

A. Terry Bahill

The Science of Baseball

Modeling Bat-Ball Collisions
and the Flight of the Ball

 Springer

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*Dedicated to my always smiling,
always laughing Karen*

Foreword

I first “met” Dr. A. Terry Bahill in 2005 while researching aerodynamic characteristics of batted baseballs as part of a personal project that would become the ESPN Home Run Tracker. I did not speak to him at the time (that would come later), but rather downloaded and read many of the papers which he had posted on his website. Dr. Bahill’s explanations and calculations were a great help to me at a time when my career in baseball analytics was just beginning, but as we have corresponded over the years, my admiration for his work, particularly his gift for communicating ideas, has only increased. His latest publication, *The Science of Baseball: Modeling Bat-Ball Collisions and the Flight of the Ball*, is a worthy contribution to his prodigious body of baseball research, compiled over four decades and presented with extraordinary clarity. It will serve as a valuable reference for scholarly fans, as well as baseball analysts who aspire to compete at the highest level.

Major League Baseball (MLB) clubs are, as of early 2017, in the midst of a revolution. The ranks of analysts employed by Major League Baseball clubs have swelled in recent years, as teams try to realize competitive advantages through the creative use of the data that is being generated and presented to teams at an unprecedented rate. Every MLB front office now employs people who scrutinize not only traditional statistics such as batting averages and home run totals but also metrics like pitch speed or batted-ball exit speed. The most analytically enthusiastic clubs study ball- and player-tracking data collected at rates as high as 100 data points per second, and disseminated by commercial vendors such as Baseball Info Solutions, Sportvision, Trackman, MLB Advanced Media and others. MLB’s demand for new forms of baseball analysis has inspired a large and rapidly growing pool of independent analysts who conduct research via publicly available sources, hoping to earn the opportunity to offer their services as consultants to or employees of Major League front offices. More people and companies are doing more baseball-related analytical work than ever before.

Throughout my dozen years of baseball-related work, both as an individual and in my current role as an analyst with the Boston Red Sox, I have found that the best research originated with people who possessed not only thorough baseball knowledge but also a solid understanding and a proper deference to the other governing principles of the situation under study. For contract and compensation issues, these principles are those of economics; for discretionary tactical moves such as stolen base or bunt attempts, or for pitch type selection, these principles are those of game theory; for issues related to the movement of the baseball, these principles are those of physics.

Unfortunately, too often these days we see analytical work that neglects, or even runs counter to, the underlying principles, because the analyst's mastery of the relevant principles is faulty or incomplete. For some, analysis of baseball data consists of arranging it in columns and performing statistical tests on it until something "pops." I was once offered a detailed analysis that rated elite closer Koji Uehara as the 16th best pitcher on the Red Sox roster, and further opined that his devastating splitter was among the weaker individual pitches on the entire team. After I stopped laughing, I asked a few questions and learned that these dubious results could be traced to a faulty premise about the value of pitch locations. It was, essentially, a lack of understanding of one of the most important elements of pitching analysis: how to judge the results of a pitch.

More knowledgeable analysts who are familiar with the applicable principles can better detect and avoid bad data, more efficiently set up and perform the most promising statistical tests, and can more reliably interpret the results. Dr. Bahill's expert dissection of the bat-ball collision (Chaps. 1, 2, 3, 4 and 5) and the flight of pitched and batted baseballs through the air (Chap. 7) should be read by all who wish to enhance their expertise at analysis of ball-tracking data by first understanding why the baseball moves the way it does. Complete derivations have been provided for those who wish to delve deeply into the equations, but they need not present a persistent barrier to those readers who prefer to skim the line-by-line mathematics and skip ahead to the conclusions. A prime example is the sensitivity analysis presented in Chap. 7, which describes the change in batted-ball range which follows a given change in various inputs such as batted-ball speed, batted-ball spin or air density.

Baseball analysts past, present and future are indebted to Dr. Bahill for the efforts he has made to make understanding of the complex underlying physics of baseball accessible to all at each person's chosen level of detail. His precise yet eminently accessible explanations of the physics of the bat-ball collision and the flight of the ball are more useful than ever in an era when MLBAM's Statcast system tells 30 and 100 times per second what has happened, but leaves to the observer the task of figuring out why it happened (which is, of course, the key to

predicting what will happen in the future, the ultimate objective of all analysts). If you wish not only to understand the game of baseball better but to contribute to the body of knowledge of the game of baseball, read this book carefully, and then read it again. For the moment, knowledge of baseball physics can still differentiate an analyst from his or her peers, but in the field of baseball analytics, no competitive advantage persists for long.

Baseball Operations Analyst,
Boston Red Sox,
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January 23, 1984

Prof. A. Terry Bahill
Electrical & Computer Engineering
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Schenley Park
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Dear Mr. Bahill:

Received your letter and have also had a chance to read your research, and I fully agree with your findings.

I always said I couldn't see a ball hit the bat except on very, very rare occasions and that was a slow pitch that I swung on at shoulder height. I am very close to seeing the ball hit the bat on those occasions.

As to participating in your other experiments; at this time, I can't tell you that I can comply with your request.

Regarding the current theories of some of the present batting coaches (with which I absolutely disagree) to watch the ball go into the catcher's mitt - by doing that, you don't give yourself a chance to swing and open up properly. Try it yourself - look down at the plate and try to make a full swing. I hope you don't throw your back out of joint!

In any event, good luck with your projects.

Sincerely,

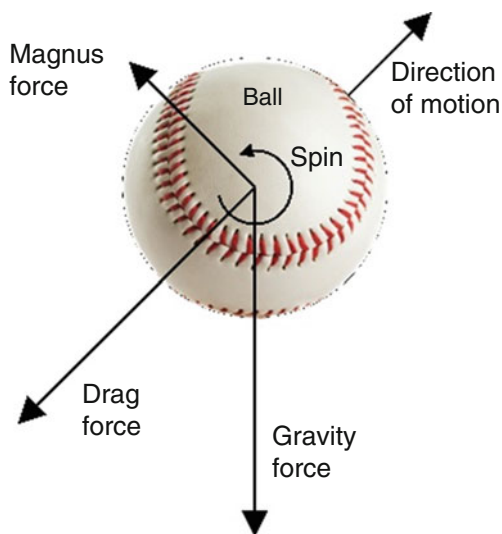


Ted Williams

TW/shg

Preface

Collisions between baseballs, softballs and bats are complex and therefore their models are complex. The first purpose of this book is to show how complex these collisions can be, while still being modeled using only Newton's principles and the conservation laws of physics. This book presents models for the speed and spin of balls and bats. These models and equations for bat-ball collisions are intended for use by high school and college physics students, engineering students, the baseball analytics community and most importantly students of the science of baseball. Unlike models in previous books and papers, these models use only simple Newtonian principles and the conservation laws to explain simple bat-ball collision configurations. It is hoped that this book will help readers develop an understanding of the modeling of bat-ball collisions. The second purpose of this book is to help batters select or create baseball or softball bats that would be optimal for them. The third purpose is to show what affects air density and how air density affects the flight of the ball.



Chapter 1 lays the groundwork for analyzing bat-ball collisions.

Chapter 2 introduces nine basic configurations of bat-ball collisions using words and figures.

Chapter 3 starts developing the equations for these configurations. It starts with the simple configurations having the ball collide with the center of mass of the bat. Then it moves on to configurations that are more complex using the same equations and development. The notation developed here will be used throughout the book.

Chapter 4 is the pinnacle of this book. It contains our most comprehensive model. It models a collision at the sweet spot of the bat with spin on the pitch. It has five equations and five unknowns. It develops equations for the bat and ball linear and angular velocities after the collision in terms of those same four parameters before the collision. This chapter contains a sensitivity analysis of the model that shows which parameters are the most important. It also has advice for selecting the optimal bat. Such a bat does not have its barrel end cupped out. This chapter is unique in the science of baseball literature. It is also self-contained. You need not read previous chapters in order to understand it. In other words, a teacher could use this chapter in a physics or engineering course and the students would only have to buy this one chapter.

This is a big deal. The BaConLaw model also describes the motion of the bat after the collision. Many models describe the motion of the ball after the collision, but few (if any) describe the motion of the bat. When you see a batter hit a ball, do you see the recoil of the bat? Can you describe it? Well these equations do.

Chapter 5 contains four alternative models for bat-ball collisions. Their purposes are different and are based on different fundamental principles. The Effective Mass model was created by physicists independent of the author of this book. Therefore, comparisons to it are important for validating the model of Chap. 4. The second and

third models are data-based, not theory-based. They use a different approach and use a different *type* of data. The fourth model considers friction during the collision. It is shown that this type of collision cannot be modeled thoroughly using only the conservation laws. Our modeling technique could not handle the Collision with Friction model because our model is only good for a point in time before the collision and a point after the collision: it cannot handle behavior during the collision. Chapter 4 fulfilled part of the first purpose of this book. It showed a complex configuration for which our technique did work. Chapter 5 completed the fulfillment of this purpose by showing a configuration for which our technique was too simple.

Nothing in Chaps. 1, 2, 3, 4 and 5 is controversial. There are no unstated assumptions. Important equations have been derived with at least two techniques. In Chaps. 2, 3, 4 and 5, the equation numbers are the same. In other words, Eq. (2.3) is the same as Eq. (3.3) is the same as Eq. (4.3) and is the same as Eq. (5.3). The equations in Chaps. 2, 3, 4 and 5 were derived using only Newton's principles and the conservation laws of physics. The equations in Chap. 7 for the drag and Magnus forces are original and are based on far more than Newton's principles.

Chapter 6 summarizes and discusses Chaps. 1, 2, 3, 4 and 5. Chapters 1, 2, 3, 4, 5 and 6 deal with bat-ball collisions. They solve equations in closed form. There are no approximations. Chapter 7 deals with messy real systems. It uses experimental data and only gives approximations.

Chapter 7 contains derivations for equations governing the flight of the ball. It shows what affects air density and how air density affects the flight of the ball. It shows that a home run ball might go 26 feet farther in San Francisco than in Denver. It also answers the question, "Which can be thrown farther a baseball or a tennis ball?" This chapter can be read independently from the rest of the book.

We need people who can explain this book to baseball managers and general managers.



Acknowledgements

I am deeply indebted to Al Nathan for preventing me from publishing a book with mistakes in it. I thank Bob Watts, Rod Cross, Bruce Gissing and Jim Close for helpful comments on the manuscript. This book is written in the first person plural. Plural because my graduate students did all of the work. Major contributions were made by Tom La Ritz, Bill Karnavas, Miguel Morna Freitas and J. Venkateswaran. Extra special thanks go to Dave Baldwin who inspired and coauthored a dozen baseball research papers with me.

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Biographical Sketch



A. Terry Bahill is an Emeritus Professor of Systems Engineering and of Biomedical Engineering at the University of Arizona in Tucson. He served as a Lieutenant in the United States Navy. He received his Ph.D. in electrical engineering and computer science from the University of California, Berkeley. He is the author of eight engineering books and over 250 papers, over 100 of them in peer-reviewed scientific journals. Bahill has worked with dozens of high-tech companies presenting seminars on Systems Engineering, working on system development teams and helping them to describe their Systems Engineering processes. He holds a US patent for the Bat Chooser™, a system that computes the Ideal Bat Weight™ for individual baseball and softball batters. He was elected to the Omega Alpha Association,

the systems engineering honor society. He received the Sandia National Laboratories Gold President's Quality Award. He is a Fellow of the Institute of Electrical and Electronics Engineers (IEEE), of Raytheon Missile Systems, of the International Council on Systems Engineering (INCOSE) and of the American Association for the Advancement of Science (AAAS). He is the Founding Chair Emeritus of the INCOSE Fellows Committee. His picture is in the Baseball Hall of Fame's exhibition "Baseball as America." You can view this picture at <http://sysengr.engr.arizona.edu/>.

Chapter 1

Types of Bat-Ball Collisions

1.1 Introduction

Purpose: This book has three primary purposes: first, to create models for bat-ball collisions using only fundamental principles of Newtonian mechanics, second, to help a batter select or create an optimal baseball or softball bat and third, to show what affects air density and how air density affects the flight of the ball.

1.2 Newton's Principles

Even though Newton formulated his principles over 300 years ago, his principles still provide the best explanations for collisions between baseballs and baseball bats. Although they are presented as equations in this book, math phobic readers can just skip the equations and read the words without loss of continuity. Newton's principles of motion can be written as follows.

I. Inertia or uniform motion. Every object either remains at rest or continues to move at a constant velocity, unless acted upon by an external force.

$$\sum F = 0 \quad \Rightarrow \quad dv/dt = 0$$

Note that force, velocity, acceleration, impulse and momentum are all vector quantities, although we do not specifically mark them as such (Table 1.1). Therefore, in the text (but not in the underlying models) we will treat ball speed and ball velocity as the synonymous terms.

Table 1.1 List of variables and parameters and their abbreviations

Symbol: This table is arranged alphabetically by the symbol	Abbreviation ball = 1 bat = 2 before = b after = a	Description, if specific, then for configuration 2b unless otherwise noted	Typical values for a C243 pro stock wooden bat and a professional major-league baseball player	
			SI ^b units	Baseball units
$\beta_{\text{bat - knob}}$	β	Angular velocity of a bat about the knob in configuration 2c	rad/s	rpm
CoE		Conservation of energy	Joules	
CoM		Conservation of momentum	kg · m/s	
$CoAM$		Conservation of angular momentum	kg · m ² /s	
CoR		Coefficient of restitution of a high-speed bat-ball collision	0.466	0.466
d_{bat}		Length of a bat	0.863 m	34 inch
$d_{\text{bat - cm - cop}}$	$d_{\text{cm - cop}}$	Distance from the center of mass (cm) to the sweet spot, which we define as the Center of Percussion (cop)	0.119 m	4.7 in
$d_{\text{bat - knob - cm}}$	$d_{\text{k - cm}}$	Distance from the center of the knob to the center of mass	0.568 m	22.4 in
$d_{\text{bat - knob - cop}}$	$d_{\text{k - cop}}$	Distance from the center of the knob to the center of percussion	0.687 m	27.0 in
$d_{\text{spine - cm}}$		Distance from the batter's spine to the center of mass of a bat, an experimentally measured value	1.05 m	41 in
$d_{\text{bat - cm - end}}$		Distance from the center of mass to the barrel end of a bat	0.281 m	11.1 in
$d_{\text{bat - cop - end}}$		Distance from the center of percussion to the barrel end of a bat	0.162 m	6.4 in
g		earth's gravitational constant (at the University of Arizona)	9.718 m/s ²	
I_{ball}	I_1	Moment of inertia of a baseball with respect to its center of mass	0.000079 kg m ²	4.3 oz in ²
$I_{\text{bat - cm}}$	$I_2 = I_{\text{cm}}$	Moment of inertia of a bat with respect to rotations about its center of mass	0.0511 kg m ²	2792 oz in ²
$I_{\text{bat - knob}}$	I_{k}	Moment of inertia of a bat with respect to rotations about the knob	0.335 kg m ²	18,315 oz in ²

(continued)

Table 1.1 (continued)

Symbol: This table is arranged alphabetically by the symbol	Abbreviation ball = 1 bat = 2 before = b after = a	Description, if specific, then for configuration 2b unless otherwise noted	Typical values for a C243 pro stock wooden bat and a professional major-league baseball player	
			SI ^b units	Baseball units
KE_{before}		Kinetic energy of a bat and a ball before the collision	370 J	
KE_{after}		Kinetic energy of a bat and a ball after the collision	175 J	
KE_{lost}		Kinetic energy lost or transformed in the collision	195 J	
m_{ball}	m_1	Mass of a baseball	0.145 kg	5.125 oz
m_{bat}	m_2	Mass of a baseball bat	0.88 kg	31 oz
$\bar{m}$		$\bar{m} = \frac{m_{\text{ball}}m_{\text{bat}}}{m_{\text{ball}} + m_{\text{bat}}}$	0.124 kg	4.4 oz
M_{eff}		Mass of a portion of the bat in the effective mass model	0.707 kg	25 oz
μ_f		Dynamic coefficient of friction for a baseball sliding on a wooden bat	0.5	
r_{ball}	r_1	Radius of a baseball	0.037 m	1.45 in
r_{bat}	r_2	Radius of a baseball bat	0.031 m	1.3 in
<i>Pitch speed</i>		Speed of a ball at the pitcher’s release point	−41 m/s	−92 ^a mph
$v_{\text{ball - before}}$	v_{1b}	Velocity of a ball immediately before the collision, 90% of pitch speed	−37 m/s	−83 ^a mph
$v_{\text{ball - after}}$	v_{1a}	Velocity of a ball after the collision, often called the launch velocity or the <i>bat-ted-ball speed</i> .	42 m/s	93 mph
v_{bat}	v_2	Velocity of a bat. If a specific place or time is intended then the subscript may contain cm (center of mass), ip (impact point), before (b) or after (a).		
$v_{\text{bat - cm - before}}$	v_{2cmb}	Velocity of the center of mass of a bat before the bat-ball collision.	23 m/s	51 mph
$v_{\text{bat - cm - after}}$	v_{2cma}	Velocity of the center of mass of a bat after the collision.	11 m/s	25 mph
$v_{t_{\text{bat - ip - before}}}$	v_{2ipb}	Total velocity of the impact point of a bat before the collision.	27 m/s	60 ^a mph

(continued)

Table 1.1 (continued)

Symbol: This table is arranged alphabetically by the symbol	Abbreviation ball = 1 bat = 2 before = b after = a	Description, if specific, then for configuration 2b unless otherwise noted	Typical values for a C243 pro stock wooden bat and a professional major-league baseball player	
			SI ^b units	Baseball units
$v_{\text{bat}} - \text{ip} - \text{after}$	$v_{2\text{ipa}}$	Total velocity of the impact point of a bat after the collision.	11 m/s	25 mph
$\omega_{\text{ball}} - \text{before}$	$\omega_{1\text{b}}$	Angular velocity of a ball about its center of mass before the collision. This spin rate depends on the particular type of pitch.	± 209 rad/s	± 2000 rpm
$\omega_{\text{ball}} - \text{after}$	$\omega_{1\text{a}}$	Angular velocity of a ball about its center of mass after the collision	± 209 rad/s	± 2000 rpm
$\omega_{\text{bat}} - \text{before}$	$\omega_{2\text{b}}$	Angular velocity of a bat about its center of mass before the collision	32 rad/s	309 rpm
$\omega_{\text{bat}} - \text{after}$	$\omega_{2\text{a}}$	Angular velocity of a bat about its center of mass after the collision	6 rad/s	56 rpm
$\omega_{\text{spine}} - \text{before}$		Angular velocity of the batter's arms about the spine	21 rad/s	201 rpm

^aThe equations of this book concern parameters right before and right after the collision, not at other times. For example, a pitcher could release a fastball with a speed of 92 mph, by the time it got to the collision zone it would have slowed down by 10% to 83 mph. Therefore, in our simulations we used 83 mph for $v_{\text{ball}} - \text{before}$

^bSI stands for the International System of Units

II. Impulse and Momentum.

Applying a force changes the momentum of a body. The *rate* of change of momentum is directly proportional to the force applied and is in the direction of the applied force.

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} \Rightarrow F = ma$$

Stated differently, the *amount* of change in momentum of a body is proportional to the impulse applied to the body and is in the direction of the impulse. An impulse J occurs when a force F acts over an interval of time Δt , and it is given by $J = \int_{\Delta t} F dt$. Since force is the time derivative of momentum, p , it follows that $J = \Delta p = m\Delta v$. Finally, for rotational systems, applying an impulsive torque changes the angular momentum about the torque axis.

III. Action/reaction. For every action there is an equal and opposite reaction.

IV. Restitution. The coefficient of restitution (*CoR*) is defined as the ratio of the relative speeds of two objects after and before a collision. This holds whether one object or the other is initially at rest or the objects are approaching each other. The *CoR* models the energy lost in a collision.

$$CoR = \frac{\text{relative speed after the collision}}{\text{relative speed before the collision}}$$

In this book, we will use these four principles of Newton. But more importantly, we will also use the overarching conservation laws that state: energy, linear momentum and angular momentum cannot be created or destroyed. These laws are more general than the principles and apply in all circumstances. Because our model is based on these Conservation Laws of physics applied to Baseball, we call it the BaConLaw model.

1.2.1 Variables and Parameters

The terms parameter and variable are often used interchangeably. Nevertheless, in this book we will try to distinguish between the terms. Our variables have equations that give them values. Our variables contain parameters that will produce different sets of equations. In this book, we will treat the following as *variables*: the inputs $v_{\text{ball-before}}$, $\omega_{\text{ball-before}}$, $v_{\text{bat-cm-before}}$, $\omega_{\text{bat-before}}$ and *CoR*, the outputs $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, $\omega_{\text{bat-after}}$ and KE_{lost} , the forces on the ball, launch velocity, launch angle and launch spin rate. The following are *parameters* of our equations: the dimensions, mass and moment of inertia of the bat and ball, the air density, the drag coefficient, the Magnus coefficient, the Reynolds number, and collision speed. For each invocation of an equation, they will have a fixed value. Sometimes we will refer to variables and parameters together as properties of the model. The following are *constants* that always have the same values: π and the earth's gravitational constant at the University of Arizona, g .

1.3 Characterizing Bat-Ball Collisions

A collision can be *elastic* or *inelastic*. In an elastic collision (such as a steel ball, or a superball, bouncing off a large steel plate), there is practically no energy lost or transformed. Whereas, in an inelastic collision (such as a bat-ball collision) energy is transformed. Most authors call this the energy *lost*, but it is not lost: it is merely transformed into a different form, such as heat in the ball, vibrations in the bat, acoustic energy in the 'crack of the bat,' friction and permanent deformations of the bat and ball. This book considers only inelastic collisions where kinetic energy is lost.

1.3.1 Collision Taxonomy

There are many kinds of collisions between two rigid bodies. One kind, where the duration of the collision is short and the area of the collision is small, is called an *impact*. Bat-ball impacts are described with the following three characteristics: dimension, location and direction. The following definitions, involving these characteristics, hold before and after the collision.

1.3.1.1 Dimension

If the equations of motion can be described in a two-dimensional (2D) plane, then the impact is *planar*. For example, the game of billiards is, for the most part, planar. Otherwise, if the equations of motion require description in three-dimensional (3D) space, then the impact is *nonplanar*.

1.3.1.2 Line of Impact

For an impact between two objects, there is a common tangent plane that is perpendicular to the radius of curvature of each object at the point of contact. The vector that is perpendicular to this plane at this point is called the *line of impact*.

1.3.1.3 Location

An impact is *central* if the centers of mass of both bodies are on the line of impact, otherwise the impact is *eccentric*.

1.3.1.4 Direction

An impact is *direct* if the directions of motion of the both bodies are on the line of impact; it is *parallel* if the direction of the center of mass one body is on the line of impact and the other is on a parallel line, otherwise the impact is *oblique*.

These terms are useful because they predict the complexity of the equations of motion. Planar-central-direct impacts are the simplest because all motions are along the same axis and there are no impulsive torques. Nonplanar-eccentric-oblique impacts have the most complicated equations. These terms also help a person to determine the type of analysis that will be necessary to study a certain collision configuration. If you are going to simulate a collision, then your first decisions involve these terms, Table 1.2.

Table 1.2 Top-level decisions for simulating bat-ball collisions

Characteristic	Allowable set of values {legal values}
Dimension of analysis	{planar, nonplanar}
Location of collision	{central, eccentric}
Direction of motion	{direct, parallel, oblique}
Spin on the pitch	{yes, no}
Point of contact	{center of mass, sweet spot}

1.4 Models for a Batter Swinging a Bat

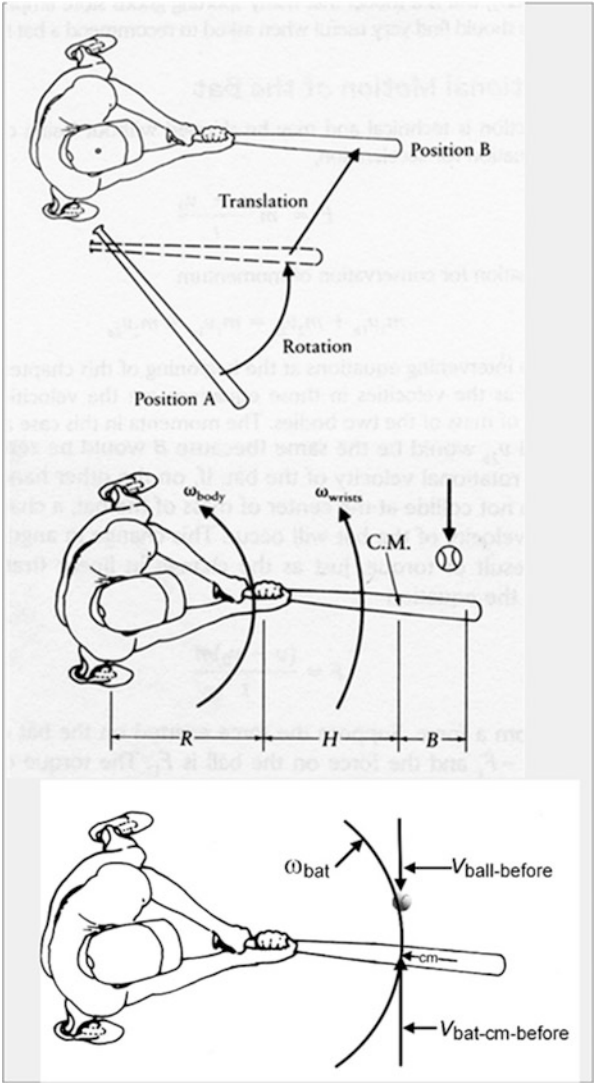
In the top panel of Fig. 1.1, the batter swings the bat with translational and rotational motions. In the middle panel, the rotation has two components, one about the batters spine ω_{body} and another about the pivot point between the hands, ω_{wrists} . In the bottom panel, the movement of the bat before the collision is modeled as the sum of $v_{\text{bat - cm - before}}$ and a line tangent to the $\omega_{\text{bat arc}}$, which is centered at a pivot point between the hands. This straight-line velocity was measured in our experiments and it is called the bat speed before the collision. It is a combination of translation and rotation. These alternative models emphasize different aspects of the swing of the bat. We will primarily use the bottom model.

Modeling philosophy note. Having several alternative models helps ensure that you understand the physical system. No model is more correct than another. They just emphasize different aspects of the physical system.

1.5 Summary

This chapter presented Newton’s laws of motion, our table of abbreviations and nomenclature for describing collisions. It also gave three simple models for a person swinging a bat.

Fig. 1.1 Three different models for the swing of the bat



Chapter 2

Configurations of Bat-Ball Collisions

2.1 Introduction

Purpose: The purpose of this chapter is to present many possible configurations of bat-ball collisions. Then to explain the configurations that we can model and those that we cannot.

This chapter presents several configurations of bat-ball collisions. For each of these configurations, we model the state of the bat and the ball at a point in time right before the collision and at another point just after the collision. We are not modeling the behavior (1) during the collision, (2) long before the collision (the pitched ball) or (3) long after the collision (the batted-ball). The flight of the pitch and the batted-ball are modeled in Chap. 7.

2.2 Collisions at the Center of Mass

2.2.1 Configuration 1a

Configuration 1a is a head-on collision at the center of mass of the bat, as shown in Fig. 2.1. Spin on the ball and bat are not considered. This simple type of analysis was done by Bahill and Karnavas (1989). It uses Conservation of Linear Momentum (CoM) and the Coefficient of Restitution (CoR).

Configurations 1a and 1b are planar, central, direct impacts.

The impact is planar because the governing equations are in the x-y plane. This collision can be drawn on a flat piece of paper.

The impact is central because the line of impact passes through the centers of mass of both the ball and the bat.

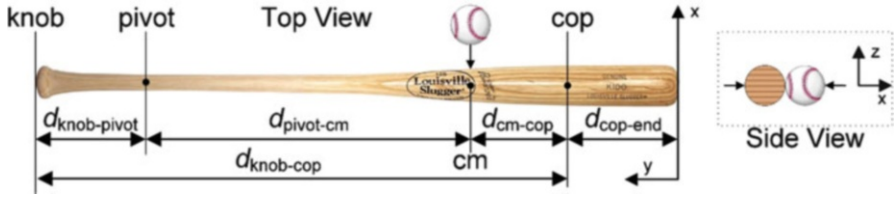


Fig. 2.1 Configurations 1a and 1b are head-on collisions at the center of mass (cm) of the bat. All figures in this book are for right-handed batters

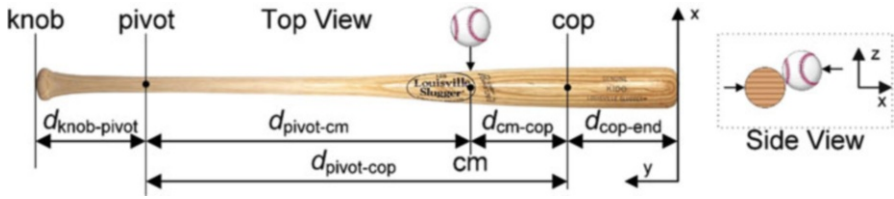


Fig. 2.2 Configuration 1c is a collision at the center of mass (cm) of the bat, but vertically it is above the long axis of the bat

The impact is direct because the centers of mass of both the bat and the ball are moving along the line of impact. This means that the initial tangential (y-axis and z-axis) velocity components are zero.

In this model, the bat does not rotate.

This type of collision would produce a line drive back to the pitcher.

2.2.2 Configuration 1b

Configuration 1b is the same as configuration 1a, except that it adds KE_{lost} and Conservation of Energy as checks on the derivations. Planar, central, direct collisions (like configurations 1a and 1b) are called *head-on* collisions.

2.2.3 Configuration 1c

Configuration 1c is a collision at the center of mass of the bat along the y-axis, but vertically it is above or below the long axis of the bat, as shown in Fig. 2.2. This is the same as configuration 1a, except that there is a vertical offset between the directions of motion of the bat and ball at the collision (the bat hits the bottom part of the ball) and the equations allow spin on the ball. Nathan et al. (2012 and Kensrud, Nathan and Smith 2016) have presented experimental data for the spin on the ball after such a collision.



Fig. 2.3 The sweet spot of the bat is centered about six inches away from the barrel end of the bat

Configuration 1c is a planar, central, oblique impact.

The impact is planar because the impact is in the x - z plane: the bat and ball will both have x - and z -axis motion after the impact, but no motion in the y direction.

The impact is central because the line of impact passes through the centers of mass of both the ball and the bat.

The impact is oblique because in the x - z plane the motion of the bat and ball are not parallel to the line of impact.

This type of collision would typically produce a flyball to center field, or maybe a pop-up. The equations for this type of impact will be considered in a future paper. Configuration 1c will not be mentioned again in this book.

2.3 Collisions at the Sweet Spot

The term *sweet spot* is a layman's term for a general area of the bat about two inches wide and one-third of an inch high centered about six inches away from the barrel end of the bat, as shown in Fig. 2.3. Section 3.3.1.1 gives nine possible definitions for the sweet spot of the bat. This is nebulous. Therefore, when we are writing about a general area of the bat, or when we are reporting on papers that used the term, we will use the term *sweet spot*. However, in our figures, we need to be more specific. Hence, we adopt the first definition in Sect. 3.3.1.1 of the sweet spot, namely the center of percussion (cop). In our simulations, we need to specify a particular point on the bat for the collision: therefore, we also use the center of percussion in our simulations. Finally, in our equations we do not restrict the collision to be at any particular point on the bat: Therefore, in equations, we state that the collision occurs at the impact point (ip).

Configurations 2 are head-on collisions at the sweet spot of the bat, which we define to be the center of percussion (Bahill 2004). This type of analysis was done by Watts and Bahill (1990). Compared to Configurations 1, they move the collision from the center of mass of the bat to the sweet spot of the bat.

2.3.1 Configurations 2a and 2b

Configuration 2a is a head-on collision at the sweet spot {center of percussion (cop)} of the bat. Compared to Configuration 1a, it adds an equation based on Newton's second principle and it adds rotation of the bat about its center of mass.

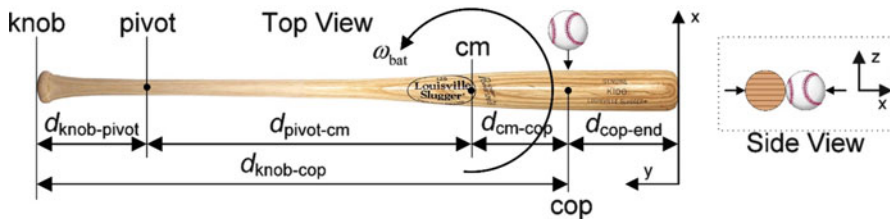


Fig. 2.4 Configurations 2 are collisions at the sweet spot {center of percussion (cop)} of the bat

Configuration 2a is a planar, eccentric, parallel impact.

The impact is planar because the equations are in the x-y plane. This collision can be drawn on a flat piece of paper.

The impact is eccentric because the line of impact does not pass through the center of mass of the bat in the x-y plane. It could be noted that the line of impact passes through the center of mass of the bat in the x-z plane. But that is irrelevant. Once the line of impact misses the center of mass in any plane, the impact is eccentric. The impact is parallel because the line of impact is parallel to the x-axis, the ball is moving along the x-axis and the bat's center of mass is moving parallel to the x-axis.

Configuration 2a would produce a line drive back to the pitcher.

Configuration 2b is a collision at the sweet spot of the bat. It is similar to configuration 2a, except that it adds Conservation of Energy as a consistency check, Conservation of Angular Momentum, spin on the ball and kinetic energy lost. Configuration 2b is the pinnacle of this book.

For configurations 2, planar, eccentric, parallel collisions are called *head-on*.

For collisions 2a and 2b, which are described with Fig. 2.4, there is no torque on the ball. Therefore, there will be no change in angular velocity of the ball. For these head-on collisions, the angular velocity of the ball before the collision is the same as the angular velocity of the ball after the collision.

2.3.2 Configuration 2c

Configuration 2c is a collision at the sweet spot of the bat with spin on the pitch. Conservation of Angular Momentum about the z-axis was successfully used. It replaces rotation about the center of mass with rotation about the knob of the bat, identified with β_{bat} .

2.3.3 Configuration 2d

Configuration 2d is a collision at the sweet spot of the bat with spin on the pitch and friction between the bat and ball, as will be shown later in Fig. 5.2. As an obvious example of what spin can do, consider a tennis player putting sidespin on a tennis ball, the ball certainly will move sideways when it hits the ground Cross (2011). Likewise, when a spinning baseball collides with a bat there will be a friction force that changes the rate of spin of the ball. This configuration uses conservation of momentum and Newton's second principle. It adds friction at the contact point and a momentum moment.

2.3.4 Configuration 3

Configuration 3 is a collision at the sweet spot of the bat, but above or below the long axis of the bat as shown in Fig. 2.5. This is the same as configuration 2b, except it adds offset to the bat-ball collision and bat twist. Nathan et al. (2012, Kensrud, Nathan and Smith 2016) gave experimental data for the spin of a baseball after collisions in this type of an impact and Sawicki, Hubbard and Stronge (2003) gave simulation results.

Configuration 3 is a nonplanar, eccentric, oblique impact.

If spin on the ball causes motion in the y -axis direction, then the impact is nonplanar because the bat and ball will both have x -, y - and z -axis motion after the impact. The impact is eccentric because the line of impact misses the center of mass of the bat in the x - y plane.

The impact is oblique because in the x - z plane the motion of the bat and ball are not parallel to the line of impact.

This type of collision would typically produce a flyball to center field. Configuration 3 will not be mentioned again in this book.

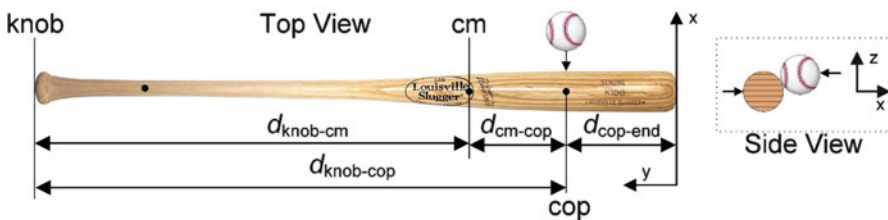


Fig. 2.5 Configuration 3 is a collision at the sweet spot (cop) of the bat, but above the horizontal axis of the bat

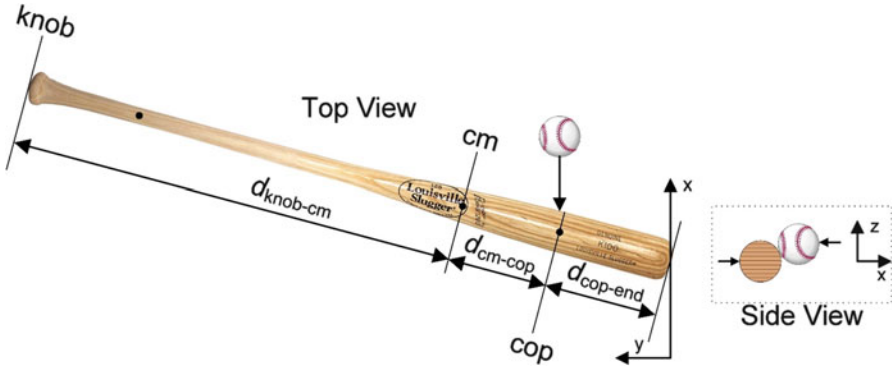


Fig. 2.6 Configuration 4 is an oblique collision at the sweet spot (cop) and above the horizontal axis of the bat

2.3.5 Configuration 4

Configuration 4 is an oblique collision at the sweet spot, but above or below the horizontal (long) axis of the bat as shown in Fig. 2.6. This is the same as configuration 3, except that it adds the bat being rotated short of (or beyond) a line parallel to the y-axis at the time of the collision.

Configuration 4 is a nonplanar, eccentric, oblique impact.

This impact is nonplanar because the bat and ball will both have x-, y- and z-axis motion after the impact.

The impact is eccentric because the line of impact misses the center of mass of the bat in the x-y plane.

The impact is oblique because the bat is not moving along the x-axis at the time of impact. This means that there will be tangential (y-axis) velocity components.

This type of collision would typically produce a flyball to right (or left) field. Configuration 4 will not be mentioned again in this book.

2.4 Summary

Abbreviations used in Table 2.1.

Abbreviation	Name
CoE	Conservation of energy
CoM	Conservation of momentum
CoAM	Conservation of angular momentum
CoR	Coefficient of restitution
Cm	Center of mass
KE_{lost}	The kinetic energy lost or transformed during the collision
μ_f	Coefficient of friction
e_m	Coefficient of moment restitution

Table 2.1 Comparison of the collision configurations

Characteristic	Configuration									
	1a	1b	1c	2a	2b	2c	2d	3	4	
Dimension	Planar	Planar	Planar	Planar	Planar	Planar	Planar	Nonplanar	Nonplanar	
Location	Central	Central	Central	Eccentric	Eccentric	Eccentric	Eccentric	Eccentric	Eccentric	
Direction	Direct	Direct	Oblique	Parallel	Parallel	Parallel	Parallel	Oblique	Oblique	
Spin on the pitch	No	No	Yes	No	Yes	Yes	Yes	Yes	Yes	
Rotation of bat	No	No	No	Yes	Yes	Yes	Yes	Yes	Yes	
Point of contact	cm	cm	cm	ip	ip	ip	ip	ip	ip	
Difference from previous configuration	Uses CoM and CoR , adds physiology	Adds CoE & KE_{lost}	Adds vertical offset at collision point	Moves collision to ip, adds Newton's second & ω_{bat}	Adds CoE, CoAM, ω_{ball} & KE_{lost}	Adds β_{bat} about knob	Adds μ_f friction	Adds vertical offset and twist (or roll) of the bat	Adds bat not parallel to y-axis	
Developed by	Bahill and Karnavas (1989)	Chapter 3		Watts and Bahill (1990)	Chapter 4	Chapter 5	Chapter 5			
Variables and parameters used to model losses	CoR	CoR , KE_{lost}	CoR , μ_f and e_m	CoR	CoR , KE_{lost}	CoR	CoR , and μ_f	CoR , μ_f and e_m	CoR , μ_f and e_m	

Table 2.2 Equations and constraints for some of the configurations

Characteristic	Configuration					
	1a	1b	2a	2b	2c	2d
<i>Location</i>						
Is the collision at the center of mass (cm) or the impact point (ip)?	cm	cm	ip	ip	ip	ip
<i>Equations</i>						
Conservation of linear momentum	y	y	y	y	y	
Coefficient of restitution	y	y	y	y	y	
Newton's second principle			y	y	y	y
Conservation of energy		y		y		
Kinetic energy lost		y		y		
Conservation of angular momentum				y	y	y
Number of equations used	2	3	3	5	4	2
Number of unknowns (outputs)	2	2	1	5	3	1
<i>Constraints</i>						
Principles of physiology	y	y	y	y	y	y
Is ω_{ball} used?				y	y	y
Is ω_{bat} used?			y	y		
Is β_{bat} used?					y	
Is friction used?						y

Table 2.1 shows the history of the development of the nine configurations mentioned in this book. It also shows how the details of the models differ. In configurations 1ab and 2abc spin is allowed, but it is not included in the equations, because in Chap. 4 we show that spin has no effect in head-on collisions.

Table 2.2 shows the equations and constraints that were used in each model. For example, configuration 2b for a collision at the sweet spot, used equations for Conservation of Linear Momentum, Coefficient of Restitution, Newton's second principle, Conservation of Energy, kinetic energy lost and Conservation of Angular Momentum. It used five equations and had five outputs (unknowns). It used principles of physiology, spin on the ball and rotation of the bat about its center of mass.

Modeling philosophy note. This chapter presented alternative models. They emphasize different aspects of the physical system. In this chapter, they got more and more complicated as they tried to cover larger and larger aspects of the real system. This chapter sets the structure for the rest of the book. In Chap. 3, we will follow this structure except that we will add equations. But once again, we will start with baby steps and then get more complicated.

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Chapter 3

Equations for Bat-Ball Collisions

3.1 Introduction

Purpose: The purpose of this chapter is to start presenting the incipient equations that we will use to model selected configurations of Chap. 2. We will allow the reader to progress slowly through the equations: take baby steps first.

Each of the next six sections in this chapter starts with a table that describes the inputs, outputs and equations that will be used in that section.

3.2 Collisions at the Center of Mass

For configurations 1a, 1b and 1c, the model for bat motion is a linear translation of the bat.

3.2.1 Configuration 1a

Configuration 1a is a head-on collision at the center of mass of the bat, as shown in Fig. 1.1 (bottom) and Fig. 2.1. This section uses a linear model (meaning there is no ω_{bat} or ω_{ball}) with two equations in two unknowns (Bahill and Karnavas 1989, 1991), which are given in Table 3.1.

We will now derive the equations for a head-on (planar, central, direct) collision at the center of mass (cm) of the bat. The abbreviations used in the following equations are described in Table 1.1. Many authors, for example (Bahill and Karnavas 1989, 1991; Watts and Bahill 1990, 2000; Brach 2007), have previously studied collisions using the Newtonian concepts of

Conservation of Momentum

Table 3.1 Equations for configuration 1a, two equations and two unknowns

Inputs	$v_{\text{ball - before}}, v_{\text{bat - cm - before}}$
Outputs (unknowns)	$v_{\text{ball - after}}, v_{\text{bat - cm - after}}$
Equations	
Conservation of Linear Momentum	$m_{\text{ball}}v_{\text{ball - before}} + m_{\text{bat}}v_{\text{bat - cm - before}} = m_{\text{ball}}v_{\text{ball - after}} + m_{\text{bat}}v_{\text{bat - cm - after}}$
Definition of CoR	$CoR_{1a} = -\frac{v_{\text{ball - after}} - v_{\text{bat - cm - after}}}{v_{\text{ball - before}} - v_{\text{bat - cm - before}}}$

$$m_{\text{ball}}v_{\text{ball - before}} + m_{\text{bat}}v_{\text{bat - cm - before}} = m_{\text{ball}}v_{\text{ball - after}} + m_{\text{bat}}v_{\text{bat - cm - after}}$$

and the Kinematic Coefficient of Restitution (CoR)

$$CoR_{1a} = -\frac{v_{\text{ball - after}} - v_{\text{bat - cm - after}}}{v_{\text{ball - before}} - v_{\text{bat - cm - before}}}$$

to derive the following equations for the velocities of the ball and bat after the collision, which were presented in Bahill and Karnavas (1989):

$$v_{\text{ball - after}} = \frac{v_{\text{ball - before}}(m_{\text{ball}} - CoR_{1a}m_{\text{bat}}) + v_{\text{bat - cm - before}}m_{\text{bat}}(1 + CoR_{1a})}{m_{\text{ball}} + m_{\text{bat}}}$$

$$v_{\text{bat - cm - after}} = \frac{v_{\text{ball - before}}m_{\text{ball}}(1 + CoR_{1a}) + v_{\text{bat - cm - before}}(m_{\text{bat}} - m_{\text{ball}}CoR_{1a})}{m_{\text{ball}} + m_{\text{bat}}}$$

After rearranging, we have the canonical form

$$v_{\text{ball - after}} = v_{\text{ball - before}} + \frac{(v_{\text{bat - cm - before}} - v_{\text{ball - before}})m_{\text{bat}}(1 + CoR_{1a})}{m_{\text{ball}} + m_{\text{bat}}}$$

$$v_{\text{bat - cm - after}} = v_{\text{bat - cm - before}} - \frac{(v_{\text{bat - cm - before}} - v_{\text{ball - before}})m_{\text{ball}}(1 + CoR_{1a})}{m_{\text{ball}} + m_{\text{bat}}}$$

Historically, these derivations started with the two-rotation model for the swing of a baseball or a softball bat (Fig. 1.1, middle) and linearized the model by finding tangents to the circular motion (Fig. 1.1, bottom). Bahill and Karnavas (1989) expanded this model by measuring the speed of the swing for a few hundred baseball and softball players and used this experimental data and model, to derive equations for the batted-ball speed for each individual person.

This derivation used the following assumptions:

1. Neglect permanent deformation of the bat and ball.
2. Assume a head-on (planar, direct, central) collision at the center of mass of the bat.
3. Ignore the change in the rotational kinetic energy of the ball: the energy stored in the spin of the ball is less than 1% of the translational energy (Bahill and Baldwin 2008). For a curveball hitting the sweet spot of the bat, the initial

kinetic energy stored in the bat and the ball is 375 J, of which 1.7 J is stored in the spin of the ball: so neglecting it seems reasonable. In the section for configuration 2b, we show that for a head on collision (without considering friction) there will be no change in the ball's angular rotation.

4. Assume that there are no tangential forces during the collision.
5. Neglect the moment of inertia of the batter's arms.
6. We assumed a free-end collision. The velocity of the bat reaches its peak at or before the collision. This means that the batters hands and arms are no longer applying acceleration forces. Hence, we neglected forces from the batters hands during the collision.

The reason for considering collisions at the center of mass is to allow the reader to progress slowly through the derivations. Take baby steps first. Configuration 1a in Chap. 3 takes two pages of easy equations. The BaConLaw model of Chap. 4 takes 40 pages of detailed equations.

This is the end of the Bahill and Karnavas (1989, 1991) model derivation.

3.2.1.1 Alternative Bat Effective Mass Model

The bat effective mass bat-ball collision modeling community, established by Nathan (2003), derives the batted-ball speed equation as follows. Fig. 2.1 and the equation for conservation of linear momentum give us

$$m_{\text{ball}} v_{\text{ball-before}} + m_{\text{bat}} v_{\text{bat-cm-before}} = m_{\text{ball}} v_{\text{ball-after}} + m_{\text{bat}} v_{\text{bat-cm-after}}$$

We can solve this for $v_{\text{bat-cm-after}}$

$$v_{\text{bat-cm-after}} = \left\{ v_{\text{bat-cm-before}} + \frac{m_{\text{ball}} v_{\text{ball-before}} - m_{\text{ball}} v_{\text{ball-after}}}{m_{\text{bat}}} \right\}$$

and substitute this into thier equation for the coefficient of restitution.

$$e = - \frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}}}$$

$$e = - \frac{v_{\text{ball-after}} - \left\{ v_{\text{bat-cm-before}} + \frac{m_{\text{ball}} v_{\text{ball-before}} - m_{\text{ball}} v_{\text{ball-after}}}{m_{\text{bat}}} \right\}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}}}$$

$$e(v_{\text{ball-before}} - v_{\text{bat-cm-before}}) = -v_{\text{ball-after}} - v_{\text{bat-cm-before}} + \frac{m_{\text{ball}}(v_{\text{ball-before}} - v_{\text{ball-after}})}{m_{\text{bat}}}$$

collecting the $v_{\text{ball-after}}$ terms on the left side yields

$$v_{\text{ball-after}} + \frac{m_{\text{ball}} v_{\text{ball-after}}}{m_{\text{bat}}} = -v_{\text{bat-cm-before}} + \frac{m_{\text{ball}} v_{\text{ball-before}}}{m_{\text{bat}}} - e(v_{\text{ball-before}} - v_{\text{bat-cm-before}})$$

grouping the terms on the right

$$v_{\text{ball-after}} + \frac{m_{\text{ball}} v_{\text{ball-after}}}{m_{\text{bat}}} = -v_{\text{bat-cm-before}} + e v_{\text{bat-cm-before}} + \frac{m_{\text{ball}} v_{\text{ball-before}}}{m_{\text{bat}}} - e v_{\text{ball-before}}$$

$$v_{\text{ball-after}} = +v_{\text{ball-before}} \left(\frac{\frac{m_{\text{ball}}}{m_{\text{bat}}} - e}{1 + \frac{m_{\text{ball}}}{m_{\text{bat}}}} \right) - v_{\text{bat-cm-before}} \frac{1 - e}{1 + \frac{m_{\text{ball}}}{m_{\text{bat}}}}$$

Multiply top and bottom by m_{bat} and we get the Bahill and Karnavas equation presented above and repeated here.

$$v_{\text{ball-after}} = \frac{v_{\text{ball-before}}(m_{\text{ball}} - m_{\text{bat}} \text{CoR}_{1a}) - v_{\text{bat-cm-before}} m_{\text{bat}}(1 - \text{CoR}_{1a})}{m_{\text{ball}} + m_{\text{bat}}}$$

In Sect. 5.2 we define $q = \left(\frac{e - \frac{m_{\text{ball}}}{M_{\text{eff}}}}{1 + \frac{m_{\text{ball}}}{M_{\text{eff}}}} \right)$ and then $v_{\text{ball-after}} = q v_{\text{ball-before}} + (1 + q) v_{\text{bat-before}}$.

The purpose of presenting this model here is to emphasize that it is important to consider alternative models. If their main results agree, then that validates both models. We will return to this bat Effective Mass model in Sect. 5.2.

3.2.2 Configuration 1b

Configuration 1b is a head-on collision at the center of mass of the bat, as shown in Fig. 2.1. Spin on the ball and bat are not considered. This is the same as configuration 1a, but it adds Conservation of Energy. It has three equations and two unknowns as shown in Table 3.2.

Although an additional equation is not needed, we will now present the Conservation of Energy equation as a consistency check. There is nothing in the system that will release energy during the collision (loaded springs or explosives). The bat swing is level so there will be no change in potential energy.

Before the collision, there is kinetic energy in the ball and the bat.

Table 3.2 Equations for configuration 1b, which adds Conservation of Energy (CoE). It has three equations and two unknowns

Inputs	$v_{\text{ball - before}}, v_{\text{bat - cm - before}}$
Outputs	$v_{\text{ball - after}}, v_{\text{bat - cm - after}}$
Equations	
Conservation of Energy	$\frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2 - \frac{\bar{m}}{2}(v_{\text{ball-before}} - v_{\text{bat-cm-before}})^2(1 - CoR_{1b}^2) = \frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2$
Conservation of Linear Momentum	$m_{\text{ball}}v_{\text{ball - before}} + m_{\text{bat}}v_{\text{bat - cm - before}} = m_{\text{ball}}v_{\text{ball - after}} + m_{\text{bat}}v_{\text{bat - cm - after}}$
Definition of CoR	$CoR_{1b} = -\frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}}}$

$$KE_{\text{before}} = \frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2$$

We modeled the bat velocity as a linear term comprising a translation and two rotations (See Fig. 1.1). This linear velocity is what we measured in our experiments.

$$KE_{\text{after}} = \frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2$$

$$KE_{\text{before}} = KE_{\text{after}} + KE_{\text{lost}} \quad (3.1)$$

Kinetic energy will be transformed to heat in the ball, vibrations in the bat and deformations of the bat and ball. The Coefficient of Restitution (CoR) models the energy that is transformed in a frictionless head-on collision between two objects. Such a collision will have no tangential velocity components. The equation for the kinetic energy lost in a bat-ball collision of configuration 1b (Dadouriam 1913, Eq. (XI), p. 248; Ferreira da Silva 2007, Eq. 23; Brach 2007, Eq. 3.7) is

$$KE_{\text{lost}} = \frac{\bar{m}}{2}(\text{collision velocity})^2(1 - CoR_{1b}^2)$$

$$\text{where } \bar{m} = \frac{m_{\text{ball}}m_{\text{bat}}}{m_{\text{ball}} + m_{\text{bat}}}.$$

$$KE_{\text{lost}} = \frac{\bar{m}}{2}(v_{\text{ball-before}} - v_{\text{bat-cm-before}})^2(1 - CoR_{1b}^2) \quad (3.2)$$

This equation will be derived in the configuration 2b section. Combining these three equations (KE_{before} , KE_{after} and KE_{lost}) yields the equation for Conservation of Energy for configuration 1b

Table 3.3 Simulation values for bat-ball collisions at the center of mass, configuration 1b

	SI units	Baseball units
Inputs		
$v_{\text{ball}} - \text{before}$	−37 m/s	−83 mph
$v_{\text{bat}} - \text{cm} - \text{before}$	23 m/s	52 mph
CollisionSpeed		135 mph
CoR_{1b}		0.475
Outputs		
$v_{\text{ball}} - \text{after}$	40 m/s	89 mph
$v_{\text{bat}} - \text{cm} - \text{after}$	11 m/s	25 mph

Table 3.4 Configuration 1b kinetic energies, J

KE ball linear velocity before=	100
KE bat linear velocity before=	246
KE before total=	346
KE ball linear velocity after=	114
KE bat linear velocity after=	55
KE after=	169
KE lost =	177
KE after + KE lost=	346

$$\begin{aligned}
 & \frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2 - \frac{\bar{m}}{2}(v_{\text{ball-before}} - v_{\text{bat-cm-before}})^2(1 - CoR_{1b}^2) \\
 & = +\frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2
 \end{aligned}
 \tag{3.3}$$

This assumes that there is no spin on the ball or the bat, meaning that we have ignored angular momentum. Using the numbers in Table 1.1 produces the results shown in Table 3.3.

3.2.3 Simulation Results

Table 3.4 shows the kinetic energies for the same simulation.

A batted-ball velocity, $v_{\text{ball}} - \text{after}$, of 89 mph is reasonable. The fact that $KE_{\text{before}} = KE_{\text{after}} + KE_{\text{lost}} = 346$ J shows that this set of equations is consistent. As a reality check, we note that the average kinetic energy in the swings of 28 members of the San Francisco Giants baseball team was 292 J (Bahill and Karnavas 1991). Given human variability and the different circumstances for the experiments, these numbers are compatible.

Let us now consider the consequences of neglecting the spin of the ball. A typical spin rate for a curveball is 2000 rpm. So the rotational kinetic energy in the ball will be about $0.5I_{\text{ball}}\omega_{\text{ball-before}}^2 = 1.7 \text{ J}$. This is small compared to the translational kinetic energies.

This is the end of the equations for configuration 1b. In the rest of this book, we will use Newtonian mechanics, to derive equations for the velocity of the bat and the ball after their collision, for collisions that do *not* occur at the center of mass of the bat.

3.2.4 The Coefficient of Restitution

The Coefficient of Restitution (CoR) models the energy lost in a collision between two objects. It is commonly defined as the ratio of the relative speed between the two objects after a collision to the relative speed before the collision. Here are the CoR definitions for some of our configuration types. The subscripts refer to the collision type given in Chap. 2.

$$\begin{aligned} CoR_{1a} &= CoR_{1b} = -\frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}}} \\ CoR_{2a} &= CoR_{2b} = -\frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}}\omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}}} \\ CoR_{2c} &= -\frac{v_{\text{ball-after}} - v_{\text{knob-after}} - d_{\text{knob-ip}}\beta_{\text{after}}}{v_{\text{ball-before}} - v_{\text{knob-before}} - d_{\text{knob-ip}}\beta_{\text{before}}} \end{aligned}$$

These equations will be explained later when they are used.

The *CoR* is used to model the energy lost during a bat-ball collision. If the *CoR* were 1.0, then all the original energy would be recovered in the motion of the system after impact. However, if there were losses due to energy dissipation or energy storage, then the *CoR* would be between 0 and 1.0. In a bat-ball collision there is energy dissipation: both the bat and the ball increase in temperature. Also both the bat and the ball store energy in vibrations. This energy is not available to be transferred to the ball and therefore the ball velocity is smaller.

The *CoR* depends on the speed of the collision. Our simulations use the following equation for a wooden bat and a baseball $CoR = 0.61 - 0.001 \text{ CollisionSpeed}$, where *CollisionSpeed* (the sum of the ball speed and the bat speed) is in mph. This equation came from unpublished data provided by Jess Heald of Worth Sports Co. and they assume a collision at the sweet spot of the bat. Table 3.5 gives CoRs measured in seven experimental studies.

Most of the data points for 60 mph collisions against flat walls show that baseballs are in conformance with the rules of major league baseball. However, for high speeds and wooden bats, there is a lot of variation in the data. Some studies say that the CoR of a collision between a ball and flat wooden wall is higher than the

Table 3.5 Experimental CoR values for colliding baseballs

Source	Baseball collides with	Equation	CoR value at 60 mph	CoR value at 120 mph
Jess Heald President of Worth Sports Co. 1986, reported in Watts and Bahill (1990)	Flat wooden wall	$CoR = 0.61 - 0.001 \text{ CollisionSpeed}$	0.550	0.490
Crisco, Greenwald, Blume and Penna (2002)	Wooden bat	$CoR = 0.67 - 0.0015 \text{ CollisionSpeed}$	0.580	0.490
Fallon and Sherwood (2000)	Flat wooden wall		0.548	
Fallon and Sherwood (2000)	Wooden bat	*At 140 mph		0.504*
Drane et al. (2008)	Flat wooden wall		0.546	
Drane et al. (2008)	Wooden bat	*At 90 mph	0.537	0.503*
Major League Baseball rules	Flat wooden wall	*At 58 mph	0.514–0.568*	
Nathan et al. (2011)	Flat steel plate	$CoR = 0.64 - 0.0014 \text{ CollisionSpeed}$	0.556	0.472
Cross (2011), Fig. 8.5	Flat wooden wall	$CoR = 0.67 - 0.0021 \text{ CollisionSpeed}$	0.544	0.418

CoR of a collision between a ball and a wooden bat, and some say that it is lower. The CoR depends on the shape of the object that the ball is colliding with. When a baseball is shot out of an air cannon onto a flat wooden wall, most of the ball's deformation is restricted to the outer layers: the cowhide cover and the four yarn shells. However, in a high-speed collision between a baseball and a cylindrical bat, the deformation penetrates into the cushioned cork center. This allows more energy to be stored and released in the ball and the CoR might be higher.

The CoR also depends on where the ball hits the bat, the speed of the collision, the relative humidity, the temperature, the deformation of the objects, the surface texture and the configuration of the collision.

Figure 3.1 shows that the CoR for baseballs is a function of the collision speed, the temperature and the relative humidity. The experiments reported in Table 3.5 did not state the temperature or humidity in which their experiments were performed. Therefore, the data in Table 3.5 must be taken with a grain of salt. The data point at 140 mph, from Fallon and Sherwood (2000), was based on 140 valid collisions with major league baseballs: so it is probably accurate. It is given to emphasize the fact that we do not know what the CoR is for high-speed

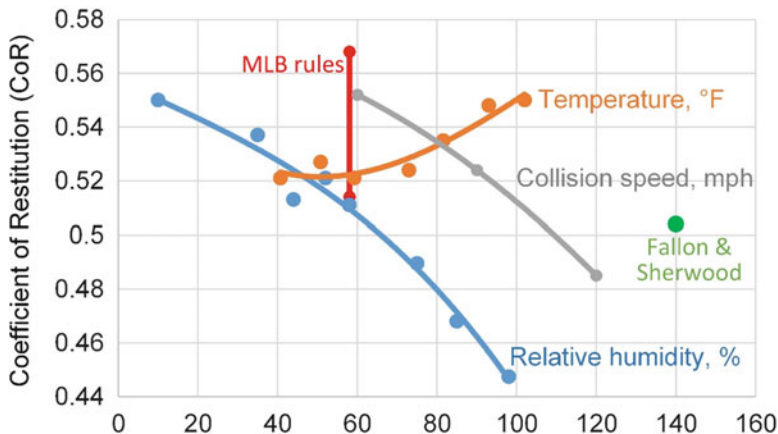


Fig. 3.1 Coefficients of Restitution (CoR) for major league baseballs as functions of temperature, collision speed and relative humidity. Data are from Nathan et al. (2011). The point at 140 mph is from Fallon and Sherwood (2000). The red line shows the major league baseball rule for a collision at 58 mph

collisions. Meaning that we cannot extrapolate the Nathan, Smith, Faber and Russell (2011) curve to speeds above 120 mph.

Therefore, for the simulations of this book, we will use the following equation from Worth Sports Co.

$$CoR = 0.61 - 0.001 \text{ CollisionSpeed}$$

and we will be cautious about using its values for speeds above 120 mph. Using this equation means that we are ignoring the effects of where the ball hits the bat (we assume that it is at the center of mass or at the sweet spot), the relative humidity, the temperature, the shape of the objects (we assume that the baseball is colliding with a flat wooden wall), the deformation of the objects and the surface texture (seams). We only consider major league baseballs.

Modeling philosophy note. George Box wrote, “All models are wrong, but some are useful (Box 1981).” In this section, we wrote that the coefficient of restitution for collisions is between zero and one, $0 \leq CoR \leq 1$. But these are not theoretical limits. For example, a baseball thrown through a window screen will have a negative CoR. Whereas a ball that releases energy on every bounce, for example one that is coated with an explosive or one that contains a spring and an escapement like a watch, can have a CoR greater than one. A model is a simplified representation of a particular view or aspect of a real system. No model can represent all views.

