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Heavy WIMP Effective Theory

Formalism and Applications for Scattering
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Mikhail P. Solon

Heavy WIMP Effective Theory

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on Nucleon Targets

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To Orville and Sophia Solon

Supervisor's Foreword

Mikhail Solon's excellent thesis formulates the new heavy WIMP effective theory. This theory allows precise determination of WIMP-nucleon scattering cross sections in the absence of detailed knowledge about underlying dark sector dynamics, essential for planning and interpreting the results of future dark matter direct detection experiments.

Exploring the possible interactions between WIMPs and standard model particles has exposed gaps in our understanding of effective quantum field theory. Besides explicit computations of dark matter properties, Solon's thesis addresses several basic theoretical questions. It describes a new formalism for implementing Lorentz invariance constraints in nonrelativistic theories with unknown, or unspecified, ultraviolet completion. The thesis also provides new perturbative QCD results that determine the impact of heavy quarks on the hadronic matrix elements involved in WIMP direct detection.

Solon's thesis is a technical tour de force, with impacts in dark matter phenomenology, field theory formalism, and precision hadronic physics. His computation of scattering rates for heavy electroweak-charged dark matter has become a benchmark for the field of direct detection, and his thesis has also spawned a new field of investigation in dark matter indirect detection, determining heavy WIMP annihilation rates using effective field theory methods. The systematic treatment of renormalization and heavy quark threshold matching conditions, the review of background field methods, and the survey of hadronic matrix elements should be useful to students and researchers interested in a variety of new physics searches and precision measurements involving weak probes of the nucleon.

Chicago, IL, USA

Richard J. Hill

Preface

The discovery of a standard model-like Higgs boson and the hitherto absence of evidence for other new states may indicate that if WIMPs comprise cosmological dark matter, they are heavy compared to electroweak scale particles, $M \gg m_{W^\pm}, m_{Z^0}$. In this limit, the absolute cross section for a WIMP of given electroweak quantum numbers to scatter from a nucleon becomes computable in terms of standard model parameters. Extending aspects of heavy particle formalism familiar from heavy quark effective theory, we develop heavy WIMP effective theory to isolate universal behavior within the WIMP paradigm.

We present ingredients necessary for this effective theory framework, including the formalism for bottom-up construction of heavy particle Lagrangians based on induced representations of the Lorentz group, the complete calculation of one- and two-loop weak-scale matching amplitudes, a consistent renormalization scheme in the presence of nontrivial residual masses, and the QCD framework for passing from the theory renormalized at the electroweak scale to the theory of quarks and gluons below the charm scale.

We analyze the heavy WIMP limit of WIMP-nucleon scattering and present the first complete calculation of the leading spin-independent cross section in standard model extensions consisting of one or two electroweak $SU(2)_W \times U(1)_Y$ multiplets, including a careful treatment of perturbative and hadronic-input uncertainties. The standard model exhibits a surprising transparency of nucleons to WIMP scattering, due to a cancellation between scalar and tensor amplitude contributions. The resulting cross-section predictions and their fractional uncertainties depend sensitively on parameter inputs, and we investigate the impact of model-independent inputs, such as perturbative QCD corrections and nucleon scalar matrix elements, and of model-dependent inputs, such as WIMP quantum numbers, additional electroweak multiplets, and extended Higgs sectors.

Berkeley, CA, USA

Mikhail P. Solon

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I lived the past 5 years with only one responsibility: to learn physics. I owe the pleasure of such a simple existence to a large number of people.

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Chapter 1

Heavy WIMP Effective Theory

1.1 Introduction

Observed anomalies in astrophysical systems, ranging from galactic to cosmological in scale, provide compelling evidence for dark matter.¹ Independent and increasingly precise measurements of the cosmic abundance of total matter and of its baryonic component (e.g., from the cosmic microwave background, big bang nucleosynthesis and large-scale structure) have converged on the picture that $\sim 85\%$ of the matter in the universe cannot be explained by the Standard Model (SM) of particle physics. While evidence from rotation curves of spiral galaxies may suggest modifications to the theory of gravity, lensing measurements indicate gravitational sources in regions with no baryonic matter, favoring the idea of unseen massive clusters. Beyond the gravitational interaction that implies its existence, there is however little known about the particle nature of dark matter. Does it have non-gravitational interactions with Standard Model particles? Is there a single particle, or an intricate structure of multiple particles similar to the Standard Model? Is it a fundamental particle, or a composite object arising from nonperturbative dynamics? What are its mass and spin?

Discovering the nature of particle dark matter is largely an experimental expedition. While searches may be motivated by theoretical visions, definitive knowledge can only come from a detection signal or from careful linkage of various empirical observations. Experiments in the present decade are exploring a broad range of processes, such as dark matter production at colliders, annihilation at the galactic center and scattering off nucleon targets, with however an absence of definite signals thus far (the LHC has discovered the Higgs boson, but no evidence for

¹For reviews see [14, 73] and references therein.

other new states). Although null results translate into tighter phenomenological constraints, the dark matter problem remains in a state too opaque for definite predictions to be made from a vast landscape of models for new particles.

