of b'. Then the condition

$$x \in D_{b',j}(\rho, v, 6) \cap D_4$$

implies that

$$2x_1|b'| + |b'|^2 + |(j+v)\delta|^2| = O(\rho^{\alpha_d}), x_1^2 + x_2^2 + \cdots + x_{d-1}^2 = O(\rho^2).$$

These equalities show that x_1 belongs to the interval of length $O(\rho^{\alpha_d})$ and

$$\mu(D_{b',j}(\rho, v, 6) \cap D_4) = O(\rho^{d-2+\alpha_d}).$$

Now using (3.4.5) and taking into account that

$$n_1 = O(\rho^{\frac{1}{2}(1+\alpha_d)}), |\Gamma_\delta(\rho^{\alpha_d})| = O(\rho^{(d-1)\alpha_d}),$$

we obtain

$$\mu(D(\rho, v, 4) \cap D_4 = O(\rho^{d-2+\frac{1}{2}+(d+\frac{1}{2})\alpha_d}) = o(\rho^{d-2+a}),$$

since $a > \frac{1}{2} + (d + \frac{1}{2})\alpha_d$ [see (3.1.11) and (3.1.9)]. This estimation with (3.4.11), (3.4.12), and (3.4.9) implies that

$$\mu(D'(\rho)) > c_{15}\rho^{d-2+a}.$$

Thus the second equality of (3.4.7) follows from (3.4.8). Now take any element β from $S_2'(\rho, b, v) \cap \Gamma_\delta$. It follows from the definitions of the sets $S_2'(\rho, b, v)$, $D_{b',j}(\rho, v, 4)$, $A(\beta, \rho)$ [see (3.4.6) and (3.2.7)] that $v \notin A(\beta, \rho)$ and (3.2.9) holds. Let us prove the inequalities in (3.4.1). By the definition of $S_2'(\rho, b, v)$ we have

$$\beta \in V_b^\delta(4\rho^a)\backslash V_b^\delta(\rho^a).$$

This means that

$$\rho^a \le |2\langle \beta, b\rangle + |b|^2| < 4\rho^a.$$

This with the obvious relations

$$|b| = O(1), |\tau| = O(1)$$

implies (3.4.1).

Now we prove (3.4.2). If $\gamma \in S(\delta, b)\backslash \delta\mathbb{R}$, then

$$\gamma = nb + a\delta, n \neq 0, n \in \mathbb{Z}, a \in \mathbb{R}, |\langle \gamma, b\rangle| = |n||b|^2 \ge |b|^2, \tag{3.4.13}$$

since each $\gamma \in \Gamma$ has a decomposition $\gamma = b' + a\delta$, where $b' \in \Gamma_\delta$, and b is a visible element of Γ_δ [see 2.3.2 of Chap. 2 and the definition of $S(\delta, b)$ in (3.1.5)]. This with the relation $\langle \beta + \tau, \delta\rangle = 0$ gives $\langle \beta + \tau, \gamma\rangle = n\langle \beta + \tau, b\rangle$. Therefore the first inequality of (3.4.1) implies (3.4.2).

Let us prove (3.4.3). If

$$\gamma \notin S(\delta, b), |\gamma| < |\rho|^\alpha$$

then $\gamma = b' + a\delta$, where $a \in \mathbb{R}$, $b' \in \Gamma_\delta(\rho^\alpha)\backslash b\mathbb{R}$, and $\langle \beta + \tau, \gamma\rangle = \langle \beta + \tau, b'\rangle$. Therefore using

$$|b'| = O(\rho^\alpha), |\tau| = O(1)$$

and arguing as in the proof of (3.4.1), we see that the relation

$$\beta \notin V_{b'}^\delta(\rho^{a+2\alpha}),$$

(see definition of $S_2'(\rho, b, v)$) implies (3.4.3). The inequality (3.4.4) follows from the definition of $f_{\delta,\beta+\tau}(x)$, (3.4.2), (3.4.3), and from the obvious relation

$$\sum_{\gamma\in\Gamma} |\gamma||q_\gamma| < c_{16}.$$

The last inequality with (3.4.13) implies the convergence of the series (3.1.5) □

Theorem 3.4.1 *Suppose $q \in W_2^s(F)$, where $s \geq 6(3^d(d+1)^2)+d$, and the band functions are known. Then the spectral invariants $\mu_j(v)$ for $j \in \mathbb{Z}$, $v \in [0,1)$ and (3.1.4), (3.1.7), (3.1.15), (3.1.16), (3.1.19) can be determined constructively.*

Proof Let $j \in \mathbb{Z}$ and $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. In Chap. 2 (see Lemma 2.3.7) we proved that

$$(\varepsilon(\rho), \frac{1}{2} - \varepsilon(\rho)) \cup (\frac{1}{2} + \varepsilon(\rho), 1 - \varepsilon(\rho)) \subset W(\rho),$$

where $W(\rho)$ is defined in (3.2.7) and $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. Therefore $v \in W(\rho)$ for $\rho \gg 1$. On the other hand, by Lemma 3.4.1, there exists $\beta \in S_2(\rho)$ such that (3.2.9), the relation $v \notin A(\beta, \rho)$ and (3.4.1)–(3.4.4) holds. Then $v \in S_3(\beta, \rho)$ [see (3.2.7)]. Thus j, β, v satisfy (3.2.3) and β satisfies (3.2.9), (3.4.1)–(3.4.4) for $\rho \gg 1$. Replacing ρ by $\rho_k \equiv 3^k\rho$ for $k = 1, 2, \ldots$, in the same way, we obtain the sequence $\beta_1, \beta_2, \ldots$, such that

$$\beta_k \in S_2(\rho_k), v \in S_3(\beta_k, \rho_k)$$

and the relations obtained from (3.2.9), (3.4.1)–(3.4.4) by replacing β, ρ with β_k, ρ_k holds. Now take τ from F_δ and consider the band functions $\Lambda_N(\beta_k + \tau + v\delta)$ for $N = 1, 2, \ldots$. Let $A_k(v)$ be the set of all $\tau \in F_\delta$ for which there exists N satisfying the conditions:

$$|\Lambda_N(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2| < (\rho_k)^{\frac{\alpha}{2}}, \tag{3.4.14}$$

$$\Lambda_N(\beta_k + \tau + v\delta) \text{ is a simple eigenvalue,} \tag{3.4.15}$$

$$||\beta_k + \tau| \frac{\partial \Lambda_N(\beta_k + \tau + v\delta)}{\partial h} - |\beta_k + \tau|^2| < \rho_k^{2-2a+\alpha}, \tag{3.4.16}$$

where $h = \frac{\beta_k + \tau}{|\beta_k + \tau|}$. By (3.1.10), (3.2.20) and Theorem 3.3.1, $\Lambda_{j',\beta_k}(v, \tau)$ for $|j'| < \rho_k^{\frac{\alpha}{5}}$ and for

$$\beta_k \in S_2(\rho_k), \beta_k \notin \bigcup_{b \in \Gamma_\delta(p\rho_k^\alpha)} V_b^\delta(\rho_k^a), v \in S_3(\beta_k, \rho_k), \tau \in S_4(\beta_k, j', v, \rho_k) \tag{3.4.17}$$

satisfy the conditions (3.4.14)–(3.4.16). Therefore

$$S_4(\beta_k, j', v, \rho_k) \subset A_k(v)$$

for $|j'| < \rho_k^{\frac{\alpha}{5}}$ and hence $A_k(v)$ is not an empty set. Moreover, it follows from (3.4.15) that $\Lambda_N(\beta_k + \tau + v\delta)$ and

$$\frac{\partial \Lambda_N(\beta_k + \tau + v\delta)}{\partial h}$$

are measurable functions of τ and hence $A_k(v)$ is a measurable set. Let

$$\Lambda_{N_1}(\beta_k+\tau+v\delta) < \Lambda_{N_2}(\beta_k+\tau+v\delta) < \cdots < \Lambda_{N_{n(k)}}(\beta_k+\tau+v\delta) \quad (3.4.18)$$

be the eigenvalues of L_t satisfying (3.4.14)–(3.4.16). Using Theorem 3.3.1 and Lemma 3.3.2, we see that if (3.4.17) holds for $j' \in S_1(\rho_k)$, then there exist (j_1, β_k), $(j_2, \beta_k), \cdots, (j_{n(k)}, \beta_k)$ such that

$N_i = N(j_i, \beta_k)$ for $i = 1, 2, \ldots, n(k)$, that is,

$$\Lambda_{N_i}(\beta_k+\tau+v\delta) = \Lambda_{j_i,\beta_k}(v,\tau),\ \forall i = 1, 2, \ldots, n(k) \quad (3.4.19)$$

(see Remark 3.2.1). Let $\mu_j(v)$ be $i(j)$th eigenvalue of the operator T_v when the eigenvalues of T_v are numbered in the increasing order. (Note that the eigenvalues of the operator T_v for $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ are simple (see [Eas]). Using (3.4.18), (3.4.19) and (3.1.10), (3.1.11) we obtain that if k is a large number and (3.4.17) holds for all j' such that $\mu_{j'} \le \mu_j$, then

$$\Lambda_{N_{i(j)}}(\beta_k+\tau+v\delta) = |\beta_k+\tau|^2 + \mu_j(v) + O(\rho_k^{-a}), \quad (3.4.20)$$

$$\Lambda_{N_{i(j)}}(\beta_k+\tau+v\delta) = |\beta_k+\tau|^2+\mu_j(v)+\frac{1}{4}\int_F |f^2_{\delta,\beta_k+\tau}|\left|\varphi_{j,v}\right|^2 dx+O(\rho_k^{-3a+2\alpha_1}\ln\rho_k), \quad (3.4.21)$$

For $\tau \in A_k(v)$ take $i(j)$th element $\Lambda_{N_{i(j)}}(\beta_k+\tau+v\delta)$ [see (3.4.18)] of the set of the eigenvalues satisfying (3.4.14)–(3.4.16) and consider the integral

$$J(A_k) = \frac{1}{\mu(F_\delta)}\int_{A_k(v)} (\Lambda_{N_{i(j)}}(\beta_k+\tau+v\delta) - |\beta_k+\tau|^2)d\tau.$$

This integral is a sum of $J(S'_4)$ and $J(A_k(v)\backslash S'_4)$, where S'_4 denotes the intersection of $S_4(\beta_k, j', v, \rho_k)$ for all j' such that $\mu_{j'} \le \mu_j$. If $\tau \in S'_4$ and k is a large number, then (3.4.20) holds. Thus using (3.4.20) and (3.2.8) for $\rho = \rho_k$, we get

$$J(S'_4) = \mu_j(v) + O(\rho_k^{-\alpha}).$$

On the other hand the inclusion $A_k(v) \subset F_\delta$, (3.2.8), and (3.4.14) imply that

$$\mu(A_k(v)\backslash S'_4) = O(\rho_k^{-\alpha})$$

and

$$J(A_k(v)\backslash S'_4) = O(\rho_k^{-\frac{\alpha}{2}}).$$

These equalities yield

$$J(A_k(v)) = \mu_j(v) + O(\rho_k^{-\frac{\alpha}{2}}).$$

Letting $k \to \infty$, we find $\mu_j(v)$ for $j \in \mathbb{Z}$ and $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Since $\mu_j(0)$ and $\mu_j(\frac{1}{2})$ are the end points of the interval $\{\mu_j(v) : v \in (0, \frac{1}{2})\}$, the invariant $\mu_j(v)$ is determined constructively for all $v \in [0, 1)$. In the Appendix 4, we constructively determine (3.1.16) from the asymptotic formulas for $\mu_j(v)$.

Now using (3.4.21) and taking into account that the invariant $\mu_j(v)$ is determined, we determine the invariant (3.1.4) as follows. Let $B(\beta_k, v)$ be the set of $\tau \in F_\delta$ for which there exists N satisfying (3.4.15), (3.4.16), and

$$|\Lambda_N(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2 - \mu_j(v)| < \rho_k^{-2a+\frac{\alpha}{2}}. \tag{3.4.22}$$

For $\tau \in B(\beta_k, v)$ take one of the eigenvalues $\Lambda_N(\beta_k + \tau + v\delta)$ satisfying (3.4.15), (3.4.16), (3.4.22) and consider

$$J'(B(\beta_k, v)) = \frac{|\langle \beta_k + \tau, b\rangle|^2}{\mu(F_\delta)|b|^4} \int_{B(\beta_k,v)} (\Lambda_N(\beta_k + \tau + v\delta) - |\beta_k + \tau|^2 - \mu_j(v))d\tau.$$

This integral is a sum of $J'(S_4)$ and $J'(B(\beta_k, v)\backslash S_4)$. If $\tau \in S_4$ and k is a large number, then arguing as above and taking into account that $\mu_j(v)$ is a simple eigenvalue, we see that only the eigenvalue $\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta)$ [see (3.4.21)] satisfies (3.4.15), (3.4.16), (3.4.22). Hence in $J'(S_4)$ instead of $\Lambda_N(\beta_k + \tau + v\delta)$ we must take $\Lambda_{N_{i(j)}}(\beta_k + \tau + v\delta)$. Therefore using (3.4.21), we get

$$J'(S_4) = \frac{|\langle \beta_k + \tau, b\rangle|^2}{4\mu(F_\delta)|b|^4} \int_{S_4}\int_F |f_{\delta,\beta_k+\tau}(x)\varphi_{j,v}(\langle \delta, x\rangle)|^2 dx d\tau + O(\rho_k^{2\alpha_1-a}\ln \rho). \tag{3.4.23}$$

Moreover using (3.4.22), (3.4.1), and

$$\mu(B(\beta_k, v)\backslash S_4) = O(\rho_k^{-\alpha})$$

[see (3.2.8)], we obtain

$$J'(B(\beta_k, v)\backslash S_4) = O(\rho_k^{-\frac{\alpha}{2}}). \tag{3.4.24}$$

Substituting the decomposition $|\delta|^{-2}\langle \gamma, \delta\rangle\delta + |b|^{-2}\langle \gamma, b\rangle b$ of γ for $\gamma \in S(\delta, b)$, $|\gamma| < |\rho_k|^\alpha$ into the denominator of the fraction in $f_{\delta,\beta_k+\tau}(x)$ [for definition of this function see (3.1.12)] and using (3.4.1), (3.4.3), we obtain

$$\lim_{k\to\infty} |b|^{-2}\langle \beta_k + \tau, b\rangle f_{\delta,\beta_k+\tau}(x) = \sum_{\gamma\in S(\delta,b)\backslash\delta\mathbb{R}} \frac{\gamma}{\langle \gamma, b\rangle} q_\gamma e^{\langle \gamma, x\rangle} \equiv q_{\delta,b}(x), \tag{3.4.25}$$

where $q_{\delta,b}(x)$ is defined in (3.1.5) and the convergence of the series (3.1.5) is proved in the proof of Lemma 3.4.1. This with (3.4.23) and (3.4.24) implies that

$$\lim_{k\to\infty} J'(B(\beta_k, v) = \int_F \left|q_{\delta,b}(x)\right|^2 \left|\varphi_{j,v}(\langle\delta, x\rangle)\right|^2 dx \equiv J(\delta, b, j, v) \qquad (3.4.26)$$

[see (3.1.4)]. In (3.4.26) letting $j \to \infty$ and using (3.1.6), we get the invariant $J_0(\delta, b)$ [see (3.1.7)]. Then we find the other invariants

$$J_1(\delta, b), J_2(\delta, b), \ldots,$$

of (3.1.7) as follows

$$J_1 = \lim_{j\to\infty} (J - J_0)j, J_2 = \lim_{j\to\infty} ((J - J_0)j^2 - J_1 j), \ldots$$

In the Appendix 4 using the asymptotic formulas for the eigenfunctions of $T_v(Q)$, we constructively determine the invariants (3.1.15), (3.1.19) from (3.1.7) and (3.1.16) □

Appendices

Appendix 1: The Proof of (3.2.28)

Here we estimate the conjugate $\overline{C_1(j', \lambda_{j,\beta})}$ of $C_1(j', \lambda_{j,\beta})$, namely we prove that

$$\sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r)} \frac{\overline{A(j', \beta, j' + j_1, \beta + \beta_1)}\,\overline{A(j' + j_1, \beta + \beta_1, j, \beta)}}{\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1}} = O(\rho^{-2a} r^2), \qquad (3.5.1)$$

[see (3.2.26)], where

$$Q(\rho^\alpha, 9r) = \{(j_1, \beta_1) : |j_1\delta| < 9r, 0 < |\beta_1| < \rho^\alpha\}, j \in S_1(\rho), |j'\delta| < r, r = O(\rho^{\frac{1}{2}\alpha_2}).$$

The conditions on indices j', j_1, j and (3.2.20) imply that

$$\mu_{j'+j_1} = O(r^2), \mu_j = O(r^2).$$

These with $\beta \notin V^\delta_{\beta_1}(\rho^a)))$, where $\beta_1 \in \Gamma_\delta(p\rho^\alpha)$, [see (3.2.9)] give

$$\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1} = -2\langle\beta, \beta_1\rangle + O(r^2), |\langle\beta, \beta_1\rangle| > \frac{1}{3}\rho^a. \qquad (3.5.2)$$

Using this, (3.2.15) and (3.5.1), we get

$$\overline{C_1(j',\lambda_{j,\beta})}=\sum_{\beta_1}\frac{C'}{-2\langle\beta,\beta_1\rangle}+O(\rho^{-2a}r^2), \tag{3.5.3}$$

where

$$C'=\sum_{j_1}\overline{A(j',\beta,j'+j_1,\beta+\beta_1)}\,\overline{A(j'+j_1,\beta+\beta_1,j,\beta)}.$$

In Chap. 2, we proved that [see (2.3.7), (2.3.21), Lemma 2.3.3]

$$\overline{A(j',\beta,j'+j_1,\beta+\beta_1)}=\sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)}c(n_1,\beta_1)a(n_1,\beta_1,j',\beta,j'+j_1,\beta+\beta_1), \tag{3.5.4}$$

$$\overline{A(j'+j_1,\beta+\beta_1,j,\beta)}=\sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^\alpha)}c(n_2,-\beta_1)a(n_2,-\beta_1,j'+j_1,\beta+\beta_1,j,\beta),$$

$$\Gamma'(\rho^\alpha)=\{(n_1,\beta_1):\beta_1\in\Gamma_\delta\backslash 0,n_1\in\mathbb{Z},\beta_1+(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\delta\in\Gamma(\rho^\alpha)\},$$

$$c(n_1,\beta_1)=q_{\gamma_1},\gamma_1=\beta_1+(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\delta\in\Gamma(\rho^\alpha), \tag{3.5.5}$$

$$a(n_1,\beta_1,j',\beta,j'+j_1,\beta+\beta_1)=(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j',v(\beta)}(\zeta),\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta)),$$

$$\begin{aligned}a(n_2,-\beta_1,j'+j_1,\beta+\beta_1,j,\beta)&=(e^{i(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta),\varphi_{j,v(\beta)}(\zeta))\\&=(\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta),e^{-i(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta))\\&=\overline{(e^{-i(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle\zeta}\varphi_{j,v(\beta)}(\zeta),\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta))},\end{aligned} \tag{3.5.6}$$

where δ^* is the element of Ω satisfying $\langle\delta^*,\delta\rangle=2\pi$.

Now, to estimate the right-hand side of (3.5.3) we prove that

$$\begin{aligned}&\sum_{j_1}a(n_1,\beta_1,j',\beta,j'+j_1,\beta+\beta_1)a(n_2,-\beta_1,j'+j_1,\beta+\beta_1,j,\beta)\\&=a(n_1+n_2,0,j',\beta,j,\beta)+O(\rho^{-p\alpha}).\end{aligned} \tag{3.5.7}$$

By definition, we have

$$\begin{aligned}a(n_1+n_2,0,j',\beta,j,\beta)&=(e^{i(n_1+n_2)\zeta}\varphi_{j',v(\beta)}(\zeta),\varphi_{j,v(\beta)}(\zeta))\\&=(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j',v(\beta)}(\zeta),e^{-i(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)).\end{aligned}$$

This, (3.5.6), and the following formulas

$$e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j',v(\beta)}(\zeta) \tag{3.5.8}$$
$$=\sum_{|j_1\delta|<9r} a(n_1,\beta_1,j',\beta,j'+j_1,\beta+\beta_1)\varphi_{j'+j_1,v(\beta+\beta_1)}(\zeta)+O(\rho^{-p\alpha}),$$
$$e^{-i(n_2-(2\pi)^{-1}\langle-\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)$$
$$=\sum_{|j_1\delta|<9r} \overline{a(n_2,-\beta_1,j',\beta,j'+j_1,\beta+\beta_1)}\varphi_{j'+j_1,v(\beta+\beta_1)}+O(\rho^{-p\alpha}),$$
$$\sum_{j_1}|a(n_1,\beta_1,j',\beta,j'+j_1,\beta+\beta_1)|=O(1)$$

[see (2.3.16), (2.3.17) of Chap. 2) give the proof of (3.5.7). Now from (3.5.7), (3.5.4) and (3.5.3) we obtain

$$C'=\sum_{n_1}\sum_{n_2}(c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j',\beta,j,\beta)+O(\rho^{-p\alpha})),$$
$$\overline{C_1(j',\lambda_{j,\beta})}=\sum_{\beta_1}\sum_{n_1}\sum_{n_2}C_1'(\beta_1,n_1,n_2)+O(\rho^{-2a}r^2),$$

where

$$C_1'(\beta_1,n_1,n_2)=\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j',\beta,j,\beta)}{-2\langle\beta,\beta_1\rangle}.$$

It is clear that

$$C_1'(\beta_1,n_1,n_2)+C_1'(-\beta_1,n_2,n_1)=0. \tag{3.5.9}$$

Therefore

$$\overline{C_1(j',\lambda_{j,\beta})}=O(\rho^{-2a}r^2).$$

Appendix 2: The Proof of (3.2.35)

Arguing as in the proof of (3.2.27), we see that

$$C_2(\Lambda_{j,\beta})=C_2(\lambda_{j,\beta})+O(\rho^{-3a})$$

and by (3.5.4)

$$\overline{C_2(\lambda_{j,\beta})}=\sum_{\beta_1,\beta_2}(\sum_{n_1,n_2,n_3}(\sum_{j_1,j_2}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)}{(\lambda_{j,\beta}-\lambda_{j(1),\beta(1)})(\lambda_{j,\beta}-\lambda_{j(2),\beta(2)})}a(n_1,\beta_1,j,\beta,j(1),\beta(1))$$
$$\times a(n_2,\beta_2,j(1),\beta(1),j(2),\beta(2))a(n_3,-\beta_1-\beta_2,j(2),\beta(2),j,\beta),$$

where

$$(j_1, \beta_1) \in Q(\rho^{\alpha}, 9r_1), (j_2, \beta_2) \in Q(\rho^{\alpha}, 90r_1), j \in S_1, \beta_1 + \beta_2 \neq 0.$$

Applying (3.5.7) two times and using (3.5.8), we get

$$\begin{aligned}
&\sum_{j_1} a(n_1, \beta_1, j, \beta, j(1), \beta(1))(\sum_{j_2} a(n_2, \beta_2, j(1), \beta(1), j(2), \beta(2))a(n_3, -\beta_1 - \beta_2, j(2), \beta(2), j, \beta)) \\
&\quad = \sum_{j_1} a(n_1, \beta_1, j, \beta, j(1), \beta(1))(a(n_2 + n_3, -\beta_1, j(1), \beta(1), j, \beta) + O(\rho^{-p\alpha})) \\
&\quad = a(n_1 + n_2 + n_3, 0, j, \beta, j, \beta) + O(\rho^{-p\alpha}).
\end{aligned}$$

Using this in the above expression for $C_2(\lambda_{j,\beta})$ and taking into account that

$$\begin{aligned}
\lambda_{j,\beta} - \lambda_{j(1),\beta(1)} &= -2\langle \beta, \beta_1 \rangle + O(\rho^{2\alpha_1}), |\langle \beta, \beta_1 \rangle| > \frac{1}{3}\rho^{a}, \\
\lambda_{j,\beta} - \lambda_{j(2)\beta(2)} &= -2\langle \beta, \beta_1 + \beta_2 \rangle + O(\rho^{2\alpha_1}), |\langle \beta, \beta_1 + \beta_2 \rangle| > \frac{1}{3}\rho^{a},
\end{aligned}$$

which can be proved as (3.5.2), we have

$$\begin{aligned}
&C_2(\lambda_{j,\beta}) = O(\rho^{-3a+2\alpha_1}) \\
&\quad + \sum_{\beta_1,\beta_2} \sum_{n_1,n_2,n_3} \frac{c(n_1, \beta_1)c(n_2, \beta_2)c(n_3, -\beta_1 - \beta_2)a(n_1 + n_2 + n_3, 0, j, \beta, j, \beta)}{4\langle \beta, \beta_1 \rangle \langle \beta, \beta_1 + \beta_2 \rangle}.
\end{aligned}$$

Grouping the terms with the equal multiplicands

$$\begin{aligned}
&c(n_1, \beta_1)c(n_2, \beta_2)c(n_3, -\beta_1 - \beta_2), c(n_2, \beta_2)c(n_1, \beta_1)c(n_3, -\beta_1 - \beta_2), \\
&c(n_1, \beta_1)c(n_3, -\beta_1 - \beta_2)c(n_2, \beta_2), c(n_2, \beta_2)c(n_3, -\beta_1 - \beta_2)c(n_1, \beta_1), \\
&c(n_3, -\beta_1 - \beta_2)c(n_1, \beta_1)c(n_2, \beta_2), c(n_3, -\beta_1 - \beta_2)c(n_2, \beta_2)c(n_1, \beta_1)
\end{aligned}$$

and using the obvious equality

$$\begin{aligned}
&\frac{1}{\langle \beta, \beta_1 \rangle \langle \beta, \beta_1 + \beta_2 \rangle} + \frac{1}{\langle \beta, \beta_2 \rangle \langle \beta, \beta_2 + \beta_1 \rangle} + \frac{1}{\langle \beta, \beta_1 \rangle \langle \beta, -\beta_2 \rangle} \\
&\quad + \frac{1}{\langle \beta, \beta_2 \rangle \langle \beta -, \beta_1 \rangle} + \frac{1}{\langle \beta, -\beta_1 - \beta_2 \rangle \langle \beta, -\beta_2 \rangle} + \frac{1}{\langle \beta, -\beta_1 - \beta_2 \rangle \langle \beta, -\beta_1 \rangle} = 0,
\end{aligned}$$

we see that

$$C_2(\lambda_{j,\beta}) = O(\rho^{-3a+2\alpha_1}).$$

Appendix 3: The Proof of (3.2.34)

By (3.2.27) we have

$$C_1(\Lambda_{j,\beta}) = C_1(\lambda_{j,\beta}) + O(\rho^{-3a}).$$

Therefore, we need to prove that

$$\overline{C_1(\lambda_{j,\beta})} = \frac{1}{4}\int_F \left|f_{\delta,\beta+\tau}(x)\right|^2 \left|\varphi^{\delta}_{j,v}(\langle\delta, x\rangle)\right|^2 dx + O(\rho^{-3a+2\alpha_1}),$$

where

$$\overline{C_1(\lambda_{j,\beta})} \equiv \sum_{\beta_1}\sum_{j_1} \frac{\overline{A(j,\beta,j+j_1,\beta+\beta_1)}\,\overline{A(j+j_1,\beta+\beta_1,j,\beta)}}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}},$$
$$(j_1,\beta_1) \in Q(\rho^{\alpha}, 9r_1),\ j \in S_1,$$

and by (3.5.4)

$$\overline{C_1(\lambda_{j,\beta})} = \sum_{\beta_1}\sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^{\alpha})}\ \sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^{\alpha})}\ \sum_{j_1} \frac{c(n_1,\beta_1)c(n_2,-\beta_1)}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}}$$
$$\times\, a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta).$$

Replacing $\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}$ by

$$-(2\langle\beta+\tau,\beta_1\rangle + |\beta_1|^2 + \mu_{j+j_1}(v(\beta+\beta_1)) - \mu_j(v(\beta)))$$

and using (3.5.7) for $j' = j$, we have

$$\overline{C_1(j,\lambda_{j,\beta})} = \sum_{\beta_1}\sum_{n_1}\sum_{n_2} \frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{-2\langle\beta+\tau,\beta_1\rangle}$$
$$+\sum_{\beta_1}\sum_{n_1}\sum_{n_2}\ \sum_{j_1} \frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)}{2\langle\beta+\tau,\beta_1\rangle(2\langle\beta+\tau,\beta_1\rangle+|\beta_1|^2+\mu_{j+j_1}-\mu_j)}$$
$$\times\, a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)(|\beta_1|^2 + \mu_{j+j_1}(v(\beta+\beta_1)) - \mu_j(v(\beta))).$$

The formula (3.5.9) shows that the first summation of the right-hand side of this equality is zero. Thus we need to estimate the second sum. For this we use the following relation

$$
\begin{aligned}
&\mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)=(e^{i(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\zeta}\varphi_{j,v(\beta)}(\zeta),T_v\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta))\\
&\quad=(T_v(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta)),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)\\
&\quad=(|n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle|^2|\delta|^2+\mu_j(v))(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi_{j,v(\beta)}(\zeta),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta))\\
&\qquad-2i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)|\delta|^2(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi'_{j,v(\beta)}(\zeta),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)).
\end{aligned}
$$

Using this, (3.5.7), and the formula

$$
\begin{aligned}
&\sum_{j_1}(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta))a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)\\
&=(e^{i(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta))+O(\rho^{-p\alpha})
\end{aligned}
$$

which can be proved as (3.5.7), we obtain

$$
\begin{aligned}
&\sum_{j_1}\mu_{j+j_1}(v(\beta+\beta_1)a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)\\
&\qquad=(|n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle|^2)|\delta|^2+\mu_j(v)a(n_1+n_2,0,j,\beta,j,\beta)\\
&\qquad\quad-2i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)|\delta|^2(e^{i(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta),\varphi_{j,v(\beta)}(\zeta)).
\end{aligned}
\tag{3.5.10}
$$

Here the last multiplicand can be estimated as follows

$$
\begin{aligned}
&\mu_j(v)(\varphi_{j,v(\beta)}(\zeta),e^{i(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta))=(\varphi_{j,v(\beta)}(\zeta),T_v(e^{i(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta)))\\
&\quad=(n_1+n_2)^2|\delta|^2(\varphi_{j,v(\beta)}(\zeta),e^{i(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta))\\
&\qquad+2i(n_1+n_2)|\delta|^2(\varphi_{j,v(\beta)}(\zeta),e^{i(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta))+\mu_j(v)(\varphi_{j,v(\beta)},e^{i(n_1+n_2)\zeta}\varphi_{j,v(\beta)}),\\
&\quad(e^{i(n_1+n_2)\zeta}\varphi'_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta))=\frac{n_1+n_2}{2i}(e^{i(n_1+n_2)\zeta}\varphi_{j,v(\beta)}(\zeta)),\varphi_{j,v(\beta)}(\zeta)).
\end{aligned}
$$

Using this, (3.5.10), and (3.5.7), we get

$$
\begin{aligned}
&\sum_{j_1}(a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta))\\
&\quad\times(|\beta_1|^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))=a(n_1+n_2,0,j,\beta,j,\beta)\\
&\quad\times(|\beta_1|^2+|n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}|^2|\delta|^2-(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})|\delta|^2(n_1+n_2))\\
&=(|\beta_1|^2+|\delta|^2(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})(-n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}))a(n_1+n_2,0,j,\beta,j,\beta).
\end{aligned}
$$

Thus

$$\overline{C_1(j,\lambda_{j,\beta})} = C + O(\rho^{-3a+2\alpha_1}),$$

where

$$C = \sum_{\beta_1,n_1,n_2} \frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{4|\langle\beta+\tau,\beta_1\rangle|^2} \times (|\beta_1|^2 + (n_1 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})(-n_2 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})|\delta|^2). \tag{3.5.11}$$

Now we consider

$$\int_F \left|f_{\delta,\beta+\tau}(x)\right|^2 \left|\varphi_{n,v}(\langle\delta,x\rangle)\right|^2 dx,$$

where $f_{\delta,\beta+\tau}$ is defined in (3.1.12). By (3.5.5)

$$f_{\delta,\beta+\tau}(x) = \sum_{(n_1,\beta_1)\in\Gamma'_\delta(\rho^\alpha)} \frac{\beta_1 + (n_1 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta}{\langle\beta+\tau,\beta_1\rangle} c(n_1,\beta_1) e^{i\langle\beta_1 + (n_1 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta, x\rangle}.$$

Here $f_{\delta,\beta+\tau}(x)$ is a vector of $\mathbb{R}^d$. Using $\langle\beta,\delta\rangle = 0$ for $\beta\in\Gamma_\delta$, we obtain

$$\left|f_{\delta,\beta+\tau}(x)\right|^2 = \sum_{(n_1,\beta_1),(n_2,\beta_2)\in\Gamma'_\delta(\rho^\alpha)} \frac{\langle\beta_1,\beta_2\rangle + (n_1 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})(n_2 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})|\delta|^2}{\langle\beta+\tau,\beta_1\rangle\langle\beta+\tau,\beta_2\rangle} \times c(n_1,\beta_1)c(-n_2,-\beta_2) e^{i\langle\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}\langle\beta_1-\beta_2,\delta^*\rangle)\delta,x\rangle}.$$

Since $\varphi_{j,v}(\langle\delta,x\rangle)$ is a function of $\langle\delta,x\rangle$, we have

$$\int_F e^{i\langle\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}\langle\beta_1-\beta_2,\delta^*\rangle)\delta,x\rangle} \left|\varphi_{j,v}(\langle\delta,x\rangle)\right|^2 dx = 0$$

for $\beta_1 \neq \beta_2$. Therefore

$$\int_F \left|f_{\delta,\beta+\tau}(x)\right|^2 \left|\varphi_{j,v}(\langle\delta,x\rangle)\right|^2 dx = \sum_{\beta_1,n_1,n_2} \frac{c(n_1,\beta_1)c(-n_2,-\beta_1)}{|\langle\beta+\tau,\beta_1\rangle|^2} \times (|\beta_1|^2 + (n_1 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})(n_2 - \frac{\langle\beta_1,\delta^*\rangle}{2\pi})|\delta|^2 a(n_1-n_2,0,j,\beta,j,\beta).$$

Replacing n_2 by $-n_2$, we get

$$\int_F \left|f_{\delta,\beta+\tau}(x)\right|^2 \left|\varphi_{n,v}(\langle\delta,x\rangle)\right|^2 dx = 4C$$

[see (3.5.11)] and (3.2.34).