Table 3.6 Equations for configuration 2a, three equations and three unknowns

Inputs	$v_{\text{ball}} - \text{before}$, $v_{\text{bat}} - \text{cm} - \text{before}$, $\omega_{\text{bat}} - \text{before}$ and CoR
Outputs	$v_{\text{ball}} - \text{after}$
Equations	
Conservation of Linear Momentum, Eq. (3.4)	$m_{\text{ball}}v_{\text{ball}} - \text{before} + m_{\text{bat}}v_{\text{bat}} - \text{cm} - \text{before} = m_{\text{ball}}v_{\text{ball}} - \text{after} + m_{\text{bat}}v_{\text{bat}} - \text{cm} - \text{after}$
Definition of CoR , Eq. (3.5)	$CoR_{2a} = - \frac{v_{\text{ball}} - \text{after} - v_{\text{bat}} - \text{cm} - \text{after} - d_{\text{cm-ip}}\omega_{\text{bat}} - \text{after}}{v_{\text{ball}} - \text{before} - v_{\text{bat}} - \text{cm} - \text{before} - d_{\text{cm-ip}}\omega_{\text{bat}} - \text{before}}$
Newton's Second Principle, Eq. (3.6)	$d_{\text{cm-ip}}m_{\text{ball}}(v_{\text{ball}} - \text{after} - v_{\text{ball}} - \text{before}) = -I_{\text{bat}}(\omega_{\text{bat}} - \text{after} - \omega_{\text{bat}} - \text{before})$

3.3 Collisions at the Sweet Spot

3.3.1 Configuration 2a

Configuration 2a is a head-on (planar, parallel) collision at the sweet spot of the bat, which we define to be the Center of Percussion (CoP). Watts and Bahill (1990) expanded the Bahill and Karnavas (1989) model to create configuration 2a. They introduced a third unknown, the rotation of the bat, ω_{bat} , after the collision and a third equation, which was based on Newton's second principle. Therefore, this section has three equations, shown in Table 3.6, but we only solved for one unknown. The model for bat movement is that of a translation and a rotation about its center of mass.

This section considers collisions for impact points (ip) that are not at the center of mass of the bat. Our objective was to derive an equation for the velocity of the ball after its collision with the bat. We expanded the previous linear model to the combined rotation plus translation model with the bat-ball impact point off of the center of mass, at the sweet spot (see Fig. 2.3). There are about a dozen definitions for the sweet spot of the bat (Bahill 2004). We will use the symbols defined in Table 1.1. Figure 2.3 is appropriate for these collisions. In the Coefficient of Restitution (CoR) equation, the bat speed is a combination of the bat translation before the collision and the rotation about the center of mass caused by rotations about the batter's spine and wrists. This velocity is what we measured in our experiments.

3.3.1.1 Definition of the Sweet Spot

For skilled batters, we assume that most bat-ball collisions occur near the sweet spot of the bat, which is, however, difficult to define precisely. The horizontal sweet spot of the bat has been defined as the center of percussion, the node of the fundamental vibrational mode, the antinode of the hoop mode, the maximum energy transfer area, the maximum-batted-ball speed area, the maximum coefficient of restitution

area, the minimum energy loss area, the minimum sensation area and the joy spot. Let us now examine each of these definitions. This section is based on Bahill and Baldwin (2008).

1. **Center of Percussion.** For most collision points, when the ball hits the bat it produces a translation of the bat and a rotation of the bat. However, if the ball hits the bat at the center of mass there will be a translation but no rotation. Whereas, if the bat is fixed at a pivot point and the ball hits the bat at the Center of Percussion (*CoP*) for that pivot point, then there will be a rotation about that pivot point but no translation (and therefore no sting on the hands). The pivot point and the *CoP* for that pivot point are conjugate points, because if instead the bat is fixed at the *CoP* and the ball hits the pivot point then there will be a pure rotation about the *CoP*. The *CoP* and its pivot point are related by the following equation derived by Sears et al. (1976), where the parameters are defined in Fig. 3.1.

$$d_{\text{pivot-cop}} = \frac{I_{\text{pivot}}}{m_{\text{bat}} d_{\text{pivot-cm}}}$$

The *CoP* is not one fixed point on the bat. There is a different *CoP* for every pivot point. If the batter chokes up on the bat, the pivot point (and consequently the *CoP*) will change. In fact, the pivot point might even change during an individual swing. In this section, we assume that the pivot point is 6 inches (15 cm) from the knob, because that is where the batter's hands are. We could assume that the pivot point is at the end of the knob (Milanovich and Nesbit 2014). This produces a different *CoP*.

There are three common experimental methods for determining the *CoP* of a bat. (Method 1) *Pendular motion*: Hang a bat at a point 6 inches (15 cm) from the knob with 2 or 3 feet (1 m) of string. Hit the bat with an impact hammer. Hitting it off the *CoP* will make it flop like a fish out of water, because there is a translational force and a rotational force at the pivot point. Hitting it near the *CoP* will make it swing like a pendulum. (Method 2) *Toothpick pivot*: Alternatively, you can pivot the bat on a toothpick through a hole at the pivot point and strike the bat at various places. When struck near the *CoP* for that pivot point the toothpick will not break. At other places, the translational forces will break the toothpick. (Method 3) *Equivalent pendulum*: A third method for measuring the distance between the pivot point and the *CoP* is to make a pendulum by putting a mass equal to the bat's mass on a string and adjusting its length until the pendulum's period and the bat's period are the same. This method has the smallest variability.

2. **Node of the fundamental mode.** The node of the fundamental bending vibrational mode is the area where this vibrational mode (roughly between 150 and 200 Hz for a wooden bat) of the bat has a null point. To find this node, with your fingers and thumb grip a bat about 6 inches from the knob. Lightly tap the barrel at various points with an impact hammer. The area where you feel no vibration

and hear almost nothing (except the secondary vibrational crack or ping at 500 to 800 Hz) is the node. A rubber mallet could be used in place of an impact hammer: the point is, the hammer itself should not produce any noise. The antinode of the third bending vibrational mode may also be important.

3. **Antinode of the hoop mode.** For hollow metal and composite baseball and softball bats, there is another type of vibration, called a hoop vibration. The walls of a hollow bat deform during a bat-ball collision. The walls are crushed in and then bounce back out. This vibration can be modeled as a hoop or a ring around the bat; this ring deforms like the vertical cross-sectional area of a water drop falling from a faucet; first the water drop is tall and skinny, in free fall it is round and when it hits the ground it becomes short and fat. The location of the antinode of the first hoop mode is another definition of the sweet spot.
4. **Maximum-batted-ball speed point.** There is a point on the bat that produces the maximum-batted-ball speed. Section 4.10 shows this point to be 9.2 cm (3.6 inches) from the center of mass, or 25.4 cm (10 inches) from the end of the barrel. This point can be computed theoretically as follows. Start with an equations for $v_{\text{ball-after}}$, such as Eq. (4.8). Take the derivative with respect to d . Set this equal to zero and solve for d . This value will depend on $v_{\text{ball-before}}$ which you obtain from, for example, Table 4.2.
5. **Maximum coefficient of restitution area.** The coefficient of restitution (CoR) is commonly defined as the ratio of the relative speed after a collision to the relative speed before the collision. In our studies, the CoR is used to model the energy transferred to the ball in a collision with a bat. If the CoR were 1.0, then all the original energy would be recovered in the motion of the system after impact. But if there were losses due to energy dissipation or energy storage, then the CoR would be less than 1.0. For example, in a bat-ball collision there is energy dissipation: both the bat and the ball increase slightly in temperature. In one experiment, 100 bat-ball collisions in rapid succession raised the temperature of a softball by 10 °F (Duris and Smith 2004). Also both the bat and the ball store energy in vibrations. Not all of this energy will be transferred to the ball. (For now, we ignore the kinetic energy stored in the ball's spin.) The maximum coefficient of restitution area is the area that produces the maximum CoR for a bat-ball collision. This area can be computed theoretically using Eq. (4.5) as described in definition (4) above.
6. **Maximum energy transfer area.** A collision at the maximum energy transfer area transfers the most energy to the ball. This definition says that the best contact area on the bat is that which loses the least amount of energy to bat translation, rotation, vibration, etc. This area can be computed theoretically using Eq. (4.11) as described in definition (4).
7. **Minimum energy loss area.** There is an area that minimizes the total (translation plus rotation plus vibration) energy lost in the bat. This area depends on the fundamental bending mode, the second mode and the center of percussion. This area can be approximated theoretically using Eq. (4.11) as described in definition (4).

8. **Minimum sensation area.** For most humans, the sense of touch is most sensitive to vibrations between 200 and 400 Hz. For each person there is a collision area on the bat that would minimize these sensations in the hands.
9. **Joy spot.** Finally, Ted Williams and Underwood (1982) stated that hitting the ball at the joy spot makes you the happiest. His joy spot was centered 5 inches (13 cm) from the end of the barrel.

These nine areas are different, but they are close together. We group them together and refer to this *region* as the sweet spot. We measured a large number of bats (youth, adult, wood, aluminum, ceramic, titanium, etc.) and found that the sweet spot was 15–20% of the bat length from the barrel end of the bat. In our Ideal Bat Weight experiments (Bahill and Karnavas 1989, 1991) and our variable moment of inertia experiments (Bahill 2004) for adult bats the center of the sweet spot was defined to be 5 inches (13 cm) from the barrel end of the bat.

It does not make sense to try getting greater precision in the definition of the sweet spot, because the concept of a sweet spot is a human concept, and it probably changes from human to human. For one example, in calculating the center of percussion, the pivot point of the bat must be known and this changes from batter to batter, and it may even change during the swing of an individual batter (Milanovich and Nesbit 2014).

Table 3.7 shows general properties for a standard Hillerich and Bradbury Louisville Slugger wooden C243 pro stock 34-inch (86 cm) bat with the barrel end cupped out to reduce weight. This is a different bat than that described in Table 1.1. These modern scientific methods of calculating the center of the sweet spot of the bat are all only a few centimeters above the true value given by Ted Williams four decades ago.

Table 3.7 Parameters for a C243 wooden bat, assuming a pivot point 6 inches from knob

	SI units	Baseball units
Length	0.863	34
Mass	0.880	31
Period (sec)	1.65	1.65
I_{knob} (kg-m ²)	0.335	
I_{cm} (kg-m ²)	0.0511	
Measured $d_{\text{knob-cm}}$	0.57	22.4
Measured $d_{\text{knob-cop}}$	0.69	27.2
Calculated $d_{\text{knob-cop}}$	0.69	27.2
Measured $d_{\text{pivot-cop}}$	0.55	21.7
Calculated $d_{\text{pivot-cop}}$	0.54	21.3
Calculated $d_{\text{pivot-cm}}$	0.42	16.5
Measured $d_{\text{knob-firstNode}}$	0.67	26.4
Calculated $d_{\text{knob-cop}}$ for a pivot point in the knob (cm)	0.66	26.0
Distance from the center of percussion to the end of the bat	0.162	6.38

There is no sweet spot of the bat: however, there is a sweet area and for a 34-inch wooden bat, it is 5–7 inches (13–18 cm) from the barrel end of the bat. We presented nine definitions for the sweet spot of the bat. Some of these definitions had a small range of experimentally measured values (e.g. 1 cm for the node of the fundamental vibration mode), whereas others had a large range of experimentally measured values (e. g. 10 cm for the maximum batted-ball speed area). But of course, none of these definitions has square sides. They are all bowl shaped. So the width depends on how far you allow the parameter to decline before you say that you are out of the sweet area. In general, the sweet area is about 2 inches wide. Our survey of retired major league batters confirmed that the sweet spot of the bat is about 2 inches (5 cm) wide. Therefore, most of the sweet-spot definitions of this chapter fall within this region. In summary, recent scientific analyses have validated Ted William's statement that the sweet spot of the bat is an area 5–7 inches from the end of the barrel.

For completeness, we note that the vertical component of the sweet spot is one-third of an inch high (Baldwin and Bahill 2004). See Fig. 4.4.

3.3.1.2 Coordinate System

We will use a right-handed coordinate system with the x-axis pointing from home plate to the pitching rubber, the y-axis points from first base to third base, and the z-axis points straight up. A torque rotating from the x-axis to the y-axis would be positive upward. Previously, in other papers describing only the pitch, we defined the x-axis as pointing from the pitching rubber to home plate and then the y-axis went from third to first base (Bahill and Baldwin 2007). Over the plate, the ball comes downward at a 10° angle and the bat usually moves upward at about 10° , so later the z-axis will be rotated back 10° .

3.3.1.3 Assumptions

- A1. The swing of the bat is as modeled in Fig. 3.1.
- A2. Collisions at the Center of Percussion will produce a rotation about the center of mass, but no translation of the bat.
- A3. For configurations 1a, 1b and 2a, we will not include the kinetic energy stored in the rotation of the baseball. That is, we assume that the pitch is a knuckleball with no spin. In later sections, we will consider a fastball and a curveball.
- A5. The collision duration is short, for example, one millisecond.
- A6. Because the collision duration is short and the swing is level, we can ignore the effects of gravity *during* the collision.
- A7. We neglect permanent deformations of the bat and ball.
- A8. The Coefficient of Restitution (*CoR*) for a baseball wooden-bat collision at major-league speeds starts at about 0.55 and decreases with collision speed.

- A9. For configuration 2d, Coulomb friction is a good model for a bat-ball collision. When colliding objects slide relative to each other, a friction force is generated, whose direction is tangential to the surface of contact and whose magnitude is proportional to the normal force at the point of contact. We assume that during impact the ball slides and does not roll on the bat, but the sliding halts before separation. The dynamic coefficient of friction, μ_f , is used to model these losses. This is called a Coulomb model. In contrast, a Coulomb model would not be appropriate for a pool cue hitting a ball of clay: a more complex model would be needed. A Coulomb model will be used in configuration 2d.
- A10. The dynamic coefficient of friction has been measured by Bahill at $\mu_f = 0.5$.
- A11. We write about kinetic energy losses during a collision: that is the way it is described in the literature. However, we should call these transformations, because, for example, kinetic energy is not lost during a collision. It might be transformed into heat in the ball, vibrations in the bat, acoustic energy in the “crack of the bat” or deformations of the bat or ball.
- A12. In this book, we do not model the moment of inertia of the batter’s arms.
- A13. Pictures of bats in this book are for wooden bats. However, the equations and conclusions are the same for wooden and aluminum bats. The differences would be in the mass, moment of inertia and dimensions.
- A14. We do not differentiate between day games and night games. We know that when the shadow of the stadium is between the pitcher and the batter, the batter’s performance is reduced. We ignore this effect.
- A15. Assume free-end collisions. For impacts at the sweet spot of the bat, the momentum transfer to the ball is complete by the time the elastic wave arrives at the handle. Therefore, any action by the hands will affect the bat at the impact point only after the ball and bat have separated, Nathan (2000).

3.3.1.4 Conservation of Linear Momentum

The law of Conservation of Linear Momentum states that linear momentum will be conserved in a collision if there are no external forces. We will approximate the bat’s motion before the collision with the tangent to the curve of its arc. For a collision anywhere on the bat, every point on the bat has the same angular velocity, but the linear velocities will be different, which means that $v_{t_{\text{bat}} - \text{before}}$ is a combination of translations and rotations unique for each point on the bat. Conservation of momentum in the direction of the x-axis states that the momentum before plus the external impulse will equal the momentum after the collision. There are no external impulses during the ball-bat collision: therefore, this is the equation for Conservation of Linear Momentum.

$$m_{\text{ball}} v_{\text{ball-before}} + m_{\text{bat}} v_{\text{bat-cm-before}} = m_{\text{ball}} v_{\text{ball-after}} + m_{\text{bat}} v_{\text{bat-cm-after}} \quad (3.4)$$

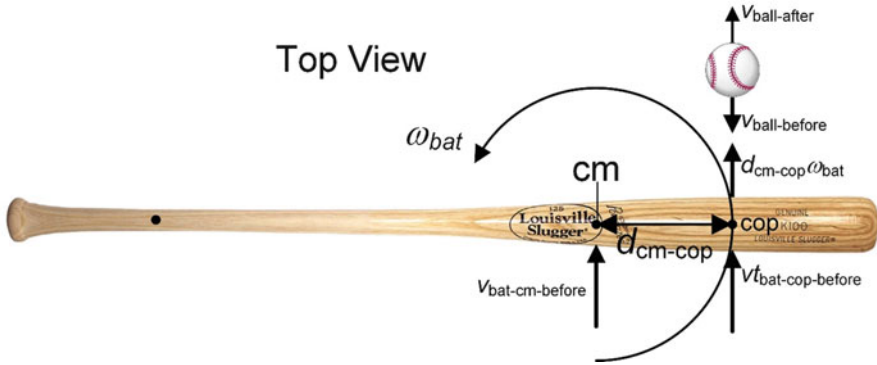


Fig. 3.2 This figure shows $v_{\text{ball-before}}$, $v_{\text{bat-cm-before}}$, $v_{\text{ball-after}}$ and $d_{\text{cm-ip}}\omega_{\text{bat}}$, which are used to define the Coefficient of Restitution for configurations 2

3.3.1.5 Definition of the Coefficient of Restitution

The kinematic Coefficient of Restitution (CoR) was defined by Sir Isaac Newton as the ratio of the relative velocity of the two objects after the collision to the relative velocity before the collision at the point of impact.

In our models, for a collision at any impact point (ip) we have

$$CoR_{2a} = - \frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}}\omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}}} \quad (3.5)$$

These variables and parameters are illustrated in Fig. 3.2. A note on notation: ω_{bat} is the angular velocity of the bat *about its center of mass*, $v_{\text{bat-cm}}$ is the linear velocity *of the center of mass* of the bat in the x-direction and $d_{\text{cm-ip}}$ is the distance between the center of mass and the point of impact. We measured $v_{\text{bat-cm-before}} + d_{\text{cm-ip}}\omega_{\text{bat-before}}$, which is experimental data that depends on our model formulation (Fig. 3.2) and the kinematics of the person swinging the bat.

3.3.1.6 Newton's Second Principle

Watts and Bahill (1990) derived the following equation from Newton's second principle that states that a force acting on an object produces acceleration in accordance with the equation $F = ma$. If an object is accelerating, then its velocity and momentum is increasing. This principle is often stated as; applying an impulsive force to an object will change its momentum. According to Newton's third principle, when a ball hits a bat at the impact point there will be a force on the bat in the direction of the negative x-axis, let us call this $-F_1$, and an equal but opposite force on the ball, called F_1 . This force will be applied during the duration of the collision, called t_c . When a force is applied for a short period of time, it is called an impulse. According to Newton's second principle, an impulse will change

momentum. The force on the bat will create a torque of $-d_{\text{cm-ip}}F_1$ around the center of mass of the bat. An impulsive torque will produce a change in angular momentum of the *bat*.

$$-d_{\text{cm-ip}}F_1t_c = I_{\text{bat}}(\omega_{\text{bat-after}} - \omega_{\text{bat-before}})$$

Now this impulse will also change the linear momentum of the *ball*.

$$F_1t_c = m_{\text{ball}}(v_{\text{ball-after}} - v_{\text{ball-before}})$$

Multiply both sides of this equation by $d_{\text{cm-ip}}$ and add these two equations to get the equation for

Newton's Second principle.

$$d_{\text{cm-ip}}m_{\text{ball}}(v_{\text{ball-after}} - v_{\text{ball-before}}) = -I_{\text{bat}}(\omega_{\text{bat-after}} - \omega_{\text{bat-before}}) \quad (3.6)$$

These equations were derived for the bat-ball system. Therefore, there were no external impulses (If the collision is at the sweet spot then the batters arms do not apply an impulse.) Equations (3.4), (3.5) and (3.6) produce the following equation for the batted-ball velocity (Watts and Bahill 1990, 2000). Its derivation will be given in the next chapter.

$$v_{\text{ball-after}} = \frac{v_{\text{ball-before}} \left(m_{\text{ball}}I_{\text{bat}} - m_{\text{bat}}I_{\text{bat}}CoR_{2a} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2 \right) + v_{\text{bat-cm-before}}m_{\text{bat}}I_{\text{bat}}(1 + CoR_{2a}) + m_{\text{bat}}d_{\text{cm-ip}}\omega_{\text{bat-before}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

or

$$v_{\text{ball-after}} = \frac{v_{\text{ball-before}} - (v_{\text{ball-before}} - v_{\text{bat-cm-before}})m_{\text{bat}}I_{\text{bat}}(1 + CoR_{2a}) + m_{\text{bat}}d_{\text{cm-ip}}\omega_{\text{bat-before}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

The output for the above equation, for typical inputs, is given in Table 3.8. Equations for $v_{\text{bat-cm-after}}$ and $\omega_{\text{bat-after}}$ were not derived by Watts and Bahill (1990, 2000). They will be derived in the next chapter.

3.3.1.7 Simulation Values Configuration 2a

Figure 1.1 (bottom) is appropriate for configurations 1a and 1b, if the bat translation and rotation are measured and modeled with one vector, $v_{\text{bat-cm}}$. With a change from the center of mass to the sweet spot, again Fig. 1.1 (bottom) is appropriate for configurations 2a and 2b, if the bat translation and rotation are

Table 3.8 Simulation values for bat-ball collisions at the sweet spot, configuration 2a

	SI units (m/s, rad/s)	Baseball units (mph, rpm)
Inputs		
$v_{\text{ball}} - \text{before}$	−37	−83
$v_{\text{bat}} - \text{cm} - \text{before}$	23	52
$\omega_{\text{bat}} - \text{before}$	32	309
$v_{\text{bat}} - \text{cop} - \text{before}$	28	62
CoR_{2a}	0.465	0.465
Output		
$v_{\text{ball}} - \text{after}$	41	92

measured and modeled with two vectors, $v_{\text{bat} - \text{cm}}$ and ω_{bat} . Later it will be shown that Fig. 1.1 (bottom) is also appropriate for configuration 2c, if the bat translation and rotation are measured and modeled with two vectors, $v_{\text{knob} - \text{trans}}$ and β_{bat} .

This is the end of the Watts and Bahill (1990, 2000) derivation, called configuration 2a. This chapter gave background, a literature review and the overarching organization of bat-ball collision configurations. The next chapter will drill into configuration 2b.

3.4 Bat Speeds

In our simulation for configuration 2a, whose results are given in Table 3.8, we used an impact point speed of 62 mph (28 m/s). Where did that number come from? Table 3.9 shows the results of several studies performed over the last few decades that have measured the speed of the baseball bat. These studies are listed in chronological order. For now, we only give the results for male collegiate and professional baseball players. This table gives the average speed of the sweet spot, which was usually defined as the center of percussion. This is the total speed of the sweet spot meaning the translational plus rotational velocities.

Table 3.9 gives average sweet-spot speeds for eight studies of male college and professional batters. When multiple bats were used, we chose the bat closest to that described in Table 1.1. In our simulations, we used 62 mph for the total bat speed, which we defined to be the linear plus rotational speed of the sweet spot of the bat.

Some studies in the literature filtered their data and only included selected batters, usually the fastest. Internet sites that are trying to sell their equipment and services typically cite bat speeds between 70 and 90 mph (31–40 m/s). We think that these numbers are bogus. The big web sites such as mlb.com, espn.com/mlb/ and hittrackeronline.com give the leaders in many categories, meaning that they have selected, for example, the 20 fastest players out of 750. This would be misleading if the reader thought that these numbers were *representative* of major league batters. In Table 3.9 we give average values for sweet-spot speeds.

Table 3.9 Average total sweet-spot speed before a collision, chronological order

Average speed of the sweet spot, m/s	Average speed of the sweet spot, mph	Subjects, only males	Reference
26	58	28 San Francisco giants	Database of Bahill and Karnavas (1989)
31	69	7 selected professional baseball players	Welch et al. (1995)
30	68	19 baseball players	Crisco et al. (2002)
27	60	16 college baseball players	Fleisig et al. (2002)
26	58	7 college baseball players	Koenig et al. (2004)
32	71	One subject	King et al. (2012)
27	60	10 collegiate baseball players	Higuchi et al. (2016)
28	62	700 swings of major league baseball players where the outcome was a hit	Willman ^a (2017)

^aThis source did not state whether these swing speeds were at the center of mass, the sweet spot or the impact point, but we assumed the impact point

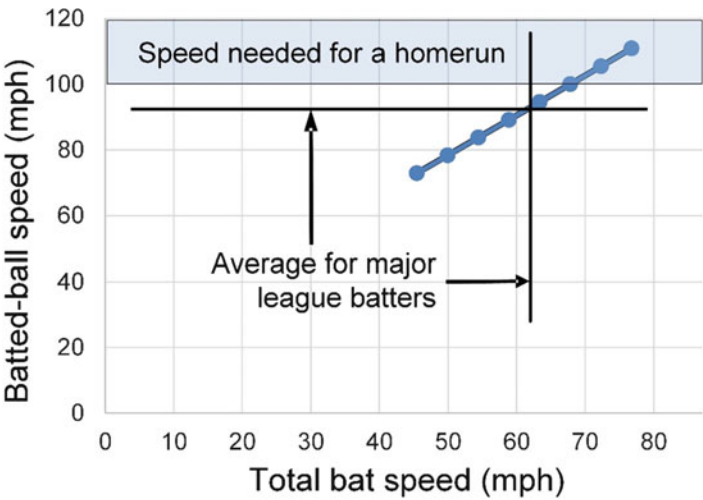


Fig. 3.3 Batted-ball speed as a function of total bat speed

Next, we wanted to know how these laboratory measurements compare to major league batters in actual games. Figure 3.3 shows the batted-ball speed as a function of the total bat speed before the collision. Using the data of Willman (2017) for the

year 2016, we found that for 15,000 base hits in major league baseball the average batted-ball speed was 91 mph. This figure shows that, given physiological variation, the average major league batter has a high enough bat speed to occasionally hit a home run, when the batted-ball has the ideal spin and launch angle. However, most major league batters seldom hit home runs. Indeed, of the 2200 active players listed by MLB.com half of them have never hit a home run in their major league careers. The simulation summarized in Table 3.8 shows that a typical ball velocity before the collision, $v_{\text{ball-before}}$, of 83 mph (37 m/s) and an average bat speed, $v_{\text{bat-cop-before}}$, of 62 mph (28 m/s) would produce an average batted-ball speed, $v_{\text{bat-after}}$, of 92 mph (41 m/s), which would not be enough for a home run in any major league stadium. Our rule of thumb is that it takes a batted-ball speed of 100 mph (45 m/s), under optimal conditions, to produce a home run.

Most *recent* studies of bat speed have used multiple video cameras and commercial prepackaged software to measure and compute bat speed (Willman 2017). Unfortunately, these systems have no calibration tests. On television, the batted-ball speed is often called the exit speed, the exit velocity or the launch speed.

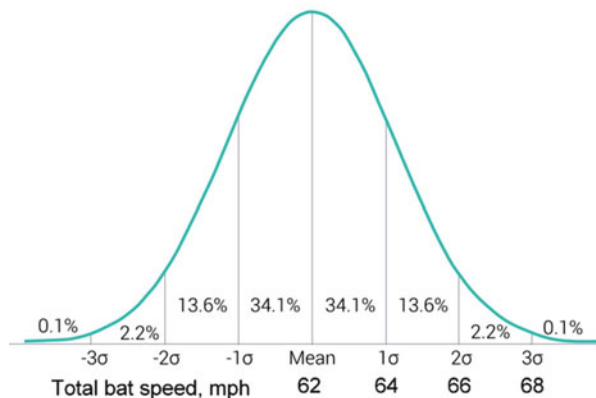
The studies of Fleisig et al. (2001, 2002), Cross (2009), Milanovich and Nesbit (2014) and King et al. (2012) decomposed the center of percussion speed into two components: the linear translation velocity and the angular rotation velocity, $v_{\text{bat-cop-before}} = v_{\text{cm}} + d_{\text{cm-cop}}\omega_{\text{cm-before}}$. A consensus of these four databases produced

$$v_{\text{bat-cop-before}} = 23 + 0.134 \times 32 = 28 \text{ m/s} = 62 \text{ mph}$$

which we used in our simulations.

Well, if the average bat speed is only 62 mph and, according to Fig. 3.3, a bat speed of 68 mph is needed for a home run, then how can anyone ever hit a home run? The answer is that 62 mph is an average for a particular batter. All of his swings are not at that speed: some of his swings will be faster and some will be slower. The distribution of the individual swing speeds will follow a curve as in Fig. 3.4. This curve shows that this batter's average bat speed is 62 mph. 34.1% of his swings will be between 62 and 64 mph. 13.6% will be between 64 and 66 mph.

Fig. 3.4 Distribution of bat speeds for an individual batter (The standard deviation was estimated from Watts and Bahill (2000) Fig. 43, Bahill (2004) and unpublished data)



2.2% will be between 66 and 68 mph. Finally, the group we want, 0.1% will be faster than 68 mph, the speed needed for a home run. Thus, for this batter, 0.1% or one in a thousand of his swings would be fast enough to produce a home run, if he launched the ball at an angle of 34° with backspin of 2000 rpm.

A similar analysis could be done for all batters in a group instead of just one batter. The analysis would be the same except that the standard deviation would be larger, as shown in Table 5.1.

3.5 Spin on the Ball

In this section, we will prove that in head-on collisions without friction (e.g. configurations 1a, 1b, 2a, 2b and 2c), for a pitch of any spin, there will be no change in the spin of the ball. First, for such collisions, simple inspection of the figures shows that there are no torques on the ball. Therefore, there should be no changes in the momenta.

Next, let us use the law of Conservation of Angular Momentum about the center of mass of the bat. When the ball contacts the bat, as shown in Fig. 3.5, the ball has linear momentum of $m_{\text{ball}}v_{\text{ball-before}}$. However, the ball does not know if it is translating or if it is tied on a string and rotating about the center of mass of the bat. Therefore, following conventional practice in physics for Conservation of Angular Momentum analyses, we will model the ball as also rotating about the bat's center of mass at a distance $d = d_{\text{cm-ip}}$. In effect, the ball has an initial angular momentum of $m_{\text{ball}}d_{\text{cm-ip}}v_{\text{ball-before}}$ about an axis through the bat's center of mass. In addition, it is possible to throw a curveball so that it spins about the vertical, z-axis, as also shown in Fig. 3.5. We call this a purely horizontal curveball (although it will still drop more due to gravity, than it will curve horizontally). The curveball will have angular momentum of $I_{\text{ball}}\omega_{\text{ball-before}}$ about an axis parallel to the z-axis. However, this is its momentum about *its* center of mass and we want the momentum about the axis through the center of mass of the *bat*. Therefore, we use the parallel axis theorem, producing $(I_{\text{ball}} + m_{\text{ball}}d^2)\omega_{\text{ball-before}}$.

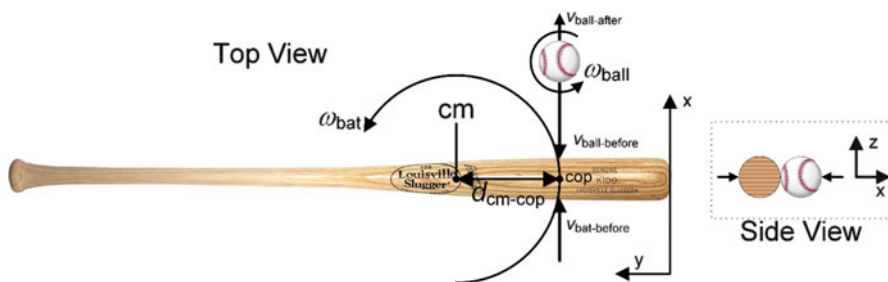


Fig. 3.5 The variables and parameters $v_{\text{ball-before}}$, $v_{\text{bat-before}}$, ω_{ball} , $d_{\text{cm-ip}}$ and ω_{bat} that are used in the Conservation of Angular Momentum equation for a bat-ball collision system

Now, the bat has an initial angular momentum of $I_{\text{bat}}\omega_{\text{bat-before}}$. It also has an angular momentum about the bat's center of mass due to the bat translational momentum $m_{\text{bat}}d v_{\text{bat-before}}$, however, in this case $d=0$ because the center of mass of the bat is passing through its center of mass. L is the symbol used for angular momentum. I guess all the cool letters (like F , m , a , v , I , ω , d , etc.) were already taken, so gray-bearded physicists were stuck with the blah symbol L . Therefore, the initial angular momentum about an axis through the center of mass of the bat is

$$L_{\text{initial}} = m_{\text{ball}} v_{\text{ball-before}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-before}} + I_{\text{bat}} \omega_{\text{bat-before}}$$

All of these momenta are positive, pointing out of the page.

For the angular momentum after the collision, we will treat the ball, as before, as an object rotating around the axis of the center of mass of the bat with angular momentum, $m_{\text{ball}} v_{\text{ball-after}} d_{\text{cm-ip}}$. Now we could treat the bat as a long slender rod with a moment of inertia of $m_{\text{bat}} d_{\text{bat}}^2 / 12$, where d_{bat} is the bat length. However, this is only an approximation and we have actual experimental data for the bat moment of inertia. Therefore, the bat angular momentum is $I_{\text{bat}} \omega_{\text{bat-after}}$. Thus, our final angular momentum about an axis through the center of mass of the bat is

$$L_{\text{final}} = m_{\text{ball}} v_{\text{ball-after}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-after}} + I_{\text{bat}} \omega_{\text{bat-after}}$$

The law of Conservation of Angular Momentum states that when no external torque acts on an object the initial angular momentum about some axis equals the final angular momentum about that axis.

$$\begin{aligned} L_{\text{initial}} &= L_{\text{final}} \\ m_{\text{ball}} v_{\text{ball-before}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-before}} + I_{\text{bat}} \omega_{\text{bat-before}} &= \\ m_{\text{ball}} v_{\text{ball-after}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-after}} + I_{\text{bat}} \omega_{\text{bat-after}} \end{aligned}$$

Newton's second law states that applying an impulsive torque changes the angular momentum about the torque axis. Here the impulsive torque is caused by the change in linear momenta. Therefore,

$$\begin{aligned} dm_{\text{ball}} (v_{\text{ball-after}} - v_{\text{ball-before}}) &= -I_{\text{bat}} (\omega_{\text{bat-after}} - \omega_{\text{bat-before}}) \\ \omega_{\text{bat-after}} &= \left\{ \omega_{\text{bat-before}} - \frac{dm_{\text{ball}}}{I_{\text{bat}}} (v_{\text{ball-after}} - v_{\text{ball-before}}) \right\} \end{aligned}$$

Let us substitute this $\omega_{\text{bat-after}}$ into our Conservation of Angular Momentum equation above.

$$\begin{aligned}
& m_{\text{ball}} v_{\text{ball-before}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-before}} \\
& + I_{\text{bat}} \omega_{\text{bat-before}} = m_{\text{ball}} v_{\text{ball-after}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-after}} \\
& + I_{\text{bat}} \left\{ \omega_{\text{bat-before}} - \frac{dm_{\text{ball}}}{I_{\text{bat}}} (v_{\text{ball-after}} - v_{\text{ball-before}}) \right\}
\end{aligned}$$

We want to solve this for the angular velocity of the ball after the collision, $\omega_{\text{ball-after}}$

$$\begin{aligned}
& (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-after}} = \\
& + m_{\text{ball}} v_{\text{ball-before}} d + (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-before}} + I_{\text{bat}} \omega_{\text{bat-before}} \\
& - m_{\text{ball}} v_{\text{ball-after}} d - I_{\text{bat}} \omega_{\text{bat-before}} + dm_{\text{ball}} (v_{\text{ball-after}} - v_{\text{ball-before}})
\end{aligned}$$

Cancel the terms in color and we get

$$\begin{aligned}
& (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-after}} = (I_{\text{ball}} + m_{\text{ball}} d^2) \omega_{\text{ball-before}} \\
& \boxed{\omega_{\text{ball-after}} = \omega_{\text{ball-before}}}
\end{aligned}$$

We have now proven that for head-on collisions, for a pitch with any spin about the z-axis, the spin of the ball before and after is the same. What about a pitch that has spin about the z-axis and also about the y-axis, like most pitches? The collision will not change ball rotation. As shown above, it will not change the spin about the z-axis. We could write another set of equations for angular momentum about the y-axis. However, the bat has no angular momentum about the y-axis, so there is nothing to affect the ball spin about the y-axis. In conclusion, a head-on collision between a bat and a ball will not change the spin on the ball. Some papers have shown a relationship between the ball spin before and the ball spin after, but they were using oblique collisions as in configuration 3 (Nathan et al. 2012; Kensrud et al. 2017). We have not considered friction in this section. It will not be covered until Sect. 5.5.

3.6 Summary

This chapter presented the equations for a collision at the center of mass of the bat and for a simple collision at the sweet spot. For configurations 1a and 1b, it gave the velocity of the bat and the ball after the collision. For configuration 1b, it also gave the equation for the kinetic energy lost in the collision. It showed how the definition of the coefficient of restitution would change, as our models got more complex. It gave nine common definitions of the sweet spot of the bat. It stated general assumptions that we will use throughout this book. It gave an equation for the velocity of the ball after the collision. Finally, it proved that for head-on collisions without friction $\omega_{\text{ball-after}} = \omega_{\text{ball-before}}$.

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Chapter 4

The BaConLaw Model for Bat-Ball Collisions

4.1 Introduction

Purpose: The purpose of this chapter is to explain bat-ball collisions with a complete, precise, correct set of equations, without jargon. The BaConLaw model describes head-on bat-ball collisions at the sweet spot of the bat. It gives the speed and spin of the bat and ball before and after collisions. It also gives advice for selecting the optimal bat.

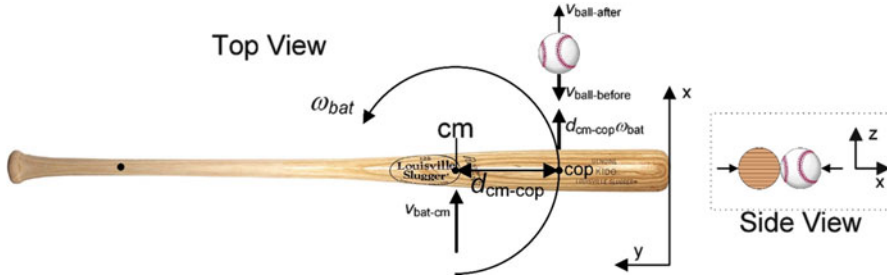
Configuration 2b is our most comprehensive model. It models a collision at the sweet spot of the bat with spin on the pitch. The model for the movement of the bat is a translation and a rotation about its center of mass. To configuration 2a, it adds Conservation of Energy, Conservation of Angular Momentum, KE_{lost} and ball spin. It has five equations and five unknowns, which are shown in Table 4.1. It is named the BaConLaw model because it is based on the Conservation Laws of physics applied to Baseball. This chapter is unique in the science of baseball literature, because no one before has derived the post-collision equations for ball speed, bat speed and bat angular velocity from basic Newtonian principles. It is also unusual in the field of mathematical modeling, because all of the intermediary steps are given. This was done to increase replicability.

One of our assumptions is that the bat-ball collision is a free-end collision. That means that the bat acts as if no one is holding onto its knob. To visualize this, imagine that the bat is laying on a sheet of ice and you are looking down on top of it, as in Fig. 4.1. Then a baseball slams into the bat at 80 mph. This collision produces a translation and a rotation of the bat about its center of mass.

A note on notation. Nothing in this chapter requires the collision be at the sweet spot of the bat. Therefore, in our equations we use the general symbol ‘ip’ to indicate the impact point as in, $d_{\text{cm} - \text{ip}}$ to denote the distance between the center of mass and the impact point. However, in our simulations we require the parameters of a particular bat. Therefore, when presenting the results of our simulations we use ‘cop’ to

Table 4.1 Equations for the BaConLaw model, five equations and five unknowns

Inputs	$v_{\text{ball}} - \text{before}$, $\omega_{\text{ball}} - \text{before}$, $v_{\text{bat}} - \text{cm} - \text{before}$, $\omega_{\text{bat}} - \text{before}$ and CoR
Outputs (unknowns)	$v_{\text{ball}} - \text{after}$, $\omega_{\text{ball}} - \text{after}$, $v_{\text{bat}} - \text{cm} - \text{after}$, $\omega_{\text{bat}} - \text{after}$, and KE_{lost}
Equations	
Conservation of Energy, Eq. (4.3)	$\frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat-before}}^2 =$ $\frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat-after}}^2 + KE_{\text{lost}}$
Conservation of Linear Momentum, Eq. (4.4)	$m_{\text{ball}}v_{\text{ball-before}} + m_{\text{bat}}v_{\text{bat-cm-before}} = m_{\text{ball}}v_{\text{ball-after}} + m_{\text{bat}}v_{\text{bat-cm-after}}$
Definition of CoR , Eq. (4.5)	$CoR_{2b} = -\frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}}\omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}}}$
Newton's Second Law, Eq. (4.6)	$d_{\text{cm-ss}}m_{\text{ball}}(v_{\text{ball-after}} - v_{\text{ball-before}}) = -I_{\text{bat}}(\omega_{\text{bat-after}} - \omega_{\text{bat-before}})$
Conservation of Angular Momentum, Eq. (4.7s)	$L_{\text{ball-before}} + L_{\text{bat-before}} = L_{\text{ball-after}} + L_{\text{bat-after}}$ $m_{\text{ball}}v_{\text{ball-before}}d + (I_{\text{ball}} + m_{\text{ball}}d^2)\omega_{\text{ball-before}} + I_{\text{bat}}\omega_{\text{bat-before}}$ $= m_{\text{ball}}v_{\text{ball-after}}d + (I_{\text{ball}} + m_{\text{ball}}d^2)\omega_{\text{ball-after}} + I_{\text{bat}}\omega_{\text{bat-after}}$

**Fig. 4.1** The BaConLaw model for configuration 2b

denote the center of percussion. For example, the symbol $d_{\text{cm-cop}}$ indicates the distance between the center of mass and the center of percussion.

The BaConLaw model comprises a translation and a rotation of the bat about its center of mass. Because a bat is a rigid object, every spot on a bat will have the same linear translational velocity and the same angular rotational velocity.

$$\omega_{\text{bat-knob}} = \omega_{\text{bat-cm}} = \omega_{\text{bat-ip}} \text{ and}$$

$$v_{\text{bat-knob-trans}} = v_{\text{bat-cm-trans}} = v_{\text{bat-ip-trans}}$$

However, each spot on the bat will have a different *total* velocity that depends on its distance from the pivot point. We will use v_t to indicate total velocity of the bat. The BaConLaw model is described with this equation.

$$vt_{\text{bat-ip}} = v_{\text{bat-cm}} + d_{\text{cm-ip}}\omega_{\text{bat}}$$

The velocity of the sweet spot is given the symbol $vt_{\text{bat-ss}}$ to emphasize that it is the *total* velocity of the sweet spot meaning the vector sum of the linear translational velocity and the angular rotational velocity. If we had measured the velocity of a bat at a particular point and that bat was being swung by a human, then we measured the total of linear velocity and angular rotational velocity. Hence, we measured vt_{cop} or vt_{cm} . In our *equations* we use the linear components, v_{cop} and v_{cm} , but in our *experiments* we actually measure the total velocities, vt_{cop} and vt_{cm} .

4.2 Definition of Variables and Parameters

Inputs $v_{\text{ball-before}}$, $\omega_{\text{ball-before}}$, $v_{\text{bat-cm-before}}$, $\omega_{\text{bat-before}}$ and CoR

$v_{\text{ball-before}}$ is the linear velocity of the *ball* in the x-direction (from home plate to the pitcher's rubber) before the collision.

$\omega_{\text{ball-before}}$ is the angular velocity of the *ball about its center of mass* before the collision.

$v_{\text{bat-cm-before}}$ is the linear velocity of the *center of mass of the bat* in the x-direction before the collision.

$\omega_{\text{bat-before}}$ is the angular velocity of the *bat about its center of mass* before the collision.

CoR_{2b} is the coefficient of restitution for configuration 2b.

Outputs $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, $\omega_{\text{bat-after}}$ and KE_{lost}

$v_{\text{ball-after}}$ is the linear velocity of the *ball* in the x-direction after the collision.

$\omega_{\text{ball-after}}$ is the angular velocity of the *ball about its center of mass* after the collision.

$v_{\text{bat-cm-after}}$ is the linear velocity of the *center of mass* of the bat in the x-direction after the collision.

$\omega_{\text{bat-after}}$ is the angular velocity of the *bat about its center of mass* after the collision.

KE_{lost} is the kinetic energy lost or transformed in the collision.

We want to solve for $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, $\omega_{\text{bat-after}}$ and KE_{lost} .

We will use the following fundamental equations of physics: Conservation of Energy, Conservation of Linear Momentum, the Definition of Kinematic CoR , Newton's Second Principle and the Conservation of Angular Momentum.

4.2.1 Condensing the Notation for the Equations

First, we want to simplify the notation by making the following substitutions. These abbreviations are contained in Table 1.1, but by repeating them here, it makes this chapter independent from the rest of the book.

$$\begin{aligned}
 d_{\text{cm-ip}} &= d \\
 I_{\text{bat}} &= I_2 = I_{\text{cm}} \\
 m_{\text{ball}} &= m_1 \\
 m_{\text{bat}} &= m_2 \\
 v_{\text{ball-before}} &= v_{1b} \\
 v_{\text{ball-after}} &= v_{1a} \\
 v_{\text{bat-cm-before}} &= v_{2b} \\
 v_{\text{bat-cm-after}} &= v_{2a} \\
 \omega_{\text{bat-before}} &= \omega_{2b} \\
 \omega_{\text{bat-after}} &= \omega_{2a}
 \end{aligned}$$

These substitutions produce the following equations

Conservation of Energy

$$\begin{aligned}
 \frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat-before}}^2 &= \\
 \frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}I_{\text{ball}}\omega_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2 + \frac{1}{2}I_{\text{bat}}\omega_{\text{bat-after}}^2 + KE_{\text{lost}} & \quad (4.3) \\
 m_1v_{1b}^2 + m_2v_{2b}^2 + I_2\omega_{2b}^2 &= m_1v_{1a}^2 + m_2v_{2a}^2 + I_2\omega_{2a}^2 + 2KE_{\text{lost}} \quad (4.3s)
 \end{aligned}$$

In the label (4.3s), “s” stands for short.

Conservation of Linear Momentum

Assume that the bat and ball are point masses with all of their mass concentrated at the center of mass.

$$m_{\text{ball}}v_{\text{ball-before}} + m_{\text{bat}}v_{\text{bat-cm-before}} = m_{\text{ball}}v_{\text{ball-after}} + m_{\text{bat}}v_{\text{bat-cm-after}} \quad (4.4)$$

$$m_1v_{1b} + m_2v_{2b} = m_1v_{1a} + m_2v_{2a} \quad (4.4s)$$

Definition of the Coefficient of Restitution (CoR)

$$CoR_{2b} = - \frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}}\omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}}} \quad (4.5)$$

$$CoR_{2b} = - \frac{v_{1a} - v_{2a} - d\omega_{2a}}{v_{1b} - v_{2b} - d\omega_{2b}} \quad (4.5s)$$

Newton’s second principle states that applying an impulsive torque changes the angular momentum about the torque axis. Therefore,

$$d_{\text{cm-ip}} m_{\text{ball}} (v_{\text{ball-after}} - v_{\text{ball-before}}) = -I_{\text{bat}} (\omega_{\text{bat-after}} - \omega_{\text{bat-before}}) \quad (4.6)$$

$$dm_1 (v_{1a} - v_{1b}) = -I_2 (\omega_{2a} - \omega_{2b}) \quad (4.6s)$$

We have ignored the angular velocity of the ball because in Sect. 3.4 we proved that for head-on collisions without friction $\omega_{\text{ball-after}} = \omega_{\text{ball-before}}$.

Conservation of Angular Momentum

The initial and final angular momenta comprise ball translation, ball rotation, bat translation and bat rotation about its center of mass.

$$L_{\text{initial}} = L_{\text{final}}$$

$$\begin{aligned} m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} = \\ + m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} \end{aligned} \quad (4.7s)$$

Summary of simplifications, with units

$$\begin{aligned} A &= \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} & \frac{1}{\text{kg}^2 \text{m} \cdot \text{s}} \\ B &= (v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b}) & \frac{\text{m}}{\text{s}} \\ C &= v_{1b} - v_{2b} - d\omega_{2b} & \frac{\text{m}}{\text{s}} \\ D &= \frac{m_1 d^2}{I_2} & \text{unitless} \\ G &= v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b}) & \text{kg}^2 \text{m}^3 / \text{s} \\ G &= (v_{2b} + \omega_{2b} d)(1 + CoR_{2b}) m_2 I_2 & \text{kg}^2 \text{m}^3 / \text{s} \\ K &= (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) & \text{kg}^2 \text{m}^2 \\ \bar{m} &= \frac{m_1 m_2}{m_1 + m_2} & \text{kg} \end{aligned}$$

Note that none of these simplifications contains the outputs $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, $\omega_{\text{bat-after}}$ and KE_{lost} . The most useful simplifications are the ones that are constants, independent of velocities after the collision. Using these simplifications allows us to print these long equations in a book. These simplifications are only used during the derivations. They are removed from the output equations. We will now use the Newtonian principles in Eqs. (4.4), (4.5) and (4.6) to find $v_{\text{ball-after}}$, $v_{\text{bat-cm-after}}$, and $\omega_{\text{bat-after}}$.

4.3 Finding Ball Velocity After the Collision

First, we will solve for the velocity of the ball after the collision, $v_{\text{ball-after}}$.

Start with Eq. (4.6) and solve for the angular velocity of the bat after the collision, ω_{2a}

$$dm_1(v_{1a} - v_{1b}) = -I_2(\omega_{2a} - \omega_{2b})$$

$$\boxed{\omega_{2a} = \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})}$$

This equation was derived from Eq. (4.6). We will use it repeatedly. Next, we use Eq. (4.5) and solve for the velocity of the bat after the collision, v_{2a}

$$\begin{aligned} CoR_{2b} &= -\frac{v_{1a} - v_{2a} - d\omega_{2a}}{v_{1b} - v_{2b} - d\omega_{2b}} \\ CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) &= -v_{1a} + v_{2a} + d\omega_{2a} \\ v_{2a} &= v_{1a} + CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2a} \end{aligned}$$

This equation was derived from Eq. (4.5). We will use this expression repeatedly. Next, substitute ω_{2a} into this v_{2a} equation. We put the substitution in squiggly braces $\{ \}$ to make it obvious what has been inserted.

$$v_{2a} = v_{1a} + CoR_{2b} (v_{1b} - v_{2b} - d\omega_{2b}) - d\left\{\omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})\right\}$$

$$\text{Let } D = \frac{m_1 d^2}{I_2} \text{ and } C = \{v_{1b} - v_{2b} - d\omega_{2b}\}$$

$$\begin{aligned} v_{2a} &= v_{1a} + \{D\}(v_{1a} - v_{1b}) + CoR_{2b}\{C\} - d\omega_{2b} \\ v_{2a} &= v_{1a}(1 + D) - v_{1b}D + CoR_{2b}C - d\omega_{2b} \end{aligned}$$

Prepare to substitute this v_{2a} into Eq. (4.4) by multiplying by the mass of the bat, m_2

$$m_2 v_{2a} = \{m_2 v_{1a}(1 + D) - m_2 D v_{1b} + m_2 CoR_{2b} C - m_2 d\omega_{2b}\}$$

Now substitute this $m_2 v_{2a}$ into Eq. (4.4)

$$\begin{aligned} m_1 v_{1b} + m_2 v_{2b} &= m_1 v_{1a} + m_2 v_{2a} \\ m_1 v_{1b} + m_2 v_{2b} &= m_1 v_{1a} + \{m_2 v_{1a}(1 + D) - m_2 D v_{1b} + m_2 CoR_{2b} C - m_2 d\omega_{2b}\} \end{aligned} \tag{4.4}$$

Put all v_{1a} terms on the left.

$$m_1 v_{1a} + m_2 v_{1a}(1 + D) = m_1 v_{1b} + m_2 v_{2b} + m_2 D v_{1b} - m_2 CoR_{2b} C + m_2 d\omega_{2b}$$

Replace the dummy variables C and D and we get

$$m_1 v_{1a} + m_2 v_{1a} \left\{ 1 + \frac{m_1 d^2}{I_2} \right\} = m_1 v_{1b} + m_2 v_{2b} + m_2 \left\{ \frac{m_1 d^2}{I_2} \right\} v_{1b} - m_2 CoR_{2b} \{v_{1b} - v_{2b} - d\omega_{2b}\} + m_2 d\omega_{2b}$$

grouping with respect to v_{1a} , v_{1b} ,

v_{2b} and ω_{2b} yields

$$v_{1a} \left[m_1 + m_2 + \frac{m_1 m_2 d^2}{I_2} \right] = v_{1b} \left[m_1 + \frac{m_1 m_2 d^2}{I_2} - m_2 CoR_{2b} \right] + v_{2b} m_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d (1 + CoR_{2b})$$

Multiply by the moment of inertia of the bat, I_2 .

$$v_{1a} [m_1 I_2 + m_2 I_2 + m_1 m_2 d^2] = v_{1b} [m_1 I_2 + m_1 m_2 d^2 - m_2 CoR_{2b} I_2] + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})$$

Rearrange

$$v_{1a} = \frac{v_{1b} (m_1 I_2 - m_2 I_2 CoR_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + d\omega_{2b} m_2 I_2 (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

Expanding the abbreviations gives

$$\begin{aligned} v_{\text{ball-after}} = v_{\text{ball-before}} & \frac{(m_{\text{ball}} I_{\text{bat}} - m_{\text{bat}} I_{\text{bat}} CoR_{2b} + m_{\text{ball}} m_{\text{bat}} d^2)}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2} \\ & + v_{\text{bat-cm-before}} \frac{m_{\text{bat}} I_{\text{bat}} (1 + CoR_{2b})}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2} \\ & + d\omega_{\text{bat-before}} \frac{m_{\text{bat}} I_{\text{bat}} (1 + CoR_{2b})}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2} \end{aligned} \quad (4.8)$$

This equation was derived from Eqs. (4.4), (4.5) and (4.6).

Now we want to rearrange this normal form equation into its canonical form.

Let $K = (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)$

$$G = v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})$$

$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2 CoR_{2b} + m_1 m_2 d^2)}{K} + \frac{G}{K}$$

add $\left(v_{1b} - \frac{v_{1b}K}{K}\right)$ to the right side

$$v_{1a} = \{v_{1b}\} + \frac{v_{1b}(m_1 I_2 - m_2 I_2 CoR_{2b} + m_1 m_2 d^2) \{-v_{1b}(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)\}}{K} + \frac{G}{K}$$

Simplify

$$v_{1a} = v_{1b} + \frac{v_{1b}(m_1 I_2 - m_2 I_2 CoR_{2b} + m_1 m_2 d^2 - m_1 I_2 - m_2 I_2 - m_1 m_2 d^2)}{K} + \frac{G}{K}$$

$$v_{1a} = v_{1b} + \frac{v_{1b}(-m_2 I_2 - m_2 I CoR_{2b})}{K} + \frac{G}{K}$$

$$v_{1a} = v_{1b} + \frac{-v_{1b} m_2 I_2 (1 + CoR_{2b}) + G}{K}$$

$$v_{1a} = v_{1b} + \frac{-v_{1b} m_2 I_2 (1 + CoR_{2b}) + v_{2b} m_2 I_2 (1 + CoR_{2b}) + \omega_{2b} m_2 d I_2 (1 + CoR_{2b})}{K}$$

Finally, we get the canonical form for the linear velocity of the ball after the collision:

$$\boxed{v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_2 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}} \quad (4.8c)$$

This equation was derived from Eqs. (4.4), (4.5) and (4.6). Expanding the abbreviations gives

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d)(1 + CoR_{2b})m_{\text{bat}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}$$

Please note that nothing in this section required the collision to be at the sweet spot of the bat. In these equations $d_{\text{cm-ip}}$ could be replaced with any positive distance to the point of impact. That is why we usually used the letter d without a subscript.

If the collision is at the center of mass of the bat instead of at the sweet spot, then $d = d_{\text{cm-ip}} = 0$. Now we replace CoR_{2b} with CoR_{1a} and the above equation reduces to

$$v_{1a} = v_{1b} + \frac{(v_{2b} - v_{1b})(1 + CoR_{1a})m_2 I_2}{m_1 I_2 + m_2 I_2}$$

cancelling I_2 yields

$$v_{1a} = v_{1b} + \frac{(v_{2b} - v_{1b})(1 + CoR_{1a})m_2}{m_1 + m_2}$$

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}})(1 + CoR_{1a})m_{\text{bat}}}{m_{\text{ball}} + m_{\text{bat}}}$$

We derived this equation previously in the section entitled “Collisions at the center of mass, Configuration 1a.” The subscripts of CoR refer to the collision configuration names not to the ball and bat before and after.

4.4 Finding Bat Velocity After the Collision

As before, we start with Eq. (4.6) and solve for the angular velocity of the bat after the collision, ω_{2a}

$$dm_1(v_{1a} - v_{1b}) = -I_2(\omega_{2a} - \omega_{2b})$$

$$\omega_{2a} = \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})$$

We will use this expression repeatedly. Next use Eq. (4.5) and solve for the velocity of the bat after the collision, v_{2a}

$$CoR_{2b} = -\frac{v_{1a} - v_{2a} - d\omega_{2a}}{v_{1b} - v_{2b} - d\omega_{2b}}$$

$$CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) = -v_{1a} + v_{2a} + d\omega_{2a}$$

$$v_{2a} = v_{1a} + CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2a}$$

Substitute ω_{2a} into this v_{2a} equation. I put the substitution in squiggly braces { } to make it obvious what has been inserted.

$$v_{2a} = v_{1a} + CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\left\{\omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})\right\}$$

So far, this derivation is identical to that in the previous section.

Now, let $C = v_{1b} - v_{2b} - d\omega_{2b}$

$$v_{2a} = v_{1a} + \frac{m_1 d^2}{I_2}(v_{1a} - v_{1b}) + CoR_{2b}\{C\} - \omega_{2b}d$$

Eq. (4.8) derived in the previous section is

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2}$$

As before, let $K = (m_1I_2 + m_2I_2 + m_1m_2d^2)$

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_2I_2}{K}$$

Let $B = (v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})$

$$v_{1a} = \left\{ v_{1b} - \frac{Bm_2I_2}{K} \right\}$$

Put this into both places for v_{1a} in the v_{2a} equation above.

$$\begin{aligned} v_{2a} = & \left\{ v_{1b} - \frac{Bm_2I_2}{K} \right\} \\ & + \frac{m_1d^2}{I_2} \left(\left\{ v_{1b} - \frac{Bm_2I_2}{K} \right\} - v_{1b} \right) \\ & + CoR_{2b} C - \omega_{2b}d \end{aligned}$$

Now multiply by K

$$\begin{aligned} v_{2a}K = & \left\{ v_{1b}K - \frac{Bm_2I_2}{K}K \right\} \\ & + \frac{m_1d^2}{I_2} \left[v_{1b}K - K \frac{Bm_2I_2}{K} - v_{1b}K \right] \\ & + CoR_{2b} CK - \omega_{2b}dK \end{aligned}$$

$$\begin{aligned} v_{2a}K = & v_{1b}K - Bm_2I_2 \\ & + \frac{m_1d^2}{I_2} [v_{1b}K - Bm_2I_2 - v_{1b}K] \\ & + CoR_{2b} CK - \omega_{2b}dK \end{aligned}$$

Cancel the terms in color

$$\begin{aligned} v_{2a}K = & v_{1b}K - Bm_2I_2 \\ & + \frac{m_1d^2}{I_2} [-Bm_2I_2] \\ & + CCoR_{2b} K - \omega_{2b}dK \end{aligned}$$

Substitute $B = (v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})$

$$\begin{aligned} v_{2a}K = & v_{1b}K - \{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})\}m_2I_2 \\ & - \frac{m_1d^2}{I_2} [\{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})\}m_2I_2] \\ & + CCoR_{2b} K - \omega_{2b}dK \end{aligned}$$

Let us write this as three variables v_{1b} , v_{2b} , and $d\omega_{2b}$ with their associated coefficients.

$$\begin{aligned}
v_{2a}K = & v_{1b}K - v_{1b}m_2I_2(1 + CoR_{2b}) + v_{2b}m_2I_2(1 + CoR_{2b}) \\
& + \omega_{2b}m_2dI_2(1 + CoR_{2b}) - v_{1b}m_1m_2d^2(1 + CoR_{2b}) \\
& + v_{2b}m_1m_2d^2(1 + CoR_{2b}) + \omega_{2b}m_1m_2d^3(1 + CoR_{2b}) \\
& + v_{1b}CoR_{2b} K - v_{2b}CoR_{2b} K - \omega_{2b}dK(1 + CoR)
\end{aligned}$$

Rearrange

$$\begin{aligned}
v_{2a}K = & v_{1b}K - v_{1b}m_2I_2(1 + CoR_{2b}) - v_{1b}m_1m_2d^2(1 + CoR_{2b}) + v_{1b}CoR_{2b} K \\
& + v_{2b}m_2I_2(1 + CoR_{2b}) + v_{2b}m_1m_2d^2(1 + CoR_{2b}) - v_{2b}CoR_{2b} K \\
& + \omega_{2b}m_2dI_2(1 + CoR_{2b}) + \omega_{2b}m_1m_2d^3(1 + CoR_{2b}) - \omega_{2b}dK(1 + CoR)
\end{aligned}$$

Now let us break up the $(1 + CoR_{2b})$ terms.

$$\begin{aligned}
v_{2a}K = & v_{1b}K - v_{1b}m_2I_2 - v_{1b}m_2I_2CoR_{2b} - v_{1b}m_1m_2d^2 \\
& - v_{1b}m_1m_2d^2CoR_{2b} + v_{1b}CoR_{2b} K + v_{2b}m_2I_2 \\
& + v_{2b}m_2I_2CoR_{2b} + v_{2b}m_1m_2d^2 + v_{2b}m_1m_2d^2CoR_{2b} \\
& - v_{2b}CoR_{2b}K + \omega_{2b}m_2dI_2 + \omega_{2b}m_2dI_2CoR_{2b} \\
& + \omega_{2b}m_1m_2d^3 + \omega_{2b}m_1m_2d^3CoR_{2b} - \omega_{2b}dK \\
& - \omega_{2b}dKCoR_{2b}
\end{aligned}$$

Are any of these terms the same? No. OK, now let's substitute

$$K = (m_1I_2 + m_2I_2 + m_1m_2d^2)$$

and hope for cancellations.

$$\begin{aligned}
v_{2a}K = & v_{1b}(m_1I_2 + m_2I_2 + m_1m_2d^2) - v_{1b}m_2I_2 - v_{1b}m_2I_2CoR_{2b} \\
& - v_{1b}m_1m_2d^2 - v_{1b}m_1m_2d^2CoR_{2b} + v_{1b}CoR_{2b}(m_1I_2 + m_2I_2 + m_1m_2d^2) \\
& + v_{2b}m_2I_2 + v_{2b}m_2I_2CoR_{2b} + v_{2b}m_1m_2d^2 + v_{2b}m_1m_2d^2CoR_{2b} \\
& - v_{2b}CoR_{2b}(m_1I_2 + m_2I_2 + m_1m_2d^2) \\
& + \omega_{2b}m_2dI_2 + \omega_{2b}m_2dI_2CoR_{2b} + \omega_{2b}m_1m_2d^3 + \omega_{2b}m_1m_2d^3CoR_{2b} - \omega_{2b}(m_1dI_2 + m_2dI_2 + m_1m_2d^3) \\
& - \omega_{2b}(m_1dI_2 + m_2dI_2 + m_1m_2d^3)CoR_{2b}
\end{aligned}$$

The terms in color cancel, leaving

$$\begin{aligned}
v_{2a}K = & v_{1b}m_1I_2(1 + CoR_{2b}) \\
& + v_{2b}(-m_1I_2CoR_{2b} + m_2I_2 + m_1m_2d^2) \\
& - \omega_{2b}m_1dI_2(1 + CoR_{2b})
\end{aligned}$$

Continuing this simplification

distribute the second term and add $-v_{2b}m_1I_2 + v_{2b}m_1I_2$

$$\begin{aligned}
 v_{2a}K &= +v_{1b}m_1I_2(1 + CoR_{2b}) - v_{2b}m_1I_2CoR_{2b} \{-v_{2b}m_1I_2 + v_{2b}m_1I_2\} + v_{2b}m_2I_2 + v_{2b}m_1m_2d^2 - \omega_{2b}m_1dI_2(1 + CoR_{2b}) \\
 &= (v_{1b} - v_{2b})m_1I_2(1 + CoR_{2b}) + v_{2b}m_1I_2 + v_{2b}m_2I_2 + v_{2b}m_1m_2d^2 - \omega_{2b}m_1dI_2(1 + CoR_{2b}) \\
 &= (v_{1b} - v_{2b})m_1I_2(1 + CoR_{2b}) + v_{2b}K - \omega_{2b}m_1dI_2(1 + CoR_{2b}) \\
 &= v_{2b}K + (v_{1b} - v_{2b})m_1I_2(1 + CoR_{2b}) - \omega_{2b}m_1dI_2(1 + CoR_{2b})
 \end{aligned}$$

Finally divide by K to get the velocity of the bat after the collision in canonical form.

$$v_{2a} = v_{2b} + \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1I_2}{(m_1I_2 + m_2I_2 + m_1m_2d^2)}$$

This equation was derived from Eqs. (4.4), (4.5), (4.6) and (4.8). Expanding our abbreviations, we get

$$\begin{aligned}
 v_{\text{bat-cm-after}} &= v_{\text{bat-cm-before}} \\
 &+ \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d)(1 + CoR_{2b})m_{\text{ball}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
 \end{aligned} \tag{4.9}$$

We can change this into our normal form by first combining the two terms over one common denominator.

$$\begin{aligned}
 v_{2a} &= v_{2b} \frac{(m_1I_2 + m_2I_2 + m_1m_2d^2)}{(m_1I_2 + m_2I_2 + m_1m_2d^2)} + \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1I_2}{(m_1I_2 + m_2I_2 + m_1m_2d^2)} \\
 &= \frac{v_{2b}(m_1I_2 + m_2I_2 + m_1m_2d^2) + (v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1I_2}{(m_1I_2 + m_2I_2 + m_1m_2d^2)}
 \end{aligned}$$

and then simplifying

$$v_{2a} = \frac{v_{2b}(-m_1I_2CoR_{2b} + m_2I_2 + m_1m_2d^2) + v_{1b}m_1I_2(1 + CoR_{2b}) - \omega_{2b}m_1dI_2(1 + CoR_{2b})}{(m_1I_2 + m_2I_2 + m_1m_2d^2)}$$

Expanding our abbreviations, we get

$$\begin{aligned}
 v_{\text{bat-cm-after}} &= v_{\text{ball-before}} \frac{m_{\text{ball}}I_{\text{bat}}(1 + CoR_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
 &+ v_{\text{bat-cm-before}} \frac{(-m_{\text{ball}}I_{\text{bat}}CoR_{2b} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
 &- d\omega_{\text{bat-before}} \frac{m_{\text{ball}}I_{\text{bat}}(1 + CoR_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
 \end{aligned}$$

4.5 Alternative Derivation of Bat Velocity After the Collision

This time, let us start with the normal form for Eq. (4.8).