Nonetheless, the role of theoretical input is paramount. Beyond models that supply a zoo of dark matter candidates to be tested at current and forthcoming facilities, it is imperative to develop theoretical formalism to delineate the possible interactions of DM with known particles, making clear which uncertainties are inherently model dependent and which can, at least in principle, be improved by further Standard Model analysis. Effective field theory techniques provide a framework for sketching visions of nature with minimal model dependence, employ only a few parameters to capture the essential physics in experimental processes, and allow for systematically improvable precision. Physics of the Standard Model appear as backgrounds to detection signals in collider and astrophysical searches, and many inputs for direct scattering at low energies require the study of perturbative QCD and of nucleon (and nuclear) properties. Experimental studies of dark matter are presenting opportunities (requirements) for understanding Standard Model physics in new regimes, involving physics at multiple scales. In this thesis, we develop such tools for controlled theoretical calculations, leading to robust predictions for dark matter-nucleon interactions relevant to direct searches in underground experiments.

We focus on the paradigm of Weakly Interacting Massive Particles (WIMPs), representing dark matter candidates with electroweak charge and a thermal history consistent with the observed present-day relic abundance. A large class of models have a WIMP as the lightest state of a new sector, e.g., neutralinos of supersymmetric extensions [73], and in this situation, the Standard Model is extended at low energies by one or a few particles transforming under definite representations of $SU(2)_W \times U(1)_Y$. The field of WIMP dark matter direct detection is by now a mature subject.² Early treatments of QCD effects in neutralino-nucleon scattering include the works of Drees and Nojiri [34]. Basic aspects of formalism may be found in the review of Jungman et al. [73]. However, the last few years have witnessed the discovery and mass measurement for a SM-like Higgs boson [1, 24], new constraints on the mass scale of particles beyond the SM [50], and important computational advances in lattice QCD [39, 51]. A complete description of dark matter interactions with Standard Model particles is now possible but require a robust and carefully linked analysis of physics at multiple scales, ranging from the ultraviolet (UV) theory of dark matter at the highest, down to nuclear theory at the lowest.

Effective field theory techniques allow us to disentangle the physics at different scales, and identify universal features in the interaction of dark matter with known particles. At the UV scale, the landscape of dark matter candidates is vast, even when restricted to the well-motivated class of electroweak-charged dark matter. Extending aspects of heavy particle formalism familiar from heavy quark effective theory, we develop heavy WIMP effective theory to isolate universal behavior within the WIMP paradigm. The universality emerges when the WIMP is much heavier than

²A subset of recent work in the field may be found in the Snowmass review [31].

the electroweak scale particles, $M \gg m_W$, such as in the case of a thermal relic electroweak triplet or doublet, and is motivated in part by the hitherto absence of new states at the LHC and null results of direct detection experiments [4, 5]. In this limit, the absolute cross section for a WIMP of given electroweak quantum numbers to scatter off a nucleon becomes computable in terms of only Standard Model parameters, gaining theoretical control in the absence of a specified UV completion. Prospects for direct detection of heavy WIMPs are more challenging, but in a precise sense more constrained due to heavy particle universality.

Independent of the heavy WIMP assumption, the analysis at and below the weak scale is generic to a large class of dark matter scenarios. This “Standard Model anatomy” of direct detection consists of input from different particle physics techniques: effective field theory techniques for weak-scale matching, perturbative QCD for renormalization group evolution and heavy quark threshold matching, lattice QCD for evaluating hadronic matrix elements, and chiral effective theory for analyzing multi-nucleon effects. We develop some aspects of this framework, providing formalism for weak-scale matching computations, and a careful treatment of QCD effects when passing from a theory renormalized at the electroweak scale to a low-energy theory of quarks and gluons where hadronic matrix elements are evaluated. Distinguishing between different dark matter candidates in direct detection experiments demands a complete treatment of perturbative and hadronic uncertainties, including the resummation of large logarithms and consideration of loop amplitudes that are typically neglected in the $m_W \sim M$ regime, but which contribute at leading order in the general case. We also identify pieces of the framework whose further development would impact our knowledge of scattering cross sections, such as a complete set of heavy quark decoupling relations in perturbative QCD, scalar quark matrix elements in $n_f = 4$ flavor QCD, and an analysis of multi-nucleon effects with tensor operators.

While our efforts are aimed at complementing the wealth of data from the experimental frontier, our approach has also led to new understanding of basic ideas in quantum field theory such as the role of spacetime symmetries in heavy particle effective theories. When the full theory for a heavy particle is known, the heavy particle effective theory can be constructed by performing a field redefinition. However, in many interesting applications such as dark matter, the full theory is not known and may not even exist as in the case of a bound state arising from strong dynamics. Is there a way to construct effective theories for heavy particles without explicitly integrating out degrees of freedom? A clear statement of relativity and its constraints in heavy particle effective theory provide explicit field transformation laws for bottom-up construction of Lorentz-invariant heavy particle Lagrangians to arbitrary order in the heavy mass expansion.

In the next two sections, we elaborate on the motivations for heavy WIMP effective theory, focusing on the universality of the heavy WIMP limit. We highlight key aspects of the formalism and results from its application to WIMP-nucleon scattering. The last section describes the chapter organization of this thesis.

1.2 Universal Heavy WIMP Limit

An underlying theory envisioned as a complete extension of the SM, such as the Minimal Supersymmetric Standard Model (MSSM), has rich structure motivated not only by the existence of dark matter, but also by other physics such as neutrinos, cosmology, and perhaps by themes such as unification and naturalness. Properties of dark matter, and hence predictions for its physical observables, are completely described within the model, but may depend on a very large number of both discrete and continuous parameter choices. Even upon imposing a multitude of existing experimental constraints (e.g., from collider searches, direct and indirect detection experiments, early universe cosmology and flavor physics), it is difficult to make definite predictions or to interpret would-be signals without assuming special cases that reduce the theory space considerably.