Appendix 4: Asymptotic Formulas for $T_v(Q)$

It is well-known that the large eigenvalues of $T_0(Q)$ lie in $O(\frac{1}{m^4})$ neighborhood of

$$|m\delta| + \frac{1}{16\pi|m\delta|^3}\int_0^{2\pi} |Q(t)|^2\, dt$$

for the large values of m (see [Eas], p. 58). This formula yields the invariant (3.1.16). Using the asymptotic formulas for solutions of the Sturm-Liouville equation (see [Eas], p. 63), one can easily obtain that

$$\varphi_{n,v}(\zeta) = e^{i(n+v)\zeta}(1 + \frac{Q_1(\zeta)}{2i(n+v)|\delta|^2} + \frac{Q(\zeta) - Q(0) - \frac{1}{2}Q_1^2(\zeta)}{4(n+v)^2|\delta|^4}) + O(\frac{1}{n^3})),$$

where

$$Q_1(\zeta) = \int_0^{\zeta} Q(t)dt.$$

From this, by direct calculations, we find $A_0(\zeta),\ A_1(\zeta),\ A_2(\zeta)$ [see (3.1.6)] and then using these in (3.1.7), we get the invariant (3.1.15).

Now we consider the eigenfunction $\varphi_{n,v}(\zeta)$ of $T_v(p)$ in the case $v \neq 0,\ \frac{1}{2}$ and

$$p(\zeta) = p_1 e^{i\zeta} + p_{-1}e^{-i\zeta}.$$

The eigenvalues and the eigenfunctions of $T_v(0)$ are $(n+v)^2|\delta|^2$ and $e^{i(n+v)\zeta}$, for $n \in \mathbb{Z}$. Since the eigenvalues of $T_v(p)$ are simple for $v \neq 0,\ \frac{1}{2}$, by the well-known perturbation formula

$$\begin{aligned}(\varphi_{n,v}(\zeta),\ e^{i(n+v)\zeta})\varphi_{n,v}(\zeta) &= e^{i(n+v)\zeta} \\ &+ \sum_{k=1,2,\ldots} \frac{(-1)^{k+1}}{2i\pi}\int_C (T_v(0)-\lambda)^{-1}p(x)^k (T_v(0)-\lambda)^{-1}e^{i(n+v)\zeta}d\lambda,\end{aligned} \tag{3.5.12}$$

where C is a contour containing only the eigenvalue $(n+t)^2|\delta|^2$. Using

$$(T_v(0)-\lambda)^{-1}e^{i(n+v)\zeta} = \frac{e^{i(n+v)\zeta}}{(n+v)^2|\delta|^2 - \lambda},$$

we see that the kth ($k = 1, 2, 3, 4$) term F_k of the series (3.5.12) has the form

$$F_1 = \frac{1}{2i\pi}\int_C \sum_{m=1,-1} \frac{p_m e^{i(n+m+v)\zeta}}{((n+v)^2|\delta|^2-\lambda)((n+m+v)^2|\delta|^2-\lambda)}d\lambda,$$

$$\begin{aligned}F_2 = \frac{-1}{2i\pi}\int_C \sum_{m,l=1,-1} &\frac{p_m p_l e^{i(n+m+l+v)\zeta}}{((n+v)^2|\delta|^2-\lambda)} \\ &\times \frac{1}{((n+m+v)^2|\delta|^2-\lambda)((n+m+l+v)^2|\delta|^2-\lambda)}d\lambda,\end{aligned}$$

$$F_3 = \frac{1}{2i\pi}\int_C \sum_{m,l,k=1,-1} \frac{p_m p_l p_k e^{i(n+m+l+k+v)\zeta}}{((n+v)^2|\delta|^2-\lambda)((n+m+v)^2|\delta|^2-\lambda)}$$
$$\times \frac{1}{((n+m+l+v)^2|\delta|^2-\lambda)((n+m+l+k+v)^2|\delta|^2-\lambda)}d\lambda,$$
$$F_4 = \frac{-1}{2i\pi}\int_C \sum_{m,l,k,r=1,-1} \frac{p_m p_l p_k p_r e^{i(n+m+l+k+r+v)\zeta}}{((n+m+l+k+r+v)^2|\delta|^2-\lambda)}$$
$$\times \frac{1}{((n+m+v)^2|\delta|^2-\lambda)((n+m+l+v)^2|\delta|^2-\lambda)}$$
$$\times \frac{1}{((n+m+l+k+v)^2|\delta|^2-\lambda)((n+v)^2|\delta|^2-\lambda)}d\lambda.$$

Since the distance between $(n+v)^2|\delta|^2$ and $(n'+v)^2|\delta|^2$ for $n' \neq n$ is greater than $c_{17}n$, we can choose the contour C such that

$$\frac{1}{|(n'+v)^2|\delta|^2-\lambda|} < \frac{c_{18}}{n},\ \forall\lambda \in C,\ \forall n' \neq n$$

and the length of C is less than c_{19}. Therefore

$$F_5 + F_6 + \cdots = O(n^{-5}).$$

Now we calculate the integrals in F_1, F_2, F_3, F_4 by the Cauchy integral formula and then decompose the obtained expression in power of $\frac{1}{n}$. Then

$$F_1 = e^{i(n+v)\zeta}((p_1e^{i\zeta} - p_{-1}e^{-i\zeta})\frac{1}{|\delta|^2}(\frac{-1}{2n} + \frac{v}{2n^2} - \frac{4v^2+1}{8n^3} + O(\frac{1}{n^4}))$$
$$+ (p_1e^{i\zeta} + p_{-1}e^{-i\zeta})\frac{1}{|\delta|^2}(\frac{v}{4n^2} - \frac{v}{2n^3} + \frac{12v^2+1}{16n^4} + O(\frac{1}{n^5}))).$$

Let $F_{2,1}$ and $F_{2,2}$ be the sum of the terms in F_2 for which $m+l = \pm 2$ and $m+l=0$ respectively, i.e.,

$$F_2 = F_{2,1} + F_{2,2},$$

where

$$F_{2,1} = e^{i(n+v)\zeta}(((p_1)^2e^{2i\zeta} + (p_{-1})^2e^{-2i\zeta})\frac{1}{|\delta|^4}(\frac{-1}{8n^2} + \frac{-v}{4n^3} - \frac{12v^2+7}{32n^4} + O(\frac{1}{n^5}))$$
$$+ ((p_1)^2e^{2i\zeta} - (p_{-1})^2e^{-2i\zeta})\frac{1}{|\delta|^4}(\frac{-3}{16n^3} + O(\frac{1}{n^4}))),$$
$$F_{2,2} = e^{i(n+v)\zeta}\,|p_1|^2\,(\frac{c_{20}}{n^2} + \frac{c_{21}}{n^3} + \frac{c_{22}}{n^4} + O(\frac{1}{n^5}))$$

and c_{20}, c_{21}, c_{22} are the known constants. Similarly,

$$F_3 = F_{3,1} + F_{3,2},$$

where $F_{3,1}$ and $F_{3,2}$ are the sum of the terms in F_3 for which $m+l+k=\pm 3$ and $m+l+k=\pm 1$ respectively. Hence

$$F_{3,1} = e^{i(n+v)\zeta}((p_1^3 e^{3i\zeta} - p_{-1}^3 e^{-i\zeta})\frac{1}{|\delta|^6}(\frac{-1}{48n^3} + O(\frac{1}{n^4}))$$
$$+ (p_1^3 e^{3i\zeta} + p_{-1}^3 e^{-3i\zeta})\frac{1}{|\delta|^6}(\frac{1}{16n^4} + O(\frac{1}{n^5}))),$$
$$F_{3,2} = e^{i(n+v)\zeta}((p_1 e^{i\zeta} - p_{-1}e^{-i\zeta})\,|p_1|^2\,(\frac{c_{23}}{n^3} + \frac{c_{24}}{n^4} + O(\frac{1}{n^5}))$$
$$+ (p_1 e^{i\zeta} + p_{-1}e^{-i\zeta})\,|p_1|^2\,(\frac{c_{25}}{n^4} + O(\frac{1}{n^5}))).$$

Similarly

$$F_4 = F_{4,1} + F_{4,2} + F_{4,3},$$

where $F_{4,1}, F_{4,2}, F_{4,3}$ are the sum of the terms in F_4 for which $m+l+k+r=\pm 4, m+l+k+r=\pm 2$, $m+l+k+r=0$ respectively. Thus

$$F_{4,1} = e^{i(n+v)\zeta}(p_1^4 e^{4i\zeta} + p_{-1}^4 e^{-4i\zeta})\frac{1}{|\delta|^8}(\frac{1}{384n^4} + O(\frac{1}{n^5})),$$
$$F_{4,2} = e^{i(n+v)\zeta}(p_1^2 e^{2i\zeta} + p_{-1}^2 e^{-2i\zeta})\,|p_1|^2\,(\frac{c_{26}}{n^4} + O(\frac{1}{n^5}))),$$
$$F_{4,3} = e^{i(n+v)\zeta}\,|p_1|^4\,(\frac{c_{27}}{n^4} + O(\frac{1}{n^5}))).$$

Since $p_{-1}^k e^{-ik\zeta}$ is conjugate of $p_1^k e^{ik\zeta}$, the real and imaginary parts of $F_k e^{-i(n+v)\zeta}$ consist of terms with multiplicands

$$p_1^k e^{ik\zeta} + p_{-1}^k e^{-ki\zeta} \quad \& \quad p_1^k e^{ik\zeta} - p_{-1}^k e^{-ik\zeta}$$

respectively. Taking into account this and using the above estimations, we get

$$|(\varphi_{n,v}, e^{i(n+v)\zeta})\varphi_{n,v}|^2 = 2(\sum_{k=1,2,3,4} \mathbf{Re}(F_k) + \mathbf{Re}(F_1F_2) + \mathbf{Re}(F_1F_3)) + |F_1|^2 + |F_2|^2 + O(n^{-5})$$
$$= 1 + \frac{1}{2n^2}\frac{1}{|\delta|^2}(p_1e^{i\zeta} + p_{-1}e^{-i\zeta} + c_{28}|p_1|^2) + \frac{1}{n^3}((p_1e^{i\zeta} + p_{-1}e^{-i\zeta})c_{29}$$
$$+ c_{30}|p_1|^2) + \frac{1}{n^4}((p_1e^{i\zeta} + p_{-1}e^{-i\zeta})c_{31} + c_{32}|p_1|^2 + c_{33}|p_1|^4$$
$$+ c_{34}|p_1|^2(p_1e^{i\zeta} + p_{-1}e^{-i\zeta}) + (c_{35} + c_{36}|p_1|^2)(p_1^2e^{2i\zeta} + p_{-1}^2e^{-2i\zeta}))$$
$$+ O(\frac{1}{n^5}),$$

where $\mathbf{Re}(F)$ denotes the real part of F. On the other hand

$$|(\varphi_{n,v}(\zeta), e^{i(n+v)\zeta})|^2 = (c_{37}\frac{1}{n^2} + c_{38}\frac{1}{n^3} + c_{39}\frac{1}{n^4})|p_1|^2 + c_{40}\frac{1}{n^4}|p_1|^4 + O(\frac{1}{n^5}).$$

These equalities imply (3.1.18). The invariant (3.1.19) is a consequence of (3.1.18), (3.1.16) and (3.1.7) for $k=2,4$ ■

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Chapter 4
Periodic Potential from the Spectral Invariants

Abstract In this chapter, we consider the inverse problem of the three-dimensional Schrödinger operator $L(q)$ with a periodic, relative to a lattice Ω of $\mathbb{R}^3$, potential q. Firstly, we construct a set D of trigonometric polynomials such that: (a) D is dense in $W_2^s(\mathbb{R}^3/\Omega)$, where $s > 3$, in the $\mathbb{C}^\infty$-topology, (b) any element q of the set D can be determined constructively and uniquely, modulo inversion $x \to -x$ and translations $x \to x + \tau$ for $\tau \in \mathbb{R}^3$, from the given spectral invariants that were determined constructively from the given Bloch eigenvalues. Then a special class V of the periodic potentials is constructed, which can be easily and constructively determined from the spectral invariants. This chapter consists of 7 sections. First section is introduction, where we describe briefly the scheme of this chapter and discuss the related papers. In the second section using the spectral invariants obtained in Chap. 3 we find the simplest invariants for the sets D and V. In the third, fourth and fifth sections we give algorithms for the unique determination of the potential $q \in D$ and $q \in V$ respectively from the simplest spectral invariants. In the sixth section we consider the stability of the algorithm for $q \in V$ with respect to the spectral invariants and Bloch eigenvalues. Finally, in the seventh section we prove that there are no other periodic potentials in the set of large class of functions whose Bloch eigenvalues coincide with the Bloch eigenvalues of $q \in V$. Thus, Chap. 4 gives some examples and ideas for finding the potential from the spectral invariants and hence from the Bloch eigenvalues. Besides it gives a theoretical base (a lot of nonlinear equations with respect to the Fourier coefficients of q) to solve numerically this problem.

4.1 Introduction

We investigate the inverse problem for the three-dimensional Schrödinger operator $L(q)$ generated in $L_2(\mathbb{R}^3)$ by the differential expression $l(u) = -\Delta u + q(x)u$, where $x \in \mathbb{R}^3$, with a real periodic, relative to a lattice Ω of $\mathbb{R}^3$, potential $q(x)$. Let $\omega_1, \omega_2, \omega_3$ be a basis of the lattice Ω and

$$F = \{c_1\omega_1 + c_2\omega_2 + c_3\omega_3 : c_k \in [0, 1), k = 1, 2, 3\}$$

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be a fundamental domain $\mathbb{R}^3/\Omega$ of Ω. Recall that the spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$ for $t \in F^*$ generated in $L_2(F)$ by the expression $l(u)$ and the conditions

$$u(x+\omega) = e^{i\langle t,\omega\rangle}u(x),\ \forall\omega \in \Omega,$$

where F^* is the fundamental domain of the lattice Γ, Γ is the lattice dual to Ω, and $\langle.,.\rangle$ is the inner product in $\mathbb{R}^3$. The eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq \cdots$ of $L_t(q)$ are called the Bloch eigenvalues of $L(q)$. These eigenvalues define the functions $\Lambda_1(t),\ \Lambda_2(t), \ldots$ of t that are called the band functions of $L(q)$. The aim of this chapter is the constructive determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the Bloch eigenvalues.

The inverse problems of the one-dimensional Schrödinger operator, that is, the Hill operator, denoted by $H(q)$, and the multidimensional Schrödinger operator $L(q)$ are absolutely different. Inverse spectral theory for the Hill operator has a long history and there exist many books and papers about it (see, for example, [Le, Mar, PoTr]). In order to determine the potential q, where $q(x+\pi) = q(x)$, of the Hill operator, in addition to the given band functions $\Lambda_1(t),\ \Lambda_2(t), \ldots$, one needs to know the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the Dirichlet boundary value problem and the signs of the numbers $u_-(\sqrt{\lambda_1}),\ u_-(\sqrt{\lambda_2}), \ldots$, where $u_-(\lambda) = c(\lambda,\pi) - s'(\lambda,\pi)$ and $c(\lambda, x)$, $s(\lambda, x)$ are the solutions of the Hill equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x)$$

satisfying $c(\lambda, 0) = s'(\lambda, 0) = 1,\ c'(\lambda, 0) = s(\lambda, 0) = 0$ (see [Mar], Chap. 3, Sect. 4). In other words, the potential q of the Hill operator can not be determined uniquely from the given band functions, since if the band functions $\Lambda_1(t),\ \Lambda_2(t), \ldots$ of $H(q)$ are given, then for every choice of the numbers $\lambda_1, \lambda_2, \ldots$ from the gaps $\Delta_1, \Delta_2, \ldots$ of the spectrum of the Hill operator and for every choice of the signs of the numbers $u_-(\lambda_1),\ u_-(\lambda_2), \ldots$, there exists a potential q having $\Lambda_1(t),\ \Lambda_2(t), \ldots$ as the band functions and $\lambda_1, \lambda_2, \ldots$ as the Dirichlet eigenvalues. In spite of this, it is possible to determine uniquely the potential q of the multidimensional Schrödinger operator $L(q)$ from only the given band functions for a certain class of potential. Because, in the case $d > 1$ the band functions give more informations. Namely, the band functions give the spectral invariants that have no meaning in the case $d = 1$. We solve the inverse problem by these spectral invariants. We will discuss this in the end of the introduction.

The inverse problem for the multidimensional Schrödinger operator $L(q)$ is investigated for the first time by Eskin et al. in the papers [EsRaTr1, EsRaTr2]. In [EsRaTr1] the following result was proved:

Assume that the lattice Ω of $\mathbb{R}^d$ is such that, for $\omega,\ \omega' \in \Omega,\ |\omega'| = |\omega|$ implies $\omega' = \pm\omega$. If $q(x)$ and $\widetilde{q}(x)$ are real analytic, then the equality

$$Spec(L_0(q)) = Spec(L_0(\widetilde{q})) \tag{4.1.1}$$

implies the equalities

$$Spec(L_t(q)) = Spec(L_t(\widetilde{q})) \tag{4.1.2}$$

for all $t \in \mathbb{R}^d$, *where* $Spec(L_t(q))$ *is the spectrum of the operator* $L_t(q)$ *and* $L_0(q)$ *is the operator* $L_t(q)$ *when* $t = (0, 0, \ldots, 0)$.

In [EsRaTr2] the following result was proved for the two-dimensional Schr ödinger operator $L(q)$:

For $\Omega \subset \mathbb{R}^2$ *satisfying the condition: if* $|\omega'| = |\omega|$ *for* $\omega, \omega' \in \Omega$, *then* $\omega' = \pm\omega$; *there is a set* $\{M_\alpha\}$ *of manifolds of potentials such that*

(a) $\{M_\alpha : \alpha \in [0, 1]\}$ *is dense in the set of smooth periodic potentials in the* C^∞*-topology,*

(b) *for each* α *there is a dense open set* $Q_\alpha \subset M_\alpha$ *such that for* $q \in Q_\alpha$ *the set of real analytic* $\widetilde{q}$ *satisfying* (4.1.1) *and the set of* $\widetilde{q} \in C^6(F)$ *satisfying* (4.1.2) *for all* $t \in \mathbb{R}^2$ *are finite modulo translations.*

Eskin [Es] extended the results of the papers [EsRaTr1, EsRaTr2] to the case of two-dimensional Schrödinger operator

$$H = (i\nabla + A(x))^2 + V(x), \; x \in \mathbb{R}^2$$

with periodic magnetic potential $A(x) = (A_1(x), A_2(x))$ and electric potential $V(x)$. The proof of the results of those papers is not constructive and does not seem to give any idea about possibility to construct explicitly a periodic potential.

In this chapter, we give an algorithm for the unique (modulo the inversion and translations) determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the spectral invariants which were determined constructively in Chap. 3 from the given Bloch eigenvalues. As a result, we determine constructively the potential from the given Bloch eigenvalues. The results of this chapter were published in [Ve11, Ve12].

To describe the brief scheme of this chapter, we begin by recalling the invariants obtained in Chap. 3 which will be used here. An element a of the lattice Γ is said to be a visible element of Γ if a is an element of Γ of the minimal norm belonging to the line $a\mathbb{R}$. Denote by S the set of all visible elements of Γ. Clearly,

$$q(x) = \frac{1}{2}\sum_{a\in S} q^a(x),$$

where

$$q^a(x) = \sum_{n\in\mathbb{Z}} z(na)e^{in\langle a,x\rangle},$$

and $z(c) =: (q, e^{i\langle c,x\rangle})$ for $c \in \Gamma$ is the Fourier coefficient of q. Here $(.,.)$ is the inner product in $L_2(F)$. The function $q^a(x)$ is known as directional potential of q

corresponding to the visible element a. Let a be a visible element of Γ, Ω_a be the sublattice $\{\omega \in \Omega : \langle \omega, a \rangle = 0\}$ of Ω in the hyperplane $H_a = \{x \in \mathbb{R}^3 : \langle x, a \rangle = 0\}$ and

$$\Gamma_a =: \{\gamma \in H_a : \langle \gamma, \omega \rangle \in 2\pi\mathbb{Z}, \quad \forall \omega \in \Omega_a\}$$

be the lattice dual to Ω_a. Let β be a visible element of Γ_a and $P(a, \beta)$ be the plane containing a, β, and the origin. Define a function $q_{a,\beta}(x)$ by

$$q_{a,\beta}(x) = \sum_{c \in (P(a,\beta) \cap \Gamma) \backslash a\mathbb{R}} \frac{c}{\langle \beta, c \rangle} z(c) e^{i\langle c, x \rangle}. \tag{4.1.3}$$

In Chap. 3, we constructively determined the following spectral invariants

$$\int_F \left| q^a(x) \right|^2 dx, \tag{4.1.4}$$

$$\int_F \left| q_{a,\beta}(x) \right|^2 q^a(x) dx \tag{4.1.5}$$

from the asymptotic formulas for the band functions of $L(q)$ obtained in Chap. 2. Moreover, in Chap. 3 we constructively determined the invariant

$$\int_F |q_{a,\beta}(x)|^2 (z^2(a) e^{i2\langle a, x \rangle} + z^2(-a) e^{-i2\langle a, x \rangle}) dx \tag{4.1.6}$$

when the directional potential $q^a(x)$ has the form

$$q^a(x) = z(a) e^{i\langle a, x \rangle} + z(-a) e^{-i\langle a, x \rangle}. \tag{4.1.7}$$

In this chapter, fixing the inversion and translations:

$$x \to -x \;\&\; x \to x + \tau, \tau \in \mathbb{R}^3, \tag{4.1.8}$$

we give an algorithm for the unique determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the invariants (4.1.4)–(4.1.6). Note that the potential q can be uniquely determined only by fixing the inversion and translations (4.1.8), since $L(q(x))$, $L(q(-x))$ and $L(q(x + \tau))$ have the same band functions and hence the same invariants (4.1.4)–(4.1.6).

First we consider the invariants (4.1.4)–(4.1.6) for the trigonometric polynomials of the form

$$q(x) = \sum_{a \in Q(N,M,S)} z(a) e^{i\langle a, x \rangle}, \tag{4.1.9}$$

where N, M, S are integers,

$$Q(N, M, S) = \{n\gamma_1 + m\gamma_2 + s\gamma_3 : |n| \le N, |m| \le M, \ |s| \le S\}\backslash\{0\},$$

and $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying $\langle \gamma_i, \omega_j \rangle = 2\pi\delta_{i,j}$. If $a = n\gamma_1 + m\gamma_2 + s\gamma_3$, then we write (n, m, s) and $z(n, m, s)$ instead of a and $z(a)$ respectively. For brevity of the notations, instead of $Q(N, M, S)$ we write Q if it is not ambiguous.

To describe the invariants (4.1.4)–(4.1.6) for (4.1.9), let us introduce some notations. If $b \in (\Gamma \cap P(a, \beta))\backslash a\mathbb{R}$, then the plane $P(a, \beta)$ coincides with the plane $P(a, b)$. Moreover every vector $b \in (P(a, \beta) \cap \Gamma)\backslash a\mathbb{R}$ has an orthogonal decomposition

$$b = s\beta + \mu a, \tag{4.1.10}$$

where s is a nonzero integer, β is a visible element of Γ_a, and μ is a real number. Therefore, for every plane $P(a, b)$, where $b \in \Gamma$, there exists a plane $P(a, \beta)$, where β is defined by (4.1.10), which coincides with $P(a, b)$.

Notation 4.1.1 *For every pair* $\{a, b\}$, *where a is visible element of* Γ *and* $b \in \Gamma$, *we denote by* $I_1(a, b)$ *and* $I_2(a, b)$ *the invariants* (4.1.5) *and* (4.1.6) *respectively, where* β *is a visible element of* Γ_a *defined by* (4.1.10).

Definition 4.1.1 A visible vector $a \in \Gamma$ is said to be long visible (with respect to Q) if $sa \in Q$ if and only if $s = \mp 1$.

If a is long visible, then the directional potential q^a of (4.1.9) has the form (4.1.7). Therefore the invariant (4.1.4) is

$$\|q^a\|^2 \equiv 2|z(a)|^2 \tag{4.1.11}$$

and hence the invariant (4.1.4) gives the absolute value of the Fourier coefficient $z(a)$. Moreover, we prove that there exist a lot of pairs $\{a, b\}$ such that the invariants (4.1.11), $I_1(a, b)$, and $I_2(a, b)$ give the following simple invariants

$$S_1(a, b) = Re(z(-a)z(a - b)z(b)), \ A_1(a, b) = \cos(-\alpha(a) + \alpha(a - b) + \alpha(b)), \tag{4.1.12}$$

$$S_2(a, b) = Re(z^2(-a)z(a + b)z(a - b)), \ A_2(a, b) = \cos(-2\alpha(a) + \alpha(a + b) + \alpha(a - b)), \tag{4.1.13}$$

where $Re(z)$ is the real part of the complex number z, $z(a) = r(a)e^{i\alpha(a)}$, $\alpha(a) \in (-\pi, \pi]$. In other words, for these pairs we have the equations

$$-\alpha(a) + \alpha(a - b) + \alpha(b) = d(a, b)e(1, a, b)(mod 2\pi), \tag{4.1.14}$$

$$-2\alpha(a) + \alpha(a + b) + \alpha(a - b) = d(a, b)e(2, a, b)(mod 2\pi), \tag{4.1.15}$$

where $e(i, a, b) =: \arccos A_i(a, b)$ for $i = 1, 2$ are the known numbers belonging to $[0, \pi]$, $d(a, b) = \pm 1$, and the equality $\theta = \varphi(mod 2\pi)$ means that $\theta - \varphi = 2k\pi$ for some integer k.

In Sect. 4.2, we consider the invariants (4.1.5) and (4.1.6) for the polynomials (4.1.9) and find a lot of pairs $\{a, b\}$ such that there exist the simple invariants $A_1(a, b)$, $A_2(a, b)$ corresponding to these pairs.

Besides in Sect. 4.2, we consider the invariants (4.1.4)–(4.1.6) when $q(x)$ has the form

$$q(x) = \sum_{a \in Q(1,1,1)} z(a) e^{i\langle a,x \rangle}, \tag{4.1.16}$$

where

$$Q(1, 1, 1) =: \{n\gamma_1 + m\gamma_2 + s\gamma_3 : |n| \le 1, |m| \le 1, \ |s| \le 1\} \backslash \{(0, 0, 0)\}, \ z(a) \ne 0 \tag{4.1.17}$$

for all $a \in Q(1, 1, 1)$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying

$$\langle \gamma_i, \gamma_j \rangle \ne 0, \ \langle \gamma_i + \gamma_j, \gamma_k \rangle \ne 0, \ |\gamma_i| \ne |\gamma_j|, \ \langle \gamma_i + \gamma_j + \gamma_k, \gamma_i - \gamma_j - \gamma_k \rangle \ne 0 \tag{4.1.18}$$

for all different indices i, j, k. Note that every lattice has a basis satisfying (4.1.18) (see Proposition 4.2.3 in Sect. 4.2). Moreover in Proposition 4.2.2 of Sect. 4.2, we prove that every element a of $Q(1, 1, 1)$ is a visible element of Γ and hence the directional potential $q^a(x)$ of (4.1.16) has the form (4.1.7). Therefore, we have the invariants (4.1.4)–(4.1.6) for all $a \in Q(1, 1, 1)$.

In Sect. 4.3 we give an algorithm for finding the Fourier coefficients $z(n, m, s)$ when $(n, m, s) \in B(N, M, S)$, where

$$B(N, M, S) = \{(n, m, s) \in Q(N, M, S) : nms(|n| - N)(|m| - M)(|s| - S) = 0\}.$$

First, we find $z(a)$ when a belongs to the boundary $\partial \widetilde{Q}$ of the parallelepiped

$$\widetilde{Q} =: \{x = (x_1, x_2, x_3) : |x_1| \le N, |x_2| \le M, \ |x_2| \le S\},$$

that is, we find the Fourier coefficients $z(n, m, s)$ if either $n = N, -N$, or $m = M, -M$, or $s = S, -S$. For this we use the following two observations.

(1) All boundary points of $\widetilde{Q}$ except the points of the set

$$A(N, M, S) = \{(\pm N, 0, 0), (0, \pm M, 0), (0, 0, \pm S)\} \cup \{(n, m, s) : |n| = |m| = |s| = N\}$$

are long visible, if N, M, S are distinct prime numbers, satisfying $N < M < S$. Hence, the absolute value $r(a)$ of $z(a)$ is known by (4.1.11).

(2) If a is a boundary point of $\widetilde{Q}$, then there are a lot of vectors b such that there exists simple invariant $A_2(a, b)$ corresponding to the pair $\{a, b\}$.

Thus, we can write a lot of equations of type (4.1.15) with respect to the argument of the Fourier coefficients. If $d(a, b)$ and the values of two summands in the left-hand side of (4.1.15) are known, then one can find the value of the third summand. To use these equations we need to know the values of the arguments of some Fourier coefficients. Three of them can be determined by fixing the translation $x \to x + \tau$, that is, by taking one of the functions $q(x + \tau)$. Namely, in the Sect. 4.3, we prove that the conditions

$$\alpha_\tau(N-1, M, S) = \alpha_\tau(N, M-1, S) = \alpha_\tau(N, M, S-1) = 0, \tag{4.1.19}$$

$$\alpha_\tau(N, M, S) \in [0, \frac{2\pi}{N+M+S-1}), \tag{4.1.20}$$

where

$$\alpha_\tau(a) = \arg(q(x+\tau), e^{i\langle a,x\rangle}),$$

determine a unique value of τ.

Thus, in Sect. 4.3, using (4.1.19) and a lot of equations of type (4.1.15) we determine $z(a)$ when $a \in \partial\widetilde{Q}$. Then, using this, we find $z(n, m, s)$, when $nms = 0$.

In Sect. 4.4 we construct a dense in $W_2^s(F)$, where $s > 3$, in the $\mathbb{C}^\infty$-topology set D of trigonometric polynomials, such that every $q \in D$ can be found by the given algorithm.

In Sect. 4.5 fixing the inversion and translations (4.1.8), we give an algorithm for the unique determination of the potential (4.1.16) of the three-dimensional Schrödinger operator $L(q)$ from the invariants (4.1.4)–(4.1.6). Moreover, we give the formulas [see (4.5.12), (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and (4.5.33)–(4.5.35)] for finding Fourier coefficients $z(a)$ of the potential (4.1.16), by using the invariants $s_1, s_2, \ldots, s_{24}$ [see (4.5.11)] obtained from (4.1.4)–(4.1.6). These formulas explicitly express the Fourier coefficients in term of the invariants $s_1, s_2, \ldots, s_{24}$. Then using these formulas we find sufficient conditions [see (4.5.2)] on the invariants that allows to find the potential of the form (4.1.16) by formulas (4.5.12), (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and (4.5.33)–(4.5.35) (see Corollary 4.5.1). Note that, the sufficient conditions on the spectral data to solve the inverse problem for the multidimensional Schrödinger operator $L(q)$ is given for the first time in the paper [Ve12], albeit in a fairly restricted set of potentials q. It is expected that, this approach may open up new horizons for inverse problem of the multidimensional Schrödinger operator with a periodic potential. Since the invariants (4.1.5) and (4.1.6) do not exist in the case $d = 1$, we do not use the investigations of the inverse problem for the one dimensional Schrödinger operator $H(q)$. For this reason, we do not discuss a great number of papers about the inverse problem of the Hill operator.

In Sect. 4.6 we study the stability of the algorithm with respect to errors both in the invariants (4.1.4)–(4.1.6) and in the Bloch eigenvalues. Note that we determine constructively the potential from the band functions in two steps. At the first step we determined the invariants from the band functions in Chap. 3. At the second step, which is given in Sect. 4.5, we find the potential from the invariants. In Sect. 4.6 we consider the stability of the problems studied in both steps. First, using the asymptotic formulas obtained in Chap. 3, we write down explicitly the asymptotic expression of the invariants (4.1.4)–(4.1.6) in terms of the band functions and consider the stability of the invariants with respect to the errors in the Bloch eigenvalues (Theorem 4.6.1 and Proposition 4.6.1). Then we prove the stability of the algorithm given in Sect. 4.5 with respect to the errors in the invariants (Theorem 4.6.2).