$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2 \text{ CoR}_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + \text{CoR}_{2b}) + \omega_{2b} m_2 d I_2 (1 + \text{CoR}_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

Let

$$C = (v_{1b} - v_{2b} - d\omega_{2b})$$

$$D = \frac{m_1 d^2}{I_2} K = (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2)$$

$$Q = (m_1 I_2 - m_2 I_2 \text{ CoR}_{2b} + m_1 m_2 d^2)$$

$$R = m_2 I_2 (1 + \text{CoR}_{2b})$$

$$S = m_2 d I_2 (1 + \text{CoR}_{2b})$$

making these substitutions yields

$$v_{1a} = \frac{v_{1b}Q + v_{2b}R + \omega_{2b}S}{K}$$

In the previous section we used Eq. (4.5) and solved for the velocity of the bat after the collision, v_{2a}

$$v_{2a} = v_{1a} + \text{CoR}_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2a}$$

Now we need to get rid of anything with a subscript of *after*, like ω_{2a} . Therefore, take Eq. (4.6) and solve for the angular velocity of the bat after the collision, ω_{2a} .

$$\omega_{2a} = \left\{ \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b}) \right\}$$

Now, substitute this into the above v_{2a} equation to get

$$v_{2a} = v_{1a} + \text{CoR}_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d \left\{ \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b}) \right\}$$

$$v_{2a} = v_{1a}(1 + D) - Dv_{1b} + \text{CoR}_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2b}$$

Substitute v_{1a} into this v_{2a} equation

$$v_{2a} = \left\{ \frac{v_{1b}Q + v_{2b}R + \omega_{2b}S}{K} \right\} (1 + D) - Dv_{1b} + \text{CoR}_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - \omega_{2b}d$$

$$\begin{aligned}
v_{2a} &= \frac{v_{1b}Q + v_{2b}R + \omega_{2b}S}{K} + \frac{v_{1b}QD + v_{2b}RD + \omega_{2b}SD}{K} \\
&\quad - Dv_{1b} + CoR_{2b} (v_{1b} - v_{2b} - d\omega_{2b}) - \omega_{2b}d \\
v_{2a}K &= v_{1b}Q + v_{2b}R + \omega_{2b}S + v_{1b}QD + v_{2b}RD + \omega_{2b}SD \\
&\quad - Dv_{1b}K + v_{1b}CoR_{2b} K - v_{2b}CoR_{2b} K - \omega_{2b}dK(1 + CoR_{2b})
\end{aligned}$$

Collect similar terms.

$$\begin{aligned}
v_{2a}K &= v_{1b}Q + v_{1b}QD - Dv_{1b}K + v_{1b}CoR_{2b} K + v_{2b}R + v_{2b}RD - v_{2b}CoR_{2b} K \\
&\quad + \omega_{2b}S + \omega_{2b}SD - \omega_{2b}dK(1 + CoR_{2b})
\end{aligned}$$

Now replace D, K and Q in the v_{1b} term.

$$\begin{aligned}
v_{2a}K &= +v_{1b} \left[\begin{aligned} &+ (m_1I_2 - m_2I_2 CoR_{2b} + m_1m_2d^2) \\ &+ \frac{m_1d^2}{I_2} (m_1I_2 - m_2I_2 CoR_{2b} + m_1m_2d^2) \\ &- \frac{m_1d^2}{I_2} (m_1I_2 + m_2I_2 + m_1m_2d^2) \\ &+ CoR_{2b} (m_1I_2 + m_2I_2 + m_1m_2d^2) \end{aligned} \right] \\
&\quad + v_{2b}[R + RD - CoR_{2b} K] + \omega_{2b}\{S + SD - dK(1 + CoR_{2b})\}
\end{aligned}$$

Simplify

$$\begin{aligned}
v_{2a}K &= +v_{1b}m_1I_2(1 + CoR_{2b}) \\
&\quad + v_{2b}[R + RD - CoR_{2b} K] \\
&\quad + \omega_{2b}(S + SD - dK(1 + CoR_{2b}))
\end{aligned}$$

Now replace D, K, R and S.

$$\begin{aligned}
v_{2a}K &= +v_{1b}m_1I_2(1 + CoR_{2b}) \\
&\quad + v_{2b} \left[\begin{aligned} &+ m_2I_2(1 + CoR_{2b}) + m_2I_2(1 + CoR_{2b})\frac{m_1d^2}{I_2} - CoR_{2b}(m_1I_2 + m_2I_2 + m_1m_2d^2) \end{aligned} \right] \\
&\quad + \omega_{2b} \left(\begin{aligned} &m_2dI_2(1 + CoR_{2b}) + m_2dI_2(1 + CoR_{2b})\frac{m_1d^2}{I_2} - (m_1dI_2 + m_2dI_2 + m_1m_2d^3)(1 + CoR_{2b}) \end{aligned} \right)
\end{aligned}$$

The terms in color cancel.

$$\begin{aligned}
 v_{2a}K = & +v_{1b} [m_1 I_2 (1 + CoR_{2b})] \\
 & + v_{2b} \left[+m_2 I_2 + m_1 m_2 d^2 + m_1 m_2 d^2 CoR_{2b} - CoR_{2b} (m_1 I_2 + m_1 m_2 d^2) \right] \\
 & + \omega_{2b} (1 + CoR_{2b}) \left(m_2 d I_2 + m_2 d I_2 \frac{m_1 d^2}{I_2} - (m_1 d I_2 + m_2 d I_2 + m_1 m_2 d^3) \right)
 \end{aligned}$$

And now these terms in color cancel.

$$\begin{aligned}
 v_{2a}K = & +v_{1b} [m_1 I_2 (1 + CoR_{2b})] \\
 & + v_{2b} [+m_2 I_2 + m_1 m_2 d^2 - m_1 I_2 CoR_{2b}] \\
 & - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})
 \end{aligned}$$

Simplify

$$\begin{aligned}
 v_{2a}K = & +v_{2b} [-m_1 I_2 CoR_{2b} + m_2 I_2 + m_1 m_2 d^2] \\
 & + v_{1b} m_1 I_2 (1 + CoR_{2b}) \\
 & - \omega_{2b} m_1 d I_2 (1 + CoR_{2b})
 \end{aligned}$$

Expanding our abbreviations gives

$$\begin{aligned}
 v_{\text{bat-cm-after}} = & v_{\text{ball-before}} \frac{m_{\text{ball}} I_{\text{bat}} (1 + CoR_{2b})}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2} \\
 & + v_{\text{bat-cm-before}} \frac{(-m_{\text{ball}} I_{\text{bat}} CoR_{2b} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2)}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2} \\
 & - d \omega_{\text{bat-before}} \frac{m_{\text{ball}} I_{\text{bat}} (1 + CoR_{2b})}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d^2}
 \end{aligned}$$

This is the same equation that we derived before.

4.6 Finding Bat Angular Velocity After the Collision

Now we want to find ω_{2a} (the angular velocity of the bat after the collision) in terms of the input variables and parameters. The following equation gives the velocity of the ball after the collision, v_{1a} from the canonical form of Eq. (4.8).

$$v_{1a} = \left\{ v_{1b} - \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\}$$

From Eq. (4.6) we solve for the angular velocity of the bat after the collision, ω_{2a}

$$\omega_{2a} = \omega_{2b} - \frac{m_1 d}{I_2} (v_{1a} - v_{1b})$$

Substitute v_{1a} into this ω_{2a} equation

$$\omega_{2a} = \omega_{2b} - \frac{m_1 d}{I_2} \left\{ v_{1b} - \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right\} + \frac{m_1 d}{I_2} v_{1b}$$

cancel the terms in red

$$\omega_{2a} = \omega_{2b} + \frac{m_1 d}{I_2} \left[\frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \right]$$

and finally we get

$$\boxed{\omega_{2a} = \omega_{2b} + \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}} \quad (4.10)$$

This equation was derived from Eqs. (4.6) and (4.8). We can change this into our normal form by first combining the two terms over one common denominator.

$$\begin{aligned} \omega_{2a} &= \omega_{2b} \frac{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} + \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})m_1 I_2}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \\ &= \frac{\omega_{2b} (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) + (v_{1b} - v_{2b})m_1 m_2 d (1 + CoR) - m_1 m_2 d^2 \omega_{2b} (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2} \end{aligned}$$

Cancel duplicate terms and we get the normal form

$$\boxed{\omega_{2a} = \frac{\omega_{2b} (m_1 I_2 + m_2 I_2 - m_1 m_2 d^2 CoR_{2b}) + (v_{1b} - v_{2b})m_1 m_2 d (1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}}$$

4.7 Three Output Equations in Three Formats

We will now summarize by giving equations for $v_{ball} - \text{after}$, $v_{bat} - \text{cm} - \text{after}$ and $\omega_{bat} - \text{after}$ in all three formats. First, we give the equation for the velocity of the ball after the collision in normal form

$$\begin{aligned}
v_{\text{ball-after}} = v_{\text{ball-before}} & \frac{(m_{\text{ball}}I_{\text{bat}} - m_{\text{bat}}I_{\text{bat}} \text{CoR}_{2b} + m_{\text{ball}}m_{\text{bat}}d^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& + v_{\text{bat-cm-before}} \frac{m_{\text{bat}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& + d\omega_{\text{bat-before}} \frac{m_{\text{bat}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
\end{aligned}$$

in canonical form

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d\omega_{\text{bat-before}})(1 + \text{CoR}_{2b})m_{\text{bat}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}$$

and in reduced canonical form

$$\begin{aligned}
\text{Let } A &= \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d\omega_{\text{bat-before}})(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
v_{\text{ball-after}} &= v_{\text{ball-before}} - Am_{\text{bat}}I_{\text{bat}}
\end{aligned}$$

Now, we give the equation for the linear velocity of the bat after the collision in normal form

$$\begin{aligned}
v_{\text{bat-after}} = v_{\text{ball-before}} & \frac{m_{\text{ball}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& + v_{\text{bat-cm-before}} \frac{(-m_{\text{ball}}I_{\text{bat}}\text{CoR}_{2b} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& - d\omega_{\text{bat-before}} \frac{m_{\text{ball}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
\end{aligned}$$

in canonical form

$$v_{\text{bat-after}} = v_{\text{bat-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d\omega_{\text{bat-before}})(1 + \text{CoR}_{2b})m_{\text{ball}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}$$

and in reduced canonical form

$$v_{\text{bat-after}} = v_{\text{bat-before}} + Am_{\text{ball}}I_{\text{bat}}$$

Finally, we give the equation for the angular velocity of the bat after the collision in normal form

$$\begin{aligned}
\omega_{\text{bat-after}} = & v_{\text{ball-before}} \frac{m_{\text{ball}}m_{\text{bat}}d(1 + CoR_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& - v_{\text{bat-cm-before}} \frac{m_{\text{ball}}m_{\text{bat}}d(1 + CoR_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& + \omega_{\text{bat-before}} \frac{(m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} - m_{\text{ball}}m_{\text{bat}}d^2CoR_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
\end{aligned}$$

in canonical form

$$\begin{aligned}
\omega_{\text{bat-after}} = & \omega_{\text{bat-before}} \\
& + \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d\omega_{\text{bat-before}})(1 + CoR_{2b})m_{\text{ball}}m_{\text{bat}}d}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
\end{aligned}$$

and in reduced canonical form

$$\omega_{\text{bat-after}} = \omega_{\text{bat-before}} + Am_{\text{ball}}m_{\text{bat}}d$$

We now want to add the equation for Conservation of Energy, Eq. (4.3).

4.8 Adding Conservation of Energy and Finding KE_{lost}

This approach, of adding Conservation of Energy to the set of bat-ball collision equations, is unique in the science of baseball literature. From configuration 1b, we had that before the collision there is kinetic energy in the ball and kinetic energy in the bat.

$$KE_{\text{before}} = \frac{1}{2}m_{\text{ball}}v_{\text{ball-before}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-before}}^2$$

And after the collision, there is also kinetic energy in the ball and bat system.

$$KE_{\text{after}} = \frac{1}{2}m_{\text{ball}}v_{\text{ball-after}}^2 + \frac{1}{2}m_{\text{bat}}v_{\text{bat-cm-after}}^2$$

However, they are not equal. In bat-ball collisions, some kinetic energy is transformed into heat, vibrations and deformations. This is called the kinetic energy lost or transformed. It is modeled with the CoR.

$$KE_{\text{lost}} = KE_{\text{before}} - KE_{\text{after}}$$

In the configuration 1b section, we stated that

$$KE_{\text{lost-config-1b}} = \frac{\bar{m}}{2} (\text{collision velocity})^2 (1 - CoR_{1b}^2)$$

$$\text{where } \bar{m} = \frac{m_{\text{ball}} m_{\text{bat}}}{m_{\text{ball}} + m_{\text{bat}}}.$$

$$KE_{\text{lost-config-1b}} = \frac{\bar{m}}{2} (v_{\text{ball-before}} - v_{\text{bat-cm-before}})^2 (1 - CoR_{1b}^2)$$

This is Eq. (3.2).

However, this equation for kinetic energy lost is not valid for the BaConLaw model because we now also have angular kinetic energy in the rotation of the bat. There are no springs in the system and the bat swing is level, therefore there is no change in potential energy. Before the collision, there is kinetic energy in the bat created by rotation of the batter's body and arms plus the translational kinetic energy of the ball.

$$KE_{\text{before}} = \frac{1}{2} m_{\text{ball}} v_{\text{ball-before}}^2 + \frac{1}{2} m_{\text{bat}} v_{\text{bat-cm-before}}^2 + \frac{1}{2} I_{\text{ball}} \omega_{\text{ball-before}}^2 + \frac{1}{2} I_{\text{bat}} \omega_{\text{bat-before}}^2$$

As always, ω means rotation about the center of mass of the object. The collision will make the bat spin about its center of mass. If the collision is at the Center of Percussion for the pivot point, then it will produce a rotation about the center of mass, but no translation.

$$KE_{\text{after}} = \frac{1}{2} m_{\text{ball}} v_{\text{ball-after}}^2 + \frac{1}{2} m_{\text{bat}} v_{\text{bat-cm-after}}^2 + \frac{1}{2} I_{\text{ball}} \omega_{\text{ball-after}}^2 + \frac{1}{2} I_{\text{bat}} \omega_{\text{bat-after}}^2$$

$$KE_{\text{before}} = KE_{\text{after}} + KE_{\text{lost}}$$

$$\begin{aligned} & \frac{1}{2} m_{\text{ball}} v_{\text{ball-before}}^2 + \frac{1}{2} m_{\text{bat}} v_{\text{bat-cm-before}}^2 + \frac{1}{2} I_{\text{ball}} \omega_{\text{ball-before}}^2 + \frac{1}{2} I_{\text{bat}} \omega_{\text{bat-before}}^2 \\ &= \frac{1}{2} m_{\text{ball}} v_{\text{ball-after}}^2 + \frac{1}{2} m_{\text{bat}} v_{\text{bat-cm-after}}^2 + \frac{1}{2} I_{\text{ball}} \omega_{\text{ball-after}}^2 + \frac{1}{2} I_{\text{bat}} \omega_{\text{bat-after}}^2 + KE_{\text{lost}} \end{aligned}$$

In our reduced notation

$$\begin{aligned} \frac{1}{2} m_1 v_{1b}^2 + \frac{1}{2} m_2 v_{2b}^2 + \frac{1}{2} I_1 \omega_{1b}^2 + \frac{1}{2} I_2 \omega_{2b}^2 &= \frac{1}{2} m_1 v_{1a}^2 \\ &+ \frac{1}{2} m_2 v_{2a}^2 + \frac{1}{2} I_1 \omega_{1a}^2 + \frac{1}{2} I_2 \omega_{2a}^2 + KE_{\text{lost}} \end{aligned}$$

The KE_{before} and the KE_{after} are easy to find. It is the KE_{lost} that is hard to find.

In Sect. 3.4, we proved that for head-on collisions without friction $\omega_{\text{ball-before}} = \omega_{\text{ball-after}}$. Therefore, the ball spin terms in these Conservation of Energy equations cancel resulting in

$$0 = m_1 v_{1b}^2 + m_2 v_{2b}^2 + I_2 \omega_{2b}^2 - m_1 v_{1a}^2 - m_2 v_{2a}^2 - I_2 \omega_{2a}^2 - 2KE_{\text{lost}}$$

From before, we have

$$A = \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

$$v_{1a} = v_{1b} - Am_2 I_2$$

$$v_{2a} = v_{2b} + Am_1 I_2$$

$$\omega_{2a} = \omega_{2b} + Am_1 m_2 d$$

Substituting A , the linear velocity of the ball after the collision, v_{1a} , the linear velocity of the bat after the collision, v_{2a} and the angular velocity of the bat after the collision, ω_{2a} into the new Conservation of Energy equation yields

$$2KE_{\text{lost}} = \left\{ \begin{aligned} &m_1 v_{1b}^2 + m_2 v_{2b}^2 + I_2 \omega_{2b}^2 - m_1 (v_{1b} - Am_2 I_2)^2 \\ &- m_2 (v_{2b} + Am_1 I_2)^2 - I_2 (\omega_{2b} + Am_1 m_2 d)^2 \end{aligned} \right\}$$

Now we want to put this into the form that we had for Eq. (3.2) in the section for configuration 1b. The following derivation is original. First, we expand the squared terms.

$$\begin{aligned} 2KE_{\text{lost}} &= \textcolor{red}{m_1} \textcolor{red}{v_{1b}^2} + \textcolor{blue}{m_2} \textcolor{blue}{v_{2b}^2} + \textcolor{violet}{I_2} \textcolor{violet}{\omega_{2b}^2} - m_1 (\textcolor{red}{v_{1b}^2} - 2v_{1b} Am_2 I_2 + A^2 m_2^2 I_2^2) \\ &\quad - m_2 (\textcolor{blue}{v_{2b}^2} + 2v_{2b} Am_1 I_2 + A^2 m_1^2 I_2^2) - I_2 (\textcolor{violet}{\omega_{2b}^2} + 2\omega_{2b} Am_1 m_2 d + A^2 m_1^2 m_2^2 d^2) \\ &\quad \text{cancel terms in the same color} \\ 2KE_{\text{lost}} &= -m_1 (-2v_{1b} Am_2 I_2 + A^2 m_2^2 I_2^2) \\ &\quad - m_2 (2v_{2b} Am_1 I_2 + A^2 m_1^2 I_2^2) - I_2 (2\omega_{2b} Am_1 m_2 d + A^2 m_1^2 m_2^2 d^2) \\ &\quad \text{distribute the leading terms} \\ 2KE_{\text{lost}} &= 2v_{1b} Am_1 m_2 I_2 - A^2 m_1 m_2^2 I_2^2 \\ &\quad - 2v_{2b} Am_1 m_2 I_2 - A^2 m_1^2 m_2 I_2^2 - 2\omega_{2b} Am_1 m_2 d I_2 - A^2 m_1^2 m_2^2 d^2 I_2^2 \end{aligned}$$

Rearrange

$$\begin{aligned} 2KE_{\text{lost}} &= 2v_{1b} Am_1 m_2 I_2 - 2v_{2b} Am_1 m_2 I_2 - A^2 m_1^2 m_2 I_2^2 - A^2 m_1 m_2^2 I_2^2 - 2\omega_{2b} Am_1 m_2 d I_2 \\ &\quad - A^2 m_1^2 m_2^2 d^2 I_2^2 \end{aligned}$$

factor

$$2KE_{\text{lost}} = Am_1 m_2 I_2 2(v_{1b} - v_{2b}) - A^2 m_1 m_2 I_2 (m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) - 2\omega_{2b} Am_1 m_2 d I_2$$

factor out $Am_1 m_2 I_2$

$$2KE_{\text{lost}} = Am_1 m_2 I_2 [2(v_{1b} - v_{2b}) - A(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) - 2\omega_{2b} d]$$

Substitute for A

$$2KE_{\text{lost}} = Am_1m_2I_2 \left[2(v_{1b} - v_{2b}) - \left\{ \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})}{m_1I_2 + m_2I_2 + m_1m_2d^2} \right\} (m_1I_2 + m_2I_2 + m_1m_2d^2) - 2\omega_{2b}d \right]$$

$$2KE_{\text{lost}} = Am_1m_2I_2 [2(v_{1b} - v_{2b}) - (v_{1b} - v_{2b})(1 + CoR_{2b}) + d\omega_{2b}(1 + CoR_{2b}) - 2\omega_{2b}d]$$

factor $(v_{1b} - v_{2b})$ out of the first two terms and combine the last two terms

$$2KE_{\text{lost}} = Am_1m_2I_2 [(v_{1b} - v_{2b})(1 - CoR_{2b}) - d\omega_{2b}(1 - CoR_{2b})]$$

factor $(1 - CoR_{2b})$

$$2KE_{\text{lost}} = Am_1m_2I_2 (1 - CoR_{2b})(v_{1b} - v_{2b} - d\omega_{2b})$$

substitute for A

$$2KE_{\text{lost}} = \left\{ \frac{(v_{1b} - v_{2b} - d\omega_{2b})(1 + CoR_{2b})}{m_1I_2 + m_2I_2 + m_1m_2d^2} \right\} m_1m_2I_2 (1 - CoR_{2b}) \times (v_{1b} - v_{2b} - d\omega_{2b})$$

$$2KE_{\text{lost}} = \frac{m_1m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2} (v_{1b} - v_{2b} - d\omega_{2b}) \times (1 + CoR_{2b})(1 - CoR_{2b})(v_{1b} - v_{2b} - d\omega_{2b})$$

Finally we get

$$KE_{\text{lost}} = \frac{1}{2} \frac{m_1m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2} (v_{1b} - v_{2b} - d\omega_{2b})^2 (1 - CoR_{2b}^2)$$

or

$$KE_{\text{lost}} = \frac{1}{2} \frac{m_{\text{ball}}m_{\text{bat}}I_{\text{bat}}(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d)^2 (1 - CoR_{2b}^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \quad (4.11)$$

This is a general result for the BaConLaw model. It is original and unique.

Now for a collision at the center of mass of the bat, like configurations 1a and 1b, $d = 0$. Therefore,

$$KE_{\text{lost}} = \frac{1}{2} \frac{m_1m_2}{m_1 + m_2} (v_{1b} - v_{2b})^2 (1 - CoR_{1b}^2)$$

When we substitute, $\bar{m} = \frac{m_1m_2}{m_1 + m_2}$ we get

$$KE_{\text{lost}} = \frac{\bar{m}}{2}(v_{1b} - v_{2b})^2(1 - CoR_{1b}^2)$$

This is the same as Eq. (3.2) that we gave in the Sect. 3.2.2 for configuration 1b where we mentioned that this is an old, well-known equation that is hard to derive.

$$KE_{\text{lost}} = \frac{\bar{m}}{2}(v_{\text{ball-before}} - v_{\text{bat-cm-before}})^2(1 - CoR_{1b}^2)$$

Likewise, if the spin of the bat about its center of mass is zero before the collision $\omega_{2b} = 0$, then our KE_{lost} equation Eq. (4.11) also reduces to that given for configuration 1b, Eq. (3.2).

In this section, we derived a general equation and showed that if the collision were at the center of mass ($d = 0$) or the bat had no spin $\omega_{2b} = 0$, then the general equation reduced to the simple equation of configuration 1b. We conclude that adding an equation for Conservation of Energy to the model proved the consistency of our set of equations.

4.9 Adding Conservation of Angular Momentum

In this section, which is almost the same as Sect. 3.5, we will prove that for a head-on collision, without friction, for a pitch of any spin there will be no change in the spin of the ball. To do this we will use the law of Conservation of Angular Momentum about the center of mass of the bat. When the ball contacts the bat, as shown in Fig. 4.1, the ball has linear momentum of $m_{\text{ball}}v_{\text{ball-before}}$. However, the ball does not know if it is translating or if it is tied on a string and rotating about the center of mass of the bat. Following conventional practice in physics, we will model the ball as rotating about the bat's center of mass at a distance $d = d_{\text{cm-ip}}$. Therefore, the ball has an initial angular momentum of $m_{\text{ball}}d_{\text{cm-ip}}v_{\text{ball-before}}$ about an axis through the bat's center of mass. In addition, it is possible to throw a curveball so that it spins about the vertical, z-axis, as also shown in Fig. 4.1. We call this a purely horizontal curveball (although it will still drop more due to gravity, than it will curve horizontally). The curveball will have angular momentum of $I_{\text{ball}}\omega_{\text{ball-before}}$ about an axis parallel to the z-axis. However, this is its momentum about *its* center of mass and we want the momentum about the axis through the center of mass of the *bat*. Therefore, we use the parallel axis theorem producing $(I_{\text{ball}} + m_{\text{ball}}d^2)\omega_{\text{ball-before}}$.

The bat has an initial angular momentum of $I_{\text{bat}}\omega_{\text{bat-before}}$. It also has an angular momentum about the bat's center of mass due to the bat translation momentum $m_{\text{bat}}v_{\text{bat-before}}$, however, in this case $d = 0$ because the center of mass of the bat is passing through its center of mass. L is the symbol used for angular momentum. I guess all the cool letters (like F , m , a , v , I , ω , d , etc.) were already taken, so old-time physicists were stuck with the blah symbol L . Therefore, the initial angular momentum about the center of mass of the bat is

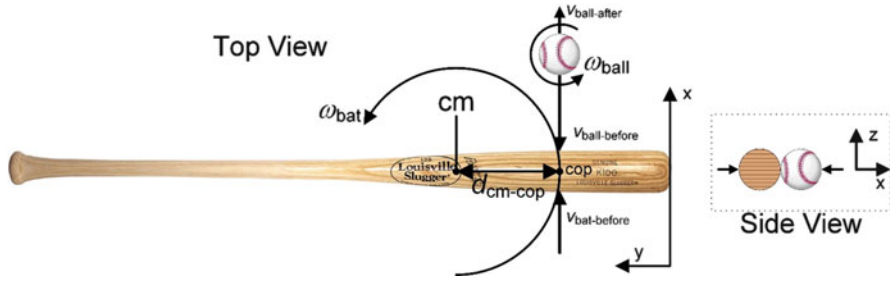


Fig. 4.2 This figure shows $v_{\text{ball-before}}$, $v_{\text{ball-after}}$, ω_{ball} , $d_{\text{cm-ip}}$ and ω_{bat} , which are used in the Conservation of Angular Momentum equation for the BaConLaw model

$$L_{\text{initial}} = m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b}$$

All of these momenta are positive, pointing out of the page (Fig. 4.2). (Remember that v_{1b} is a negative number.)

For the final angular momentum, we will treat the ball, as before, as an object rotating around the axis of the center of mass of the bat with angular momentum, $m_{\text{ball}} v_{\text{ball-after}} d_{\text{cm-ip}}$. Now we could treat the bat as a long slender rod with a moment of inertia of $m_{\text{bat}} d_{\text{bat}}^2 / 12$, where d_{bat} is the bat length. However, this is only an approximation and we have actual experimental data for the bat moment of inertia. Therefore, the bat angular momentum is $I_{\text{bat}} \omega_{\text{bat-after}}$. Thus, our final angular momentum about the center of mass of the bat is

$$L_{\text{final}} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a}$$

The law of Conservation of Angular Momentum states that when no external torque acts on an object the initial angular momentum about some axis equals the final angular momentum about that axis.

$$\boxed{L_{\text{initial}} = L_{\text{final}}}$$

$$m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a}$$

Previously we used Eq. (4.6), Newton's second principle and solved for the angular velocity of the bat after the collision, ω_{2a} .

$$dm_1 (v_{1a} - v_{1b}) = -I_2 (\omega_{2a} - \omega_{2b}) \quad (4.6)$$

$$\omega_{2a} = \left\{ \omega_{2b} - \frac{dm_1}{I_2} (v_{1a} - v_{1b}) \right\}$$

So let us substitute this into our Conservation of Angular Momentum equation above.

$$m_1 v_{1b} d + I_1 \omega_{1b} + m_1 \omega_{1b} d^2 + I_2 \omega_{2b} = m_1 v_{1a} d + I_1 \omega_{1a} + m_1 \omega_{1a} d^2 + I_2 \left\{ \omega_{2b} + \frac{dm_1}{I_2} (v_{1b} - v_{1a}) \right\}$$

We want to solve this for the angular velocity of the ball after the collision, ω_{1a}

$$-I_1 \omega_{1a} - m_1 \omega_{1a} d^2 = -m_1 v_{1b} d - I_1 \omega_{1b} - I_2 \omega_{2b} - m_1 \omega_{1b} d^2 + m_1 v_{1a} d + I_2 \omega_{2b} + dm_1 (v_{1b} - v_{1a})$$

Cancel the terms in color and rearrange

$$\omega_{1a} (I_1 + m_1 d^2) = \omega_{1b} (I_1 + m_1 d^2)$$

$\begin{aligned} \omega_{1a} &= \omega_{1b} \\ \omega_{\text{ball-after}} &= \omega_{\text{ball-before}} \end{aligned}$	(4.12)
---	--------

For the BaConLaw model, we have now proven that for a pitch with any spin about the z-axis, the spin before and after is the same. What about a pitch that has spin about the z-axis and also about the y-axis, like most pitches? The collision will not change ball rotation. As shown above, it will not change the spin about the z-axis. We could write another set of equations for angular momentum about the y-axis. However, the bat has no angular momentum about the y-axis, so there is nothing to affect the ball spin about the y-axis. {We are neglecting bat swings described as chops or uppercuts and friction. The effects of friction will be examined in Sect. 5.5, Collision with Friction.} In conclusion, a head-on collision between a bat and a ball will not change the spin on the ball (Table 4.2). Some papers have shown a relationship between ball spin before and ball spin after, but they were using oblique collisions as in configuration 3 (Nathan et al. 2012; Kensrud et al. 2017).

4.10 Simulation Results

The Excel simulation satisfies the following checks: (1) Conservation of Energy, (2) Kinetic energy lost, (3) Conservation of Linear Momentum, (4) Coefficient of Restitution, (5) Newton's second principle, namely an impulse changes momentum and (6) Conservation of Angular Momentum. Table 4.3 shows the kinetic energies for the same simulation.

We note that the total kinetic before (372 J) equals the kinetic energy after (176 J) plus the kinetic energy lost (196 J). However, if we set $d_{\text{cm-ip}} = 0$ in the simulation so that the impact point is at the center of mass of the bat, then Tables 4.2 and 4.3 change and produce the results of Tables 3.3 and 3.4 for configuration 1b, where the total kinetic before (346 J) equaled the kinetic energy after (169 J) plus the kinetic energy lost (177 J). This means that the whole BaConLaw model (equations, simulations, sensitivity analyses, etc.) can be reduced to be appropriate

Table 4.2 Simulation values for bat-ball collisions at the sweet spot, the BaConLaw model

	SI units	Baseball units
Inputs		
$v_{ball} - \text{before}$	−37 m/s	−83 mph
$\omega_{ball} - \text{before}$	209 rad/s	2000 rpm
$v_{bat} - \text{cm} - \text{before}$	23 m/s	52 mph
$\omega_{bat} - \text{before}$	32 rad/s	309 rpm
$vt_{bat} - \text{cop} - \text{before}$	28 m/s	62 mph
Collision speed	65 m/s	145 mph
CoR_{2b}	0.465	0.465
Outputs		
$v_{ball} - \text{after}$	41 m/s	92 mph
$\omega_{ball} - \text{after}$	$= \omega_{ball} - \text{before}$	
$v_{bat} - \text{cm} - \text{after}$	11 m/s	24 mph
$\omega_{bat} - \text{after}$	1 rad/s	7 rpm
KE_{lost}	196 J	

Table 4.3 The BaConLaw model kinetic energies, J

KE ball linear velocity before=	100
KE bat linear velocity before=	246
KE ball angular velocity before=	1.7
KE bat angular velocity before=	25
KE before total=	372
KE ball linear velocity after=	122
KE bat linear velocity after=	53
KE ball angular velocity after	1.7
KE bat angular velocity after=	0.01
KE after=	176
KE lost =	196
KE after + KE lost=	372

for configurations 2a, 1a and 1b by zeroing appropriate values. This is an important validation point.

4.11 Sensitivity Analysis

This section contains equations and it can be skipped without loss of continuity.

This book is about the science of baseball. So why does it have this section on sensitivity analysis? In order to understand the science of baseball, we make models. In order to validate these models we do sensitivity analyses.

A second purpose of this book is to show how the batter can buy or make an optimal baseball or softball bat. From the viewpoint of the batter, an optimum bat would produce the maximum batted-ball velocity. The larger the batted-ball

velocity, the more likely the batter will get on base safely (Baldwin and Bahill 2004). Therefore, we made the batted-ball velocity our performance criterion. We will now find the sensitivity of the batted-ball velocity, $v_{\text{ball-after}}$, with respect to the eight model variables and parameters, namely $v_{\text{ball-before}}$, m_{ball} , I_{bat} , m_{bat} , CoR_{2b} , $d_{\text{cm-ip}}$, $v_{\text{bat-cm-before}}$ and $\omega_{\text{bat-before}}$. We will start with the equation for the ball velocity after the collision, v_{1a} , Eq. (4.8).

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - \omega_{2b}d)(1 + CoR_{2b})m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2}$$

In order to perform an analytic sensitivity analysis we need the partial derivatives of v_{1a} with respect to the eight variables and parameters. These partial derivatives are often called the absolute sensitivity functions.

Let

$$B = (v_{1b} - v_{2b} - \omega_{2b}d)(1 + CoR_{2b})$$

$$K = (m_1I_2 + m_2I_2 + m_1m_2d^2)$$

Therefore,

$$v_{1a} = v_{1b} - \frac{Bm_2I_2}{K}$$

The following partial derivatives with respect to the variables are easy to derive.

$$\frac{\partial v_{1a}}{\partial v_{1b}} = 1 - \frac{(1 + CoR_{2b})m_2I_2}{K} \quad \text{unitless}$$

$$\frac{\partial v_{1a}}{\partial v_{2b}} = \frac{(1 + CoR_{2b})m_2I_2}{K} \quad \text{unitless}$$

$$\frac{\partial v_{1a}}{\partial \omega_{2b}} = \frac{(1 + CoR_{2b})dm_2I_2}{K} \quad \text{m}$$

$$\frac{\partial v_{1a}}{\partial CoR_{2b}} = -\frac{(v_{1b} - v_{2b} - \omega_{2b}d)m_2I_2}{K} \quad \text{m/s}$$

In the above partial derivatives, units on the left and right sides of the equations are the same. This is a simple, but important accuracy check. We perform such a dimensional analysis on all of our equations.

For the following partial derivatives with respect to the parameters, we will need the derivative of a quotient, defined as

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}$$

Using this differential equation we get the following partial derivatives.

$$\frac{\partial v_{1a}}{\partial d} = \frac{K\omega_{2b}(1 + CoR_{2b})m_2I_2 + 2BI_2m_1m_2^2d}{K^2} \quad 1/s$$

$$\begin{aligned}\frac{\partial v_{1a}}{\partial m_1} &= \frac{Bm_2I_2(I_2 + m_2d^2)}{K^2} \quad \text{m/kg} \cdot \text{s} \\ \frac{\partial v_{1a}}{\partial m_2} &= \frac{-Bm_1I_2^2}{K^2} \quad \text{m/kg} \cdot \text{s} \\ \frac{\partial v_{1a}}{\partial I_2} &= \frac{Bm_1m_2^2d^2}{K^2} \quad \text{1/kg} \cdot \text{m} \cdot \text{s}\end{aligned}$$

4.11.1 Semirelative Sensitivity Functions

Now that we have the partial derivatives, we want to form the *semirelative-sensitivity functions*, which are defined as

$$\tilde{S}_\alpha^F = \left. \frac{\partial F}{\partial \alpha} \right|_{\text{NOP}} \alpha_0$$

where NOP and the subscript 0 mean that all variables and parameters assume their nominal operating point values (Smith, Szidarovszky, Karnavas and Bahill 2008).

$$\begin{aligned}\tilde{S}_\alpha^F &= \left. \frac{\partial F}{\partial \alpha} \right|_{\text{NOP}} \alpha_0 \\ \tilde{S}_{v_{1b}}^{v_{1a}} &= 1 - \left. \frac{(1+CoR_{2b})m_2I_2}{K} \right|_{\text{NOP}} v_{1b0} \\ \tilde{S}_{v_{2b}}^{v_{1a}} &= \left. \frac{(1+CoR_{2b})m_2I_2}{K} \right|_{\text{NOP}} v_{2b0} \\ \tilde{S}_{\omega_{2b}}^{v_{1a}} &= \left. \frac{(1+CoR_{2b})m_2dI_2}{K} \right|_{\text{NOP}} \omega_{2b0} \\ \tilde{S}_{CoR}^{v_{1a}} &= \left. \frac{-(v_{1b}-v_{2b}-\omega_{2b}d)m_2I_2}{K} \right|_{\text{NOP}} CoR_0 \\ \tilde{S}_d^{v_{1a}} &= \left. \frac{K\omega_{2b}(1+CoR_{2b})m_2I_2 + 2Bm_2I_2m_1m_2d}{K^2} \right|_{\text{NOP}} d_0 \\ \tilde{S}_{m_1}^{v_{1a}} &= \left. \frac{Bm_2I_2(I_2+m_2d^2)}{K^2} \right|_{\text{NOP}} m_{10} \\ \tilde{S}_{m_2}^{v_{1a}} &= \left. \frac{-Bm_1I_2^2}{K^2} \right|_{\text{NOP}} m_{20} \\ \tilde{S}_{I_2}^{v_{1a}} &= \left. \frac{Bm_1m_2^2d^2}{K^2} \right|_{\text{NOP}} I_{20}\end{aligned}$$

Table 4.4 gives the nominal values, along with the range of physically realistic values for collegiate and professional baseball batters, and the semirelative sensitivity values computed analytically. The bigger the sensitivity is, the more important the variable or parameter is for maximizing batted-ball velocity.

The right column of Table 4.4 shows that the most important property (the largest value), in terms of maximizing batted-ball velocity, is the linear velocity of the center of mass of the bat before the collision, $v_{\text{bat-cm-before}}$. This is certainly no

Table 4.4 Typical values and first-order sensitivities with respect to the batted-ball velocity for the BaConLaw model

Variables and parameters	Nominal values		Range of realistic values		$\tilde{S}_\alpha^F = \left. \frac{\partial F}{\partial \alpha} \right _{\alpha_0}$ semirelative sensitivity values
	SI units	Baseball units	SI units	Baseball units	
Inputs					
$v_{\text{ball}} - \text{before}$	−37 m/s	−83 mph	−27 to −40 m/s	−60 to −90 mph	8
$\omega_{\text{ball}} - \text{before}$	209 rad/s	2000 rpm	209 ± 21 rad/s	2000 ± 200 rpm	0
$v_{\text{bat}} - \text{cm} - \text{before}$	23 m/s	52 mph	23 ± 5 m/s	52 ± 10 mph	28
$\omega_{\text{bat}} - \text{before}$	32 rad/s	309 rpm	32 ± 11 rad/s	300 ± 100 rpm	5
$v_{\text{fbat}} - \text{cop} - \text{before}$	28 m/s	62 mph			
Parameters					
CoR_{2b}	0.465		0.465 ± 0.05		25
$d_{\text{cm}} - \text{cop}$	0.134 m	5.3 in	0.134 ± 0.05 m	5.3 ± 2 in	−2
m_{ball}	0.145 kg	5.125 oz	0.145 ± 0.004 kg	5.125 ± 0.125 oz	−14
m_{bat}	0.905 kg	32 oz	0.709–0.964 kg	25–34 oz	10
$I_{\text{bat}} - \text{cm}$	0.048 kg m ²	2624 oz in ²	0.036–0.06 kg m ²	1968–3280 oz in ²	3

surprise. The second most important property is the coefficient of restitution, CoR_{2b} . The least important properties are the angular velocity of the ball, $\omega_{\text{ball}} - \text{before}$, the distance between the center of mass and the impact point, $d_{\text{cm}} - \text{ip}$, and the moment of inertia of the bat, I_{bat} . The sensitivities to the distance between the center of mass and the impact point, $d_{\text{cm}} - \text{ip}$, and the mass of the ball, m_{ball} , are negative, which merely means that as they increase the batted-ball speed decreases. Cross (2011) wrote that in his model the most sensitive properties were also the bat speed followed by the CoR . His sensitivity to the mass of the ball was also negative.

For this operating point {meaning the nominal values given in Table 4.4 where $d_{\text{cm}} - \text{ip} = 0.134$ m}, the sensitivity of the batted-ball speed with respect to the impact point, the distance $d_{\text{cm}} - \text{ip}$, was negative. This means that as the impact point gets farther away from the center of mass the batted-ball speed falls off. This is true for all values where $d_{\text{cm}} - \text{ip} > 0.1$ m. For smaller values, the sensitivity coefficient is positive. This means that there is a point of impact that produces the maximum batted-ball speed. This is not surprising and is a well-known fact Nathan (2003).

4.11.2 Interactions

We will now discuss interactions, or second-order partial derivatives. Once my Mother cleaned the toilet with Clorox bleach. She was pleased with the result. The next week she cleaned the toilet with ammonia. She was even happier. So then, she

decided that if bleach by itself worked so well and ammonia by itself worked so well, then surely both of them together would be wonderful. She created chloramine gas and we had to get out of the house and spend the rest of the day in the desert, because this gas kills people (<https://www.thoughtco.com/bleach-and-ammonia-chemical-reaction-609280>). Next, don't drink ethyl alcohol and take barbiturates or acetaminophen (Tylenol) at the same time, unless you are trying to commit suicide. Finally, because grapefruit juice contains furanocoumarins it increases the absorption rate of cholesterol-lowering statins such as Zocor, which could lead to serious side effects. Interactions can amplify or attenuate the effects of drugs and chemicals. Now let us look at some interactions in the BaConLaw model.

Because $\frac{\partial v_{1a}}{\partial m_2}$ contains both I_2 and v_{2b} ,

and $\frac{\partial v_{1a}}{\partial I_2}$ contains both m_2 and v_{2b} ,

and $\frac{\partial v_{1a}}{\partial v_{2b}}$ contains both m_2 and I_2 ,

we see that there are interactions. How important are they? To find out, let us calculate the second-order, interaction functions for the three terms above. The first two are easy.

$$\frac{\partial^2 v_{1a}}{\partial v_{2b} \partial m_2} = \frac{(1 + CoR_{2b})I_2 [K - m_2(I_2 + m_1 d^2)]}{K^2} \quad 1/\text{kg}$$

$$\frac{\partial^2 v_{1a}}{\partial v_{2b} \partial I_2} = \frac{(1 + CoR_{2b})m_2 [K - I_2(m_1 + m_2)]}{K^2} \quad 1/\text{kg} \cdot \text{m}^2$$

Here, we choose the interactions of the bat mass, the moment of inertia and the bat speed, because they were expected to be large based on principles of physiology. Additionally, the forthcoming discussion on optimizing the bat suggests an interaction between the bat mass and its moment of inertia. The above two second-order partial derivatives were easy to calculate. However, it will take a bit more work to get the third part of this triad. We will now derive the interaction between bat mass and its moment of inertia, m_{bat} and I_{bat} . From before, we had

$$\frac{\partial v_{1a}}{\partial m_2} = \frac{BI_2 [-K + m_2(I_2 + m_1 d^2)]}{K^2}$$

To find $\frac{\partial^2 v_{1a}}{\partial I_2 \partial m_2}$ we must first simplify $\frac{\partial v_{1a}}{\partial m_2}$. We will be dealing with I_2 so let us isolate it. But first replace K in the numerator and we get

$$\begin{aligned}
K^2 \frac{\partial v_{1a}}{\partial m_2} &= -(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) B I_2 + B m_2 I_2 (I_2 + m_1 d^2) \\
&= -B \left[(m_1 I_2 + m_2 I_2 + m_1 m_2 d^2) I_2 - m_2 I_2 (I_2 + m_1 d^2) \right] \\
&= -B \left[m_1 I_2^2 + m_2 I_2^2 + m_1 m_2 d^2 I_2 - m_2 I_2^2 - m_1 m_2 d^2 I_2 \right]
\end{aligned}$$

Cancel the terms in color

$$\begin{aligned}
K^2 \frac{\partial v_{1a}}{\partial m_2} &= -B m_1 I_2^2 \\
\frac{\partial v_{1a}}{\partial m_2} &= \frac{-B m_1 I_2^2}{K^2}
\end{aligned}$$

Expand B and this simplification results.

$$\frac{\partial v_{1a}}{\partial m_2} = \frac{-(v_{1b} - v_{2b} - \omega_{2b} d)(1 + CoR_{2b}) m_1 I_2^2}{K^2}$$

Now, we will take the partial derivative of this function with respect to I_2 .

$$\begin{aligned}
\frac{\partial^2 v_{1a}}{\partial I_2 \partial m_2} &= \frac{-K^2 2B m_1 I_2 + B m_1 I_2^2 2K (m_1 + m_2)}{K^4} \\
&= \frac{2B m_1 I_2 K [-K + I_2 (m_1 + m_2)]}{K^4}
\end{aligned}$$

substitute for the second K in the numerator

$$\begin{aligned}
&= \frac{2B m_1 I_2 K [-\{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\} + I_2 (m_1 + m_2)]}{K^4} \\
&= \frac{2B m_1 I_2 K [-m_1 I_2 - m_2 I_2 - m_1 m_2 d^2 + m_1 I_2 + m_2 I_2]}{K^4}
\end{aligned}$$

cancel the terms in color

$$= \frac{-2B m_1^2 m_2 d^2 I_2 K}{K^4}$$

finally substitute B and K

$$\frac{\partial^2 v_{1a}}{\partial I_2 \partial m_2} = \frac{-2(v_{1b} - v_{2b} - \omega_{2b} d)(1 + CoR_{2b}) m_1^2 m_2 d^2 I_2}{\{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2\}^3} \quad 1/(\text{kg}^2 \cdot \text{m} \cdot \text{s})$$

This demonstrates that if we have equations for the functions, then we can do an analytic sensitivity analysis. However, for some functions it may take some effort. Fortunately, it takes no effort to calculate the following interaction terms using the partial derivatives in the previous section.

$$\frac{\partial v_{1a}}{\partial v_{1b}} = 1 - \frac{(1 + CoR_{2b})m_2 I_2}{K}$$

$$\frac{\partial^2 v_{1a}}{\partial v_{1b} \partial CoR} = \frac{m_2 I_2}{K} \quad \text{unitless}$$

and

$$\frac{\partial v_{1a}}{\partial v_{2b}} = \frac{(1 + CoR_{2b})m_2 I_2}{K}$$

$$\frac{\partial^2 v_{1a}}{\partial v_{2b} \partial CoR} = \frac{m_2 I_2}{K} \quad \text{unitless}$$

These two partial derivatives are the same, but their semirelative sensitivity functions will be different. Let us derive one more second-order partial derivative.

$$\frac{\partial v_{1a}}{\partial m_2} = \frac{-(v_{1b} - v_{2b} - \omega_{2b}d)(1 + CoR_{2b})m_1 I_2^2}{K^2}$$

$$\frac{\partial^2 v_{1a}}{\partial CoR \partial m_2} = \frac{-(v_{1b} - v_{2b} - \omega_{2b}d)m_1 I_2^2}{K^2}$$

This function does not look interesting.

Now for the above five second-order partial derivatives we can form the following semirelative sensitivity functions for interactions.

$$\tilde{S}_{v_{2b}-m_2}^{v_{1a}} = \frac{(1+CoR_{2b})I_2[K-m_2(I_2+m_1d^2)]}{K^2} \Big|_{\text{NOP}} v_{2b_0} m_{2_0}$$

$$\tilde{S}_{v_{2b}-I_2}^{v_{1a}} = \frac{(1+CoR_{2b})m_2[K-I_2(m_1+m_2)]}{K^2} \Big|_{\text{NOP}} v_{2b_0} I_{2_0}$$

$$\tilde{S}_{I_2-m_2}^{v_{1a}} = \frac{-2(v_{1b}-v_{2b}-\omega_{2b}d)(1+CoR_{2b})m_1^2 m_2 d^2 I_2}{K^3} \Big|_{\text{NOP}} I_{2_0} m_{2_0}$$

$$\tilde{S}_{v_{1b}-CoR}^{v_{1a}} = \frac{m_2 I_2}{K} \Big|_{\text{NOP}} v_{1b_0} CoR_0$$

and

$$\tilde{S}_{v_{2b}-CoR}^{v_{1a}} = \frac{m_2 I_2}{K} \Big|_{\text{NOP}} v_{2b_0} CoR_0$$

Table 4.5 shows values for a few of the 28 possible second-order interaction functions. They are small, which means that the model is well behaved. However, let's ask again, "What exactly what are interaction terms?" It means that the numerical value of the sensitivity of a function f to parameter α depends on the numerical value of parameter β . Often the interaction can be seen in the sensitivity function equations. In the BaConLaw model, the sensitivity of the batted-ball

Table 4.5 Interaction sensitivities with respect to the batted-ball velocity for the BaConLaw model computed numerically for +1% variable and parameter changes

Interacting variables and parameters	$v_{\text{ball-after}}$ with the first parameter α increased by 1%, m/s	$v_{\text{ball-after}}$ with the second parameter β increased by 1%, m/s	$\Delta v_{\text{ball-after}}^{\alpha} + \Delta v_{\text{ball-after}}^{\beta}$ Sum of columns 2 & 3	$v_{\text{ball-after}}$ with both parameters increased by 1%, m/s	$\tilde{S}_{\alpha}^F = \left. \frac{\partial F}{\partial \alpha} \right _{\text{NOP}} \alpha_0$ semirelative sensitivity values
Nominal batted-ball velocity $v_{\text{ball-after}} = 41.079$ m/s = 91.89 mph					
$v_{\text{ball-before}}$ interacting with CoR_{2b}	41.156	41.327		41.405	14
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	0.077	0.248	0.325	0.326	
$v_{\text{bat-before}}$ interacting with CoR_{2b}	41.361	41.327		41.610	9
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	0.282	0.248	0.530	0.531	
$v_{\text{ball-before}}$ interacting with m_{ball}	41.156	40.942		41.017	-8
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	0.077	-0.137	-0.060	-0.062	
m_{bat} interacting with m_{ball}	41.182	40.942		41.044	7
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	0.103	-0.137	-0.034	-0.035	
m_{bat} interacting with I_{bat}	41.182	41.114		41.216	1
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	0.103	0.035	0.138	0.137	
$v_{\text{ball-before}}$ interacting with $v_{\text{bat-before}}$	41.156	41.361		41.438	0
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	0.077	0.282	0.359	0.359	
$d_{\text{cm-ss}}$ interacting with I_{bat}	41.061	41.114		41.097	7
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	-0.018	0.035	0.017	0.018	
CoR interacting with m_{ball}	41.327	40.942		41.189	-4
$v_{\text{ball-after-nominal}} - v_{\text{ball-after-perturbed}}$ m/s	-0.248	0.137	-0.111	-0.110	

velocity to the mass of the bat depends on the numeric value of the moment of inertia of the bat, because it appears in the numerator of this sensitivity function.

$$\tilde{S}_{m_2}^{v_{1a}} = \left. \frac{-(v_{1b} - v_{2b} - \omega_{2b}d)(1 + CoR_{2b})m_1 I_2^2}{K^2} \right|_{\text{NOP}} m_{2_0}$$

However, from Table 4.5, the numeric value of m_{bat} interacting with I_{bat} is only 1, which is smaller than the magnitude of the sensitivity of the batted-ball velocity to the mass of the bat by itself, which is 10 (from Table 4.4), or to the magnitude of the sensitivity of the batted-ball velocity to the moment of inertia of the bat by itself, which is 3. Thus, this interaction is unexpectedly *not* important. This model has many interactions, but fortunately, most of them are small. Interactions are hard to

detect. And if they are big, they can ruin a system or a model. The most and least important interaction functions of this model are shown in Table 4.5.

Can we use this information to increase bat performance? For wooden bats, it is legal to drill a one to 2 inch hole into the barrel end of the bat up to 1¼ inches deep. It is also legal to taper the last 3 inches of the barrel say from 2.61 inches (6.6 cm) down to 1¼ of an inch (4.4 cm). Both of these modifications would decrease the bat weight, decrease the moment of inertia about the center of mass and would move the sweet spot closer to the knob. According to Table 4.4, the first two changes would decrease batted-ball speed, whereas the third would increase batted-ball speed. So, what is the right answer? We will not know until after we consider physiology in Sect. 4.12.4.

To complete this section on sensitivity analysis, we will now look at interactions using semirelative sensitivity functions that we will compute with *numerical* techniques instead of using the analytic equations derived above.

4.11.2.1 Empirical (or Numerical) Sensitivity Analysis

If you do not have equations for the model's functions {or for heuristic reasons as in this section}, then you can do a sensitivity analysis using numerical techniques. To estimate values for the second-partial derivatives we start with

$$\frac{\partial^2 f(\alpha_0, \beta_0)}{\partial \alpha \partial \beta} \approx \frac{f(\alpha, \beta) - f(\alpha_0, \beta) - f(\alpha, \beta_0) + f(\alpha_0, \beta_0)}{\Delta \alpha \Delta \beta}$$

from Bahill and Madni (2017). Then for a 1% increase in the parameter α $\Delta \alpha = 0.01\alpha_0$. Likewise $\Delta \beta = 0.01\beta_0$. Therefore

$$\frac{\partial^2 f(\alpha_0, \beta_0)}{\partial \alpha \partial \beta} \approx \frac{f(\alpha, \beta) - f(\alpha_0, \beta) - f(\alpha, \beta_0) + f(\alpha_0, \beta_0)}{0.01\alpha_0 \times 0.01\beta_0}$$

Now to get the semirelative-sensitivity function we multiply this mixed-second-partial derivative by the nominal values α_0 and β_0

$$\begin{aligned} \tilde{S}_{\alpha-\beta}^f &= \left. \frac{\partial^2 f}{\partial \alpha \partial \beta} \right|_{\text{NOP}} \alpha_0 \beta_0 \\ \tilde{S}_{\alpha-\beta}^f &\approx \left. \frac{f(\alpha, \beta) - f(\alpha_0, \beta) - f(\alpha, \beta_0) + f(\alpha_0, \beta_0)}{0.01\alpha_0 \times 0.01\beta_0} \right|_{\text{NOP}} \alpha_0 \beta_0 \\ \tilde{S}_{\alpha-\beta}^f &\approx |f(\alpha, \beta) - f(\alpha_0, \beta) - f(\alpha, \beta_0) + f(\alpha_0, \beta_0)|_{\text{NOP}} \times 10,000 \end{aligned}$$

We used this equation to get the values for Table 4.5. The column heading $\Delta v_{\text{ball-after}}^\alpha + \Delta v_{\text{ball-after}}^\beta$ means find the change in the velocity of the batted ball after the

perturbation of the parameter α and add to it the change in the velocity of the ball after the perturbation of the parameter β .

To form Table 4.5, we defined the performance criterion, chose a pair of parameters, changed the first by a fixed percentage, calculated the new performance criterion value, calculated the change in the performance criterion value, reset the first parameter, changed the second parameter by the same percentage, calculated the new performance criterion value, calculated the change in the performance criterion value, added these two changes in the performance criterion values together, then changed both parameters at the same time, calculated the performance criterion value and calculated the change in the performance criterion value.

Let us now explain the top rows of Table 4.5, the interaction of $v_{\text{ball-before}}$ with CoR_{2b} . If you increase $v_{\text{ball-before}}$ (call it α) by 1%, then the batted-ball speed will increase from its nominal value of 41.079 m/s to its modified value of 41.156 m/s. This is an increase of 0.077 m/s. Now reset $v_{\text{ball-before}}$ and then increase CoR_{2b} (call it β) by 1%. The batted-ball speed will increase from the nominal value to its modified value of 41.327 m/s. This is an increase of 0.248 m/s. Therefore, these two changes, when performed individually, produce a total change of 0.325, highlighted in blue in Table 4.5. Now comes the important part, if you increase both $v_{\text{ball-before}}$ and CoR_{2b} by 1% at the same time, then the batted-ball speed increases from the nominal value to a modified value of 41.405 m/s. This is an increase of 0.326 m/s, highlighted in green in Table 4.5. Therefore, we can see that when these two changes are performed individually they produce a total increase of 0.325, however when performed together they produce an increase of 0.326 m/s. The whole is greater than the sum of its parts.

However, interactions do not always accentuate changes. Here is one that goes in the opposite direction; the interaction of m_{bat} with I_{bat} . Suppose that someone tells you that Eq. (4.8) shows that increasing bat mass will increase batted-ball speed. And someone else tells you that increasing the bat moment of inertia will increase your batted-ball speed. Well if each is good by itself why not do both? For instance, if you increase m_{bat} by 1%, then the batted-ball speed will increase from its nominal value of 41.079 m/s to its modified value of 41.182 m/s. This is an increase of 0.103 m/s. Now reset m_{bat} and then increase I_{bat} by 1%. The batted-ball speed will increase from the nominal value to its modified value of 41.114 m/s. This is an increase of 0.035 m/s. Therefore, these two changes, when performed individually, produce a total increase of 0.138, highlighted in blue. Now comes the important part, if you increase both m_{bat} and I_{bat} by 1% at the same time, then the batted-ball speed increases from the nominal value to a modified value of 41.216 m/s. This is an increase of 0.137 m/s, highlighted in green. Therefore, we can see that when these two changes are performed individually they produce a total increase of 0.138, however when performed together they produce an increase of 0.137 m/s. The whole is less than the sum of its parts. Here the interaction attenuates the individual changes.

Figure 4.3 shows the interaction of bat weight and bat moment of inertia (MoI) graphically. If you increase the bat weight, the batted-ball speed goes up. However,

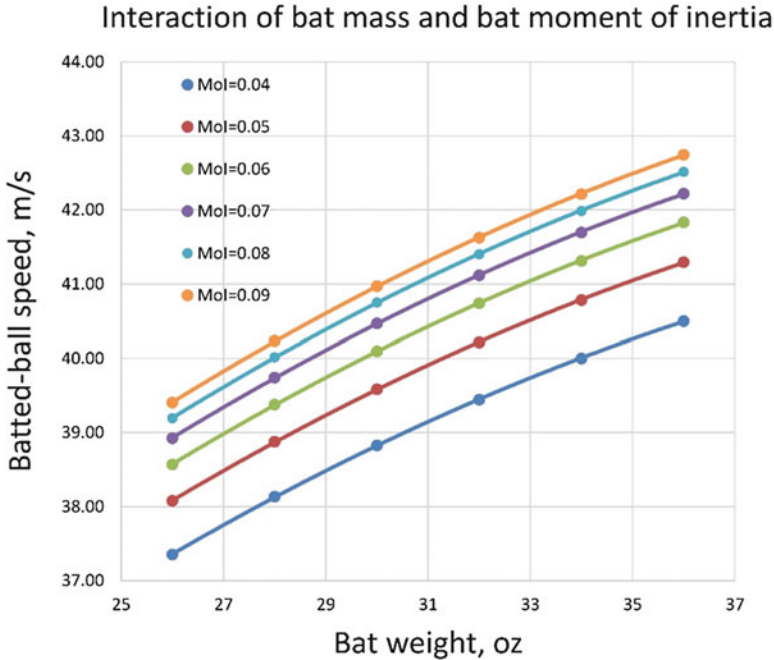


Fig. 4.3 Interaction of bat mass and bat moment of inertia

these six curves do not have the same shape. The curve for $MoI = 0.4$ starts to saturate at the right side. However, the curve for $MoI = 0.9$ does not flatten as much at the right side. This is the effect of the interaction. The difference in spacing of the lines is not the effect of the interaction. That is merely the dependence of the batted-ball speed on the moment of inertia (MoI).