In the absence of definitive signals for dark matter, a model-independent approach with effective theories, such as contact interactions in the case of a heavy mediator, is an alternative that describes a wide range of behavior without the complicated parameter space. While effective operators are not complete visions of nature and may only relate to key aspects of phenomenology, gauge and Lorentz symmetry constraints imply a minimum set of interactions that capture the physics of a particular observable. Within the assumed hierarchy of scales, a complete basis of contact operators provides a scheme for interpretation and correlation of signals from experiments probing different energies. However, the generality of this approach may also imply a lack of predictive power.

Let us consider WIMP-nucleon scattering to illustrate the relation between these two options (underlying model vs effective contact operators) for describing dark matter interactions. In particular, consider the case of a self-conjugate dark matter particle (e.g., real scalar or Majorana fermion) with mass $M \gtrsim m_W$ and arbitrary spin, transforming in a representation of electroweak $SU(2)_W \times U(1)_Y$. For investigating direct scattering at low energies, we require the effective theory describing dark matter interactions with quarks and gluons in $n_f = 5$ flavor QCD. We will see in Sect. 3.4 that 12 parameters c_i , corresponding to 12 contact operators, represent the most general effective lagrangian at leading order in $1/m_W$, relevant for spin-independent, low-velocity scattering with nucleons. Upon integrating out weak-scale particles (top quark, Higgs and electroweak gauge bosons) we may match a specified model onto the 12 coefficients representing the effective lagrangian valid for energies below the weak scale, $E \ll m_W$. Figure 1.1 shows a schematic of the 12-dimensional space with axes given by the coefficients c_i (only three are explicit). Model predictions are shown as square blobs, reflecting a range of c_i values depending on parameter choices. The blobs labelled m_i represent known models such as Standard Model extensions with supersymmetry or additional strongly interacting sectors. The blob labelled n represents known or unknown models whose predictions are difficult to calculate, perhaps requiring nonperturbative physics such as in the case where the dark matter particle is a bound state of strong dynamics. The blob labelled u represents models that

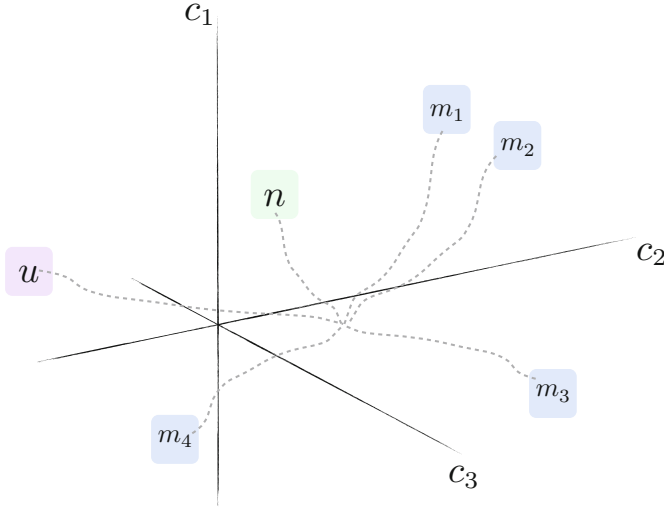


Fig. 1.1 Schematic theory space for spin-independent WIMP-nucleon scattering parameterized in terms of the 12 coefficients c_i of the effective lagrangian. Model predictions are represented by the different blobs. The heavy WIMP limit is depicted by the convergence of model points following the *dashed lines*. For a WIMP of given electroweak quantum numbers, the universal point only depends on standard model inputs

are unknown. The 12-dimensional space is inclusive, capturing the experimental signatures for all models (known, incalculable or unknown), but offers no guidance as to what particular values the coefficients take. Conversely, if the model is known and calculable, then definite values, although subject to model dependence and parameter choices, may be given.

Distinct from the construction of specific models, or of contact operators assuming heavy mediators, is the framework of heavy WIMP effective theory developed in this thesis. Regardless of the origin for the $SU(2)_W$ multiplet, e.g. whether it is a composite or fundamental particle, universal behavior emerges in the limit where the WIMP mass is large compared to the electroweak scale, $M \gg m_W$. The emergence of these universal properties, and corrections to them, can be systematically analyzed using techniques of heavy particle effective theories [21, 70]. Let us consider the heavy WIMP limit, where the interactions of a WIMP with Standard Model particles are simply

$$\mathcal{L} = \bar{h}_v i v \cdot D h_v + \dots, \quad (1.1)$$

where h_v is a heavy-particle field transforming in a representation of electroweak $SU(2)_W \times U(1)_Y$, and $iD_\mu = i\partial_\mu + g_1 Y B_\mu + g_2 W_\mu^a T^a$ is the gauge covariant derivative. Above, the ellipsis denotes higher order operators in the $1/M$ expansion, and the vector v^μ is an arbitrary time-like reference vector appearing in the

construction of heavy particle lagrangians and labeling the heavy-particle field (e.g., we may choose $v^\mu = (1, 0, 0, 0)$, corresponding to a heavy particle in its rest frame). The lagrangian in Eq. (1.1) does not depend on the WIMP mass, spin, or other properties beyond the choice of gauge quantum numbers. Model dependence is systematically encoded in operator coefficients representing $1/M$ corrections.

Returning to our discussion of the parameter space for spin-independent WIMP-nucleon scattering (cf. Fig. 1.1), the heavy WIMP limit implies that each of the 12 coefficients may be expanded as

$$c_i = c_{i,0} + c_{i,1} \frac{m_W}{M} + \dots, \quad (1.2)$$

where, for a WIMP of given electroweak quantum numbers, the leading coefficients $c_{i,0}$ only depend on Standard Model parameters. Convergence of predictions in the heavy WIMP limit, for models known, incalculable, and unknown, are depicted in Fig. 1.1 by the dashed lines meeting at the universal point in theory space. Heavy WIMP effective theory provides theoretical control in the absence of a specified UV completion, giving definite predictions with the minimum parameters, for a large class of models (known, incalculable or unknown).