In Sect. 4.7 we prove some uniqueness theorems. First, we prove a theorem about Hill operator $H(p)$ when $p(x)$ is a trigonometric polynomial (see Theorem 4.7.1). Then we construct a set W of all periodic functions $q(x)$ whose directional potentials $q^a(x)$ for all $a \in S\backslash\{\gamma_1, \gamma_2, \gamma_3\}$ are arbitrary continuously differentiable functions, where S is the set of all visible elements of Γ, $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying (4.1.18), and the directional potentials $q^{\gamma_1}(x)$, $q^{\gamma_2}(x)$, $q^{\gamma_3}(x)$ satisfy some conditions. At the end we prove that if q has the form (4.1.16), $\widetilde{q} \in W$ and the band functions of $L(q)$ and $L(\widetilde{q})$ coincide, then $\widetilde{q}$ is equal to q modulo inversion and translations (4.1.8) (see Theorem 4.7.2).

4.2 On the Simple Invariants

First, let us consider the invariants (4.1.5) and (4.1.6) for the trigonometric polynomial (4.1.9).

Definition 4.2.1 A pair $\{a, b\}$, where a is a long visible element of Q and $b \in Q$, is said to be a canonical pair of type 1 if $\langle b, a-b\rangle \neq 0$ and the following implication holds

$$\{c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R} \Leftrightarrow c \in \{b, a-b\}. \tag{4.2.1}$$

A pair $\{a, b\}$, where a is a long visible element of Q and $b \in Q$, is said to be a canonical pair of type 2 if $\langle a+b, a-b\rangle \neq 0$ and the following implication holds

$$\{a+c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R} \Leftrightarrow c \in \{\pm b\}. \tag{4.2.2}$$

Theorem 4.2.1 *If the potential q has the form* (4.1.9) *and a is long visible element of Q, then the invariants $I_1(a, b)$, $I_2(a, b)$, defined in* Notation 4.1.1*, yield the invariants*

$$Re(z(-a)(\sum_{c\in G_1} g(a,c)z(a-c)z(c))), \tag{4.2.3}$$

$$Re(z^2(-a)(\sum_{c\in G_2} h(a,c)z(a-c)z(a+c))), \tag{4.2.4}$$

where

$$g(a,c) = \frac{\langle c, c-a\rangle}{(\langle c, \beta\rangle)^2},\quad h(a,c) = \frac{\langle c+a, c-a\rangle}{(\langle c, \beta\rangle)^2},$$

G_1 and G_2 are the set of all c such that $\{c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R}$ and $\{a+c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R}$ respectively.

If $\{a, b\}$ *is a canonical pair of type* k, *where* $k = 1, 2$, *then* (4.2.3) *and* (4.2.4) *give the simple invariants* $S_k(a, b)$, $A_k(a, b)$ *defined in* (4.1.12) *and* (4.1.13).

Proof If the potential $q(x)$ has the form (4.1.9), then (4.1.3) becomes

$$q_{a,\beta}(x) = \sum_{c \in (P(a,\beta) \cap Q) \backslash a\mathbb{R}} \frac{c}{\langle \beta, c \rangle} z(c) e^{i\langle c, x \rangle}. \tag{4.2.5}$$

Using this and (4.1.7) in (4.1.5), we get

$$I_1(a, b) = \int_F \left| q_{a,\beta}(x) \right|^2 q^a(x) dx = \Sigma_1 + \Sigma_2, \tag{4.2.6}$$

where $I_1(a, b)$ is defined in Notation 4.1.1,

$$\Sigma_1 = \sum_{c \in (P(a,b) \cap Q) \backslash a\mathbb{R}} \frac{\langle c, c + a \rangle}{\langle c, \beta \rangle \langle c + a, \beta \rangle} z(c) z(-a - c) z(a), \tag{4.2.7}$$

$$\Sigma_2 = \sum_{c \in (P(a,b) \cap Q) \backslash a\mathbb{R}} \frac{\langle c, c - a \rangle}{\langle c, \beta \rangle \langle c - a, \beta \rangle} z(c) z(a - c) z(-a).$$

Since $Q(N, M, S)$ is symmetric with respect to the origin, the substitution $c' = -c$ in (4.2.7) does not change Σ_1. Using this substitution in (4.2.7) and then taking into account that $z(-b) = \overline{z(b)}$, $\langle a, \beta \rangle = 0$, we obtain

$$\Sigma_1 = \overline{\Sigma_2}, \ \Sigma_1 + \Sigma_2 = Re(2\Sigma_2).$$

This with (4.2.6) shows that the invariant $I_1(a, b)$ gives the invariant (4.2.3).

Replacing a by $2a$, in the same way, we obtain the invariant

$$Re(z^2(-a)(\sum_{c \in G} \frac{\langle c, c - 2a \rangle}{(\langle c, \beta \rangle)^2} z(2a - c) z(c))) \tag{4.2.8}$$

from the invariant $I_2(a, b)$, where G is the set of all c such that $\{c, 2a - c\} \subset (P(a, b) \cap Q) \backslash a\mathbb{R}$, $\langle c, c - 2a \rangle \neq 0$. Thus, in (4.2.8) replacing c by $a + c$ and using the obvious equality $\langle a, \beta \rangle = 0$, we get (4.2.4).

Now suppose that $\{a, b\}$ is a canonical pair of type 1. Then it follows from the definition of G_1 and from the definition of the canonical pair of type 1 that $G_1 = \{b, a - b\}$. Therefore (4.2.3) has the form

$$Re(z((-a)(\frac{\langle b, b - a \rangle}{(\langle b, \beta \rangle)^2} z(a - b) z(b) + \frac{\langle a - b, -b \rangle}{(\langle a - b, \beta \rangle)^2} z(b) z(a - b))).$$

The invariant $S_1(a,b)$ can be obtained from this invariant, because $\langle b, b-a\rangle = \langle a-b, -b\rangle$ and $\langle a, \beta\rangle = 0$. The invariant (4.1.11) and $S_1(a,b)$ imply $A_1(a,b)$. In the same way we obtain the invariants $S_2(a,b)$ and $A_2(a,b)$ from (4.2.4) □

Now we determine a lot of canonical pairs of the types 1 and 2.

Condition 4.1 *Suppose* $\Gamma = \mathbb{Z}^3$ *and* $z(n,m,s) \neq 0$ *for* $(n,m,s) \in B(N,M,S)$, *where* N, M, S *are prime numbers satisfying* $S > 2M,\ M > 2N, N \gg 1$.

Proposition 4.2.1 *Suppose the* Condition 4.1 *holds.*
(a) *The pair* $\{a, b\}$ *is a canonical pair of type* 2 *in each of the following cases:*
(1) $a = (N, M-1, s)$, $b = (0, \pm 1, p)$, *where* $s+p, s-p \in [-S, S]$, $|p| \le M-1$.
(2) $a = (N, m, S-1)$, $b = (0, q, \pm 1)$, *where* $m+q, m-q \in [-M, M]$.
(3) $a = (N, m, s)$, $b = (0, \pm 1, p)$, *where* $m \in [-M+1, M-1]$, $s+p, s-p \in [-S, S]$, $s-2p \notin [-S, S]$, $(N, m, s) \notin A(N, M, S)$ *and* $N^2+m^2-1+s^2-p^2 \neq 0$.
(b) *The pair* $\{a, b\}$ *is a canonical pair of type* 1 *in each of the following cases*
(1) $a = (N, M-1, s)$, $b = (0, -1, N)$, $S - N < s \le S$, $s \neq kN$,*where* $k \in Z$.
(2) $a = (N, M, 0)$, $b = (N, 0, S))$.
(c) *If* n *and* m *are the relatively prime nonnegative integers and* $(n, m, 0) \in Q$, *then*

$$Q \cap (P((0,-M,S),(n,m,0))) = (Q_{-1} \cup Q_0 \cup Q_1) \cap Q, \tag{4.2.9}$$

where $P((0,-M,S),(n,m,0))$ *is the plane passing through* $(0,0,0)$, $(0,-M,S)$, $(n,m,0)$ *and* $Q_k = \{l(n,m,0) + k(0,-M,S) : l \in \mathbb{Z}\}$ *for* $k = -1, 0, 1$.

Proof (a) The conditions of Condition 4.1 on N, M, S and the conditions of this proposition on s, p, q, m imply the inequality $\langle a+b, a-b\rangle \neq 0$. Now, by the Definition 4.2.1, we need to show that (4.2.2) holds. Let $c = (n_1, m_1, s_1)$ be any vector satisfying

$$\{a+c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R}. \tag{4.2.10}$$

Since, in all of the above cases, the first coordinate of a is N, the implication (4.2.10) and the definition of $Q(N, M, S)$ imply that $n_1 = 0$ for the all cases (1)–(3). Hence

$$c \in \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0\} =: \{x_1 = 0\}. \tag{4.2.11}$$

On the other hand it follows from (4.2.10) that $c \in P(a,b)$. Thus c belongs to the line intersection of the planes $P(a,b)$ and $\{x_1 = 0\}$. Since b also belongs to this line and b is a visible element of Γ, we have $c = kb$ for some nonzero integer k. Clearly, if k is not ± 1, then either $a+c$ or $a-c$ does not belongs to $Q(N, M, S)$, which means that (4.2.2) holds.

(b) First let us consider the case (1). It is clear that $\langle b, a-b\rangle = N(s-N) - M \neq 0$, since N and M are the distinct prime numbers. Therefore, we need to prove that (4.2.1) holds (see Definition 4.2.1). Let $c = (n_1, m_1, s_1)$ be any vector satisfying

$$\{c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R}. \tag{4.2.12}$$

If the vector c lies on plane $P(a, b)$, then the determinant of the matrix with rows a, b and c is zero. It gives the equality

$$N(s_1 + m_1 N) = n_1(s + (M - 1)N). \tag{4.2.13}$$

Since N is prime number and $s + (M - 1)N$ is not a multiple of N, we have $n_1 = kN$ for $k \in \mathbb{Z}$. Then $c = (kN, m_1, s_1)$. The set $Q(N, M, S)$ contains the vector c only in the following three cases: $k = 0$, $k = 1$, $k = -1$. In the case $k = 0$ we have $n_1 = 0$. Then from (4.2.13) one sees that $s_1 = -Nm_1$, that is, $c = m_1(0, 1, -N)$, where $m_1 \neq 0$. If $m_1 \neq -1$, i.e., $c \neq b$, then the conditions $S - N < s \leq S$ of the proposition imply that $a - c \notin Q$. Thus, in the case $k = 0$, we obtain that $c = b$. If $k = -1$, then one can readily see that $c = (-N, m_1, s_1)$, $a - c \notin Q$. It remains to consider the case $k = 1$, that is, $n_1 = N$. In this case we use the following obvious implication:

$$a \in \{x_k = n\},\ b \in \{x_k = 0\} \Rightarrow P(a, b) \cap \{x_k = 0\} = b\mathbb{R},\ P(a, b) \cap \{x_k = n\} = a + b\mathbb{R}. \tag{4.2.14}$$

Since

$$c = (N, m_1, s_1) \in \{x_1 = N\},\ a \in \{x_1 = N\},\ b \in \{x_1 = 0\},\ c \in P(a, b)\backslash a\mathbb{R}$$

[see (4.2.12)], the relation (4.2.14) yields that $c \in a + b\mathbb{R}$. Moreover, $c = a + kb$ for some nonzero integer k, since b is the visible element of Γ. Using this and taking into account that $a + kb$, where $a = (N, M - 1, s)$, $b = (0, -1, N)$, lies in Q if and only if $k = -1$, we obtain $c = a - b$.

Now consider the case (2). First let us prove that in this case the plane $P(a, b)$ contains only the vectors $\pm(N, M, 0)$, $\pm(N, 0, S)$, $\pm(0, M, -S)$ of Q. In fact, every element (n, m, s) of this plane satisfies the equation

$$S(nM - mN) = sNM. \tag{4.2.15}$$

First let us consider the case $s = 0$, i.e., the case $nM = mN$. Since N and M are distinct prime numbers and $-N \leq n \leq N$, $-M \leq m \leq M$, it follows that either $n = \pm N$, $m = \pm M$ or $n = m = 0$. Now consider the case $s \neq 0$. Then the right-hand side of (4.2.15) is a multiple of S. Therefore taking into account that S is a prime number satisfying Condition 4.1 and $-S \leq s \leq S$, we have $s = \pm S$. This together with (4.2.15) gives the relation $(n \pm N)M = mN$. From this relation one sees that either $n = \mp N$, $m = 0$ or $n = 0$, $m = \pm M$. Thus

$$P(a, b) \cap Q = \{\pm(N, M, 0), \pm(N, 0, S), \pm(0, M, -S)\}.$$

Using this and taking into account that $a = (N, M, 0)$, $b = (N, 0, S)$, we get

$$(P(a, b) \cap Q)\backslash a\mathbb{R} = \{\pm b, \pm(a - b)\}. \tag{4.2.16}$$

Now suppose that c is a vector satisfying (4.2.12). If $c = -b$, then

$$a - c = a + b \notin (P(a, b) \cap Q)\backslash a\mathbb{R}$$

due to (4.2.16). Similarly, if $c = -(a - b)$, then

$$a - c = 2a - b \notin (P(a, b) \cap Q)\backslash a\mathbb{R}$$

again due to (4.2.16). Therefore (4.2.12) and (4.2.16) imply the proof of (4.2.1).

(c) The relation $(n_1, m_1, s_1) \in P((0, -M, S), (n, m, 0))$ holds if and only if

$$S(mn_1 - m_1 n) = s_1 M n. \tag{4.2.17}$$

If $(mn_1 - m_1 n) = 0$, then $s_1 = 0$. If $(mn_1 - m_1 n) \neq 0$, then $s_1 = \pm S$, since S is prime satisfying Condition 4.1, and $(n, m, 0) \in Q$. Hence, (n_1, m_1, s_1) belongs to either $\{x_3 = 0\}$ or $\{x_3 = S\}$ or $\{x_3 = -S\}$. Therefore (4.2.9) follows from (4.2.14) □

Now let us consider the invariants (4.1.5) and (4.1.6) for the simple trigonometric polynomial (4.1.16). To describe the invariant (4.1.4) let us prove the following proposition.

Proposition 4.2.2 *Every element a of the set $Q(1, 1, 1)$, defined in (4.1.17), is a visible element of Γ and the corresponding directional potential q^a has the form (4.1.7).*

Proof Let a be element of $Q(1, 1, 1)$. By the definition of $Q(1, 1, 1)$

$$a = n\gamma_1 + m\gamma_2 + s\gamma_3, \ \ |n| \le 1, |m| \le 1, \ \ |s| \le 1, \ \ a \neq 0. \tag{4.2.18}$$

If a is not a visible element of Γ, then there exists a visible element b of Γ such that $a = kb$ for some integer $k > 1$. This with (4.2.18) implies that

$$b = \tfrac{1}{k}(n\gamma_1 + m\gamma_2 + s\gamma_3). \tag{4.2.19}$$

Since $b \in \Gamma$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ we have $b = n_1\gamma_1 + m_1\gamma_2 + s_1\gamma_3$, where n_1, m_1, s_1 are integers. Combining this with (4.2.19) and taking into account the linearly independence of the vectors $\gamma_1, \gamma_2, \gamma_3$, we get

$$(n_1 - \tfrac{n}{k})\gamma_1 + (m_1 - \tfrac{m}{k})\gamma_2 + (s_1 - \tfrac{s}{k})\gamma_3 = 0, \text{ and } n_1 - \tfrac{n}{k} = m_1 - \tfrac{m}{k} = s_1 - \tfrac{s}{k} = 0.$$

This is impossible, since $|n| \le 1, |m| \le 1, |s| \le 1$, at least one of the numbers n, m, s is not zero [see (4.2.18)], $k > 1$ and the numbers n_1, m_1, s_1 are integers. This contradiction shows that any element a of $Q(1, 1, 1)$ is a visible element of Γ. Therefore, it follows from the definition of $Q(1, 1, 1)$ [see (4.1.17)] that the line

$a\mathbb{R}$ contains only two elements a and $-a$ of the set $Q(1, 1, 1)$. This means that the directional potential q^a has the form (4.1.7) □

By Proposition 4.2.2 the invariant (4.1.4) for the potential (4.1.16) has the form

$$I(a) = |z(a)|^2, \ \forall a \in Q(1, 1, 1), \tag{4.2.20}$$

that is, we determine the absolute value of $z(a)$ for all $a \in Q(1, 1, 1)$.

To investigate the invariants (4.1.5) and (4.1.6), we use the conditions in (4.1.18). Therefore, first, let us consider these conditions.

Proposition 4.2.3 *Any lattice Γ has a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ satisfying (4.1.18). In particular, if*

$$\Gamma = \{(na, mb, sc) : n, m, s \in \mathbb{Z}\}, \tag{4.2.21}$$

where $a, b, c \in \mathbb{R}\backslash\{0\}$, then at least one of the bases $\{(a, 0, 0), (a, b, 0), (a, b, c)\}$ and $\{(-a, 0, 0), (a, b, 0), (a, b, c)\}$ of Γ satisfies (4.1.18).

Proof Suppose that a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of Γ does not satisfy (4.1.18). Define $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$ by

$$\widetilde{\gamma}_1 = \gamma_1, \ \widetilde{\gamma}_2 = n\gamma_1 + \gamma_2, \ \widetilde{\gamma}_3 = m\gamma_1 + s\gamma_2 + \gamma_3,$$

where n, m, s are integers. Since $\gamma_1 = \widetilde{\gamma}_1$, $\gamma_2 = \widetilde{\gamma}_2 - n\widetilde{\gamma}_1$, $\gamma_3 = \widetilde{\gamma}_3 - m\widetilde{\gamma}_1 - s(\widetilde{\gamma}_2 - n\widetilde{\gamma}_1)$, the triple $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$ is also a basis of Γ. In (4.1.18) replacing $\{\gamma_1, \gamma_2, \gamma_3\}$ by $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$, we obtain 12 inequalities with respect to n, m and s. Since n, m and s are arbitrary integers one can readily see that there exists n, m and s for which these inequalities hold. For example, let

$$\widetilde{\gamma}_1 = \gamma_1, \ \widetilde{\gamma}_2 = n\gamma_1 + \gamma_2, \ \widetilde{\gamma}_3 = n^2\gamma_1 + \gamma_3, \tag{4.2.22}$$

where n is a large positive number, that is, $n \gg 1$. Then it follows from (4.2.22) that

$$\left\langle \widetilde{\gamma}_i, \ \widetilde{\gamma}_j \right\rangle \gg 1, \ \left\langle \widetilde{\gamma}_i + \widetilde{\gamma}_j, \widetilde{\gamma}_j \right\rangle \gg 1, \ \forall i \neq j,$$

that is, the first and second inequalities in (4.1.18) hold. Besides, by (4.2.22), we have

$$|\widetilde{\gamma}_1|^2 \sim 1, \ |\widetilde{\gamma}_2|^2 \sim n^2, \ |\widetilde{\gamma}_3|^2 \sim n^4, \ \left\langle \widetilde{\gamma}_i, \ \widetilde{\gamma}_j \right\rangle = O(n^3), \tag{4.2.23}$$

where $a_n \sim b_n$ means that there exist positive constants c_1 and c_2 such that

$$c_1|b_n| < |a_n| < c_2|b_n| \tag{4.2.24}$$

for $n = 1, 2, \ldots$. The third inequality of (4.1.18) holds due to (4.2.23). By (4.2.23) the term $\pm|\widetilde{\gamma}_3|^2$ in the fourth inequality of (4.1.18) can not be cancelled by the other terms of this inequality. Thus, we proved that any lattice Γ has a basis $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3\}$ satisfying (4.1.18).

Note that, for the given lattice, one can easily find the basis satisfying (4.1.18). For example, in the case (4.2.21), one can readily see that the basis $\{(a, 0, 0), (a, b, 0), (a, b, c)\}$ satisfies (4.1.18) if $c^2 \neq 3a^2$ and the basis $\{(-a, 0, 0), (a, b, 0), (a, b, c)\}$ satisfies (4.1.18) if $c^2 \neq a^2$. Thus at least one of the bases

$\{(a, 0, 0), (a, b, 0), (a, b, c)\}$ and $\{(-a, 0, 0), (a, b, 0), (a, b, c)\}$ satisfies (4.1.18) □

Now to describe the invariants (4.1.5) and (4.1.6) for (4.1.16) let us introduce some notations. If $b \in (\Gamma \cap P(a, \beta))\backslash a\mathbb{R}$, then the plane $P(a, \beta)$ coincides with the plane $P(a, b)$. Moreover, every vector $b \in (P(a, \beta) \cap \Gamma)\backslash a\mathbb{R}$ has an orthogonal decomposition (4.1.10). Therefore, as we noted in introduction, for every plane $P(a, b)$, where $b \in \Gamma$, there exists a plane $P(a, \beta)$, where β is defined by (4.1.10), coinciding with $P(a, b)$. For every pair $\{a, b\}$, where a is visible element of Γ and $b \in \Gamma$, we redenote by $I_1(a, b)$ and $I_2(a, b)$ the invariants $I_1(a, \beta)$ and $I_2(a, \beta)$ defined in (4.1.5) and (4.1.6) respectively, where β is a visible element of Γ_a defined by (4.1.10).

Theorem 4.2.2 *The following equalities for the invariant* (4.1.5) *hold:*

$$I_1(\gamma_i + \gamma_j, \gamma_i) = A_1(\gamma_i + \gamma_j, \gamma_i) Re(z(-\gamma_i - \gamma_j) z(\gamma_j) z(\gamma_i)), \tag{4.2.25}$$

$$I_1(\gamma_i - \gamma_j, \gamma_i) = A_1(\gamma_i - \gamma_j, \gamma_i) Re(z(-\gamma_i + \gamma_j) z(-\gamma_j) z(\gamma_i)), \tag{4.2.26}$$

$$I_1(\gamma, \gamma_i) = A_1(\gamma, \gamma_i) Re(z(-\gamma) z(\gamma - \gamma_i) z(\gamma_i)), \tag{4.2.27}$$

$$I_1(2\gamma_i - \gamma, \gamma_i) = A_1(2\gamma_i - \gamma, \gamma_i) Re(z(\gamma - 2\gamma_i) z(\gamma_i - \gamma) z(\gamma_i)), \tag{4.2.28}$$

where $A_1(\gamma_i \pm \gamma_j, \gamma_i)$, $A_1(\gamma, \gamma_i)$, $A_1(2\gamma_i - \gamma, \gamma_i)$ *are nonzero numbers defined by*

$$A_1(a, b) = 2\left((\langle b, \beta\rangle)^{-2} + (\langle a - b, \beta\rangle)^{-2}\right) \langle a - b, b\rangle, \tag{4.2.29}$$

$\{\gamma_1, \gamma_2, \gamma_3\}$ *is a basis of* Γ *satisfying* (4.1.18), $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ *and* $Re(z)$ *is the real part of* z.

Proof If the potential $q(x)$ has the form (4.1.16), then (4.1.3) becomes

$$q_{a,\beta}(x) = \sum_{c \in (P(a,\beta) \cap Q)\backslash a\mathbb{R}} \frac{c}{\langle \beta, c\rangle} z(c) e^{i\langle c, x\rangle}, \tag{4.2.30}$$

where, for brevity, $Q(1, 1, 1)$ is denoted by Q. Using this and (4.1.7) in (4.1.5) and taking into account that the invariant $I_1(a, \beta)$ defined by (4.1.5) is redenoted by $I_1(a, b)$, we get

$$I_1(a, b) = \Sigma_1 + \Sigma_2, \tag{4.2.31}$$

where

$$\Sigma_1 = \sum_{c \in (P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle c, c+a\rangle}{\langle c, \beta\rangle \langle c+a, \beta\rangle} z(c)z(-a-c)z(a),$$

$$\Sigma_2 = \sum_{c \in (P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle c, c-a\rangle}{\langle c, \beta\rangle \langle c-a, \beta\rangle} z(c)z(a-c)z(-a)$$

and β is a visible element of Γ_a defined by (4.1.10). Since $Q(1, 1, 1)$ is symmetric with respect to the origin, the substitution $\widetilde{c} = -c$ in Σ_1 does not change Σ_1. Using this substitution in Σ_1 and then taking into account that $z(-b) = \overline{z(b)}$, $\langle a, \beta\rangle = 0$, we obtain $\Sigma_1 = \overline{\Sigma_2}$. This with (4.2.31) gives

$$I_1(a, b) = 2Re\left(z(-a)\left(\sum_{c \in (P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle a-c, c\rangle}{(\langle c, \beta\rangle)^2} z(a-c)z(c)\right)\right). \tag{4.2.32}$$

Since a, β, $(0, 0, 0)$ belong to the plane $P(a, b)$ and β orthogonal to the line $a\mathbb{R}$, we have

$$\langle c, \beta\rangle \neq 0,\ \forall c \in (P(a, b) \cap Q)\backslash a\mathbb{R}. \tag{4.2.33}$$

Now using (4.2.32) we obtain the invariants (4.2.25) and (4.2.26) as follows. First let us consider (4.2.25). Let $a = \gamma_i + \gamma_j$ and $b = \gamma_i$. Then

$$(P(a, b) \cap Q)\backslash a\mathbb{R} = \{\pm\gamma_i,\ \pm\gamma_j,\ \pm(\gamma_i - \gamma_j)\}.$$

On the other hand, if $c \in \{-\gamma_i, -\gamma_j, \pm(\gamma_i - \gamma_j)\}$, then $a - c \notin Q$. Therefore, the summation in the formula (4.2.32) for the case $a = \gamma_i + \gamma_j$, $b = \gamma_i$ is taken over $c \in \{\gamma_i, \gamma_j\}$ and hence (4.2.25) holds. It follows from (4.2.33) and from the first inequality in (4.1.18) that $A_1(\gamma_i + \gamma_j, \gamma_i) \neq 0$.

Replacing $a = \gamma_j$ by $-\gamma_j$ and arguing as in the proof of (4.2.25), we get (4.2.26).

Now let us consider (4.2.27). Let $a = \gamma = \gamma_1 + \gamma_2 + \gamma_3$ and $b = \gamma_1$. Then

$$(P(a, b) \cap Q)\backslash a\mathbb{R} = \{\pm\gamma_1,\ \pm(\gamma_2 + \gamma_3)\}.$$

If $c = -\gamma_1$, or $c = -\gamma_2 - \gamma_3$, then $a - c \notin Q$. Therefore, the summation in the formula (4.2.32) for this case is taken over $c \in \{\gamma_1, \gamma_2 + \gamma_3\}$ and hence (4.2.27) holds for $i = 1$. In the same way, we obtain (4.2.27) for $i = 2, 3$.

Now let us consider (4.2.28). Let $a = 2\gamma_i - \gamma$ and $b = \gamma_i$. Then

$$(P(a, b) \cap Q)\backslash a\mathbb{R} = \{\pm\gamma_i,\ \pm(\gamma_i - \gamma)\}.$$

On the other hand, if $c = -\gamma_i$, or $c = \gamma - \gamma_i$, then $a - c \notin Q$. Therefore, the summation in the formula (4.2.32) for this case is taken over $c \in \{\gamma_i, \gamma_i - \gamma\}$ and hence (4.2.28) holds. Since $\gamma_i - \gamma = -(\gamma_j + \gamma_k)$, it follows from the second inequality in (4.1.18) that $A_1(2\gamma_i - \gamma, \gamma_i) \neq 0$. □

Theorem 4.2.3 *The following equalities for the invariant* (4.1.6) *hold:*

$$I_2(\gamma_i, \gamma_j) = A_2(\gamma_i, \gamma_j) Re(z^2(-\gamma_i) z(\gamma_i + \gamma_j) z(\gamma_i - \gamma_j)), \tag{4.2.34}$$

$$I_2(\gamma_i, \gamma - \gamma_i) = A_2(\gamma_i, \gamma - \gamma_i) Re(z^2(-\gamma_i) z(\gamma) z(2\gamma_i - \gamma)), \tag{4.2.35}$$

where $A_2(\gamma_i, \gamma_j)$, $A_2(\gamma_i, \gamma - \gamma_i)$ *are nonzero numbers defined by* $A_2(a, b) = 2(a-b, a+b)(b, \beta)^{-2}$ *and* $\gamma, \gamma_1, \gamma_2, \gamma_3$ *are defined in* Theorem 4.2.2.

Proof Replacing a by $2a$, and arguing as in the proof of (4.2.32), we get

$$I_2(a, b) = 2Re\left(z^2(-a)\left(\sum_{c \in (P(a,b) \cap Q) \backslash a\mathbb{R}} \frac{\langle 2a - c, c\rangle}{(\langle c, \beta\rangle)^2} z(2a - c) z(c)\right)\right). \tag{4.2.36}$$

In (4.2.36) replacing c by $a + c$ and taking into account that $\langle a, \beta\rangle = 0$, we obtain the invariant

$$I_2(a, b) = 2Re\left(z^2(-a)\left(\sum_{c \in (P(a,b) \cap Q) \backslash a\mathbb{R}} \frac{\langle a + c, a - c\rangle}{(\langle c, \beta\rangle)^2} z(a + c) z(a - c)\right)\right). \tag{4.2.37}$$

Now using this, we obtain the invariants (4.2.34) and (4.2.35) as follows. First let us consider (4.2.34). Let $a = \gamma_i$, $b = \gamma_j$. Then

$$(P(a, b) \cap Q) \backslash a\mathbb{R} = \{\pm\gamma_j, \ \pm(\gamma_i - \gamma_j), \ \pm(\gamma_i + \gamma_j)\}.$$

One the other hand, if $c = \pm(\gamma_i - \gamma_j)$, or $c = \pm(\gamma_i + \gamma_j)$, then at least one of the vectors $a - c$ and $a + c$ does not belong to Q. Therefore, the summation in (4.2.37) for this case is taken over $c \in \{\pm\gamma_j\}$ and hence (4.2.34) holds. By the third inequality in (4.1.18) we have $A_2(\gamma_i, \gamma_j) \neq 0$.

Now let us consider (4.2.35). Let $a = \gamma_i$ and $b = \gamma - \gamma_i$. Then

$$(P(a, b) \cap Q) \backslash a\mathbb{R} = \{\pm\gamma, \ \pm(\gamma - \gamma_i), \ \pm(\gamma - 2\gamma_i)\}.$$

If $c = \gamma$, then $c + a = \gamma + \gamma_i \notin Q$. If $c = -\gamma$, then $c - a = -\gamma - \gamma_i \notin Q$. If $c = \gamma - 2\gamma_i$, then $c - a = \gamma - 3\gamma_i \notin Q$. If $c = -(\gamma - 2\gamma_i)$, then $c + a = -\gamma + 3\gamma_i \notin Q$. Therefore, the summation in the formula (4.2.37) for this case is taken over $c \in \{\pm(\gamma - \gamma_i)\}$ and hence (4.2.35) holds. Since $\gamma = \gamma_i + \gamma_j + \gamma_k$, it follows from the last inequality in (4.1.18) that $A_2(\gamma_i, \gamma - \gamma_i) \neq 0$. □

4.3 Finding the Fourier Coefficients Corresponding to the Boundary

First we prove the following simple theorem:

Theorem 4.3.1 *There exists a unique value of* $\tau \in F$ *such that the conditions* (4.1.19), (4.1.20) *hold.*

Proof It follows from (4.1.9) and from the definition of F that

$$\alpha_\tau(a) = \langle a, \tau\rangle + \alpha(a), \ \ \tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3, \tag{4.3.1}$$

where $\alpha_\tau(a)$ is defined in (4.1.20), and

$$\alpha(a) = \alpha_0(a) = \arg(q(x), e^{i\langle a,x\rangle}), \ \ c_k \in [0, 1), k = 1, 2, 3.$$

Using this one sees that (4.1.19) is equivalent to the following system of equations

$$\begin{aligned} 2\pi((N-1)c_1 + Mc_2 + Sc_3) &= -\alpha(N-1, M, S)(mod 2\pi),\\ 2\pi(Nc_1 + (M-1)c_2 + Sc_3) &= -\alpha(N, M-1, S)(mod 2\pi),\\ 2\pi(Nc_1 + Mc_2 + (S-1)c_3) &= -\alpha(N, M, S-1)(mod 2\pi). \end{aligned}$$

The determinant of the coefficient matrix of this system with respect to the unknowns c_1, c_2, c_3 is $8\pi^3(N+M+S-1)$. Therefore this system has a solution. Let $c_1, \ c_2, c_3$ and $c_1', c_2', \ c_3'$ be different solutions of this system corresponding to the different values of the right-hand side. Introduce the unknowns $x = c_1 - c_1', \ y = c_2 - c_2', z = c_3 - c_3'$. It is clear that x, y, z are the solution of the system

$$\begin{aligned} (N-1)x + My + Sz &= k,\\ Nx + (M-1)y + Sz &= m,\\ Nx + My + (S-1)z &= n. \end{aligned}$$

where k, m, n are integers. The solutions of this system has the form

$$x = \frac{f(k, m, s)}{N+M+S-1}, \ x = \frac{g(k, m, s)}{N+M+S-1}, \ x = \frac{h(k, m, s)}{N+M+S-1},$$

where $f(k, m, s), \ g(k, m, s), \ h(k, m, s)$ are integers and $f(1, 1, 1) = g(1, 1, 1) = h(1, 1, 1) = 1$. Therefore the above system of equations with respect to the unknowns $c_1, c_2, c_3 \in [0, 1)$ has $N + M + S - 1$ solutions $(c_{1,l}, c_{2,l}, \ c_{3,l})$ satisfying

$$c_{j,l+1} - c_{j,l} = \frac{1}{N+M+S-1}, \ j = 1, 2, 3 \text{ and } l = 1, 2, \ldots, N+M+S-2.$$

Thus using (4.3.1), the equality $\langle \omega_i, \gamma_j \rangle = 2\pi\delta_{i,j}$ and taking into account the notations

$z(n\gamma_1 + m\gamma_2 + s\gamma_3) =: z(n, m, s)$, $z(a) = r(a)e^{i\alpha(a)}$, $\alpha(a) \in [-\pi, \pi)$, one sees that there exist $\tau_1, \tau_2, \ldots, \tau_{N+M+S-1}$ such that

$$\tau_{l+1} - \tau_l = \frac{\omega_1 + \omega_2 + \omega_3}{N + M + S - 1},$$

$$\alpha_{\tau_{l+1}}(N, M, S) - \alpha_{\tau_l}(N, M, S) = \frac{2\pi}{N + M + S - 1}.$$

This implies that there exists a unique value of τ satisfying (4.1.19), (4.1.20) □

By Theorem 4.3.1, without loss of generality, it can be assumed that

$$\alpha(N - 1, M, S) = \alpha(N, M - 1, S) = \alpha(N, M, S - 1) = 0. \tag{4.3.2}$$

On the other hand, the invariant (4.1.11) determines the modulus of

$$z(N - 1, M, S), z(N, M - 1, S), z(N, M, S - 1), \tag{4.3.3}$$

since the vectors $(N - 1, M, S)$, $(N, M - 1, S)$, $(N, M, S - 1)$ are the long visible elements of $Q(N, M, S)$. Therefore, the Fourier coefficients in (4.3.3) are known.