4.11.2.2 Humidor

The Colorado Rockies store their baseballs in a humidor at 50% relative humidity and 70 °F. According to the appendix of Chap. 7, on a typical July afternoon in Denver the relative humidity is 34% and the average temperature is 88 °F. According to Alan Nathan (<http://www.baseballprospectus.com/article.php?articleid=13057>), compared to storing the balls in an outdoor environment, storing the balls in a humidor decreases the coefficient of restitution (because the balls get mushier, see Fig. 3.1) and increases the weight of the balls (because they absorb water): these two effects reduce the number of home runs in this stadium by 25%. However, this conclusion must be tempered, because there is an interaction between changes in CoR and m_{ball} . You cannot just say if $CoR \downarrow, v_{ball-after} \downarrow$ and if $m_{ball} \uparrow, v_{ball-after} \downarrow$ therefore if $CoR \downarrow$ and $m_{ball} \uparrow$, then $v_{ball-after} \downarrow$.

In our sensitivity analysis, we increased the value of each parameter by 1%. It told us that if you increase the CoR by 1%, then, according to the bottom rows of Table 4.5, the batted-ball speed will increase from its nominal value of 41.079 m/s to its modified value of 41.327 m/s. This is an increase of 0.248 m/s. Now if you increase m_{ball} by 1%, then the batted-ball speed will decrease from the nominal value to its modified value of 40.942 m/s. This is a decrease of 0.137 m/s. Therefore, these two changes, when performed individually, produce a total increase of 0.111, highlighted in blue in Table 4.5. Now comes the important part, if you increase both CoR and m_{ball} by 1% at the same time, then the batted-ball speed increases from the nominal value to a modified value of 41.189 m/s. This is an increase of 0.110 m/s, highlighted in green. Therefore, when these two changes are performed individually they produce a total increase of 0.111, however when performed together they produce an increase of only 0.110 m/s. The whole is less than the sum of its parts. Here the interaction attenuates the changes.

Therefore, to do a proper analysis, you cannot change one parameter, change the other parameter and then add the results. In your simulation, you must change both parameters at the same time.

Okay, that is the end of the sensitivity analysis of the BaConLaw model. Now let's go back to Coors Field in Denver. From the appendix in Chap. 7, we see that the relative humidity on an average July afternoon in Denver is 34%. Alan Nathan wrote that the difference between the 50% relative humidity in the humidor and the outside air in Denver causes a decrease of 3.7% in the CoR and an increase of 1.6% in the weight of the ball.

When those changes (and the parameters of a perfect home run ball) are put into the BaConLaw model *and* the Ball in Flight model of Chap. 7, we find that decreasing the CoR by 3.7% percent decreases the range of the batted ball by 8.5 feet. Increasing the weight of the ball by 1.6% increases the range of the batted ball by 1.6 feet. Summing these two changes gives a range decrease of 6.9 feet. But if the changes are made in the models at the same time the result is a range decrease of 9.1 feet. The whole is greater than the sum of its parts.

The Arizona Diamondbacks are considering installing a similar humidor in their stadium in Phoenix. Therefore, we should do a similar analysis for them. In addition, we should also do an analysis for the temperature differences. We should analyze the effects of storing the balls at 70 °F versus storing them at the average daily high temperature in Phoenix in July of 104 °F. But of course, this depends on where the balls are stored if they are not in a humidor and whether the dome is open or closed.

Interactions can amplify or attenuate the effects of drugs, chemicals and parameters in a model. Interactions mean that the numerical value of the sensitivity of a function to a particular parameter depends on the numerical value of another parameter. In a well-behaved model, the interaction terms are small. If the interaction terms are large, they warn that in your analysis you cannot change one

parameter, change another and then sum the results. You must have a model and simulation. And in you must change both parameters at the same time.

4.11.3 Accuracy

An important point about this section is that we computed the semirelative sensitivity values with two techniques: analytic equations and empirical (or numerical) estimates. To compare these two techniques, we note that using the empirical method the estimate for the +1% increment of the bat speed is

$$\tilde{S}_{v_{2b}}^{v_{1a}} = \frac{(1 + CoR_{2b})m_2I_2}{K} \Big|_{\text{NOP}} \quad v_{2b0} = 28.203485470404$$

Whereas, the analytic method as in Table 4.4 gives the following exact value.

$$\tilde{S}_{v_{2b}}^{v_{1a}} = \frac{(1 + CoR_{2b})m_2I_2}{K} \Big|_{\text{NOP}} \quad v_{2b0} = 28.203485470399$$

With a 10% change in the variable values, the match would be worse. With a 0.1% change, the match would be better.

This analysis has only included the equations of physics. Later, in Sect. 4.12, we will consider principles of physiology. In that section we will recommend that batters choose lightweight end-loaded bats.

A second purpose of this book is to show how the batter can buy or make an optimal baseball or softball bat. From the viewpoint of the batter, the batted-ball speed is the most important output. The larger it is the more likely the batter will get on base safely (Baldwin and Bahill 2004).

4.11.4 Optimizing with Commercial Software

We applied *What'sBest!*, a subset of the LINGO solvers, to our model. We constrained each variable to stay within physically realistic limits under natural conditions. Such values are shown in Table 4.4. We have previously gotten good results using this technique when doing empirical sensitivity analyses (Bahill et al. 2009). Then we asked the optimizer to give us the set of values that would maximize batted-ball speed. The optimizer applied a nonlinear optimization program. Surprisingly, the results were almost the same as in Table 4.4! That is, for variables and parameters with positive sensitivities, the optimizer chose the maximum values. For variables and parameters with negative sensitivities, the

Table 4.6 Sensitivities with nominal and optimal values for the variables and parameters

Semirelative sensitives of the batted-ball velocity with respect to	With nominal values	With optimal values
Inputs		
$v_{\text{ball}} - \text{before}$	8	12
$\omega_{\text{ball}} - \text{before}$	0	0
$v_{\text{bat}} - \text{cm} - \text{before}$	28	36
$\omega_{\text{bat}} - \text{before}$	5	5
Parameters		
CoR_{2b}	25	32
$d_{\text{cm} - \text{ip}}$	-2	+0.4
m_{ball}	-14	-13
m_{bat}	10	12
$I_{\text{bat}} - \text{cm}$	3	1

optimizer chose the minimum values. For the parameter with both negative and positive sensitivities, the optimizer chose the optimal value.

Using all of the optimal values at the same time increased the batted-ball speed from 92 to 117 mph (41–52 m/s). Using this optimal set of values changed the sensitivities, as shown in Table 4.6.

1. The numerical sensitivity values mostly increased. This is a direct result of the definition of the semirelative sensitivity function where the partial derivative is multiplied by the variable or parameter value. If the variable or parameter value increases, then the sensitivity value also increases.
2. The rank order stayed the same except that the output became more sensitive to the linear velocity of the ball before the collision, $v_{\text{ball}} - \text{before}$ than to the mass of the bat, m_{bat} . In the optimal set, both of these sensitivities increased, but because the value of the linear velocity of the ball before the collision, $v_{\text{ball}} - \text{before}$ changed from 37 to 40 m/s (Table 4.4) whereas the value of the mass of the bat, m_{bat} only changed from 0.905 to 0.964 kg, the change in the sensitivity to the linear velocity of the ball before the collision, $v_{\text{ball}} - \text{before}$ became bigger.
3. The optimizer found the optimum value for $d_{\text{cm} - \text{ip}}$ to be 10 cm (4 inches). Above this value, the semirelative sensitivity was negative; below this value, the sensitivity was positive. This is important. We could have found the same result if we had used the partial derivative of the batted-ball velocity with respect to the distance d , taken the derivative with respect to d and set it equal to zero, as in the following derivation by Ferenc Szidarovszky. We start with Eq. (4.8).

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - \omega_{2b}d)(1 + CoR_{2b})m_2I_2}{m_1I_2 + m_2I_2 + m_1m_2d^2}$$

Let

$$E = (1 + CoR_{2b})m_2I_2 \text{ and } F = m_1I_2 + m_2I_2$$

Then

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - \omega_{2b}d)E}{F + m_1m_2d^2}$$

and the

$$\text{numerator of } \left\{ \frac{\partial v_{1a}}{\partial d} \right\} = -E[m_1 m_2 d^2 \omega_{2b} + 2m_1 m_2 d(v_{1b} - v_{2b}) + B\omega_{2b}]$$

Now we set this equal to zero and solve for $d_{\text{max-velocity}}$.

$$d_{\text{max-velocity}} = \frac{(v_{1b} - v_{2b})}{\omega_{2b}} + \sqrt{\left[\frac{(v_{1b} - v_{2b})}{\omega_{2b}} \right]^2 + \frac{m_1 I_2 + m_2 I_2}{m_1 m_2}}$$

The batted-ball velocity has a maximum or a minimum at this value of d . To determine which, we derive the second partial derivative. The

$$\text{numerator of } \left\{ \frac{\partial^2 v_{1a}}{\partial d^2} \right\} = -E \left[m_1^2 m_2^2 d^3 \omega_{2b} - 2F m_1 m_2 d \omega_{2b} - \frac{F^2 \omega_{2b}}{d} \right] < 0$$

This is negative. Therefore, this value of $d_{\text{cm-ip}}$ gives the *maximum* batted-ball velocity, not the minimum. Using the numbers in Table 1.1, the optimum value for $d_{\text{cm-ip}}$ is 9.2 cm (3.6 inches).

This all means that the sensitivity analysis is robust. Its results remain basically the same after big changes in the variables and parameters.

We then tried a different optimization technique. Instead of constraining each variable to stay within realistic physical limits, we allowed the optimizer to change each variable by at most $\pm 10\%$ and then give us the set of values that maximized batted-ball speed. The numerical values of the sensitivities changed but the rank order stayed the same, except for $v_{\text{ball-before}}$ and m_{bat} just as it did with the realistic values technique.

Both empirical sensitivity analyses and optimization can constrain each variable to stay within specified realistic physical limits or change each variable by a certain percentage. Both techniques gave the same results. However, we prefer the former technique (Bahill et al. 2009).

We found an interesting relationship between the sensitivity analyses and optimization: they gave the same results! Because the interaction terms are small, for variables and parameters with positive sensitivities, the optimizer chose the maximum values and for variables and parameters with negative sensitivities, the optimizer chose the minimum values. Where the sensitivity function had both positive and negative slopes, it found the optimal value. But of course, this finding is not original. Sensitivity analyses are commonly used in optimization studies (Choi and Kim 2005). These studies typically apply sensitivity analysis after optimization. They try to find values or limits for the objective function or the right-hand sides of the constraints that would change the decisions. However, in our study, we applied optimization after the sensitivity analysis and we had only one variable in our objective function. Therefore, our problem was much simpler than sensitivity analyses in the optimization literature.

4.12 Optimizing the Bat

The following paragraphs are from *Major League Baseball 2016 Official Baseball Rules*.

3.02 (1.10) The Bat

- (a) The bat shall be a smooth, round stick not more than 2.61 inches in diameter at the thickest part and not more than 42 inches in length. The bat shall be one piece of solid wood.
- (b) Cupped Bats. An indentation in the end of the bat up to 1¼ inches in depth is permitted and may be no wider than 2 inches and no less than 1 inch in diameter. The indentation must be curved with no foreign substance added.
- (c) The bat handle, for not more than 18 inches from its end, may be covered or treated with any material or substance. Any such material or substance that extends past the 18-inch limitation shall cause the bat to be removed from the game.
- (d) No colored bat may be used in a professional game unless approved by the Rules Committee.

The second purpose of this chapter is to help the batter acquire an optimal baseball or softball bat. Therefore, we ask, “How can the batter use these sensitivity and optimization results to select or customize a bat that would be optimal for him or herself?” First, it is no surprise that the speed of the bat before the collision is the most important variable in Table 4.4. Its effect is shown in Fig. 4.4, where the slope of the line is the absolute sensitivity.

For Fig. 4.4, we computed the batted-ball velocity with $v_{\text{ball-after}} = v_{\text{ball-before}} - A m_{\text{bat}} I_{\text{bat}}$ and then we plotted the batted-ball speed as a function of the total bat speed before the collision. Remember that A is not a constant, it depends on

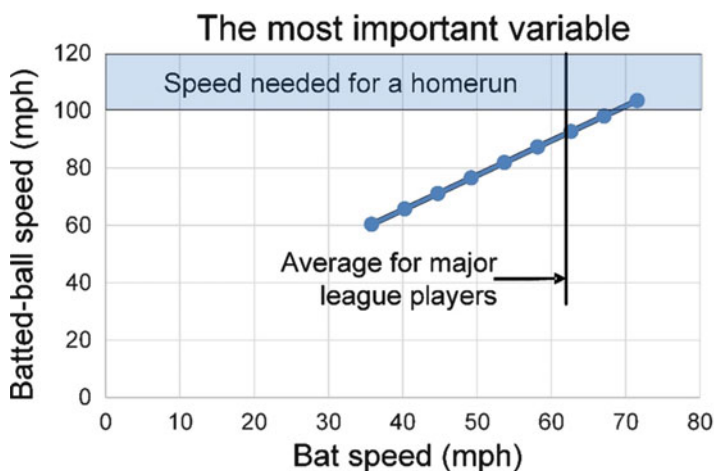


Fig. 4.4 Total bat speed before the collision is the most important variable in the BaConLaw model

the three inputs: the velocity of the ball and bat before the collision and the angular velocity of the bat before the collision. Figure 4.4 is for a pitch speed of 92 mph {a ball speed at contact of 83 mph}. This figure shows that an average hit is not a home run.

For decades, Little League coaches have taught their boys to practice and gain strength so that they could increase their bat speeds. They also said that it is very important to reduce the variability in the bat swings: Every swing should be the same. “Don’t try to kill the ball.” Given our new information, we now recommend that Little League coaches continue to give the same advice: increase bat speed and reduce variation. Practice is the key. Dave Baldwin (2007), a major-league pitcher with a career 3.08 ERA, sagaciously wrote that if you lose a game, don’t blame the umpire or your teammates; just go home and practice harder.

Using the Bat Chooser™, our measurements of over 300 batters showed that variability in the speed of the swing decreases as level of performance increases from Little League to Major League Baseball. For major leaguers the bat speed standard deviations were typically around $\pm 5\%$ (Bahill and Karnavas 1989), which is a small value for physiological data.

The variable with the second largest sensitivity is the coefficient of restitution (*CoR*). The *CoR* of a bat-ball collision depends on where the ball hits the bat. It is difficult, but absolutely essential, for the batter to control this. He or she must consistently hit the ball with the sweet spot of the bat. The *CoR* also depends on the manufacturing process. The NCAA now measures the Bat-ball Coefficient of Restitution (BBCOR) for sample lots coming off the manufacturing line. Therefore, amateurs are all going to get similar BBCORs. However, a lot can still be done with the *CoR* for aluminum and composite bats during their useful life. For example, the performance of composite bats typically improves with age because of the break-in process; repeatedly hitting the bat eventually breaks down the bat’s composite fibers and resinous glues. ‘Rolling’ the bat also increases its flexibility. Rolling the bat stretches the composite fibers and accelerates the natural break-in process simulating a break-in period of hitting, say, 500 balls.

For wooden bats, the batter could try to influence the *CoR* by choosing the type of wood that the bat is made of. Throughout history, the most popular woods have been white ash, sugar maple and hickory. However, hickory is heavy, so most professionals now use ash or maple. A new finding about bat manufacturing is that the slope of the grain has an effect on the strength and elasticity of the bat. As a result, the wood with the straightest grain is reserved for professionals and wood with the grain up to five degrees off from the long-axis of the bat is relegated to amateurs. Furthermore, the manufacturer’s emblem is stamped on the flat grain side of ash bats so that balls collide with edge grain as shown in Fig. 4.1, whereas the emblem is stamped on the edge grain side of maple wood bats (Fig. 4.6) because they are stronger when the collision is on the flat grain side.

The next largest sensitivities are for the mass of the ball and its velocity before the collision, m_{ball} and $v_{ball - before}$. However, the batter can do nothing to influence the mass of the ball or the ball velocity before the collision, so we will not concern ourselves with them. Likewise, the batter has no control over the ball

spin, $\omega_{\text{ball-before}}$, so we will ignore it when selecting bats. Now if this discussion were being written from the perspective of the pitcher (Baldwin 2007), then these three parameters would be important.

The next most important variable in Table 4.4 is the mass of the bat. Therefore, we will now consider the mass and other related properties of the bat. The sensitivity of the batted-ball speed with respect to the mass of the bat is positive, meaning (if everything else is held constant) as the mass goes up so does the batted-ball speed. However, the heavier bat cannot be swung as fast (Bahill and Karnavas 1989) due to the force-velocity relationship of human muscle, to be discussed in conjunction with Fig. 4.10. This physiological relationship was not included so far in the equations of this book because so far we only modeled the *physics* of the collision, notwithstanding physiology trumping physics in this case. The net result of physics *in conjunction with physiology* is that lighter bats are better for almost all batters (Bahill 2004).

Perhaps due to this general feeling that lighter bats are better, many professionals have ‘corked’ their bats. This reduced the mass of the bat, but because it also reduced the moment of inertia, it did not improve performance significantly (Nathan et al. 2011). However, it is now legal to make a one to two-inch diameter hole 1¼ inches deep into the barrel end of the bat (see Fig. 4.6). Most batters do this because it makes the bat lighter with few adverse effects. Other bat parameters that are being studied include the type of wood (density, strength, elasticity, straightness of the grain, etc.) and type of materials (density, strength, elasticity, break-in period, durability, type of Al alloy, etc.).

For an aluminum bat, some batters reduce the thickness of the barrel wall by shaving the inside of the barrel. This reduces the bat mass, which according to physics *and* physiology, increases batted-ball speed. However, it also reduces durability.

The distance between the center of mass of the bat and the center of percussion, $d_{\text{cm-cop}}$, is the next most important parameter. We presumed that the sweet spot of the bat was the center of percussion (CoP) of the bat. All batters try to hit the ball on the sweet spot of the bat. To help the batter, manufacturers of aluminum bats have been moving the CoP by moving the internal weight from the end of the bat toward the knob <http://www.acs.psu.edu/drussell/bats/cop.html>. It is now an annual game of cat and mouse. The manufacturers move the CoP, then the rule makers change their rules, then the manufacturers move . . . etc.

Finally, we come to the moment of inertia of the bat, I_{bat} , with respect to its center of mass. The physics, revealed with the sensitivity analysis, states that although the moment of inertia is one of the least important parameters, it would help to increase its value. More importantly, physiology showed that all batters would profit from using end-loaded bats (Bahill 2004). There are many ways to change the moment of inertia of a bat. Most aluminum bats start with a common shell and then the manufacturer adds a weight inside to bring the bat up to its labeled weight. The important question then becomes, *where* should the weight be added? It has been suggested that they add weight in the knob because this would comply with the regulations and would not decrease bat speed (Brancazio 1984).

However, the results of Bahill (2004) show that they should add the weight in the barrel end of the bat making it *end-loaded*. This will increase the batted-ball speed. For a wooden bat, the moment of inertia can be changed by cupping out the barrel end, adding weight to the knob or tapering the barrel end. Assume that the end of the barrel of a bat is only used to “protect” the outside edge of the plate: no one hits home runs on the end of the bat. Therefore, a professional could use a bat where the last 3 in (7 cm) were tapered from 2.61 inches (6.6 cm) down to $1\frac{3}{4}$ of an inch (4.4 cm). This would decrease the weight, decrease the moment of inertia about the knob and would move the center of mass closer to the knob: these changes would probably benefit some batters. However, such modifications would have to be individually designed for each player.

At this point, it may be useful to reiterate that an end-loaded bat is not a normal bat with a weight attached to its end. Adding a weight to the end of a normal bat would increase both the weight and the moment of inertia. This would *not* be likely to help anyone. In the design and manufacture of an end-loaded bat, the weight is distributed so that the bat has a normal weight but a larger than normal moment of inertia.

Most people can feel the difference between bats with different moments of inertia. A coach with the San Francisco Giants showed us a legal custom-made bat with a large moment of inertia created by leaving it with a huge knob. He presumed that his players already understood the influence of bat weight on bat speed so he was trying to expand their understanding to the influence of bat moment of inertia on the speed of the swing. One of our University of Arizona softball players described our biggest moment of inertia bat with, “That’s the one that pulls your arms out.”

Our best generalization is that almost all batters would profit from using end-loaded bats. Smith and Kensrud (2014) concluded their paper with “Batter swing speed decreased with increasing bat inertia, while ... the hit-ball speed increases with bat inertia.”

Summarizing, these are the most important factors for understanding bat performance: bat weight, the coefficient of restitution, the moment of inertia and characteristics of humans swinging the bats.

In the future, once equations for configurations 3 and 4 have been derived, it will be possible to see how the coefficient of friction μ_f affects the batted-ball speed. Then we will be able to decide if the varnish or paint on the bat should be made rough-textured or smooth, or if bats should be rubbed or oiled in order to improve bat performance.

To confuse fielders who are trying to locate the bat-ball collision point, perhaps the bat could be painted white with random thin red lines. Or perhaps bats could be painted pink supposedly to promote breast cancer research.

Figure 4.5 shows the outcomes of hitting the ball at different places on the front surface of the bat. We used this figure to help determine the size of the vertical sweet spot of the bat. It also suggests that putting oil on the top surface of the bat could change short pop-ups (sure outs) into innocuous foul tips.

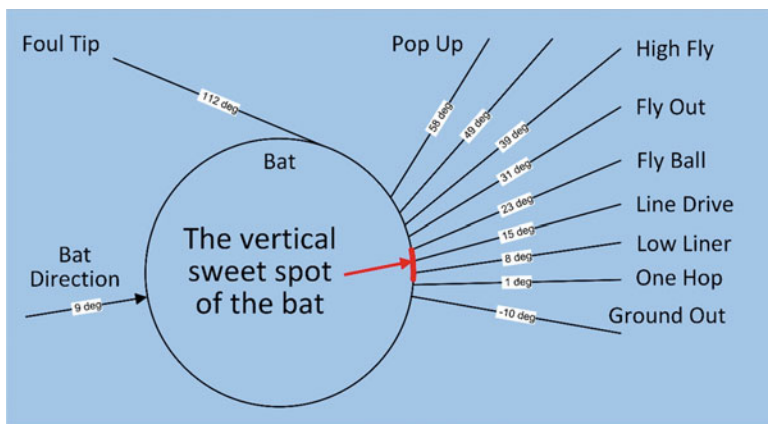


Fig. 4.5 Direction of the batted-ball as a result of hitting the ball at different places on the front surface of the bat

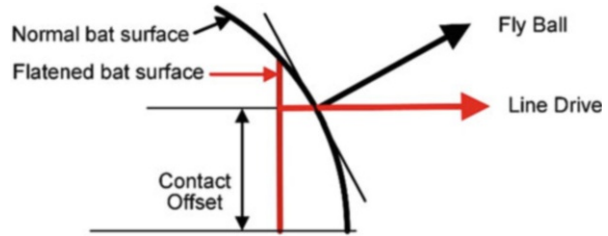
Fig. 4.6 Bat with an exaggerated flat front side. The actual size of the flat surface should not be larger than the vertical sweet spot, which is one-third of an inch high



What if the front side of the bat were flat, as in Fig. 4.6, instead of round? Then fly balls and grounders would be line drives instead. What could make the front side of the bat flat? Using sandpaper or a plane on the front side of a bat would make that front surface flat. Figure 4.7 shows that if the front of the bat were flat then a fly ball (black lines) would be changed into a line drive (red lines). This would increase performance. Most umpires would probably not notice or might accept a bat whose front surface had been sanded or planed. This would also reduce the variability in the batted-ball trajectories, because plus or minus 1 mm would yield the same result every time.

To improve bat performance manufacturers could reduce the variability of bat and ball parameters. Major-league bats were custom made for us by Hillerich and Bradsby Co. The manufacturing instructions were “Professional Baseball Bat, R161, Clear Lacquer, 34 inch, 32oz, make as close to exact as possible, end

Fig. 4.7 Enlarged schematic view of a bat with a flat front and resulting ball trajectories



brand - genuine model R161 pro stock, watch weights” emphasis added. The result was six bats with an average weight of 32.1 ounces and a standard deviation of 0.5! This large standard deviation surprised us. We assume there is the same variability in bats used by major-league players.

There is also variability in the ball. We might assume that the center of mass of the ball is coincident with the geometric center of the ball. However, put a baseball or softball in a bowl of water. Let the movement subside. Then put an X on the top the ball. Now spin it and let the motion subside again. The X will be on top again. This shows that for most baseballs and softballs the center of mass is *not* coincident with the geometric center of the ball.

4.12.1 Summary of Bat Selection

These sensitivity and optimality analyses showed that the most important variable, in terms of increasing batted-ball speed, is bat speed before the collision. This is in concert with ages of baseball folklore and principles of physiology. Therefore, batters should develop strength, increase coordination and practice so that their swings are fast and with low variability.

These analyses showed that the next most important variable is the coefficient of restitution, the *CoR*. Engineers and bat regulators are free to play their annual cat and mouse game of increasing *CoR* then writing rules and making tests that inhibit these changes. Indeed, most recent bat research has gone into increasing the *CoR* of bat-ball collisions.

Pitch speed, ball spin and the mass of the ball are important. However, the batter cannot control them. Therefore, they cannot help the batter to choose or modify a bat.

The next most important parameter is the bat mass, m_{bat} . However, physics recommends heavy bats, whereas the force-velocity relationship of muscle recommends light bats. In this case, physiology trumps physics. Each person’s preferred bat should be as light as possible while still fitting within baseball needs, regulations and availability.

The last interesting parameter from the sensitivity analysis and the optimization study is the bat moment of inertia, I_{bat} . The sensitivity analysis suggested that a

larger bat moment of inertia would be better. However, old studies in the physics of baseball literature recommended smaller moments of inertia. An experimental physiology study stated that all players would profit from using end-loaded bats (Bahill 2004). Since then most studies have recommended bats with higher moments of inertia (Cross 2011; Smith and Kensrud 2014; Crisco et al. 2014).

The second purpose of this book is to show what the batter can do to achieve optimal bat performance. The most important thing is practice. Next, batters should select lightweight bats. They should then select bats that increase the *CoR* by all legal means. Finally, they should choose end-loaded bats.

4.12.2 *The Ideal Bat Weight*

So far, the equations in this book were equations of physics. However, we repeatedly mentioned physiology. Now it is time to look at physiology. This section is based on Bahill and Karnavas (1991).

Our instrument for measuring bat speed, the¹ Bat Chooser™, had two vertical laser beams, each with associated light detectors. Our subjects swung bats through the laser beams. A computer recorded the time between interruptions of the light beams. Knowing the distance between the light beams and the time required for the bat to travel that distance, the computer calculated the speed of the sweet spot, which we defined as the center of percussion.

The computer told the batters to swing each bat as fast as they could while still maintaining control. It said, “Pretend you are trying to hit a Nolan Ryan fastball.”

In our experiments, each batter swung six bats through the light beams. The bats ran the gamut from super-light to super-heavy; yet they had similar lengths and weight distributions. In our developmental experiments, we tried about four dozen bats. We used aluminum bats, wooden bats, plastic bats, heavy metal warm-up bats, bats with holes in them, bats with lead in them, major league bats, college bats, softball bats, Little League bats, brand-new bats and bats made in the 1950s.

In one of our first set of experiments (Bahill and Karnavas 1989), we used six bats of significantly different weights but which were all about 34 inches (89 cm) long, with a center of mass about 23 inches (58 cm) from the end of the handle. They are described in Table 4.7 and Fig. 4.8.

In a 20-min time interval, each subject swung each bat through the instrument five times. The order of presentation was randomized. The selected bat was announced by a speech synthesizer, for example: “Please swing bat Hank Aaron, that is, bat A.” (We named our bats after famous baseball players who had names starting with the letter assigned to the bat.)

¹Bat Chooser and Ideal Bat Weight are trademarks of Bahill Intelligent Computer Systems.

Table 4.7 Test bats used by the major league batters

Name	Weight (oz)	Mass (kg)	Distance from knob to center of mass (in)	Distance from knob to center of mass (m)	Average sweet-spot speed (mph) also given in Fig. 4.10	Average sweet-spot speed (m/s)	Description of the bat
D	49.0	1.39	22.5	0.57	60	27	Aluminum bat filled with water
C	42.8	1.21	24.7	0.63	61	27	Wooden bat with lead in the barrel
A	33.0	0.94	23.6	0.60	65	29	Wooded bat
B	30.6	0.87	23.3	0.59	65	29	Wooden bat
E	23.6	0.67	23.6	0.60	74	33	Wooden fungo bat
F	17.9	0.51	21.7	0.55	88	40	Wooden handle mounted on a light steel pipe with a six ounce weight at the end

Fig. 4.8 Our first set of experimental bats (Photo credit Richard Harding)



For each swing, we recorded the bat weight and the speed of the center of mass, which we converted to the speed of the center of percussion. That was as far as physics could take us; we then had to look to the principles of physiology.

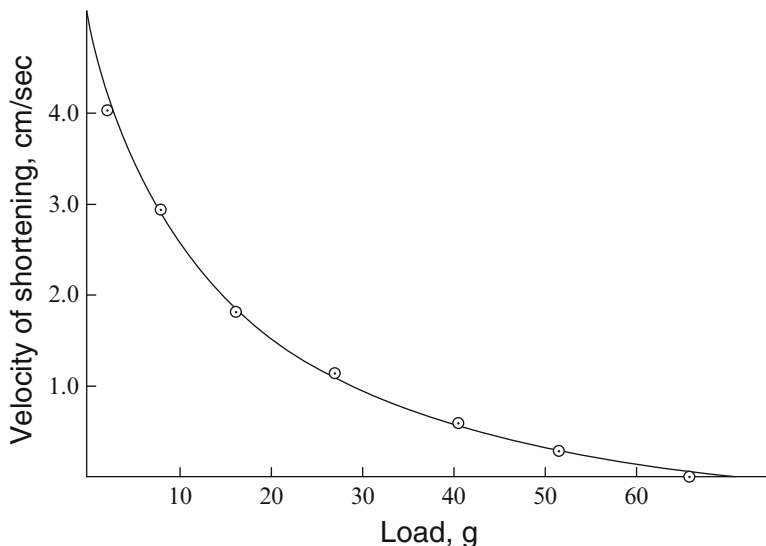


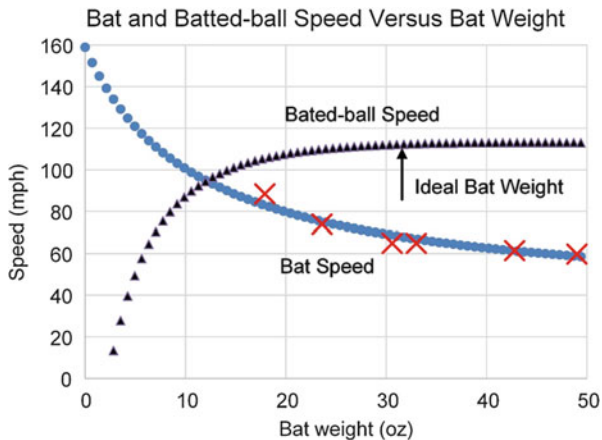
Fig. 4.9 Hill's original force-velocity relationship figure. He fit the following equation to his data: $(P + 14.35)(v + 1.03) = 87.6$ where P is the load in grams and v is the velocity in cm/s (Hill 1938)

Physiologists have long known that muscle speed decreases with increasing load as shown in Fig. 4.9. This is why bicycles have gears; gears enable riders to maintain the muscle speed that imparts maximum power through the pedals, while the load, as reflected by the bicycle speed, varies greatly. To discover how the muscle properties of individual baseball players affect their Ideal Bat Weights, for each batter, we plotted bat speed as a function of bat weight to produce a graphical model known as a muscle force-velocity relationship as shown in Fig. 4.10. The red Xs represent the average of the five swings of each bat; the standard deviations were small for physiological data (smaller than the red Xs). These standard deviations were shown in Bahill and Karnavas (1991).

Traditionally, physiologists have used three types of equations to describe the force-velocity relationship of muscles: straight lines, hyperbolas and exponentials. Each type of equation has produced the best fit for some experimenters, under certain conditions and with certain muscles. However, usually the hyperbola fits the data best. In our experiments, we tried all three equations and chose the one that had the best fit to the data of each batter's 30 swings. For the data of the force-velocity relationship illustrated in Fig. 4.10, we found that a hyperbola provided the best fit.

These curves indicate how bat speed varies with bat weight. We now want to find the bat weight that will make the ball leave the bat with the highest speed and thus have the greatest chance of eluding the fielders (Baldwin and Bahill 2004). We call this the maximum-batted-ball-speed bat weight. To calculate this bat weight we must couple the muscle force-velocity relationships to the equations of physics.

Fig. 4.10 Measured bat speed (red Xs), a hyperbolic fit to these data (blue dots) and the calculated batted-ball speed (black triangles) for a 90 mph pitch to one of the fastest San Francisco Giants



For the major league batter whose data are shown in Fig. 4.10, the best fit for his force-velocity data was the hyperbola, $(m_{\text{bat}} + 11) \times (v_{\text{bat-before}} - 36) = 1350$ units are ounces and mph, that is shown with blue dots. This batter had some of the fastest swing speeds on the team. When we substituted this equation into the batted-ball velocity equation, Eq. (4.8), we were able to plot the ball speed after a collision as a function of bat weight, shown with black triangles in Fig. 4.10.

$$(m_{\text{bat}} + 11) \times (v_{\text{bat-before}} - 36) = 1350$$

Solve for the bat velocity

$$v_{\text{bat-before}} = \left\{ \frac{36m_{\text{bat}} + 1746}{m_{\text{bat}} + 11} \right\}$$

Now we substitute this into Eq. (4.8)

$$v_{1a} = \frac{v_{1b}(m_1 I_2 - m_2 I_2 \text{ CoR}_{2b} + m_1 m_2 d^2) + v_{2b} m_2 I_2 (1 + \text{CoR}_{2b}) + \omega_{2b} m_2 d I_2 (1 + \text{CoR}_{2b})}{m_1 I_2 + m_2 I_2 + m_1 m_2 d^2}$$

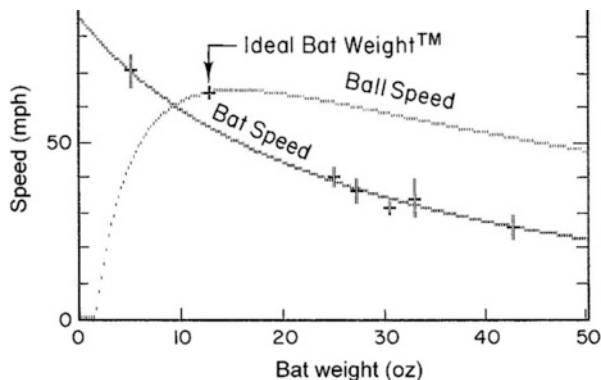
to get the batted-ball velocity

$$v_{\text{ball-after}} = v_{\text{ball-before}} \frac{\left(m_{\text{ball}} I_{\text{cm}} - m_{\text{bat}} I_{\text{cm}} \text{ CoR}_{2b} + m_{\text{ball}} m_{\text{bat}} d_{\text{cm-ip}}^2 \right)}{K} + \left\{ \frac{36m_{\text{bat}} + 1746}{m_{\text{bat}} + 11} \right\} \frac{m_{\text{bat}} I_{\text{cm}} (1 + \text{CoR}_{2b})}{K} + \omega_{\text{bat-before}} \frac{m_{\text{bat}} d I_{\text{cm}} (1 + \text{CoR}_{2b})}{K}$$

In this equation, I_{cm} is also a function of m_{bat} .

This equation produced the curve composed of black triangles in Fig. 4.10. This curve shows that the maximum-batted-ball-speed bat weight for this batter is about 45 ounces, which is heavier than that used by most batters. However, this batted-ball

Fig. 4.11 Bat speed and calculated batted-ball speed after the collision both as functions of bat weight for a 40 mph pitch to Alex, a 10-year old Little League player. The dots represent the average of the five swings of each bat; the vertical bars on each dot represent the standard deviations



speed curve is almost flat between 30 and 49 ounces. Notably, this player normally used a 32-ounce bat. Evidently the greater control permitted by the 32-ounce bat outweighed the 1% increase in speed that could be achieved with the 45-ounce bat.

The maximum-batted-ball-speed bat weight is not the best bat weight for any player because a lighter bat will give a batter better control and more accuracy. Obviously, a trade-off must be made between batted-ball speed and control. Because the batted-ball speed curve is so flat around the point of the maximum-batter-ball-speed, we believe there is little advantage in using a bat as heavy as the maximum-batter-ball-speed bat weight. Therefore, we defined the *1**Ideal Bat Weight*TM to be the weight where the ball speed curve drops 1% below the maximum-batted-ball speed. Using this criterion, the Ideal Bat Weight for this batter is 31.75 ounces. We believe this gives a good trade-off between batted-ball speed and accuracy. For this batter, the batted-ball speed is nearly flat around the ideal bat weight. So it does not seem to be critical. But for most other batters this was not true, as is shown in Fig. 4.11.

The Ideal Bat Weight is specific to each individual; it is not correlated with height, weight, age, circumference of the upper arm, or any combination of these factors, nor is it correlated with any other obvious physical factors. Nevertheless, Bahill and Morna Freitas (1995) mined their database of 163 subjects and 36 factors and determined some rules of thumb that could make suggestions. For example, for a general 9 or 10 year old Little Leaguer, the recommended bat weight in ounces would be height in inches divided by three plus four ounces, $recommended\ bat\ weight = \frac{height}{3} + 4$. Table 4.8 shows their recommendations.

In conclusion, there is an ideal bat weight for each batter. It can be measured in a laboratory or it can be estimated using rules of thumb like those in Table 4.8.

Table 4.8 Simple integer models for recommending bat weights

Group	Recommended bat weight (oz)
Baseball, major league	Height/3 + 7
Baseball, amateur	Height/3 + 6
Softball, fast pitch	Height/7 + 20
Softball, slow pitch	Weight/115 + 24
Junior league (13 & 15 years)	Height/3 + 1
Little league (11 & 12 years)	Weight/18 + 16
Little league (9 & 10 years)	Height/3 + 4
Little league (7 & 8 years)	Age*2 + 4

Age (years); height (inches); body weight (pounds)

4.12.3 Bat Moment of Inertia

The bat moment of inertia is an enigma because for most, but not all, batters as the bat moment of inertia goes up the bat speed goes down (Bahill 2004; Cross 2011; Smith and Kensrud 2014; Crisco et al. 2014). For Bahill’s (2004) women softball batters, 80% had negative slopes for bat speed versus the moment of inertia.

Now we need a model for these data. Because of the positive and negative slopes, averaging the data makes no sense. Therefore, we chose one of the All Americans in our database as our model. Her data were fit with the equation

$$v_{\text{sweet spot-before}} = -22I_{\text{bat-center of mass}} + 30$$

(4.12)

where the bat velocity has units of m/s and the inertia has units of kg · m². The eight bats in our variable inertia experiments had moments of inertia about the center of mass in the range of 0.03–0.09 kg · m². Typical bats used by players on this team had moments of inertia of around 0.05.

In these experiments, we used the bats described in Table 4.9. They decoupled the mass and moment of inertia, because they had nearly identical masses but different moments of inertia. That is, in each set the masses were close to the same value, although the moments of inertia varied widely.

4.12.4 Modifying the Bat

Previously we mentioned that for wooden bats, it is legal to taper the last 3 inches of the barrel from 2.61 inches (6.6 cm) down to 1¾ of an inch (4.4 cm). This modification would decrease the bat weight, decrease the moment of inertia and move the center of mass closer to the knob.

Table 4.9 Properties of the variable moment of inertia bats

Name	Period of oscillation (sec)	Mass (kg)	Distance from knob to center of mass, d_{k-cm} (m)	Moment of inertia with respect to the knob, I_{knob} (kg-m ²)	Moment of inertia with respect to the center of mass, I_{cm} (kg-m ²)
Aluminum bat set					
A	1.648	0.824	0.496	0.275	0.072
B	1.682	0.824	0.494	0.286	0.085
C	1.689	0.824	0.520	0.303	0.080
D	1.702	0.833	0.526	0.316	0.086
Bats with a wooden handle and a brass disk mounted on a threaded rod, similar to bat F in Fig. 4.8					
Red bat	1.443	0.799	0.427	0.176	0.030
Blue bat	1.493	0.807	0.458	0.204	0.035
Green bat	1.563	0.801	0.493	0.239	0.044
Yellow bat	1.631	0.805	0.509	0.270	0.061

4.12.4.1 The R161 Bats

Hillerich and Bradbury made six such R161 Louisville Slugger wooden bats for us. When we compared these six bats to six of their unmodified R161 bats, we found that, on average, this modification reduced the mass by 5%, reduced the moment of inertia about the center of mass by 5.2% and reduced the distance from the knob to the center of mass by 1.6%.

That last paragraph described the measured physical changes to the bat. Next, we wanted to see how those changes coupled with human physiology to affect the bat speed. First, we used the data of Fig. 4.10 at its nominal bat weight of 31.75 ounces and found that a 5% decrease in bat mass increased the bat speed by 1.7%. Next, we used Eq. (4.12) and found that a 5.2% decrease in the moment of inertia about the center of mass increased the bat speed by 0.2%. Summing these changes gave a total increase in bat speed of 1.9%. Figure 4.12 shows these numbers.

That takes care of the physical changes of the bat and how those changes couple with physiology to affect bat speed. Now, we are finally ready to use the physics captured in the BaConLaw model.

According to the sensitivity analysis of the BaConLaw model summarized in Table 4.4, decreasing m_{bat} and I_{cm} would decrease batted-ball speed, whereas decreasing the distance between the center of mass and the center of percussion (the sweet spot) would increase batted-ball speed.

Semirelative sensitivity values from Table 4.4		
m_{bat}	10	$m_{bat} \downarrow, v_{ball} - after \downarrow$
I_{bat-cm}	3	$I_{bat-cm} \downarrow, v_{ball} - after \downarrow$
d_{cm-cop}	-2	$d_{cm-cop} \downarrow, v_{ball} - after \uparrow$

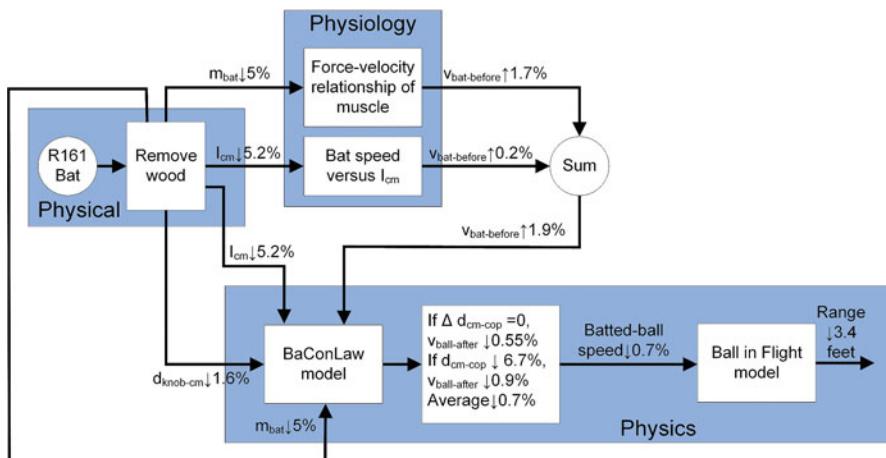


Fig. 4.12 Analysis process and numerical values for the tapered R161 Bat

To see the changes in bat speed, we modified the inputs to the BaConLaw model for the modified R161 bat. We decreased the mass by 5%, decreased the moment of inertia about the center of mass by 5.2% and increased the bat speed by 1.9%. This gave us a new smaller batted-ball speed. We will now show how the distance between the knob and the center of mass affects this smaller batted-ball speed. We know that the distance from the knob to the center of mass decreased by 1.6%. However, we do not have data for the change in distance from the center of mass to the center of percussion as wood is removed from the barrel end of the bat. However, we can bracket that change. If we assume that the *distance* from the center of mass to the center of percussion stays fixed, then the batted-ball speed decreases by 0.55%. On the other hand, if we assume that the *center of percussion* stays fixed, while the distance from the knob to the center of mass decreases by 1.6%, then the distance between the center of mass and the center of percussion increases by 6.7% and the batted-ball speed decreases by 0.9%. Let us average the results of those two assumptions, and say that the new batted-ball speed decreases by an additional 0.73%.

When we put this decreased batted-ball speed into the Ball in Flight model of Chap. 7, we found that the distance of a perfectly hit home run ball *decreased by three feet!*

Vedula and Sherwood (2004) performed a finite element analysis of wooden baseball bats. They found that if they reduced the mass in the barrel end of the bat by 10%, then the distance between the center of mass and the center of percussion increases by 5% and the batted-ball speed decreases by 1.7%. This matches our results quite well.

This is a very surprising result. It states that tapering the last 3 inches of the barrel will *not* increase the batted-ball speed or the ball's range.

4.12.4.2 The C243 Bat

Because this result was so surprising, we repeated the analysis with another bat that had its barrel end cupped out, as shown in Fig. 4.6. We measured the volume of the cupped out hole in the end of a Louisville Slugger C243 bat. It was 25 cc. The density of white ash is 0.6 that of water. Therefore, cupping the bat reduced its mass by 15 g, or 1.7%. Theoretically, using $I_{\text{cm-after-cupping}} = I_{\text{cm-before-cupping}} - m_{\text{cup}}d_{\text{cm-end}}^2$, this should reduce the inertia at the center of mass by $0.0012 \text{ kg} \cdot \text{m}^2$ or 2.2%. Finally, the last of the three parameters changed by cupping, the measured distance from the knob to the center of mass, was reduced by 1.7%.

That last paragraph described the measured physical changes to the bat. Now we want to see how those changes couple with physiology to affect the bat speed. First, we used the data of Fig. 4.10 at its nominal operating point of 31.75 ounces, and found that an 1.7% decrease in bat mass increased the bat speed by 0.57%. Next, we used Eq. (4.12) and found that a 2.3% decrease in the moment of inertia about the center of mass produced an 0.1% increase in bat speed. Summing these increases gives a total increase in bat speed of 0.67%. This is probably why bat manufacturers cup the ends of their bats. Because they know that cupping the end of the bat increases the bat speed.

That takes care of the physical changes of the bat and how those changes couple with physiology to effect bat speed. Now, we are finally ready to use the physics captured in the BaConLaw model. To see the changes in bat speed, we modified the inputs to the BaConLaw model for the C243 bat. We decreased the mass by 1.7%, decreased the moment of inertia about the center of mass by 2.3% and increased the bat speed by 0.67%. The measured distance from the center of mass to the center of percussion increased by 4.3%. When we changed these four parameters in the BaConLaw model, the batted-ball speed decreased by 0.25%.

Finally, when we put this decreased batted-ball speed into the Ball in Flight model of Chap. 7, we found that the distance of a perfectly hit home run ball *decreased by one foot*. Figure 4.13 shows our process and captures these numbers.

Both of the bat modifications described here {tapering the barrel and cupping the barrel end}, remove wood from the end of the bat. This decreases the bat mass, moment of inertia and distance from the knob to the center of mass. This should be true for any wooden bat. Physiology shows that the first two changes {reducing the mass and the moment of inertia} increase the bat speed. This is the main reason for making these modifications. Increasing the bat speed will increase the batted-ball speed.

For the tapered bat, decreasing the distance from the knob to the center of mass by 1.6% increased the distance between the center of mass and the center of percussion somewhere between zero and 6.7%. For the cupped bat, this distance was measured as an increase of 4.3%. Both methods increased the distance between the center of mass and the center of percussion. We are satisfied with these approximations, because the model is not very sensitive to this distance. Changing this distance by $\pm 4\%$ only changed the batted-ball speed by, on average, $\pm 0.08\%$.

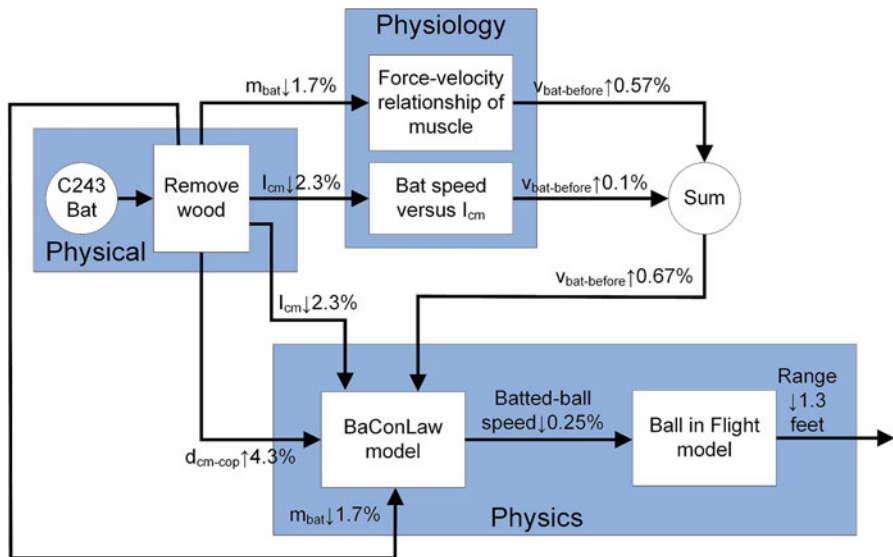


Fig. 4.13 Analysis process and numerical values for the modified C243 Bat

There are four inputs to the BaConLaw model. When wood is removed from the end of the bat the first two (m_{bat} and $I_{\text{bat-cm}}$) decrease, which decreases the batted-ball speed. However, the bat speed increases, which increases the batted-ball speed. The last input, the distance from the center of mass to the center of percussion, probably increases, which also decreases the batted-ball speed. Which of these four changes wins? We can only tell by deriving values for the parameters and using those in the equations of the BaConLaw model.

In the two modified bat examples that we examined in this section, the modified bats caused the batted-ball speed and therefore the ball's range to go down.

In conclusion, both tapering the barrel and cupping the barrel end of the bat decrease the batted-ball speed and subsequently decreases the range of the batted-ball. Why then would batters choose bats with the end cupped out? Perhaps it is because they are more comfortable with the cupped bat, they don't understand the interaction of the parameters or the decrease in performance is small.

4.13 Outline of the BaConLaw Model Derivations

We started with Eq. (4.6) and solved for the bat angular velocity after the collision

$$dm_1(v_{1a} - v_{1b}) = -I_2(\omega_{2a} - \omega_{2b})$$

$$\omega_{2a} = \left\{ \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b}) \right\}$$

Next we used Eq. (4.5) and solved for the bat linear velocity after the collision

$$CoR_{2b} = -\frac{v_{1a} - v_{2a} - d\omega_{2a}}{v_{1b} - v_{2b} - d\omega_{2b}}$$

$$v_{2a} = v_{1a} + CoR_{2b}(v_{1b} - v_{2b} - d\omega_{2b}) - d\omega_{2a}$$

Then we substituted ω_{2a} into the above v_{2a} equation to get

$$v_{2a} = v_{1a} + CoR_{2b} (v_{1b} - v_{2b} - d\omega_{2b}) - d \left\{ \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b}) \right\}$$

Finally, we used this v_{2a} in Eq. (4.4) to get Eq. (4.8) for the ball linear velocity after the collision, in terms of only the before collision variables and parameters. The linear velocity of the ball after the collision is

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d_{\text{cm-ip}})(1 + CoR_{2b})m_{\text{bat}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

Then we solved Eqs. (4.4), (4.5) and (4.6) for the velocity of the bat after the collision in terms of only the before collision variables and parameters. The linear velocity of the bat after the collision is

$$v_{\text{bat-cm-after}} = v_{\text{bat-cm-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d_{\text{cm-ip}})(1 + CoR_{2b})m_{\text{ball}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

Lastly, we solved Eqs. (4.4), (4.5) and (4.6) for the angular velocity of the bat after the collision, ω_{2a} , in terms of only the before collision variables and parameters. The angular velocity of the bat after the collision is

$$\omega_{\text{bat-after}} = \omega_{\text{bat-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}})(1 + CoR_{2b})m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

To get a final equation for the angular velocity of the ball after the collision, ω_{1a} , we put $\omega_{2a} = \omega_{2b} - \frac{dm_1}{I_2}(v_{1a} - v_{1b})$ into the Conservation of Angular Momentum equation, Eq. (4.7s), and showed that for a head-on collision (with no friction or external forces) like this BaConLaw model, $\omega_{\text{ball-after}} = \omega_{\text{ball-before}}$.

4.14 Summary

In this chapter, we successfully incorporated Conservation of Energy into the set of bat-ball collision equations for the BaConLaw model. This Conservation of Energy equation confirmed the consistency of our set of derived equations. We also used Conservation of Energy to derive an equation for the kinetic energy lost in the collision. We derived a general equation for KE_{lost} , Eq. (4.11), and showed that if the collision were at the center of mass ($d_{\text{cm-ip}} = 0$), then this general equation reduced to an old well-known result, Eq. (3.2).

We did a sensitivity analysis on the set of equations for the BaConLaw model. It showed that the most important variable, in terms of increasing batted-ball speed, is the bat speed before the collision. Today in the sporting world, the coefficient of restitution and the bat mass are experiencing the most experimentation for improving bat performance. However, in the future, bat moments of inertia allow for the most improvement of bat performance. Most importantly, future studies must include physics in conjunction with physiology.

The following equations comprise our BaConLaw model for bat-ball collisions. First, the kinetic energy lost or transformed.

$$KE_{\text{lost}} = \frac{1}{2} \frac{m_{\text{ball}}m_{\text{bat}}I_{\text{bat}}(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d_{\text{cm-ip}})^2(1 - CoR_{2b}^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

where $d_{\text{cm-ip}}$ is the distance between the bat's center of mass and the impact point. The linear velocity of the ball after the collision is

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d_{\text{cm-ip}})(1 + CoR_{2b})m_{\text{bat}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

The linear velocity of the bat after the collision is

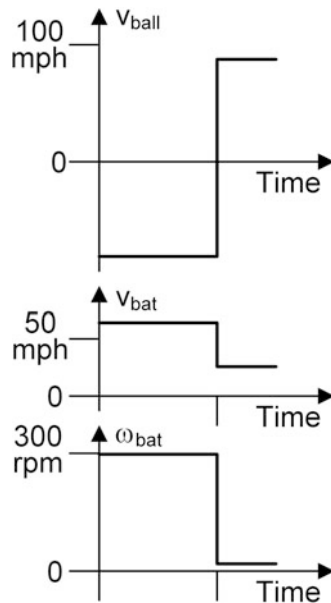
$$v_{\text{bat-cm-after}} = v_{\text{bat-cm-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}}d_{\text{cm-ip}})(1 + CoR_{2b})m_{\text{ball}}I_{\text{bat}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

The angular velocity of the bat after the collision is

$$\omega_{\text{bat-after}} = \omega_{\text{bat-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}})(1 + CoR_{2b})m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d_{\text{cm-ip}}^2}$$

Our most succinct formulation of the BaConLaw model is

Fig. 4.14 Linear and angular velocities of the ball and bat



$$A = \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}} \omega_{\text{bat-before}})(1 + CoR_{2b})}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d_{\text{cm-ip}}^2}$$

$$CoR_{2b} = - \frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}} \omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}} \omega_{\text{bat-before}}}$$

$$v_{\text{ball-after}} = v_{\text{ball-before}} - A m_{\text{bat}} I_{\text{bat}}$$

$$v_{\text{bat-after}} = v_{\text{bat-before}} + A m_{\text{ball}} I_{\text{bat}}$$

$$\omega_{\text{bat-after}} = \omega_{\text{bat-before}} + A m_{\text{ball}} m_{\text{bat}} d_{\text{cm-ip}}$$

$$\omega_{\text{ball-after}} = \omega_{\text{ball-before}}$$

The BaConLaw model describes the motion of the bat after the collision. This is a big deal. Many models describe the motion of the ball after the collision, but few (if any) describe the motion of the bat. When you see a batter hit a ball, do you see the jerk of the bat? Can you describe it? Well these equations do, as shown in Fig. 4.14.

This model for bat-ball collisions gives the linear and angular velocity of the bat and ball after the collision in terms of the linear and angular velocity of the bat and ball before the collision. It uses only the fundamental principles of Newtonian mechanics and the conservation laws. This chapter also fulfills the second purpose of this book, namely to show what the batter can do to achieve an optimally performing bat, namely select lightweight, end-loaded bats. Finally, cupping the barrel end of the bat does not increase the ball's range.

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Chapter 5

Alternative Models

5.1 Introduction

Purpose: The purpose of this chapter is to present four alternatives to the BaConLaw model, explain their different purposes and explain why each might be used for a different purpose.

This chapter contains four models that are more complicated than our BaConLaw model of Chap. 4. The first one, the Effective Mass model, is an analog to the BaConLaw model. The bat Effective Mass model and the BaConLaw model both start with Newton's principles; then they diverge. They are different; however, they yield the same rule of thumb for the batted-ball speed! This should strengthen and give people more confidence in both models. The second and third models in this chapter allow movement of the knob. The Spiral Center of Mass model shows the movement of the center of mass of the bat before the collision. The Sliding Pin model analyzes the movement of the bat with a translation and a rotation about its *knob*. It illustrates the concept of using different models for different purposes. The fourth model challenges our simple technique of using only Newton's principles and the conservation laws. It is for a collision at the center of percussion of the bat with spin on the pitch and with consideration of friction between the bat and ball. Its purpose is to show a situation that cannot be modeled using only the conservation laws.

Modeling philosophy note Having several alternative models helps ensure that you understand the physical system. No model is more correct than another. They just emphasize different aspects of the physical system. They are not competing models they are synergetic.

5.2 Bat Effective Mass Model

Purpose: The purpose of this section is to present the bat Effective Mass model. Presently, it is the most popular physics of baseball model for bat-ball collisions. It will be compared to the BaConLaw model.

The bat effective mass bat-ball collision modeling community, established by Nathan (2003) and summarized nicely by Cross (2011), base their model on the concept of the effective mass of a bat. This section on the Effective Mass model is excerpted from Cross (2011). Consider Fig. 5.1 where a ball of mass m_{ball} collides with a stationary bat at a distance B from the center of mass (cm) of the bat. {Previously in this book, we have used the symbol $d_{\text{cm-ip}}$ to represent the distance between the center of mass and the impact point, or $d_{\text{cm-ss}}$ if the collision were at the sweet spot. However, in this section, to avoid confusion with the derivative operator used by Cross, we will use the letter B , as was done by Watts and Bahill (1990).} Let the mass of the bat be m_{bat} and suppose that the bat is initially at rest and freely supported: that is, no one is holding the handle. In that case, the ball will bounce off the bat and the bat will be set in motion. The center of mass of the bat and the impact point on the bat both recoil. Because a bat is a rigid object, every spot on a bat will have the same linear translational velocity and the same angular rotational velocity. But each spot will have a different *total* velocity that depends on its distance from the pivot point. We define that total velocity of the bat as the sum of its linear translational velocity and its weighted angular rotation velocity: $v_{\text{bat}} = v_{\text{cm}} + B\omega_{\text{bat}}$. We have used the symbol vt to represent the total velocity, e.g. $vt_{\text{bat-impact-after}}$, to differentiate from it from $v_{\text{bat-cm-after}}$ used in the rest of this book to indicate only the translational component of velocity. Because the bat rotates about its center of mass when it is struck by the ball, the speed of the impact point will be greater than the speed of the center of mass, $vt_{\text{impact}} > v_{\text{cm}}$. The impact point therefore accelerates faster than the center of mass of the bat, as if it were an isolated mass separate from the rest of the bat.

The whole bat is involved in the collision, but the effect on the ball is equivalent to a collision with an isolated *effective mass* M_{eff} that is less than the mass of the whole bat. Additionally, the impact point recoils as if it were a mass of M_{eff} . In other words, we can treat the collision as being equivalent to one between a ball of

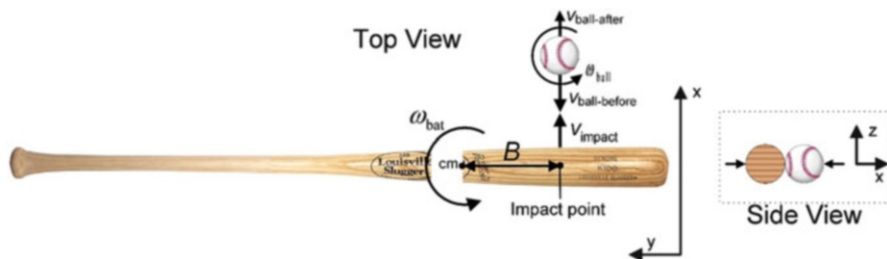


Fig. 5.1 The Effective Mass model for bat-ball collisions

mass m_{ball} and an object of mass M_{eff} . We will now derive a formula for the effective mass of a bat.

Let M_{eff} be the effective mass of the bat at the impact point. A force F acting at the impact point will cause this point and the center of mass to accelerate according to these relationships $F = M_{\text{eff}} \frac{dv_{\text{bat-impact}}}{dt}$ and $F = m_{\text{bat}} \frac{dv_{\text{bat-cm}}}{dt}$ respectively. The torque $F \times B$ causes the whole bat to rotate about its center of mass according to $F \times B = I_{\text{bat-cm}} \frac{d\omega_{\text{bat-cm}}}{dt}$ where $\omega_{\text{bat-cm}}$ is the angular velocity of the bat about its center of mass. Therefore,

$$F = M_{\text{eff}} \frac{dv_{\text{bat-impact}}}{dt} = m_{\text{bat}} \frac{dv_{\text{bat-cm}}}{dt} = \frac{I_{\text{bat-cm}}}{B} \frac{d\omega_{\text{bat-cm}}}{dt}$$

The impact point rotates at a speed of $B\omega_{\text{bat-cm}}$ with respect to the bat's center of mass. So now, the impact point has a linear translational motion and an angular rotational motion. Hence, $v_{\text{bat-impact}} = v_{\text{bat-cm}} + B\omega_{\text{bat-cm}}$. Taking the derivative with respect to time, we get

$$\frac{dv_{\text{bat-impact}}}{dt} = \frac{dv_{\text{bat-cm}}}{dt} + B \frac{d\omega_{\text{bat-cm}}}{dt}$$

which can be written as

$$\frac{F}{M_{\text{eff}}} = \frac{F}{m_{\text{bat}}} + \frac{B^2 F}{I_{\text{bat-cm}}}$$

Dividing by F produces

$$\frac{1}{M_{\text{eff}}} = \frac{1}{m_{\text{bat}}} + \frac{B^2}{I_{\text{bat-cm}}}$$

which can be rearranged to give

$$M_{\text{eff}} = \frac{m_{\text{bat}}}{1 + \frac{m_{\text{bat}} B^2}{I_{\text{bat-cm}}}} \quad (5.1)$$

In summary, Fig. 5.1 suggests that a ball impacting a stationary bat, at distance B from the center of mass of the bat will cause the bat to rotate about the center of mass. However, the speed and acceleration of the impact point is greater than that for the bat's center of mass, so the effective mass at the impact point is less than the mass of the whole bat. For the bat of Table 1.1 $M_{\text{eff}} = 0.707$ kg.