The recent discovery of the Higgs boson reminds us of the remarkably simple character of physical law. We appeal to this principle, modifying the Standard Model by only one or two electroweak multiplets to account for dark matter, and further simplifying the analysis of WIMP-nucleon scattering by using heavy-particle methods. Those familiar with so-called “minimal dark matter” [28] may wonder if our formalism is any different. Minimal dark matter extends the Standard Model with the “minimal” lagrangian for the WIMP,

$$\mathcal{L}_\phi = \frac{1}{2}(D_\mu \phi)^2 - \frac{1}{2}M^2 \phi^2, \quad \mathcal{L}_\psi = \frac{1}{2}\bar{\psi}(i\not{D} - M)\psi, \quad (1.3)$$

where $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ describe the scalar and fermion cases, respectively. The following points argue that heavy WIMP effective theory is a distinct approach, superior in the regime $M \gg m_W$:

- The universality that emerges in the limit $M \gg m_W$ is obscured in Eq. (1.3); e.g., it is not obvious that employing $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ yield the same results for $c_{i,0}$.
- While $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ may represent unknown models in the case where the WIMP is a fundamental particle, they do not obviously apply for WIMPs that arise as a bound state of strong dynamics (a proton is not described using $\mathcal{L} = \bar{p}(i\not{D} - m_p)p$ in the Standard Model).
- Heavy WIMP Feynman rules are simple, allowing for efficient calculation of the one- and two-loop amplitudes required for determining $c_{i,0}$. Previous calculations of $c_{i,0}$ have been incorrect or incomplete, perhaps due to complicated amplitudes when using $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$. For mixtures of two multiplets, the minimal approach, extending $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$, has been used to explore cross sections in the

highly-mixed regime where the tree-level contribution is dominant. Using heavy-particle effective theory, we present here the first complete parameterization of the mixed-state cross sections.

- The simplified models $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ do not parameterize additional structure of ultraviolet completions in a universal way. Nor do they give guidance as to which structures are most relevant in the heavy WIMP limit. In the scalar case, e.g., the form of Eq. (1.1) does not depend on whether the operator $\sim \phi^2 H^\dagger H$ is appended to $\mathcal{L}_\phi$. Conversely, the operator $\sim \frac{1}{M} \bar{h}_v h_v H^\dagger H$ appears at higher order in the heavy mass expansion (cf. Eq. (1.4) in the next section), accounting for $\sim \phi^2 H^\dagger H$ and the analogous higher dimension operator in the fermion case.
- The separation of scales M and m_W required for resumming large logarithms $\sim \log M/m_W$ can be achieved by using heavy WIMP effective theory.

1.3 Motivations for Heavy WIMP Effective Theory

Search strategies and detection potential are highly dependent on the WIMP's properties including its spin, its mass and its Standard Model (SM) gauge quantum numbers. In the absence of signals for physics beyond the SM, it is important to identify plausible cross section targets to guide and interpret next generation searches. Direct detection experimental constraints [4, 5], together with other phenomenological bounds such as LHC searches, may plausibly indicate that new particles must have mass somewhat above the mass of electroweak-scale particles ($M \gg m_W$). In this limit, heavy WIMP effective theory provides theoretical control without assuming a particular ultraviolet completion, allowing us to predict the absolute cross section for a WIMP of given electroweak quantum numbers to scatter from nucleons in terms of SM parameters.

Heavy WIMP effective theory provides a framework for investigating dark matter interactions, even in the case where it arises as a bound state of strong dynamics in the UV. This is analogous to the use of heavy quark effective theory for computing meson properties, and nonrelativistic quantum electrodynamics for computing nucleon properties (despite knowing the underlying theory of QCD). For models where the WIMP is a fundamental particle, its interactions may be computed within the underlying theory using perturbative methods. Nonetheless, we may employ heavy WIMP effective theory to reduce a multi-dimensional parameter space down to the few parameters relevant for heavy particle interactions. In the heavy WIMP limit given in Eq. (1.1), only the electroweak quantum numbers appear. At next order, the lagrangian for a self-conjugate WIMP of arbitrary spin depends on the mass M and coupling c_H :

$$\mathcal{L} = \bar{h}_v \left\{ i v \cdot D - \frac{D_\perp^2}{2M} + c_H \frac{H^\dagger H}{M} + \dots \right\} h_v, \quad (1.4)$$

where the ellipsis denotes operators higher order in the $1/M$ expansion. In the MSSM, e.g., c_H depends on parameters of the neutralino mass matrix, and may be computed upon integrating out the heavier neutralino components. It is straightforward to consider higher order operators encoding additional microscopic properties of the WIMP (e.g., spin, generalized electroweak dipole moments), and to include additional states such as a second multiplet in the case of mixed-state WIMPs.