In this sections, using Theorem 4.2.1, Proposition 4.2.1 and taking into account that the Fourier coefficients in (4.3.3) are known, we find all the Fourier coefficients $z(a)$ for $a \in B(N, M, S)$, where $B(N, M, S)$ is defined in the introduction. To formulate these results we use the following remark.

Remark 4.3.1 Let $a_1, a_2, \ldots, a_n$ be nonzero elements of Γ. Assign to every polynomial

$$\sum_{k=1,2,\ldots,n} z(a_k)e^{i\langle a_k, x\rangle} \tag{4.3.4}$$

the vector $(x(a_1), y(a_1), x(a_2), y(a_2), \ldots, x(a_n), y(a_n))$ of $\mathbb{R}^{2n}$, where $x(a_k)$ and $y(a_k)$ are the real and imaginary part of the Fourier coefficient $z(a_k)$. There exists one to one correspondence between the polynomials of the form (4.3.4) and elements of $\mathbb{R}^{2n}$. Further, we assume the following types of conditions on the Fourier coefficients:

Type 1. Assume that $z(a_j) \neq 0$ for some index j. In other words, we eliminate the finite number of subspaces $z(a_j) = 0$ of dimension $2n - 2$.

Type 2. Assume that some linear combinations of the invariants $e(i, a, b)$ defined in (4.1.14), (4.1.15) are not $0(mod \pi)$.

Type 3. Assume that some homogenous polynomials depending on

$x(a_1), y(a_1), x(a_2), y(a_2), \ldots$. are not zero.

These conditions mean that we eliminate some sets of dimensions less than $2n$. In any case, the $2n$ dimensional measures of the eliminated sets are zero. We named these conditions as zero measure conditions. This means that we consider almost all polynomials of the form (4.3.4). In order to avoid eclipsing the essence by technical details, we prefer to formulate the theorems for almost all the potentials of the form (4.3.4) instead of listing the eliminated sets. Note that the separated potentials show that, to determine the potential uniquely (modulo translations) from spectral invariants, it is necessary to eliminate some of these subspaces. Thus, the sufficient conditions to solve the inverse problem by these method are close to the necessary conditions.

First let us consider

$$z^2(N, M-1, l),\ z(N, M, l), z(N, M-2, l),\ \forall l. \tag{4.3.5}$$

Theorem 4.3.2 *Suppose* Condition 4.1 *holds. Then the spectral invariants* (4.1.11)–(4.1.13) *determine constructively and uniquely, modulo inversion and translation* (4.1.8)*, the numbers in* (4.3.5) *for almost all the potentials of the form* (4.1.9).

Proof Since the vectors (N, M, l), $(N, M-1, l)$, $(N, M-2, l)$ are long visible, the absolute values of the numbers in (4.3.5) are known. Therefore we need to find

$$2\alpha(N, M-1, l), \alpha(N, M, l), \alpha(N, M-2, l),\ \forall l. \tag{4.3.6}$$

To find (4.3.5) for $l = S, S-1, S-2$ we use the equation (4.1.15) for the following pairs:

$P_1 = \{(N, M, S-1), (0, 0, 1)\}$, $P_2 = \{(N, M-1, S), (0, 1, 0)\}$,
$P_3 = \{(N, M-1, S-1), (0, 1, -1)\}$, $P_4 = \{(N, M-1, S-1), (0, 1, 0)\}$,
$P_5 = \{(N, M-1, S-1), (0, 0, 1)\}$, $P_6 = \{(N, M-1, S-1), (0, 1, 1)\}$,
$P_7 = \{(N, M-2, S-1), (0, 0, 1)\}$, and $P_8 = \{(N, M-1, S-2), (0, 1, 0)\}$.

Note that it follows from Proposition 4.2.1(a) that the pairs $P_1, P_2, \ldots, P_8$ are the canonical pairs of type 2. Therefore, by Theorem 4.2.1, we have the invariant $A_2(a, b)$ and hence there corresponds equation of type (4.1.15) to each of the pairs $P_1, P_2, \ldots, P_8$. For simplicity of the notation, in (4.1.15) for P_i, instead of $e(2, a, b)$ and $d(a, b)$ we write e_i and d_i respectively. Denote $\alpha(N, M, S)$ by α. Using this notation and (4.3.2) one sees that the equality (4.1.15) for the pairs $P_1, P_2, \ldots, P_8$ has the form

$$\begin{aligned}
\alpha + \alpha(N, M, S-2) &= d_1 e_1 (mod 2\pi),\\
\alpha + \alpha(N, M-2, S) &= d_2 e_2 (mod 2\pi),\\
-2\alpha(N, M-1, S-1) + \alpha(N, M, S-2) + \alpha(N, M-2, S) &= d_3 e_3 (mod 2\pi),\\
-2\alpha(N, M-1, S-1) + \alpha(N, M-2, S-1) &= d_4 e_4 (mod 2\pi),\\
-2\alpha(N, M-1, S-1) + \alpha(N, M-1, S-2) &= d_5 e_5 (mod 2\pi),\\
-2\alpha(N, M-1, S-1) + \alpha + \alpha(N, M-2, S-2) &= d_6 e_6 (mod 2\pi),\\
-2\alpha(N, M-2, S-1) + \alpha(N, M-2, S) + \alpha(N, M-2, S-2) &= d_7 e_7 (mod 2\pi),\\
-2\alpha(N, M-1, S-2) + \alpha(N, M, S-2) + \alpha(N, M-2, S-2)) &= d_8 e_8 (mod 2\pi).
\end{aligned} \tag{4.3.7}$$

From the first and second equations of (4.3.7) we obtain

$$\alpha(N, M, S-2) = (d_1e_1 - \alpha)(mod 2\pi), \ \alpha(N, M-2, S) = (d_2e_2 - \alpha)(mod 2\pi). \tag{4.3.8}$$

These equalities with the third equation of (4.3.7) yield

$$-2\alpha(N, M-1, S-1) = (d_3e_3 - d_2e_2 - d_1e_1 + 2\alpha)(mod 2\pi).$$

Now using the last equality in the fourth, fifth and sixth equations of (4.3.7), we get

$$\begin{aligned} \alpha(N, M-2, S-1) &= (d_4e_4 + d_2e_2 + d_1e_1 - d_3e_3 - 2\alpha)(mod 2\pi), \\ \alpha(N, M-1, S-2) &= (d_5e_5 + d_2e_2 + d_1e_1 - d_3e_3 - 2\alpha)(mod 2\pi), \\ \alpha(N, M-2, S-2) &= (d_6e_6 + d_2e_2 + d_1e_1 - d_3e_3 - 3\alpha)(mod 2\pi). \end{aligned} \tag{4.3.9}$$

Writing the obtained value for $\alpha(N, M-2, S-1)$, $\alpha(N, M-2, S)$, $\alpha(N, M-2, S-2)$, $\alpha(N, M-1, S-2)$ into seventh and eighth equation of (4.3.7) we obtain

$$d_7e_7 - (d_6e_6 - 2d_4e_4 + d_3e_3 - d_1e_1) = 0(mod 2\pi), \ d_8e_8 + 2d_5e_5 + d_2e_2 = d_6e_6 + d_3e_3(mod 2\pi). \tag{4.3.10}$$

Introduce the notations $V = (d_1, d_3, d_4, d_6, d_7)$, $U = (d_8, d_5, d_2)$,

$$f_1(V) \equiv d_7e_7 - (d_6e_6 - 2d_4e_4 + d_3e_3 - d_1e_1), \ f_2(U) \equiv d_8e_8 + 2d_5e_5 + d_2e_2.$$

In these notations (4.3.10) has the form

$$f_1(V) = 0(mod 2\pi), \ f_2(U) = d_6e_6 + d_3e_3(mod 2\pi). \tag{4.3.11}$$

Since d_i is either 1 or -1, the vector V takes 32 distinct values

$$V_1, V_2, \ldots, V_{16} \text{ and } -V_1, -V_2, \ldots, -V_{16}.$$

Then the function $f_1(V)$ takes 32 values

$$f_1(V_1), f_1(V_2), \ldots, f_1(V_{16}) \text{ and } f_1(-V_1), f_1(-V_2), \ldots, f_1(-V_{16}).$$

Similarly, the vector U takes 8 distinct values $U_1, U_2, \ldots, U_8$ and the function $f_2(U)$ takes 8 values $f_2(U_1), f_2(U_2), \ldots, f_2(U_8)$. Suppose

$$f_1(V_k) - f_1(Vj) \neq 0(mod 2\pi)$$

for $k \neq j$. Then there are only one index k and two values $V_k, -V_k$ of V satisfying

$$f_1(V_k) = -f_1(-V_k) = 0(mod 2\pi).$$

On the other hand, the arguments of the Fourier coefficients of $q(x)$ and $q(-x)$ take the opposite values. Therefore, for fixing the translation $q(x) \longrightarrow q(-x)$, we take one of these two remaining values $V_k, -V_k$ of V. Thus, one can find the signs of d_1, d_3, d_4, d_6, d_7 from the first equality in (4.3.11). Since the signs of d_3 and d_6 are already known, we find d_8, d_5, d_2 from the second equality in (4.3.11) if

$$d_6 e_6 + d_3 e_3 \neq 0(mod 2\pi) \text{ and } f_2(U_k) - f_2(U_j) \neq 0(mod 2\pi).$$

Thus the numbers $d_1, d_2, \ldots, d_8$ are known. Since $e_1, e_2, \ldots, e_8$ are known invariants, the numbers in (4.3.6) for $l = S, S-1, S-2$ can be expressed in terms of α. Moreover we have the formulas [see (4.3.8), (4.3.9)]

$$\begin{aligned} -2\alpha(N, M-1, S-p) &= E_1 + 2p\alpha, \\ \alpha(N, M, S-p) &= E_2 - (p-1)\alpha, \\ \alpha(N, M-2, S-p) &= E_3 - (p+1)\alpha \end{aligned} \tag{4.3.12}$$

for $p = 0, 1, 2$, where by E_i for $i = 1, 2, \ldots$ we denote the linear combinations of $e_1, e_2, \ldots$ with known coefficients.

Now let us consider (4.3.6) for all l. For this we use the Eq. (4.1.15) for the canonical pairs $P_9(s) = \{(N, M-1, s),(0, 1, 1)\}$, $P_{10}(s) = \{(N, M-1, s-1), (0, 1, 0)\}$,

$P_{11}(s) = \{(N, M-1, s), (0, 1, -1)\}$ of type 2 (see Proposition 4.2.1(a)). The Eq. (4.1.15) for these pairs are

$$\begin{aligned} -2\alpha(N, M-1, s) + \alpha(N, M, s+1) + \alpha(N, M-2, s-1) &= d_9(s)e_9(s)(mod 2\pi), \\ -2\alpha(N, M-1, s-1) + \alpha(N, M, s-1) + \alpha(N, M-2, s-1) &= d_{10}(s)e_{10}(s)(mod 2\pi), \\ -2\alpha(N, M-1, s) + \alpha(N, M, s-1) + \alpha(N, M-2, s+1) &= d_{11}(s)e_{11}(s)(mod 2\pi), \end{aligned} \tag{4.3.13}$$

where $d_9(s), d_{10}(s), d_{11}(s)$ are either 1 or -1. Using the equations

$$\begin{aligned} -2\alpha(N, M-1, s) + \alpha(N, M, s+2) + \alpha(N, M-2, s-2) &= d_{12}e_{12}(mod 2\pi), \\ -2\alpha(N, M-1, s) + \alpha(N, M, s-2) + \alpha(N, M-2, s+2) &= d_{13}e_{13}(mod 2\pi) \end{aligned}$$

which are the Eq. (4.1.15) for the pairs $\{(N, M-1, s), (0, 1, 2)\}$, $\{(N, M-1, s), (0, 1, -2)\}$, and arguing as in the determinations of the signs of d_8, d_5, d_2, one can find the signs of $d_9(s), d_{10}(s), d_{11}(s)$. Then from the equations (4.3.13), we can find (4.3.6) for $l = s-1$ if (4.3.6) is known for $l = s+1, s$. Moreover as we proved above they satisfy the formulae (4.3.12) for $p = 0, 1, 2$. The formulas in (4.3.12) for all p can easily be obtained from (4.3.13) by induction. In the same way, we obtain the formulas

$$\alpha(N, M-p, S) = E_4 - (p-1)\alpha, \;\; \alpha(0, M, -S) = E_5 - (2S+N-1)\alpha. \tag{4.3.14}$$

By Proposition 4.2.1(b), the pair $\{(N, M, 0), (N, 0, S)\}$ is a canonical pair of type 1. Hence, using the invariant $A_1(a, b)$ [see (4.1.12)] for $a = (N, M, 0), b = (N, 0, S)$ and formulas (4.3.12), (4.3.14), we get the value of

$$\cos((N + M + S - 1)\alpha + E_6).$$

Similarly, using the pair $\{(N, M, 0), (N, 0, -S)\}$, we find

$$\cos((N + M + S - 1)\alpha + E_7).$$

By these two values of the cosine, we find $(N + M + S - 1)\alpha$ under condition $E_6 \neq E_7(mod\pi)$. This with (4.1.20) gives us the unique value of α and we find the numbers in (4.3.6) under some zero measure conditions in the sense of Remark 4.3.1 □

To find the Fourier coefficient $z(a)$ for all $a \in \partial\widetilde{Q}$, where $\partial\widetilde{Q}$ is defined in the introduction, we use the following lemmas

Lemma 4.3.1 *Let $\{a_1, b\}$ and $\{a_2, b\}$, where a_1 and a_2 are the long visible elements of $Q(N, M, S)$, be the canonical pairs of type 1. Then the invariants*

$$S_1(a_1, b) = Re(z(-a_1)z(a_1 - b)z(b)), S_1(a_2, b) = Re(z(-a_2)z(a_2 - b)z(b)), \tag{4.3.15}$$

defined in (4.1.12), uniquely determine $z(b)$ if $z(a_k - b)$and $z(a_k)$ for $k = 1, 2$ are known and

$$Im(z(a_1 - b)z(a_1)z(-(a_2 - b))z(-a_2)) \neq 0. \tag{4.3.16}$$

Proof The equations in (4.3.15) is a system of the linear equations with respect to the unknowns $x(b)$, $y(b)$ and the inequality (4.3.16) shows that the determinant of the coefficient matrix of this system is not zero. Therefore (4.3.16) has a unique solution □

Lemma 4.3.2 *Suppose $c \in Q$ has two different decompositions*

$$c = a_1 + b_1, c = a_2 + b_2, \text{ where } \{a_1, b_1, a_2, b_2\} \subset Q_N,$$

such that $z^2(a_k)$ and $z(a_k - b_k)$for $k = 1, 2$ are known and

$$Im(z^2(a_1)z(a_1 - b_1)z^2(-(a_2))z(-(a_2 - b_2))) \neq 0. \tag{4.3.17}$$

If $\{a_1, b_1\}$ and $\{a_2, b_2\}$, where a_1 and a_2 are the long visible elements of $Q(N, M, S)$, are the canonical pairs of type 2, then the invariants

$$S_2(a_k, b_k) = Re(z^2(-a_k)z(a_k - b_k)z(a_k + b_k)),$$

defined by (4.1.13), where $k = 1, 2$, uniquely determine $z(c)$.

The proof is the same with the proof of Lemma 4.3.1.

Theorem 4.3.3 *Suppose that the* Condition 4.1 *holds. Then the spectral invariants* (4.1.11)–(4.1.13) *and* (4.2.4) *determine constructively and uniquely, modulo inversion and translation* (4.1.8)*, the Fourier coefficients* $z(a)$ *for all* $a \in \partial\widetilde{Q}$ *for almost all the potentials of the form* (4.1.9).

Proof Step 1. In this step we find the Fourier coefficient $z(N, M-1, s)$ for $s = S-2p$. Since $z^2(N, M-1, s)$ is known due to Theorem 4.3.2, $z(N, M-1, s)$ is known up to the sign:

$$z(N, M-1, s) = k_s v_s,$$

where v_s is known and k_s is either 1 or -1. Moreover, k_S is known [see (4.3.2)]. To find k_s for $s = S-2p$, where $p = 1, 2, \ldots, S-1$ we use the invariant (4.2.4) for the pair $\{a, b\}$, where $a = (N, M-1, S-p)$, $b = (0, 0, 1)$. To write the invariant (4.2.4) for this pair, we need to determine the set G_2, defined in Theorem 4.2.1, for this pair. By the definition, G_2 is the set of all c such that

$$\{a+c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R}.$$

Clearly, if this inclusion holds, then c has the form $(0, m, s)$. Hence c belongs to the line intersection of the planes $P(a, b)$ and $\{x_1 = 0\}$. By (4.2.14) this line is $b\mathbb{R}$. It means that $c = (0, 0, q)$ for some integer q. Thus G_2 is the set of all $(0, 0, q)$ such that

$$\{(N, M-1, S-p-q), (N, M-1, S-p+q)\} \subset Q.$$

This inclusion implies that $-p \le q \le p$. Therefore the invariant (4.2.4) for the pair $\{(N, M-1, S-p)$,$(0, 0, 1)\}$ has the form

$$Re(z^2(N, M-1, S-p) \sum_{q=-p}^{p} \frac{\langle (N, M-1, S-p-q), (N, M-1, S-p+q)\rangle}{\langle (N, M-1, S-p), \beta\rangle} h_q V_q), \tag{4.3.18}$$

where $V_q =: v_{S-p+q} v_{S-p-q}$ is known number and $h_q =: k_{S-p+q} k_{S-p-q}$ is either 1 or -1. Let

$$H = (h_{-p}, h_{-p+1}, \ldots, h_p)$$

and f be a function taking H to (4.3.18). Assume that f takes distinct nonzero values at distinct points. Then (4.3.18) determines

$$h_q = k_{S-p+q} k_{S-p-q} \tag{4.3.19}$$

if $\langle (N, M-1, S-p-q), (N, M-1, S-p+q)\rangle \neq 0$. Thus (4.3.19) is known. Taking $p = q$ in (4.3.19), we find $k_S k_{S-2p}$. Since k_S is known [see (4.3.2)], we find k_{S-2p} if

$$A(p) =: \langle (N, M-1, S-2p), (N, M-1, S) \rangle \neq 0.$$

Since the equation $A(p) = 0$ may have only one integer root p_0 we have defined k_{S-2p} for all p except $p = p_0$. It is clear that there exists p and q such that $p_0 = p+q$ and $\langle (N, M-1, S-p-q), (N, M-1, S-p+q) \rangle \neq 0$. Therefore, using (4.3.19), we define k_{S-p_0}, since k_{S-p-q} is known.

Step 2. To find $z(N, M-1, S-2p+1)$, we use the Lemma 4.3.1. Let

$$a_1 = (N, M-1, S), a_2 = (N, M-1, S-2), b = (0, -1, N).$$

Without loss of generality, it can be assumed that $S-2 \neq kN$. Otherwise, we consider $a_2 = (N, M-1, S-4)$. By Proposition 4.2.1(b) the pairs $\{a_1, b\}$ and $\{a_2, b\}$ are the canonical pairs of type 1. Therefore applying Lemma 4.3.1 and taking into account that $z(a_k - b)$ and $z(a_k)$ for $k = 1, 2$ are known due to Theorem 4.3.2 and Step 1, we find $z(b)$. Now, without loss of generality, we assume that $S-1 \neq kN$. Otherwise we consider $S-3$ instead of $S-1$. By Proposition 4.2.1(b) the pair $\{a, b\}$, where $a = (N, M-1, S-1)$ and $b = (0, -1, N)$, is the canonical pair of type 1. Hence using the invariant (4.1.12) and taking into account that $z(b)$ and $z(a-b)$ are known, we determine the sign of k_{S-1}. From the knowledge of the sign of k_S, we have found the sign of k_{S-2p} by (4.3.18). In the same way, from the knowledge of the sign of k_{S-1}, we find the sign of k_{S-2p-1}. Thus, we have found $z(N, M-1, s)$ for all s.

Step 3. Now using Lemma 4.3.2, we find $z(N, m, s)$ for all m, s by induction. They were found in Theorem 4.3.2 and in steps 1,2 of this theorem for $m = M, M-1, M-2$. Let us find $z(N, m, s)$ assuming that we have already found the $z(N, q, s)$ for $q = M, M-1, \ldots, m+1$. Clearly, for any $s \in [S, -S]$ there are different pairs $(s_1, p_1), (s_2, p_2)$ such that

$$s_k + p_k = s;\ s_k, p_k, s_k - p_k \in [S, -S];\ s_k - 2p_k \notin [-S, S];\ N^2 + m^2 - 1 + s_k^2 - p_k^2 \neq 0$$

$s_k \neq \pm N$, $s_k - p_k \neq \pm N$ for $k = 1, 2$. Then, by Proposition 4.2.1(a) (see case 3), the pair $\{a_k, b_k)\}$ for $k = 1, 2$, where $a_k = (N, m+1, s_k)$, $b_k = (0, -1, p_k)$, are the canonical pairs of type 2. Moreover $z(a_k), z(a_k - b_k)$ are known by the assumption of the induction. Hence the application of Lemma 4.3.2 yields $z(N, m, s)$. Interchanging the roles of the first and second coordinates and then the roles of the first and third coordinates, we find $z(a)$ for all $a \in \partial \widetilde{Q}$ under some zero measure conditions in the sense of Remark 4.3.1 □

Theorem 4.3.4 *Suppose* Condition 4.1 *holds. Then the spectral invariants* (4.1.11)–(4.1.13)*,* (4.2.3) *determine constructively and uniquely, modulo inversion and translation* (4.1.8)*, the Fourier coefficients*

$$z(n, m, 0), z(n, 0, s), z(0, m, s)$$

for all n, m, s and for almost all the potentials of the form (4.1.9).

Proof Let us find $z(n, m, 0)$. Since $(n, m, 0) \neq (0, 0, 0)$ and $z(-a) = \overline{z(a)}$, without loss of generality, it can be assumed that $m > 0, n \geq 0$. Moreover, for the simplicity of the notations, it can be assumed that n, m are relatively prime numbers, since we find $z(l(n, m, 0))$ for all l. To find $z(n, m, 0)$, we use the invariant (4.2.3) for the pair $\{a_q, (n, m, 0)\}$, where $a_q = (0, -M, S) + q(n, m, 0)$. To write the invariant (4.2.3) for this pair we need to investigate the set G_1, defined in Theorem 4.2.1. By the definition, G_1 is the set of all c such that

$$\{c, a_q - c\} \subset (P(a_q, (n, m, 0)) \cap Q)\backslash a_q\mathbb{R}.$$

Using this, the obvious equality $P(a_q, (n, m, 0)) = P((0, -M, S), (n, m, 0))$ and (4.2.9), we obtain that G_1 is the set of all c such that

$$\{c, a_q - c\} \subset ((Q_{-1} \cup Q_0 \cup Q_1) \cap Q)\backslash a_q\mathbb{R}.$$

If $c \in Q_{-1}$ then

$$a_q - c = (q - l)(n, m, 0) + (0, -2M, 2S) \notin Q.$$

If $c \in Q_0$, then $c = l(n, m, 0)$ for some l. Let p be the greatest integer satisfying $pn \leq N$, $pm \leq M$. Then $l(n, m, 0) \in Q$ if and only if $-p \leq l \leq p$. Moreover

$$a_q - c = (0, -M, S) + (q - l)(n, m, 0) \in Q_1.$$

Similarly, if $c \in Q_1$, i.e., $c = (0, -M, S) + (q - l)(n, m, 0)$ for some l, then $a_q - c = l(n, m, 0)$. Therefore, the invariant (4.2.3) for the pair $\{a_q, (n, m, 0)\}$, has the form

$$Rez(-(a_q)) \sum_l c_l z(a_q - l(n, m, 0)) z(l(n, m, 0)), \tag{4.3.20}$$

where $q = 1, 2, \ldots, p$ and $c_l = g(a_q, l(n, m, 0))$. Similarly, the invariant (4.2.3) for the pair $\{b_q, (n, m, 0)\}$, where $b_q = (0, M, S) + q(n, m, 0)$, has the form

$$Rez(-(b_q)) \sum_l d_l z(b_q - l(n, m, 0)) z(l(n, m, 0)), \tag{4.3.21}$$

where $q = -1, -2, \ldots, -p$ and $d_l = g(b_q, l(n, m, 0))$. Since the Fourier coefficients

$$z(a_q), z(a_q - l(n, m, 0)), z(b_q), z(b_q - l(n, m, 0))$$

are known due to Theorem 4.3.3, we have $2p$ linear form [see (4.3.20) and (4.3.21)] with respect to $2p$ unknowns

$$x(n, m, 0), x(2(n, m, 0)), \ldots, x(p(n, m, 0)) \text{ and } y(n, m, 0), y(2(n, m, 0)), \ldots, x(q(n, m, 0)).$$

Since the invariant (4.2.3) is known number, (4.3.20) and (4.3.21) give $2p$ linear equations with respect to these unknowns. One can find these unknowns if the determinant $T(2p)$ of the coefficient matrix of the system of these linear equations is not zero. Let us show that this determinant is not identically zero. Let $x(l(n,m,0))$ be the lth and $y(l(n,m,0))$ be the $(p+l)$th unknown of the system, where $l = 1, 2, \ldots, p$. Similarly, let the lth equation of the system be given by the lth linear form of (4.3.20) and the $(p+l)$th equation of the system be given by the lth linear form of (4.3.21). Then $T(2p)$ can be written in the form

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,p} & b_{1,1} & b_{1,2} & \ldots & b_{1,p} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,p} & b_{2,1} & b_{1,2} & \ldots & b_{2,p} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{p,1} & a_{p,2} & \ldots & a_{p,p} & b_{p,1} & b_{p,2} & \ldots & b_{p,p} \\ c_{1,1} & c_{1,2} & \ldots & c_{1,p} & d_{1,1} & d_{1,2} & \ldots & d_{1,p} \\ c_{2,1} & c_{2,2} & \ldots & c_{2,p} & d_{2,1} & d_{2,2} & \ldots & d_{2,p} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{p,1} & c_{p,2} & \ldots & c_{p,p} & d_{p,1} & d_{p,2} & \ldots & d_{p,p} \end{vmatrix},$$

where

$$a_{q,l} = x(a_q)(c_l x(a_{q-l}) + c_{-l} x(a_{q+l})) + y(a_q)(c_l y(a_{q-l}) + c_{-l} y(a_{q+l})),$$

$$b_{q,l} = x(a_q)(c_l y(a_{q-l}) - c_{-l} y(a_{q+l})) + y(a_q)(c_l x(a_{q-l}) - c_{-l} x(a_{q+l})),$$

$$c_{q,l} = x(b_{-q})(d_l x(b_{-q-l}) + d_{-l} x(b_{-q+l})) + y(b_{-q})(d_l y(b_{-q-l}) + d_{-l} y(b_{-q+l})),$$

$$d_{q,l} = x(b_{-q})(d_l y(b_{-q-l}) - d_{-l} y(b_{-q+l})) + y(b_{-q})(d_l x(b_{-q-l}) - d_{-l} x(b_{-q+l})).$$

The qth and $(p+q)$th diagonal elements $a_{q,q}$ and $d_{q,q}$ of the determinant contain the summand $x(a_q)c_q x(a_0)$ and $x(b_{-q})d_{-q} y(b_0)$ respectively. The nondiagonal elements do not contain these summands. Therefore, the determinant $T(2p)$ contains the summand

$$\sqcap_{q=1,2,\ldots,p}(c_q x(a_q)x(a_0)d_{-q}x(b_{-q})y(b_0))$$

which can not be cancelled by the other summand of the determinant. Moreover, the multiplicands c_q and d_{-q} are not zero since

$$\langle q(n,m,0), a_q - q(n,m,0)\rangle = -qmM \neq 0, \ \langle q(n,m,0), b_q - q(n,m,0)\rangle = qmM \neq 0$$

Therefore the zero set of the determinant $T(2p)$ of the coefficient matrix of the system has zero measure. Thus solving this system we find $z(n,m,0)$ under some zero measure conditions in the sense of Remark 4.3.1. In the same way we find $z(n,0,s)$ and $z(0,m,s)$ □

4.4 Inverse Problem in a Dense Set

In this section, we construct a dense in $W_2^s(F)$, where $s > 3$, in $\mathbb{C}^\infty$-topology set D of trigonometric polynomials and prove that one can determine constructively and uniquely (module inversion and translation (4.1.8)) the potential $q \in D$ from the spectral invariants (4.1.4)–(4.1.6). For this we use the following condition:

Condition 4.2 *Suppose $z(n,m,s) \neq 0$ for $(n,m,s) \in C(\sqrt{N})$, where*

$$C(\sqrt{N}) = \{(n,m,s) : 0 < |n| < \frac{1}{2}\sqrt{N}, 0 < |m| < \frac{1}{2}\sqrt{N}, 0 < |s| < \frac{1}{2}\sqrt{N}\}$$

and $z(n,m,s) = 0$ for $(n,m,s) \in (Q(N,M,S))\backslash(C(\sqrt{N}) \cup B(N,M,S))$.

To find $z(n,m,s)$ for $(n,m,s) \in C(\sqrt{N})$ we use the following proposition.

Proposition 4.4.1 *If the* Condition 4.2 *holds, then the invariant* (4.2.3) *for $a \in B(N,M,S)$, $b \in C(\sqrt{N})$ yields the invariant*

$$Re(z(-a)(\sum_{c \in G} g(a,c)z(a-c)z(c))), \tag{4.4.1}$$

where $g(a,c) = \frac{\langle c, c-a \rangle}{(\langle c,\beta \rangle)^2}$, G is the set of all c such that

$$\{c, a-c\} \subset ((P(a,b) \cap Q)\backslash a\mathbb{R}) \cap (C(\sqrt{N}) \cup B(N,M,S)) \tag{4.4.2}$$

and at least one of the points c and $a-c$ belongs to $C(\sqrt{N})$.

Proof By Condition 4.2, if $\{c, a-c\}$ is not a subset of $C(\sqrt{N}) \cup B(N,M,S)$ then $z(a-c)z(c) = 0$. Therefore, it follows from the definition of G_1 that the summation in (4.2.3) is taken over all c satisfying (4.4.2). On the other hand, if both c and $a-c$ belong to $B(N,M,S)$, then the summand $z(-a)g(a,c)z(a-c)z(c)$ of (4.2.3) is known due to Theorems 4.3.3 and 4.3.4. Therefore, (4.2.3) implies the invariant (4.4.1), if Condition 4.2 holds □

Theorem 4.4.1 *The invariants* (4.1.11)–(4.1.13) *and* (4.4.1) *determine constructively and uniquely, modulo inversion and translation* (4.1.8)*, the Fourier coefficients $z(n,m,s)$, where $(n,m,s) \in C(\sqrt{N})$, for almost all the potentials of the form* (4.1.9) *satisfying* Conditions 4.1 *and* 4.2*.*

Proof To find $z(n,m,s)$ for $(n,m,s) \in C(\sqrt{N})$, we use the invariant (4.4.1) for the pair $a = (-N+n, 0, j)$, $b = (n,m,s)$, where j is a prime number satisfying

$$M < j \leq S - \sqrt{N}. \tag{4.4.3}$$

Since $n \neq 0$ and $z(-n,-m,-s) = \overline{z(n,m,s)}$, without loss of generality, it can be assumed that $n > 0$. To use (4.4.1), we prove that

$$G = \{b,\ a - b\}, \text{ where } b = (n, m, s),\ \ a - b = (-N, -m, j - s), \tag{4.4.4}$$

where G is defined in Proposition 4.4.1. Since the inclusion $\{b, a-b\} \subset G$ is obvious, we need to prove that $G \subset \{b, a - b\}$. For this we use the following inequalities

$$0 < |n|, |m|, |s| < \frac{1}{2}\sqrt{N},\ 2N < M < j \le S - \sqrt{N} \tag{4.4.5}$$

which follows from (4.4.3), Condition 4.1, and the assumption $(n, m, s) \in C(\sqrt{N})$. Thus, to prove (4.4.4) we need to show that any element $c = (n_1, m_1, s_1)$ of G is either b or $a - b$. First let us prove that $n_1m_1s_1 \neq 0$. Indeed, using the definition of $C(\sqrt{N})$ and the inequalities in (4.4.5) one can readily verify that the following three statements are true.