There are three important differences between this model and the BaConLaw model developed in Chap. 4. (1) In Fig. 5.1, $v_{\text{ball-before}}$ is pointing down and it is *positive* in that direction: for the rest of the book $v_{\text{ball-before}}$ was positive in the x -direction. {However, $v_{\text{ball-after}}$ is still defined to be positive in the direction of the

x-axis.} That is why there is a minus sign in front of the $m_{\text{ball}}v_{\text{ball-before}}$ term in conservation of momentum Eq. (5.2). (2) In this model, vt_{impact} is the *total* velocity of the impact point. That is, it is the sum of the translational velocity and the velocity due to rotation about the center of mass. (3) Because vt_{impact} is the sum of the translational and rotational velocities, the coefficient of restitution equation has only two terms on top and bottom, that is $e = \frac{v_{\text{ball-after}} - vt_{\text{bat-after}}}{v_{\text{ball-before}} + vt_{\text{bat-before}}}$, instead of three as in Eqs. (3.5) and (4.5).

Our next task is to get an equation for the velocity of the ball after the collision. We will start with an equation for the conservation of momentum. From here on, we no longer require a stationary bat before the collision.

$$M_{\text{eff}}vt_{\text{bat-before}} - m_{\text{ball}}v_{\text{ball-before}} = M_{\text{eff}}vt_{\text{bat-after}} + m_{\text{ball}}v_{\text{ball-after}} \quad (5.2)$$

Note this is different from the conservation of momentum equation used in the rest of this book because of the different definition of the direction of the ball before the collision.

Next, we need the coefficient of restitution.

$$e = \frac{v_{\text{ball-after}} - vt_{\text{bat-after}}}{v_{\text{ball-before}} + vt_{\text{bat-before}}}$$

We use this expression to eliminate $vt_{\text{bat-after}}$ in Eq. (5.2). Substitute the coefficient of restitution into Eq. (5.2) and we get

$$v_{\text{ball-after}} = \left(\frac{e - \frac{m_{\text{ball}}}{M_{\text{eff}}}}{1 + \frac{m_{\text{ball}}}{M_{\text{eff}}}} \right) v_{\text{ball-before}} + \left(\frac{1 + e}{1 + \frac{m_{\text{ball}}}{M_{\text{eff}}}} \right) vt_{\text{bat-before}}$$

Plug in M_{eff}

$$v_{\text{ball-after}} = \left(\frac{e - \frac{m_{\text{ball}}}{\frac{m_{\text{bat}}}{1 + \frac{m_{\text{bat}}B^2}{I_{\text{bat-cm}}}}}}{1 + \frac{m_{\text{ball}}}{\frac{m_{\text{bat}}}{1 + \frac{m_{\text{bat}}B^2}{I_{\text{bat-cm}}}}}} \right) v_{\text{ball-before}} + \left(\frac{1 + e}{1 + \frac{m_{\text{ball}}}{\frac{m_{\text{bat}}}{1 + \frac{m_{\text{bat}}B^2}{I_{\text{bat-cm}}}}}} \right) vt_{\text{bat-before}}$$

Ten algebraic steps yield our final expression for the *batted-ball* velocity.

$$\begin{aligned} v_{\text{ball-after}} = & -v_{\text{ball-before}} \left(\frac{m_{\text{ball}}I_{\text{bat-cm}} - m_{\text{bat}}I_{\text{bat-cm}}e + m_{\text{ball}}m_{\text{bat}}B^2}{m_{\text{ball}}I_{\text{bat-cm}} + m_{\text{bat}}I_{\text{bat-cm}} + m_{\text{ball}}m_{\text{bat}}B^2} \right) \\ & + vt_{\text{bat-before}} \left(\frac{m_{\text{bat}}I_{\text{bat-cm}}(1 + e)}{m_{\text{ball}}I_{\text{bat-cm}} + m_{\text{bat}}I_{\text{bat-cm}} + m_{\text{ball}}m_{\text{bat}}B^2} \right) \end{aligned} \quad (5.3)$$

This is the end of the derivation of the batted-ball velocity equation using the Effective Mass model. Now, compare Eq. (5.3) to Eq. (4.8) from our BaConLaw model.

$$\begin{aligned}
v_{\text{ball-after}} = v_{\text{ball-before}} & \frac{(m_{\text{ball}}I_{\text{bat}} - m_{\text{bat}}I_{\text{bat}} \text{ CoR}_{2b} + m_{\text{ball}}m_{\text{bat}}d^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& + v_{\text{bat-cm-before}} \frac{m_{\text{bat}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\
& + \omega_{\text{bat-before}} \frac{m_{\text{bat}}dI_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2}
\end{aligned}$$

The differences are that in the Effective Mass model, the first term on the right of Eq. (5.3) has a minus sign because the initial ball velocity was defined to be positive downward in Fig. 5.1. (2) In the Effective Mass model, the second term on the right is equivalent to two terms in the BaConLaw model because of the definition $v_{t_{\text{bat}} - \text{impact} - \text{before}} = v_{\text{bat} - \text{cm} - \text{before}} + B\omega_{\text{bat} - \text{cm} - \text{before}}$. (3) Because of that definition, $e \neq \text{CoR}_{2b}$.

Returning to the exposition of Cross (2011), he then states that if a ball with velocity $v_{\text{ball-before}}$ collides with a stationary bat and bounces back with a velocity $v_{\text{ball-after}}$ then

$$q = \frac{v_{\text{ball-after}}}{v_{\text{ball-before}}}$$

Now, and most importantly,

$$q = \left(\frac{e - \frac{m_{\text{ball}}}{M_{\text{eff}}}}{1 + \frac{m_{\text{ball}}}{M_{\text{eff}}}} \right) \text{ and } 1 + q = \frac{1 + e}{1 + \frac{m_{\text{ball}}}{M_{\text{eff}}}}$$

Using this new symbol, Eq. (5.3) for $v_{\text{ball-after}}$ becomes

$$v_{\text{ball-after}} = qv_{\text{ball-before}} + (1 + q)v_{t_{\text{bat}} - \text{before}}$$

This equation holds for bats that are freely suspended and rotate about their centers of mass, as shown in Fig. 5.1. Rod Cross continues with, “This is the primary physics equation that describes the outgoing speed of a struck ball, regardless of whether the ball is struck by a bat or a racquet or a club. The performance of any given striking implement depends mainly on the value of q for that implement.” However, $v_{\text{ball-before}}$ and $v_{t_{\text{bat}} - \text{before}}$ require some considerations. For example, $v_{t_{\text{bat}} - \text{before}}$ depends on the impact point, the mass of the ball, the mass of the bat, the moment of inertia of the bat and characteristics of the person swinging the bat. In addition, the coefficient of restitution, e , is not a constant. It depends on the impact point and the pivot point, as well as the speed of the collision, the relative humidity, the temperature, the deformation of the objects, the surface texture and the type of ball. However, in spite of these variabilities, Nathan (2003) and Cross (2011) found that for most baseball collisions

$$v_{\text{ball-after}} = 0.2v_{\text{ball-before}} + 1.2v_{\text{bat-before}}$$

On the other hand, the following is Eq. (4.8), the batted-ball velocity equation from the BaConLaw model.

$$\begin{aligned} v_{\text{ball-after}} = v_{\text{ball-before}} & \frac{(m_{\text{ball}}I_{\text{bat}} - m_{\text{bat}}I_{\text{bat}} \text{ CoR}_{2b} + m_{\text{ball}}m_{\text{bat}}d^2)}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\ & + v_{\text{bat-cm-before}} \frac{m_{\text{bat}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \\ & + d_{\text{cm-ip}}\omega_{\text{bat-before}} \frac{m_{\text{bat}}I_{\text{bat}}(1 + \text{CoR}_{2b})}{m_{\text{ball}}I_{\text{bat}} + m_{\text{bat}}I_{\text{bat}} + m_{\text{ball}}m_{\text{bat}}d^2} \end{aligned}$$

If we substitute parameter values for a major league wooden bat, as described in Table 1.1, into this equation, then the velocity of the ball after the collision becomes

$$v_{\text{ball-after}} = -0.217v_{\text{ball-before}} + 1.217\{v_{\text{bat-cm-before}} + d_{\text{cm-ip}}\omega_{\text{bat-before}}\}$$

where the units are m/s and rad/s. Remember that the velocity of the ball before the collision is a negative number. So far, we have made no approximations; everything has been exactly according to Newton's principles and the conservation laws. In contrast, we will now create our *rule of thumb* by using pitch speed instead of the ball-collision speed and using total bat speed instead of its two components.

$$\text{batted.ball speed} = -0.19 \text{ pitch speed} + 1.22 \text{ total bat speed}$$

The units could be m/s or mph. The pitch speed would be that determined by a radar gun focused near the pitcher's release point and announced on television. The bat speed would come from Tables such as 3.9 and 4.2. Using our typical data of Table 4.2, we have an average pitch speed of -92 mph and a total bat speed of 62 mph. Putting these numbers into our rule of thumb yields

$$\text{batted.ball speed} = 0.19 \times 92 + 1.22 \times 62 = 93 \text{ mph}$$

which is just about the average for major league hits. Using the data of Willman (2017), we found that for the 15,000 base hits in major league baseball in 2016, the average batted-ball speed was 91 mph.

The bat Effective Mass model and the BaConLaw model both start with Newton's principles: then they diverge. They are different: however, they yield the same rule of thumb for the batted-ball speed! This should strengthen and give people more confidence in both models.

Modeling philosophy note Having several alternative models helps ensure that you understand the physical system. No model is more correct than another. They just

emphasize different aspects of the physical system. They are not competing models they are synergetic.

5.3 The Sliding Pin Model

Purpose: The purpose of this section is to present the Sliding Pin model, the moving pivot point data and the Spiral Center of Mass model. These all use a different *type* of data from the rest of the book, namely translation of the knob and rotation about the knob.

With the advent of low-cost multiple-video-camera systems for making three-dimensional (3D) measurements of the movement of bats and balls, a new source of data became available. Using these data, the Sliding Pin model models the movement of the bat with a translation and a rotation about its *knob*. It is shown in Fig. 5.2. The bat is pinned through the knob, so it is forced to rotate about the knob. But the pin is allowed to slide along the x-axis (up and down in Fig. 5.3) to allow for the translational velocity of the bat.

Consider a bat that is pinned through its knob, but the pinned point is allowed to slide up and down, as in Fig. 5.3. Because the bat is a rigid body, every spot on the bat will have the same linear translational velocity and the same angular rotational velocity.

$$\begin{aligned} v_{\text{bat-knob-trans}} &= v_{\text{bat-cm-trans}} = v_{\text{bat-cop-trans}} \\ \text{and } \beta_{\text{bat-knob}} &= \beta_{\text{bat-cm}} = \beta_{\text{bat-cop}} \end{aligned}$$

However, each spot on the bat will have a different *total* velocity that depends on the location of the pivot point and the spot's distance from the pivot point.

$$\begin{aligned} vt_{\text{cm}} &= v_{\text{knob}} + d_{\text{knob-cm}}\beta_{\text{bat}} \\ vt_{\text{cop}} &= v_{\text{knob}} + d_{\text{knob-cop}}\beta_{\text{bat}} \end{aligned}$$

5.3.1 Moving Pivot Point Data

Table 3.9 gave the bat sweet-spot speeds as numbers that combined the speed of the center of mass and the rotation of the bat about that point. However, data by from (Fleisig et al. 2001, 2002; Cross 2009; Milanovich and Nesbit 2014; King et al. 2012) gave us a new type of data to model. They gave us simultaneous independent measurements of linear velocity and angular rotational velocity.

Milanovich and Nesbit (2014) studied 14 female collegiate softball players. They used multiple video cameras to collect data and they created three-

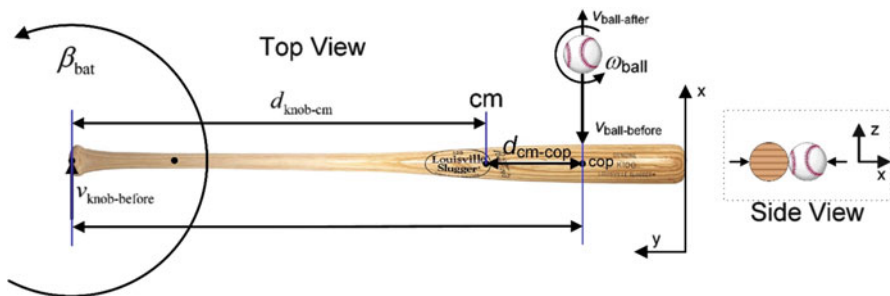
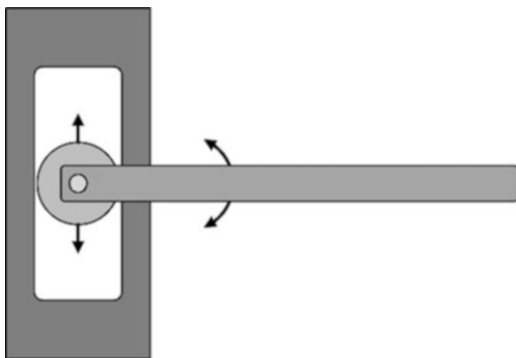


Fig. 5.2 Sliding Pin model for a bat pivoting about its knob

Fig. 5.3 Detail of a sliding pin joint



dimensional reconstructions of bat swings. Using their Table II, which averages the data of all subjects and all swings of their aluminum bat, we will now solve for the velocity of the center of mass.

Using data from their measurements at the pivot point (the knob)

$$\beta_{\text{before}} = 29.6$$

$$vt_{\text{cm}} = v_{\text{kno-b}} + d_{\text{kno-b-cm}}\beta_{\text{before}}$$

$$vt_{\text{cm}} = 3.6 + 0.48 \times 29.6 = 17.8 \text{ m/s}$$

Using data from their measurements at the sweet spot (the center of percussion)

$$vt_{\text{cop}} = v_{\text{kno-b}} + d_{\text{k-cop}}\beta_{\text{before}}$$

$$vt_{\text{cm}} = vt_{\text{cop}} \frac{d_{\text{k-cm}}}{d_{\text{k-cop}}}$$

$$vt_{\text{cm}} = 15.9 \text{ m/s}$$

Finally, using data from their measurements at the center of mass

$$vt_{\text{cm}} = 16.1 \text{ m/s}$$

So their data for vt_{cm} , 17.8, 15.9 and 16.1 m/s, are reasonably consistent.

Now, let us move on to other studies that made simultaneous independent measurements of the linear translational velocity and the angular rotational velocity. The average sweet-spot speeds from the study of Fleisig et al. (2001, 2002) for 16 male baseball players were

$$v_{\text{ss}} = 27 \text{ m/s and } \beta = 38 \text{ rad/s}$$

Cross (2009) had a single male subject with

$$v_{\text{cm}} = 16.5 \text{ m/s and } \beta = 33 \text{ r/s}$$

King et al. (2012) had one male subject with

$$v_{\text{kno}} = 6 \text{ m/s and } \beta = 36 \text{ r/s}$$

There are several reasons for differences in the experimental values. (1) Men swing the bat faster than women do. The average sweet-spot speed at impact of Milanovich and Nesbit (2014) was 20 m/s. Table 3.9 shows that male baseball players typically have higher speeds than this. Fleisig et al. (2002) measured 17 college women at 21 m/s and 16 college men at 27 m/s. Bahill (2004) measured 20 university women at 21 m/s and 28 major leaguers at 26 m/s. (2) The aluminum bat swung by the Milanovich and Nesbit (2014) subjects was lighter than the wooden bats used in the other studies. (3) Averaging data from many subjects produced slower results, particularly when the women were not elite athletes. (4) The low frame rate of the motion capture cameras low-pass filtered the data and attenuated the velocities. Further smoothing and processing reduced the velocities even more. (5) The bat rotates about a point in or nearby the knob, but there is variability in this point. Indeed, in early phases of the swing the pivot point is outside of the knob. But when the bat reaches the collision point, the pivot point has come inside the knob. (6) They gave data for the movement of the grip, which was 6 inches away from the knob. (7) Configuration 2c is not a free-end collision. The hands are still holding the bat at the collision point and they might be applying forces to the bat. However, all of their variables (yaw, pitch, roll and v_{kno}) reach their peak values before the collision point. Therefore, if the hands were applying forces, these forces were not accelerating the bat in the x-direction. Furthermore, if the collision were at the center of percussion, then the collision would not create forces at the pivot point. (8) Experimental data are always subject to noise and measurement error.

Our most comprehensive data for bat swings come from William Clark, Founder of Diamond Kinetics (personal communication, 2017). Table 5.1 shows their data for 200 male professional baseball players swinging 33-inch wooden bats.

Table 5.1 Linear, angular and total bat speeds for 20,000 swings by male professional batters

Variable	SI units	Baseball units
Linear knob speed, $v_{\text{bat} - \text{knob} - \text{before}}$	4.5 m/s, $\sigma = 1.7$	10.1 mph, $\sigma = 3.9$
Angular rotation speed, $\beta_{\text{bat} - \text{before}}$	41 rad/s, $\sigma = 5$	387 rpm, $\sigma = 51$
Total speed at the center of mass $v_{t_{\text{cm} - \text{before}}} = v_{\text{knob} - \text{before}} + d_{\text{knob} - \text{cm}}\beta_{\text{bat} - \text{before}}$	27.9 m/s, $\sigma = 3.7$	62.3 mph, $\sigma = 8.2$
Total speed at the sweet spot $v_{t_{\text{cop} - \text{before}}} = v_{\text{knob} - \text{before}} + d_{\text{knob} - \text{cop}}\beta_{\text{bat} - \text{before}}$	33.3 m/s, $\sigma = 4.3$	74.5 mph, $\sigma = 9.6$

In this table, the variance in the angular velocity of the center of mass and sweet spot was small, smaller than the variance in the linear speed of the knob. These data produced this equation $v_{t_{\text{bat} - \text{cop} - \text{before}}} = v_{\text{knob}} + d_{\text{k} - \text{cop}}\beta_{\text{b}} = 4.5 + 0.7 \times 41 = 33 \text{ m/s} = 74 \text{ mph}$, which we used in our simulations, whose outputs are shown in Tables 5.3 and 5.4. This equation has the same six to one ratio of $d_{\text{k} - \text{cop}}\beta_{\text{before}}$ and $v_{t_{\text{knob}}}$ as our frame-by-frame analysis of the swing of a major league batter.

5.3.2 Spiral Center of Mass Model

Cross (2009) developed an intriguing model for the swing of the bat. It is based on data from a video-camera system that measured the translation and rotation about the center of mass for a collision at the sweet spot. The pivot point of the bat moved during the swing. In the Cross Spiral Center of Mass model, the center of mass of the bat followed a logarithmic spiral pathway described with this equation $R = 0.25e^{0.23\theta}$. Figure 5.4 shows this movement.

5.3.3 Back to the Sliding Pin Model

Purpose: The purpose of the Sliding Pin model is to model a new type of data different from the rest of the book. Previously the input data for our models were the translational and rotational velocities at the center of mass of the bat. The Sliding Pin model will use the translational and rotational velocities at the *knob*.

The Sliding Pin model is unique in the science of baseball literature. It has four equations and four unknowns. This new model is described in Fig. 5.5 and Table 5.2. Its purposes are (1) to show the limits of the conservation law modeling technique and (2) to model some unique new experimental data. Unlike the BaConLaw model and the Effective Mass model, it is data-driven not theory-driven.

At the beginning of this section, we must emphasize that the BaConLaw model given in Fig. 4.1 and Table 4.1 is not equivalent to the Sliding Pin model of Fig. 5.5

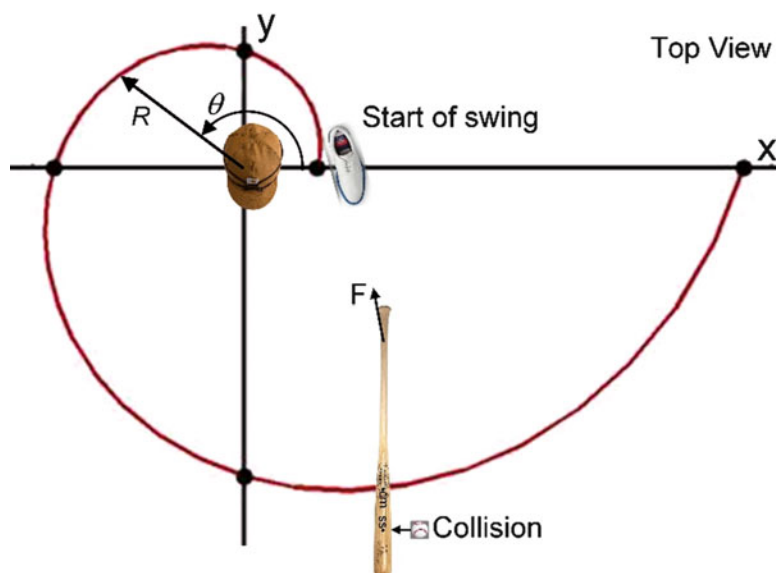


Fig. 5.4 The Spiral Center of Mass model of Cross (2009). In this top view, the batter's head is at the intersection of the x- and y-axes and his left foot is to the right

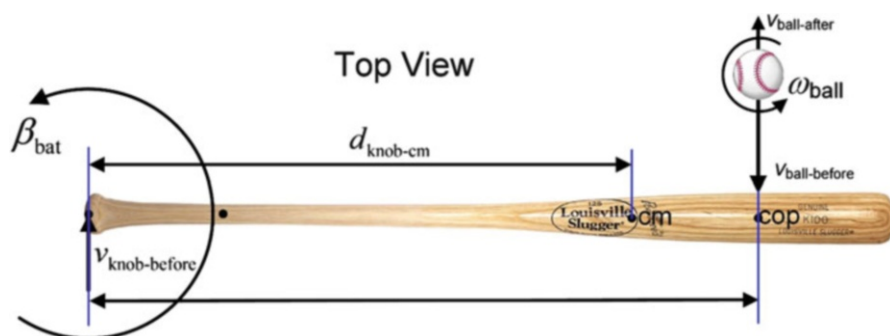


Fig. 5.5 The Sliding Pin model for bat-ball collisions

and Table 5.2. Although the equations may look analogous, many of them are different, because they are modeling different things. The BaConLaw model is for a free-end collision of an unsupported bat that will translate and rotate about its center of mass. The Sliding Pin model is for the collision of a restrained bat. The bat is being forced to rotate about its knob. The human is doing the restraining by applying forces on the handle during the swing. To make this perfectly clear, let us simplify the situation by ignoring translations and consider only rotations. The bat of Fig. 4.1 will rotate about its center of mass with an initial angular velocity of

Table 5.2 Synopsis of equations for the Sliding Pin, four equations and four unknowns

Inputs	$v_{\text{ball-before}}, \omega_{\text{ball-before}}, v_{\text{knob-before}}, \beta_{\text{knob-before}}$ and CoR_{2c}
Outputs	$v_{\text{ball-after}}, \omega_{\text{ball-after}}, v_{\text{bat-ip-after}}, \beta_{\text{knob-after}}$ and KE_{lost}
Equations	
Conservation of Linear Momentum, Eq. (5.4)	$m_{\text{ball}}v_{\text{ball-before}} + m_{\text{bat}}v_{\text{bat-before}} = m_{\text{ball}}v_{\text{ball-after}} + m_{\text{bat}}v_{\text{bat-after}}$
Definition of CoR , Eq. (5.5)	$CoR_{2c} = -\frac{v_{\text{ball-after}} - v_{\text{knob-after}} - d_{\text{kno-b-ip}}\beta_{\text{before}}}{v_{\text{ball-before}} - v_{\text{kno-b-before}} - d_{\text{kno-b-ip}}\beta_{\text{after}}}$
Newton's Second Law, Eq. (5.6)	$d_{\text{k-ip}}m_{\text{ball}}(v_{\text{ball-after}} - v_{\text{ball-before}}) = -I_{\text{kno-b}}(\beta_{\text{after}} - \beta_{\text{before}})$
Conservation of Angular Momentum about the z-axis, Eq. (5.7s)	$m_{\text{ball}}d_{\text{k-ip}}v_{\text{ball-before}} + (I_{\text{ball}} + m_{\text{ball}}d_{\text{k-ip}}^2)\omega_{\text{ball-before}} + (I_{\text{kno-b}})\beta_{\text{before}} =$ $+m_{\text{ball}}d_{\text{k-ip}}v_{\text{ball-after}} + (I_{\text{ball}} + m_{\text{ball}}d_{\text{k-ip}}^2)\omega_{\text{ball-after}} + (I_{\text{kno-b}})\beta_{\text{after}}$

$\omega_{\text{bat-before}}$. This will give it an initial kinetic energy of $\frac{I_{\text{cm}}\omega_{\text{bat-before}}^2}{2}$. Whereas, the bat of Fig. 5.5 will rotate about its knob with an initial angular velocity of β_{before} . This will give it a kinetic energy of $\frac{I_{\text{kno-b}}\beta_{\text{before}}^2}{2}$. If the models were equivalent then $\frac{I_{\text{cm}}\omega_{\text{bat-before}}^2}{2} = \frac{I_{\text{kno-b}}\beta_{\text{before}}^2}{2}$. By the parallel axis theorem $I_{\text{kno-b}} = I_{\text{cm}} + m_{\text{bat}}d_{\text{k-cm}}^2$. Which means that this equation would have to be true $I_{\text{cm}}\omega_{\text{bat-before}}^2 = (I_{\text{cm}} + m_{\text{bat}}d_{\text{k-cm}}^2)\beta_{\text{before}}^2$. This would require $\omega_{\text{bat-before}} = \sqrt{\beta_{\text{before}}^2 + \frac{m_{\text{bat}}d_{\text{k-cm}}^2}{I_{\text{cm}}}}$. Clearly, $\omega_{\text{bat-before}} \neq \beta_{\text{before}}$ and therefore the

BaConLaw model is not equivalent to the Sliding Pin model. The cause of this difference is that the BaConLaw model is for a free-end collision. Whereas in the Sliding Pin model the batter is applying forces to the handle during the swing. The Sliding Pin model is more complicated than the BaConLaw model. Therefore, the Sliding Pin model takes our bat-ball collision modeling community a baby step upwards.

Configuration 2c is for a collision at the sweet spot of the bat with spin on the pitch. It adds a new model for bat motion: the movement of the bat comprises a translation and a rotation about its knob. Because of this, we need a different equation for the CoR . This model is original. Our previous configurations, 2a and 2b, measured and used the total velocity (translational plus angular velocity) for the velocity of the sweet spot before and after the collision. However, the experimental studies examined in the previous sections gave independent linear and angular speeds of the bat about the *knob* right before the collision. We will now see if our modeling approach can accommodate this new data.

Modeling philosophy note In general, there are two common techniques for modeling systems: the first is theory-based and the second is data-based. Here are some steps for theory-based system models. Find appropriate physical and/or

physiological principles, then design, build and test a model. Design experiments to collect new data. Use these data to verify and validate the model. Use the model to make predictions and guide future data collection activities. The BaConLaw model was theory-based. The theories were the conservation laws. We found the theories first and then we gathered experimental data to support the model. The second technique for modeling a system is data-based. With this technique, the modeler starts with collecting and organizing the data and then he or she makes a model that fits that measured data. The Sliding Pin and Spiral Center of Mass models are data-based. We found the experimental data first and then we created the model to match the data. In Chap. 5, we give four different models for bat-ball collisions. They have different purposes and different outputs. The point is to explain to the reader that it is good to have alternative models.

5.3.4 Coefficient of Restitution

The Coefficient of Restitution (*CoR*) was defined by Sir Isaac Newton as the ratio of the relative velocity of the two objects after a collision to the relative velocity before the collision. The *CoR* models the energy lost in the collision.

In our models for a collision at the sweet spot (ss) of the bat we have

$$CoR = - \frac{\text{relative velocity after collision}}{\text{relative velocity before collision}}$$

For the Sliding Pin model, we define the *CoR* with this equation (Fig. 5.4).

$$CoR_{2c} = - \frac{v_{\text{ball-after}} - v_{\text{knob-after}} - d_{\text{knob-ip}}\beta_{\text{after}}}{v_{\text{ball-before}} - v_{\text{knob-before}} - d_{\text{knob-ip}}\beta_{\text{before}}}$$

This *CoR* is a variation of the *CoRs* that we have used in previous sections.

Definition of variables

Inputs $v_{\text{ball-before}}$, $\omega_{\text{ball-before}}$, $v_{\text{bat-before}}$, $\beta_{\text{bat-before}}$ and CoR_{2c}

$v_{\text{ball-before}}$ is the linear velocity of the ball in the x-direction before the collision.

$\omega_{\text{ball-before}}$ is the angular velocity of the *ball about its center of mass* before the collision.

$v_{\text{bat-before}}$ is the linear translational velocity of the *knob of the bat* in the x-direction before the collision.

$\beta_{\text{bat-before}}$ is the angular velocity of the *bat about its knob* before the collision.

CoR_{2c} is the coefficient of restitution for this configuration.

Outputs $v_{\text{ball-after}}$, $v_{\text{bat-after}}$, $\beta_{\text{bat-after}}$

$v_{\text{ball-after}}$ is the linear velocity of the ball in the x-direction after the collision.

$\omega_{\text{ball-after}}$ is the angular velocity of the *ball about its center of mass* after the collision.

$v_{\text{bat-after}}$ is the translational velocity of the knob of the bat in the x-direction after the collision.

$\beta_{\text{bat-after}}$ is the angular velocity of the bat *about its knob* after the collision.

We want to solve for $v_{\text{ball-after}}$, $\omega_{\text{ball-after}}$, $v_{\text{bat-after}}$, $\beta_{\text{bat-after}}$.

We will use the following fundamental equations of physics: Conservation of Linear Momentum, the Definition of *CoR*, Newton's Second Principle and the Conservation of Angular Momentum.

5.3.5 Condensing Equation Notation

First, we want to simplify our notation. We will make the following substitutions.

$$d_{\text{knoB-impact-point}} = d_{\text{k-ip}}$$

$$d_{\text{knoB-cm}} = d_{\text{k-cm}}$$

$$I_{\text{ball}} = I_1$$

$$I_{\text{bat-cm}} = I_2 = I_{\text{cm}}$$

$$I_{\text{bat-knob}} = I_{\text{k}}$$

$$I_{\text{knoB}} - m_2 d_{\text{k-cm}}^2 = I_2$$

$$m_{\text{ball}} = m_1$$

$$m_{\text{bat}} = m_2$$

$$v_{\text{ball-before}} = v_{1b}$$

$$v_{\text{ball-after}} = v_{1a}$$

$$v_{\text{bat-knob-before}} = v_{2b}$$

$$v_{\text{bat-after}} = v_{2a}$$

$$\beta_{\text{bat-before}} = \beta_b$$

$$\beta_{\text{bat-after}} = \beta_a$$

These substitutions produce the following equations.

5.3.5.1 Conservation of Linear Momentum

Assume that the bat and ball are point masses with all of their mass concentrated at the centers of mass. For now, neglect angular rotations.

$$m_{\text{ball}} v_{\text{ball-before}} + m_{\text{bat}} v_{\text{bat-cm-before}} = m_{\text{ball}} v_{\text{ball-after}} + m_{\text{bat}} v_{\text{bat-cm-after}} \quad (5.4)$$

However, from Sect. 5.3, for the linear velocity, we have

$$v_{\text{bat-cm}} = v_{\text{bat-knob}} = v_{\text{bat}} = v_2$$

Therefore,

$$m_1 v_{1b} + m_2 v_{2b} = m_1 v_{1a} + m_2 v_{2a} \quad (5.4s)$$

5.3.5.2 Definition of Coefficient of Restitution (CoR)

$$CoR_{2c} = - \frac{v_{\text{ball-after}} - v_{\text{bat-knob-after}} - d_{\text{kno}b\text{-ip}} \beta_{\text{after}}}{v_{\text{ball-before}} - v_{\text{bat-knob-before}} - d_{\text{kno}b\text{-ip}} \beta_{\text{before}}} \quad (5.5)$$

$$CoR_{2c} = - \frac{v_{1a} - v_{2a} - d_{k\text{-ip}} \beta_a}{v_{1b} - v_{2b} - d_{k\text{-ip}} \beta_b} \quad (5.5s)$$

5.3.5.3 Newton's Second Principle

If we were following the development in Chap. 4, we would now apply Newton's Second Principle, which states that applying an impulsive torque about an axis of rotation changes the angular momentum about that axis. However, the Sliding Pin model is not a theory-based model: it is data-based and right now we need some experimental data because the batter's hands might be applying a torque to the handle. Although, the Sliding Pin model of Fig. 5.5 shows that at the point of impact the force applied by the batters hands is perpendicular to the direction of motion (Cross 2009). Therefore, the hands would not apply a torque to the bat. Furthermore, Milanovich and Nesbit (2014) showed that the bat's linear velocity (Fig. 6), angular velocity their (Fig. 7) and forces (Fig. 9) were all decreasing at the time of impact. Moreover, the torques had already reached zero by the time of impact (Fig. 9). In summary, because of the experimental data, we will ignore the possibility of the hands applying a torque to the bat at the time of impact and we will continue our derivation with Newton's second principle.

Newton's Second Principle states that applying an impulsive torque about an axis of rotation changes the angular momentum about that axis. We can apply this principle to a collision at the sweet spot with rotation about the knob of the bat.

$$d_{\text{kno}b\text{-ip}} m_{\text{ball}} (v_{\text{ball-after}} - v_{\text{ball-before}}) = -I_{\text{kno}b} (\beta_{\text{after}} - \beta_{\text{before}}) \quad (5.6)$$

$$d_{k\text{-ip}} m_1 (v_{1a} - v_{1b}) = -I_k (\beta_a - \beta_b) \quad (5.6s)$$

Solve for β_a

$$\beta_a = \beta_b - \frac{(v_{1a} - v_{1b})m_1 d_{k-ip}}{I_k}$$

$$\beta_{\text{after}} = \beta_{\text{before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}})m_{\text{ball}}d_{k-ip}}{I_{\text{kno}}}$$

5.3.5.4 Abbreviations

For simplicity (especially when doing derivations by hand), the following temporary simplifications will be used in the derivations. Because they are analogous to the abbreviation used in Chap. 4, these will have a bar over the letter.

$$\bar{A} = \frac{(v_{1b} - v_{2b} - d_{k-ip}\beta_b)(1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2} \quad \frac{1}{\text{kg}^2 \text{m} \cdot \text{s}}$$

$$\bar{C} = v_{1b} - v_{2b} - d_{k-ip}\beta_b \quad \text{m/s}$$

$$\bar{D} = \frac{m_1 d_{k-ip}^2}{I_k} \quad \text{unitless}$$

$$\bar{G} = +v_{2b}m_2 I_k(1 + CoR_{2c}) + \beta_b m_2 I_k d_{k-ip}(1 + CoR_{2c}) \quad \text{kg}^2 \text{m}^3/\text{s}$$

$$\bar{K} = (m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2) \quad \text{kg}^2 \text{m}^2$$

The units are for dimensional analysis. Note that none of these constants contains the outputs $v_{\text{ball-after}}$, $v_{\text{bat-after}}$ or $\beta_{\text{bat-after}}$. One of the purposes of this book is to show how complex these collisions can be, while still being modeled using only Newton's principles and the conservation laws. The most useful simplifications are the ones that are constants independent of velocities after the collision. These simplifications are only used during the derivations. They are removed from the output equations. We will now use the Newtonian principles in equations (5.4), (5.5) and (5.6) and the conservation laws to find $v_{\text{ball-after}}$, $v_{\text{bat-after}}$ and $\beta_{\text{bat-after}}$.

5.3.5.5 Conservation of Angular Momentum

We will now use the law of Conservation of Angular Momentum about the axis through the knob of the bat. When the ball contacts the bat, as shown in Fig. 5.2, the ball has linear momentum of $m_{\text{ball}}v_{\text{ball-before}}$. Therefore, following tradition, we will model the ball as rotating about the bat's knob at a distance $d = d_{k-ip}$. Thus, the ball has an initial angular momentum of $m_{\text{ball}}d_{\text{kno}}d_{k-ip}v_{\text{ball-before}}$. In addition, it is possible to throw a curveball so that it spins about the vertical, z-axis, as also shown in Fig. 5.5. We call this a purely horizontal curveball (although it will still drop due to gravity, more than it will curve horizontally). The curveball will have angular momentum of $I_{\text{ball}}\omega_{\text{ball-before}}$ about an axis parallel to the z-axis. However, this is

momentum about the center of mass of the *ball* and we want the equivalent momentum about the knob of the *bat*. So we use the parallel axis theorem producing $(I_{\text{ball}} + m_{\text{ball}}d_{\text{k-ip}}^2)\omega_{\text{ball-before}}$.

The bat has an initial angular momentum reflecting the rotation about the knob. The symbol used for angular momentum is L . Therefore, the initial angular momentum for the bat-ball system about the axis through the knob of the bat is

$$L_{\text{initial}} = m_1 d_{\text{k-ip}} v_{1b} + (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1b} + I_{\text{knob}} \beta_b$$

All of these momenta are positive, pointing out of the page. (Remember that v_{1b} is a negative number.) Please refer to Fig. 5.5 now.

For the final angular momentum, we will treat the ball, as before, as an object rotating around the axis through the knob of the bat with angular momentum, $m_{\text{ball}} d_{\text{k-ip}} v_{\text{ball-before}}$. Now we could treat the bat as a long slender rod with a moment of inertia of $m_{\text{bat}} d_{\text{bat}}^2/12$ where d_{bat} is the bat length. However, this is only an approximation and we have actual experimental data for the bat moment of inertia. Thus, our final angular momentum about the knob of the bat is

$$L_{\text{final}} = m_1 d_{\text{k-ip}} v_{1a} + (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1a} + I_{\text{knob}} \beta_a$$

As we did in the section on Newton's Second Principle, we will ignore the possibility of the hands applying a torque to the bat handle at the time of impact. So now, we apply the law of Conservation of Angular Momentum, which states that when no external torque acts on an object the initial angular momentum about some axis equals the final angular momentum about that axis.

$$L_{\text{initial}} = L_{\text{final}}$$

$$\begin{aligned} m_1 d_{\text{k-ip}} v_{1b} + (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1b} + I_k \beta_b = \\ + m_1 d_{\text{k-ip}} v_{1a} + (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1a} + I_k \beta_a \end{aligned} \quad (5.7s)$$

Now, we solve this Conservation of Angular Momentum equation for the angular velocity about the knob after the collision, β_a .

$$-I_k \beta_a = m_1 d_{\text{k-ip}} v_{1a} + (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1a} - m_1 d_{\text{k-ip}} v_{1b} - (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1b} - I_k \beta_b$$

divide by minus I_k

$$\begin{aligned} \beta_a = \frac{-m_1 d_{\text{k-ip}} v_{1a} - (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1a} + m_1 d_{\text{k-ip}} v_{1b} + (I_1 + m_1 d_{\text{k-ip}}^2) \omega_{1b} + I_k \beta_b}{I_k} \\ \beta_a = \beta_b - \frac{m_1 d_{\text{k-ip}} (v_{1a} - v_{1b}) + (I_1 + m_1 d_{\text{k-ip}}^2) (\omega_{1a} - \omega_{1b})}{I_k} \end{aligned}$$

This equation was derived from Eq. (5.7s) Conservation of Angular Momentum. In Sect. 4.9, we showed that for a head-on bat-ball collision the ball spin before the collision is the same as the ball spin after the collision. Well, this is a head-on collision. Therefore ($\omega_{1a} = \omega_{1b}$) and the above equation reduces to

$$\beta_a = \beta_b - \frac{(v_{1a} - v_{1b})m_1 d_{k-ip}}{I_k}$$

$$\beta_{after} = \beta_{before} - \frac{(v_{ball-before} - v_{bat-cm-before})m_{ball} d_{k-ip}}{I_{knob}}$$

which is the same equation that we derived from Eq. (5.6), Newton's Second Principle.

5.3.6 Ball Velocity After the Collision

We will now find the ball velocity after the collision. We start with Eq. (5.5) and solve for the bat translational velocity after the collision, v_{2a}

$$CoR_{2c} = -\frac{v_{1a} - v_{2a} - d_{k-ip}\beta_a}{v_{1b} - v_{2b} - d_{k-ip}\beta_b}$$

$$\text{Let } \bar{C} = v_{1b} - v_{2b} - d_{k-ip}\beta_b$$

$$v_{2a} = v_{1a} + CoR_{2c}\bar{C} - d_{k-ip}\beta_a$$

Now we substitute the β_a that we just derived.

$$v_{2a} = v_{1a} + CoR_{2c}\bar{C} - d_{k-ip}\left\{\beta_b - \frac{d_{k-ip}m_1(v_{1a} - v_{1b})}{I_k}\right\}$$

$$v_{2a} = v_{1a} + CoR_{2c}\bar{C} + \frac{m_1 d_{k-ip}^2 (v_{1a} - v_{1b})}{I_k} - d_{k-ip}\beta_b$$

$$v_{2a} = v_{1a} \left(1 + \frac{m_1 d_{k-ip}^2}{I_k}\right) - \frac{m_1 d_{k-ip}^2 v_{1b}}{I_k} + CoR_{2c}\bar{C} - d_{k-ip}\beta_b$$

$$\text{Let } \bar{D} = \frac{m_1 d_{k-ip}^2}{I_k}$$

$$v_{2a} = v_{1a}(1 + \bar{D}) - v_{1b}\bar{D} + CoR_{2c}\bar{C} - d_{k-ip}\beta_b$$

Use this v_{2a} in Eq. (5.4) to get the ball velocity after the collision, v_{1a} .

Prepare to substitute this v_{2ka} into Eq. (5.4) by multiplying by the bat mass, m_2

$$v_{2a}m_2 = v_{1a}m_2(1 + \bar{D}) - v_{1b}m_2\bar{D} + m_2CoR_{2c}\bar{C} - \beta_b m_2 d_{k-ip}$$

Now substitute this $v_{2a}m_2$ into Eq. (5.4)

$$\begin{aligned} m_1 v_{1b} + m_2 v_{2b} &= m_1 v_{1a} + m_2 v_{2a} \\ v_{1b}m_1 + v_{2b}m_2 &= v_{1a}m_1 + \{v_{1a}m_2(1 + \bar{D}) - v_{1b}m_2\bar{D} + m_2CoR_{2c}\bar{C} - \beta_b m_2 d_{k-ip}\} \end{aligned}$$

Rearrange

$$+v_{1a}m_1 + v_{1a}m_2(1 + \bar{D}) = +v_{1b}m_1 + v_{1b}m_2\bar{D} + v_{2b}m_2 - m_2CoR_{2c}\bar{C} + \beta_b m_2 d_{k-ip}$$

Replace the dummy variables $\bar{C}$ and $\bar{D}$

$$\begin{aligned} +v_{1a}m_1 + v_{1a}m_2 \left(1 + \left\{ \frac{m_1 d_{k-ip}^2}{I_k} \right\} \right) &= \\ +v_{1b}m_1 + v_{1b}m_2 \left\{ \frac{m_1 d_{k-ip}^2}{I_k} \right\} & \\ +v_{2b}m_2 & \\ -m_2CoR_{2c} \{v_{1b} - v_{2b} - d_{k-ip}\beta_b\} & \\ +\beta_b m_2 d_{k-ip} & \end{aligned}$$

Rearrange

$$\begin{aligned} v_{1a} \left[m_1 + m_2 \left(1 + \left\{ \frac{m_1 d_{k-ip}^2}{I_k} \right\} \right) \right] &= \\ +v_{1b} \left[m_1 + m_2 \left\{ \frac{m_1 d_{k-ip}^2}{I_k} \right\} - m_2CoR_{2c} \right] & \\ +v_{2b}m_2(1 + CoR_{2c}) & \\ +\beta_b m_2 d_{k-ip}(1 + CoR_{2c}) & \end{aligned}$$

Multiply by the moment of inertia of the bat, I_k .

$$\begin{aligned} v_{1a} \left[m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2 \right] &= \\ +v_{1b} \left[m_1 I_k - m_2 I_k CoR_{2c} + m_1 m_2 d_{k-ip}^2 \right] & \\ +v_{2b}m_2 I_k (1 + CoR_{2c}) & \\ +\beta_b m_2 I_k d_{k-ip} (1 + CoR_{2c}) & \end{aligned}$$

$$v_{1a} = \frac{v_{1b} \left[m_1 I_k - m_2 I_k CoR_{2c} + m_1 m_2 d_{k-ip}^2 \right] + v_{2b}m_2 I_k (1 + CoR_{2c}) + \beta_b m_2 I_k d_{k-ip} (1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2}$$

This is the normal form of the equation for v_{1a} . However, we now want to rearrange this equation into our canonical form. Let

$$\bar{K} = (m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2)$$

$$\bar{G} = +v_{2b} m_2 I_k (1 + CoR_{2c}) + \beta_b m_2 I_k d_{k-ip} (1 + CoR_{2c})$$

Then

$$v_{1a} = \frac{v_{1b} [m_1 I_k - m_2 I_k CoR_{2c} + m_1 m_2 d_{k-ip}^2] + \bar{G}}{\bar{K}}$$

add $\left\{ v_{1b} - \frac{v_{1b} \bar{K}}{\bar{K}} \right\}$ to the right side

$$v_{1a} = \{v_{1b}\} + \frac{v_{1b} (m_1 I_k - m_2 I_k CoR_{2c} + m_1 m_2 d_{k-ip}^2)}{\bar{K}} \left\{ - \frac{v_{1b} (m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2)}{\bar{K}} \right\} + \frac{\bar{G}}{\bar{K}}$$

$$v_{1a} = v_{1b} + \frac{v_{1b} (m_1 I_k - m_2 I_k CoR_{2c} + m_1 m_2 d_{k-ip}^2)}{\bar{K}} - \frac{v_{1b} (m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2)}{\bar{K}} + \frac{\bar{G}}{\bar{K}}$$

$$v_{1a} = v_{1b} + \frac{v_{1b} (m_1 I_k - m_2 I_k CoR_{2c} + m_1 m_2 d_{k-ip}^2 - m_1 I_k - m_2 I_k - m_1 m_2 d_{k-ip}^2)}{\bar{K}} + \frac{\bar{G}}{\bar{K}}$$

$$v_{1a} = v_{1b} + \frac{v_{1b} (-m_2 I_k - m_2 I_k CoR_{2c})}{\bar{K}} + \frac{\bar{G}}{\bar{K}}$$

$$v_{1a} = v_{1b} + \frac{-v_{1b} m_2 I_k (1 + CoR_{2c}) + \bar{G}}{\bar{K}}$$

Replace the dummy variable $\bar{G}$ and we get the following equation.

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b}) m_2 I_k (1 + CoR) - \beta_b m_2 I_k d_{k-ip} (1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2}$$

Simplify and our final equation for the batted-ball velocity becomes

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - \beta_b d_{k-ip}) (1 + CoR_{2c}) m_2 I_k}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2}$$

$$v_{ball-after} = v_{ball-before} - \frac{(v_{ball-before} - v_{bat-before} - \beta_{before} d_{k-ip}) (1 + CoR_{2c}) m_{bat} I_{knob}}{m_{ball} I_{knob} + m_{bat} I_{knob} + m_{ball} m_{bat} d_{k-ip}^2}$$

or if we let

$$\bar{A} = \frac{(v_{1b} - v_{2b} - d_{k-ip} \beta_b) (1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2} \quad v_{1a} = v_{1b} - \bar{A} m_2 I_k$$

Nothing in this derivation depended on the collision being at the sweet spot of the bat. Therefore, d_{k-ip} could be replaced with the distance from the knob to any arbitrary impact point. This equation was derived from Eqs. (5.4), (5.5) and (5.6).

This $\bar{A}$ differs from the A of the BaConLaw model in that it uses I_k instead of I_2 , d_{k-ss} instead of d_{cm-ss} and CoR_{2c} instead of CoR_{2b} .

5.3.7 Bat Translational Velocity After the Collision

Now, we will derive an equation for the translational velocity of the bat after the collision. We start with Eq. (5.5) and solve for the bat translational velocity after the collision, v_{2a}

$$CoR_{2c} = -\frac{v_{1a} - v_{2a} - d_{k-ip}\beta_a}{v_{1b} - v_{2b} - d_{k-ip}\beta_b}$$

$$v_{2a} = v_{1a} + CoR_{2c}(v_{1b} - v_{2b} - d_{k-ip}\beta_b) - d_{k-ip}\beta_a$$

First, get rid of β_a by substituting this β_a that we derived above.

$$\beta_a = \beta_b - \frac{d_{k-ip}m_1(v_{1a} - v_{1b})}{I_k}$$

$$v_{2a} = v_{1a} + CoR_{2c}(v_{1b} - v_{2b} - d_{k-ip}\beta_b) - d_{k-ip}\left\{\beta_b - \frac{d_{k-ip}m_1(v_{1a} - v_{1b})}{I_k}\right\}$$

$$v_{2a} = v_{1a} + CoR_{2c}(v_{1b} - v_{2b} - d_{k-ip}\beta_b) + \frac{m_1d_{k-ip}^2(v_{1a} - v_{1b})}{I_k} - d_{k-ip}\beta_b$$

$$v_{2a} = v_{1a}\left(1 + \frac{m_1d_{k-ip}^2}{I_k}\right) - v_{1b}\left(\frac{m_1d_{k-ip}^2}{I_k} - CoR_{2c}\right) - v_{2b}CoR_{2c} - d_{k-ip}\beta_b(1 + CoR_{2c})$$

$$\text{Let } \bar{D} = \frac{m_1d_{k-ip}^2}{I_k}$$

$$v_{2a} = v_{1a}(1 + \bar{D}) - v_{1b}(\bar{D} - CoR_{2c}) - v_{2b}CoR_{2c} - d_{k-ip}\beta_b(1 + CoR_{2c})$$

Now get rid of v_{1a} by substituting this v_{1a} that we derived above.

$$v_{1a} = \left\{v_{1b} - \frac{(v_{1b} - v_{2b} - d_{k-ip}\beta_b)(1 + CoR_{2c})m_2I_k}{m_1I_k + m_2I_k + m_1m_2d_{k-ip}^2}\right\}$$

$$v_{2a} = \left\{v_{1b} - \frac{(v_{1b} - v_{2b} - d_{k-ip}\beta_b)(1 + CoR_{2c})m_2I_k}{m_1I_k + m_2I_k + m_1m_2d_{k-ip}^2}\right\}(1 + \bar{D}) - v_{1b}(\bar{D} - CoR_{2c}) - v_{2b}CoR_{2c} - d_{k-ip}\beta_b(1 + CoR_{2c})$$

$$\begin{aligned}\bar{K}v_{2a} = & \bar{K}v_{1b}(1 + \bar{D}) - (v_{1b} - v_{2b} - d_{k-ip}\beta_b)(1 + CoR_{2c})(1 + \bar{D})m_2I_k \\ & - v_{1b}\bar{K}(\bar{D} - CoR_{2c}) - v_{2b}\bar{K}CoR_{2c} - d_{k-ip}\beta_b\bar{K}(1 + CoR_{2c})\end{aligned}$$

$$\begin{aligned}\bar{K}v_{2a} = & v_{1b}[\bar{K}(1 + \bar{D}) - MM(1 + \bar{D}) - \bar{K}\bar{D} + \bar{K}CoR_{2c}] \\ & + v_{2b}[MM(1 + \bar{D}) - \bar{K}CoR_{2c}] \\ & + d_{k-ip}\beta_b[MM(1 + \bar{D}) - \bar{K}(1 + CoR_{2c})]\end{aligned}$$

$$\begin{aligned}\bar{K}v_{2a} = & v_{1b}[\bar{K} - MM(1 + \bar{D}) + \bar{K}CoR_{2c}] \\ & + v_{2b}[MM(1 + \bar{D}) - \bar{K}CoR_{2c}] \\ & + \beta_b d_{k-ip}[MM(1 + \bar{D}) - \bar{K}(1 + CoR_{2c})]\end{aligned}$$

$$\begin{aligned}\bar{K}v_{2a} = & v_{1b}[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})] \\ & - v_{2b}[\bar{K}CoR_{2c} - MM(1 + \bar{D})] \\ & - d_{k-ip}\beta_b[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})]\end{aligned}$$

Add $\{+v_{2b}\bar{K} - v_{2b}\bar{K}\}$ to the right side

$$\begin{aligned}\bar{K}v_{2a} = & v_{1b}[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})] \\ & - v_{2b}[\bar{K}CoR_{2c} - MM(1 + \bar{D})] + \{v_{2b}\bar{K} - v_{2b}\bar{K}\} \\ & - d_{k-ip}\beta_b[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})]\end{aligned}$$

$$\begin{aligned}\bar{K}v_{2a} = & v_{1b}[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})] \\ & - v_{2b}[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})] + v_{2b}\bar{K} \\ & - d_{k-ip}\beta_b[\bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})]\end{aligned}$$

Let $Q = \bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})$

$$\bar{K}v_{2a} = v_{2b}\bar{K} + v_{1b}Q - v_{2b}Q - \beta_b d_{k-ip}Q$$

divide by $\bar{K}$

$$v_{2a} = v_{2b} + \frac{(v_{1b} - v_{2b} - d_{k-ip}\beta_b)Q}{\bar{K}}$$

That looks good. So, let's work on Q for a while.

$$Q = \bar{K}(1 + CoR_{2c}) - MM(1 + \bar{D})$$

$$MM = m_2 I_k (1 + CoR_{2c})$$

$$\bar{D} = \frac{m_1 d_{k-ip}^2}{I_k}$$

$$MM(1 + \bar{D}) = m_2 I_k (1 + CoR_{2c}) \left(1 + \frac{m_1 d_{k-ip}^2}{I_k} \right)$$

$$MM(1 + \bar{D}) = [m_2 I_k (1 + CoR_{2c}) + m_1 m_2 d_{k-ip}^2 (1 + CoR_{2c})]$$

$$Q = \bar{K}(1 + CoR_{2c}) - [m_2 I_k (1 + CoR_{2c}) + m_1 m_2 d_{k-ip}^2 (1 + CoR_{2c})]$$

$$\bar{K} = m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2$$

$$Q = (m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2)(1 + CoR_{2c}) - [m_2 I_k (1 + CoR_{2c}) + m_1 m_2 d_{k-ip}^2 (1 + CoR_{2c})]$$

cancel equal terms

$$Q = m_1 I_k (1 + CoR_{2c})$$

$$\frac{v_{2a} = v_{2b} + (v_{1b} - v_{2b} - d_{k-ip} \beta_b) m_1 I_k (1 + CoR_{2c})}{\bar{K}}$$

$$v_{2a} = v_{2b} + \frac{(v_{1b} - v_{2b} - d_{k-ip} \beta_b) m_1 I_k (1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2}$$

If we let

$$\bar{A} = \frac{(v_{1b} - v_{2b} - d_{k-ip} \beta_b)(1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2}$$

$$v_{2a} = v_{2b} + \bar{A} m_1 I_k$$

$$v_{bat-after} = v_{bat-before} + \frac{(v_{ball-before} - v_{bat-before} - \beta_{before} d_{k-ip}) m_{ball} I_{knob} (1 + CoR_{2c})}{m_{ball} I_{knob} + m_{bat} I_{knob} + m_{ball} m_{bat} d_{k-ip}^2}$$

5.3.8 Bat Angular Velocity After the Collision

Now, we will derive an equation for the rotational velocity of the bat after the collision. We start with the previously derived equation for β_a .

$$\beta_a = \beta_b - \frac{d_{k-ip} m_1 (v_{1a} - v_{1b})}{I_k}$$

Now we must get rid of the term with the *after* subscript. Multiply by I_k

$$\beta_a I_k = \beta_b I_k - d_{k-ip} m_1 (v_{1a} - v_{1b})$$

Substitute the previously derived expression for v_{1a} .

$$v_{1a} = \left\{ \frac{v_{1b} - (v_{1b} - v_{2b} - \beta_b d_{k-ip})(1 + CoR_{2c})m_2 I_k}{\bar{K}} \right\}$$

$$\beta_a I_k = \beta_b I_k - d_{k-ip} m_1 \left(\left\{ v_{1b} - \frac{(v_{1b} - v_{2b} - \beta_b d_{k-ip})(1 + CoR_{2c})m_2 I_k}{\bar{K}} \right\} - v_{1b} \right)$$

Cancel $(v_{1b} - v_{1b})$ and multiply by $\bar{K}$

$$\beta_a I_k \bar{K} = \beta_b I_k \bar{K} - d_{k-ip} m_1 (- (v_{1b} - v_{2b} - \beta_b d_{k-ip})(1 + CoR_{2c})m_2 I_k)$$

Distribute the $-d_{k-ip} m_1$ term

$$\beta_a I_k \bar{K} = \beta_b I_k \bar{K} + (v_{1b} - v_{2b} - \beta_b d_{k-ip}) d_{k-ss} m_1 m_2 I_k (1 + CoR_{2c})$$

Collect similar terms

$$\begin{aligned} \beta_a I_k \bar{K} &= \beta_b I_k \bar{K} \\ &+ v_{1b} m_1 m_2 d_{k-ip} I_k (1 + CoR_{2c}) \\ &- v_{2b} m_1 m_2 d_{k-ip} I_k (1 + CoR_{2c}) \\ &- \beta_b d_{k-ss} m_1 m_2 d_{k-ip} I_k (1 + CoR_{2c}) \end{aligned}$$

Divide by $I_k \bar{K}$

$$\begin{aligned} \beta_a &= \beta_b \\ &+ v_{1b} \frac{(1 + CoR_{2c})m_1 m_2 d_{k-ip}}{\bar{K}} \\ &- v_{2b} \frac{(1 + CoR_{2c})m_1 m_2 d_{k-ip}}{\bar{K}} \\ &- \beta_b d_{k-ip} \frac{(1 + CoR_{2c})m_1 m_2 d_{k-ip}}{\bar{K}} \\ \beta_a &= \beta_b + \frac{(v_{1b} - v_{2b} - \beta_b d_{k-ss})(1 + CoR_{2c})m_1 m_2 d_{k-ip}}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2} \end{aligned}$$

Let

$$\begin{aligned} \bar{A} &= \frac{(v_{1b} - v_{2b} - \beta_b d_{k-ip})(1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k-ip}^2} \\ \beta_a &= \beta_b + \bar{A} m_1 m_2 d_{k-ip} \end{aligned}$$

5.3.9 Conservation of Energy

The following equation is for the kinetic energy lost.

$$0 = m_1 v_{1b}^2 + m_2 v_{2b}^2 + I_k \beta_b^2 - m_1 v_{1a}^2 - m_2 v_{2a}^2 - I_k \beta_a^2 - 2KE_{\text{lost}}$$

These are our equations for the outputs.

$$\begin{aligned} v_{1a} &= v_{1b} - \bar{A} m_2 I_k \\ v_{2a} &= v_{2b} + \bar{A} m_1 I_k \\ \beta_a &= \beta_b + \bar{A} m_1 m_2 d_{k\text{-ip}} \\ \omega_{1a} &= \omega_{1b} \end{aligned}$$

Substituting the linear velocity of the ball after the collision, v_{1a} , the linear velocity of the bat after the collision, v_{2a} and the angular velocity of the bat after the collision, ω_{2a} into this Conservation of Energy equation yields

$$KE_{\text{lost}} = \frac{1}{2} \left\{ m_1 v_{1b}^2 + m_2 v_{2b}^2 + I_k \beta_b^2 - m_1 (v_{1b} - \bar{A} m_2 I_k)^2 - m_2 (v_{2b} + \bar{A} m_1 I_k)^2 - I_k (\beta_b + \bar{A} m_1 m_2 d_{k\text{-ip}})^2 \right\}$$

Substitute for $\bar{A}$

$$\bar{A} = \frac{(v_{1b} - v_{2b} - d_{k\text{-ip}} \beta_b)(1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k\text{-ip}}^2}$$

After a little bit of algebra that follows the development in chapter 4 we get

$$KE_{\text{lost}} = \frac{1}{2} \frac{m_1 m_2 I_k}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k\text{-ip}}^2} \left[(v_{1b} - v_{2b} - \beta_b d_{k\text{-ip}})^2 (1 - CoR_{2c}^2) \right]$$

or expanding the abbreviations gives

$$KE_{\text{lost}} = \frac{1}{2} \frac{m_{\text{ball}} m_{\text{bat}} I_{\text{knob}}}{m_{\text{ball}} I_{\text{knob}} + m_{\text{bat}} I_{\text{knob}} + m_{\text{ball}} m_{\text{bat}} d_{k\text{-ip}}^2} \left[(v_{\text{ball-before}} - v_{\text{bat-before}} - \beta_{\text{before}} d_{k\text{-ip}})^2 (1 - CoR_{2c}^2) \right]$$

5.3.10 Summary: The Output Equations

Our final equation for the batted-ball velocity is

$$v_{1a} = v_{1b} - \frac{(v_{1b} - v_{2b} - \beta_b d_{k\text{-ip}})(1 + CoR_{2c})m_2 I_k}{m_1 I_k + m_2 I_k + m_1 m_2 d_{k\text{-ip}}^2}$$

Expanding the subscripts, we get

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \frac{(v_{\text{ball-before}} - v_{\text{bat-before}} - \beta_{\text{before}} d_{\text{k-ip}}) m_{\text{bat}} I_{\text{kno}} (1 + CoR_{2c})}{m_{\text{ball}} I_{\text{kno}} + m_{\text{bat}} I_{\text{kno}} + m_{\text{ball}} m_{\text{bat}} d_{\text{k-ip}}^2}$$

Our final equation for the translational bat velocity after the collision is

$$v_{2a} = v_{2b} + \frac{(v_{1b} - v_{2b} - d_{\text{k-ip}} \beta_b)(1 + CoR_{2c}) m_1 I_k}{m_1 I_k + m_2 I_k + m_1 m_2 d_{\text{k-ip}}^2}$$

or

$$v_{\text{bat-after}} = v_{\text{bat-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-before}} - \beta_{\text{before}} d_{\text{k-ip}})(1 + CoR_{2c}) m_{\text{ball}} I_{\text{kno}}}{m_{\text{ball}} I_{\text{kno}} + m_{\text{bat}} I_{\text{kno}} + m_{\text{ball}} m_{\text{bat}} d_{\text{k-ip}}^2}$$

Our final equation for the rotational velocity of the bat after the collision is

$$\beta_a = \beta_b + \frac{(v_{1b} - v_{2b} - \beta_b d_{\text{k-ip}})(1 + CoR_{2c}) m_1 m_2 d_{\text{k-ip}}}{m_1 I_k + m_2 I_k + m_1 m_2 d_{\text{k-ip}}^2}$$

$$\beta_{\text{bat-after}} = \beta_{\text{bat-before}} + \frac{(v_{\text{ball-before}} - v_{\text{bat-before}} - \beta_{\text{before}} d_{\text{k-ip}})(1 + CoR_{2c}) m_{\text{ball}} m_{\text{bat}} d_{\text{k-ip}}}{m_{\text{ball}} I_{\text{kno}} + m_{\text{bat}} I_{\text{kno}} + m_{\text{ball}} m_{\text{bat}} d_{\text{k-ip}}^2}$$

These three equations have a common term

$$\bar{A} = \frac{(v_{\text{ball-before}} - v_{\text{bat-before}} - \beta_{\text{before}} d_{\text{k-ip}})(1 + CoR_{2c})}{m_{\text{ball}} I_{\text{kno}} + m_{\text{bat}} I_{\text{kno}} + m_{\text{ball}} m_{\text{bat}} d_{\text{k-ip}}^2}$$

We can summarize with the following.