The simplicity of heavy-WIMP interactions are reflected in its Feynman rules, leading to efficient calculation of physical amplitudes. In the case of spin-independent scattering, e.g., the determination of coefficients $c_{i,0}$ in Eq.(1.2) requires intricate one- and two-loop amplitudes. Using the heavy WIMP approach, we have performed the first complete calculation of these 12 coefficients at leading order in $1/M$. Combined with a careful analysis of both perturbative and hadronic uncertainties, this provides robust predictions for spin-independent cross section. For an $SU(2)_W$ triplet and doublet, corresponding to the wino and Higgsino of supersymmetric models, we obtain the parameter-free predictions

$$\sigma_{\text{SI}}^T = 1.3_{-0.5}^{+1.2+0.4} \times 10^{-47} \text{ cm}^2, \quad \sigma_{\text{SI}}^D \lesssim 10^{-48} \text{ cm}^2 \quad (95\% \text{ C.L.}), \quad (1.5)$$

where the superscripts T and D denote triplet and doublet, respectively. A simple dimensional estimate for these cross sections yields $\sigma_{\text{SI}} \sim \alpha_2^4 m_N^4 / m_W^6 \sim 10^{-45} \text{ cm}^2$, however destructive interference between scalar and tensor amplitudes leads to anomalously small predictions. The results in Eq.(1.5) inform phenomenological studies of the detectability of these dark matter candidates at current and future detectors,³ and present a challenge to experiments in face of large backgrounds from neutrino scattering appearing for cross sections of $\mathcal{O}(10^{-47} \text{ to } 10^{-48}) \text{ cm}^2$.

Predictions in the heavy WIMP limit have minimal model dependence, and therefore provide an opportunity to identify uncertainties that can be improved by further Standard Model analysis. As remarked above, the Standard Model exhibits a surprising transparency of nucleons to WIMP scattering, due to destructive interference between scalar and tensor amplitudes [59, 60, 65]. We find that such cancellations are generic, but the severity depends sensitively on WIMP quantum numbers, Standard Model parameters and perturbative corrections, as illustrated in Fig. 1.2. Robust conclusions, even in the simple heavy WIMP limit, demand a careful analysis of perturbative and input uncertainties. We investigate the impact of perturbative contributions from weak-scale matching, renormalization group evolution, heavy quark threshold corrections, and nucleon matrix elements. We identify the leading contributions and dominant uncertainties, and thus determine strategies for improved determination of cross sections. For example, the largest uncertainty in the scalar amplitude comes from variation of the charm matching scale, and a more precise determination could therefore be obtained given higher-order heavy quark

³Results consistent with the naive estimate were obtained in previous works missing the destructive interference [28, 42].

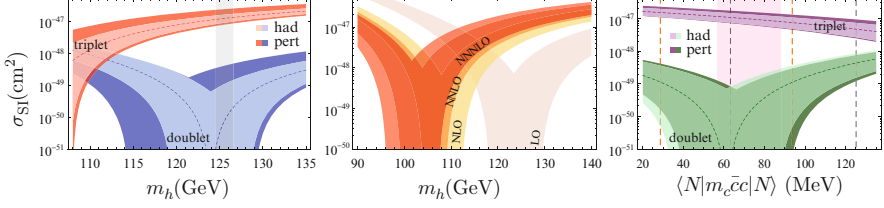


Fig. 1.2 Spin-independent cross sections for low-velocity scattering on the proton as a function of either the Higgs mass m_h or the charm matrix element $\langle N|m_c \bar{c}c|N \rangle$. The sensitivity of the cross section predictions and their fractional uncertainty to WIMP quantum numbers (triplet vs doublet) and the Higgs mass m_h , to higher order perturbative QCD corrections (LO vs NLO vs NNLO vs NNNLO), and to hadronic inputs such as the charm scalar matrix element, are shown in the first, second and third panels, respectively. These plots are reproduced here for convenience from Figs. 6.1, 6.2 and 6.6, where complete descriptions are given

decoupling relations or a measurement of the charm scalar matrix element from the lattice. We also find that gluon contributions are large, and may not be neglected from the set of weak-scale matching amplitudes.

In this thesis, we focus on the application of heavy WIMP effective theory for direct scattering on nucleon targets. As in the case of heavy quark effective theory for QCD, heavy WIMP effective theory may also be employed for other processes that require analysis of multiple scales, with the WIMP mass M as the hard scale. Recent investigations of experimental bounds from indirect detection searches for gamma rays point to a need for precise determination of WIMP annihilation rates [30]. Heavy WIMP effective theory extended with hard-collinear modes provides the tools for such an analysis, allowing, by a separation of scales M and m_W , the resummation of large logarithms, $\sim \log \frac{M}{m_W}$.

The analysis of dark matter observables is a new application of heavy particle effective theory, which hitherto has been applied for investigations within the Standard Model. While the basic techniques of heavy particle effective theory are well known, we have developed several nontrivial aspects, including a clear statement of Lorentz invariance leading to precise formalism for bottom-up construction of heavy particle lagrangians, the introduction of a consistent renormalization scheme for heavy WIMP-SM vertices, and the computation of new heavy-particle loop integrals with nonzero residual masses. The new integral basis evaluated here may be applied to other processes such as low-energy lepton-nucleon scattering [62].

1.4 Chapter Organization

In the remaining chapters, we present a detailed exposition of the formalism and application of heavy WIMP effective theory for WIMP-nucleon scattering. Figure 1.3 supplements the chapter descriptions given below.