1. If $n_1 = 0$, then $(n_1, m_1, s_1) \notin C(\sqrt{N})$, $a - c = (-N + n, -m_1, j - s_1) \notin C(\sqrt{N})$.

2. If $m_1 = 0$, then $(n_1, m_1, s_1) \notin C(\sqrt{N})$, $a - c = (-N + n - n_1, 0, j - s_1) \notin C(\sqrt{N})$.

3. If $s_1 = 0$, then $(n_1, m_1, s_1) \notin C(\sqrt{N})$, $a - c = (-N + n - n_1, -m_1, j) \notin C(\sqrt{N})$.

Therefore the relation $(n_1, m_1, s_1) \in G$ and the definition of G (see Proposition 4.4.1) imply that $n_1m_1s_1 \neq 0$. Since $c \in G$ we have $c \in P(a, b) \cap Q$. The point $c = (n_1, m_1, s_1)$ belongs to the plane $P(a, b)$ if and only if

$$(n - N)(ms_1 - sm_1) = j(mn_1 - nm_1). \tag{4.4.6}$$

This equation holds in the following two cases:

Case 1. $(ms_1 - sm_1) = 0$. Then $(mn_1 - nm_1) = 0$. These two equalities imply that the point $c = (n_1, m_1, s_1)$ lies on the line $(n, m, s)\mathbb{R}$. Therefore we have

$$c = (n_1, m_1, s_1) = k(n_0, m_0, s_0),\ \ (n, m, s) = k_0(n_0, m_0, s_0), \tag{4.4.7}$$

where k and k_0 are the integers and (n_0, m_0, s_0) is a visible element of $\mathbb{Z}^3$ lying in $(n, m, s)\mathbb{R}$. Moreover, it follows from (4.4.5) and from the above relation $n_1m_1s_1 \neq 0$ that

$$0 < |n_0|, |m_0|, |s_0| < \frac{1}{2}\sqrt{N} \text{ and } kk_0 \neq 0 \tag{4.4.8}$$

Using this let us prove that $k(n_0, m_0, s_0) \in G$ if and only if $k = k_0$. If $k = k_0$, then by (4.4.7) we have $(n_1, m_1, s_1) = (n, m, s) = b \in G$. Now we prove that if $k \neq k_0$, then $c = k(n_0, m_0, s_0) \notin G$. Suppose at least one of the inequalities

$$|kn_0| > \frac{1}{2}\sqrt{N}, |km_0| > \frac{1}{2}\sqrt{N}, |ks_0| > \frac{1}{2}\sqrt{N} \tag{4.4.9}$$

holds. Then using (4.4.8), the definitions of $C(\sqrt{N})$ and $B(N, M, S)$, and taking into account that N, M, and S are the prime numbers, we see that

$$c = k(n_0, m_0, s_0) \notin C(\sqrt{N}) \cup B(N, M, S),$$

and hence $c \notin G$. Now suppose that all the inequalities in (4.4.9) do not hold. Then using (4.4.5), (4.4.7), (4.4.8) and the assumption $k \neq k_0$, one can easily verify that

$$-N + n - kn_0 \neq 0, \pm N;\ \ km_0 \neq 0, \pm M;\ \ j - ks_0 \neq 0, \pm S;\ \ j - ks_0 > \sqrt{N}.$$

These relations and the definitions of $C(\sqrt{N})$ and $B(N, M, S)$ imply that

$$a - c = a - k(n_0, m_0, s_0) = (-N + n - kn_0, -km_0, j - ks_0) \notin C(\sqrt{N}) \cup B(N, M, S),$$

which means that $c \notin G$ (see the definition of G in the Proposition 4.4.1). Hence, it is proved that if $k \neq k_0$, then $(n_1, m_1, s_1) \notin G$. Thus, in Case 1, the inclusion $c \in G$ implies the equality $c = b$.

Case 2. $(ms_1 - sm_1) \neq 0$. Then it follows from (4.4.6) that

$$(ms_1 - sm_1) = pj, \tag{4.4.10}$$

where p is a nonzero integer, since j is a prime number satisfying $j > N - n$ [see (4.4.5)]. The formulas (4.4.10) and (4.4.6) imply that

$$(n - N)p = mn_1 - nm_1. \tag{4.4.11}$$

Using (4.4.10) and (4.4.5) one can readily verify that at least one of the inequalities

$$|m_1| > \sqrt{N},\ \ |s_1| > \sqrt{N} \tag{4.4.12}$$

holds. If the first inequality of (4.4.12) holds, then

$$c = (n_1, m_1, s_1) \notin C(\sqrt{N}),\ \ a - c = (-N + n - n_1, -m_1, j - s_1) \notin C(\sqrt{N})$$

and hence $c \notin G$.

Now assume that $|s_1| > \sqrt{N}$ and $|m_1| \leq \sqrt{N}$. Then $c = (n_1, m_1, s_1) \notin C(\sqrt{N})$. Therefore the relation $c \in G$ and the definition of G give

$$a - c = (-N + n - n_1, -m_1, j - s_1) \in C(\sqrt{N}).$$

Using this, the definition of $C(\sqrt{N})$, and (4.4.5), we obtain

$$| - N - n_1| < \sqrt{N}, 0 < |m_1| < \sqrt{N},\ \ |j - s_1| < \sqrt{N}. \tag{4.4.13}$$

Since $c \in G$, we have $c \in C(\sqrt{N}) \cup B(N, M, S)$. On the other hand $c \notin C(\sqrt{N})$. Hence $c = (n_1, m_1, s_1) \in B(N, M, S)$, that is, at least one of the following inclusions hold

$$n_1 \in \{0, N, -N\}, m_1 \in \{0, M, -M\}, s_1 \in \{0, S, -S\}.$$

This with (4.4.13) and (4.4.3) implies that $n_1 = -N$. Using this in (4.4.11), we get

$$N(p - m) = n(p + m_1). \tag{4.4.14}$$

We assumed that $|m_1| \le \sqrt{N}$. Besides, by (4.4.5) we have $|m| \le \sqrt{N}$, $|n| \le \sqrt{N}$. From these inequalities and (4.4.14) one can easily conclude that $|p + m_1| < N$. Thus N is a prime number and is greater than $|n|$ and $|p + m_1|$. Therefore from (4.4.14) we obtain that $p + m_1 = 0$, $p - m = 0$, and hence $p = m = -m_1$. Using this in (4.4.10), we obtain

$$(ms_1 + sm) = mj, \;\; s_1 = j - s, \;\; c = (n_1, m_1, s_1) = (-N, -m, j - s) = a - b.$$

Thus, we proved that any element c of the set G is either b (see Case 1) or $a - b$. Hence $G \subset \{b, a - b\}$ and (4.4.4) is proved.

Now it follows from (4.4.4) that the invariant (4.4.1) has the form

$$2\mathbf{Re} z(-a)g(a, b)z(a - b)z(b)). \tag{4.4.15}$$

Clearly, there exist two numbers j_1 and j_2 such that they satisfy the conditions of j and

$$\langle(-N + n, 0, j_1), (n, m, s)\rangle \ne 0, \;\; \langle(-N + n, 0, j_1), (n, m, s)\rangle \ne 0,$$

which implies that the multiplicand $g(a, b)$ in (4.4.15) for $a = (-N + n, 0, j_i)$, where $i = 1, 2$, is not zero. Hence (4.4.15) gives the invariants

$$Re(z(-(-N + n, 0, j_i))z(-N, -m, j_i - s)z(n, m, s))), \tag{4.4.16}$$

where $z(-(-N + n, 0, j_i))$ and $z(-N, -m, j_i - s)$ for $i = 1, 2$ are known (see Theorems 4.3.3 and 4.3.4). By Lemma 4.3.1 the invariants (4.4.16) give the Fourier coefficient $z(n, m, s)$ under some zero measure conditions in the sense of Remark 4.3.1 $\square$

Thus, we considered the set of the polynomials of the form

$$p(x) = \sum_{a \in B(N,M,S) \cup C(\sqrt{N})} z(a)e^{i\langle a,x\rangle} \tag{4.4.17}$$

(see Conditions 4.1, 4.2 and the Theorems 4.3.3, 4.3.4 and 4.4.1), where $B(N, M, S)$ and $C(\sqrt{N})$ are defined in the introduction and in Condition 4.2 respectively and

$z(a) \neq 0$. By $E(N, M, S)$ denote the subspace of $L_2(F)$ generated by functions $e^{i\langle a,x\rangle}$ for $a \in (B(N, M, S) \cup C(\sqrt{N}))$. Let $D(N, M, S)$ be the set of all polynomial of the form (4.4.17) satisfying the zero measure conditions, in the sense of Remark 4.3.1, used in the proof of the Theorems 4.3.2, 4.3.3, 4.3.4 and 4.4.1. Due to Remark 4.3.1, the set $D(N, M, S)$ is obtained from $E(N, M, S)$ by eliminating the sets whose n dimensional measure is zero, where n is the number of the elements of $B(N, M, S) \cup C(\sqrt{N})$. Therefore, for every positive ε and for each $f_N \in E(N, M, S)$ the ball

$$\{h \in E(N, M, S) : \sup |h(x) - f_N(x)| < \varepsilon\}$$

contains an element p_N of $D(N, M, S)$, that is,

$$\sup_{x\in F} |p_N(x) - f_N(x)| < \varepsilon. \tag{4.4.18}$$

Now consider a triple sequence $\{(N_k, M_k, S_k)\}$ such that for all k the triple (N_k, M_k, S_k) satisfies the conditions which are satisfied for (N, M, S) (see Condition 4.1) and $N_k \to \infty$ as $k \to \infty$. Thus N_k, M_k, S_k are the prime numbers satisfying

$$M_k > 2N_k,\ S_k > 2M_k,\ N_1 \gg 1,\ \lim_{k\to\infty} N_k = \infty \tag{4.4.19}$$

Denote by $D(N_k, M_k, S_k)$ the set obtained from $D(N, M, S)$ by substitution (N_k, M_k, S_k) for (N, M, S). Let

$$D = \cup_{k=1}^{\infty} D(N_k, M_k, S_k). \tag{4.4.20}$$

Theorem 4.4.2 (a) *The set D is dense in $W_2^s(F)$, where $s > 3$, in $\mathbb{C}^\infty$-topology.*

(b) *The invariants (4.1.4)–(4.1.6) determine constructively and uniquely, modulo inversion and translations (4.1.8), the potentials q of the set D.*

Proof (a) Note that $f \in W_2^s(F)$ means that

$$f(x) = \sum_{a\in\Gamma} (f, e^{i\langle a,x\rangle}) e^{i\langle a,x\rangle},\quad \sum_{a\in\Gamma} |(f, e^{i\langle a,x\rangle})|^2 (1 + |a|^{2s}) < \infty. \tag{4.4.21}$$

Without loss of generality, it can be assumed that $(f, 1) = 0$. If $s > 3$, then

$$\sup_{x\in F} \Big| \sum_{a\in R(\sqrt{N})} (f, e^{i\langle a,x\rangle}) e^{i\langle a,x\rangle} \Big| \leq \sum_{a\in R(\sqrt{N})} |(f, e^{i\langle a,x\rangle})| = O((\sqrt{N})^{-(s-3)}), \tag{4.4.22}$$

where $R(\sqrt{N}) = \{a \in \Gamma : |a| \geq \sqrt{N}\}$. It follows from the definitions of $B(N, M, S)$ and $C(\sqrt{N})$ that

$$\Gamma\backslash(B(N, M, S) \cup C(\sqrt{N}) \cup \{(0, 0, 0)\}) \subset R(\sqrt{N}), \tag{4.4.23}$$

By (4.4.22) and (4.4.23) $f(x)$ has an orthogonal decomposition $f(x) = f_N(x) + r_N(x)$, where

$$f_N(x) = \sum_{a \in (B(N,M,S) \cup C(\sqrt{N})} (f, e^{i\langle a,x\rangle}) e^{i\langle a,x\rangle}, \quad \sup_{x\in F} |r_N(x)| = O((\sqrt{N})^{-(s-3)}), \tag{4.4.24}$$

$f_N \in E(N, M, S)$. Therefore for any $\varepsilon > 0$ there exists N such that

$$\sup |f(x) - f_N(x)| < \varepsilon. \tag{4.4.25}$$

From (4.4.18) and (4.4.25) we obtain that for any $f \in W_2^s(F)$ and for any $\varepsilon > 0$ there exists N and $p_N(x) \in D(N, M, S)$ such that

$$\sup_{x\in F} |f(x) - p_N(x)| < 2\varepsilon$$

which means that D is dense in $W_2^s(F)$ in $\mathbb{C}^\infty$-topology.

(b) Let q be an element of D. Since the vector $(N_k, 1, 0)$ is a visible element of $\mathbb{Z}^3$ for each N_k, the invariants

$$\|q^{(N_k,1,0)}\|$$

for $k = 1, 2, \ldots$[see (4.1.4)] are given. By the definition of D, the number

$$k =: \{\max s : \ \|q^{(N_s,1,0)}\| \neq 0\}$$

is finite. Therefore q belongs to the set $D(N_k, M_k, S_k)$. The statement of Theorem 4.4.2(b) for this set follows from the definition of $D(N, M, S)$ and from the Theorems 4.3.3, 4.3.4, and 4.4.1 □

4.5 Finding the Simple Potential from the Invariants

In this section, we give an algorithm and formulas for finding the all Fourier coefficients $z(a)$ of the potential (4.1.16) from the invariants (4.2.25)–(4.2.28), (4.2.34) and (4.2.35). First, let us introduce some notations. The number of elements of the set

$$\{n\gamma_1 + m\gamma_2 + s\gamma_3 : |n| \le 1, |m| \le 1, |s| \le 1\}$$

is 27, since the numbers n, m, s take 3 values $-1, 0, 1$ independently. The set $Q(1, 1, 1)$ [see (4.1.17)] is obtained from this set by eliminating the element $(0, 0, 0)$, and hence consist of 26 elements. Moreover, if $\gamma \in Q(1, 1, 1)$, then $-\gamma \in Q(1, 1, 1)$ and $\gamma \neq -\gamma$. Hence the elements of $Q(1, 1, 1)$ can be denoted by $\gamma_1, \gamma_2, \ldots, \gamma_{13}$ and $-\gamma_1, -\gamma_2, \ldots, -\gamma_{13}$. Let us denote the elements $\gamma_1, \gamma_2, \ldots \gamma_{13}$ as following: $\gamma_1, \gamma_2, \gamma_3$ be a basis of Γ satisfying (4.1.18) and

$$\begin{aligned}&\gamma_4 = \gamma_2 + \gamma_3,\ \gamma_5 = \gamma_1 + \gamma_3,\ \gamma_6 = \gamma_1 + \gamma_2,\ \gamma_7 = \gamma_1 + \gamma_2 + \gamma_3,\\ &\gamma_8 = \gamma_1 - \gamma_2,\ \gamma_9 = \gamma_1 - \gamma_3,\ \gamma_{10} = \gamma_2 - \gamma_3\\ &\gamma_{11} = \gamma_2 + \gamma_3 - \gamma_1,\ \gamma_{12} = \gamma_1 + \gamma_3 - \gamma_2,\ \gamma_{13} = \gamma_1 + \gamma_2 - \gamma_3\end{aligned} \tag{4.5.1}$$

Introduce the notations

$$z(\gamma_j) = a_j + ib_j = r_j e^{i\alpha_j}, \tag{4.5.2}$$

where $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}$, $r_j = |z(\gamma_j)| \in (0, \infty)$, and $\alpha_j = \alpha(\gamma_j) = \arg(z(\gamma_j)) \in [0, 2\pi)$ for

$i = 1, 2, \ldots, 13$. Since the modulus r_j of the Fourier coefficients $z(\gamma_j)$ are known due to (4.2.20), we need to know the values of the arguments α_j of $z(\gamma_j)$. For this we use the following conditions on the arguments $\alpha_1, \alpha_2, \ldots, \alpha_7$:

$$\begin{aligned}&\alpha_7 - \alpha_1 - \alpha_2 - \alpha_3 \neq \pi k,\ \alpha_7 - \alpha_{s+3} - \alpha_s \neq \pi k,\ \alpha_{m+3} - \alpha_{j+3} + \alpha_m - \alpha_j \neq \pi k,\\ &\alpha_4 - \alpha_2 - \alpha_3 \neq \frac{\pi}{2}k,\ \alpha_5 - \alpha_1 - \alpha_3 \neq \frac{\pi}{2}k,\ \alpha_6 - \alpha_1 - \alpha_2 \neq \frac{\pi}{2}k,\\ &\alpha_4 + \alpha_5 - \alpha_1 - \alpha_2 - 2\alpha_3 \neq \pi k,\ \alpha_4 + \alpha_6 - \alpha_1 - \alpha_3 - 2\alpha_2 \neq \pi k,\\ &\alpha_5 + \alpha_6 - \alpha_2 - \alpha_3 - 2\alpha_1 \neq \pi k,\end{aligned} \tag{4.5.3}$$

where $s = 1, 2, 3$; $k \in \mathbb{Z}$ and m, j are integers satisfying $1 \leq m < j \leq 3$. In this section, we give an algorithm for the unique (modulo (4.1.8)) determination of the potentials q of the form (4.1.16) satisfying (4.5.3) from the invariants (4.1.4)–(4.1.6). In the following remark we consider geometrically the set of all potentials of the form (4.1.16) satisfying (4.5.3).

Remark 4.5.1 Since $z(\gamma) = \overline{z(-\gamma)}$, there exists one to one correspondence between the trigonometric polynomials of the form (4.1.16) and the vectors $(r_1, \alpha_1, r_2, \alpha_2, \ldots, r_{13}, \alpha_{13})$ of the subset

$$S =: (0, \infty)^{13} \otimes [0, 2\pi)^{13}$$

of the space $\mathbb{R}^{26}$. We use conditions (4.5.3) as restrictions on the potential (4.1.16) and hence on the set S. Denote by S' the subset of S corresponding to the set of the potential (4.1.16) satisfying conditions (4.5.3). The conditions (4.5.3) means that we eliminate from the subset

$$D =: \{(\alpha_1, \alpha_2, \ldots, \alpha_7) : \alpha_1 \in [0, 2\pi), \alpha_1 \in [0, 2\pi),\ \alpha_2 \in [0, 2\pi), \ldots,\ \alpha_7 \in [0, 2\pi)\}$$

of $\mathbb{R}^7$ the following six-dimensional hyperplanes

$$\begin{aligned}&\{\alpha_7 - \alpha_1 - \alpha_2 - \alpha_3 = \pi k\},\ \{\alpha_7 - \alpha_{s+3} - \alpha_s = \pi k\},\ \{\alpha_{m+3} - \alpha_{j+3} + \alpha_m - \alpha_j = \pi k\},\\ &\{\alpha_4 - \alpha_2 - \alpha_3 = \frac{\pi}{2}k\},\ \{\alpha_5 - \alpha_1 - \alpha_3 = \frac{\pi}{2}k\},\ \{\alpha_6 - \alpha_1 - \alpha_2 = \frac{\pi}{2}k\},\\ &\{\alpha_4 + \alpha_5 - \alpha_1 - \alpha_2 - 2\alpha_3 = \pi k\},\ \{\alpha_4 + \alpha_6 - \alpha_1 - \alpha_3 - 2\alpha_2 = \pi k\},\\ &\{\alpha_5 + \alpha_6 - \alpha_2 - \alpha_3 - 2\alpha_1 = \pi k\}\end{aligned}$$

of $\mathbb{R}^7 = \{(\alpha_1, \alpha_2, \ldots, \alpha_7)\}$, where $s = 1, 2, 3$; $k \in \mathbb{Z}$ and m, j are integers satisfying $1 \leq m < j \leq 3$. In this notation we have

$$S = (0, \infty)^{13} \otimes [0, 2\pi)^6 \otimes D, \;\; S' = (0, \infty)^{13} \otimes [0, 2\pi)^6 \otimes D',$$

where D' is obtained from D by eliminating the above six-dimensional hyperplanes. It is clear that the 26 dimensional measure of the set $S \backslash S'$ is zero. Since the main result (Theorem 4.5.2) of this section is concerned to the potentials corresponding to the set S', we investigate the almost all potentials of the form (4.1.16).

Since the operators $L(q(x-\tau))$ for $\tau \in F$ have the same Bloch eigenvalues, we may fix τ, that is, take one of the functions $q(x-\tau)$, which determines three of the arguments.

Theorem 4.5.1 *There exists a unique value of $\tau \in F$ such that the following conditions hold*

$$\alpha(\tau, \gamma_1) = \alpha(\tau, \gamma_2) = \alpha(\tau, \gamma_3) = 0, \tag{4.5.4}$$

where $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of the lattice Γ and $\alpha(\tau, \gamma) = \arg(q(x-\tau), e^{i\langle\gamma, x\rangle})$.

Proof Let $\omega_1, \omega_2, \omega_3$ be a basis of Ω satisfying

$$\langle\gamma_i, \omega_j\rangle = 2\pi\delta_{i,j} \tag{4.5.5}$$

and $F = \{c_1\omega_1 + c_2\omega_2 + c_3\omega_3 : c_k \in [0, 1), k = 1, 2, 3\}$ be a fundamental domain $\mathbb{R}^3/\Omega$ of Ω. If $\tau \in F$, then we have $\tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3$. Therefore, using the notations of (4.1.16) and (4.5.4) one can readily see that

$$\alpha(\tau, \gamma) = \arg(q(x-\tau), e^{i\langle\gamma, x-\tau\rangle} e^{i\langle\gamma, \tau\rangle}) = \alpha(\gamma) - \langle\gamma, \tau\rangle. \tag{4.5.6}$$

This with (4.5.5) yields $\alpha(\tau, \gamma_k) = \alpha(\gamma_k) - 2\pi c_k$ which means that (4.5.4) is equivalent to

$2\pi c_k = \alpha(\gamma_k)$, where $\alpha(\gamma_k) \in [0, 2\pi)$, $2\pi c_k \in [0, 2\pi)$ and $k = 1, 2, 3$. Thus, there exists a unique value of $\tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3 \in F$ satisfying (4.5.4) $\square$

By Theorem 4.5.1 and by (4.1.17), without loss of generality, it can be assumed that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0, \;\; z(\gamma_i) = a_i > 0, \;\; \forall i = 1, 2, 3. \tag{4.5.7}$$

Using (4.5.6) one can easily verify that the expressions in the left-hand sides of the inequalities in (4.5.3) do not depend on τ. Therefore, using the assumption (4.5.7) one can readily see that the condition (4.5.3) has the form

$$\alpha_7 \neq \pi k, \;\; \alpha_j \neq \frac{\pi}{2}k, \;\; \alpha_7 - \alpha_j \neq \pi k, \;\; \alpha_m \pm \alpha_j \neq \pi k, \tag{4.5.8}$$

where $k \in \mathbb{Z}$; $j = 4, 5, 6$; $m = 4, 5, 6$ and $m \neq j$. Using the notation of (4.5.2) and taking into account that $r_j r_m \sin(\alpha_j \pm \alpha_m) = b_j a_m \pm b_m a_j$, $r_j r_m \neq 0$ [see (4.1.17)], we see that (4.5.8) can be written in the form

$$b_7 \neq 0,\ a_j b_j \neq 0,\ b_7 a_j - a_7 b_j \neq 0,\ b_j a_m \pm b_m a_j \neq 0, \tag{4.5.9}$$

where $j = 4, 5, 6$; $m = 4, 5, 6$ and $m \neq j$.

The equality $(q(-x), e^{i\langle a,x\rangle}) = \overline{(q(x), e^{i\langle a,x\rangle})}$ shows that the imaginary part of the Fourier coefficients of $q(x)$ and $q(-x)$ take the opposite values. Therefore, taking into account the first inequality of (4.5.9), for fixing the inversion $q(x) \longrightarrow q(-x)$, in the set of potentials of the form (4.1.16) satisfying (4.5.3), we assume that

$$b_7 > 0. \tag{4.5.10}$$

Now using (4.5.7), (4.5.9), (4.5.10) and the invariants (4.2.20), (4.2.25)–(4.2.28), (4.2.34), (4.2.35), we give an algorithm for finding the Fourier coefficients $z(a)$ for all $a \in Q$. Let us emphasize the main points of the reconstruction algorithm and the relevant data for this algorithm. By (4.2.20), $|z(a)|$ for $a \in Q(1, 1, 1)$ is an invariant. Since the first multiplicands $A_1(a, b)$ and $A_2(a, b)$ of the right-hand sides of the invariants (4.2.25)–(4.2.28) and (4.2.34), (4.2.35) are nonzero known numbers (see the Theorems 4.2.2 and 4.2.3), we can also use the second multiplicands of them as invariants too. Namely, we use the following 24 invariants, denoted by $s_1, s_2, \ldots, s_{24}$, as relevant data:

$$\begin{aligned}
s_i &= |z(\gamma_i)|,\ s_4 = |z(\gamma_2+\gamma_3)|,\ s_5 = |z(\gamma_1+\gamma_3)|,\ s_6 = |z(\gamma_1+\gamma_2)|,\\
s_7 &= Re(z(-\gamma_1-\gamma_2)z(\gamma_1)z(\gamma_2)),\ s_8 = Re(z(-\gamma_1-\gamma_3)z(\gamma_1)z(\gamma_3)),\\
s_9 &= Re(z(-\gamma_2-\gamma_3)z(\gamma_2)z(\gamma_3)),\ s_{9+i} = Re(z(-\gamma)z(\gamma-\gamma_i)z(\gamma_i)),\\
s_{12+i} &= Re(z(\gamma-2\gamma_i)z(\gamma_i-\gamma)z(\gamma_i)),\ s_{15+i} = Re(z^2(-\gamma_i)z(\gamma)z(2\gamma_i-\gamma)),\\
s_{19} &= Re(z^2(-\gamma_1)z(\gamma_1+\gamma_2)z(\gamma_1-\gamma_2)),\ s_{20} = Re(z^2(-\gamma_2)z(\gamma_2+\gamma_1)z(\gamma_2-\gamma_1)),\\
s_{21} &= Re(z^2(-\gamma_1)z(\gamma_1+\gamma_3)z(\gamma_1-\gamma_3)),\ s_{22} = Re(z^2(-\gamma_3)z(\gamma_3+\gamma_1)z(\gamma_3-\gamma_1)),\\
s_{23} &= Re(z^2(-\gamma_2)z(\gamma_2+\gamma_3)z(\gamma_2-\gamma_3)),\ s_{24} = Re(z^2(-\gamma_3)z(\gamma_3+\gamma_2)z(\gamma_3-\gamma_2)),
\end{aligned} \tag{4.5.11}$$

where $i = 1, 2, 3$, $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, the invariants s_k are obtained from (4.2.20), (4.2.25), (4.2.27), (4.2.28), (4.2.35) and (4.2.34) for $k = 1, 2, \ldots 6$, $k = 7, 8, 9$, $k = 10, 11, 12$, $k = 13, 14, 15$, $k = 16, 17, 18$ and $k = 19, 20, \ldots, 24$ respectively. The main point of the reconstruction is the following. It follows from the definition of s_1, s_2, s_3 [see the first row of (4.5.11)] and from (4.5.7) that

$$z(\gamma_i) = s_i > 0,\ \forall i = 1, 2, 3. \tag{4.5.12}$$

Thus the Fourier coefficients $z(\gamma_1)$, $z(\gamma_2)$ and $z(\gamma_3)$ are expressed in terms of s_1, s_2 and s_3 respectively. In the following theorem, using (4.5.12) and the invariants $s_4, s_5, \ldots, s_{24}$, we find formulas (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and

(4.5.33)–(4.5.35) that express all the other Fourier coefficients in terms of the invariants $s_1, s_2, \ldots, s_{24}$.

Theorem 4.5.2 *The invariants* (4.5.11) *determine constructively and uniquely, modulo inversion and translation* (4.1.8)*, all the potentials of the form* (4.1.16) *satisfying* (4.5.3).

Proof To determine the potential (4.1.16), we find the all Fourier coefficients step by step by using the invariants (4.5.11).

Step 1. In this step using the invariants $s_4, s_5, \ldots, s_9$ and the relations (4.5.12), (4.5.9) and (4.5.10), we find

$$z(\gamma_1 + \gamma_2),\ z(\gamma_1 + \gamma_3),\ z(\gamma_2 + \gamma_3),\ z(\gamma_1 + \gamma_2 + \gamma_3). \tag{4.5.13}$$

The invariants s_7, s_8, s_9 and formula (4.5.12) give the real parts a_4, a_5, a_6 [see (4.5.2)] of the Fourier coefficients $z(\gamma_2 + \gamma_3), z(\gamma_1 + \gamma_3), z(\gamma_1 + \gamma_2)$:

$$a_4 = \frac{s_9}{s_2 s_3},\ a_5 = \frac{s_8}{s_1 s_3},\ a_6 = \frac{s_7}{s_1 s_2}. \tag{4.5.14}$$

Then, using the invariants s_4, s_5, s_6, we find the absolute values of the imaginary parts of these Fourier coefficients. Thus due to the notations of (4.5.1) and (4.5.2), we have

$$z(\gamma_2 + \gamma_3) = a_4 + it_4|b_4|, z(\gamma_1 + \gamma_3) = a_5 + it_5|b_5|, z(\gamma_1 + \gamma_2) = a_6 + it_6|b_6|, \tag{4.5.15}$$

where $|b_m|$ for $m = 4, 5, 6$ are known real numbers and t_m is the sign of b_m, i.e., is either -1 or 1. To determine t_4, t_5, t_6, we use the invariants s_{9+i} for $i = 1, 2, 3$ [see (4.5.11)]. Using (4.5.12), the invariant s_{10}, which is s_{9+i} for $i = 1$, the obvious relations $z(a) = \overline{z(-a)}$, and the notations $\gamma = \gamma_1 + \gamma_2 + \gamma_3 = \gamma_7$, $z(\gamma_j) = a_j + ib_j$ [see (4.5.1), (4.5.2)], we obtain the equation

$$a_4 a_7 + t_4|b_4|b_7 = \frac{s_{10}}{s_1} \tag{4.5.16}$$

with respect to the unknowns a_7 and b_7. In the same way, from the invariant s_{9+i} for $i = 2, 3$, we obtain

$$a_5 a_7 + t_5|b_5|b_7 = \frac{s_{11}}{s_2}, \tag{4.5.17}$$

$$a_6 a_7 + t_6|b_6|b_7 = \frac{s_{12}}{s_3}. \tag{4.5.18}$$

By (4.5.9) $t_5|b_5|a_4 - t_4|b_4|a_5 \neq 0$, $t_6|b_6|a_4 - t_4|b_4|a_6 \neq 0$, $t_6|b_6|a_5 - t_5|b_5|a_6 \neq 0$. Therefore finding b_7 from the systems of equations generated by pairs {(4.5.16), (4.5.17)}, {(4.5.16), (4.5.18)}, {(4.5.17), (4.5.18)}, and taking into account (4.5.10), we get the inequalities

$$\frac{a_4\frac{s_{11}}{s_2}-a_5\frac{s_{10}}{s_1}}{t_5|b_5|a_4-t_4|b_4|a_5}>0,\ \frac{a_4\frac{s_{12}}{s_3}-a_6\frac{s_{10}}{s_1}}{t_6|b_6|a_4-t_4|b_4|a_6}>0,\ \frac{a_5\frac{s_{12}}{s_3}-a_6\frac{s_{11}}{s_2}}{t_6|b_6|a_5-t_5|b_5|a_6}>0 \tag{4.5.19}$$

respectively. Now we prove that the relations (4.5.16)–(4.5.19) determines uniquely the unknowns a_7,b_7, t_4, t_5, t_6. Suppose to the contrary that there exists to different solutions $(a_7, b_7, t_4, t_5, t_6)$ and $(a_7', b_7', t_4', t_5', t_6')$ of (4.5.16)–(4.5.19). Clearly, if 2 components of the triple (t_4', t_5', t_6') take the opposite values of the corresponding components of the triple (t_4, t_5, t_6) then all the inequalities in (4.5.19) do not hold simultaneously. Therefore, at least, two component of (t_4', t_5', t_6') must be the same with the corresponding two components of (t_4, t_5, t_6). It can be assumed, without loss of generality, that $t_4' = t_4$ and $t_5' = t_5$. Then it follows from the system of equation (4.5.16), (4.5.17) that $a_7' = a_7$, $b_7' = b_7$. Since $b_6 b_7 \neq 0$ due to (4.5.9) it follows from (4.5.18) that $t_6' = t_6$. Thus without loss of generality, we can assume that $b_4 > 0, b_5 > 0, b_6 > 0$, that is, $t_4 = t_5 = t_6 = 1$. Then the Fourier coefficients in (4.5.13) can be determined from (4.5.16)–(4.5.19). Namely, by [see (4.5.1), (4.5.2)], we have

$$z(\gamma_2+\gamma_3) = a_4 + ib_4,\ z(\gamma_1+\gamma_3) = a_5 + ib_5,\ z(\gamma_1+\gamma_2) = a_6 + ib_6, \tag{4.5.20}$$

$$z(\gamma_1+\gamma_2+\gamma_3) = a_7 + ib_7, \tag{4.5.21}$$

where, it follows from the invariants s_4, s_5, s_6 and (4.5.14) that

$$b_4 = \frac{\sqrt{(s_2s_3s_4)^2 - s_9^2}}{s_2s_3} > 0,\ b_5 = \frac{\sqrt{(s_1s_3s_5)^2 - s_8^2}}{s_1s_3} > 0,\ b_6 = \frac{\sqrt{(s_1s_2s_6)^2 - s_7^2}}{s_1s_2} > 0 \tag{4.5.22}$$

and it follows from (4.5.16), (4.5.17) that

$$a_7 = \frac{b_4\frac{s_{11}}{s_2} - b_5\frac{s_{10}}{s_1}}{b_5a_4 - b_4a_5},\ b_7 = \frac{a_4\frac{s_{11}}{s_2} - a_5\frac{s_{10}}{s_1}}{b_5a_4 - b_4a_5}. \tag{4.5.23}$$

Step 2. In this step using the invariants $s_{19}, s_{20}, \ldots, s_{24}$, and (4.5.9), we find

$$z(\gamma_1-\gamma_2),\ z(\gamma_1-\gamma_3),\ z(\gamma_2-\gamma_3). \tag{4.5.24}$$

From s_{19} and the equalities $z(-\gamma_1) = z(\gamma_1) = s_1$ [see (4.5.12)], $z(\gamma_1+\gamma_2) = a_6 + ib_6$, $z(\gamma_1-\gamma_2) = a_8 + ib_8$ [see (4.5.1), (4.5.2)], we obtain an equation

$$a_6a_8 - b_6b_8 = s_1^{-2}s_{19}, \tag{4.5.25}$$

with respect to the unknowns a_8 and b_8, since a_6 and b_6 are known due to (4.5.14) and (4.5.22). From s_{20}, in the same way, we get

$$a_6a_8 + b_6b_8 = s_2^{-2}s_{20}. \tag{4.5.26}$$

Since $a_6b_6 \neq 0$ due to (4.5.9), from (4.5.25) and (4.5.26), we find a_8 and b_8 :

$$a_8 = \frac{s_1^{-2}s_{19} + s_2^{-2}s_{20}}{2a_6},\ b_8 = \frac{s_2^{-2}s_{20} - s_1^{-2}s_{19}}{2b_6}. \tag{4.5.27}$$

Now instead of the pair $\{s_{19}, s_{20}\}$ using the pair $\{s_{21}, s_{22}\}$ we obtain

$$a_9 = \frac{s_1^{-2}s_{21} + s_3^{-2}s_{22}}{2a_5},\ b_9 = \frac{s_3^{-2}s_{22} - s_1^{-2}s_{21}}{2b_5}, \tag{4.5.28}$$

where $z(\gamma_1 - \gamma_3) = a_9 + ib_9$ [see (4.5.1), (4.5.2)] and then using the pair $\{s_{23}, s_{24}\}$ we obtain

$$a_{10} = \frac{s_2^{-2}s_{23} + s_3^{-2}s_{24}}{2a_4},\ b_{10} = \frac{s_3^{-2}s_{24} - s_2^{-2}s_{23}}{2b_4}, \tag{4.5.29}$$

where $z(\gamma_2 - \gamma_3) = a_{10} + ib_{10}$.