If we let $\bar{A} = \frac{(v_{1b} - v_{2b} - d_{\text{k-ip}} \beta_b)(1 + CoR_{2c})}{m_1 I_k + m_2 I_k + m_1 m_2 d_{\text{k-ip}}^2}$ Then our set of equations becomes $CoR_{2c} = - \frac{v_{1a} - v_{2a} - d_{\text{k-ip}} \beta_a}{v_{1b} - v_{2b} - d_{\text{k-ip}} \beta_b}$ $v_{1a} = v_{1b} - \bar{A} m_{\text{bat}} I_k$ $v_{2a} = v_{2b} + \bar{A} m_{\text{ball}} I_k$ $\beta_a = \beta_b + \bar{A} m_{\text{ball}} m_{\text{bat}} d_{\text{k-ip}}$ $\omega_{1a} = \omega_{1b}$
--

5.4 Differences Between the BaConLaw and Sliding Pin Models

The purpose of Chap. 4 was to develop the BaConLaw model that explains bat-ball collisions with precise, correct equations, without jargon. The BaConLaw model described head-on bat-ball collisions at the sweet spot of the bat. It gave the speed and spin of the bat and ball before and after collisions. The purpose of the Sliding Pin model of Chap. 5 was to model a new type of data. Previously the input data for our models were the translational and rotational velocities at the *center of mass* of the bat. However, the Sliding Pin model used the translational and rotational velocities at the *knob*. The experimental data produced different nominal values for the inputs. Because these two models had different purposes and inputs, we would not expect them to be equivalent. And they are not. Here are some of the differences between these two models.

The BaConLaw model If you toss a bat into the air, it will have linear motion and it will rotate about its center of mass. Because a bat is a rigid object, every spot on a bat will have the same linear translational velocity and the same angular rotational velocity.

$$v_{\text{bat-knob-trans}} = v_{\text{bat-cm-trans}} = v_{\text{bat-cop-trans}} \text{ and} \\ \omega_{\text{bat-knob}} = \omega_{\text{bat-cm}} = \omega_{\text{bat-cop}}$$

However, each spot on the bat will have a different *total* velocity that depends on the pivot point and the spot's distance from that pivot point.

$$v_{t\text{cop}} = v_{\text{cm}} + d_{\text{cm-cop}}\omega_{\text{bat}}$$

If a bat tossed into the air were hit by a ball, it would be a free-end collision because there are no other forces acting on the bat. The BaConLaw model uses a free-end collision because of the simplicity. We need not search for other forces on the bat, because there are none. The BaConLaw model and the Effective Mass model both assume free-end collisions, with no external forces and rotations about the center of mass.

The Sliding Pin model Now imagine a bat that is pinned through its knob, but the pinned point is allowed to slide along the x-axis, as in Fig. 5.2. This bat will have linear motion and it will rotate about its knob. Because a bat is a rigid object, every spot on a bat will have the same linear translational velocity and the same angular rotational velocity.

$$v_{\text{bat-knob-trans}} = v_{\text{bat-cm-trans}} = v_{\text{bat-cop-trans}} \\ \text{and } \beta_{\text{bat-knob}} = \beta_{\text{bat-cm}} = \beta_{\text{bat-cop}}$$

However, each spot on the bat will have a different *total* velocity that depends on the pivot point and the spot's distance from that pivot point.

$$vt_{cm} = v_{knob} + d_{knob-cm}\beta_{bat}$$

$$vt_{cop} = v_{knob} + d_{knob-cop}\beta_{bat}$$

If the pinned bat were hit by a ball, it would produce forces on the pin. This makes the Sliding Pin model more complicated than the BaConLaw model. Forces in the x-direction are not worrisome: they are known to be small (Milanovich and Nesbit 2014) and they would merely accelerate the bat in the x-direction. Two of the forces on the pin will be along the y-axis. The centrifugal force due to the bats rotation about the pin will be in the negative y-direction. The human will be applying an approximately equal and opposite centripetal force in the positive y-direction, as shown in Fig. 5.3. But at the time of the collision these forces will not affect the bat's velocity $\bar{v}_{bat}$ because they are perpendicular to it. The Sliding Pin model assumes negligible forces on the pin and rotations about the knob.

Consider the BaConLaw model of Fig. 4.1 and the Sliding Pin model of Fig. 5.2. For the time being let us ignore the translational movements and consider only rotational movements. Suppose you want to move the sweet spot forward a distance x . The BaConLaw model of Fig. 4.1 would require a rotation through an angle θ_{cm} where the $\tan \theta_{cm} = \frac{x}{d_{cm-ip}}$. Whereas, the Sliding Pin model of Fig. 5.2 would require a rotation through an angle θ_{knob} where the $\tan \theta_{knob} = \frac{x}{d_{knob-ip}}$. Now the angular velocity of the BaConLaw model is $\omega_{bat} = \frac{d\theta_{cm}}{dt}$ whereas the angular velocity of the Sliding Pin model is $\beta_{bat} = \frac{d\theta_{knob}}{dt}$. Clearly $\omega_{bat} \neq \beta_{bat}$.

The Sliding Pin model is analogous to the BaConLaw model, but it is not equivalent.

$$A \neq \bar{A}$$

$$A = \frac{(v_{1b} - v_{2b} - d_{cm-ip}\omega_{2b})(1 + CoR_{2b})}{m_1 I_{bat-cm} + m_2 I_{bat-cm} + m_1 m_2 d_{cm-ip}^2}$$

$$\bar{A} = \frac{(v_{1b} - v_{2b} - d_{k-ip}\beta_b)(1 + CoR_{2c})}{m_1 I_{knob} + m_2 I_{knob} + m_1 m_2 d_{k-ip}^2}$$

Therefore, the two $v_{ball-after}$ equations yield different numerical values. From the BaConLaw model of Chap. 4 we have

$$v_{ball-after} = v_{ball-before} - Am_{bat}I_{bat-cm}$$

And from the Sliding Pin model of this chapter we have

$$v_{\text{ball-after}} = v_{\text{ball-before}} - \bar{A}m_{\text{bat}}I_{\text{knob}}$$

Because $Am_{\text{bat}}I_{\text{bat-cm}} \neq \bar{A}m_{\text{bat}}I_{\text{knob}}$ the v_{1a} of the BaConLaw model is not the same as the $v_{\text{ball-after}}$ of the Sliding Pin model.

For the angular momentum $L_{\text{initial}} = L_{\text{final}}$ for both models, but the numerical values are different. Numerically the CoRs are the same although their equations are different.

$$CoR_{2b} = - \frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}}\omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}}\omega_{\text{bat-before}}}$$

$$CoR_{2c} = - \frac{v_{\text{ball-after}} - v_{\text{knob-after}} - d_{\text{knob-ip}}\beta_{\text{after}}}{v_{\text{ball-before}} - v_{\text{knob-before}} - d_{\text{knob-ip}}\beta_{\text{before}}}$$

where the subscript ‘ip’ stands for the impact point.

Inputs and outputs for the BaConLaw model from Table 4.1

Inputs	$v_{\text{ball-before}}, \omega_{\text{ball-before}}, v_{\text{bat-cm-before}}, \omega_{\text{bat-before}}$ and CoR_{2b}
Outputs	$v_{\text{ball-after}}, \omega_{\text{ball-after}}, v_{\text{bat-ip-after}}, \omega_{\text{bat-after}}$, and KE_{lost}

Inputs and outputs for the Sliding Pin model from Table 5.1

Inputs	$v_{\text{ball-before}}, \omega_{\text{ball-before}}, v_{\text{knob-before}}, \beta_{\text{knob-before}}$ and CoR_{2c}
Outputs	$v_{\text{ball-after}}, \omega_{\text{ball-after}}, v_{\text{bat-ip-after}}, \beta_{\text{knob-after}}$ and KE_{lost}

The BaConLaw and Sliding Pin models are analogous, but they are not equivalent. The derivations followed the same processes and the outputs have similar forms but the numbers are different.

The BaConLaw model states that the maximum batted-ball speed will occur for a collision 0.66 m from the knob, while the Sliding Pin model states that the maximum batted-ball speed will occur for a collision 0.68 m from the knob. Once again, the models are different.

For the BaConLaw model

$$\frac{\partial v_{1a}}{\partial I_2} = \frac{Bm_1m_2^2d_{\text{cm-cop}}^2}{K^2}$$

For the Sliding Pin model

$$\frac{\partial v_{1a}}{\partial I_k} = \frac{Bm_1m_2^2d_{\text{k-ip}}^2}{K^2}$$

5.4.1 *Simulation Results*

Tables 5.3 and 5.4 show the results of our Excel simulation of the Sliding Pin model equations using the Diamond Kinetics input data from Table 5.1. These results are similar to those in Tables 4.2 and 4.3 for the BaConLaw model except that the batted-ball speed $v_{\text{ball-after}}$ is smaller, 83 mph (37.2 m/s) versus 92 mph (41 m/s). The probable cause is that the BaConLaw model used input values that were appropriate for major leaguers, whereas the Diamond Kinetics data were for professionals, but not major leaguers.

The kinetic energies of the bat linear velocity and the bat angular velocity in Table 5.4 are different from those in Table 4.3, because the experimental data for

Table 5.3 Simulation values for bat-ball collisions of the Sliding Pin model

	SI units (m/s, rad/s)	Baseball units (mph, rpm)
Inputs		
$v_{\text{ball-before}}$	−37.1	−83.0
$\omega_{\text{ball-before}}$	209	2000
$v_{\text{knob-before}}$	4.5	10
β_{before}	41	387
CoR_{2c}	0.453	0.453
Outputs		
$v_{\text{ball-after}}$	37.2	83.4
$\omega_{\text{ball-after}}$	$=\omega_{\text{ball-before}}$	
$v_{\text{knob-after}}$	−7	−17
β_{after}	18	175

Table 5.4 Comparison of inputs and outputs of the Sliding Pin model and the BaConLaw model

<i>Sliding Pin model</i>		
Inputs		
$v_{\text{knob-before}}$	4.5	10
β_{before}	41	387
$v_{\text{tbat-ip-before}}$	33.4 m/s	75 mph
Outputs		
$v_{\text{knob-after}}$	−7	−17
β_{after}	18	175
<i>BaConLaw model</i>		
Inputs		
$v_{\text{bat-cm-before}}$	23 m/s	52 mph
$\omega_{\text{bat-before}}$	32 rad/s	309 rpm
$v_{\text{tbat-ip-before}}$	28 m/s	62 mph
Outputs		
$v_{\text{bat-cm-after}}$	11 m/s	24 mph
$\omega_{\text{bat-after}}$	1 rad/s	7 rpm

Table 5.5 Kinetic energies for the Sliding Pin model collision, Joules

KE of ball linear velocity before, $v_{\text{ball}} - \text{before}$	100
KE of bat linear translational velocity before, $v_{\text{bat}} - \text{trans} - \text{before}$	9
KE of ball angular velocity before, $\omega_{\text{ball}} - \text{before}$	1.7
KE of bat angular velocity before, β_b	280
KE before, total	391
KE of ball linear velocity after, $v_{\text{ball}} - \text{after}$	100
KE of bat linear translational velocity after, $v_{\text{bat}} - \text{trans} - \text{after}$	25
KE of ball angular velocity after, $\omega_{\text{ball}} - \text{after}$	1.7
KE of bat angular velocity after, β_a	57
KE after, total	184
KE lost	207
KE before minus (KE after plus KE lost)	391

It is just a coincidence that the KE of the ball linear velocity before and after are nearly the same

these variables are different. Otherwise, the numbers in Table 5.4 are comparable to those of Table 4.3. This shows that our analysis and equations are consistent.

Modeling philosophy note Earlier we noted that, if we set $d_{\text{cm-ip}} = 0$ in the simulation of the BaConLaw model so that the impact point was at the center of mass of the bat, then Tables 4.2 and 4.3 changed and produced the results of Tables 3.3 and 3.4 for configuration 1b. This means that the whole BaConLaw model (equations, simulations, sensitivity analyses, results, etc.) can be reduced to be appropriate for configurations 1a, 1b and 2a by zeroing appropriate values. However, this does not work for all models. For example, we cannot set variables and parameters in the Sliding Pin model so that it is equivalent to the BaConLaw model or the Effective Mass model. The Sliding Pin model is analogous to the BaConLaw model, but it is not equivalent.

5.5 Collisions with Friction

Purpose: The purpose of this section is to present the Collision with Friction model. Our modeling technique could not handle this configuration because our model is only good for a point before the collision and a point after the collision. It cannot handle behavior during the collision. The BaConLaw model of Chap. 4 fulfilled part of the first purpose of this book. It showed a complex configuration for which our technique did work. This section completes the fulfillment of this purpose by showing a configuration for which our technique is too simple.

One of the purposes of this book is to find how complicated our configurations can be and still be solvable using only Newton's principles and the conservation laws. The BaConLaw model passed this test. So now, let us try configuration 2d, the

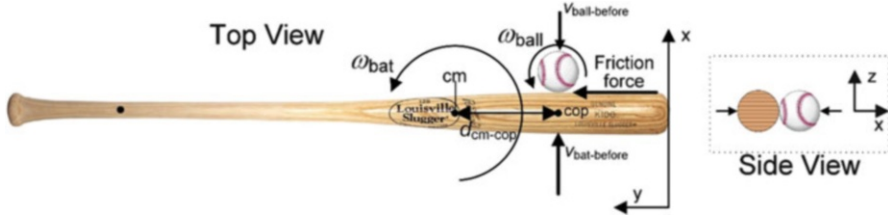


Fig. 5.6 Model of the bat-ball collision with the addition of friction between the bat and ball. The arrows show that angular momenta are positive when pointing out of the page

Table 5.6 Equations for the Collision with Friction model, two equations and one unknown

Inputs	$v_{\text{ball-before}}, \omega_{\text{ball-before}}, v_{\text{bat-before}}, \omega_{\text{bat-before}}$
Outputs	$\omega_{\text{ball-after}}$
Equations	
Newton's Second Law, Eq. (5.6)	$\Delta t (r_{\text{ball}} \times F_{\text{friction}}) = -(I_{\text{ball}} \omega_{\text{ball-after}} - I_{\text{ball}} \omega_{\text{ball-before}})$
Conservation of Angular Momentum, Eq. (5.7s)	$L_{\text{initial}} = L_{\text{final}}$ $m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - 0.1 \mu_f m_1 r_1 v_{1b} + m_1 d^2 \omega_{1b}$ $= +m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + 0.1 \mu_f m_1 r_1 v_{1a} - m_1 d^2 \omega_{1a}$

Collision with Friction model. This model is for a collision at the sweet spot of the bat with spin on the pitch and with consideration of friction between the bat and ball, as shown in Fig. 5.6. The inputs, outputs and equations are given in Table 5.6.

5.5.1 Using Newton's Principles

During the collision, the ball velocity changes from $v_{\text{ball-before}}$ to $v_{\text{ball-after}}$. Assume that the ball velocity reaches zero somewhere in the middle of the collision. Therefore, during this first part of the collision the velocity changes from $v_{\text{ball-before}}$ to 0. By Newton's second principle, we can write the force that the ball exerts on the bat normal to the tangent plane of the collision is $F_{\text{normal}} = \frac{m_{\text{ball}} v_{\text{ball-before}}}{\Delta t}$. We postulate that during the first part of the collision the ball is sliding across the bat. Therefore the friction force acting on the ball is

$$F_{\text{friction}} = F_{\text{normal}} \mu_f$$

$$F_{\text{friction}} = \left| \frac{m_1 v_{1b} \mu_f}{\Delta t} \right|$$

Fig. 5.7 A bat-ball collision showing how much the baseball is deformed during a collision. The collision lasts about 1 ms (Photo Credit: UMass Lowell Baseball Research Center. From <https://student.societyforscience.org/sites/student.societyforscience.org/files/main/articles/batball.jpg>)



The absolute value sign is necessary because friction always opposes motion. I experimentally measured the dynamic coefficient of friction between a wooden baseball bat and a baseball to be $\mu_f = 0.5$. We will use this numerical value in the simulation. According to Newton's second law, this friction force, shown in Fig. 5.6, creates a torque that reduces the angular momentum of the ball. The amount depends on how long we apply the torque, $\Delta t'$.

$$\Delta t'(r_{\text{ball}} \times F_{\text{friction}}) = -(I_1 \omega_{1a} - I_1 \omega_{1b})$$

As always, omega, ω , stands for the rotational velocity of an object about its center of mass. This friction force only exists when the ball is sliding across the surface of the bat, not when it is rolling or gripping. Figure 5.7 shows how the ball is deformed during the collision. This suggests that the ball is sliding on the bat during only a short part of the collision (maybe the first 10% of the total collision duration), then it grips the bat tightly.

We can solve the above equation for ω'_{1a} . This omega has a prime symbol on it because it is not the omega after the whole collision. It is the omega after only the first part of the collision where the ball is sliding on the bat. Let $\Delta t'$ be the duration of sliding and Δt be the duration of this part of the collision.

$$I_1 \omega'_{1a} = I_1 \omega_{1b} - \Delta t' F_{\text{friction}} r_{\text{ball}}$$

$$I_1 \omega'_{1a} = I_1 \omega_{1b} - \Delta t' \left| \frac{m_1 v_{1b} \mu_f}{\Delta t} \right| r_1$$

$$\text{Assume that } \Delta t' = \Delta t / 10$$

$$I_1 \omega'_{1a} = I_1 \omega_{1b} - |0.1 m_1 v_{1b} \mu_f r_1|$$

$$\omega'_{1a} = \omega_{1b} - \frac{0.1 \mu_f m_1 r_1}{I_1} |v_{1b}|$$

This result does not depend on Δt . Near the end of the collision the friction force rearises, but in the opposite direction (Cross 2011; Kensrud et al. 2017). This increases the ball spin.

$$F_{\text{normal}} = \frac{m_1 v_{1a}}{\Delta t}$$

$$F_{\text{friction}} = \left| \frac{m_1 v_{1a} \mu_f}{\Delta t} \right|$$

This time the ω'_{1b} has the prime symbol because it is not the omega before the whole collision. It is the omega before only this part of the collision.

$$I_1 \omega_{1a} = I_1 \omega'_{1b} + \Delta t' F_{\text{friction}} r_{\text{ball}}$$

$$I_1 \omega_{1a} = I_1 \omega'_{1b} + \Delta t' \left| \frac{m_1 v_{1a} \mu_f}{\Delta t} \right| r_1$$

Again assume $\Delta t' = \Delta t / 10$

$$I_1 \omega_{1a} = I_1 \omega'_{1b} + |0.1 m_1 v_{1a} \mu_f r_1|$$

$$\omega_{1a} = \omega'_{1b} + \frac{0.1 \mu_f m_1 r_1}{I_1} |v_{1a}|$$

Now, we ignore all of the time when the ball is not sliding across the bat and ω'_{1a} becomes ω'_{1b} and we can combine these equations to get

$$\omega_{1a} = \omega_{1b} + \frac{\mu_f m_1 r_1}{10 I_1} (|v_{1a}| - |v_{1b}|)$$

or by expanding the subscripts

$$\omega_{\text{ball-after}} = \omega_{\text{ball-before}} + \frac{\mu_{\text{friction}} m_{\text{ball}} r_{\text{ball}}}{10 I_{\text{ball}}} (|v_{\text{ball-after}}| - |v_{\text{ball-before}}|)$$

However, this whole analysis depends on how long the ball slides on the bat before it switches to rolling or griping.

5.5.2 Conservation of Angular Momentum

Most of the equations for the BaConLaw model also apply to the Collision with Friction model. The exceptions are Conservation of Energy and kinetic energy lost. As in the BaConLaw model, at the instant when the ball contacts the bat, as shown in Fig. 5.2, the ball has a linear translational velocity of $v_{\text{ball-before}}$ that, as before, we model as the ball rotating about the bat's center of mass at a distance $d = d_{\text{cm-ip}}$. When it comes time to substitute a value for d we will use either $d = d_{\text{cm-ip}}$ or $d = \sqrt{d_{\text{cm-ip}}^2 + r_{\text{bat}}^2}$. However, the sensitivity analysis has shown that this is one of the least significant parameters in the model. Therefore, which we use is not important. The ball also has angular momentum because of its spin: we use the parallel axis theorem to compute the moment of inertia with respect to the center of mass of the bat, $(I_{\text{ball}} + m_{\text{ball}}d^2)\omega_{\text{ball-before}}$. The bat has initial angular momentum, $I_2\omega_{2b}$. Now we add a new term due to the friction between the bat and ball, $r_{\text{ball}} \times F_{\text{friction}}$. This term exists during the collision, not before. Nevertheless, we will lump it in with the initial angular momentum. Therefore, we can write the sum of the initial angular momenta of the bat-ball system about an axis through the center of mass of the bat parallel to the z-axis. In Fig. 5.2, positive moments will be pointing out of the page.

$$L_{\text{initial}} = m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - F_{\text{friction}} r_1 \Delta t$$

$$L_{\text{initial}} = m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - |0.1 \mu_f m_1 v_{1b} r_1|$$

Assume that the last term is $I_1 \omega_{1b}$ about the center of mass of the ball. To relate it to an axis through the center of mass of the bat, use the parallel axis theorem.

$$L_{\text{initial}} = m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - (I_1 - m_1 d^2) \omega_{1b}$$

$$L_{\text{initial}} = m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - I_1 \omega_b + m_1 d^2 \omega_{1b}$$

$$L_{\text{initial}} = m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - |0.1 \mu_f m_1 v_{1b} r_1| + m_1 d^2 \omega_{1b}$$

For the final angular momentum after the collision, we will treat the ball, as before, as an object orbiting the center of mass of the bat with angular momentum, $m_{\text{ball}} v_{\text{ball-after}} d_{\text{cm-ip}}$. The ball also has angular momentum because of its spin: we use the parallel axis theorem to compute the moment of inertia with respect to an axis through the center of mass of the bat, $(I_{\text{ball}} + m_{\text{ball}}d^2)\omega_{\text{ball-after}}$. The bat angular momentum is $I_{\text{bat}}\omega_{\text{bat-after}}$. The sum of the angular momenta after the collision is

$$L_{\text{final}} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + F_{\text{friction}} r_1 \Delta t$$

$$L_{\text{final}} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + |0.1 \mu_f m_1 v_{1a} r_1|$$

Table 5.7 Simulation values for bat-ball collisions at the sweet spot for the Collision with Friction model

	SI units (m/s or rad/s)	mph or rpm
Inputs		
$v_{\text{ball}} - \text{before}$	-37	-83
$v_{\text{bat}} - \text{ip} - \text{before}$	28	62
Results		
$\omega_{\text{ball}} - \text{before}$	209	2000
$\omega_{\text{ball}} - \text{after}$	222	2126
$\omega_{\text{ball}} - \text{before}$	0	0
$\omega_{\text{ball}} - \text{after}$	13	126
$\omega_{\text{ball}} - \text{before}$	-209	-2000
$\omega_{\text{ball}} - \text{after}$	-196	-1874

Assume that the last term is $I_1\omega_{1b}$ about the center of mass of the ball. To relate it to an axis through the center of mass of the bat, use the parallel axis theorem.

$$L_{\text{final}} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + (I_z - m_1 d^2) \omega_{1a}$$

$$L_{\text{final}} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + I_z \omega_a - m_1 d^2 \omega_{1a}$$

$$L_{\text{final}} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + |0.1\mu_f m_1 v_{1a} r_1| - m_1 d^2 \omega_{1a}$$

Now for the whole bat-ball collision, we know that the initial angular momentum must equal the final angular momentum.

$$L_{\text{initial}} = L_{\text{final}}$$

$$\begin{aligned} m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - 0.1\mu_f m_1 r_1 |v_{1b}| + m_1 d^2 \omega_{1b} \\ = +m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \omega_{2a} + 0.1\mu_f m_1 r_1 |v_{1a}| - m_1 d^2 \omega_{1a} \end{aligned}$$

Previously we used Newton's Second Law, $dm_1(v_{1a} - v_{1b}) = -I_2(\omega_{2a} - \omega_{2b})$, and solved for ω_{2a} , $\omega_{2a} = \omega_{2b} + \frac{dm_1}{I_2}(v_{1b} - v_{1a})$. So let us substitute this into our Conservation of Angular Momentum equation above.

$$\begin{aligned} m_1 v_{1b} d + (I_1 + m_1 d^2) \omega_{1b} + I_2 \omega_{2b} - |\mu_f m_1 v_{1b} r_1| \\ + m_1 d^2 \omega_{1b} = m_1 v_{1a} d + (I_1 + m_1 d^2) \omega_{1a} + I_2 \left\{ \omega_{2b} + \frac{dm_1}{I_2} (v_{1b} - v_{1a}) \right\} + |\mu_f m_1 v_{1a} r_1| \\ - m_1 d^2 \omega_{1a} \end{aligned}$$

We want to solve this for the angular velocity of the ball after the collision, ω_{1a}

$$\begin{aligned} & -(I_1 + m_1 d^2) \omega_{1a} + m_1 d^2 \omega_a \\ & = -m_1 v_{1b} d - (I_1 + m_1 d^2) \omega_{1b} - I_2 \omega_{2b} - |0.1 \mu_f m_1 v_{1b} r_1| + m_1 d^2 \omega_b \\ & + m_1 v_{1a} d + I_2 \omega_{2b} + dm_1 (v_{1b} - v_{1a}) + |0.1 \mu_f m_1 v_{1a} r_1| \end{aligned}$$

Cancel the terms in color, multiply by -1 and rearrange

$$\begin{aligned} \omega_{1a} (I_1 + m_1 d^2) - m_1 d^2 \omega_{1a} &= \omega_{1b} (I_1 + m_1 d^2) + |0.1 \mu_f m_1 v_{1b} r_1| - m_1 d^2 \omega_{1b} - |0.1 \mu_f m_1 v_{1a} r_1| \\ \omega_{1a} I_1 &= \omega_{1b} (I_1 + m_1 d^2) + |0.1 \mu_f m_1 v_{1b} r_1| - m_1 d^2 \omega_{1b} - |0.1 \mu_f m_1 v_{1a} r_1| \\ \omega_{1a} I_1 &= \omega_{1b} I_1 + |0.1 \mu_f m_1 v_{1b} r_1| - |0.1 \mu_f m_1 v_{1a} r_1| \\ \boxed{\omega_{1a} = \omega_{1b} + \frac{\mu_f m_1 r_1}{10 I_1} (|v_{1a}| - |v_{1b}|)} \end{aligned}$$

This is the same equation that we derived earlier using Newton's principles. This result does not depend on d . Table 5.7 shows the simulation results using this equation. The top two rows show the nominal input values. The next two rows show before and after values for an initial ball spin of 209 rad/s. The next two rows show before and after values for an initial ball spin of 0 rad/s. The final two rows show before and after values for an initial ball spin of -209 rad/s. We know that these numbers are not exact, but they are probably within an order of magnitude. If we put $\omega_{\text{ball}} = -1874$ rpm into the simulation for Fig. 7.13, we find that the difference in range is 1%. The purpose of this table is to estimate the magnitude of error introduced by our Sect. 3.5 derivation of $\omega_{\text{ball-after}} = \omega_{\text{ball-before}}$.

The equations for (1) $v_{\text{bat-ss-after}}$ the linear velocity of the ball after the collision, (2) $v_{\text{bat-ss-after}}$ the linear velocity of the sweet spot of the bat after the collision, (3) $\omega_{\text{bat-after}}$ the angular velocity of the bat about its center of mass after the collision and (4) CoR the coefficient of restitution are the same as those derived for the BaConLaw model.

This section on the Collision with Friction model assumed that the ball slides (does not roll) across the surface of the bat during the collision. However, that is a bad assumption because the ball could slip, slide, roll or grip, or flip from one mode to another during the collision (Cross 2011; Kensrud et al. 2017). To make matters even worse, Rod Cross (personal communication 2016) pointed out that when a ball grips the bat as in Fig. 5.7 there is a large static friction force acting and it can even reverse direction during the impact. Furthermore, presently, the behavior of the bat and ball *at game speeds* is not known. Therefore, although the equations are consistent, we are going to say that the analysis is not valid because we know so little about the actual bat and ball behavior during the collision.

Modeling philosophy note The Collision with Friction model includes friction during the collision. Our modeling technique cannot handle this configuration because our model is only good for a point before the collision and a point after

the collision. It cannot handle behavior *during* the collision. Chapter 4 fulfilled part of the first purpose of this book. It showed a complex configuration for which our technique did work. Chapter 5 completed the fulfillment of this purpose by showing a configuration for which our technique was too simple. From a modeling perspective, this is an important section because few studies show failures. In this section, I show a failure. I tried to model an event, but was unsuccessful. Then I explain why I was unsuccessful.

5.6 Summary

The bat Effective Mass model and the BaConLaw model both start with Newton's principles: then they diverge. They are different: however, they yield the same rule of thumb for the batted-ball speed! This should strengthen and give people more confidence in both models.

Modeling philosophy note Having several alternative models helps ensure that you understand the physical system. No model is more correct than another. They just emphasize different aspects of the physical system. They are not competing models they are synergetic.

This chapter presented alternative models. The Effective Mass model (Fig. 5.1) was similar to the BaConLaw model of Chap. 4, except that it did not have the algebraic equations. The fundamental model for both was that of a free-end collision of a bat and ball that produced a translation and a rotation of the bat about its center of mass. They produced the same rule of thumb for the speed of the batted ball. For a major league wooden baseball bat the speed of the ball after the collision is

$$\text{batted.ball speed} = -0.19 \text{ pitch speed} + 1.22 \text{ total bat speed}$$

The units could be either m/s or mph.

The next two alternative models in this chapter were data-based models. They allowed forces on the bat handle. The Spiral Center of Mass model (Fig. 5.3) matched data for the swing of the bat where the center of mass of the bat followed a spiral trajectory. The Sliding Pin model (Fig. 5.5) used a translation and a rotation about the knob of the bat. It also allowed the batter to apply forces on the handle during the swing. These three models modeled different aspects of the swing and collision. Therefore, they gave different results for outputs such as batted-ball speed.

The last model in this chapter included friction during the collision. Our modeling technique could not handle this configuration because our model is only good for a point before the collision and a point after the collision. It cannot handle behavior during the collision.

Chapter 4 fulfilled part of the first purpose of this book. It showed a complex configuration for which our technique did work. Chapter 5 completed the fulfillment of this purpose by showing a configuration for which our technique was too simple.

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Chapter 6

Synopsis of Chapters 1 to 5

6.1 Introduction

Purpose: The purpose of this chapter is to compare the models presented in the first five chapters, show links to other studies in the physics of baseball literature and answer the question, “Could Ted Williams see his bat hit the ball?”

Chapter 1 presented Newton’s principles and laid the groundwork for analyzing bat-ball collisions. Using text and figures, Chap. 2 explained nine common configurations of bat-ball collisions. In Chap. 3, we started developing sets of equations for those configurations. Configuration 1b was for a very simple collision at the center of mass of a translating bat. Configuration 2a added a rotation of the bat and moved the collision point to the sweet spot of the bat.

In Chap. 4, we developed our complete model for bat-ball collisions. The following equations comprise our BaConLaw model for bat-ball collisions.

$$\begin{aligned}
 KE_{\text{lost}} &= \frac{1}{2} \frac{m_{\text{ball}} m_{\text{bat}} I_{\text{bat}} (v_{\text{ball-before}} - v_{\text{bat-cm-before}} - \omega_{\text{bat-before}} d_{\text{cm-ip}})^2 (1 - CoR_{2b}^2)}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d_{\text{cm-ip}}^2} \\
 A &= \frac{(v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}} \omega_{\text{bat-before}}) (1 + CoR_{2b}) m_{\text{bat}} I_{\text{bat}}}{m_{\text{ball}} I_{\text{bat}} + m_{\text{bat}} I_{\text{bat}} + m_{\text{ball}} m_{\text{bat}} d_{\text{cm-ip}}^2} \\
 CoR_{2b} &= - \frac{v_{\text{ball-after}} - v_{\text{bat-cm-after}} - d_{\text{cm-ip}} \omega_{\text{bat-after}}}{v_{\text{ball-before}} - v_{\text{bat-cm-before}} - d_{\text{cm-ip}} \omega_{\text{bat-before}}} \\
 v_{\text{ball-after}} &= v_{\text{ball-before}} - A m_{\text{bat}} I_{\text{bat}} \\
 v_{\text{bat-after}} &= v_{\text{bat-before}} + A m_{\text{ball}} I_{\text{bat}} \\
 \omega_{\text{bat-after}} &= \omega_{\text{bat-before}} + A m_{\text{ball}} m_{\text{bat}} d_{\text{cm-ip}} \\
 \omega_{\text{ball-after}} &= \omega_{\text{ball-before}}
 \end{aligned}$$

This BaConLaw model for bat-ball collisions gives the linear and angular velocity of the bat and ball after the collision in terms of these same variables before the collision. Its development used only Newtonian mechanics and the conservation

laws. It was assumed that there are no external forces and no friction. The fundamental principle and limiting condition for the BaConLaw model was our assumption that the bat-ball collision is a free-end collision. That means that the bat acts as if no one is holding onto its knob. To visualize this, imagine that the bat is laying on a sheet of ice and you are looking down on top of it, as in Fig. 4.1. Then a baseball slams into the bat at 80 mph. This collision produces a translation and a rotation of the bat about its center of mass.

Chapter 5 contained four alternative models for bat-ball collisions. The bat Effective Mass model used the same fundamental principles of Newtonian mechanics as the BaConLaw model and the same limiting assumption that the bat-ball collision is a free-end collision. Therefore, its equation and results are similar to the BaConLaw model. For the BaConLaw model the inputs, outputs and states are the same: namely the linear velocity of the bat, the angular rotational velocity of the bat and the velocity of the ball. Whereas, for the bat Effective Mass model the input is the total velocity (meaning translation plus rotation) of the bat and the output is usually only the velocity of the ball.

The Spiral Center of Mass model and the Sliding Pin model are data-based, not theory-based. They use a different *type* of data from the previous models. The inputs to these models are the independently calculated translations and rotations about a specified point on the bat during the swing. They allow rotation about the knob of the bat. Most distinctively, they do not assume a free-end collision. The Spiral Center of Mass model represents the movement of the bat through three-dimensional space during the swing. This motion is not the simple translation and rotation about the center of mass used by the BaConLaw and Effective Mass models. The Spiral Center of Mass model stops when the collision begins. The Sliding Pin model starts when the collision begins.

The purpose of the BaConLaw model was to describe head-on bat-ball collisions at the sweet spot of the bat. It gave the speed and spin of the bat and the ball before and after collisions. The inputs for the BaConLaw model were the translational and rotational velocities at the center of mass of the bat. The purpose of the Sliding Pin model was to model a new type of data. The Sliding Pin model used the translational and rotational velocities at the knob. Because these two models had different purposes and different inputs, they are not equivalent.

Finally, the Collision with Friction model considered friction during the collision. It was shown that this type of collision cannot be modeled precisely using only the conservation laws. Therefore, this model completes the fulfillment of the first purpose of this book, to show a configuration that is too complex for our simple technique.

6.2 Limitations

We showed that the BaConLaw model for bat-ball collisions could be modeled using only Newton's principles and the conservation laws. Whereas, configurations 2d, 3 and 4 will have to use additional details such as those presented in physics of

baseball papers: Adair (2002), Branch (2007), Cross (2011), Hubbard (<http://faculty.engineering.ucdavis.edu/hubbard/>), Nathan (<http://baseball.physics.illinois.edu/>), Russell (<http://www.acs.psu.edu/drussell/>), Sherwood (<https://www.uml.edu/Engineering/Mechanical/faculty/sherwood-james.aspx>), and Smith (<http://www.mme.wsu.edu/people/faculty/faculty.html?smith>). This current book is at a higher level of abstraction (Bahill et al. 2008) than those physics of baseball papers, because it ignores details during the collision, such as (1) the ball can slip, slide, roll or grip the bat and the ball switches between these modes, (2) the coefficient of friction can change from dynamic to static, (3) the bat and ball deform (Mustone and Sherwood 2003) (4) the collision has normal and tangential components and (5) the bat has a twist or a rotation about its long axis. This book ignores the difference between a half-dozen parameters that have commonly been used for collision analysis such as the *kinetic* coefficient of restitution, the *energetic* coefficient of restitution, μ or e_T that models the energy loss due to tangential forces, and e_m that models the losses in angular momentum. This book grouped all of the energy losses into one parameter, the *kinematic* Coefficient of Restitution (*CoR*). This book models the variables of the bat and ball at a time just before the collision and at a time just after the collision, not during the collision.

The authors mentioned in the previous paragraph are, for the most part, members of the bat Effective Mass modeling community. The bat Effective Mass model for bat-ball collisions was developed by Al Nathan. The people in this community think that it is an intuitive model. It was presented at the beginning of Chap. 5. The bat Effective Mass model usually produces only the batted-ball speed, whereas the BaConLaw model also gives equations for the bat linear and angular velocities after the collision. However, I come from a different background. I am an engineer and a modeler. Back in the 1970s, we engineers would not design with integrated circuits that did not have a second source. Therefore, integrated circuit manufacturers *gave* their masks to their competitors! That way there would be a second source for the integrated circuits in case the first manufacturer's process went sour. From that experience, I learned to cherish alternative models. Chapters 1 to 5 of this book provide alternative models for bat-ball collisions. The BaConLaw model of Chap. 4 was based on the conservation laws. Its derivations are completely different, yet it yields similar results to the bat Effective Mass model. This should allow people to put more faith in both models. They are not competing models: they are synergistic.

A model is a simplified representation of a particular view of a real system. No model matches all views of its real system perfectly. If it did, then there would be no advantage to using the model. In modeling theory, there is *never one* correct model. Good modelers always embrace alternative models. This enhances the probability of the models being useful.

The terms in Table 1.1 should be understandable by high-school students, undergraduates and all other students of the science of baseball. These terms are all you need to know to understand this book. This book does not obfuscate with jargon, rules of thumb or esoteric terms such as *swing weight* (moment of inertia about a pivot point 6 in. from the knob), *swing speed* (the angular velocity of the bat), the *trampoline effect* (hollow aluminum and composite bats are more elastic

than wooden bats), *hoop frequency* (vibration of the barrel), the ball-bat coefficient of restitution (BBCOR), collision efficiency, rebound power, intrinsic power, bounce factor and recoil factor. By using only fundamental principles and no jargon, it is hoped that the reader will gain intuition about the behavior of the bat and ball before and after collisions.

6.2.1 Seeing the Collision

P. O. BOX 481
ISLAMORADA, FLORIDA KEYS
33036

January 23, 1984

Prof. A. Terry Bahill
Electrical & Computer Engineering
Carnegie-Mellon University
Schenley Park
Pittsburgh, PA 15213

Dear Mr. Bahill:

Received your letter and have also had a chance to read your research, and I fully agree with your findings.

I always said I couldn't see a ball hit the bat except on very, very rare occasions and that was a slow pitch that I swung on at shoulder height. I am very close to seeing the ball hit the bat on those occasions.

As to participating in your other experiments; at this time, I can't tell you that I can comply with your request.

Regarding the current theories of some of the present batting coaches (with which I absolutely disagree) to watch the ball go into the catcher's mitt - by doing that, you don't give yourself a chance to swing and open up properly. Try it yourself - look down at the plate and try to make a full swing. I hope you don't throw your back out of joint!

In any event, good luck with your projects.

Sincerely,



Ted Williams

TW/shg

When a baseball bat moving at 62 mph (28 m/s) hits a baseball traveling in the opposite direction at 83 mph (37 m/s) there is a violent collision, which was shown in Fig. 5.7. Table 4.3 showed that during the collision the kinetic energy in the motion of the bat changed by 218 Joules (J): a loss of 193 J in linear translational kinetic energy and a loss of 25 J in angular kinetic energy. Notably, 218 J is

equivalent to dropping a 50-pound (24 kg) weight from your waist onto your toe or having a one-pound rock (0.5 kg) hit your windshield while you are driving down a highway at 70 mph (113 km/h).

Frame-by-frame analysis of a high-speed video of a major-league batter showed that at the beginning of the collision there was (1) a big abrupt change in the ball velocity as it swung from negative to positive, (2) a sudden drop in the linear velocity of the sweet spot of the bat and (3) a sharp change in the angle of the bat, β . These last two changes can be measured in a frame-by-frame analysis, but they cannot be visualized well, even in slow motion, because of the limited frame rate and resolution of the cameras.

Now, imagine a film of Ted Williams hitting a baseball. His swing is smooth and graceful although the kinetic energy of his bat changes by 218 Joules during a collision. The reason his swing seems so smooth is that we mainly visualize the movement of his body, arms, hands and the bat. We model this movement with the bat's angular rotation about the knob, β . The change in this angular motion is not visually obvious because it is just a short small jerk in the middle of a big swinging motion. Hence, what we see does not change much. On the other hand, the bat's linear translational motion, $v_{\text{bat-cm}}$, decreases from 52 to 24 mph (23 to 11 m/s). However, we do not visualize this translational motion well, because his swing looks like a big rotation: it does not look like a translation. As a result, the movement that we visualize well, β , does not change much. Whereas, the movement that changes a lot, $v_{\text{bat-cm}}$, is not visualized well. This explains why people do not perceive an abrupt jerk when the bat and ball collide.

What about the batter? Would he be able to see the effects of this violent collision? Probably not. Bahill and LaRitz (1984) showed that no batter could keep his eye on the ball from the pitcher's release point to the bat-ball collision. Their graduate students fell behind when the ball was 9 feet (2.7 m) in front of the plate. Comparatively, their major-league baseball player was able to keep his position error below two degrees until the ball was 5.5 feet (1.7 m) from the plate. Then he fell behind. This finding runs contrary to baseball's hoary urban legend that Ted Williams could see the ball hit his bat. However, in reality, Ted Williams could not see the ball hit his bat. In a letter that he sent to Bahill dated January 23, 1984 he wrote,

"Received your letter and have also had a chance to read your research, and I fully agree with your findings.

I always said I couldn't see a ball hit the bat except on very, very rare occasions and that was a slow pitch that I swung on at shoulder height. I cam[e] very close to seeing the ball hit the bat on those occasions."

In summary, the bat-ball collision is violent. Everyone can see its effect on the ball: the ball is squashed (Fig. 5.3) and changes its velocity by 175 mph. However, no one sees the collision's effect on the bat. Because, first of all, the bat-ball collision only lasts one millisecond, which is much too fast for visual pattern recognition. Second, even in slow motion, the spectator only sees the smooth movement of the batter's body, arms, hands and bat, which glide continuously.

The spectator cannot see movements that change abruptly, such as the bat's linear translational velocity, $v_{\text{bat} - \text{cm}}$. Finally, the batter is not able to see the bat-ball collision at all. This explains why nobody sees an abrupt jerk of the bat when the bat hits the ball, not even Ted Williams.

6.3 Summary

One purpose of this book was to show how complicated bat-ball collisions could be while still being modeled using only Newton's principles and the conservation laws. We were successful. The BaConLaw model was the pinnacle of our models. Whereas, the Collision with Friction model involved actions *during* the collision. Because our technique is only valid for points before and after the collision, we concluded that the Collision with Friction model is inappropriate for our simple Newtonian technique. Therefore, the BaConLaw model is the most complex configuration for which our technique, based only on Newton's principles and the conservation laws, are valid. Our configurations were explained in Chap. 2. The five equations that we used were listed in Table 4.1. These equations were used for configurations 2a, 2b, 2c and 2d. Additionally, all of these results can be simplified to be appropriate for previous configurations. We derived these equations for the BaConLaw model of configuration 2b. But most importantly, if we set the initial ball spin equal to zero, then they satisfy configuration 2a. If we let $d_{\text{cm} - \text{ip}} = 0$ the resultant equations are the same as those we derived for configurations 1a and 1b.

A second purpose of this book was to show how the individual batter could select or create the optimal baseball or softball bat for him or herself. The sensitivity analysis and optimization study of this book showed that the most important variable, in terms of increasing batted-ball speed, is bat speed before the collision. However, in today's world, the coefficient of restitution and the bat mass are experiencing the most experimentation in trying to improve bat performance. Although, the bat moment of inertia provides more room for future improvement. Above all, future studies must include physics in conjunction with physiology in order to improve bat performance.

Finally, we noted that, "You can't keep your eye on the bat."

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Chapter 7

The Ball in Flight Model

7.1 Introduction

Purpose: One purpose of this chapter was to derive equations and develop the Ball in Flight model. This model was then used to show how altitude, temperature, barometric pressure and relative humidity affect air density and consequently how air density affects the flight of the ball.

7.2 Movement of the Ball in Flight

Baseball batters say that the pitch hops, drops, curves, breaks, rises, sails or tails away. Baseball pitchers say that they throw fastballs, screwballs, curveballs, drop curves, flat curves, knuckle curveballs, sliders, change ups, palm balls, split fingered fastballs, splitters, forkballs, sinkers, cutters, two-seam fastballs and four-seam fastballs. This sounds like a lot of variation. However, no matter how the pitcher grips or throws the ball, once it is in the air its motion depends only on gravity, its velocity and its spin. In engineering notation, these pitch characteristics are described respectively by a gravity vector, a linear velocity vector and an angular velocity vector, each with magnitude and direction. The magnitude of the linear velocity vector is called pitch speed and the magnitude of the angular velocity vector is called the spin rate. These vectors produce a force acting on the ball that causes a deflection of the ball's trajectory. This chapter is based on Bahill, Baldwin and Ramberg, (2009).

Figures 7.1 and 7.2 show the effects of spin on the pitch. During the pitch of a major-league baseball, the ball falls about 3 feet due to gravity ($d = \frac{1}{2} at^2$). However, the fastball has backspin that opposes gravity and the curveball has top spin that aids the fall due to gravity. The simulations for these figures were run at standard temperature and pressure (STP).

Fastball

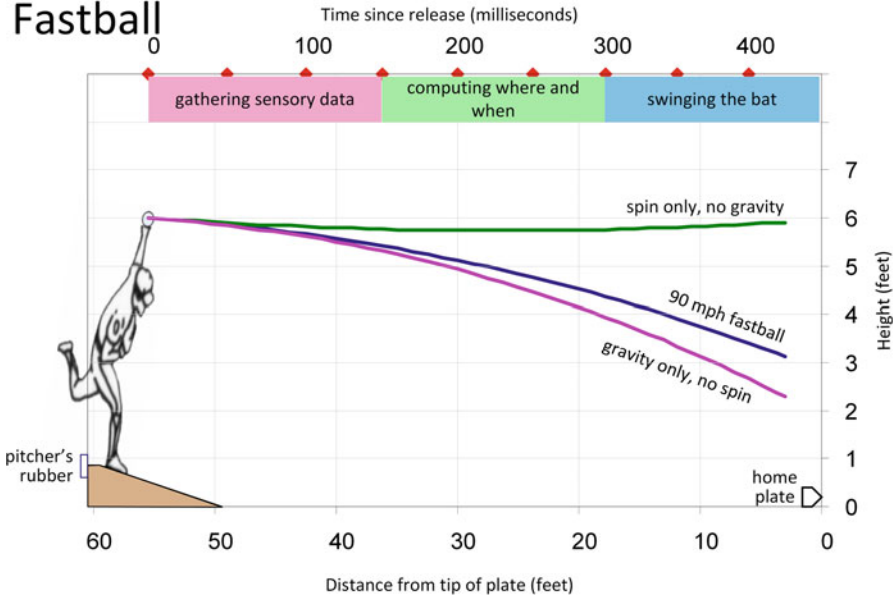


Fig. 7.1 A 90 mph (40 m/s) overhand fastball launched 1° downward with 1200 rpm of backspin

Curveball

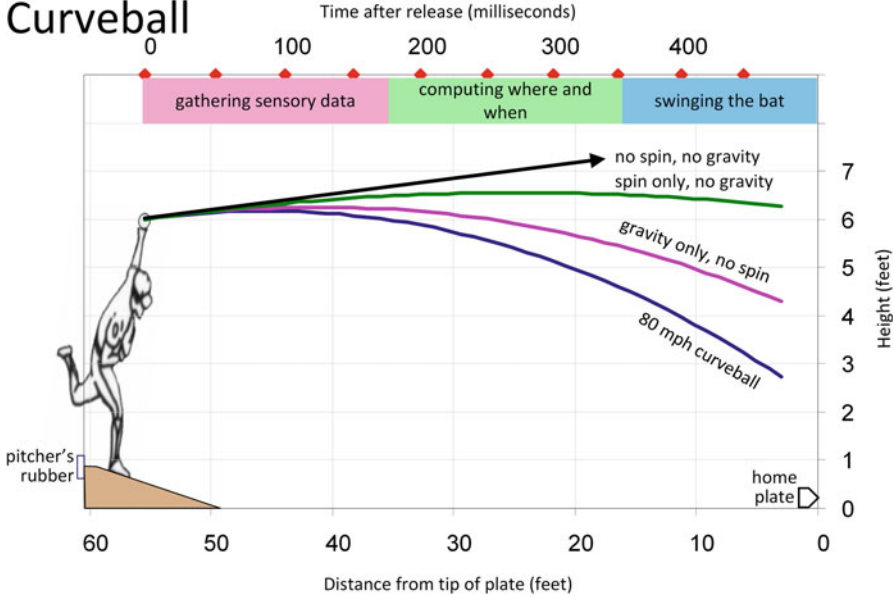


Fig. 7.2 An 80 mph (36 m/s) overhand curveball launched 2° upward with 2000 rpm of topspin

Table 7.1 Values for representative major-league pitches from Willman (2017)

Type of pitch	Speed of the pitch at the pitcher’s release point			Pitch spin rate, absolute values		Number of pitches
	Average, mph	Standard deviation	Average, m/s	Average, rpm	Standard deviation	
4-seam fastball	93.6	2.3	41.8	2169	363	10,215
2-seam fastball	92.7	2.4	41.4	2148	321	2959
Slider	85	3.1	38	745	346	4072
Changeup	85	3.5	38	1714	419	2370
Curveball	79	3.8	35	1286	461	1865

In the simulations of Figs. 7.1 and 7.2, the pitcher releases the ball 5 feet (1.5 m) in front of the pitcher’s rubber at a height of 6 feet (1.8 m). The batter hits the ball 1.5 feet (0.5 m) in front of home plate. These figures also show what the batter’s brain is doing during the pitch. During the first third of the pitch, he is gathering sensory information (mostly with his eyes) about the velocity and spin of the pitch. During the middle third of the pitch, he is computing where and when the ball will cross the plate. During the last third, he is swinging the bat and can do little to alter its trajectory.

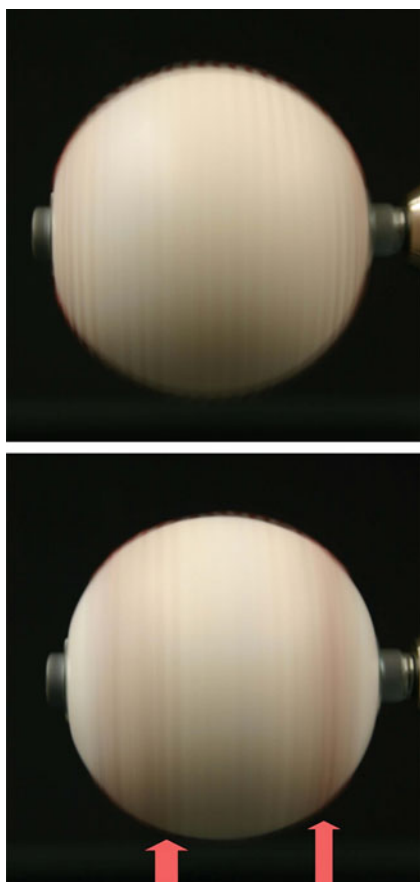
For a half-century, our models were hampered by limited data for the spin of the ball. The best, published experimental data for the spin rate of different pitched baseballs came from Selin’s cinematic measurements of baseball pitches (Selin 1959). But now there is a plethora of data. The two biggest surprises from these new data were that the average fastball has a bigger spin rate than the average curveball and that the change up is not really slow.

Table 7.1 presents data for pitches thrown in 2016 in the Arizona Diamond-back’s stadium. These numbers came from Willman (2017) BaseballSavant. The numbers for the changeup were surprising because their glossary states, “A changeup is one of the slowest pitches thrown in baseball...” Therefore, I computed several datasets and consulted several sources. The numbers were similar. However, please note that the standard deviations for both the velocity and the spin rate are large.

For readers who are familiar with statistics, please allow me this aside for those that are not. By using the term standard deviation in Table 7.1, I assumed that, for example, the velocity data for curve balls were normally distributed. This means that 68% of the data points were within plus or minus one standard deviation of the average, 79 mph: meaning that 68% of the curve ball velocities were between 83 mph and 75 mph. However, this also means that 16% of major league curveballs had velocities below 75 mph.

The number of pitches column shows the relative popularity of each type of pitch. Of the dozen types of pitches listed in Willman (2017), Table 7.1 only gives data for five. The two-seam and the four-seam fastballs are both listed, just to show

Fig. 7.3 Photographs of spinning balls simulating a fastball thrown with (*top*) a four-seam grip and (*bottom*) a two-seam grip. The balls are being rotated at 1200 rpm (20 times per second). The camera exposures are about 0.25 s



that there is little physical difference between the two. The difference must be psychological (meaning visual) (Bahill et al. 2005).

Figure 7.3 shows photographs of spinning baseballs. The simulated four-seam fastball in the top of Fig. 7.3 appears to be a grey blur with thin vertical red lines about 1/7 of an inch apart. These are the individual stitches of the baseball. In contrast, the two-seam fastball (bottom) seems to exhibit two big red vertical stripes about 3/8 of an inch wide. These stripes are evident because they represent seams rather than individual stitches. They provide easily perceived information to the batter for determining the angle of the spin and the direction of the resultant deflection. In an experiment with 104 laypeople, our subjects could distinguish the pink stripes of the two-seam fastball on average 43 feet from the ball, whereas they could only see the pink lines of the four-seam fastball on average 17 feet away. In the bottom of Fig. 7.3, the red stripes are vertical. Were the stripes at an angle, they would indicate the horizontal direction in which the ball would curve. Therefore, the big difference between four-seam and two-seam fastballs is that (because

Table 7.2 Major league averages from Statcast

Type of pitch	Average, speed at the release point, mph	Average spin rate, rpm
4-seam fastball	92.9	2226
2-seam fastball	91.9	2123
Slider	84.6	2090
Changeup	83.9	1746
Curveball	78.2	2308

of the visibility of vertical red stripes) the batter may be able to perceive the spin on the two-seam fastball. Videos of these simulated fastballs are available at <http://sysengr.engr.arizona.edu/baseball/index.html>.

Table 7.2 presents 2015 data from the Statcast system (Petriello 2016). It has higher spin rates for the slider and the curveball than Table 7.1. This reiterates the fact that the numbers given in Tables 7.1 and 7.2 are still just estimated values subject to theoretical and measurement errors. We presume that these systems measured the air density and wind speed at field level for every pitch.

7.3 Right-Hand Rules for a Spinning Ball in Flight

We will now apply the right-hand rules to the linear velocity vector and the angular velocity vector in order to describe the direction of the spin-induced deflection of the a spinning ball in flight. First, we use the angular right-hand rule to find the direction of the spin axis. As shown in Fig. 7.4, if you curl the fingers of your right hand in the direction of spin, your extended thumb will point in the direction of the spin axis.

Next, we use the coordinate right-hand rule to determine the direction of the spin-induced deflection force. Point the thumb of your right hand in the direction of the spin axis (as determined from the angular right-hand rule), and point your index finger in the direction of forward motion (Fig. 7.4). Bend your middle finger so that it is perpendicular to your index finger. Your middle finger will be pointing in the direction of the spin-induced deflection (of course, the ball also drops due to gravity). The spin-induced deflection force will be in a direction represented by the cross product of the angular velocity vector (the spin axis) and the linear velocity vector of the ball: $\text{Angular velocity} \times \text{Linear velocity} = \text{Spin-induced deflection force}$. Or mnemonically, **Spin axis** $\times$ **Direction** = **Spin-induced deflection** (SaD Sid). This acronym only gives the direction of spin-induced deflection. The equations yielding the magnitude of the spin-induced deflection force are discussed in Sect. 7.6.

The right-hand rules apply to all spinning balls whether thrown by a right-handed pitcher or a left-handed pitcher. They apply to baseballs, softballs, golf balls, soccer balls, tennis balls and even bocce balls.

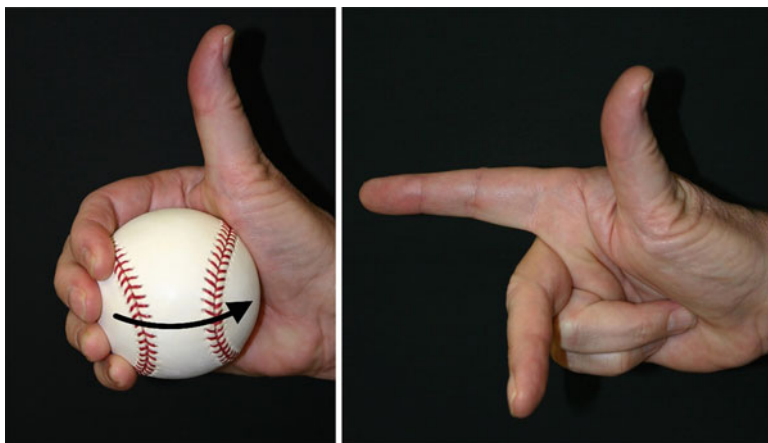


Fig. 7.4 The angular right-hand rule (left). When the fingers are curled in the direction of rotation, the thumb points in the direction of the spin axis. The coordinate right-hand rule (right). If the thumb points in the direction of the spin axis and the index finger points in the direction of forward motion, then the middle finger will point in the direction of the spin-induced deflection (Photographs by Zach Bahill)

7.4 Direction of Forces on Specific Pitches

Figures 7.5 and 7.6 show the directions of spin (circular red arrows) and spin axes (straight black arrows) of some common pitches from the perspective of a camera in center field or the pitcher (Fig. 7.5 represents a right-hander's view and Fig. 7.6 a left-hander's view). We will now consider the direction of the spin-induced deflection of each of these pitches.

The spin on the ball is produced by the grip of the fingers and the motion of the pitcher's arm and wrist. This is the difference between all types of pitches (Kindall 1983). When a layperson throws a ball, the fingers are the last part of the hand to touch the ball. If the ball is thrown with an overhand motion, then the fingertips touching the bottom of the ball will impart backspin to the ball. The overhand fastball shown in Fig. 7.6 has predominantly backspin, which gives it lift, thereby decreasing its fall due to gravity as shown in Fig. 7.1. However, most pitchers throw the fastball with a three-quarter arm delivery, which means the arm does not come straight over-the-top, but rather it is in between over-the-top and sidearm. This delivery rotates the spin axis from the horizontal as shown for the fastball in Fig. 7.5. This rotation of the axis reduces the lift and also introduces lateral deflection, to the right for a right-handed pitcher.

The curveball can also be thrown with an overhand delivery, but this time the pitcher rolls his wrist and causes the fingers to sweep in front of the ball. This produces a spin axis as shown for the overhand curveball of Fig. 7.5. This pitch will curve at an angle from upper right to lower left as seen by a right-handed pitcher or a camera in center field. Thus, the ball curves diagonally. The advantage of the drop

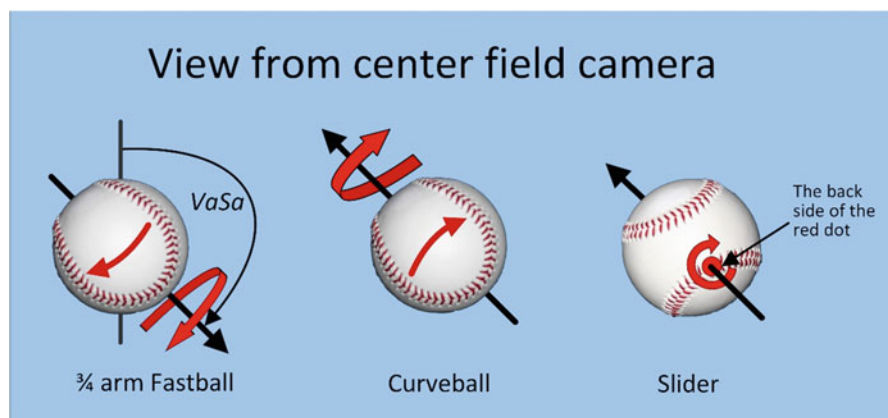


Fig. 7.5 The direction of spin (circular red arrows) and the spin axes (straight black arrows) of a three-quarter arm fastball, an overhand curveball and a slider, all from the perspective of a right-handed pitcher, meaning the ball is moving into the page. $VaSa$ is the angle between the Vertical axis and the Spin axis. The spin axes could be labelled spin vectors, because they suggest both direction and magnitude from Table 7.2

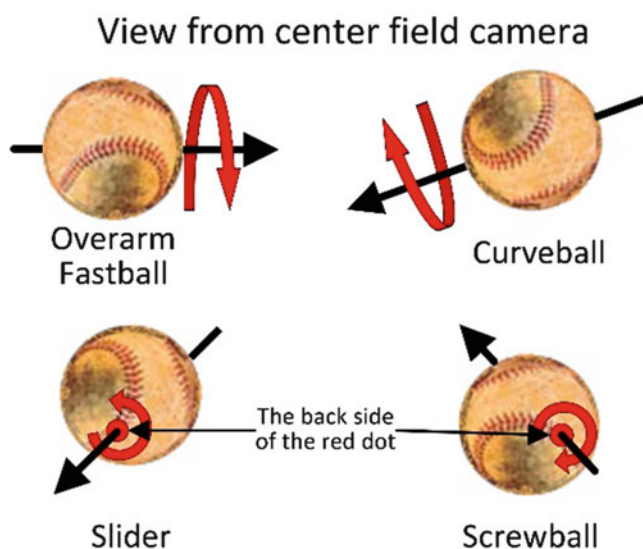


Fig. 7.6 The direction of spin (circular arrows) and the spin axes (straight arrows) of an overhand fastball, an overhand curveball, a slider and a screwball thrown by a left-handed pitcher. The ball would be moving into the page

in a pitch is that the sweet area of the bat is about 2 inches long (5 cm), see Sect. 3.3.1.1 (Bahill 2004) but only one-third of an inch (8 mm) high, see Fig. 4.5 (Bahill and Baldwin 2003; Baldwin and Bahill 2004). Thus, when the bat is swung in a horizontal plane, a vertical drop is more effective than a horizontal curve at taking the ball away from the bat's sweet area.