(2) (3)	M	$\mathcal{L}_{\text{UV}}$	$\mathcal{L}'_{h_v, \text{SM}} = \bar{h}_v \{iv \cdot D + \dots\} h_v$
(3) (4)	m_W	$\mathcal{L}_{h_v, \text{SM}} = \bar{h}_v \{iv \cdot D + \dots\} h_v$	$\mathcal{L}_{\chi_v, \text{QCD}} = \bar{\chi}_v \chi_v \sum_i c_i \mathcal{O}_i$ $c_i^{(5)}(\mu_t)$
(5) (6)	m_b	$c_i^{(5)}(\mu_b)$	$c_i^{(4)}(\mu_b)$
	m_c	$c_i^{(4)}(\mu_c)$	$c_i^{(3)}(\mu_c)$
(5) (6)	m_N	$c_i^{(3)}(\mu_0)$	$\sigma_{\text{SI}} \sim \langle c_i \mathcal{O}_i \rangle ^2$

Fig. 1.3 Physical scales in the analysis of WIMP-nucleon scattering for the regime $M \gg m_W$. We show the relevant quantities defined at each scale and the chapter (*encircled number*) where the corresponding results are presented. Matching between two theories is denoted by *dashed lines*; e.g., at the weak scale we match $\mathcal{L}_{h_v, \text{SM}}$ and $\mathcal{L}_{\chi_v, \text{QCD}}$ by integrating out weak-scale particles $W^\pm, Z^0, h, t$, while at the bottom scale, we match $n_f = 5$ flavor QCD onto $n_f = 4$ flavor QCD by integrating out the bottom quark

We may integrate out physics at scales M and above by employing a heavy particle description for the WIMP. In the absence of a specified UV theory, we use the formalism described in Chap. 2 for bottom-up construction of Lorentz invariant heavy particle lagrangians. In that chapter, we give a brief introduction to heavy particle effective theory, and present a clear statement of Lorentz invariance based on the method of induced representations. Formalism for arbitrary spin fields and for self-conjugate fields are presented, and for illustration, we construct the parity and time-reversal invariant effective lagrangian for a heavy fermion interacting with an Abelian gauge field.

In Chap. 3, we construct heavy WIMP effective theories for Standard Model extensions consisting of one or two electroweak multiplets, focusing on the lagrangians $\mathcal{L}_{h_v, \text{SM}}$ and $\mathcal{L}_{\chi_v, \text{QCD}}$ at the weak scale. In place of a UV theory, we consider the lagrangian $\mathcal{L}_{h_v, \text{SM}}$ as our starting point, describing interactions of the multiplet h_v with SM gauge and Higgs bosons in the electroweak symmetric phase. For mixtures of two multiplets, WIMP couplings require renormalization, and we define an extension of the onshell renormalization scheme for the electroweak SM in the presence of nontrivial residual masses.

If a UV theory $\mathcal{L}_{\text{UV}}$ is specified at scale M , we may similarly construct $\mathcal{L}'_{h_v, \text{SM}}$ at that scale, and perform a matching between $\mathcal{L}_{\text{UV}}$ and $\mathcal{L}'_{h_v, \text{SM}}$. We illustrate this in Chap. 3 with two examples: first, we construct the heavy scalar triplet lagrangian up to order $1/M^3$ and perform its matching to a toy UV theory; second, we show an alternative derivation of the singlet-doublet heavy particle lagrangian by introducing field redefinitions in the relativistic theory. In the heavy WIMP limit, the heavy-particle lagrangian is not renormalized. For the general case including, e.g., operators higher order in the $1/M$ expansion or additional new states, the relation between $\mathcal{L}'_{h_v, \text{SM}}$ and $\mathcal{L}_{h_v, \text{SM}}$ may be obtained using renormalization group methods.

For investigating direct scattering at low energies, we also construct $\mathcal{L}_{\chi_v, \text{QCD}}$ at the weak scale, describing interactions of the lightest, electrically neutral WIMP, χ_v , with quarks and gluons in $n_f = 5$ flavor QCD.

In Chap. 4, we present formalism necessary to determine weak-scale matching coefficients, i.e., $c_{i,0}$ in Eq. (1.2), in the computation of scattering cross sections. In Fig. 1.3, these are the coefficients of $\mathcal{L}_{\chi_v, \text{QCD}}$, written as $c_i^{(5)}(\mu_t)$ with superscript denoting the number of active quark flavors at the weak scale μ_t where the coefficients are defined. Careful computation of competing Standard Model contributions in the matching of $\mathcal{L}_{h_v, \text{SM}}$ onto $\mathcal{L}_{\chi_v, \text{QCD}}$ is necessary to estimate the correct order of magnitude of scattering cross sections in many simple and motivated models of DM. We review relevant aspects of techniques such as the background field method for matching to gluon operators and the treatment of effective theory subtractions. Within the heavy WIMP framework, we present a complete reduction of the required one- and two-loop amplitudes into a basis of heavy-particle loop integrals with nonzero residual mass.

Having encoded physics of the heavy WIMP sector in matching coefficients $c_i^{(5)}(\mu_t)$, the remaining analysis, presented in Chap. 5, is independent of the $M \gg m_W$ assumption, and consists of renormalization group (RG) running to a low scale $\mu_0 < m_c$, matching at heavy quark thresholds, and evaluating hadronic matrix elements. This module is systematically improvable in subleading corrections and is applicable to generic direct detection calculations. We present the required renormalization analysis of bare coefficients obtained from matching, solutions for the RG evolution of coefficients between particle thresholds (e.g., mapping $c_i^{(5)}(\mu_t)$ onto $c_i^{(5)}(\mu_b)$), and the matching between $(n_f + 1)$ -flavor and n_f -flavor QCD at heavy quark scales (e.g., mapping $c_i^{(5)}(\mu_b)$ onto $c_i^{(4)}(\mu_b)$). We also perform a careful analysis of the necessary hadronic matrix elements $\langle \mathcal{O}_i \rangle$ at the low scale $\mu_0 \sim m_N$. This represents a robust treatment of QCD corrections when passing from a theory renormalized at the electroweak scale, given by $c_i^{(5)}(\mu_t)$, to a low-energy theory of quarks and gluons, given by $c_i^{(3)}(\mu_0)$.