Step 3. In this step using the invariants s_{12+i} and s_{15+i} for $i = 1, 2, 3$ we find

$$z(\gamma_2 + \gamma_3 - \gamma_1),\ z(\gamma_1 + \gamma_3 - \gamma_2),\ z(\gamma_1 + \gamma_2 - \gamma_3). \tag{4.5.30}$$

Using s_{12+i} and s_{15+i} for $i = 1$ and taking into account that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, $z(a) = \overline{z(-a)}$, $z(\gamma_2 + \gamma_3 - \gamma_1) = a_{11} + ib_{11}$ [see (4.5.1), (4.5.2)], we obtain the equations

$$a_4a_{11} + b_4b_{11} = s_1^{-1}s_{13}, \tag{4.5.31}$$

$$a_7a_{11} + b_7b_{11} = s_1^{-2}s_{16}. \tag{4.5.32}$$

with respect to the unknowns a_{11} and b_{11}, where a_4, b_4 and a_7, b_7 are defined by (4.5.14), (4.5.22) and (4.5.23). Since $a_4b_7 - b_4a_7 \neq 0$, due to (4.5.9), from this system of equations we get

$$a_{11} = \frac{b_4s_1^{-2}s_{16} - b_7s_1^{-1}s_{13}}{a_4b_7 - b_4a_7},\ b_{11} = \frac{a_4s_1^{-2}s_{16} - a_7s_1^{-1}s_{13}}{a_4b_7 - b_4a_7}. \tag{4.5.33}$$

In the same way, using s_{12+i}, s_{15+i} for $i = 2$ and for $i = 3$, we find the following formulas for $z(\gamma_1 + \gamma_3 - \gamma_2) = a_{12} + ib_{12}$ and $z(\gamma_1 + \gamma_2 - \gamma_3) = a_{13} + ib_{13}$:

$$a_{12} = \frac{b_5s_2^{-2}s_{16} - b_7s_2^{-1}s_{13}}{a_5b_7 - b_5a_7},\ b_{12} = \frac{a_5s_2^{-2}s_{16} - a_7s_2^{-1}s_{13}}{a_5b_7 - b_5a_7}, \tag{4.5.34}$$

$$a_{13} = \frac{b_6s_3^{-2}s_{16} - b_7s_3^{-1}s_{13}}{a_6b_7 - b_6a_7},\ b_{13} = \frac{a_6s_3^{-2}s_{16} - a_7s_3^{-1}s_{13}}{a_6b_7 - b_6a_7}. \tag{4.5.35}$$

The theorem is proved □

Formulas (4.5.12), (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and (4.5.33)–(4.5.35) shows that the conditions (4.5.9) can be written in term of the spectral invariants $s_1, s_2, \ldots, s_{24}$. Using the notations $p = s_1s_2s_3,\ p_1 = \sqrt{(s_2s_3s_4)^2 - s_9^2}$, $p_2 = \sqrt{(s_1s_3s_5)^2 - s_8^2}$, $p_3 = \sqrt{(s_1s_2s_6)^2 - s_7^2}$, one can easily verify that

$$a_{3+i} = \frac{s_{10-i}s_i}{p},\ b_{3+i} = \frac{p_is_i}{p}$$

and the relations $b_7 \neq 0,\ b_7a_j - a_7b_j \neq 0$ for $j = 4, 5, 6,$ are equivalent to

$$s_1^2s_9s_{11} - s_2^2s_8s_{10} \neq 0,\ (s_1^2s_9s_{11} - s_2^2s_8s_{10})s_{10-i} - (s_1^2s_{11}p_1 - s_2^2s_{10}p_2)p_i \neq 0$$

for $i = 1, 2, 3$ [see (4.5.14) and (4.5.22)]. Therefore (4.5.9), in term of the invariants, has the form:

$$\begin{gathered} s_1^2s_9s_{11} - s_2^2s_8s_{10} \neq 0,\ s_is_{10-i}p_i \neq 0, \\ (s_1^2s_9s_{11} - s_2^2s_8s_{10})s_{10-i} - (s_1^2s_{11}p_1 - s_2^2s_{10}p_2)p_i \neq 0,\ s_{10-k}p_i \pm p_ks_{10-i} \neq 0, \end{gathered} \tag{4.5.36}$$

where $i = 1, 2, 3;\ k = 1, 2, 3$ and $k \neq i$. Thus from Theorem 4.5.2 we obtain the following

Corollary 4.5.1 *If the spectral invariants $s_1, s_2, \ldots, s_{24}$ of $L(q)$, where q is a potential of the form* (4.1.16)*, are given and satisfy* (4.5.36)*, then one can determine the potential q constructively and uniquely, modulo* (4.1.8)*, by formulas* (4.5.12), (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) *and* (4.5.33)–(4.5.35).

4.6 On the Stability of the Algorithm

We determine constructively the potential from the Bloch eigenvalues in two steps. In the first step we have determined the invariants from the Bloch eigenvalues in Chap. 3. In the second step we found the potential from the invariants in Sect. 4.5 of this chapter. In this section we consider the stability of the problems studied in both steps.

First, we consider the stability of the invariants (4.1.4)–(4.1.6) with respect to the errors in the Bloch eigenvalues for the potential of the form (4.1.16). For this let us recall the formulas of Chap. 3 that will be used here. In Chap. 3 the spectral invariants are expressed by the Bloch eigenvalues of the Schrödinger operator $L(q^\delta)$ with the directional potential $q^\delta(x)$, where δ is a visible element of Γ. The function q^δ depends on only one variable $s = \langle \delta, x \rangle$ and can be written as

$$q^{\delta}(x) = Q^{\delta}(\langle \delta, x \rangle), \ Q^{\delta}(s) = \sum_{n \in \mathbb{Z}} z(n\delta) e^{ins}. \tag{4.6.1}$$

The Bloch eigenvalues and the Bloch functions of the operator $L(q^{\delta})$ are

$$\lambda_{j,\beta}(v, \tau) = |\beta + \tau|^2 + \mu_j(v), \ \Phi_{j,\beta}(x) = e^{i\langle \beta+\tau, x\rangle} \varphi_{j,v}(s),$$

where $\beta \in \Gamma_\delta$, $\tau \in F_\delta =: H_\delta/\Gamma_\delta$, $j \in \mathbb{Z}$, $v \in [0, 1)$, $\mu_j(v)$ and $\varphi_{j,v}(s)$ are the eigenvalues and eigenfunctions of the operator $T_v(Q^{\delta})$ generated by the boundary value problem:

$$-|\delta|^2 y''(s) + Q^{\delta}(s) y(s) = \mu y(s), \ y(2\pi) = e^{i2\pi v} y(0), \ y'(2\pi) = e^{i2\pi v} y'(0).$$

In Chap. 3, we constructed a set of eigenvalue, denoted by $\Lambda_{j,\beta}(v, \tau)$, of $L_t(q)$ satisfying

$$\Lambda_{j,\beta}(v, \tau) = |\beta + \tau|^2 + \mu_j(v) + \frac{1}{4} \int_F \left| f_{\delta,\beta+\tau} \right|^2 \left| \varphi_{j,v} \right|^2 dx + O(\rho^{-3a+2\alpha_1} \ln \rho), \tag{4.6.2}$$

where $\beta \sim \rho$, $j = O(\rho^{\alpha_1})$, $\alpha_1 = 3\alpha$, $a = 406\alpha$, $\alpha = \frac{1}{432}$, $-3a + 2\alpha_1 = -\frac{101}{36}$ and

$$f_{\delta,\beta+\tau}(x) = \sum_{\gamma:\gamma \in Q(1,1,1)\backslash \delta\mathbb{R}} \frac{\gamma}{\langle \beta + \tau, \gamma\rangle} z(\gamma) e^{i\langle \gamma, x\rangle}. \tag{4.6.3}$$

To consider the stability of the invariants (4.1.4)–(4.1.6) with respect to the errors in the band functions, we use (4.6.2) and the following asymptotic decomposition of $\mu_j(v)$ and $\left|\varphi_{j,v}(s)\right|^2$:

$$\mu_j(v) = |j\delta|^2 + \frac{c_1}{j} + \frac{c_1}{j^2} + \cdots + \frac{c_n}{j^n} + O(\frac{1}{j^{n+1}}), \tag{4.6.4}$$

$$\left|\varphi_{j,v}(s)\right|^2 = A_0 + \frac{A_1(s)}{j} + \frac{A_2(s)}{j^2} + \cdots + \frac{A_n(s)}{j^n} + O(\frac{1}{j^{n+1}}), \tag{4.6.5}$$

where

$$c_1 = c_2 = 0, \ c_3 = \frac{1}{16\pi|\delta|^3} \int_0^{2\pi} \left| Q^{\delta}(t) \right|^2 dt \tag{4.6.6}$$

(see [Mar] and [Eas]). In Chap. 3, we proved that if $q^{\delta}(x)$ has the form (4.1.7), then

$$A_0 = 1, \ A_1 = 0, \ A_2 = \frac{Q^{\delta}(s)}{2} + B_1 |z(\delta)|^2, \ A_3 = B_2 Q^{\delta}(s) + B_3 |z(\delta)|^2, \tag{4.6.7}$$

$$A_4 = B_4 Q^{\delta}(s) + B_5((z(\delta))^2 e^{i2\langle \delta, x\rangle} + (z(-\delta))^2 e^{-i2\langle \delta, x\rangle}) + B_6,$$

where $B_1, B_2, \ldots, B_6$ are the known constants.

Theorem 4.6.1 *Let $q(x)$ be the potential of the form* (4.1.16)*, satisfying* (4.5.3)*. If the Bloch eigenvalues of order ρ^2 of $L(q)$ are given with accuracy $O(\rho^{-\frac{101}{36}} \ln \rho)$, then one can determine the spectral invariants* (4.1.4)–(4.1.6)*, constructively and uniquely, with accuracy $O(\rho^{-\frac{97}{108}} \ln \rho)$.*

Proof First, using the asymptotic formula (4.6.2), we write explicitly the asymptotic expression of the invariants

$$\mu_j(v),\ J(\delta, b, j, v) = \int_F |q_{\delta,b}(x)\varphi_{j,v}(\langle \delta, x\rangle)|^2 dx \tag{4.6.8}$$

determined constructively in Chap. 3, where $v \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $j \in \mathbb{Z}$, $q_{\delta,b}(x)$ is defined in (4.1.3), $\delta \in Q(1, 1, 1)$ and b is a visible element of Γ_δ, in terms of the Bloch eigenvalues with an estimate of the remainder term. Let $s_1 b_1, s_2 b_2, \ldots, s_m b_m$ be projections of the vectors of the set $Q(1, 1, 1)\backslash\delta\mathbb{R}$ onto the plane H_δ, where $s_i \in \mathbb{R}$ and $b_i \in \Gamma_\delta$. If $b_i \in b_j\,\mathbb{R}$, where $i > j$, then we do not include b_i to the list of projections, that is, $b_1, b_2, \ldots, b_m$ are pairwise linearly independent. Consider the planes $P(\delta, b_k)$ for $k = 1, 2, \ldots, m$. It is clear that the set $Q(1, 1, 1)\backslash\delta\mathbb{R}$ is the union of the pairwise disjoint sets $P(\delta, b_k) \cap (Q\backslash\delta\mathbb{R})$ for $k = 1, 2, \ldots, m$. To find the spectral invariants (4.6.8), we write $f_{\delta,\beta+\tau}(x)$ [see (4.6.3)] in the form

$$f_{\delta,\beta+\tau}(x) = \sum_{k=1}^{m} F_{\delta,b_k,\beta+\tau}(x), \tag{4.6.9}$$

where

$$F_{\delta,b_k,\beta+\tau}(x) = \sum_{\gamma:\gamma\in P(\delta,b_k)\cap(Q\backslash\delta\mathbb{R})} \frac{\gamma}{\langle \beta+\tau, \gamma\rangle} z(\gamma) e^{i\langle\gamma,x\rangle}. \tag{4.6.10}$$

Clearly, if $\gamma \in P(\delta, b_k)\backslash\delta\mathbb{R}$ and $\gamma' \in P(\delta, b_l)\backslash\delta\mathbb{R}$ for $l \neq k$, then $\gamma' + \gamma \notin \delta\mathbb{R}$. Therefore taking into account that $\varphi_{j,v}(\langle \delta, x\rangle)$ is a function of $\langle \delta, x\rangle$, we obtain

$$\int_F \left\langle F_{\delta,b_k,\beta+\tau}(x),\ F_{\delta,b_l,\beta+\tau}(x)\right\rangle |\varphi_{j,v}(\langle \delta, x\rangle)|^2 dx = 0,\ \forall l \neq k.$$

This with (4.6.9) implies that

$$\int_F |f_{\delta,\beta+\tau}|^2 \left|\varphi_{j,v}\right|^2 dx = \sum_{k=1}^{m} \int_F |F_{\delta,b_k,\beta+\tau}|^2 \left|\varphi_{j,v}\right|^2 dx. \tag{4.6.11}$$

In Chap. 3 we proved that for each $b_0 \in \Gamma_\delta$ there exists $\beta_0 + \tau$ such that

$$|\beta_0 + \tau| \sim \rho,\ \frac{1}{3}\rho^a < |\langle \beta_0 + \tau, b_0\rangle| < 3\rho^a,$$

and $\Lambda_{j,\beta_0}(v,\tau)$ satisfies (4.6.2). Since $b_k \in \Gamma_\delta$, there exist $\beta_k + \tau$ such that

$$\frac{1}{3}\rho^a < |\langle \beta_k + \tau, b_k\rangle| < 3\rho^a \tag{4.6.12}$$

and $\Lambda_{j,\beta_0}(v,\tau)$ satisfies (4.6.2). From (4.6.12) we see that $\cos\theta_{k,k} = O(\rho^{a-1}) = o(1)$, where $\theta_{s,k}$ is the angle between $\beta_s+\tau$ and b_k. Therefore $\cos\theta_{s,k} \sim 1$ for $s \neq k$ and hence

$$\langle \beta_s + \tau, b_k\rangle \sim \rho \tag{4.6.13}$$

for all $s \neq k$. If $b_0 \notin b_1\mathbb{R} \cup b_2\mathbb{R} \cup \cdots \cup b_m\mathbb{R}$, then (4.6.13) holds for $k = 0$ and $s = 1, 2, \ldots, m$.

Now substituting the orthogonal decomposition $|\delta|^{-2}\langle\gamma,\delta\rangle\delta + |b_k|^{-2}\langle\gamma, b_k\rangle b_k$ of γ for $\gamma \in P(\delta, b_k) \cap (Q\backslash\delta\mathbb{R})$ into the denominator of the fraction in (4.6.10), and taking into account that $\beta+\tau \in H_\delta$, $\langle\beta+\tau,\delta\rangle = 0$, we obtain

$$F_{\delta,b_k,\beta+\tau}(x) = \frac{|b_k|^2}{\langle\beta+\tau, b_k\rangle} q_{\delta,b_k}(x),$$

where $q_{\delta,b_k}(x)$ is defined in (4.1.3). This with (4.6.8) implies that

$$\int_F |F_{\delta,b_k,\beta+\tau}|^2 \left|\varphi_{j,v}\right|^2 dx = \frac{|b_k|^4}{(\langle\beta+\tau, b_k\rangle)^2} J(\delta, b_k, j, v). \tag{4.6.14}$$

Substituting (4.6.11) and (4.6.14) in (4.6.2) and then instead of β writing β_s for $s = 0, 1, \ldots, m$, we get the system of $m+1$ equations

$$\mu_j(v) + \sum_{k=1}^{m} \frac{|b_k|^4}{4(\langle\beta_s+\tau, b_k\rangle)^2} J(\delta, b_k, j, v) = \Lambda_{j,\beta_s}(v,\tau) + |\beta_s+\tau|^2 + O(\rho^{-3a+2\alpha_1}\ln\rho), \tag{4.6.15}$$

with respect to the unknowns $\mu_j(v)$, $J(\delta, b_1, j, v)$, $J(\delta, b_2, j, v), \ldots, J(\delta, b_m, j, v)$. By (4.6.12) and (4.6.13) the coefficient matrix of (4.6.15) is $(a_{i,j})$, where $a_{i1} = 1$ for $i = 1, 2, \ldots, m+1$ and

$$a_{k,k} \sim \rho^{-2a},\ a_{s,k} \sim \rho^{-2},\ \forall k > 1,\ \forall s \neq k. \tag{4.6.16}$$

Expanding the determinant Δ of the matrix $(a_{i,j})$, one can readily see that the highest order term of this expansion is the product of the diagonal elements of the matrix $(a_{i,j})$ which is of order ρ^{-2ma} and the other terms of this expansions are $O(\rho^{-2m})$. Therefore, we have

$$\Delta \sim \rho^{-2ma} \tag{4.6.17}$$

Now we are going to use the fact that the right-hand side of (4.6.15) is determined with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$, if the Bloch eigenvalues of order ρ^2 of $L(q)$ are given with accuracy $O(\rho^{-3a+2\alpha_1}\ln\rho)$. Let Δ_k, $\Delta_{k,0}$ and $\Delta_{k,1}$ be determinant obtained from Δ by replacing sth elements of the kth column by

$$\Lambda_{j,\beta_s}(v,\tau)+|\beta_s+\tau|^2+O(\rho^{-3a+2\alpha_1}\ln\rho),\ \ \Lambda_{j,\beta_s}(v,\tau)+|\beta_s+\tau|^2)$$

and $O(\rho^{-3a+2\alpha_1}\ln\rho)$ respectively. One can easily see that

$$\Delta_1-\Delta_{1,0}=\Delta_{1,1}=O(\rho^{-2ma-3a+2\alpha_1}\ln\rho),$$
$$\Delta_k-\Delta_{k,0}=\Delta_{k,1}=O(\rho^{-2ma-a+2\alpha_1}\ln\rho) \tag{4.6.18}$$

for $k>1$. Therefore, solving the system (4.6.15) by the Cramer's rule and using (4.6.17), (4.6.18), we find $\mu_j(v)$ and $J(\delta,b_k,j,v)$ with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$ and $O(\rho^{-a+2\alpha_1}\ln\rho)$ respectively.

Now using (4.6.4) for $j\sim\rho^{\alpha_1}$, where n is chosen so that $j^{n+1}>\rho^{3a}$, and taking into account that $\mu_j(v)$ is determined with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$, we consider the invariant (4.1.4). In (4.6.4) replacing j by kj, for $k=1,2,\ldots,n$, we get the system of n equations

$$\frac{c_1}{jk}+\frac{c_2}{(jk)^2}+\cdots+\frac{c_n}{(jk)^n}=\mu_{jk}(v)+|jk\delta|^2+O(\frac{1}{j^{n+1}}), \tag{4.6.19}$$

with respect to the unknowns $c_1,c_2,\ldots,c_n$. The coefficient matrix of this system is $(a_{i,k})$, where $a_{i,k}=\frac{1}{(ji)^k}$ for $i,k=1,2,\ldots,n$. Therefore, the determinant of $(a_{i,k})$ is

$$\frac{1}{j}\frac{1}{j^2}\cdots 1 j^n\frac{1}{n!}\det(v_{i,k}),$$

where $v_{i,k}=v_i^{k-1}$, $v_i=\frac{1}{i}$, that is, $(v_{i,k})$ is the Vandermonde matrix and

$$\det(v_{i,k})=\prod_{1\le j<i\le n}\left(\frac{1}{i}-\frac{1}{j}\right).$$

Now solving the system (4.6.19) by the Cramer's rule and using the arguments used for the solving of (4.6.15), we find c_3 with an accuracy $O(\rho^{-3a+5\alpha_1}\ln\rho)$, since the elements of the third column is of order $\rho^{3\alpha_1}$ and the right-hand side of (4.6.19) is determined with error $O(\rho^{-3a+2\alpha_1}\ln\rho)$. Thus formula (4.6.6) gives the invariant (4.1.4) with error $O(\rho^{-3a+5\alpha_1}\ln\rho)$.

To consider the invariant (4.1.5) and (4.1.6), we use (4.6.5), where $j\sim\rho^{\alpha_1}$ and n can be chosen so that $j^{n+1}>\rho^a$. In (4.6.5) replacing j by kj, for $k=1,2,\ldots,n+1$, and using it in $J(\delta,b_s,j,v)$ [see (4.6.8)], we get the system of $n+1$ equations

$$J_0(\delta, b_s) + \frac{J_1(\delta, b_s)}{jk} + \frac{J_2(\delta, b_s)}{(jk)^2} + \cdots + \frac{J_n(\delta, b_s)}{(jk)^n} = J(\delta, b_s, j, v), \quad (4.6.20)$$

with respect to the unknowns $J_0(\delta, b_s), J_1(\delta, b_s), \ldots, J_n(\delta, b_s)$, where

$$J_k(\delta, b_s) = \int_F |q_{\delta,b_s}(x)|^2 A_k(\langle\delta, x\rangle)dx.$$

In the above we proved that the right-hand side of (4.6.20) is determined with error $O(\rho^{-a+2\alpha_1}\ln\rho)$. Therefore, instead of (4.6.19) using (4.6.20) and repeating the arguments used in finding of c_3, we find $J_0(\delta, b_s), J_1(\delta, b_s), \ldots, J_4(\delta, b_s)$ with accuracy $O(\rho^{-a+6\alpha_1}\ln\rho)$. Then using (4.6.7), we determine the invariants (4.1.5) and (4.1.6) with the accuracy $O(\rho^{-a+6\alpha_1}\ln\rho)$, where $a - 6\alpha_1 = \frac{97}{108}$ □

Proposition 4.6.1 *Let $q(x)$ be the potential of the form* (4.1.16)*, satisfying* (4.5.3)*. If the Bloch eigenvalues of order ρ^2 of $L(q)$ are given with accuracy ε, where $1 \gg \varepsilon \geq \rho^{-\frac{151}{54}}$, then one can determine the invariants* (4.1.4)–(4.1.6)*, constructively and uniquely, with accuracy $\rho^{\frac{103}{54}}o(\varepsilon)$.*

Proof Since $O(\rho^{-3a+2\alpha_1}\ln\rho) = o(\varepsilon)$ (see (4.6.2), (4.6.15) can be written in the form

$$\mu_j(v) + \sum_{k=1}^{m} \frac{|b_k|^4}{4(\langle\beta_s + \tau, b_k\rangle)^2} J(\delta, b_k, j, v) = \Lambda_{j,\beta_s}(v, \tau) + |\beta_s + \tau|^2 + o(\varepsilon). \quad (4.6.21)$$

Instead of (4.6.15) using (4.6.21), that is, instead of $O(\rho^{-3a+2\alpha_1}\ln\rho)$ using $o(\varepsilon)$, and repeating the arguments that were used in solving of (4.6.15) we find $\mu_j(v)$ and $J(\delta, b_k, j, v)$ with error $o(\varepsilon)$ and $o(\rho^{2a}\varepsilon)$ respectively. In the same way from (4.6.19) and (4.6.20) (everywhere instead of $O(\rho^{-3a+2\alpha_1}\ln\rho)$ using $o(\varepsilon)$, and repeating the arguments that were used in the proof of Theorem 4.6.1) we find the spectral invariants (4.1.4)–(4.1.6) with accuracy $\rho^{2a+4\alpha_1}o(\varepsilon)$, where $2a + 4\alpha_1 = \frac{824}{432} = \frac{103}{54}$ □

Note that Proposition 4.6.1 differs from Theorem 4.6.1. In the former one the error ε does not depend on the order ρ^2 of the given eigenvalues. We expect that this simplifies the real applications.

Now using formulas (4.5.12), (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and (4.5.33)–(4.5.35) (see the proof of Theorem 4.5.2), we prove that if the spectral invariants $s_1, s_2, \ldots, s_{24}$ [see (4.5.11)] of $L(q)$, where q is a potential of the form (4.1.16), are given and satisfy (4.5.36), then one can determine the potential q constructively and uniquely, modulo (4.1.8), with error $(M + h)\varepsilon$, where M is explicitly expressed by $s_1, s_2, \ldots, s_{24}$ and $h \to 0$ as $\varepsilon \to 0$. To determine M we introduce the following notations. By (4.5.2) the Fourier coefficients of (4.1.16) are $z(\gamma_j) = a_j + ib_j$, where, by formulas (4.5.12), (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and (4.5.33)–(4.5.35), a_j and b_j are explicitly expressed by $s_1, s_2, \ldots, s_{24}$. Indeed, using (4.5.14) and (4.5.22) in (4.5.23) and (4.5.27)–(4.5.29) we write a_j and b_j for $j = 7, 8, 9, 10$ in term of $s_1, s_2, \ldots, s_{24}$. Then using (4.5.14), (4.5.22) and (4.5.23) in (4.5.33)–(4.5.35) we write a_j and b_j for $j = 11, 12, 13$ in term of $s_1, s_2, \ldots, s_{24}$. Thus a_j and b_j are the functions of $s_1, s_2, \ldots, s_{24} : a_j = a_j(s_1, s_2, \ldots, s_{24})$, $b_j = b_j(s_1, s_2, \ldots, s_{24})$ for $j = 1, 2, \ldots, 13$. Introduce the functions

$$f_j(\varepsilon) = a_j(s_1 + \varepsilon, s_2 + \varepsilon, \ldots, s_{24} + \varepsilon),\ \ g_j(\varepsilon) = b_j(s_1 + \varepsilon, s_2 + \varepsilon, \ldots, s_{24} + \varepsilon)$$

and define M by

$$M = 2 \sum_{j=1,2,\ldots,13} \sqrt{(\frac{df_j(0)}{d\varepsilon})^2 + (\frac{dg_j(0)}{d\varepsilon})^2}.$$

Theorem 4.6.2 *If the spectral invariants* (4.5.11) *of* $L(q)$, *where* q *is a potential of the form* (4.1.16)*, are given with error* ε *and satisfy* (4.5.36)*, then the potential* q *can be determined constructively and uniquely, modulo* (4.1.8)*, with error* $(M + h)\varepsilon$ *in the* C^∞ *metric, where* $\varepsilon \ll 1$ *and* $h \to 0$ *as* $\varepsilon \to 0$.