Fig. 7.7 The batter's view of a slider thrown by a right-handed pitcher: the ball is coming out of the page. The red dot alerts the batter that the pitch is a slider



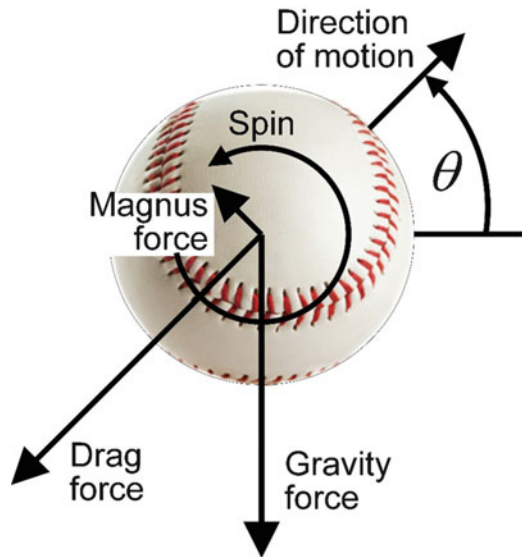
The slider is an enigmatic pitch. It is thrown somewhat like a football. Unlike the fastball and curveball, the spin axis of the slider is not perpendicular to the direction of forward motion. As the angle between the spin axis and the direction of motion decreases, the magnitude of deflection decreases, but the direction of deflection remains the same. If the spin axis is coincident with the direction of motion, as for the backup slider (Bahill and Baldwin 2007, footnote 3), the ball spins like a bullet and experiences no deflection. Therefore, a right-handed pitcher usually throws the slider so that he sees the axis of rotation pointed up and to the left. This causes the ball to drop and curve from the right to the left. Rotation about this axis allows some batters to see a red dot at the spin axis on the upper-right-side of the ball (See Fig. 7.7). Baldwin et, Bahill and Nathan (2007) and Bahill et al. (2005) show pictures of this spinning red dot. Videos of this spinning red dot are on Bahill's web site <http://sysengr.engr.arizona.edu/baseball/index.html>. Seeing this red dot is important — if the batter can see this red dot, then he will know the pitch is a slider and he can better predict its trajectory.

7.5 Magnitude of Forces on a Spinning Ball in Flight

Watts and Baroni (1989) proposed that three forces affect the ball in flight, as shown in Fig. 7.8: gravity pulls the ball downward, air resistance or drag operates in the opposite direction of the ball's motion and, if the ball is spinning, there is a Magnus force perpendicular to the direction of motion. Equations for these forces are often written as (Fig. 7.9)

$$\begin{aligned} F_{\text{gravity}} &= m_{\text{ball}} g \\ F_{\text{drag}} &= 0.5 \pi \rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D \\ F_{\text{Magnus}} &= 0.5 \pi \rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M \end{aligned}$$

Fig. 7.8 The forces acting on a spinning ball flying through the air



7.5.1 The Force of Gravity

The force of gravity is downward, $F_{\text{gravity}} = m_{\text{ball}}g$, where m_{ball} is the mass of the ball and g is the acceleration due to gravity (9.718 m/s^2 at the University of Arizona): the magnitude of F_{gravity} is the ball's weight, as in Table 7.3a.

Our tactics are to use baseball units (e. g. feet, mph and pounds, Table 7.3a) for inputs, SI units (e. g. meters, kilograms and seconds, Table 7.3b) for computations, and baseball units for outputs.

7.5.2 The Magnus Force

In 1671, Sir Isaac Newton (1671) noted that spinning tennis balls experienced a lateral deflection mutually perpendicular to the direction of flight and to the direction of spin. Later, in 1742, Benjamin Robins (1742) bent the barrel of a musket to produce spinning musket balls and also noted that the spinning balls experienced a lateral deflection perpendicular to the direction of flight and to the direction of spin. In 1853, Gustav Magnus studied spinning artillery shells fired from rifled artillery pieces and found that the range depended on crosswinds. A crosswind from the right lifted the shell and gave it a longer range: a crosswind from the left made it drop short. In 1902, the Polish born Martin Kutta and independently in 1906 Nikolai Joukowski studied cylinders spinning in an airflow. They were the first to model this force with an equation. Although these four experiments sound quite different (and they did not know about each other's

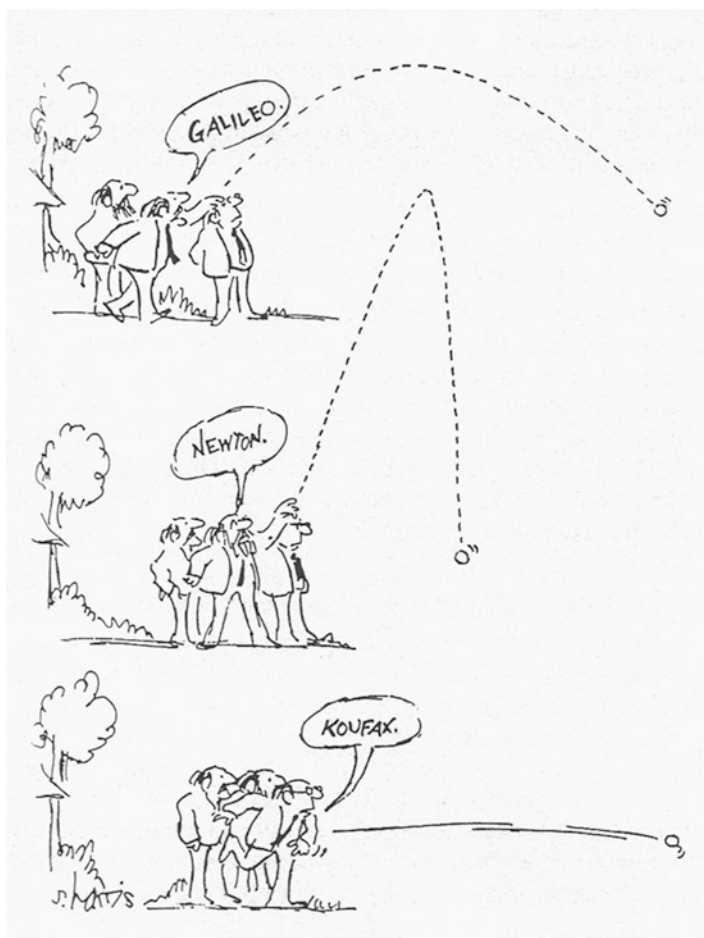


Fig. 7.9 A Sydney Harris (1986) cartoon (© ScienceCartoonsPlus.com, used with permission)

work), they were all investigating the same underlying force. This force, now commonly called the Magnus force, operates when a spinning object (like a baseball) moves through a fluid (like air) which results in it being pushed sideways.

The earliest empirical equation for this transverse force on a spinning object moving in a fluid is the Kutta-Joukowski Lift Theorem.

$$\mathbf{L} = \rho \mathbf{U} \times \mathbf{\Gamma} \quad (7.1)$$

where $\mathbf{L}$ is the lift force per unit length of a cylinder, ρ is the fluid density, $\mathbf{U}$ is the fluid velocity and $\mathbf{\Gamma}$ is the circulation around the cylinder, which is analogous to the angular velocity. The boldface font indicates that $\mathbf{L}$, $\mathbf{U}$ and $\mathbf{\Gamma}$ are vectors. The original Sikorsky and Lightfoot 1949 lift and circulation data were given in

Table 7.3a Typical baseball and softball parameters for line drives using baseball units (Bahill and Baldwin 2007)

	Major-league Baseball	Little League	^a NCAA Softball
Ball type	Baseball	Baseball	Softball
Ball weight (oz)	5.125	5.125	6.75
Ball weight, F_{gravity} , (lb)	0.32	0.32	0.42
Ball radius (in)	1.45	1.45	1.9
Ball radius, r_{ball} (ft)	0.12	0.12	0.16
Pitch speed (mph)	85	50	65
Pitch speed, v_{ball} (ft/s)	125	73	95
Distance from front of rubber to tip of plate (ft)	60.5	46	43
Pitcher's release point: (distance from tip of plate, height), (ft)	(54.5, 6) ^b	(42.5, 5)	(40.5, 2.5)
Bat-ball collision point: (distance from tip of plate, height), (ft)	(3, 3)	(3, 3)	(3, 3)
Bat type	Wooden C243	Aluminum	Aluminum
Typical bat weight (oz)	32	23	25
Maximum bat radius (in)	1.3	1.125	1.125
Speed of sweet spot (mph)	57–69 ^c	45	50
Backspin of batted-ball (rpm)	1800–2500 ^c	1800–2500	1800–2500
Launch angle (degrees)	8–20 ^c	8–20	8–20
Initial batted-ball velocity, v_{ball} (mph)	85–100 ^c	70–80	70–80
Coefficient of Restitution (CoR)	0.55–0.49	0.5	0.44
Desired ground contact point from the plate (ft)	120–240	80–140	80–150
^d Air mass density, ρ (lb-s2/ft4)	0.0023	0.0023	0.0023

^aNCAA stands for the National Collegiate Athletic Association, which is the governing body for university sports in the United States

^b<http://m.mlb.com/statcast/leaderboard#avg-pitch-velo> calls this point the “extension”

^cFrom Willman (2017)

^dAir density depends on altitude, temperature, barometric pressure and humidity

Always (2008). A little bit of mathematics can change this equation for the force on a cylinder to the force on a sphere [NASA <https://www.grc.nasa.gov/WWW/K-12/airplane/beach.html>] $F_{\text{Magnus}} = 0.5\pi\rho r_{\text{ball}}^3(\boldsymbol{\omega}_{\text{ball}} \times \mathbf{v}_{\text{ball}})C_M$, where C_M is a constant. This is an experimental, not a theoretical equation. This is the form that is given in Watts and Ferrer (1987), Watts and Bahill (2000 p. 80) and Sarafian (2015).

A second approach for deriving an equation for the force on a spinning object in a moving fluid stream is to use balls thrown through the air or spun in a wind tunnel. This approach usually starts with an equation of the form $F_{\text{lift}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_{\text{lift}}$. Then the experimenters try to find relationships between the parameters by measuring forces on a ball in a wind tunnel or by measuring the trajectory of a ball in free flight with cameras and then estimating the forces. From these forces, the lift coefficient can be calculated, if you know the air density. The lift coefficient is usually plotted as a function of the spin parameter. The spin parameter is defined as

Table 7.3b Typical baseball and softball parameters for line drives (SI units)

	Major-league baseball	Little League	NCAA Softball
Ball type	Baseball	Baseball	Softball
Ball mass, m_{ball} (kg)	0.145	0.145	0.191
Ball radius, r_{ball} (m)	0.037	0.037	0.048
Pitch speed, v_{ball} (m/s)	38	22	29
Distance from front of rubber to tip of plate (m)	18.4	14.0	13.1
Pitcher's release point: distance from tip of plate and height	17 m out 2 m up	13 m out 1.5 m up	12 m out 0.8 m up
Bat-ball collision point: distance from tip of plate and height	1 m out 1 m up	1 m out 1 m up	1 m out 1 m up
Bat type	Wooden C243	Aluminum	Aluminum
Typical bat mass (kg)	0.9	0.6	0.7
Maximum bat radius (m)	0.033	0.029	0.029
Speed of sweet spot (m/s)	25–31	20	22
Backspin of batted-ball, ω_{ball} (rad/s)	188–262	188–262	188–262
Launch angle (degrees)	8–20	8–20	8–20
Initial batted-ball velocity, v_{ball} (m/s)	38–45	31–36	31–36
CoR	0.55–0.49	0.5	0.44
Desired ground contact point: distance from the plate (m)	37–73	24–43	24–46
Air density, ρ (kg/m ³) This is the average air density for a game played in a major-league stadium on a July afternoon.	1.045	1.045	1.045

Air density depends on altitude, temperature, barometric pressure and humidity

the ratio of the spin velocity to the linear velocity, $SP = \left| \frac{r_{\text{ball}}\omega_{\text{ball}}}{v_{\text{ball}}} \right|$. We use the symbol SP for the spin parameter, whereas some other authors use the symbol S . Using typical values from Tables 7.1 and 7.2 the spin parameters for a major-league fastball and curveball are respectively $SP = 0.2$ and $SP = 0.25$. Table 7.4 shows spin parameters for other flying baseballs.

There is a large literature showing the lift coefficient for a variety of experimental conditions. We are only interested those that used spinning baseballs. Those with cricket balls, golf balls, smooth balls or nonspinning baseballs are of little use to us. Furthermore, we are only interested in data where the spin parameter was between 0.1 and 0.3. Other values are outside our game of baseball. The knuckle ball and the pop up are governed by effects that are not covered in this book. They are covered respectively by Watts and Sawyer (1975) and McBeath et al. (2008). Clanet (2015) analyzed both.

Experimental data for spinning major-league baseballs, with $0.1 < SP < 0.3$, show $C_{\text{lift}} \approx 1.2 \times SP$ (Watts and Ferrer 1987; Sawicki et al. 2003; Nathan 2008; Kensrud 2010). We called the numerical value in this equation C_M . Therefore,

Table 7.4 Spin parameter and Reynolds number for average balls in flight

Type of launch	Initial speed (mph)	Spin rate (rpm), absolute values	Spin parameter, SP	^a Reynolds number, Re, times 10^{-5}
Fastball	93	2200	0.20	1.685
Slider	85	2000	0.20	1.540
Curveball	79	2300	0.25	1.431
Change-up	85	1700	0.17	1.540
Knuckle ball	65	30	0.00	1.178
Batted-ball, home run, initial values	98	2000	0.18	1.776
Home run, ball hitting the ground	55	1760	0.28	0.996
Slow line drive	85	2500	0.25	1.540
Fast line drive	100	1800	0.16	1.812
Extreme pop-up	70	6000	0.74	1.268
NCAA softball pitch	65	1200	0.21	1.538

^athe Reynolds number will be discussed in the next section

$C_{\text{lift}} = C_M SP = \frac{C_M r_{\text{ball}} \omega_{\text{ball}}}{v_{\text{ball}}}$. Remember, we started with $F_{\text{lift}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_{\text{lift}}$. These experiments contained primarily horizontal motions, so $F_{\text{Magnus}} \approx F_{\text{lift}}$. Substituting $C_{\text{lift}} = \frac{C_M r_{\text{ball}} \omega_{\text{ball}}}{v_{\text{ball}}}$ into this lift force equation produces

$$F_{\text{Magnus}} = 0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M \quad (7.2)$$

where C_M is a constant around 1.2. This is the same equation that we derived above from the Kutta-Joukowski Lift Theorem. This is our final equation for the Magnus force.

7.5.3 The Drag Force

Figure 7.8 also shows a force directly opposite to the direction of motion. This force is called the drag force, or air resistance. The magnitude of this drag force is

$$F_{\text{drag}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D \quad (7.3)$$

where ρ is air mass density, v_{ball} is the ball velocity and r_{ball} is the radius of the ball (Watts and Bahill 2000, p. 161). Typical values for these parameters are given in Table 7.3a. For the aerodynamic drag coefficient, C_D , we use a value of 0.4.

Kagan and Nathan (2014) analyzed data from the Pitchf/x system (the camera computer system that overlays pitch trajectories on television replays.). For one particular pitch that was analyzed in detail, they computed $C_D = 0.34$. They stated that Nathan's website had Pitchf/x data for 8000 pitches. The C_D values varied from

0.28 to 0.58. Kensrud (2010), Figs. 4.35 and 4.38, showed spinning baseball C_D s at 0.4 and 0.3 respectively. Kensrud et al. (2015) measured $C_D \approx 0.35$ for a major league baseball at 98 mph. There is a lot of variability in these data because the drag coefficient depends on ball speed, ball spin, roughness of the ball surface, height of the seams (Kensrud et al. 2015), orientation of the seams and for a golf ball the shape and number of dimples.

Thrown and batted-balls can achieve speeds above 100 mph (147 m/s) and at high speeds the drag coefficient gets smaller (Frohlich 1984; Watts and Bahill 2000, p. 157; Adair 2002; Sawicki et al. 2003, 2004). There are no wind-tunnel data showing the drag coefficient of spinning baseballs over the entire range of velocities and spin rates that characterize major-league pitches and hits. Data taken from a half-dozen studies of spinning baseballs, nonspinning baseballs and other balls showed C_D between 0.15 and 0.55 (Sawicki et al. 2003). In the data of Nathan et al., (2006), the drag coefficient can be fit with a straight line of $C_D = 0.45$, although there is considerable scatter in these data. The drag force causes the ball to lose about 10% of its speed en route to the plate. The simulations of Alaways et al. (2001) also studied this loss in speed. Data shown in their figure 9 for the speed lost en route to the plate can be nicely fit with $PercentSpeedLost = 20C_D$, which implies $C_D = 0.5$. Clanet (2015). gives a value of 0.38 for baseballs. In summary, the literature has a lot of variation in the coefficient of drag for a spinning baseball. However, most of the numbers are between 0.3 and 0.5 (Fig. 7.10).

7.5.3.1 The Reynolds Number

The drag on an object in a moving airflow depends on how the air flows around the object. For example, the boundary layer flow around the object could be laminar or turbulent. The drag also depends on the points where the airflow separates from the surface of the object. How the air flows around an object is a function of how fast the air flows. More specifically, the drag is a function of the Reynolds number as shown in Fig. 7.11.

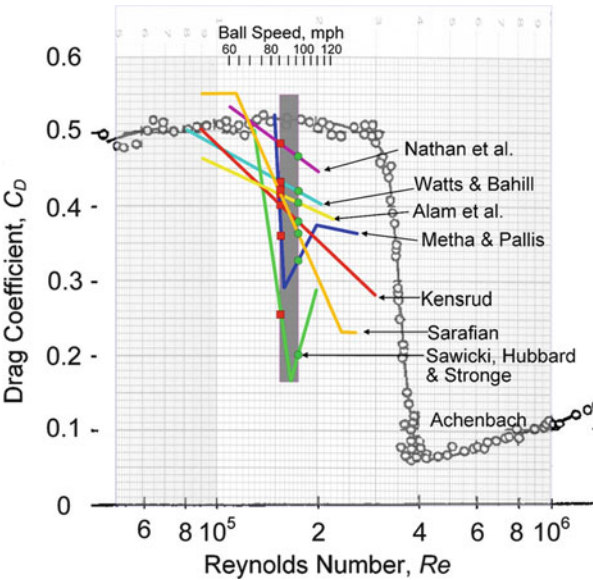
The Reynolds number is defined as $Re = \frac{2r_{ball}v_{ball}}{\nu}$ where ν is the *kinematic* viscosity of air. We use $1.8 \times 10^{-5} \text{ m}^2/\text{s}$ or $2 \times 10^{-4} \text{ ft}^2/\text{s}$ for a baseball at 85 °F. The Reynolds number is also written as $Re = \frac{2\rho_{ball}v_{ball}}{\mu}$ where μ is the *dynamic* viscosity of air in $\text{kg}/\text{m} \cdot \text{s}$. The Reynolds number is used to assert whether a flow is laminar or turbulent. Where the flow is laminar, viscous forces dominate and Re is low. Where the flow is turbulent, inertial forces dominate and Re is high. The Reynolds number is named after the British physicist and engineer Osborne Reynolds who discovered the relationship in 1883. This might all sound complicated, therefore, when I write, Reynolds number, you the reader should think scaled ball velocity.

In Fig. 7.11, for *smooth* balls, the circles of Achenbach (1972), the drop in C_D starts at $Re = 3 \times 10^5$ and ends at 4×10^5 . This sharp change in drag is bound to arouse curiosity. Frohlich (1984) wrote that if the pitch went through

Fig. 7.10 He even dreams about that stupid ball



Fig. 7.11 The drag coefficient varies with the Reynolds number. The circles and the line fit to them are copies of Achenbach's original figure (1972). The green circles represent the initial ball speeds at the pitcher's release point for a 95 mph fastball and the red squares show the final ball speeds when the ball crosses the plate. The gray box is then the region for the flight of the pitch



this ‘drag crisis,’ en route to the plate, then the ball would surely exhibit a strange trajectory. Figure 7.11 also shows linearized drag coefficient data for seven other data analyses. The Watts and Bahill (2000) analysis for spinning baseballs (Fig. 52, data of Gonzalez) did not have a sharp drop in the drag coefficient. Metha and Pallis (2001) showed a critical Reynolds number at $Re = 1.7 \times 10^5$ for nonspinning baseballs in wind tunnels. Sawicki et al. (2004) referenced the flight of baseballs in Olympic baseball games. Their calculated drag coefficient for a spinning baseball decreased precipitously at $Re = 1.6 \times 10^5$. Nathan, Hopkins, Chong and Kaczmariski (2006) show drag coefficients around 0.45 for all Reynolds numbers. Kensrud (2010, Fig. 4.50) shows the minimum drag coefficient for nonspinning MLB baseballs at $Re = 2.3 \times 10^5$. The newest experimental data, by Alam et al. (2012), show the minimum drag coefficient for nonspinning MLB baseballs at $Re = 2 \times 10^5$. Sarafian (2015) has a theoretical curve with a minimum drag coefficient at $Re = 2.6 \times 10^5$. Good linear fits to these sets of data are given in Fig. 7.11. Many studies have shown that roughening the surface of the ball or spinning the ball moves the middle and the right parts of the Achenbach curve up and to the left.

Now comes the most important part of this analysis. How much would the drag coefficient change *during* a variety of pitches? If a major-league fastball started with a speed of 95 mph, then it would cross the plate with a speed of 85.5 mph. (This 10% reduction in ball speed from the pitchers release point until the ball crosses the plate is universal.) Parameters of such a pitch are displayed in Table 7.5 and in the gray box of Fig. 7.11.

The replication crisis The results shown in Fig. 7.11 and Table 7.5 are quite different. Such failures to replicate previous findings are common in science, particularly in the psychological literature, where half of the important findings could not be replicated (https://en.wikipedia.org/wiki/Replication_crisis; also Kahnemen 2014). This is called the *replication crisis*. However, in the physical sciences, we would expect a much greater replication rate. Therefore, in this physics of baseball endeavor, either some fundamental physical parameter is misunderstood or there is no desire to replicate previous experimental results.

To enhance replicability, Stodden et al. (2016) recommended that we “share data, software, workflows, and details of the computational environment that generate published findings in open trusted repositories.” Details include things like the treatment of outliers and missing data values. A good counter example to this is the Major League Baseball database based on Pitchf/x etc. that not only does not share data, software and workflows, but it also hides computations and calls them proprietary. From the viewpoint of the scientific community, this is awful behavior by Major League Baseball.

The papers cited in Fig. 7.11 and Table 7.5 generally reference the previous papers, but they do not explain why their new results are different from the old results. Maybe the physics of baseball is too immature to expect replicability. For the most part, the experimental procedures are different and many fundamental details, such as air density, are not even given.

Table 7.5 Range of drag coefficient values for a fastball

Authors	Characteristics of the ball	C_D for a 95 mph fastball at the pitcher's release point, green circles	C_D when the ball crosses home plate at 85.5 mph, red squares	Percent change in drag coefficient en route to the plate
Achenbach (1972)	Smooth, nonspinning balls			
Watts and Bahill (1990), Fig. 52	Nonspinning baseballs	0.42	0.43	2%
Metha and Pallis (2001)	Nonspinning baseballs	0.33	0.36	9%
Sawicki et al. (2004)	Spinning baseballs	0.20	0.25	25%
Kensrud, Nathan and Smith (2017)	Spinning baseballs	0.47	0.48	2%
Kensrud (2010), Fig. 7.50	Nonspinning baseballs	0.38	0.40	5%
Alam et al. (2012)	Nonspinning baseballs	0.40	0.41	1%
Sarafian (2015)	Theoretical calculations	0.36	0.42	17%

The percent increases in drag the coefficient en route to the plate were small, averaging 9%

So far, this discussion of the drag coefficient has been in terms of the pitch. During the pitch, the drag coefficient changes only by 9%, on average. Now we want to consider the batted-ball. The home run is the batted-ball that will be affected the most by changes in the drag coefficient, because it will be in the air the longest, it will have the biggest changes in velocity and it will therefore have the biggest changes in the drag coefficient.

Major League Baseball (MLB) is releasing many new data that show ball speeds above 100 mph. In 2016, MLB measured over 700,000 pitches Willman (2017). Of these 1400 or 0.2% had initial speeds over 100mph. It also had 140,000 balls hit into play (30% of these were base hits). Of these batted-balls, 3.6% had initial batted-ball speeds (exit velocities) greater than 100 mph. These comprise 0.2% of pitches and 3.6% of batted-balls. To accommodate these high velocities we could consider the following alternative models for the drag coefficient.

$C_D = 0.4$ as a simple model or we could let

$$C_D = \begin{cases} 0.5 & \text{for ball speed} \leq 85 \text{ mph} \\ 0.3 & \text{for ball speed} > 85 \text{ mph} \end{cases}$$

A more complicated model following a gestalt of Fig. 7.11 would be

$$C_D = \begin{cases} 0.5 & \text{for } v_{\text{ball}} \leq 60 \text{ mph} \\ -0.005v_{\text{ball}} + 0.8 & \text{for } 60 < v_{\text{ball}} < 100 \text{ mph} \\ 0.3 & \text{for } v_{\text{ball}} \geq 100 \text{ mph} \end{cases}$$

Another complicated model using the data of Alam et al. (2012) is

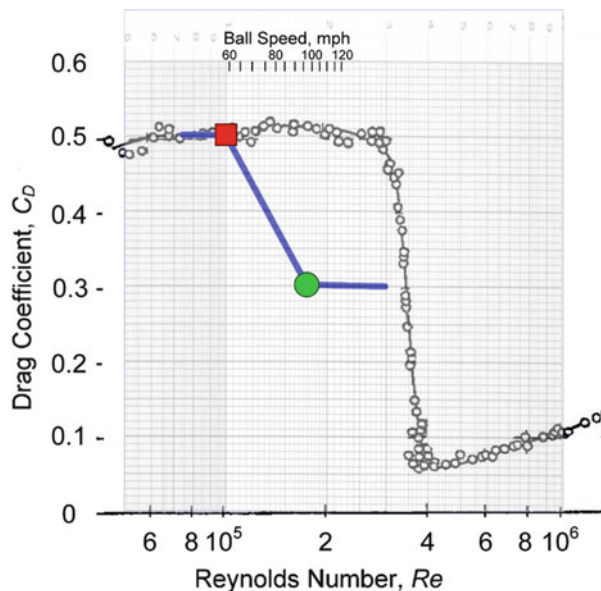
$$C_D = \begin{cases} 0.5 & \text{for } v_{\text{ball}} \leq 30 \text{ mph} \\ -0.004v_{\text{ball}} + 0.6 & \text{for } 30 < v_{\text{ball}} < 90 \text{ mph} \\ 0.25 & \text{for } v_{\text{ball}} \geq 90 \text{ mph} \end{cases}$$

However, by Ockham's razor, this added complexity without added validation is useless (Jefferys and Berger 1992). Therefore, until we get better data, we will continue to use $C_D = 0.4$.

Figure 7.12 shows the drag coefficient as a function of the Reynolds number for a simulated home run. The green circle in Fig. 7.12 represents the initial batted-ball velocity at the launch point where $v_{\text{ball}} = 97$ mph and $C_D = 0.3$. The red square indicates the coordinates when the ball hits the ground with a *Range* = 380 feet, $v_{\text{ball}} = 55$ mph and $C_D = 0.5$. Compared to pitches where the average change of C_D was 9%, for this batted-ball the change of C_D is 40%. Therefore, the change in the drag coefficient during the *pitch* is not likely to be important, however for the *batted-ball*, it might be more significant.

The BaConLaw model of Chap. 4 is linked to the Ball in Flight model of this chapter. Eq. (4.8) from Chap. 4 showed that the *batted-ball velocity* depends on the

Fig. 7.12 Drag coefficient as a function of the Reynolds number for a home run



pitch speed, pitch spin and bat speed. Now this chapter shows that the distance that the batted-ball will travel depends on the *batted-ball velocity*, the batted-ball spin rate, the launch angle, the Magnus coefficient, the drag coefficient and air density.

7.6 Sensitivity Analysis

In Sect. 4.11, we performed both an analytic and an empirical (or numerical) sensitivity analysis for the BaConLaw model. First, we chose our performance criterion, the batted-ball speed. Then we calculated the partial derivatives of that performance criterion with respect to the eight model parameters. Finally, we multiplied the partial derivatives by the nominal values of those parameters and evaluated those semirelative sensitivity functions. In that section, our performance criteria, the batted-ball speed, was the result of one of our equations. Therefore, it was easy to calculate the partial derivatives. However, in this chapter for our Ball in Flight model, our chosen performance criterion, the range, is not a result of any single equation. It would be possible but difficult to create such an equation. Therefore, in this chapter, we run the model by simulation and we do an empirical (or numerical) sensitivity analysis.

To do a sensitivity analysis of a model we first select a performance criterion. For the Ball in Flight model, we chose the range, meaning how far the batted-ball travels before it hits the ground. We used our standard pitch and swing of Chap. 4 that produced a batted-ball speed of 92 mph, a backspin rate of 2000 rpm, a launch angle of 34° and a launch height of 3 feet. We used the midlevel or average air density for major-league stadiums, $\rho = 0.00205 \text{ lb-s}^2/\text{ft}^4$ (or slugs/ft³) or $\rho = 1.0582 \text{ kg/m}^3$. We changed each variable by +1% and computed the new range. Our results are shown in Table 7.6. The range numbers are large because our parameter values are for optimal athletes performing optimally. Few major-league batting events would have values as large as these. Laypeople could not come close.

The right column of Table 7.6 shows that the most important variable, in terms of maximizing the batted-ball range, is the batted-ball speed. This is certainly no surprise. The second most important variable is the diameter of the ball. The least important parameters are the launch height and the launch angle. As you can remember, we did not have very good data for the Magnus lift coefficient, so we are happy that its sensitivity is small. The sensitivities to some of the variables and parameters are negative, which merely means that as they increase the range decreases. The results of this sensitivity analysis show that the model is well behaved. The most and least important variables and parameters are as expected. There are no unexpectedly large or small sensitivities. Comparing Tables 4.4 and 7.6, we see that the Ball in Flight model is more sensitive to its parameters than the BaConLaw model is.

Of the 36 possible interaction sensitivities the most important are (1) the batted-ball speed, v_{1a} , and the ball diameter; (2) the ball weight and the drag coefficient, C_d ;

Table 7.6 Results of a numerical sensitivity analysis of the Ball in Flight model for a +1% increase in the parameter values

Parameters	Nominal values	Nominal values increased by +1%	Altered range, ft	Change in range, ft	Semirelative sensitivity values
Range, ft	384.87				
Batted-ball speed, mph	91.9	92.819	389.56	4.69	469
Ball diameter, inches	2.90	2.9336	382.59	−2.28	−228
Drag coefficient, C_d	0.4	0.404	383.16	−1.71	−171
Ball weight, oz	5.125	5.1763	386.18	1.31	131
Air density, ρ , kg/m^3	1.0582	1.0688	383.65	−1.22	−122
Slope of lift coefficient curve, CM	1.2	1.212	385.37	0.50	50
Ball spin, rpm	−2000	−2020	385.37	0.50	50
Ball spin, rpm	−2000	−1980	384.37	−0.50	−50
Launch angle, $^\circ$	34	34.34	384.39	−0.48	−48
Launch height, feet	3	3.03	384.90	0.03	3

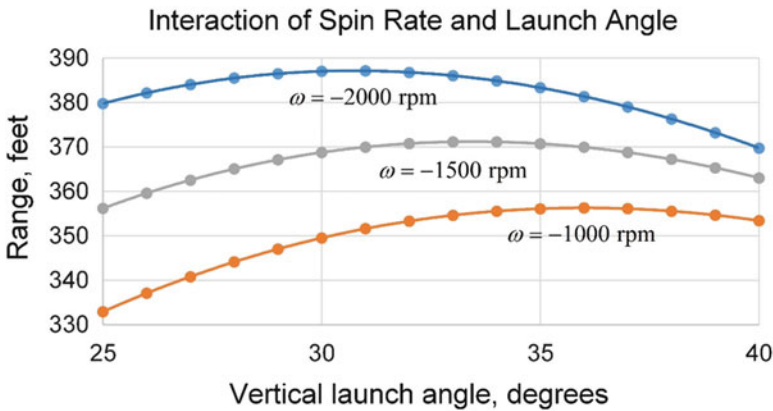


Fig. 7.13 Interaction of spin rate and launch angle

(3) the ball weight and air density, ρ ; (4) and the ball weight and the Magnus coefficient, CM .

The interaction of the ball spin and the launch angle is small. Figure 7.13 shows its effect graphically. Because of the interactions, the three lines are not the same shape and they peak at different values for the launch angle. These curves are very

flat near their peak values, illustrating the small sensitivity to the launch angle at the nominal operating point. For this figure, I choose to use the batted-ball spin rate and the launch angle, because that matches Fig. 55 of Watts and Bahill (2000) and is analogous to Nathan (2016).

Figure 7.13 shows the interaction of the spin rate and the vertical launch angle. On the left side of this figure, when the launch angle increases, the range goes up. However, these three curves do not have the same shape. The curve for the 2000 rpm spin rate has a steeper drop on the right side. This is the effect of the interaction. The difference in spacing of the lines is not the effect of the interaction. That is merely the dependence of the batted-ball speed on spin rate.

7.7 Numerical Values

This section presents numerical values for the three forces that act on the ball in flight. Its purpose is merely to create familiarity with the numbers. If US customary units are to be used in Eqs. (7.1) to (7.7), then ρ should be in $\text{lb}\cdot\text{s}^2/\text{ft}^4$ (or slugs/ft^3), v_{ball} should be in ft/s , r_{ball} should be in ft , and ω_{ball} should be in rad/s , then F_{drag} would be in lb . Let us now present a simple numerical example. Let us use the average fastball from Table 7.4. When the pitcher releases the ball is going 93 mph (136 ft/s) with 2200 rpm (230 rad/s) of backspin.

$$F_{\text{drag}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D$$

$$F_{\text{drag}} = (0.5)(3.14)(0.002)(0.12)^2(136)^2(0.4) = 0.35 \text{ lb}$$

Near the beginning of the pitch, the Magnus force will be straight up in the air, that is, pure lift.

$$F_{\text{Magnus}} = 0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M$$

$$F_{\text{Magnus}} = (0.5)(3.14)(0.002)(0.12)^3(230)(136)(1.2) = 0.07 \text{ lb}$$

The force of gravity is

$$F_{\text{gravity}} = m_{\text{ball}}g = 0.32 \text{ lb}$$

For this fastball, the Magnus force is about one-fifth the force of gravity and one-fifth of the drag force. This is consistent with Table 7.7a where the sixth column shows the drop due to drag and spin. This drop is due to a combination of $F_{\text{drag}} \sin \theta + F_{\text{Magnus}} \cos \theta$.

These simulations were run at standard temperature and pressure (STP). Therefore, the numerical values are different from those in other tables.

Using SI units and Table 7.7b, produces

Table 7.7a Gravity-induced and spin-induced drop for overhand pitches (with United States customary units) (Bahill and Baldwin 2007)

Pitch speed and type	Spin rate (rpm)	Duration of flight (msec)	Drop due to gravity (ft)	Spin-induced vertical drop (ft)	Total drop (ft)
95 mph fastball	−1200	404	2.63	−0.91	1.72
90 mph fastball	−1200	426	2.92	−0.98	1.94
85 mph slider	+1400	452	3.29	+0.74	4.03
80 mph curveball	+2000	480	3.71	+1.40	5.11
75 mph curveball	+2000	513	4.24	+1.46	5.70

Table 7.7b Gravity-induced and spin-induced drop for overhand pitches (with SI units)

Pitch speed and type	Spin rate (rad/s)	Duration of flight (msec)	Drop due to gravity (m)	Spin-induced vertical drop (m)	Total drop (m)
42.5 m/s fastball	−126	404	0.80	−0.28	0.52
40.2 m/s fastball	−126	426	0.89	−0.30	0.59
38.0 m/s slider	+147	452	0.95	+0.23	1.23
35.8 m/s curveball	+209	480	1.13	+0.43	1.56
33.5 m/s curveball	+209	513	1.29	+0.45	1.74

$$F_{\text{drag}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D$$
$$F_{\text{drag}} = (0.5)(3.14)(1.06)(0.037)^2(42)^2(0.4) = 1.56 \text{ N}$$

and

$$F_{\text{Magnus}} = 0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M$$
$$F_{\text{Magnus}} = (0.5)(3.14)(1.06)(0.037)^3(230)(42)(1.2) = 0.32 \text{ N}$$

For this fastball, the Magnus force is about one-fifth the force of gravity, which is

$$F_{\text{gravity}} = m_{\text{ball}}g = 0.145 \times 9.718 = 1.41 \text{ N}$$

As a rule of thumb, we offer the following, over a wide range of conditions, the drag force and the force of gravity have about the same magnitude and the Magnus force is about one-fifth as large.

When the ball's spin axis is not horizontal, the Magnus force should be decomposed into a force lifting the ball up and a lateral force pushing it sideways.

$$F_{\text{upward}} = 0.5\pi\rho r_{\text{ball}}^3 \omega v_{\text{ball}} C_M \sin VaSa \quad (7.4)$$

where $VaSa$ is the angle between the vertical axis and the spin axis (Fig. 7.5). The magnitude of the lateral force is

$$F_{\text{sideways}} = 0.5\pi\rho r_{\text{ball}}^3 \omega v_{\text{ball}} C_M \cos VaSa \quad (7.5)$$

Finally, if the spin axis is not perpendicular to the direction of motion (as in the case of the slider), the magnitude of the cross product of these two vectors will depend on the angle between the Spin axis and Direction of motion, this angle is called SaD (Fig. 7.14). In aeronautics, it is called the angle of attack. Finally, we get

$$F_{\text{lift}} = 0.5\pi\rho r_{\text{ball}}^3 \omega v_{\text{ball}} C_M \sin VaSa \sin SaD \quad (7.6)$$

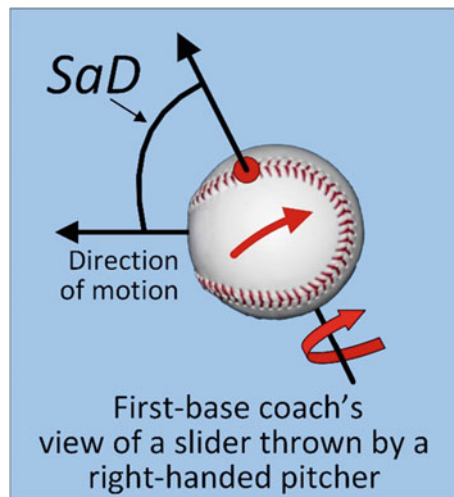
$$F_{\text{lateral}} = 0.5\pi\rho r_{\text{ball}}^3 \omega v_{\text{ball}} C_M \cos VaSa \sin SaD \quad (7.7)$$

These equations comprise our Ball in Flight model.

The spin-induced force on the ball changes during the pitch. Its magnitude decreases, because the drag force slows the ball down by about 10%. Its direction changes, because gravity is continuously pulling the ball downward, which changes the direction of motion of the ball by 5–10°. However, the ball acts like a gyroscope, so the spin axis does not change. This means that, for a slider, the angle SaD increases and partially compensates for the drop in velocity in Eqs. (7.6) and (7.7).

The right-hand rules for the lateral deflection of a spinning ball and Eqs. (7.1) to (7.7) apply to pitched and also batted-balls, except it is harder to make predictions

Fig. 7.14 The first-base coach's view of a slider thrown by a right-handed pitcher. This illustrates the definition of the angle SaD



about the magnitude of deflection of batted-balls, because the data about the spin of batted-balls are poor. The right-hand rules and these equations can also be applied to soccer, tennis and golf, where speeds, spins and deflections are similar to baseball. However, the right-hand rules and these equations would be inappropriate for American football, because the spin axis of a football is almost coincident with the direction of motion. Therefore the angle SaD is near zero and consequently the spin-induced deflections of a football are small (Rae 2004).

7.8 Effects of Air Density on a Spinning Ball in Flight

The distance that a fly ball travels is inversely related the air density. However, the explanation for this is not straightforward. Equations (7.1) and (7.3) show that both the drag and Magnus forces are directly proportional to the air density. Therefore, if air density gets smaller, the drag force gets smaller, this allows the ball to go farther. But at the same time, as air density gets smaller, the Magnus force also gets smaller, which means that the ball will not be held aloft as long and will therefore not go as far. So these two effects are in opposite directions. We have built a computer simulation that implements the above equations. This simulation shows that the change in the drag force has a greater influence on the trajectory of the ball than the change in the Magnus force does; therefore, as air density goes down, the range of a potential home run ball increases. A 10% decrease in air density produces a 4% increase in the distance of a home run ball; however, the increase is less than this for pop-ups and greater than this for line drives.

Air density is inversely related to altitude, temperature and humidity, and is directly related to barometric air pressure. We derived an equation for these relationships. It came from the WeatherLink Software (2017) and the CRC Handbook of Chemistry & Physics (1980–81) with a correction from Al Nathan (personal correspondence, 2016). It agrees with the results from Shelquist (2017). Equation (7.8) shows how air density depends on altitude, temperature, humidity and barometric air pressure.

$$\begin{aligned} \text{Air Density} &= \rho \\ &= 1.2929 \\ &\quad \times \frac{273}{\text{Temp} + 273} \times \frac{\text{Air Pres} - 0.379(\text{SVP} \times \text{RH}/100)}{760} \end{aligned} \quad (7.8)$$

where

Air Density is in kg/m³.

Temp is temperature in degrees Celsius.

Air Pres is the pressure of the air in mm of Hg and is given in Eq. (7.9).

SVP is saturation vapor pressure in mm Hg and is given in Eq. (7.10).

RH is relative humidity as a percentage.

This equation uses the absolute (or actual) atmospheric air pressure, which is also called station pressure because it is the air pressure at a particular weather station. It can be computed from the U. S. Weather Service sea-level corrected barometric pressure (which is given in newspapers, on television and on personal computers) with the following formula.

$$\text{Air Press} = \text{Barometric Pressure} \left[e^{\frac{-gM}{R(Temp+273.15)} \text{Altitude}} \right] \quad (7.9)$$

where

g is the Earth's gravitational acceleration (9.80665 m/s^2 at sea level)

M is the molecular mass of air ($0.0289644 \text{ kg/mole}$)

R is the Universal Gas Constant ($8.31447 \text{ joules/}^\circ\text{K mole}$)

Altitude is the altitude of the ballpark in meters

and Temp is the temperature in $^\circ\text{C}$

However, what is Temp the temperature of? As a simple approximation in the following examples, we have used the temperature of the baseball stadium. But the above equation should be integrated with respect to the time-averaged temperature from the baseball stadium to mean sea level. Because this is impossible, the National Weather Service (2001) uses nine different approximations: about them they write, "There is no single true, correct solution of Sea Level Pressure . . . only estimates." For any given time and place the most accurate measure of air pressure for Eq. (7.8) would be a local barometer that is not corrected to sea level (i. e. with its altitude set to 0), which is what a household barometer usually indicates.

Dozens of equations have been fit to the experimental saturation vapor pressure (SVP) data. Here is one by Buck (1981), that was updated in 1996.

$$\text{SVP} = 4.5841 e^{\frac{(18.687 - \frac{\text{Temp}}{234.5}) * \text{Temp}}{257.14 + \text{Temp}}} \quad (7.10)$$

As before, Temp is in degrees Celsius and SVP is in mmHg.

Air density is inversely related to altitude, temperature and humidity, and is directly related to barometric pressure. For the range of values in major-league ballparks, the altitude is the most important of the four input parameters. Table 7.8 gives values for a typical late-afternoon summer game, assuming that the stadium roofs are open and there are no storms. For these examples, baseball units are used instead of SI units. A more comprehensive table is given in the appendix.

Weather data such as these can be obtained from <http://www.weather.com> and <http://www.wunderground.com/>. The multi-year average July afternoon relative humidity and barometric pressure data came from internet databases that are no longer accessible. Estimates of barometric pressure are also available at <http://www.usairnet.com/weather/maps/current/barometric-pressure/>. The multi-year average July maximum daily temperatures came from <http://hurricane.ncdc.noaa.gov/cgi-bin/climatenormals/climatenormals.pl?>

Table 7.8 Air density in some typical baseball stadiums

	Altitude (feet above sea level)	Average daily high temperature (degrees Fahrenheit) in July	Relative humidity, on an average July afternoon	Average barometric pressure in July (inch of Hg)	Air density (kg/m ³)
Denver	5190	88	34%	29.98	0.96
Houston	45	94	63%	29.97	1.11
Minneapolis	815	83	59%	29.96	1.11
Phoenix	1086	104	20%	29.81	1.07
San Francisco	0	68	65%	29.99	1.19
Seattle	10	75	49%	30.04	1.18

`directive = prod_select2&prodtype = CLIM81&subrnum=`. Programs that calculate air density can be downloaded from Linric Company (<http://www.linric.com/>) or they can be used on-line at <http://www.uigi.com/WebPsych.html> or https://wahiduddin.net/calc/calc_da_rh.htm.

In physics, we typically reference constants at standard temperature and pressure (STP). However, this reference point is not as common a condition as one might think. It is actually unusual. The density of dry air at STP of 0 °C (32 ° F) and sea level is 1.2754 kg/m³. At the International Standard Atmosphere, (dry air at 15 °C, 59 ° F, at sea level) the density of air is 1.225 kg/m³. Both of these are bigger than for any baseball game, as shown in Table 7.9. In our computer programs, the default air density is that at midlevel in Table 7.9, namely 1.0582 kg/m³, or 0.00205 slug/ft³.

For a potential home run ball, both the drag and the lift (Magnus) forces are the greatest in San Francisco, where the park is just at sea level, and smallest in the “mile high” city of Denver. However, as previously stated, the drag force is more important than the Magnus force. Therefore, if all collision parameters (e.g. pitch speed, bat speed, collision point, etc.) are equal, a potential home run will travel the farthest in Denver and the shortest in San Francisco.

These values were chosen to show realistic numbers with natural variation. On any given afternoon in July, it is almost certain that baseball games will be played at the high and low ends of all these ranges.

To understand how the four fundamental variables, altitude, temperature, humidity and barometric pressure, determine the air density, these equations were evaluated at eighty-one experimental points in an Excel spreadsheet. These points were selected at the low, middle and high values of the fundamental variables, or at 3⁴ or 81 points. An edited regression output is given in Table 7.10.

Surprisingly, a simple linear equation explains most of the changes, or variability, in the air density values. The linear algebraic equation for air density obtained by least squares analysis is

Table 7.9 Values used in the simulations

	Altitude (feet above sea level)	Temperature (degrees Fahrenheit)	Relative Humidity (percent)	Barometric pressure (inch Hg)	Air density (kg/m ³)	Air density, percent change from midlevel
Low altitude	0	85	50	29.92	1.16	9.4
Low temperature	2600	70	50	29.92	1.09	2.9
Low humidity	2600	85	10	29.92	1.06	0.7
Low barometric pressure	2600	85	50	29.33	1.04	−2.0
Lowest density	5200	100	90	29.33	0.91	−14.0
Midlevel	2600	85	50	29.92	1.06	0.0
Highest density	0	70	10	30.51	1.22	15.5
High barometric pressure	2600	85	50	30.51	1.08	2.0
High humidity	2600	85	90	29.92	1.05	−0.7
High temperature	2600	100	50	29.92	1.03	−2.9
High altitude	5200	85	50	29.92	0.97	−8.6

$$\begin{aligned}
\Delta \text{Air density}(\text{percent change from mean level}) = & \\
& -0.0035 (\text{Altitude} - 2600) \\
& -0.2422 (\text{Temperature} - 85) \\
& -0.0480 (\text{Relative Humidity} - 50) \\
& + 3.4223 (\text{Barometric Pressure} - 29.92)
\end{aligned}
\tag{7.11}$$

where Δ Air density is stated as a percent change from mean level of 1.045, Altitude is in feet, Temperature is in degrees Fahrenheit, Relative Humidity is in percent and Barometric Pressure is in inches of Hg. The parameter estimates are taken from Table 7.10. This equation can be re-expressed to give the air density in kg/m³

$$\begin{aligned}
\text{Air density} = \rho = 1.045 + 0.01045 \{ & \\
& -0.0035 (\text{Altitude} - 2600) \\
& -0.2422 (\text{Temperature} - 85) \\
& -0.0480 (\text{Relative Humidity} - 50) \\
& + 3.4223 (\text{Barometric Pressure} - 29.92) \}
\end{aligned}
\tag{7.12}$$

This Air density is ρ in Table 7.2b and Eqs. (7.1) to (7.8).

Table 7.10 Edited regression summary for linear approximation (From JMP and Excel)

Summary of fit				
RSquare				0.993
RSquare adjusted				0.993
Root mean square eError				0.71
Observations (or sum weights)				81
Analysis of variance				
Source	DF	Sum of squares	Mean square	F ratio
Model	4	5662	1415	2783
Error	76	39	0.51	
C. Total	80	5701		
Parameter estimates				
Term	Estimate	Standard error	t ratio	
Intercept	0.0			
Altitude (ft) – 2600	–0.0035	0.0000	–94	
Temperature (°F) – 85	–0.2422	0.0065	–37	
Relative humidity (%) – 50	–0.0480	0.0024	–20	
Sea level corrected arometric pressure (inch Hg) – 29.92	3.4223	0.1643	21	

Note that the factors are in different dimensions with different ranges. Hence, the magnitudes of the coefficients should be interpreted in this light. That is, a coefficient with a larger magnitude does not necessarily mean it has a greater impact on the response. Also, keep in mind that the equations that yield the air density values are deterministic. That is, there is no random variation. Hence, the sum of squares residual is the variation remaining after predicting the response from the linear approximation. There is no pure error, but rather simply lack of fit to the true model. The least squares analysis differentiates between the variables for the range of the 81 observations as follows. Altitude explains 80% of the variation between the equation and the 81 data points; temperature explains 13%, barometric pressure accounts for 4% and relative humidity accounts for 3%.

Since Eq. (7.11) is linear, the impact of each factor can be shown graphically. Figure 7.15 shows the changes in air density that should be expected over the range of parameter values that would be typical for a baseball stadium on an afternoon in July in North America. It shows that altitude is the most important factor, followed by temperature, barometric pressure and relative humidity. Since the factor ranges given are indicative of their natural variation, larger absolute slopes means stronger effects. These results are for baseball and should not be used for other purposes, such as calculating safe takeoff parameters for a small airplane.

The linear Eq. (7.11) explains 99.3% of the variation in air density across our 81 setting. However, the unexplained variation, as given by the prediction standard error is 0.71%, suggesting that a further minor improvement is possible. (It is

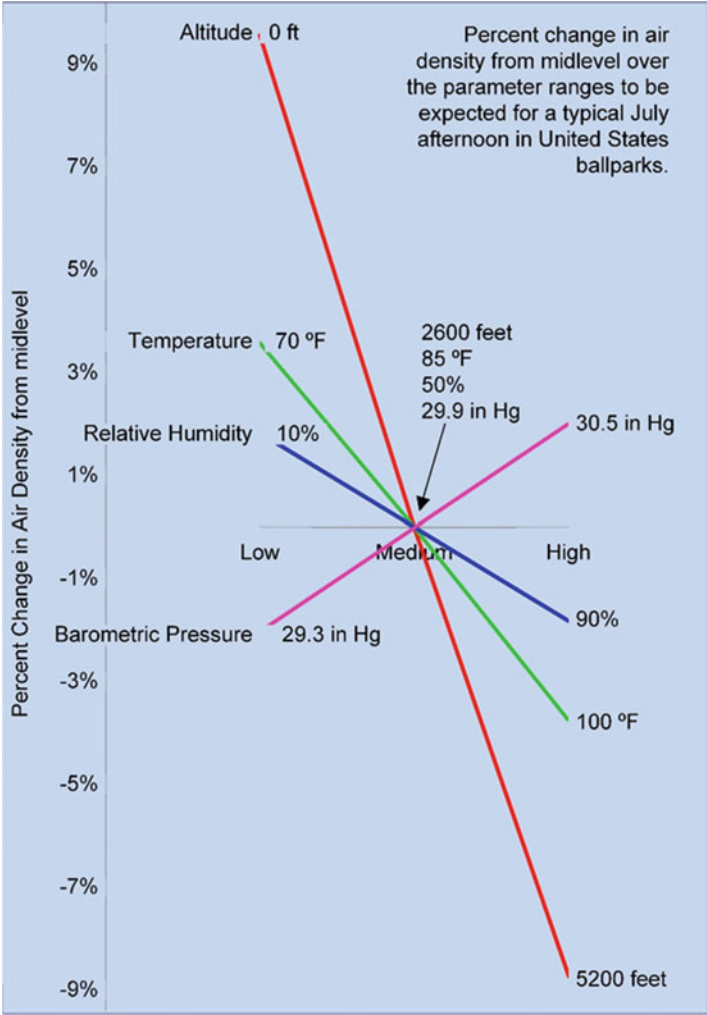
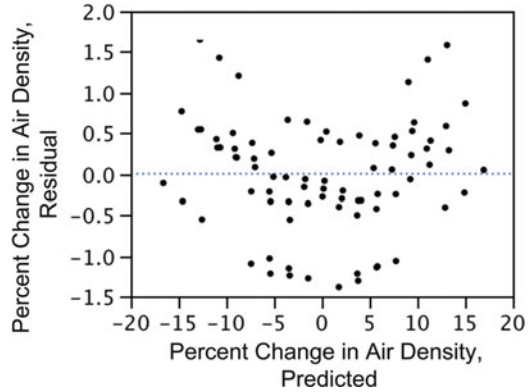


Fig. 7.15 Air density depends on altitude, temperature, barometric pressure and relative humidity

possible to obtain a very high R^2 and still have unexplained variability.) Figure 7.16 shows a quadratic pattern between the residuals and the predicted values of the linear approximation, suggesting that second-order terms might be helpful. Since altitude is the most important factor, the square of its value is a likely candidate. After fitting a regression to the complete quadratic model, that also includes four pure square terms and six cross product terms, the conjecture is confirmed, the square of altitude does play a role. In addition, the cross product term between altitude and temperature, is even more important, although they are a magnitude smaller than the linear altitude and temperature terms in their effect.

Fig. 7.16 Residuals versus predicted air density for the linear approximation



The impact of augmenting the model with these two second-order terms raises the percentage of explained variation only slightly (from 99.3% to 99.5%), but it decreases the unexplained variation, as measured by the prediction standard error from 0.71 to 0.61. The corresponding model is given by Eq. (7.13).

$$\begin{aligned}
 \Delta \text{ Air density}(\text{percent change from mean level}) = & \\
 & -0.0035 (\text{Altitude} - 2600) \\
 & -0.2422 (\text{Temperature} - 85) \\
 & -0.0480 (\text{Relative Humidity} - 50) \\
 & +3.4223 (\text{Barometric Pressure} - 29.92) \\
 & +0.000000061 \left\{ (\text{Altitude} - 2600)^2 - 4506667 \right\} \\
 & +0.000012 (\text{Altitude} - 2600) \cdot (\text{Temperature} - 85)
 \end{aligned} \tag{7.13}$$

This is a confirmation of the correctness of our model. It shows that increasing the complexity of our model, increases the accuracy of the model, slightly. This should be true for all good models.

Please note that this section is not a traditional sensitivity analysis. In a sensitivity analysis, each parameter would be changed by a certain percent and then the resulting changes in the output would be calculated (Smith et al. 2008). For baseball, if we change each parameter by 5% we find that the semirelative sensitivity of air density with respect to barometric pressure, temperature, altitude and relative humidity are respectively 1.07, -0.21 , -0.1 and -0.02 . The reason for the different results is that the high, medium and low barometric pressures that could be expected on a July afternoon in a major-league baseball stadium are 775, 760 and 745 mmHg. These changes are much less than 5%. Whereas, the high, medium and low altitudes that could be expected in a major-league baseball stadium are 5200, 2600 and 0 feet. These changes are much more than 5%. Stated simply, there would be a greater change in air density due to moving from San Francisco to Denver, than there would be due to moving from fair weather to stormy weather.

Table 7.11 Range as a function of air density

Air density (kg/m ³)	Range (ft)			Range (m)		
	Home run	Pop up	Line drive	Home run	Pop up	Line drive
1.3	372	59	266	113	18	81
1.2	382	67	268	117	20	82
1.1	394	75	269	120	23	82
1.0	406	84	271	124	26	83
0.9	418	94	272	128	29	83
0.8	432	104	274	132	32	84

The *range* of a batted-ball is defined as the distance from home plate to the spot where the ball first hits the ground. Table 7.11 shows the range for perfectly hit simulated baseballs. The pitch, from Table 4.2, was a fastball with 1200 rpm backspin that was going 85 mph (38 m/s) when it hit the sweet spot of the bat, which was going 58 mph (26 m/s): the CoR was 0.55. From Table 4.2 we can see that (if all other things were equal) such a collision could produce a home run ball launched optimally at 34° at 97 mph with 2000 rpm of backspin. This is a potential home run ball. Reducing the air density by 10% from 1.0 to 0.9 increased the range of this potential home run ball by 12 feet or 3%.

For Table 7.11, the home run was launched at 97 mph (43 m/s) at an upward angle of 34° with a backspin of 2000 rpm. The pop-up was launched at 70 mph (31 m/s) at an upward angle of 70° with a backspin of 5000 rpm. The line drive was launched at 90 mph (40 m/s) at an upward angle of 15° with a backspin of 2000 rpm.

Replication crisis To make our results more replicable, we should have deposited our 81-point Excel spreadsheet into an on-line repository. The sources of our weather data were given, but our workflows were not. We gave names of the software packages we used for the statistical regression analysis of the 81-point spreadsheet, namely JMP and Excel, but we did not give details. We gave atmospheric conditions for most of our simulations.

In this section, average values were used. Of course, ball games are not played at average values and the actual values are not constant throughout the game. In particular, wind speed and direction could change on a minute-by-minute basis. In this section, the effects of prevailing winds or height and distance of the outfield walls were not modeled. Chambers et al. (2003) have written that for most games played at Colorado Rockies stadium in Denver there was a light breeze (e. g. 5 mph, 2.2 m/s) blowing from center field toward home plate. They further stated that the outfield walls at in Denver were farther back than in most stadiums. They concluded that these two factors together reduced the number of home runs by 3–4%, which nearly compensated for Denver’s high altitude. The greatest wind effects in major-league stadiums are in San Francisco where the average is a gentle breeze blowing from home plate into the right-center field stands at 10 mph (4.5 m/s).

7.9 Vertical Deflections of Specific Pitches

The magnitude of the gravity and spin-induced drops for three kinds of pitches at various speeds are shown in Table 7.3a. Our simulations included air density, the force of gravity, the drag force and the vertical and horizontal spin-induced forces (Bahill and Karnavas 1993; Watts and Bahill 2000; Bahill and Baldwin 2004). Looking at one particular row of Table 7.3a, a 90 mph (40.2 m/s) fastball is in the air for 426 msec, so it drops 2.92 feet (0.89 m) due to gravity ($\frac{1}{2}gt^2$, where the gravitational constant g is 32.2 ft/sec² or 9.8 m/sec² and t is the time from release until the point of bat-ball collision). But the backspin lifts this pitch 0.98 ft (0.3 m), producing a total drop of 1.94 ft (0.59 m) as shown in Table 7.3a. In the spin rate column, negative numbers are backspin and positive numbers are top spin. In the spin-induced vertical drop column, negative numbers mean the ball is being lifted up by the Magnus force. All of the pitches in Table 7.3a were launched horizontally – that is, with a launch angle of zero: that is why they are different from the pitches in Figs. 7.1 and 7.2. The angle $VaSa$ was also set to zero (simulating an overhand delivery): therefore pitches thrown with a three-quarter arm delivery would have smaller spin-induced deflections than given in Table 7.3a.

A batter's failure to hit safely is most likely caused by his fallibility in predicting where and when the ball will reach the bat-ball contact point. Vertical misjudgment of this potential bat-ball contact point is the most common cause of batters' failure (Bahill and Baldwin 2003; Baldwin and Bahill 2004). The vertical differences between the curveballs and fastballs in Table 7.3a are greater than 3 feet (1 m), whereas the difference produced by the two speeds of fastballs is around 3 inches (7 cm) and the difference produced by the two speeds of curveballs is around 7 inches (18 cm). However, the batter is more likely to make a vertical error because speed has been misjudged than because the kind of pitch has been misjudged (Bahill and Baldwin 2003; Baldwin and Bahill 2004). A vertical error of as little as one-third of an inch (8 mm) in the batter's swing will generally result in a failure to hit safely (Bahill and Baldwin 2003; Baldwin and Bahill 2004); see Fig. 4.5.

The spin on the pitch causes both vertical and horizontal deflections of the ball's path. When a batter is deciding whether to swing, the horizontal deflection is more important than the vertical, because the umpire's judgment with respect to the distinct sides of the plate may have more precision than his or her judgment regarding the fuzzy top and bottom of the strike zone. However, after the batter has decided to swing and is trying to track and hit the ball, the vertical deflection becomes more important, because the sweet spot of the bat is wider than it is tall.

7.10 Effects of Air Density on Specific Pitches

A reduction in air density would reduce the drag and the Magnus forces on the pitch. Table 7.12 shows the speed and the height of the ball when it crosses the front edge of the plate for a 93 mph (42 m/s) fastball launched downward at 1.5° from a point 6 feet high with 2200 rpm of backspin using an over arm delivery and for a 79 mph (35 m/s) curveball launched upward at 1° with 2300 rpm of pure top spin. In Table 7.12 the speed is the vector velocity, meaning it is the sum of the horizontal and vertical velocities.

A 10% decrease in air density, for example from 1.0 to 0.9, produces a fastball that is 1% faster when it crosses the plate and 2% lower. Such a change in air density produces a curve ball that is also 1% faster when it crosses the plate with a drop that is 7% smaller. Earlier in this chapter we wrote, if all other things were equal, a 10% decrease in air density would produce a 3% increase in the distance of a home run ball. Now it can be seen that all other things will not be equal: the ball collision speed will be larger (the bat speed will not change). Using the higher ball collision speed increases the range of the home run ball by one foot.

Table 7.11 showed that decreasing the air density by 10%, for example from 1.0 to 0.9, could increase the distance of a home run ball by, for example, 12 feet. Now Table 7.12 shows that decreasing the air density from 1.0 to 0.9, could allow the fastball to retain more of its speed when it crosses the plate. This higher speed (86.5 compared to 85.7) allows the home run ball to travel one foot farther. Considering both of these effects, reducing the air density from 1.0 to 0.9, would allow the home run ball to travel 13 feet or 3% farther.

I hate to use extreme examples because people tend to latch onto them and consider them typical. However, our readers might not relate to a 10%change in air density. So regrettably, I will now present in Tables 7.13 and 7.14 the most extreme example for major league stadiums.

For Table 7.14, we used an average major league home run as described by Willman (2017): it was launched at 97 mph (43 m/s) at an upward angle of 28° with a backspin of 2000 rpm. In 2016, the computed range of typical home runs (meaning if the stands and the fans were not there) was between 340 and 430 feet, so the ranges in Table 7.14 are realistic.

Table 7.12 Pitch variations with air density

Air density (kg/m ³)	Fastball released at 93 mph		Curveball released at 79 mph	
	Speed at the plate (mph)	Height above the plate (ft)	Speed at the plate (mph)	Height above the plate (ft)
1.3	83.5	3.18	71.3	1.84
1.2	84.2	3.08	71.9	1.97
1.1	84.9	2.98	72.5	2.1
1.0	85.7	2.93	73.1	2.24
0.9	86.5	2.86	73.7	2.39
0.8	87.3	2.81	74.6	2.52

Table 7.13 Two rows from Table 7.7a for an average July afternoon in two major league baseball stadiums

City	Altitude (feet above sea level)	Average daily high temperature (°F)	Average relative humidity	Average barometric pressure (inch of Hg)	Average air density (kg/m ³)
Denver	5190	88	34%	29.98	0.96
San Francisco	0	68	65%	29.99	1.19

Table 7.14 A tale of two cities

City	Air density (kg/m ³)	Computed range in feet for a home run ball	Computed range in meters for a home run ball
Denver	0.96	423	129
San Francisco	1.19	399	122

Because of the difference air densities, if all other things were equal, the optimally launched home run ball would travel about 24 feet farther in Denver than in San Francisco. However, in Denver the pitch would not slow down as much. The difference in pitch speeds would add another 2 feet to the range in Denver. We hope there is enough detail in this section to make our result replicable.