In Chap. 6, we assemble the formalism and results of Chaps. 4 and 5 for making robust cross section predictions for heavy WIMPs scattering with nucleons. We study the various perturbative contributions and their uncertainties, and trace the evolution of coefficients from the high to low scale, paying close attention to

the effects of renormalization, heavy quark thresholds and to the sizes of matrix elements. We present cross section predictions, σ_{SI} , for spin-independent low-velocity scattering on a proton for pure- and mixed-state WIMPs. As in Fig. 1.2, we investigate the impact on scattering cross sections of model assumptions, such as the choice of WIMP quantum numbers and extended Higgs sectors, and of Standard Model inputs, such as perturbative QCD corrections and nucleon matrix elements.

Chapter 2

Heavy-Particle Spacetime Symmetries and Building Blocks

What is heavy particle effective theory? Imagine a charged particle ϕ with mass M in its rest frame, interacting with other particles of energy $\Lambda \ll M$ through exchange of photons. The momentum transfer in such collisions are of order Λ , allowing the heavy particle to be off-shell by only that amount. Hence, to a good approximation, the heavy particle remains at rest and appears to the light degrees of freedom as a static source of charge. In this situation, we may expand the momentum of ϕ about its large component as

$$p^\mu = Mv^\mu + k^\mu, \quad (2.1)$$

where v^μ is the velocity of ϕ (e.g., $v^\mu = (1, 0, 0, 0)$ in the rest frame), and the residual momentum k^μ is $\mathcal{O}(\Lambda)$ and accounts for the dynamics of ϕ off-shell. Hence, the “hard” energy scale M (equivalently, the momentum scale Mv^μ) is irrelevant to the system, and we may integrate it out by passing from a fully relativistic description of ϕ to a heavy particle description where only the “soft” momentum mode, k^μ , remains.

At leading-order in $1/M$, the theory describing ϕ is

$$\mathcal{L} = \bar{\phi}_v i v \cdot D \phi_v + \dots, \quad (2.2)$$

where D^μ is the covariant derivative including the gauge interaction. The heavy particle field ϕ_v , derived from the full relativistic field ϕ , is labelled by its velocity v (and has mass dimension $3/2$). The ellipsis in Eq. (2.2) denotes interactions higher order in the heavy mass expansion, which may be systematically included to meet the demands of precision, depending on the size of Λ/M . For example, ϕ ’s interactions with photons, through $\mathcal{O}(1/M)$, are given by

$$\mathcal{L} = \bar{\phi}_v \left\{ i v \cdot D + c_1 \frac{D_\perp^2}{2M} + c_F \frac{\sigma_{\mu\nu}^\perp F^{\mu\nu}}{4M} + c'_F \frac{\tilde{\sigma}_{\mu\nu}^\perp F^{\mu\nu}}{4M} + \dots \right\} \phi_v, \quad (2.3)$$

where $\tilde{T}^{\mu\nu} = T_{\alpha\beta}\epsilon^{\alpha\beta\mu\nu}$ and $a_{\perp}^{\mu} = a^{\mu} - v^{\mu}v \cdot a$. The higher-order kinetic, magnetic dipole and electric dipole terms are parameterized by Wilson coefficients c_1 , c_F and c'_F , respectively. From a low-energy perspective, these higher-dimension operators represent new physics, albeit not new particles but new interactions. The higher dimension operators depend on additional microscopic properties of the particle ϕ such as its spin, its mass and its electromagnetic moments.

The purpose of such a framework, as with any useful effective theory, is for controlled calculations of physical observables. Heavy particle effective theory systematically captures the dominant interactions at each order in $1/M$, and at lower orders, in particular, the few operators appearing are remarkably simple. These simplifications are tied to new symmetries arising in the heavy particle limit, and allow for tractable analysis of otherwise complicated processes. Precise calculations, including the resummation of large logarithms $\sim \log \frac{M}{\Lambda}$, are also made possible by the separation of scales Λ and M through the effective theory. Heavy particle methods find a wide range of applications in particle, nuclear and atomic physics [21, 40, 70, 89].

So far we have sketched heavy particle effective theory as a framework for describing a massive particle of energy M interacting with degrees of freedom having much smaller energy $\Lambda \ll M$, by systematic expansion in Λ/M . To fully answer “what is heavy particle effective theory?” we must precisely state how such a theory is constructed. A closely related question is “what are the spacetime symmetries of the Lagrangian in Eq. (2.3)?” From the intuitive picture of a heavy particle in its rest frame, rotational invariance is manifest, but what about boost invariance? With the appearance of the vector v^{μ} in the Lagrangian, it is unclear how to implement Lorentz transformations.