Proof In Corollary 4.5.1 we proved that if the invariants $s_1, s_2, \ldots, s_{24}$ satisfy the conditions (4.5.36) then one can determine the potential q. Due to the above notation, q has the form

$$q(x) = \sum_{j=1}^{13} ((f_j(0) + i g_j(0))e^{i\langle \gamma_j, x\rangle} + (f_j(0) - i g_j(0))e^{i\langle -\gamma_j, x\rangle}).$$

Since the expressions in (4.5.36) continuously depend on the invariants $s_1, s_2, \ldots, s_{24}$, the conditions on (4.5.36) hold if these invariants are replaced by $s_1 + \varepsilon, s_2 + \varepsilon, \ldots, s_{24} + \varepsilon$ for $\varepsilon \ll 1$. Therefore by Corollary 4.5.1 one can construct the potential

$$q_\varepsilon(x) =: \sum_{j=1}^{13} ((f_j(\varepsilon) + i g_j(\varepsilon))e^{i\langle \gamma_j, x\rangle} + (f_j(\varepsilon) - i g_j(\varepsilon))e^{i\langle -\gamma_j, x\rangle})$$

from the dates $s_1 + \varepsilon, s_2 + \varepsilon, \ldots, s_{24} + \varepsilon$. Thus to prove the theorem, that is, to prove the inequality $\sup_x |q_\varepsilon(x) - q(x)| < (M + h)\varepsilon$, it is enough to show that

$$f_j(\varepsilon) - f_j(0) = (\frac{df_j(0)}{d\varepsilon} + h_j)\varepsilon,\ \ g_j(\varepsilon) - g_j(0) = (\frac{dg_j(0)}{d\varepsilon} + \widetilde{h_j})\varepsilon \qquad (4.6.22)$$

for $j = 1, 2, \ldots, 13$, where $h_j \to 0$ and $\widetilde{h_j} \to 0$ as $\varepsilon \to 0$. It follows from (4.5.12) that (4.6.22) holds for $j = 1, 2, 3$. To prove (4.6.22) for $j > 3$ we use the mean-value formulas

$$f_j(\varepsilon) - f_j(0) = \frac{df_j(\varepsilon_j)}{d\varepsilon}\varepsilon,\ \ g_j(\varepsilon) - g_j(0) = \frac{dg_j(\widetilde{\varepsilon_j})}{d\varepsilon}\varepsilon,$$

where $\varepsilon_j \in (0, \varepsilon)$, $\widetilde{\varepsilon_j} \in (0, \varepsilon)$ and prove that

$$\frac{df_j(\varepsilon_j)}{d\varepsilon} = \frac{df_j(0)}{d\varepsilon} + h_j, \; \frac{dg_j(\widetilde{\varepsilon_j})}{d\varepsilon} = \frac{dg_j(0)}{d\varepsilon} + \widetilde{h_j} \tag{4.6.23}$$

Let us prove (4.6.23). It follows from (4.5.7), (4.5.9) that the denominators of the fractions in the formulas (4.5.14), (4.5.22), (4.5.23), (4.5.27)–(4.5.29) and (4.5.33)–(4.5.35) are nonzero numbers. Therefore the denominators of the fractions taking part in the expressions of $f_j(\varepsilon)$ and $g_j(\varepsilon)$ are nonzero numbers for $\varepsilon \ll 1$. Moreover, by direct calculations one can readily see $\frac{df_j(\varepsilon)}{d\varepsilon}$ and $\frac{dg_j(\varepsilon)}{d\varepsilon}$ are continuous functions at $\varepsilon = 0$. It means that (4.6.23) holds and the theorem is proved □

The consequence of Theorems 4.6.1 and 4.6.2 is the following:

Corollary 4.6.1 *Let $q(x)$ be the potential of the form* (4.1.16) *satisfying* (4.5.3). *If the Bloch eigenvalues of order ρ^2 of $L(q)$ are given with accuracy $O(\rho^{-\frac{101}{36}} \ln \rho)$, then one can determine the potential q constructively and uniquely, modulo* (4.1.8), *with accuracy $O(\rho^{-\frac{97}{108}} \ln \rho)$*

The consequence of Proposition 4.6.1 and Theorem 4.6.2 is the following:

Corollary 4.6.2 *Let $q(x)$ be the potential of the form* (4.1.16) *satisfying* (4.5.3). *If the Bloch eigenvalues of order ρ^2 of $L(q)$ are given with accuracy ε, where $1 \gg \varepsilon \geq \rho^{-\frac{151}{54}}$, then one can determine the potential q, constructively and uniquely, with error $\rho^{\frac{103}{54}} o(\varepsilon)$ in the C^∞ metric*

Proof It follows from Proposition 4.6.1 that the invariants (4.1.4)–(4.1.6) can be determined with accuracy $\rho^{\frac{103}{54}} o(\varepsilon)$. Then using (4.5.11) and taking into account that the first multiplicands $A_1(a, b)$ and $A_2(a, b)$ of the right-hand sides of the invariants (4.2.25)–(4.2.28) and (4.2.34), (4.2.35) are nonzero constants of order 1, we conclude that the invariants $s_1, s_2, \ldots, s_{24}$ can be determined with accuracy $\rho^{\frac{103}{54}} o(\varepsilon)$. Therefore the proof follows from Theorem 4.6.2 □

4.7 Uniqueness Theorems

First we consider the Hill operator $H(p)$ generated in $L_2(\mathbb{R})$ by the expression

$$l(q) =: -y''(x) + p(x)y(x)$$

when $p(x)$ is a real-valued trigonometric polynomial

$$p(x) = \sum_{s=-N}^{N} p_s e^{2isx}, \; p_{-s} = \overline{p_s}, \; p_0 = 0. \tag{4.7.1}$$

Let the pair $\{\lambda_{k,1}, \lambda_{k,2}\}$ denote, respectively, the kth eigenvalues of the operator generated in $L_2[0, \pi]$ by the expression $l(q)$ and the periodic boundary conditions for even k and the anti-periodic boundary conditions for odd k. It is well-known that (see [Eas], Theorem 4.2.4)

$$\lambda_{0,1} = \lambda_{0,2} < \lambda_{1,1} \le \lambda_{1,2} < \lambda_{2,1} \le \lambda_{2,2} < \lambda_{3,1} \le \lambda_{3,2} < \cdots < \lambda_{n,1} \le \lambda_{n,2} < \cdots.$$

The spectrum $Spec(H(p))$ of $H(p)$ is the union of the intervals $[\lambda_{n-1,2}, \lambda_{n,1}]$ for $n = 1, 2, \ldots$. The interval $\gamma_n =: (\lambda_{n,1}, \lambda_{n,2})$ is the nth gaps in the spectrum of $H(p)$. Since the spectrum of the operators $H(p(x))$ and $(H(p(x + \tau))$, where $\tau \in (0, \pi)$, are the same, we may assume, without loss of generality, that $p_{-N} = p_N = \mu > 0$. We use the following formula obtained in the paper [Gri] (see Theorem 2 in [Gri]) for the length $|\gamma_n|$ of the gap γ_n:

$$|\gamma_n| = \frac{4n}{\mu}\left(\frac{\mu e^2}{8n^2}\right)^{\frac{n}{N}} \left|\sum_{k=0}^{N-1} A_k(n)\left(1 + O\left(\frac{\ln n}{n}\right)\right)\right|, \tag{4.7.2}$$

where

$$A_k(n) = \exp\left[\frac{2ink\pi}{N} + 2n\sum_{k=0}^{N-1}\lambda_j\left(\left(\tfrac{1}{2}\mu n^{-2}\right)^{\frac{1}{N}} e^{2ik\pi/N}\right)^j\right] \tag{4.7.3}$$

and λ_j algebraically depends on the Fourier coefficients of $p(x)$.

From (4.7.3) one can readily see that

$$|A_k(n)| < \exp(an^{1-\frac{2}{N}}),\ |A_k(n)| > \exp(-an^{1-\frac{2}{N}}),\ \forall k = 0, 1, \ldots, (N-1), \tag{4.7.4}$$

where

$$a = \sum_{j=1}^{N-1} a_j,\ a_j = \sup_k \left|Re(2\lambda_j\left(\left(\tfrac{1}{2}\mu\right)^{\frac{1}{N}} e^{2ik\pi/N}\right)^j\right|. \tag{4.7.5}$$

This and (4.7.2) imply that

$$|\gamma_n| < \frac{4n}{\mu}\left(\frac{\mu e^2}{8n^2}\right)^{\frac{n}{N}} 2Ne^{an^{1-\frac{2}{N}}}. \tag{4.7.6}$$

Using (4.7.4)–(4.7.6) we prove the following:

Theorem 4.7.1 *Let $\widetilde{p}(x)$ be a real-valued trigonometric polynomial of the form*

$$\widetilde{p}(x) = \sum_{s=-K}^{K} \widetilde{p}_s e^{2isx},\ \widetilde{p}_{-s} = \overline{\widetilde{p}_s},\ \widetilde{p}_{-K} = \widetilde{p}_K = \nu > 0.$$

If $Spec(H(p)) = Spec(H(\widetilde{p}))$, *then* $K = N$, *where* $p(x)$ *is defined in* (4.7.1).

Proof Suppose $K \neq N$. Without less of generality, it can be assumed that $K < N$. We consider the following two cases:

Case 1: Assume that $\lambda_j = 0$ for all values of j. Then from (4.7.3) for $n = lN$, $l \in \mathbb{N}$ we obtain that $A_k(n) = 1$ for all k. Therefore, by (4.7.2), we have

$$|\gamma_n| = \frac{4n}{\mu}\left(\frac{\mu e^2}{8n^2}\right)^l N\left(1 + O\left(\frac{\ln n}{n}\right)\right), \forall n = lN. \tag{4.7.7}$$

Applying (4.7.6) for the length $|\delta_n|$ of the n th gap δ_n in the $Spec(H(\widetilde{p}))$, that is, replacing N and μ by K and ν respectively and arguing as in the proof of (4.7.6), we see that there exists a positive number b such that

$$|\delta_n| < \frac{4n}{\nu}\left(\frac{\nu e^2}{8n^2}\right)^{\frac{n}{K}} 2K e^{bn^{1-\frac{2}{K}}}. \tag{4.7.8}$$

Since the fastest decreasing multiplicands of (4.7.7) and (4.7.8) are n^{-2l} and $n^{-\frac{2n}{K}}$ respectively and $K < N$, it follows from (4.7.7) and (4.7.8) for $n = lN$ that $|\gamma_{lN}| > |\delta_{lN}|$ for $l \gg 1$, which contradicts to the equality $Spec(H(q)) = Spec(H(\widetilde{p}))$.

Case 2: Assume that $\lambda_j \neq 0$ for some values of j. Let us prove that the equalities

$$|\gamma_{lN}| = |\delta_{lN}|, |\gamma_{lN+1}| = |\delta_{lN+1}|, \ldots, |\gamma_{lN+N-1}| = |\delta_{lN+N-1}| \tag{4.7.9}$$

for $l \gg 1$ can not be satisfied simultaneously. Suppose to the contrary that all equalities in (4.7.9) hold. Using (4.7.2), (4.7.8) and taking into account that

$$\left(\frac{\nu e^2}{8n^2}\right)^{\frac{lN+m}{K}} \left(\frac{\mu e^2}{8n^2}\right)^{-\frac{lN+m}{N}} e^{bn^{1-\frac{2}{K}}} = O(n^{-\alpha n})$$

for $0 < \alpha < \frac{lN+m}{K} - \frac{lN+m}{N}$, from (4.7.9) we obtain

$$\sum_{k=0}^{N-1} A_k(lN+m)\left(1 + O\left(\frac{\ln l}{l}\right)\right) = O(l^{-\alpha l}),\ \forall m = 0, 1, \ldots (N-1). \tag{4.7.10}$$

Let us consider $A_k(lN+m)$ in detail. It can be written in the form

$$A_k(lN+m) = \exp\left(\frac{2imk\pi}{N}\right) e^{c_k(lN+m)},\ c_k(lN+m) = \sum_{j=1}^{N-1} M_j(k)(lN+m)^{1-\frac{2j}{N}}, \tag{4.7.11}$$

where $M_j(k)$ is a complex number. Using the mean value theorem, we get

$$c_k(lN+m)-c_k(lN)=m\sum_{j=1}^{N-1}M_j(k)(lN+\theta(k))^{-\frac{2j}{N}}=O(l^{-\frac{2}{N}}),\qquad(4.7.12)$$

where $\theta(k)\in[0,m]$ for all k. Now using (4.7.11), (4.7.12) and taking into account that
$e^z=1+O(z)$ as $z\to 0$, we obtain

$$A_k(lN+m)=\exp\left(\frac{2imk\pi}{N}\right)A_k(lN)(1+O(l^{-\frac{2}{N}})).\qquad(4.7.13)$$

Therefore (4.7.10) has the form

$$\sum_{k=0}^{N-1}\exp\left(\frac{2imk\pi}{N}\right)A_k(lN)\,(1+o(1))=O(l^{-\alpha l}),\ m=0,1,\dots(N-1).\qquad(4.7.14)$$

Consider (4.7.14) as a system of equations with respect to the unknowns
$A_0(lN),\ A_1(lN),\dots,\ A_{N-1}(lN)$. Using the well-known formula for the determinant of the Vandermonde matrix $(v_{m,k})$, where $v_{m,k}=v_m^k$, $v_m=\exp(\frac{2im\pi}{N})$, we see that the main determinant of this system is

$$(1+o(1))\det\left(e^{\frac{2imk\pi}{N}}\right)_{k,m=0}^{N-1}=(1+o(1))\prod_{0\le m<k\le(N-1)}(e^{\frac{2ik\pi}{N}}-e^{\frac{2im\pi}{N}}).$$

Thus solving (4.7.14) by the Cramer's rule we obtain $A_k(lN)=O(l^{-\alpha l})$, for $k=0,1,\dots(N-1)$ which contradicts the second inequality in (4.7.4). The theorem is proved. □

Now using this theorem we prove a uniqueness theorem for the three-dimensional Schrödinger operator. For this, first, we prove the following lemma.

Lemma 4.7.1 *Let $\widetilde{q}(x)$ be infinitely differentiable periodic potential of the form*

$$\widetilde{q}(x)=\sum_{a\in Q(1,1,1)}\widetilde{q}^a(x),\qquad(4.7.15)$$

where

$$\widetilde{q}^a(x)=\sum_{n\in\mathbb{Z}}\widetilde{z}(na)e^{in\langle a,x\rangle},\ \widetilde{z}(0)=0\qquad(4.7.16)$$

and $\widetilde{z}(na)=:(\widetilde{q}(x),e^{in\langle a,x\rangle})$ is the Fourier coefficients of $\widetilde{q}$. If the equalities

$$\widetilde{z}(n\gamma_i)=0,\ \widetilde{z}(n\gamma_j)=0,\ \forall n\in\mathbb{Z}\backslash\{-1,1\}\qquad(4.7.17)$$

hold, then

$$\widetilde{I}_1(\gamma_i+\gamma_j,\gamma_i)=A_1(\gamma_i+\gamma_j,\gamma_i)Re(\widetilde{z}(-\gamma_i-\gamma_j)\widetilde{z}(\gamma_i)\widetilde{z}(\gamma_j)), \tag{4.7.18}$$

$$\widetilde{I}_1(\gamma_i-\gamma_j,\gamma_i)=A_1(\gamma_i-\gamma_j,\gamma_i)Re(\widetilde{z}(-\gamma_i+\gamma_j)\widetilde{z}(\gamma_i)\widetilde{z}(-\gamma_j)), \tag{4.7.19}$$

$$\widetilde{I}_2(\gamma_i,\gamma_j)=A_2(\gamma_i,\gamma_j)Re(\widetilde{z}(-\gamma_i))^2\widetilde{z}(\gamma_i+\gamma_j)\widetilde{z}(\gamma_i-\gamma_j)) \tag{4.7.20}$$

for $i=1,2,3;\ j=1,2,3;\ i\neq j$, *where* $\gamma_1,\gamma_2,\gamma_3$, $A_1(a,b)$ *and* $A_2(a,b)$ *are defined in* Theorems 4.2.2 *and* 4.2.3 *respectively,* $\widetilde{I}_1(a,b)$ *and* $\widetilde{I}_2(a,b)$ *are the invariants* (4.1.5) *and* (4.1.6) *for the operator* $L(\widetilde{q})$.

Proof By definition of $\widetilde{I}_1(\gamma_i+\gamma_j,\gamma_i)$ [see (4.1.5) and (4.1.3)] we have

$$\widetilde{I}_1(\gamma_i+\gamma_j,\gamma_i)=\int_F\left|\widetilde{q}_{\gamma_i+\gamma_j,\beta}(x)\right|^2(\widetilde{q})^{\gamma_i+\gamma_j}(x)dx, \tag{4.7.21}$$

where β is defined by (4.1.10),

$$\widetilde{q}_{\gamma_i+\gamma_j,\beta}(x)=\sum_{c\in D}\frac{c}{\langle\beta,c\rangle}\widetilde{z}(c)e^{i\langle c,x\rangle}, \tag{4.7.22}$$

and $D=\{c\in(P(\gamma_i,\gamma_j)\cap\Gamma)\backslash(\gamma_i+\gamma_j)\mathbb{R}:\widetilde{z}(c)\neq 0\}$. It follows from (4.7.15) that if $c\in D$, then $c=ka$, where k is an integer, and a belongs to the set $P(\gamma_i,\gamma_j)\cap Q)\backslash(\gamma_i+\gamma_j)\mathbb{R}$. Since this set is $\{\gamma_i,\gamma_j,-\gamma_i,-\gamma_j,\gamma_i-\gamma_j,-(\gamma_i-\gamma_j)\}$ and (4.7.17) holds, we have

$$D=\{\gamma_i,\gamma_j,-\gamma_i,-\gamma_j\}\cup\{k(\gamma_i-\gamma_j):k\in\mathbb{Z}\}. \tag{4.7.23}$$

Therefore, repeating the proof of (4.2.32), we see that

$$\widetilde{I}_1(\gamma_i+\gamma_j,\gamma_i)=2Re\left(\sum_{n=1}^{\infty}\widetilde{z}(-n(\gamma_i+\gamma_j))\sum_{c\in D}\frac{\langle n(\gamma_i+\gamma_j)-c,c\rangle}{(\langle c,\beta\rangle)^2}\widetilde{z}(n(\gamma_i+\gamma_j)-c)\widetilde{z}(c)\right). \tag{4.7.24}$$

It follows from (4.7.23) that if $n>1$ and $c\in D$, then $n(\gamma_i+\gamma_j)-c\notin D$ and $\widetilde{z}(n(\gamma_i+\gamma_j)-c)=0$. Hence, from (4.7.24) we obtain

$$\widetilde{I}_1(\gamma_i+\gamma_j,\gamma_i)=2Re\left(\widetilde{z}(-(\gamma_i+\gamma_j))\sum_{c\in D}\frac{\langle(\gamma_i+\gamma_j)-c,c\rangle}{(\langle c,\beta\rangle)^2}\widetilde{z}((\gamma_i+\gamma_j)-c)\widetilde{z}(c)\right). \tag{4.7.25}$$

Using this instead of (4.2.32) and repeating the proof of (4.2.25), we get (4.7.18). In (4.7.18) replacing γ_j by $-\gamma_j$, we get (4.7.19).

Now let us prove (4.7.20). It follows from (4.7.17) that

$$(\widetilde{q})^{\gamma_i}(x) = \widetilde{z}(\gamma_i)e^{i\langle\gamma_i,x\rangle} + \widetilde{z}(-\gamma_i)e^{-i\langle\gamma_i,x\rangle}).$$

Therefore $\widetilde{I}_2(\gamma_i, \gamma_j)$ has the form

$$\widetilde{I}_2(\gamma_i, \gamma_j) = \int_F \left|\widetilde{q}_{\gamma_i,\beta}(x)\right|^2 (((\widetilde{z}(\gamma_i))^2 e^{i2\langle\gamma_i,x\rangle} + (\widetilde{z}(-\gamma_i))^2 e^{-i2\langle\gamma_i,x\rangle})dx \quad (4.7.26)$$

[see (4.1.6)], where β is defined by (4.1.10),

$$\widetilde{q}_{\gamma_i,\beta}(x) = \sum_{c\in E} \frac{c}{\langle\beta, c\rangle}\widetilde{z}(c)e^{i\langle c,x\rangle}, \quad (4.7.27)$$

$E = \{c \in (P(\gamma_i, \gamma_j) \cap \Gamma)\backslash\gamma_i\mathbb{R} : \widetilde{z}(c) \neq 0\}$. Arguing as in the proof of (4.7.24), (4.7.25), we see that

$$E = \{\gamma_j, -\gamma_j\} \cup \{k(\gamma_i - \gamma_j) : k \in \mathbb{Z}\} \cup \{n(\gamma_i + \gamma_j) : n \in \mathbb{Z}\}. \quad (4.7.28)$$

$$\widetilde{I}_2(\gamma_i, \gamma_j) = 2Re\left(\widetilde{z}^2(-\gamma_i)\sum_{c\in E} \frac{\langle\gamma_i + c, \gamma_i - c\rangle}{(\langle c, \beta\rangle)^2}\widetilde{z}(\gamma_i + c)\widetilde{z}(\gamma_i - c)\right). \quad (4.7.29)$$

If $c = k(\gamma_i - \gamma_j)$, where $k \neq 0$, or $c = n(\gamma_i + \gamma_j)$, where $n \neq 0$, then at least one of the vectors $\gamma_i - c$ and $\gamma_i + c$ does not have the form $c = sa$, where $s \in \mathbb{Z}$, $a \in P(\gamma_i, \gamma_j) \cap Q)\backslash\gamma_i\mathbb{R}$, and hence by (4.7.15) we have $\widetilde{z}(\gamma_i + c)\widetilde{z}(\gamma_i - c) = 0$. Therefore, the summation in (4.7.29) is taken over $c \in \{\pm\gamma_j\}$ and (4.7.20) holds. □

Now we prove a uniqueness theorem for the periodic, with respect to the lattice Ω, potentials $q(x)$ of $C^1(\mathbb{R}^3)$ subject to some constraints only on the directional potentials $q^{\gamma_1}(x)$, $q^{\gamma_2}(x)$ and $q^{\gamma_3}(x)$, where $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying (4.1.18). Note that the directional potential $q^a(x)$ is a function $Q^a(s)$ of one variable $s =: \langle x, a\rangle \in \mathbb{R}$, where the function $Q^a(s)$ is defined in (4.6.1). Let U be the set of all periodic,with period 2π, functions $f \in C^1(\mathbb{R})$ such that $spec(H(f)) = spec(H(\mu \cos s))$ for some positive μ. Denote by W the set of all periodic, with respect to the lattice Ω, functions $q(x)$ of $C^1(\mathbb{R}^3)$ whose directional potentials $q^{\gamma_k}(x)$ for $k = 1, 2, 3$ satisfy the conditions

$$Q^{\gamma_k} \in (C^1(\mathbb{R})\backslash U) \cup T, \ \forall k = 1, 2, 3, \quad (4.7.30)$$

where T is the set of all trigonometric polynomial. Thus we put condition only on the directional potentials $q^{\gamma_1}(x)$, $q^{\gamma_2}(x)$ and $q^{\gamma_3}(x)$. The all other directional potentials, that is, $q^a(x)$ for all $a \in S\backslash\{\gamma_1, \gamma_2, \gamma_3\}$, where S is the set of all visible elements of Γ, are arbitrary continuously differentiable functions.

Theorem 4.7.2 *Let q be the potential of the form* (4.1.16)*, satisfying* (4.5.3)*. If $\widetilde{q} \in W$ and the Bloch eigenvalues of the operators $L(q)$ and $L(\widetilde{q})$ coincide, then $\widetilde{q}$ is equal to q modulo* (4.1.8)*.*

Proof Let $\widetilde{q}$ be a function of W whose Bloch eigenvalues coincides with the Bloch eigenvalues of q. By Theorem 6.1 of [EsRaTr2] the Bloch eigenvalues of $L(\widetilde{q}^a)$ coincides with the Bloch eigenvalues of $L(q^a)$. It implies that the spectrum of $H(\widetilde{Q}^a)$ coincides with the spectrum of $H(Q^a)$, where $\widetilde{Q}^a(\langle x, a\rangle) = \widetilde{q}^a(x)$. Since the length of the nth gap in the spectrum of $H(Q^a)$ satisfies (4.7.6), the same inequality holds for the nth gap of $H(\widetilde{Q}^a)$. It implies that $\widetilde{q}^a$ is an infinitely differentiable function for all visible elements a of Γ (see [Mar]). Thus $\widetilde{q}(x)$ is an infinitely differentiable function and due to Chap. 3 the operator $L(\widetilde{q})$ has the invariants (4.1.4)–(4.1.6) denoted by $\widetilde{I}(a)$, $\widetilde{I}_1(a, b)$, $\widetilde{I}_2(a, b)$. Since the Bloch eigenvalues of $L(q)$ and $L(\widetilde{q})$ coincide, we have

$$Spec(H(\widetilde{Q}^a)) = Spec(H(Q^a)),\ \ \widetilde{I}(a) = I(a), \widetilde{I}_1(a, b) = I_1(a, \beta), \widetilde{I}_2(a, b) = I_2(a, b) \tag{4.7.31}$$

We need to prove that

$$\widetilde{q}(x) \in \{q(sx + \tau) : \tau \in F, s = \pm 1\}.$$

For this, it is enough to show that there exist $\tau \in F$, $s \in \{-1, 1\}$ such that

$$\widetilde{q}(sx - \tau) = q(x).$$

The draft scheme of the proof is the followings. In Theorem 4.5.2 we proved that if $q(x)$ has the form (4.1.16), then its Fourier coefficients $z(a)$ for $a \in Q(1, 1, 1)$ can be defined uniquely, modulo (4.1.8), from the invariants (4.2.25)–(4.2.28), (4.2.34) and (4.2.35). Here we prove that if the band functions of the operators $L(q)$ and $L(\widetilde{q})$ coincide, then $\widetilde{q}$ has the form (4.1.16) and the operator $L(\widetilde{q})$ has the spectral invariants obtained from the formulas (4.2.25)–(4.2.28), (4.2.34), (4.2.35) respectively by replacing everywhere $z(a)$ with $\widetilde{z}(a)$. Then, using the arguments of the proof of Theorem 4.5.2 and fixing the inversion and translations (4.1.8), we prove that $\widetilde{z}(a) = z(a)$ for $a \in Q(1, 1, 1)$.

Since $q^a(x) = 0$ for $a \in S\backslash Q(1, 1, 1)$, the equality (4.1.4) and the second equality of (4.7.31) imply that $\widetilde{q}$ has the form (4.7.15). Now, to show that $\widetilde{q}(x)$ has the form (4.1.16), we prove that

$$\widetilde{z}(na) = 0,\ \ \forall |n| > 1, a \in Q(1, 1, 1). \tag{4.7.32}$$

By (4.5.7) we have $Q^{\gamma_k}(s) = a_k \cos s$, where $a_k > 0$ and $k = 1, 2, 3$. Therefore, by the first equality of (4.7.31), $\widetilde{Q}^{\gamma_k} \in U$. On the other hand, we have $\widetilde{Q}^{\gamma_k} \in (C^1(\mathbb{R})\backslash U) \cup T$ (see (4.7.30). Thus $\widetilde{Q}^{\gamma_k} \in T$. Then, it follows from Theorem 4.7.1 that (4.7.32) holds for $a \in \{\gamma_1, \gamma_2, \gamma_3\}$. Hence the all conditions of Lemma 4.7.1 hold and we have the formulas (4.7.18), (4.7.19) and (4.7.20). Besides, it follows from the second equality of (4.7.31) that $|\widetilde{z}(\gamma_i)| = |z(\gamma_i)|$. By Theorem 4.5.1 there exists $\tau \in F$ such that

$$\arg(\widetilde{q}(x-\tau), e^{-i\langle\gamma_k,x\rangle}) = 0,\ \forall k = 1, 2, 3.$$

Without loss of generality, we denote $\widetilde{q}(x-\tau)$ by $\widetilde{q}$ and its Fourier coefficients by $\widetilde{z}(a)$. Thus

$$\widetilde{z}(\gamma_i) = z(\gamma_i) = a_i > 0,\ \forall i = 1, 2, 3. \tag{4.7.33}$$

Therefore (4.2.25), (4.2.26), (4.7.18), (4.7.19) and (4.7.31) imply that

$$Re(\widetilde{z}(\gamma_i \pm \gamma_j)) = Re(z(\gamma_i \pm \gamma_j)). \tag{4.7.34}$$

From this using the obvious equalities (see (4.1.4) and the second equality of (4.7.31))

$$\sum_{n=1}^{\infty} 2|\widetilde{z}(n(\gamma_i \pm \gamma_j))|^2 = \widetilde{I}(\gamma_i \pm \gamma_j) = I(\gamma_i \pm \gamma_j) = 2|z(\gamma_i \pm \gamma_j)|^2, \tag{4.7.35}$$

we obtain

$$|Im(\widetilde{z}(\gamma_i \pm \gamma_j))| \le |Im(z(\gamma_i \pm \gamma_j))|. \tag{4.7.36}$$

On the other hand, using (4.2.34), (4.7.20), (4.7.33) and (4.7.31), we get

$$Re(\widetilde{z}(\gamma_i + \gamma_j)\widetilde{z}(\gamma_i - \gamma_j)) = Re(z(\gamma_i + \gamma_j)z(\gamma_i - \gamma_j)).$$

This with (4.7.34) and (4.7.36) implies that

$$|Im(\widetilde{z}(\gamma_i \pm \gamma_j))| = |Im(z(\gamma_i \pm \gamma_j))|. \tag{4.7.37}$$

Thus by (4.7.34) and (4.7.37), we have

$$|\widetilde{z}(\gamma_i \pm \gamma_j)| = |z(\gamma_i \pm \gamma_j)|. \tag{4.7.38}$$

Therefore, from (4.7.35) we see that (4.7.32) holds for $a = \gamma_i \pm \gamma_j$. Hence we have

$$\widetilde{z}(n(\gamma_i \pm \gamma_j) = 0,\ \widetilde{z}(n\gamma_m) = 0,\ \forall n \in \mathbb{Z}\backslash\{-1, 1\}, \tag{4.7.39}$$

where i, j, m are different integers satisfying $1 \le i, j, m \le 3$. Now instead of (4.7.17) using (4.7.39), that is, instead γ_i and γ_j in (4.7.17) taking $\gamma_i \pm \gamma_j$ and γ_m respectively, and repeating the proof of Lemma 4.7.1, we obtain that

$$\widetilde{I}_1(\gamma, \gamma_i) = A_1(\gamma, \gamma_i)Re(\widetilde{z}(-\gamma)\widetilde{z}(\gamma - \gamma_i)\widetilde{z}(\gamma_i)), \tag{4.7.40}$$

$$\widetilde{I}_1(2\gamma_i - \gamma, \gamma_i) = A_1(2\gamma_i - \gamma, \gamma_i)Re(\widetilde{z}(\gamma - 2\gamma_i)\widetilde{z}(\gamma_i - \gamma)\widetilde{z}(\gamma_i)), \tag{4.7.41}$$

$$\widetilde{I}_2(\gamma_i, \gamma - \gamma_i) = A_2(\gamma_i, \gamma - \gamma_i) Re(\widetilde{z}(-\gamma_i))^2 \widetilde{z}(\gamma) \widetilde{z}(2\gamma_i - \gamma)) \tag{4.7.42}$$

for $i = 1, 2, 3$; $i \neq j$, where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$.

One can readily see that the formulas (4.7.18), (4.7.19), (4.7.40), (4.7.41), (4.7.20), (4.7.42) are obtained from the formulas (4.2.25)–(4.2.28), (4.2.34), (4.2.35) respectively by replacing everywhere $z(a)$ with $\widetilde{z}(a)$. Moreover, by (4.7.33), (4.7.34) and (4.7.37), we have

$$\widetilde{a_i} = a_i, \ \forall i = 1, 2, \ldots, 6; \ \widetilde{b_i} = \pm b_i, \ \forall i = 4, 5, 6, \tag{4.7.43}$$

where $\widetilde{a_i} + i\widetilde{b_i} = \widetilde{z}(\gamma_i)$. As in Step 1 in the proof of Theorem 4.5.2, using (4.7.40) for $i = 1, 2, 3$ and taking into account (4.7.43) we obtain the equations

$$a_4\widetilde{a}_7 + \widetilde{t}_4 |b_4| \widetilde{b}_7 = s_{10} s_1^{-1} \tag{4.7.44}$$

$$a_5\widetilde{a}_7 + \widetilde{t}_5 |b_5| \widetilde{b}_7 = s_{11} s_2^{-1}, \tag{4.7.45}$$

$$a_6\widetilde{a}_7 + \widetilde{t}_6 |b_6| \widetilde{b}_7 = s_{12} s_3^{-1}, \tag{4.7.46}$$

where $\widetilde{t}_m$ is the sign of $\widetilde{b}_m$, i.e., is either -1 or 1 and $s_1, s_2, \ldots,$ are the invariants defined in (4.5.11). It follows from (4.5.9) that the main determinants of the systems of equations, with respect to the unknowns $\widetilde{a}_7$, $\widetilde{b}_7$, generated by pairs $\{(4.7.44), (4.7.45)\}$, $\{(4.7.44), (4.7.46)\}$, $\{(4.7.45), (4.7.46)\}$ are not zero. Finding $\widetilde{b}_7$ from (4.7.44), (4.7.45) and taking into account (4.5.19), we see that $\widetilde{b}_7 \neq 0$. Therefore, for fixing the inversion $\widetilde{q}(x) \longrightarrow \widetilde{q}(-x)$, we assume that $\widetilde{b}_7 > 0$. Using this and finding $\widetilde{b}_7$ from the systems generated by pairs $\{(4.7.44), (4.7.45)\}$, $\{(4.7.44), (4.7.46)\}$, $\{(4.7.45), (4.7.46)\}$, we get the inequalities

$$\frac{a_4 s_{11} s_2^{-1} - a_5 s_{10} s_1^{-1}}{\widetilde{t}_5 |b_5| a_4 - \widetilde{t}_4 |b_4| a_5} > 0,$$
$$\frac{a_4 s_{12} s_3^{-1} - a_6 s_{10} s_1^{-1}}{\widetilde{t}_6 |b_6| a_4 - \widetilde{t}_4 |b_4| a_6} > 0,$$
$$\frac{a_5 s_{12} s_3^{-1} - a_6 s_{11} s_2^{-1}}{\widetilde{t}_6 |b_6| a_5 - \widetilde{t}_5 |b_5| a_6} > 0. \tag{4.7.47}$$

One can readily see that the relations (4.7.44)–(4.7.47) with respect to the unknowns $\widetilde{a}_7$, $\widetilde{b}_7$, $\widetilde{t}_4$, $\widetilde{t}_5$, $\widetilde{t}_6$ are obtained from (4.5.16)–(4.5.19) by replacing the unknowns a_7, b_7, t_4, t_5, t_6 with $\widetilde{a}_7$, $\widetilde{b}_7$, $\widetilde{t}_4$, $\widetilde{t}_5$, $\widetilde{t}_6$. Since we proved that (4.5.16)–(4.5.19) has a unique solution, we have:

$a_7 = \widetilde{a}_7$, $b_7 = \widetilde{b}_7$, $t_4 = \widetilde{t}_4$, $t_5 = \widetilde{t}_5$, $t_6 = \widetilde{t}_6$. This with (4.7.43) implies that

$$\widetilde{a_i} = a_i, \ \widetilde{b_i} = b_i, \ \forall i = 1, 2, \ldots, 7. \tag{4.7.48}$$

In Step 2 and Step 3 of Theorem 4.5.2 using the invariants (4.2.28), (4.2.34), (4.2.35) we have determined the all other Fourier coefficients of q provided that a_i and b_i for $i = 1, 2, \ldots, 7$ are known. Since the invariants (4.7.41), (4.7.20), (4.7.42) are obtained from the invariants (4.2.28), (4.2.34), (4.2.35) by replacing everywhere a_i and b_i with $\widetilde{a_i}$ and $\widetilde{b_i}$ respectively, and (4.7.48) holds, we have

$$\widetilde{z}(a) = z(a), \ \forall a \in Q(1,1,1). \tag{4.7.49}$$

This with the equalities (4.1.4), (4.2.20), (4.7.31) and (4.7.16) imply that (4.7.32) holds for all $a \in Q(1,1,1)$. Therefore, it follows from (4.7.15), (4.7.32) and (4.7.49) that □

$$\widetilde{q}(x) = q(x)$$

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Chapter 5
Conclusions

Abstract In this chapter we summarize the results and methods of the book in the descriptive way. In this book the following **three problems** of the spectral theory of the multidimensional Schrödinger operator $L(q)$ with a periodic potential q are investigated.