7.11 Modeling Philosophy

A model is a simplified representation of a particular view of a real system. No model perfectly matches all views of its real system. If it did, then there would be no advantage to using the model. Although the equations and numerical values in this chapter might imply great confidence and precision in our numbers, it is important to note that our equations are only models. The Kutta-Joukowski lift equation and subsequent derivations are not theoretical equations, they are only approximations fit to experimental data.

There are many models for the flight of the baseball. The models of Frohlich (1984), Watts and Bahill (1990, 2000), Adair (2002, 2004), Sawicki et al. (2003, 2004), Nathan (2006), Bahill and Baldwin (2007), and McBeath et al. (2008) give different numerical results. However, we believe, they all give the same comparative results. Meaning they all should show that a 10% decrease in air density produces about a 3% increase in the distance of a home run ball with the increase being less for pop-ups and greater for line drives.

Our models only considered certain aspects of the baseball in flight. We ignored the possibility that air flowing around certain areas of the ball (due to perhaps a scuffmark) might change from laminar to turbulent flow *en route* to the plate. Our

equations did not include effects of *shifting* the wake of turbulent air behind the ball *during* the flight. En route to the plate, the ball loses 10% of its linear velocity (Watts and Bahill 2000) and 2% of its angular velocity (McBeath et al. 2008): we did not include this reduction in angular velocity in our simulation. We ignored the stabilizing gyroscopic effect and the precession of the spin axis. Furthermore, we ignored the difference between the center of mass and the geometrical center of the baseball. We ignored possible differences in the moments of inertia of different balls. In computing velocities due to bat-ball collisions, we ignored deformations of the bat and ball, and energy dissipated when the ball grips the bat. Finally, as we have already stated, we treated the drag coefficient as a constant.

The implied precision suggested by the home run trajectories shown by Willman (2017) would need to answer all of the above issues as well accommodate wind velocity and its changes with height and perhaps even temperature gradients.

Our numerical values were only estimates, because so many factors affect them. For example, the outputs of the BaConLaw and Ball in Flight models vary with the particular bat that was used. In Sect. 4.12.4, we discussed C243 and R161 bats. They were similar in length, weight and moment of inertia, yet with our standard pitch speed of 83 mph and swing speed of 61 mph, the C243 bat produced a batted ball range of 387 feet whereas the R161 bat drove the ball 389 feet.

Table 4.12 was for head-on collisions. However, a launch angle of 34° would require an oblique collision that would produce a lower launch speed (Kensrud et al. 2017). Consequently, producing a launch speed of 97 mph would require a higher swing speed. Obliging, Willman (2017) shows many swing speeds that are higher.

The importance of this present chapter lies in comparisons rather than in absolute numbers. Our model emphasizes that the right-hand rules show the direction of forces acting on a spinning ball in flight. The model provides predictive power and comparative evaluations of the behavior of different types of pitches.

The Order of Determining Values Variables and parameters used in Chaps. 1 to 6, but not used in Chap. 7, include bat mass, bat inertia, ball inertia, CoR and the location of the collision point. Outputs of Chaps. 1 to 6 that are inputs for Chap. 7 include launch velocity, launch angle and launch spin. Now we had to find numerical values for the other Chap. 7 variables and parameters. The order of determining them is important because it is impossible to correctly derive the values in the wrong order. The correct order is shown in Table 7.15.

We first had to choose a default state: we used the midlevel values given in Table 7.9. Of course, in our simulations, particular variables and parameters were changed for particular stadiums or circumstances, but the default values were usually used. The biggest mistake that we made in the last 30 years was using standard temperature and pressure (STP) as the default for air density in the early years. Next, we needed values for altitude, temperature, relative humidity and barometric pressure; they were given in Tables 7.8 and 7.9 and the appendix of this chapter. These values were then used to compute air density, the dynamic viscosity of air (μ) and the kinematic viscosity of air (ν). The dynamic viscosity is also called the absolute viscosity or just the viscosity: it depends on temperature.

Table 7.15 The order of determining numerical values for the variables and parameters

Variable		Value in SI units	Value in US customary units	
Default state		Midlevel	Midlevel	
Altitude		2600 ft	792 m	
Temperature		29.4 °C	85°F	
Relative Humidity		50 %	50 %	
Barometric pressure		760 mm Hg	29.92 inch Hg	
		⇒ divide by 515.4		
Air density		$\rho = 1.0582 \text{ kg/m}^3$	$\rho = 0.00205 \text{ slugs/ft or lb}\cdot\text{s}^2/\text{ft}^4$	
		⇒ multiply by 2.09×10^{-2}		
Dynamic viscosity of air	⇒ divide by $\rho = 1.0582$	$\bar{\mu} = 1.922 \times 10^{-5} \text{ kg/m}\cdot\text{s or N}\cdot\text{s/m}^2$	$\mu = 4.017 \times 10^{-7} \text{ lbf}\cdot\text{s/ft}^2$	⇒ divide by $\rho = 0.00205$
		⇒ multiply by 10.7638		
Kinematic viscosity of air		$\nu = 1.816 \times 10^{-5} \text{ m}^2/\text{s}$	$\nu = 1.955 \times 10^{-4} \text{ ft}^2/\text{s}$ down then right $\nu = 1.959 \times 10^{-4} \text{ ft}^2/\text{s}$ right then down	
Diameter of a baseball		0.07366 m	2.9 in	
Mass of a baseball		0.145 kg	5.125 oz	
Launch speed		43 m/s	97 mph	
Launch angle		34 degrees	34 degrees	
Launch spin		-209 rad/s	-2000 rpm	
Reynolds number		$10^5 < Re < 2 \times 10^5$		
Spin parameter		$0.1 < SP < 0.3$		
C_M		1.2		
C_D		0.4		

The kinematic viscosity of air depends on both temperature and pressure. In the early years, we used the kinematic viscosity of air, but it was difficult to get good values for all stadiums, therefore we switched to the dynamic viscosity. We found internet sites that gave authoritarian values for the dynamic viscosity of air. This is

the statistical summary for the dynamic viscosity of air at 85 °F and 2600 feet altitude.

$$\bar{\mu} = 1.922 \times 10^{-5} \text{ kg/m} \cdot \text{s or N} \cdot \text{s/m}^2$$

$$\sigma = 0.13 \times 10^{-5}$$

$$n = 8$$

We used the official rules of major league baseball for the mass and diameter of the ball, which were given in Table 1.1. Typical ball speeds and spins came from Table 4.2. Now we had enough data to compute the spin parameter and the Reynolds number for particular pitches and hits as given in Table 7.4. We determined that for major league pitches and hits (with the exception of knuckleballs and extreme pop-ups) the

$$0.1 < SP < 0.3$$

and

$$10^5 < Re < 2 \times 10^5$$

We used these numbers to access the literature and find lift, drag and Magnus coefficients, as given in Sect. 7.6. Although it was not important, we tried to get the earth's gravitational constant at each home plate. It would have been nice to also have had the wind speed at home plate for each pitch.

The order in which these values were gathered is important because, for example, the air density cannot be computed until after the altitude, temperature, humidity and barometric pressure are known, furthermore small mistakes in the beginning would propagate throughout the whole process. Once we had values for the variables and parameters, we could start developing and running the model. First, we needed input values for the launch velocity, launch angle and launch spin. For one of the longest possible batted-ball examples, we used the following inputs from Table 4.12. The home run was launched at 97 mph (43 m/s) at an upward angle of 34° with a backspin of 2000 rpm. The results are given in Table 7.11.

Not only are numerical values important, but their variability is also important. For example, the variation in the earth's gravitation constant is small between stadiums, whereas the variation in the diameter of actual balls in play is comparatively large. Furthermore, the time scale of change is important. The wind speed changes from pitch to pitch, the temperature and barometric pressure change from inning to inning and the altitude and the earth's gravitational constant vary on a geological scale.

The numerical values used for the parameters in our equations have uncertainty. However, the predictions of the equations match baseball trajectories quite well. When better experimental data become available for parameters such as the drag coefficient and spin rate, then the equations or the values of *other* parameters will have to be adjusted to maintain the match between the equations and actual baseball

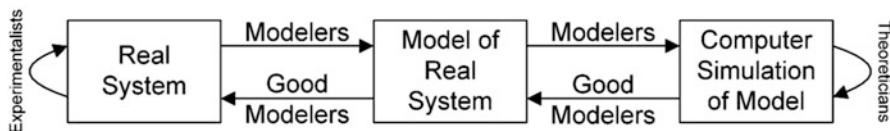


Fig. 7.17 The modeling process

trajectories. A well-developed model is an interconnected system. You should not try to improve one parameter at a time.

So far, experimental data have driven the development of the model and the simulation. However, as shown in Fig. 7.17, modeling is not a one-way street. The theorists should be sending advice to the experimentalists. For example, the worst data used in developing the model is probably that for determining values for the drag coefficient, C_D . The results of the seven studies shown in Fig. 7.11 are different: none replicated earlier results. Therefore, there is a need for someone to show the drag coefficient varying as a function of the Reynolds number for spinning baseballs with the care and precision exhibited by Achenbach (1972). It is important that they explicitly cover the realistic range of spin parameters, $0.1 < SP < 0.3$. As a second example, modelers should point out that the huge Major League Baseball databases contradict each other. In particular, the spin rates for curveballs must be wrong. Table 7.1, using data from Willman (2017), gives an average spin rate for the curveball of 1300 rpm with a standard deviation of 500: this is a huge standard deviation. Whereas, Table 7.2, with data from Statcast (Petriello 2016), gives an average spin rate of 2300 rpm for the curveball. Each of these databases contains an entire year of data. Therefore, the differences are not due to an inadequate sample size. Both of these are for the curveball, which should be easy to identify and hard to confuse with other types of pitches. Therefore, the discrepancy is significant. It suggests a fundamental flaw in the system.

As for replicability, we acknowledge that some areas of science are more difficult to study and are less mature than other areas because of the lack of basic theory to guide us. However, this is definitely not the case for the science of baseball.

Larry Stark (1968) explained that models are ephemeral: they are created, they explain a phenomenon, they stimulate discussion, they foment alternatives and then they are replaced by new models. When there are better wind-tunnel data for the forces on a spinning baseball, then our equations for the lift and drag forces on a baseball might be updated with newer parameters. However, we think our models, based on the right hand rules showing the direction of the spin-induced deflections, will have permanence: they are not likely to be superseded.

Max Planck (1948) wrote, “A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.”

7.12 Which Can Be Thrown Farther a Baseball or a Tennis Ball?

We now have all the tools necessary to analyze the different flight paths of heavy and light balls. If you type, “Which can be thrown farther a heavy ball or a light ball?” into a Google search box, you will get over a million answers: most of them are probably wrong. So let’s try to answer this question now.

The force-velocity relationship of muscle shown in Fig. 4.9 does *not* suggest that a light ball can be thrown farther than a heavy ball. For example, given a tennis ball, a baseball, a softball, a bocce ball and a woman’s shot put, we suspect that the baseball can be thrown the farthest. The tennis ball with a low weight would be at the left side of a force-velocity diagram like Fig. 4.10. It would have a high speed, but the force applied to it by the muscles would be small. Whereas, the shot put would be at the right side of Fig. 4.10. It would have a large force applied to it, but its speed would be small.

Figure 4.9 gives the force-velocity relationship for a single isolated muscle in a laboratory. Fig. 4.10 gives the force-velocity relationship for a whole intact human being swinging a bat. This similarity has been reproduced in many physiological experiments. In Fig. 7.18, we apply it to humans throwing balls.

Physics textbooks state that an ideal projectile-launch on the moon at a 45° angle would yield a maximum range of $R_{\max} = \frac{v_0^2}{g}$, which does not depend on the mass of the projectile. Therefore, let us see if we can be more realistic. The *range* of a batted-ball is defined as the distance from home plate to the spot where the ball first

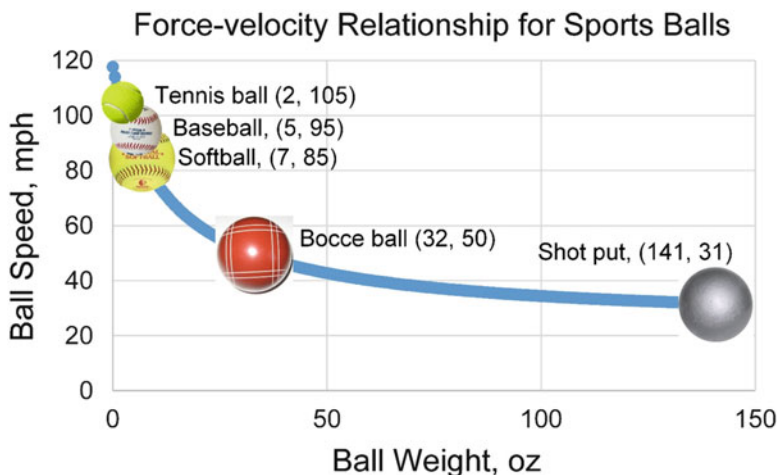


Fig. 7.18 Launch speed versus weight for different sports balls. The equation for the blue line is $(\text{weight} + 12.5) \times (\text{speed} - 24) = 1171$ where *weight* is in ounces and *speed* is in mph. These five balls are about the same size. Therefore, they could all be thrown with an overhand motion producing backspin

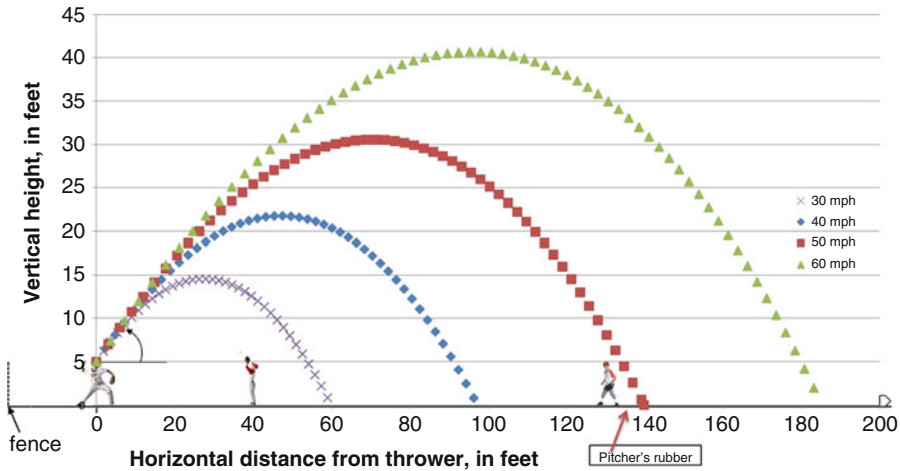


Fig. 7.19 Simulated trajectories for balls thrown from the outfield by a Little Leaguer at various launch velocities. The launch angle is 34° , but it does not look like that on the figure, because the horizontal and vertical scales are not the same

hits the ground. What determines the range of a batted-ball? In a major league baseball stadium, the range depends on the time that the ball is in the air and that depends on the vertical component of the velocity. The height of ball is given by $z = z_0 + \dot{z}t + 0.5\ddot{z}t^2$ where $\dot{z}$ means the derivative of z with respect to time, $\frac{dz}{dt}$, the vertical velocity, and $\ddot{z}$ means the second derivative of z with respect to time, $\frac{d^2z}{dt^2}$, the vertical acceleration. Typical ball trajectories derived from equations like this are shown in Fig. 7.19.

When the ball is going up, from Fig. 7.8, we have

$$F_{\text{down}} = -F_{\text{lift}} + F_{\text{gravity}} + F_{\text{drag}} \sin \theta$$

where θ is the angle between the direction of motion and the horizontal. The lift force is the vertical component of the Magnus force. Therefore,

$$F_{\text{down}} = -F_{\text{Magnus}} \cos \theta + F_{\text{gravity}} + F_{\text{drag}} \sin \theta$$

From Sect. 7.6.2 we have

$$F_{\text{Magnus}} = 0.5\pi\rho r_{\text{ball}}^3\omega_{\text{ball}}v_{\text{ball}}C_M$$

assuming that the spin axis is perpendicular to direction of motion, that is pure backspin.

$$F_{\text{gravity}} = m_{\text{ball}}g$$

$$F_{\text{drag}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_{\text{drag}}$$

Therefore,

$$F_{\text{down}} = -0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M \cos \theta + m_{\text{ball}}g + 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D \sin \theta$$

Now the vertical acceleration is related to the downward force by

$$\ddot{z} = \frac{-F_{\text{down}}}{m_{\text{ball}}}$$

Therefore,

$$z = z_0 + \dot{z}t - \frac{t^2}{2m_{\text{ball}}} (-0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M \cos \theta + m_{\text{ball}}g + 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D \sin \theta)$$

In order to simulate this equation we needed values for C_D and C_L . We took these values from Fig. 7.20, which came from Clanet (2015). We also used $C_M = 0.7$.

Once we had values for these three constants, we ran our simulation and produced the numbers in Table 7.16. We ran the simulation in dry air at 85 °F at sea level, yielding $\rho = 1.14 \text{ kg/m}^3$. This detail aids replicability.

The weak link in this section is the launch speeds. The reliability of these data decreases from baseballs to shot puts to softballs to bocce balls to tennis balls. We have the most confidence in the launch speed of the baseball at 95 mph. Many television viewer are familiar with this number. Many professional baseball players can throw a baseball at this speed: few laypeople can. Regardless, we do not want the average speed of a thousand random people, nor do we want outliers like Rocky Colavito who routinely threw the baseball over 400 feet. To state it differently, we are studying optimal athletes doing what they do optimally. This removes a lot of variability. Therefore, we are comfortable with a 95 mph launch speed for a baseball. With this speed, the baseball is in the low drag region of Fig. 7.11, around 0.38.

The data for the shot put are for Michelle Carter who won the gold medal in the 2016 Olympics with a throw of 20.63 m (68 feet). Using our simulation, we found values that would produce the actual output of 20.63 m. These values were a 30.81 mph launch speed at a 43° angle. She could also have produced that throw with a higher speed and a different launch angle. However, if this throw won the gold medal, then it was probably close to optimal. The shot put has little spin and

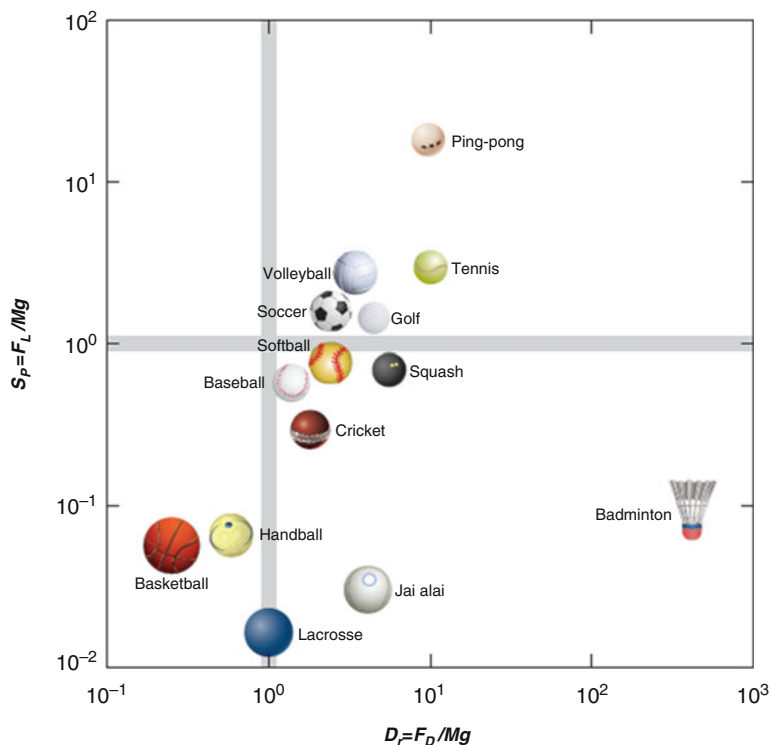


Fig. 7.20 Normalized lift and drag coefficients for various sports balls (From Clanet (2015) ©. *Annual Reviews of Fluid Mechanics*, used with permission)

therefore little lift, but it does have drag (as indicated by the optimal launch angle of 43° instead of 45°). These numbers are not apocryphal or outliers. They represent an optimal athlete performing optimally. Therefore, a launch speed of 31 mph is realistic.

Premier women pitchers throwing the softball *underhand* have maximum speeds that range between 70 and 75 mph. We ignore outliers like Eddie Feigner and Ty Stofflet who supposedly threw the softball over 100 mph. Additionally there are internet sites showing overhand softball throws of over 300 feet. Therefore, we chose a launch speed of 85 mph for men throwing a softball overhand.

The size of the bocce ball is similar to the other balls, so we expect it to be gripped the same. However, estimated speed and spin for the bocce ball are a wag.

The least reliable data are for the launch speed of the tennis ball. The 105 mph value was derived from several internet videos and Clanet (2015). The tennis ball has a fuzzy surface, which produces a high drag coefficient of 0.56 (Clanet 2015).

This difference in drag may be the main reason that the baseball can be thrown farther despite the tennis ball's higher launch speed.

In summary, the baseball can be thrown farther than the tennis ball. This conclusion depends on the force-velocity relationship of muscle and properties of the ball such as mass, drag coefficient and coefficient of lift. However, that really does not answer the original question, "Which can be thrown farther a heavy ball or a light ball?" The sensitivity analysis of Table 7.5 suggested that a heavier ball would go farther. To answer this question thoroughly we ran the simulation with only the mass being different. The results in Table 7.16 show that the heavier ball can go slightly farther.

How is it even possible for a heavy ball to go farther than a light ball? There are two explanations based on physics. First, if the balls were launched with the same velocity, then the heavier ball must have been given more energy. Therefore, it will have more momentum and it will take more force and time to slow it down. Second, the only terms in our equations that depend on mass are the acceleration terms. At the beginning of motion, the ball with the bigger mass has smaller accelerations:

$$\ddot{z} = \frac{F_{\text{down}}}{m_{\text{ball}}} \text{ and } \ddot{x} = \frac{F_{\text{retard}}}{m_{\text{ball}}}$$

Both of these will be smaller for the heavier ball. Which means that the horizontal and vertical velocities will not slow down as fast. Both of these effects will make the heavier ball go farther. However, the system is dynamic. Both the horizontal and vertical velocities decrease with time. And both the Magnus and the drag forces are functions of velocity. Therefore, for the rest of the trajectory we will drop the textural argument and revert to the simulation. We increased the mass of the baseball by 10% as shown in column 3 of Table 7.16. While we kept the launch speed the same. The heavier ball traveled 384 instead of 372 feet.

Now it is time to look at physiology. Recall the force-velocity relationship of muscle. Our muscles will produce a lower velocity for a heavier load than for a lighter load. According to Fig. 7.18, the 10% heavier baseball will be launched at 93 mph instead of 95 mph. As shown in column 4 of Table 7.16, this reduced launch velocity will reduce the range from 384 to 374 feet. In conclusion, increasing the baseball's mass by 10% increased the range by 12 feet. However, the subsequent reduction in launch velocity caused by the force-velocity relationship of muscle decreased the range by 10 feet. Therefore, if a human is throwing balls of about the same mass, then the heavier ball might go *slightly* farther.

At this point in our experiments, someone objected and said, "Yah, but you launched the normal ball and the heavy ball at the same angle. What if you were to launch each at its optimal angle?" Therefore, we reran our simulations and found the optimal angle for the normal ball was 34° producing a range of 372 feet. Whereas the optimal angle for the heavy ball launched at 93 mph was 35°, which

Table 7.16 Simulation parameters for baseballs, tennis balls, softballs, bocce balls and shot puts thrown by elite athletes

Parameter	Baseball, nominal values	Baseball, mass increased by 10%	Baseball, mass +10% and reduced launch velocity	Tennis ball	Softball	Bocce ball	Women's shot put
Ball weight, oz	5.125	5.637	5.637	2.03	6.75	32.45	141
Ball mass, m_{ball} , kg	0.15	0.16	0.16	0.06	0.19	0.92	4.00
Launch speed, mph, from Fig. 7.16	95	95	93	105	85	55	30.81
Launch speed, m/s, from Fig. 7.16	42.5	42.5	41.6	46.9	38.0	24.6	13.8
Ball diameter, in	2.90	2.90	2.90	2.51	3.84	4.21	4.04
Ball diameter, m	0.07	0.07	0.07	0.06	0.10	0.11	0.10
Nominal drag coefficient, C_D , from Clanet (2015) for baseball, tennis and softball	0.38	0.38	0.38	0.56	0.4	0.4	0.4
Air density, ρ , slugs/ft ³	0.002	0.002	0.002	0.002	0.002	0.002	0.002
Air density, ρ , kg/m ³	1.045	1.045	1.045	1.045	1.045	1.045	1.045
Nominal lift coefficient, C_M from Nathan (2008) and C_L from Clanet (2015)	0.7	0.7	0.7	1	0.75	0.8	0.8
Ball spin, ω_{ball} , rpm	-2000	-2000	-2000	-2200	-1800	-1200	-12
Ball spin, ω_{ball} , rad/s	-209	-209	-209	-209	-157	-63	-0.6
Launch angle, degrees	34	34	34	34	34	34	43
Launch height, feet	5	5	5	5	5	5	5
Launch height, m	1.5	1.5	1.5	1.5	1.5	1.5	1.5
Flight duration, seconds	5.26	5.24	5.15	5.49	5.13	3.02	2.06
Range, ft	372	384	374	250	297	186	68
Range, m	113	117	114	76	90	57	20.67

Table 7.17 Summary lines from Table 7.16

Parameter	Baseball	Heavy baseball	Tennis ball	Softball	Bocce ball	Women's shot put
Launch speed, mph	95	93	105	85	55	31
Range, feet	372	374	250	297	186	68

increased the range from 373.9 to 374.0. Our conclusion remained the same: if a human is throwing balls of about the same mass, then they will go about the same distance.

Of course, there are other physiological factors that could affect this conclusion, such as the size of the hands, the size of the ball, the grip, the throwing motion and familiarity. For example, most American males, who grew up playing baseball, thought that they could throw a baseball farther than a tennis ball. Whereas most others thought the opposite.

In conclusion, because of the difference in the drag coefficient, the baseball can definitely be thrown farther than a tennis ball. In addition, if all other parameters are held constant, a lighter ball cannot be thrown farther than a heavier ball.

7.13 Summary

According to our Ball in Flight Model, during its flight, the ball is subjected to the following forces

$$F_{\text{gravity}} = m_{\text{ball}}g$$

$$F_{\text{drag}} = 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D$$

$$F_{\text{Magnus}} = 0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M$$

For major-league baseball stadiums, the air density is inversely related to altitude, temperature and humidity, and is directly related to barometric pressure, according to this equation.

$$\begin{aligned} \text{Air density} = & 1.045 + 0.01045 \{ -0.0035 (\text{Altitude} - 2600) \\ & - 0.2422 (\text{Temperature} - 85) \\ & - 0.0480 (\text{Relative Humidity} - 50) + 3.4223 (\text{Barometric Pressure} - 29.92) \}. \end{aligned}$$

A plea was made for science of baseball experimentalists to try to replicate previous experiments and explain the reasons if they cannot.

Both the drag force Eq. (7.3) and the Magnus force Eq. (7.2) are directly proportional to the air density. Therefore, if air density gets smaller, the drag force gets smaller, this allows the ball to go farther: But at the same time, as air

density gets smaller, the Magnus force also gets smaller, which means that the ball will not be held aloft as long and will therefore not go as far. These two effects are in opposite directions. Simulation shows that the change in the drag force affects the trajectory of the ball more than the change in the Magnus force. Therefore, as air density goes down, the range of a potential home run ball increases. On a typical July afternoon in a major-league baseball stadium, a 10% decrease in air density can produce a 3% increase in the distance of a home run ball. A home run ball might go 26 feet farther in San Francisco than in Denver.

Finally, we note that a baseball can be thrown farther than a tennis ball. Additionally, if a human is throwing balls of about the same mass, then the heavier ball might go *slightly* farther.

Appendix. Weather Data for Major-League Baseball Stadiums

City/State	Team name	Altitude of home plate, feet	Altitude of home plate, m	Average daily high temperature in July °F	Average daily high temperature in July °C	SVP, mm Hg	Relative Humidity on an average July afternoon, percent	Average sea level corrected Barometric Pressure, inch Hg	Average sea level corrected barometric pressure, Pisa	Average sea level corrected barometric pressure, mm Hg	Average absolute pressure, mm Hg. Not sea-level corrected.	Air density, kg/m ³
Arizona	Diamondbacks	1061	323	104	40	55	20	29.81	14.64	757	730	1.077
Atlanta	Braves	942	287	90	32	36	59	29.99	14.73	762	738	1.111
Baltimore	Orioles	36	11	87	31	33	53	29.99	14.73	762	761	1.154
Boston	Red Sox	16	5	82	28	28	57	29.93	14.70	760	760	1.164
Chicago	Cubs	601	183	84	29	30	60	29.93	14.70	760	745	1.135
Chicago	White Sox	595	181	84	29	30	60	29.93	14.70	760	745	1.135
Cincinnati	Reds	490	149	86	30	32	58	30.00	14.74	762	748	1.136
Cleveland	Indians	653	199	81	27	27	57	30.05	14.76	763	746	1.145
Colorado	Rockies	5186	1581	88	31	34	34	29.98	14.72	761	638	0.967
Detroit	Tigers	578	176	84	29	30	54	30.06	14.76	763	748	1.141
Houston	Astros	21	6	94	34	41	63	29.97	14.72	761	760	1.133
Kansas City	Royals	857	261	90	32	36	64	29.97	14.72	761	739	1.111
Los Angeles	Angels	146	45	84	29	30	52	29.94	14.70	760	756	1.155
Los Angeles	Dodgers	501	153	84	29	30	52	29.94	14.70	760	747	1.140
Miami	Marlins	5	2	91	33	37	63	30.00	14.73	762	762	1.143
Milwaukee	Brewers	618	188	81	27	27	64	30.00	14.74	762	746	1.144
Minnesota	Twins	827	252	83	28	29	59	29.96	14.71	761	740	1.130
New York	Mets	12	4	84	29	30	55	30.02	14.74	763	762	1.163
New York	Yankees	33	10	84	29	30	55	30.00	14.73	762	760	1.160
Oakland	Athletics	0	0	73	23	21	55	29.95	14.71	761	761	1.188
Philadelphia	Phillies	0	0	86	30	32	54	30.01	14.74	762	762	1.157
Pittsburgh	Pirates	726	221	83	28	29	54	30.00	14.73	762	742	1.134
Saint Louis	Cardinals	438	134	90	32	36	60	29.98	14.73	762	750	1.128
San Diego	Padres	15	5	76	24	23	67	29.88	14.68	759	758	1.175
San Francisco	Giants	8	2	68	20	18	65	29.99	14.73	762	762	1.201
Seattle	Mariners	17	5	75	24	22	49	30.04	14.75	763	763	1.187
Tampa Bay	Rays	44	13	90	32	36	64	30.08	14.77	764	764	1.149
Texas	Rangers	543	166	96	36	44	53	29.97	14.72	761	747	1.112
Toronto	Blue jays	268	82	80	27	26	55	30.00	14.73	762	755	1.161
Washington	Nationals	6	2	88	31	34	53	30.00	14.73	762	761	1.152

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Chapter 8

Dénouement

8.1 Introduction

Purpose: This chapter consolidates the insights and wisdom in this book.

This book is about engineering the sport of baseball. It may have seemed to be about physics, but we were always on the lookout for instances where physiology or psychology should have come into play. For example, in Chap. 7 it was essential to use physiology's force-velocity relationship of muscle. We also continually looked for fatigue and warm-up effects in data. Some of our studies like that of the rising fastball, which were *not* included in this book, dealt with issues that could only be explained using physiological psychology (Bahill and Karnavas 1993; Bahill and Baldwin 2003 and 2004).

We made sure that we studied papers that used models that were different from ours. We did not want confirmation bias to restrict the papers that we chose to incorporate. Trying to hit a baseball with a bat is a task that is very attention demanding. Therefore, we looked for effects of cognitive overload. We were also on the alert for outliers that might challenge principles of physics that we used. For example, when we stated that energy cannot be created or destroyed, we were aware that in nuclear reactions mass can be converted into energy and vice versa. So we contemplated possible effects in our baseball environment and decided that there were none. The point of this paragraph is that while it might have seemed that we were merely modeling the physics of baseball, we were, and you the reader should have been, continually looking for seemingly extraneous factors that could have affected our conclusions.

What does a person need to be a successful batter? Obviously, he or she must have good coordination, excellent vision, athleticism, desire and a strong work ethic. When concentrating on the pitch, he or she must be able to ignore peripheral vision, the auditory system, the olfactory system and pain sensors. In addition, tracking the ball from the pitcher's release point to where it crosses the plate requires suppression of the vestibulo-ocular reflex (Bahill and LaRitz 1984),

above average smooth pursuit eye tracking capability (Bahill and LaRitz 1984), an exemplary ability to learn to track unique smooth pursuit visual targets (McHugh and Bahill 1985) and a tremendous amount of cognitive effort (Kahneman 2011).

In a classic psychology experiment summarized by (Kahneman 2011), four-year old children were exposed to a cruel experiment. They were given a choice of one Oreo cookie, which they could eat at any time, or two Oreo cookies if they could wait 15 min for the reward. About half the children managed the task of waiting 15 min. A dozen years later, a large gap had opened between those who had resisted temptation and those who had not. The resisters had higher measures of executive functions in cognitive tasks and especially the ability to allocate their attention efficiently (Mischel et al. 1989). My conjecture is that children who can control their impulses and concentrate on the task at hand will have the potential to become more successful baseball and softball players because they have and will develop their executive functions more fully. This will allow them to be good at deciding *when* to do what. The following poem, which is explained in the appendix, is analogous to the third chapter of the book of Ecclesiastes, with apologies to Pete Seeger.

There is a season for everything,
 a time for every action under heaven:
 a time for thinking, and a time for reacting;
 a time for planning, and a time for executing plans;
 a time for exercising, and a time for relaxing;
 a time for dreaming, and a time for studying;
 a time for chitchat, and a time for negotiation;
 a time for playing, and a time for practice;
 a time for cheers, and a time for tears.

In writing this book, we were aware of differences in human cognitive processing. When we evaluated data in peer-reviewed journals, we strove to discern the authors' recognition of how these human differences affected the authors' evaluation of their data. For example, we expected much less variability in the data of major league baseball players compared to collegiate players. If this was not apparent in the data, then the data were suspect.

In creating a model, we were always cognizant of the reader. We continually worried about whether or not a reader could replicate our experiments and equations.

Now, let us stop discussing what we were thinking while writing this book and return to what was actually stated in the chapters of this book.

Collisions between baseballs, softballs and bats are complex and therefore their models are complex. One purpose of this book was to show how complex these collisions could be, while still being modeled using only Newton's principles and the conservation laws of physics. Accordingly, this book presented the BaConLaw model for the speed and spin of balls and bats after the bat-ball collision in terms of these same four variables before the collision.

Chapter 1 presented Newton's laws and laid the groundwork for analyzing bat-ball collisions. Using text and figures, Chap. 2 explained nine common

configurations of bat-ball collisions. Chap. 3 started the development of the sets of equations for these configurations.

The workhorse of this book, Chapter 4, contains our most comprehensive model, the BaConLaw model. It models a collision at the sweet spot of the bat with spin on the pitch. It has five equations and five unknowns. The equations are complete and comprehensive. This chapter contains a sensitivity analysis of the complete model, which shows that the most important variable in the model, in terms of maximizing batted-ball speed, is the bat speed before the collision. This chapter starts the fulfillment of the first purpose of this book by showing what may be the most complex model that is compatible with our simple technique and Newtonian physics. It also fulfills the second purpose of this book, namely, to help batters select or create baseball or softball bats that would be the best for them. Cupping the barrel end of the bat does not help. This chapter is unique in the science of baseball literature.

This is a big deal. The BaConLaw model describes the motion of the *bat* after the collision. Many models describe the motion of the ball after the collision, but few (if any) describe the motion of the bat. When you see a batter hit the ball, do you see the jerk of the bat? Can you describe it? Well these equations do.

Chapter 5 contains four alternative models for bat-ball collisions. Their purposes are different and they are based on different fundamental principles. The Effective Mass model was created by physicists independent of the author of this book. Therefore, comparisons to it are important for validating the BaConLaw model of Chapter 4. The Spiral Center of Mass model and the Sliding Pin model are data-based, not theory-based. They use a different approach and they use a different *type* of data. Finally, the Collision with Friction model considers friction during the collision. It is shown that this type of collision cannot be modeled using only the conservation laws. Therefore, this model completes the fulfillment of the first purpose of this book, by showing a configuration that is too complex for our simple technique.

Chapter 6 recapitulates Chaps. 1 to 5. These chapters are at a higher level of abstraction than typical physics of baseball papers, because they ignored details of the collision, such as (1) during the collision the ball can slip, slide, roll or grip the bat, and the ball switches between these modes, (2) the coefficient of friction changes from dynamic to static and back, (3) the bat and ball deform during the collision (4) some collisions have normal and tangential velocity components and (5) the bat has a twist or a rotation about its long axis. This book ignored the difference between the kinetic, energetic and kinematic coefficients of restitution, the energy loss due to tangential forces and losses in angular momentum: it grouped all of the energy losses into one parameter, the kinematic Coefficient of Restitution. We modeled the parameters of the bat and ball only at a time just before the collision and at a time just after the collision. Because the equations are at a high level, it was possible to verify each major equation by at least two techniques. This book used simple terms that were presented in Table 1.1 that should be understandable by all students of the science of baseball. This book did not obfuscate with jargon, rules of thumb or esoteric terms. By using only fundamental principles, it is

hoped that the reader gained intuition about the behavior of the bat and ball before and after collisions.

One purpose of the Ball in Flight model of Chap. 7 was to show how altitude, temperature, barometric pressure and relative humidity affect air density and consequently how air density affects the flight of the ball. To do this, we needed equations for the flight of the ball that included air density. Therefore, the first challenge of this chapter was to derive equations for the flight of the ball that included the dependence on air density. These equations were not restricted to Newton's principles and they relied heavily on experimental data. Next, this chapter showed that air density is inversely related to altitude, temperature and humidity, and is directly related to barometric pressure. Air density affects how far a batted-ball travels. As shown by this model, on a typical July afternoon in a major-league baseball stadium, altitude is the most important factor, explaining 80% of the variability. This is followed by temperature (13%), barometric pressure (4%) and relative humidity (3%). A simple linear algebraic equation was presented that predicts air density in terms of these four variables. A different model showed how the batted-ball's range depends on both the drag force and the Magnus force and considered the relative importance of each. As an aside, this chapter answered the question of whether a person could throw a heavy ball or a light ball farther. If all other parameters are held constant, a heavier ball might be thrown *slightly* farther than a lighter ball.

Chapters 1 to 6 modeled bat-ball collisions and Chap. 7 modeled the flight of the ball, whether thrown, pitched or hit. Let's see if we can bring our technique together in one example.

To understand new puzzles we use physics first, then physiology and finally psychology. As an example, let's apply the lessons learned from this book to an interesting anomaly in baseball statistics. In major league baseball (MLB), there seems to have been more home runs in 2017 than in previous years. When pondering a new problem, we first we try to solve it using physics. Physicists have investigated the baseball's contribution to the coefficient of restitution, but it seems to have remained constant throughout recent years. Others have suggested that the flatter seams on the major league baseball would reduce the drag coefficient and thereby increase the range of the batted ball. However, at the same time, the flatter seams would also decrease the Magnus lift force and thereby *decrease* the range. Evidentially the effect of drag reduction is greater than the effect of Magnus lift force reduction, because experiments have shown that MLB's flatter seams increase the range by about 20 feet. But this is all irrelevant, because MLB's switch to the flat seam ball occurred years ago, not in the summer of 2016. Therefore, physics does not provide an answer for the increased number of home runs in 2017. Accordingly, let's try physiology. Today's players are more athletic due to conditioning and nutrition. So, they have higher bat swing speeds. But this did not happen suddenly in 2016. So, physiology does not provide an answer. Okay, so let's try psychology. MLB has created game-changing new metrics for performance. For example, pitchers are no longer evaluated solely on won-loss record and earned-run average. Instead, everyone is talking about pitch speed. Pitchers seem to be

throwing faster with many of them throwing at 100 mph. This would produce more home runs, because the faster the ball comes in, the faster the ball goes out. At the same time, MLB and television are glamorizing the home run. Trajectories are continually displayed on television and on the internet. This has probably caused batters to *decide* to try hitting more home runs by swinging faster and launching the ball at a higher angle. Therefore, psychology may be the reason for the increase in home runs. Both the pitcher and the batter gain more publicity because of higher pitch speeds and more home runs. Finally after pursuing physics, physiology and psychology, we should “follow the money.” High-speed pitchers and prolific home run hitters draw in the crowds (and their money) and hence are paid more. This probably caused them to decide to throw harder and swing faster.

Appendix

We concluded our paragraph about the four-year old children resisting their urge to eat the Oreo cookies with, “The resisters had higher measures of executive functions in cognitive tasks and especially the ability to allocate their attention efficiently.” Let us now analyze that sentence.

The field of cognitive neuroscience proposes that executive functions reside in a particular area of the brain named the prefrontal cortex. The basic executive functions include cognitive processes such as impulse control, use of working memory, attention control, resistance to interference and cognitive flexibility.

The first two of these functions develop in early childhood. *Impulse control*, also known as response inhibition, is an executive function that permits people to inhibit their impulses in order to select behaviors that are more likely to satisfy their goals. *Use of working memory* is an executive function that holds and processes information for a short time.

The last three of these functions develop later in life. *Attention control* is an executive function that allows people to allocate their attention, to choose what they pay attention to and what they ignore. Attention control can be described as a person’s ability to concentrate or focus. *Resistance to interference* is an executive function that allows people to shutout stimuli that are irrelevant to the task at hand or to the mind’s current state. *Cognitive flexibility* is an executive function that allows people to switch between thinking about two different concepts and perhaps to control multiple tasks concurrently.

Multiple basic executive functions create high-order executive functions, which include planning, scheduling, negotiating, tradeoff studies and problem solving.

The executive functions that are most important for baseball players are arguably impulse control, use of working memory, attention control, resistance to interference and planning.

My conjecture is that children who can control their impulses and concentrate on the task at hand will probably become better baseball and softball players because they have and will develop their executive functions more fully. This will allow

them to be good at deciding *when* to do what. The following poem is analogous to the third chapter of the book of Ecclesiastes, with apologies to Pete Seeger.

There is a season for everything,
 a time for every action under heaven:
 a time for thinking, and a time for reacting;
 a time for planning, and a time for executing plans;
 a time for exercising, and a time for relaxing;
 a time for dreaming, and a time for studying;
 a time for chitchat, and a time for negotiation;
 a time for playing, and a time for practice;
 a time for cheers, and a time for tears.

A time for thinking, and a time for reacting.

It takes exceptional concentration for a batter to track a pitch and predict the ball's position at the time of the collision. The batter must shutout distractions. On the other hand, the swing of the bat is merely a reaction. It is an over-practiced reaction with little variability.

A time for planning, and a time for executing plans.

Before each pitch, every fielder plans what he or she will do for every contingency. For instance, assume that the game is tied in the bottom of the sixth inning. There are no balls, no strikes and no outs. There are runners on first and third. Each fielder must formulate a plan. For example, on a deep flyball, an outfielder will throw the ball to the cutoff man (the second baseman or the short stop depending on where the ball was hit). On a lazy flyball, if the runner on third is tagging, the outfielder will throw to home plate through the cutoff man. On a shallow hit, if the runner on third is advancing, then the outfielder will throw to the catcher; if not, then the fielder must throw to second base. All of these plans must be in the fielder's working memory before the pitch. Because, when the ball is hit, there is no time for thinking: there is only time for executing the plan.

A time for exercising, and a time for relaxing.

Athletes must be in good physical shape. Regular planned exercise can help achieve this. However, all work and no play makes Jack a dull boy.

A time for dreaming, and a time for studying.

There is no time for daydreaming during a game. Whereas, between games, there is plenty of time for study; to study the opposition, to read books like this one and to learn about the world around us.

A time for chitchat, and a time for negotiation.

Talking about personal lives helps players understand how their teammates will react during a game. It is important that outfielders, for example, know each other very well. On a line drive between them, one runs in and one runs out: this prevents

broken bones. In contrast, most players will not talk with their agents during the season because they do not want the distraction.

A time for playing, and a time for practice.

When you are playing a game, your brain must be totally engaged in the game. In Chap. 4, we quoted Dave Baldwin as saying that if you lose a game, don't blame the umpire or your teammates; just go home and practice harder.

A time for cheers, and a time for tears.

After every win, all players cheer. However, when a team is eliminated from the championship tournament, many players cry.

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Chapter 9

General Modeling Principles

9.1 Introduction

The following statement was on Richard Feynman's white board when he died. By *create* he meant derive equations mathematically, on a white board, in real-time, in front of an audience, without notes.

What I cannot create,
I do not understand.
Richard Feynman

What I cannot model,
I do not understand.
Terry Bahill

Purpose: This chapter extracts the modeling lessons learned throughout this book into one cohesive whole. It is based on Bahill (2016).

9.2 Why Model?

This book is about modeling and simulation of bat-ball collisions and the flight of the ball. A *model* is a simplified representation of some aspect of a real system. A *simulation* is an implementation of a model, often on a digital computer. Models are ephemeral: they are created, they explain a phenomenon, they stimulate discussion,

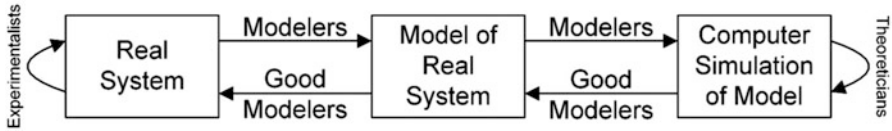


Fig. 9.1 The modeling process

they foment alternatives and then they are replaced by new models. Everyone knows how to make a model, but most researchers miss a few tasks. Therefore, we wrote this chapter that presents a succinct description of the modeling process shown in Fig. 9.1.

Most systems are impossible to study in their entirety, but they are made up of hierarchies of smaller subsystems that can be studied. Nobel Laureate Herb Simon (1962) explained the necessity for such hierarchies in complex systems. He wrote that complex systems are decomposable, enabling subsystems to be studied outside of the entire hierarchy. For example, when modeling the movement of a pitched baseball, it is sufficient to apply Newtonian mechanics considering only gravity, the ball's velocity and the ball's spin. One need not be concerned about electron orbits in the cowhide cover or the motions of the sun and the moon. Forces that are important when studying objects at one level seldom affect objects at another level. When modeling baseball systems we are fortunate that we do not have to consider how black holes form or entanglement, which Einstein mocked as “spooky action at a distance.”

Table 9.1 shows a sampling of the levels of two of the hierarchies that were used in this book. Items at the bottom of the table are at the lowest level considered in this book. Items that are higher up in the table are at a higher level of abstraction. The point is, in Chap. 7 for example, we studied altitude, temperature, humidity and barometric pressure and derived equations for them. Later we studied the equations for F_{gravity} , F_{drag} and F_{Magnus} . Later still, we studied the range of the batted ball. We studied them independently. We never had them on the same page.

9.2.1 Purpose of Models

Models can be used for many reasons, such as understanding or improving an existing system (done in this book), creating a new design or system, controlling a system, suggesting new experiments, guiding data collection activities (done in this book), allocating resources, identifying cost drivers, increasing return on investment, helping to sell the product, and reducing risk. Running business process models clarifies requirements, reveals bottlenecks, reduces cost, identifies fragmented activities and exposes duplication of efforts.

Table 9.1 Bat-ball modeling hierarchy

Chapter 7 the Ball in Flight model
What effects air density and what does air density effect?
Sensitivity analysis
Range of the batted-ball
The order of determining numerical values for the parameters (Table 7.15)
$F_{gravity} = m_{ball}g$
$F_{drag} = 0.5\pi\rho r_{ball}^2 v_{ball}^2 C_D$
$F_{Magnus} = 0.5\pi\rho r_{ball}^3 \omega_{ball} v_{ball} C_M$
The right-hand rules
Air density
Altitude, temperature, humidity and barometric pressure
Chapter 4 the BaConLaw model
Advice for choosing a bat
Sensitivity analysis
KE_{lost} , Eq. (4.11)
Output equations, Eqs. (4.8), (4.9), (4.10) and (4.12)
Conservation laws, Eqs. (4.3), (4.4) and (4.7s)
Newton's principles, <i>CoR</i> , Eqs. (4.5) and (4.6)
$v_{ball-before}$, $\omega_{ball-before}$, $v_{bat-cm-before}$, $\omega_{bat-before}$, $v_{ball-after}$, $\omega_{ball-after}$, $v_{bat-cm-after}$ and $\omega_{bat-after}$

9.2.2 Kinds of Models

There are different kinds of models: there are models of behavior, of structure, of performance and for analysis. *Models of behavior* describe how the system responds to external excitation: that is, how the system-functions transform the inputs into outputs. The BaConLaw model is a model of behavior. It describes the linear and angular speed of the bat and the ball after the collision in terms of these same parameters before the collision. *Models of structure* describe the components and their interactions. Three-dimensional CAD/CAM images check the buildability of structures. *Models of performance* describe units, values and tolerances for properties such as weight, speed of response, power required, etc. These might be captured in requirements. Typical baseball performance measures include batting average, slugging percentage and safe on-base percentage. *Models for analysis* are used to calculate properties of the whole system from the properties of its parts. For example, the time for a car to accelerate from 0 to 60 mph can be calculated from the mass of the car, the power of the drive train, the aerodynamic drag coefficient and the friction of the tires on the pavement.

9.2.3 *Types of Models*

There are many types of models. Most people use only a few and think that is all there are. Here is a partial list of some of the most commonly used types of models: physiological and physical laws and principles, differential equations, difference equations, algebraic equations, geometric representations of physical structure, computer simulations and animations, Laplace transforms, transfer functions, linear systems theory, state space models e. g. $\dot{x} = Ax + Bu$, state machine diagrams, charts, graphs, drawings, pictures, functional flow block diagrams, object-oriented models, UML and SysML diagrams, Markov processes, time-series models, physical analogs, Monte Carlo simulations, optimization equations, statistical distributions, mathematical programming, financial models, Pert charts, Gantt charts, risk analyses, tradeoff studies, mental models, scenarios and use cases.

To understand how people think we would use the models confirmation bias, attribute substitution, and representativeness. For biological domains, we must first choose a virus, a bacterium, a plant or an animal. Once we have chosen our subject, we could then derive its genome. For social domains, we might use a novel, an encyclical, a song, a poem, or perhaps even a joke.

Most models require a combination of these types. For example, in this book we used Newton's principles, the conservation laws of physics, algebraic equations, Excel spreadsheets, figures, tables, simulations, an optimization package, design of experiments and statistics. Hence, our BaConLaw model and our Ball in Flight model used many types of models.

9.2.4 *Tasks in the Modeling Process*

The following checklist contains the principle tasks that should be performed in a modeling study. The modelers should look at each item on the list and ask if they have done that task. If not, they should state why they did not do it. In this checklist, we describe {in squiggly braces} the parts of the BaConLaw model that implement the individual tasks.

- Describe the system to be modeled {The BaConLaw model describes head-on collisions between bats and balls. It gives the speed and spin of the bat and ball before and after collisions. It does not describe the dynamics during the collision nor the swing of the bat.}
- State the purpose of the model {To explain bat-ball collisions with precise, correct equations, without jargon}. This includes defining the performance criterion function.
- Determine the level of the model {The level for the BaConLaw model encompasses the ball speed, the bat speed and the bat angular velocity after the collision in terms of those same parameters before the collision. The time scale is in milliseconds.}

- State the assumptions and at every review reassess the assumptions {Our assumptions were stated in Sects. 3.2.1 and 3.3.1.3}.
- Investigate alternative models {Alternative collision configurations were explained in Chaps. 2 and 3. The BaConLaw model was given in Chap. 4 and alternative models were given in Chap. 5. Having alternative models helps ensure that you understand the physical system. No model is more correct than another. Alternative models just emphasize different aspects of the physical system. They are not competing models they are synergetic.}
- Select a tool or language for the model and simulation {We used the What'sBest! optimizer and Excel spreadsheets.} This should not be a casual decision. You should not merely accept the default. You should use a tradeoff study.
- Make the model {The BaConLaw model was created in Chap. 4.}
- Integrate with models for other systems {The outputs of the BaConLaw model became inputs to the Ball in Flight model of Chap. 7 in order to show how the range of batted-balls is affected by air density and by cupping the barrel end of the bat.}
- Gather data describing system behavior {We used data from our internal databases, from peer reviewed journal papers and from the following databases.
<http://www.hittrackeronline.com/>
<http://mlb.com/statcast/>
<https://baseballsavant.mlb.com/>
- Show that the model behaves like the real system {The outputs of the simulations were compared to the data listed in the above paragraph.}
- Verify and validate the model {Verification means, Did you build the system right? For the BaConLaw model, the outputs of the simulations agree with data listed in the above paragraph. The double checks in the simulation ensure correctness of the spreadsheets. For example, the kinetic energy lost is computed with Eq. (4.11) and also by summing individual kinetic energy components as shown in Tables 4.3 and 5.3. The conservation laws were used in the derivations and the final outputs of the simulation were inserted into the conservation law equations to ensure consistency of the spreadsheet. The main output of the BaConLaw model was compared to the output of the Effective Mass model of Sect. 5.1. The physics was peer-reviewed by two anonymous physics professors. Each of the main BaConLaw equations were derived using at least two techniques. Finally, the equations were checked by an independent mathematician. Validation means, Did you build the right system? Our customer wanted a system that described head-on collisions between bats and balls. They wanted a system that would give ball speed, bat speed and the bat angular velocity after the collision in terms of those same parameters before the collision. This is what our system does. Finally, we performed a sensitivity analysis, which is a powerful validation tool (Smith et al. 2008). It warns if something is wrong with the model.}
- Explain a discovery that was not planned in the model's design {(1) We were surprised when the equation for the kinetic energy lost in the collision,

Eq. (4.11), fell right out of BaConLaw set of equations. (2) Before writing this book, we did not expect to prove that cupping the barrel end of the bat does little good. (3) Although it seems intuitive, we were surprised when the mathematics showed that the baseball could be thrown farther than a tennis ball.)

- Perform a sensitivity analysis of the model {The most important parameters, in terms of maximizing batted-ball speed, are the velocity of the center of mass of the bat before the collision and the coefficient of restitution, CoR_{2b} . The least important parameters are the angular velocity of the ball and the distance between the center of mass and the impact point. The second-order interaction terms are small, which is good.}
- Perform a risk analysis {*Risk to our publisher.* The biggest risk is that people might be reluctant to buy and read a book with equations in it. Also, Springer would be disappointed if sales were low. Therefore, by writing with the reader in mind, we tried to ensure that sales would not be below expectations. We anticipate no copyright problems, because most of the material is original and we have permissions for the two figures that are not. *Risk to our reader.* Someone could modify their bat and hurt himself or herself working with tools or they could be ejected from a game. *Risk to the author.* If our equations were wrong, we would confuse our readers and tarnish our reputations. *Risk to quality.* The book is produced in India. Typographical and editing mistakes that occur are hard to correct because of poor communication channels. *Risk to baseball managers, general managers and umpires.* It will put a burden on these people to learn to understand the results of mathematical modeling. *Risk to MLB.* It could embarrass MLB into disclosing their algorithms. Some of these risks may seem trivial. But a risk analysis is supposed to uncover unlikely risks.}
- Analyze the performance of the model {This was described above in the verification paragraph.}
- Re-evaluate and improve the model {In the future, we will derive equations for configurations 3 and 4. We will explain why the curveball curves. We will also investigate the cognitive processing and decision making of the batter (Bahill and Baldwin 2004; Bahill et al. 2005; McBeath et al. 2008; Bahill and Madni 2017).}
- Suggest new experiments and measurements for the real system that might challenge existing models {Major League Baseball (MLB) is providing copious amounts of new data. Next, scientists need MLB's actual algorithms and measurements for the spin on the batted-ball, particularly for the home run trajectories that are so popular. Another proposed area of measurement and display involves the erratic meandering of fielders trying to catch pop-ups. This behavior and the paper by McBeath et al. (2008) show that the ball's trajectory must contain bizarre loops and cusps. MLB should show these trajectories on the television screen to help laypeople understand the fielders' wanderings.}

Choose a cute name for your model. You want people to relate to the name of your model. This will enhance financial support. In the following couplets, we give the original model name and then an alternative name. Which do you think is best?

Would you rather have had your taxpayer dollars

- support research on the Big Bang or on a theory of the origin of the universe?
- support research on dark energy or on the existence of transparent matter?
- support the Superconducting Super Collider or the search for the God particle?
- get rid of weapons of mass destruction or a tyrannical despot?

Would you rather

- go to the opera *The Marriage of Figaro* or to Mozart's opera in D major Köchel No 492?
- listen to Beethoven's *Ode to Joy* or his Symphony Number 9 in D minor, Opus 125?
- listen to Mussorgsky's *Night on Bald Mountain* or his musical picture in D minor?
- listen to Wagner's Overture to Act III of *Lohengrin* or see the snow skiing scene in the Beatles' movie *Help*?

Would you rather

- read an article about $E = mc^2$ or a paper about mass-energy equivalence?
- study the DNA double helix or chromosomal structure?
- see a grand slam or a bases-loaded home run in baseball?
- see a Hail Mary or a fourth-quarter fourth-down forty-yard pass in football?

Are you more likely to

- have wished for the fall of the Iron Curtain or of communism in eastern Europe?
- order Baked Alaska or ice cream covered with roasted merengue?
- watch the World Series or the MLB championship games?
- watch the Super Bowl or the NFL championship game?

In Kahneman's model for human thinking, which can you relate to easiest

- System 1 or the fast, instinctive and emotional system?
- System 2 or the slow, deliberative and logical system?

Are you more likely to say

- Navy SEALs or navy sea air land forces?
- SWAT team or special weapons and tactics team?

The point of this page is to convince you that a distinctive name for your model will help people remember it and relate to it. This will be aided if your name is iconic. Examples of iconic images include the flag of the United States of America, the Statue of Liberty, a crucifix, a Star of David, the Nazi swastika, the Apple Computer Company logo and the Mona Lisa. Examples of iconic smells include Hydrogen Sulfide (rotten eggs) and Methyl Mercaptan (the odor in natural gas). Our favorite perfume fragrances are eau de Wet Dog and Impending Desert Rain. You want your name to be as memorable as the eight notes at the beginning of Beethoven's fifth symphony, the three dramatic notes in Neil Diamond's *Sweet*

Caroline and the opening measures of Stanley Kubrick's 1968 film *2001: A Space Odyssey* (aka Strauss' *Also sprach Zarathustra*, opus 30, 1896).

9.2.5 Model-Based Design

There are two common techniques for designing a system or making a model: the first is model-based or theory-based (Bahill and Szidarovszky 2009) and the second is data-based (Bahill 2016). Here are some steps for model-based system design. Find appropriate physical, physiological and psychological principles, then using the tasks listed in the above section, design, build and test a model, then design and conduct experiments to collect data. Use these data to verify and validate the model. Use the model to make predictions and guide future data collection activities.

Example 1

Chapter 4 started with the following fundamental equations of physics: Conservation of Energy, Conservation of Linear Momentum, the Definition of the *CoR*, Newton's Second Principle and the Conservation of Angular Momentum. These conservation laws are the models (or theories) that the BaConLaw model and the Effective Mass model were based on.

Example 2

Chapter 7 started with the right-hand rules and the three forces that affect the ball in flight: gravity pulls the ball downward, air resistance or drag operates in the opposite direction of the ball's motion and, if the ball is spinning, there is a Magnus force perpendicular to the direction of motion. Watts and Bahill (1990) wrote equations for these forces like this:

$$\begin{aligned} F_{\text{gravity}} &= m_{\text{ball}}g \\ F_{\text{drag}} &= 0.5\pi\rho r_{\text{ball}}^2 v_{\text{ball}}^2 C_D \\ F_{\text{Magnus}} &= 0.5\pi\rho r_{\text{ball}}^3 \omega_{\text{ball}} v_{\text{ball}} C_M \end{aligned}$$

These equations are the models (or theories) that the Ball in Flight model of Chap. 7 was based on.

9.2.6 Data-Based Design

The second technique for designing a system or making a model is data-based. With this technique, the modeler starts with measuring and organizing the data and then he or she makes a model that fits that measured data. The Spiral Center of Mass and the Sliding Pin models of Chap. 5 were data-based. We found the experimental data first and then we created the model to match that data.

The BaConLaw and the Effective Mass models started with the model of a free-end collision involving the velocity of the center of mass of the bat and the bat's angular rotation about the center of mass. In contrast, for the Sliding Pin model, we first found experimental studies that gave the linear velocity of the knob and the angular velocity of the bat about the knob right before the collision. We then used that data to make our model.

9.2.7 *Second Sourcing*

It is good practice to make sure that anything you buy has a second source. That way if your first source disappears, you can continue to function.

Modelers should entice other scientists to create different models for the same physical system. This will help validate their models.

Many fields of science are demanding replication of important experiments and results. Failures to replicate previous findings are common in science, particularly in the psychological literature, where half of the important findings cannot be replicated [https://en.wikipedia.org/wiki/Replication_crisis].

If you are going to remodel your house, which faucet manufacturer are you likely to specify, Moen or LightInTheBox? Think about repair and maintenance of the system in 10 years.

Would you buy a chandelier with incandescent light bulbs and a dimmer control? Keep in mind that they do not make incandescent replacement bulbs anymore.

Missile manufacturers will not specify a part if there is not a second source. They want to ensure that they can continue manufacturing even if their first source goes bankrupt.

The atomic bombs dropped in WWII, Little Boy and Fat Man, were of different designs and used different fissionable elements. If one design did not work, then they still had a second source.

Apollo 13 was not a disaster because they had a second source of power: the lunar lander.

The county directors of elections would like to have a second source for their hardware and software. Because on Election Day, they only have one chance to get it right. A second source would also ameliorate cyber-attacks.

Suppose your new medical doctor tells you that some test has just revealed cancer and she recommends that you start radiation treatment immediately. Would you ask for a second opinion?

When asking for driving directions to an event, it makes sense to ask for alternative routes (perhaps the quickest, the shortest and also the cheapest, i. e. no tolls), so that you have alternatives, in case of a massive traffic jam.

It is a good idea to have two e-mail accounts. That way if Comcast decides to block one your correspondents because he or she fits their profile of a 'bad person' or if Microsoft is 'upgrading' their e-mail system, you can still communicate with the world.

I am sure that all readers of this book can access it from more than one place: you have multiple sources for this book. I am also confident that you back up your hard disk every day. So now, dear reader, please put down this book and go back up your hard disk.

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