1. First problem is the perturbation theory of $L(q)$ (Chap. 2).

2. Second problem is the constructive determination of the spectral invariants of $L(q)$ from the given Bloch eigenvalues (Chap. 3).

3. Third problem is the constructive determination of the potential q from the spectral invariants (Chap. 4).

We describe all these three problems that can be considered as unique features. Moreover these problems are investigated as a whole and in the pertinent form in the sense that **First problem** $\Rightarrow$ **Second problem** $\Rightarrow$ **Third problem**. Besides in Chap. 1 we present some definitions, statements and discussions to be used in the next chapters from the point of view of both the physicists and mathematicians.

Conversation and Notations

First let us discuss the results of Chap. 2. Recall that the Bloch eigenvalues and Bloch functions are the eigenvalues and eigenfunctions of the operator

$$L_t(q) = -\Delta + q \tag{5.1}$$

in the primitive unit cell F of the period lattice Ω of the potential q, with t-periodic boundary conditions, where t is a quasimomentum in the primitive unit cell F^* of the reciprocal lattice Γ. Because the eigenvalue problem is set in a fixed finite volume F, the spectrum of $L_t(q)$ consists of the eigenvalues $\Lambda_1(t)$, $\Lambda_2(t)$, ..., such that $\Lambda_n(t) \to \infty$ as $n \to \infty$. Each of the energy levels $\Lambda_n(t)$ varies continuously as t varies. In this way we arrive at a description of the levels of an electron in a periodic potential in terms of a family of continuous functions $\Lambda_n(t)$. For each n, the set of electronic levels specified by $\Lambda_n(t)$ is called an energy band. Thus the eigenvalues $\Lambda_n(t)$ are labeled with the band index n. The Bloch function is also denoted by $\Psi_{n,t}(x)$ which indicates that each value of the band index n and the vector t specify an electron state, or orbital with the energy $\Lambda_n(t)$. The information

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contained in these functions for different n and t is referred to as the band structure of the solid. Since for the general reciprocal lattice in $\mathbb{R}^d$ the determination of the nth Brillouin zone is complicated, it is hard to find $\Lambda_n(t)$ for large n in the case $q = 0$. However, in this case the Bloch eigenvalues is expressed by $\gamma \in \Gamma$ in a simple way as $|\gamma + t|^2$. In Chap. 2, (for the first time in [Ve2, Ve3, Ve4]) to observe the moving of the eigenvalues $|\gamma + t|^2$ of the free operator $L_t(0)$ for

$$\gamma + t \in \{x \in \mathbb{R}^d : \rho^2 \le |x|^2 < \rho^2 + 1\} =: W(\rho), \tag{5.2}$$

where $\rho \gg 1$, under the periodic perturbation q, instead of the traditional labeling by band index n, we label the Bloch eigenvalues and Bloch functions of the perturbed operator $L_t(q)$ by $\gamma + t$ and denote them as $\Lambda(\gamma + t)$ and $\Psi_{\gamma+t}(x)$. Then we find the values of the quasimomenta $\gamma + t$ in the non-resonance domain for which the corresponding Bloch eigenvalues $\Lambda(\gamma+t)$ of $L_t(q)$ are simple and close to $|\gamma+t|^2$, that is,

$$\Lambda(\gamma + t) = |\gamma + t|^2 + O(|\gamma + t|^{-\alpha}) \tag{5.3}$$

for some positive α and the corresponding Bloch wave $\Psi_{\gamma+t}(x)$ is close to the plane wave $e^{i\langle\gamma+t,x\rangle}$, that is,

$$\Psi_{\gamma+t}(x) = e^{i\langle\gamma+t,x\rangle} + O(|\gamma + t|^{-\alpha}). \tag{5.4}$$

As we stressed in Sect. 1.4 of Chap. 1 [see (1.4.2)] in the multidimensional case the Bloch eigenvalues $|\gamma+t|^2$, where $\gamma \in \Gamma$, of the free operator $L_t(0)$ for fixed $t \in F^*$ are densely situated in a high energy region. Moreover, there are in average $D(\rho)$ diffraction hyperplanes D_δ passing through the washer $W(\rho)$, where $D(\rho) \sim \rho^d$, and all these planes may reflect the wave $e^{i\langle\gamma+t,x\rangle}$ if the corresponding eigenvalue $|\gamma+t|^2$ lies in the interval $[\rho^2, \rho^2+1)$. Therefore in order to get the formula (5.4) we have to construct the set of quasimomenta for which the plane wave $e^{i\langle\gamma+t,x\rangle}$ under the perturbation q goes through the crystal without the essential influence of all these diffraction hyperplanes. That is why, the regular perturbation theory is ineffective and the mathematical difficulties have a physical nature—a complicated picture of the diffraction inside the crystal.

Recall that (see Sect. 1.4 of Chap. 1) in the **First case** (one-dimensional Schrödinger operator) and the **Second case** (small eigenvalues for the multidimensional Schrödinger operator with the small potential εq) the regular perturbation theory is effective due to the fact that the potential is smaller than the distance between the eigenvalues of the unperturbed operator. In the big opposite to the **First and Second cases,** in the case of the large eigenvalues of the multidimensional $L(q)$, the potential q is greater than the distance between the eigenvalues [see (1.4.3) in Sect. 1.4 of Chap. 1]. In this case the regular perturbation theory is ineffective even if the potential q is replaced by εq, where ε is a small parameter, due to the following reason. The distance between the large eigenvalues lying in $[\rho^2, \rho^2 + 1)$ is, in average $O(\rho^{2-d})$, and we can not assume that $\varepsilon < \rho^{2-d}$, since ρ^{2-d} tends to zero as $\rho \to \infty$ if $d > 2$. Thus for the multidimensional case and for the large values of the energy, the perturbation theory becomes extremely difficult. Therefore in Chap. 2 we develop a new mathematical approach to construct a perturbation theory for the multidimensional Schrödinger operator $L(q)$ for the large values of energy. This approach for the first time is given in [Ve1, Ve2, Ve3, Ve4]. Before discussing in detail the steps of this approach let us consider the **Second case** in detail by the approach of Chap. 2, since it helps to explain these steps. Namely, let us have a look to the following well-known model which demonstrates the influence of the potential εq to the eigenvalue $|\gamma + t|^2$ of order 1 of the free electron and to the plane wave $e^{i\langle \gamma+t,x\rangle}$.

Small Potential Model

In this model we discuss the well-known formulas for the Bloch eigenvalue $\Lambda_n(t)$ and the corresponding Bloch function $\Psi_{n,t}$ of the operator $L_t(\varepsilon q)$, where ε is a small parameter, $n = O(1)$ and $q \in L_2(F)$. Consider the following two cases:

Case1. **Isolated eigenvalue** (see Sect. 1.4 of Chap. 1). The crystal momentum $\gamma + t$ is far from the diffraction planes D_δ for $\delta \in \Gamma$. It means that the distance a of the eigenvalue $|\gamma + t|^2$ from the nearest eigenvalue of $L_t(0)$ is of order 1, that is,

$$\min_{\delta \in \Gamma \backslash \{0\}} \left| |\gamma + t|^2 - |\gamma + t + \delta|^2 \right| = a \sim 1. \tag{5.5}$$

First we show that there exists a Bloch eigenvalue $\Lambda_n(t)$ of the operator $L_t(\varepsilon q)$ satisfying the inequality

$$\left| \Lambda_n(t) - |\gamma + t|^2 \right| \leq \varepsilon \left\| q \right\|. \tag{5.6}$$

Suppose to the contrary that (5.6) does not hold for all $n \in \mathbb{N}$. Then using the formula

$$(\Lambda_n(t) - |\gamma + t|^2)(\Psi_{n,t}, e^{i\langle \gamma+t,x\rangle}) = \varepsilon(q\Psi_{n,t}, e^{i\langle \gamma+t,x\rangle}), \tag{5.7}$$

obtained from the formula (2.1.8) of Chap. 2 by replacing q with εq, and Parseval's equality for the orthonormal basis $\left\{\Psi_{n,t} : n \in \mathbb{N}\right\}$ we obtain the following contradiction.

$$1 = \sum_{n \in \mathbb{N}} \left| (\Psi_{n,t}, e^{i\langle \gamma+t,x\rangle}) \right|^2 = \sum_{n \in \mathbb{N}} \frac{\varepsilon^2 \left| (\Psi_{n,t}, q e^{i\langle \gamma+t,x\rangle}) \right|^2}{|\Lambda_n(t) - |\gamma + t||^2} < 1.$$

It follows from (5.5) and (5.6) that

$$\left| \Lambda_n(t) - |\gamma + t + \delta|^2 \right| \geq a - \varepsilon \left\| q \right\| \tag{5.8}$$

for all $\delta \in \Gamma \backslash \{0\}$. Therefore using Bessel's inequality for the orthonormal system

$$\left\{e^{i\langle\gamma+t+\delta,x\rangle} : \delta \in \Gamma \backslash \{0\}\right\}$$

and the formula

$$(\Lambda_n(t) - |\gamma + t + \delta|^2)(\Psi_{n,t}, e^{i\langle\gamma+t+\delta,x\rangle}) = \varepsilon(q\Psi_{n,t}, e^{i\langle\gamma+t+\delta,x\rangle}), \tag{5.9}$$

we obtain

$$\sum_{\delta\in\Gamma\backslash\{0\}} \left|(\Psi_{n,t}, e^{i\langle\gamma+t+\delta,x\rangle})\right|^2 = \sum_{\delta\in\Gamma\backslash\{0\}} \frac{\varepsilon^2 \left|(q\Psi_{n,t}, e^{i\langle\gamma+t+\delta,x\rangle})\right|^2}{\left|\Lambda_n(t) - |\gamma + t + \delta|^2\right|^2} \le \frac{\varepsilon^2 \|q\|^2}{|a - \varepsilon \|q\||^2} = O(\varepsilon^2).$$

This with the Parseval's equality gives

$$\Psi_{n,t} = e^{i\langle\gamma+t,x\rangle} + O(\varepsilon) \tag{5.10}$$

which means that the plane wave $e^{i\langle\gamma+t,x\rangle}$ is almost not reflected by the crystal. Using this in (5.7) and then taking into account that

$$\int_F q(x)dx = 0,$$

we get

$$\Lambda_n(t) = |\gamma + t|^2 + O(\varepsilon^2). \tag{5.11}$$

Case 2. **Isolated pair of eigenvalues** (see Sect. 1.4 of Chap. 1). The crystal momentum $\gamma + t$ is close to the diffraction plane D_δ and far from the other diffraction planes. In other words

$$\left||\gamma + t|^2 - |\gamma + t + \delta|^2\right| \ll 1 \tag{5.12}$$

and

$$\left||\gamma + t|^2 - |\gamma + t + \delta^{'}|^2\right| \ge b \sim 1 \tag{5.13}$$

for $\delta^{'} \neq 0, \delta$, that is, $\gamma + t$ is close only to the diffraction plane D_δ. Replacing $\Gamma \backslash \{0\}$ by $\Gamma \backslash \{0, \delta\}$ and repeating the proof of (5.10) we obtain

$$\Psi_{n,t}(x) = b(n, \gamma)e^{i\langle\gamma+t,x\rangle} + b(n, \gamma + \delta)e^{i\langle\gamma+t+\delta,x\rangle} + g(x), \tag{5.14}$$

where

$$|b(n, \gamma)|^2 + |b(n, \gamma + \delta)|^2 = O(\varepsilon^2), \quad \|g\| = O(\varepsilon),$$

and g is a function orthogonal to $e^{i\langle\gamma+t,x\rangle}$ and $e^{i\langle\gamma+t+\delta,x\rangle}$. Thus by (5.14) if $\gamma+t$ is close to D_δ then under the small perturbation εq the plane waves $e^{i\langle\gamma+t,x\rangle}$ and $e^{i\langle\gamma+t+\delta,x\rangle}$ interface each other, that is, the Bragg reflection can occur. Using (5.14) in (5.7) and (5.9) we obtain

$$(\Lambda_n(t)-|\gamma+t|^2)b(n,\gamma)=\varepsilon b(n,\gamma+\delta)q_{-\delta}+O(\varepsilon^2) \tag{5.15}$$

and

$$(\Lambda_n(t)-|\gamma+t+\delta|^2)b(n,\gamma+\delta)=\varepsilon b(n,\gamma)q_{\delta}+O(\varepsilon^2) \tag{5.16}$$

from which we estimate $\Lambda_n(t)$ and then $\Psi_{n,t}$ in a standard way.

Thus if (5.12) and (5.13) hold, then the quasimomentum $\gamma+t$ must be close only to the Bragg plane D_δ determined by δ and we get the Eqs. (5.15) and (5.16) to estimate the corresponding Bloch eigenvalues, which mean that a weak periodic potential εq has its major effect on those free electron levels whose wave vectors are close to ones at which the Bragg reflection can occur. In order words, if $\gamma+t$ lies near the Bragg plane D_δ, then in order to find the energy levels and the wave functions we have the equations including only the two levels: one corresponds to $\gamma+t$ and the other one corresponds to $\gamma+t+\delta$.

Discussion of the Approach of Chap. 2

Here we discuss the problems of the construction of the perturbation theory for the Bloch eigenvalues and Bloch functions (Bloch waves) corresponding to the large values of the energy. First let us explain why the **Small potential model** and similar classical perturbation theory are ineffective for the proof of the formulas (5.3) and (5.4). If $|\gamma+t|\sim\rho\to\infty$, then the numbers a and b defined in (5.5) and (5.13) depend on ρ and as we stressed in Sect. 1.4 of Chap. 1, in general,

$$\lim_{\rho\to\infty}a(\rho)=0,\quad \lim_{\rho\to\infty}b(\rho)=0.$$

To apply the argument of the above model we need to assume that $\varepsilon\ll a(\rho)$ and $\varepsilon\ll b(\rho)$. On the other hand, (5.3) and (5.4) have a meaning only if $|\gamma+t|\sim\rho\to\infty$. Therefore for any nonzero ε the classical perturbation theory is ineffective for the proofs of (5.3) and (5.4). Moreover instead of one inequality (5.12) we obtain $k(\rho)$ inequalities

$$\left||\gamma+t|^2-|\gamma+t+\delta_i|^2\right|\ll 1 \tag{5.17}$$

for $i=1,2,...,k(\rho)$, where, in general $k(\rho)\to\infty$ as $\rho\to\infty$ and hence instead of two equalities (5.15) and (5.16) we need to consider $k(\rho)+1$ equalities. This situation also shows the complexity of the perturbation theory in the high energy region in the multidimensional case.

Note that the concept: "the crystal momentum $\gamma + t$ is far from the diffraction planes D_δ or close only to one diffraction plane" was used very much in the physical literature and as we have seen in the small potential model that it easifies the perturbation theory for the quasimomentum $\gamma+t$ of order 1. However, this concept breaks down for the quasimomentum $\gamma + t$ with $|\gamma + t| \sim \rho \gg 1$ due to the following. To construct the set of quasimomenta $\gamma + t \in W(\rho)$ lying on the distance greater than ε from the diffraction plane D_δ, we need to discard from $W(\rho)$ the ε neighborhood of D_δ. As we noted in Sect. 1.4 of Chap. 1 there are $D(\rho)$ diffraction planes D_δ intersecting $W(\rho)$ for large ρ, where $D(\rho) \sim \rho^d$. Therefore we must take care that the remaining set does not become empty after all these discarding. For this we should choose very small $\varepsilon(\rho)$ depending on ρ and such that $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. However, then the formulas (5.3) and (5.4) have no any sense for the potential εq for any ε, since $\gamma + t \in W(\rho)$ and $\rho \to \infty$.

Thus in the big contrary of the **Small potential model** and similar examples in physics the concepts and arguments mentioned above do not help us seriously for the investigation of the perturbation theory in the high energy region. That is why, in Chap. 2, we give a new approach for this problem.

Now let us describe the steps of the perturbation theory given in Chap. 2. The rigorous proofs of all steps are given in Chap. 2. Here we give only the brief comments regarding the steps of the construction of the perturbation theory. The first step is the classification of the Bloch eigenvalues of the free operator.

Classifications of the Eigenvalues

To avoid the technical details let us discuss this step for the multidimensional Schrödinger operator $L(P)$ with a trigonometric polynomial potential

$$P(x) = \sum_{\delta \in A} q_\delta e^{i\langle \delta, x\rangle}, \tag{5.18}$$

where A is a finite subset of the reciprocal lattice Γ. We consider the eigenvalue $|\gamma + t|^2$ as a vector $\gamma + t$ of the washer $W(\rho)$ defined in (5.2) for large ρ. Using (5.18) in

$$(\Lambda_N - |\gamma + t|^2)(\Psi_{N,t}, e^{i\langle \gamma+t, x\rangle}) = (\Psi_{N,t} P, e^{i\langle \gamma+t, x\rangle}), \tag{5.19}$$

we obtain

$$(\Lambda_N(t) - |\gamma + t|^2) b(N, \gamma) = \sum_{\delta \in A} q_\delta b(N, \gamma - \delta) \tag{5.20}$$

where

$$b(N, \gamma) = (\Psi_{N,t}, e^{i\langle \gamma+t, x\rangle}).$$

Let us consider the right-hand side of (5.20). By (5.19) we have

$$b(N, \gamma - \delta) = \frac{(\Psi_{N,t} P, e^{i(\gamma-\delta+t,x)})}{\Lambda_N(t) - |\gamma - \delta + t|^2}. \tag{5.21}$$

If Λ_N is close to $|\gamma + t|^2$ and $\gamma + t$ does not belong to any of the sets

$$V_\delta(\rho^\alpha) =: \{x \in \mathbb{R}^d : ||x|^2 - |x + \delta|^2| \leq \rho^\alpha\} \tag{5.22}$$

for $\delta \in A$, that is, $\gamma + t$ is far from the diffraction planes D_δ then

$$||\gamma + t|^2 - |\gamma - \delta + t|^2| > \rho^\alpha, \ |\Lambda_N(t) - |\gamma - \delta + t|^2| > \frac{1}{2}\rho^\alpha. \tag{5.23}$$

Therefore, it follows from (5.21) and (5.23) that

$$b(N, \gamma - \delta) =: (\Psi_{N,t}, e^{i(\gamma-\delta+t,x)}) = O(\rho^{-\alpha_1}) \tag{5.24}$$

for all $\delta \in A$ and hence (5.20) has the form.

$$(\Lambda_N(t) - |\gamma + t|^2)b(N, \gamma) = O(\rho^{-\alpha}). \tag{5.25}$$

From (5.25), by the technical investigation, we obtain that if

$$\gamma + t \in U =: W(\rho)\backslash V \tag{5.26}$$

where

$$V = \left(\bigcup_{\delta \in A} V_\delta(\rho^\alpha) \right) \cap W(\rho) \tag{5.27}$$

then there exists an eigenvalue $\Lambda(\gamma + t)$ satisfying (5.3). Thus if $\gamma + t \in U$ then the corresponding eigenvalue $\Lambda(\gamma + t)$ of the perturbed operator $L_t(P)$ is close to the eigenvalue $|\gamma + t|^2$ of the free operator $L_t(P)$. If $\gamma + t \in V$ then, in general, the corresponding eigenvalue of the perturbed operator is not close to the eigenvalue of the free operator $L_t(0)$, and the eigenvalue $|\gamma + t|^2$ under the perturbation P may move of order 1. Therefore, in the papers [Ve1, Ve2, Ve3, Ve4], for the first time the eigenvalues $|\gamma + t|^2$, for large $\gamma \in \Gamma$, were divided into two groups: the non-resonance ones if $\gamma + t \in U$ and the resonance ones if $\gamma + t \in V$ and various asymptotic formulae were obtained for the perturbations of each groups. The sets U and V are named non-resonance and resonance domains respectively. Then Karpeshina [Ka1, Ka2, Ka3] and Feldman-Knorrer-Trubowitz [FeKnTr1, FeKnTr2] entitled the non-resonance (resonance) eigenvalues as nonsingular (singular) and stable (unstable) eigenvalues respectively.

The formula (5.24) shows that the influence of the plane waves $e^{i\langle\gamma-\delta+t,x\rangle}$ for $\delta \in A$ to the wave $e^{i\langle\gamma+t,x\rangle}$ is very small. However it is far from to prove (5.4), since for this we have to prove that the total influence of the plane waves $e^{i\langle\gamma+\delta+t,x\rangle}$ for all $\delta \in \Gamma\backslash\{0\}$ to the wave $e^{i\langle\gamma+t,x\rangle}$ is small. For this and to solve all the **three problems (a), (b) and (c) described in Sect. 1.4 of Chap. 1 (Simplicity, Bragg diffraction and Isoenergetic surface problems)** we constructed the simple set B. In other words, the construction and investigation of the simple set B solve simultaneously the problems of the perturbation theory (simplicity of the Bloch eigenvalues, asymptotic formulas for the Bloch eigenvalues and Bloch Functions) and isoenergetic surfaces. Therefore let us discuss the geometric construction and estimation of the simple set B as the main step of the perturbation theory.

Geometric Constructions

This construction was done in Chap. 2. Here we give only some description. To prove the simplicity of $\Lambda_N(t) =: \Lambda(\gamma+t)$ and (5.4), we construct a set B of quasimonenta such that if $\gamma + t \in B$, then the total influence of the plane waves $e^{i\langle\gamma+\delta+t,x\rangle}$ for all $\delta \in \Gamma\backslash\{0\}$ to the wave $e^{i\langle\gamma+t,x\rangle}$ is small. Since

$$\left\{e^{i\langle\gamma+t,x\rangle} : \gamma \in \Gamma\right\}$$

is an orthonormal basis we have

$$\Psi_{\gamma+t}(x) = b(N,\gamma)e^{i\langle\gamma+t,x\rangle} + \sum_{\delta\in\Gamma\backslash\{0\}} b(N,\gamma+\delta)e^{i\langle\gamma+\delta+t,x\rangle} \tag{5.28}$$

Therefore if

$$\sum_{\delta\in\Gamma\backslash\{0\}} |b(N,\gamma+\delta)|^2 = O(\rho^{-2\alpha}) \tag{5.29}$$

then the total influence mentioned above is $O(\rho^{-\alpha})$. To prove (5.24), that is, to show that the influence of the plane waves $e^{i\langle\gamma-\delta+t,x\rangle}$ for $\delta \in A$ to the wave $e^{i\langle\gamma+t,x\rangle}$ is small, we discard from $W(\rho)$ the neighbourhood $V_\delta(\rho^\alpha)$ of the diffraction planes D_δ. The set $V_\delta \cap W(\rho)$ is the part of the washer $W(\rho)$ which is contained between the two parallel hyperplanes

$$\{x : |x|^2 - |x+\delta|^2 = -\rho^\alpha\} \,\&\, \{x : |x|^2 - |x+\delta|^2 = \rho^\alpha\}.$$

This is the small part of $W(\rho)$. Since A contains finite number of elements δ after eliminating the sets V_δ for $\delta \in A$ the remaining part $\widetilde{W}(\rho)$ of $W(\rho)$ is the essential part of the washer. However we can not do this operation (these eliminations) for all $\delta \in \Gamma$, since then the remaining part may be becomes empty set. Therefore the construction of the simple set for multidimensional $L(q)$ in high energy region becomes very complicated. In [Ve3] for the construction of the simple set B of quasimomenta in case $d = 3$ we eliminated the very small vicinities of the diffraction planes, and the sets connected with the directional potential, and the intersection of the two resonance sets $V_{\delta_1}(\rho^\alpha)$ and $V_{\delta_2}(\rho^\alpha)$.

As the dimension d increases, the geometrical structure of B becomes more complicated for the following reason. Since the denseness of the eigenvalues of the free operator increases as d increases we need to use the asymptotic formulas of high accuracy and investigate the intersections of the higher order of the resonance sets. Then the functions $F(\gamma + t)$, $\lambda_j(\gamma + t)$ [see (2.1.28), (2.1.29) of Chap. 2] taking part in the construction of B becomes more complicated. Therefore surfaces and sets defined by these functions become more intricate. Instead of the vicinities of the diffraction planes we use the vicinities of some surfaces. Thus for the dimensions $d > 3$ these surfaces play the role of the diffraction planes.

Moreover the simple set B constructed in the non-resonance domain contains the main part

$$\{\gamma + t : \ \Lambda(\gamma + t) = \rho^2\}$$

of the isoenergetic surfaces $I_\rho(q)$ of $L(q)$ corresponding to ρ^2 for large ρ. We prove that this part of $I_\rho(q)$ consist of the union of smooth surfaces and the total measure of these surfaces asymptotically equals to the measure of the sphere

$$\{x \in \mathbb{R}^d : |x| = \rho\}.$$

For this we find the derivatives of the Bloch eigenvalues $\Lambda(\gamma + t)$. These derivatives and asymptotic formulas have the following applications.

Some Applications

In the above notations the diagonal and non-diagonal elements of the current matrix can be written as

$$S(\gamma+t,\gamma+t) = -\frac{ie\,\hbar}{2m}\int_F (\Psi^*_{\gamma+t}(x)\ grad\ \Psi_{\gamma+t}(x) - \Psi_{\gamma+t}(x)\ grad\ \Psi^*_{\gamma+t}(x))dx$$

and

$$S(\gamma+t,\widetilde{\gamma}+t) = -\frac{ie\,\hbar}{2m}\int_F (\Psi^*_{\widetilde{\gamma}+t}(x)\ grad\ \Psi_{\gamma+t}(x) - \Psi_{\gamma+t}(x)\ grad\ \Psi^*_{\widetilde{\gamma}+t}(x))dx$$

respectively, where ħ is Planck's constant, m and e are the mass and charge of the electron. Therefore using the formulas

$$\Psi_{\gamma+t}(x) = e^{i\langle\gamma+t,x\rangle} + \sum_{\gamma_1\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1}e^{i\langle\gamma+t+\gamma_1,x\rangle}}{|\gamma+t|^2 - |\gamma+\gamma_1+t|^2} + O(|\gamma|^{-2\alpha_1}), \quad (5.30)$$

and

$$grad\ \Psi_{\gamma+t}(x) = i(\gamma+t)e^{i\langle\gamma+t,x\rangle} + O(|\gamma|^{1-2\alpha_1})$$

for $(\gamma+t) \in B$ obtained in Chap. 2 we get the following asymptotic formulas for the diagonal and non-diagonal elements of the current matrix

$$S(\gamma+t,\gamma+t) = \frac{e\,\hbar}{m}(\gamma+t) + O(|\gamma|^{1-2\alpha_1}),$$

$$S(\gamma+t,\widetilde{\gamma}+t) = \left(\frac{eh}{m}\right)\frac{(\widetilde{\gamma}-\gamma)q_{\gamma-\widetilde{\gamma}}}{|\gamma+t|^2-|\widetilde{\gamma}+t|^2} + O(|\gamma|^{1-2\alpha_1}),$$

where $(\gamma+t) \in B$ and $(\widetilde{\gamma}+t) \in B$.

It readily follows from the formula

$$\frac{\partial}{\partial t_j}\Lambda(\gamma+t) = \frac{\partial}{\partial t_j}|\gamma+t|^2 + O(\rho^{1-2\alpha_1}),\ \forall j = 1,2,...,d.$$

of Chap. 2 [see (2.5.7) of Chap. 2] that

$$grad\ \Lambda(\gamma+t) = (\gamma+t) + O(\rho^{1-2\alpha_1})$$

from which we obtain the asymptotic formulas for the velocity and impulse of the electron.

Summarizing the results of Chap. 2 we note that the chapter gives the complete perturbation theory of the periodic Schrödinger operator of arbitrary dimension. Note that the method of this book and hence of the papers [Ve2, Ve3, Ve4, Ve5, Ve9] is unique which gives asymptotic formulas for Bloch eigenvalues and Bloch functions for arbitrary dimension. Moreover, in case of the resonance domain we constructed the simple set so that it can be easily used for the constructive determination (in Chap. 3) a family of the spectral invariants by the given Bloch eigenvalues. Thus Chap. 2 is also a base for the constructive determinations of the spectral invariants.

On the Spectral Invariants and Inverse Problems

First, recall that a functional f in the space of the periodic, with respect to the lattice Ω, functions is said to be spectral invariants if it has the following property: if the Bloch eigenvalues of the Schrödinger operators $L(q)$ and $L(p)$ with the potentials q and p coincide, then $f(q) = f(p)$. Here the spectral invariants play the intermediate role between the Bloch eigenvalues and potentials. Since the influence of the potential q is essential in the resonance domain, one can get a lot of informations about potential q from the Bloch eigenvalues corresponding to the quasimomenta lying in the high energy region and near to the diffraction hyperplanes. Therefore in Chap. 3 first we improve the asymptotic formulas for the Bloch eigenvalues and Bloch functions corresponding to those quasimomenta and get the formulas, where there are sharp estimations for the first and second terms of the asymptotic decomposition. Then using this and the behavior of the derivatives of the band functions we constructively determine the spectral invariants. Some of these invariants are explicitly expressed by the Fourier coefficients of the potential which present the possibility of determining the potential constructively by using the Bloch eigenvalues as input data due to the following arguments. If the potential q is a trigonometric polynomial, then the spectral invariants give us nonlinear equations with respect to the Fourier coefficients of q and the number of the independent equations is greater than the number of the Fourier coefficients. Moreover, most of these equations are explicitly expressed by either 1 or 2 or 3 Fourier coefficients of q. This situation allows us to give an algorithm for finding the potential q from these spectral invariants. Besides solving these nonlinear equations by the numerical methods one can determine the potential q in the set of the trigonometric polynomial. Then taking some limit process one can find the smooth potentials from the given spectral invariants and hence from the given Bloch eigenvalues. Hence in Chap. 3, we constructively determined a family of spectral invariants of $L(q)$ from the given Bloch eigenvalues that is enough to determine the potential q. Since this book is theoretical, it seems that the theoretical part of the inverse problem by spectral invariants is complete, in the sense that the book gives the full theoretical base and possibility to solve numerically this problem. Thus Chap. 3 describes the constructive determination of the spectral invariants explicitly expressed with respect to the Fourier coefficients of the potential by using the Bloch eigenvalues as input data. At the same time, it gives a rich set of invariants that is enough to determine the potential q.

Chapter 4 gives some examples and algorithms for finding the potential from the spectral invariants and hence from the Bloch eigenvalues. We consider the inverse problems of the three-dimensional Schrödinger operator with a periodic potential q by the spectral invariants obtained in the third chapter. Note that the inverse problems of the one-dimensional Schrödinger operator, the Hill operator, and the multidimensional Schrödinger operator $L(q)$ are absolutely different. In order to determine the potential q, of the Hill operator, in addition to the given band functions $\Lambda_1(t)$, $\Lambda_2(t)$, ..., one needs to know the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the Dirichlet boundary value problem and some other informations. In other words, the potential q of the Hill operator can not be determined uniquely from the given band functions, since if the band functions $\Lambda_1(t)$, $\Lambda_2(t)$, ... of $H(q)$ are given, then for every choice of the numbers $\lambda_1, \lambda_2, \ldots$ from the gaps $\Delta_1, \Delta_2, \ldots$ of the spectrum of the Hill operator there exists a potential q having $\Lambda_1(t)$, $\Lambda_2(t)$, ... as the band functions and $\lambda_1, \lambda_2, \ldots$ as the Dirichlet eigenvalues. In spite of this, it is possible to determine uniquely the potential q of the multidimensional Schrödinger operator $L(q)$ from only the given band functions. Because, in the case $d > 1$ the band functions give more informations. Namely, the band functions give the spectral invariants that have no meaning in the case $d = 1$. We solve the inverse problem by these spectral invariants.

In Chap. 4, firstly, we construct a set D of the trigonometric polynomials which is dense in the Sobolev spaces and every element can be determined constructively and uniquely from the invariants obtained in Chap. 3. More precisely, fixing the inversion

$x \to -x$ and translations: $x \to x+\tau$ for $\tau \in \mathbb{R}^3$, we give an algorithm for the unique determination of the potential $q \in D$ of the three-dimensional Schrödinger operator $L(q)$ from the given spectral invariants that were determined constructively from the given Bloch eigenvalues. Note that the potential q can be uniquely determined only by fixing the above inversion and translations, since $L(q(x))$, $L(q(-x))$ and $L(q(x+\tau))$ have the same band functions and hence the same invariants. Then a special class V of the periodic potentials is constructed, which can be easily and constructively determined from the spectral invariants and hence from the given Bloch eigenvalues. Besides, we consider the stability of the algorithm for the unique determination of the potential $q \in V$ of the three-dimensional Schrödinger operator with respect to the spectral invariants and Bloch eigenvalues.

Thus Chap. 4 give some ideas and algorithms for finding the potential from the spectral invariants and hence from the Bloch eigenvalues which may open up new horizons for the inverse problems of the important operators of the mathematical physics. Since this book gives a constructive description of the direct (perturbation theory-asymptotic formulas for Bloch eigenvalues and Bloch functions) and inverse problems (constructive determinations of the periodic potential from the given Bloch eigenvalues) of $L(q)$, it seems that it will be used as an introduction to the topic as well as the theoretical base for solving the inverse problems. Moreover the approach used in this book may be used for the spectral analysis of the important operators of the quantum mechanics and solid state physics.

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