When the underlying UV theory for the heavy particle is known, we may derive the effective theory Lagrangian by introducing a field redefinition in the full theory. For example, in terms of an arbitrary (spacetime independent) time-like unit vector v^{μ} , the decomposition of a quark field $Q(x)$ of mass M ,

$$Q(x) = e^{-iMv \cdot x} [h_v(x) + H_v(x)] , \quad (2.4)$$

with $\not{v}h_v = h_v$ and $\not{v}H_v = -H_v$, defines an effective heavy quark field $h_v(x)$, and after integrating out the antiparticle field $H_v(x)$, we arrive at the effective Lagrangian for a heavy quark,

$$\bar{Q}(i\not{D} - M)Q \rightarrow \bar{h}_v i v \cdot D h_v + \mathcal{O}(1/M). \quad (2.5)$$

Invariance of observables under small changes of v , so-called “reparameterization invariance”, enforces certain constraints on the coefficients of the effective Lagrangian [84]. These constraints are consistent with the requirements of Lorentz invariance, e.g. as imposed by matching effective theory S matrix elements to Lorentz-invariant full theory S matrix elements. However, this construction raises several questions. Is reparameterization invariance a sufficient condition for Lorentz invariance? How do we derive a reparameterization transformation law without

first constructing the underlying theory and explicitly integrating out degrees of freedom? For applications involving a composite particle such as the proton, or hypothetical new particles such as dark matter whose underlying theory is unknown, we cannot in an obvious way introduce v as a parameter inside of a field redefinition. What is the significance of v in such cases? What is the general method for constructing a Lorentz invariant heavy particle effective field theory?

In this chapter we present the formalism of induced representations of the Lorentz group, Wigner’s “little group” construction [103], for application to field transformation laws. The parameter v enters as an arbitrary reference vector in the little group construction. The relationship between Lorentz invariance and reparameterization invariance is stated precisely, and a class of allowable reparameterization transformations is obtained. We find that a standard ansatz for implementing reparameterization invariance breaks down starting at order $1/M^4$. We explain this subtlety and its resolution. A large literature exists on topics relating to reparameterization invariance, especially as applied to heavy quark Lagrangians [16, 44, 75, 80, 83, 84, 86, 98]. We aim to present a conceptually clear statement of the constraints imposed by Lorentz invariance on heavy particle effective theories.

Recent investigations using heavy particle effective theory demand high orders in the $1/M$ expansion (see e.g. [17, 58, 62]). To avoid a proliferation of undetermined constants, and to enable efficient computations, it is important to recognize that many Wilson coefficients are linked by Lorentz invariance to coefficients appearing at lower orders. At a practical level, we derive the explicit field transformation laws that can be consistently used to build Lorentz invariant Lagrangians, providing a complete solution for Wilson coefficient constraints, to arbitrary order in $1/M$.

While a top-down derivation of heavy particle effective theory, e.g., starting from QCD to derive heavy quark effective theory (HQET) through the field redefinition in (2.4), must map onto the framework presented here, our construction does not rely on knowing the underlying ultraviolet completion, or on explicitly integrating out degrees of freedom. A bottom-up construction is key for applications of heavy particle effective theory to dark matter. Let us summarize the main points for general construction of heavy particle effective theories:

1. A heavy-particle field h_v is identified with a representation of the little group for massive particles, determining its field transformation laws under rotations and boosts. It carries a label v associated with the time-like unit vector v^μ that defines the little group (cf. Sects. 2.2 and 2.3).
2. The little group for massive particles is isomorphic to $SO(3)$, and therefore has field representations carrying spin $s = 0, 1/2, 1, \dots$. A heavy particle field of arbitrary spin may be represented in covariant notation using a Dirac spinor-vector with appropriate constraints. For example, a spin-1/2 heavy-particle field has $2(1/2) + 1$ degrees of freedom and can be written as a Dirac spinor, h_v , obeying $\not{v}h_v = h_v$ as a projection constraint (cf. Sect. 2.4).
3. For a self-conjugate heavy particle we impose an additional parity equivalent to a modified CPT transformation (cf. Sect. 2.4).

Having determined the building blocks and their transformation laws under symmetries, interactions with heavy-particle fields can be constructed in the usual way: write down the most general set of gauge- and Lorentz-invariant operators in terms of heavy fields $h_v, h'_v, \dots$, the time-like unit vector v^μ , and other relativistic degrees of freedom (e.g., gauge fields, SM matter fields) up to a given order in the $1/M$ power counting. For self-conjugate fields the additional parity is imposed.

This rest of this chapter is organized as follows. In Sect. 2.1 we briefly review the construction of Lorentz invariant field theories based on finite dimensional representations of the Lorentz group. In Sect. 2.2 we introduce the formalism of induced representations and investigate the necessary conditions for a Lorentz invariant S matrix. Section 2.3 establishes the connection between Lorentz invariance and reparameterization invariance. A subtlety in the identification of allowable reparameterization transformations is explained, and a correct solution to the invariance equation (2.52) is found for applications to $1/M^4$ heavy fermion Lagrangians. Section 2.4 presents formalism for arbitrary-spin heavy particles and for heavy particles derived from self-conjugate relativistic fields. In Sects. 2.5–2.8 we apply the formalism for construction of the parity and time-reversal invariant effective Lagrangian for a heavy fermion interacting with an Abelian gauge field, i.e., NRQED, through order $1/M^4$. The implementation of Lorentz invariance in the effective theory becomes nontrivial at this order, and we obtain a complete solution for Wilson coefficient constraints using both variational and invariant operator methods (cf. Sect. 2.6). We also present results of one-photon matching which verify the coefficient constraints (cf. Sect. 2.7), and Lagrangians describing pure photon and four-fermion interactions (cf. Sect. 2.8). We close the chapter with a discussion in Sect. 2.9.

2.1 Finite Dimensional Representations of the Lorentz Algebra

The standard method for constructing Lorentz invariant Lagrangians postulates the field transformation law

$$\phi_a(x) \rightarrow M(\Lambda)_{ab} \phi_b(\Lambda^{-1}x), \quad (2.6)$$

where $M(\Lambda)$ is a finite dimensional (coordinate-independent and, in general, non-unitary) representation of the Lorentz group. In infinitesimal form, including also spacetime translations $\phi(x) \rightarrow \phi(x - a)$, we have

$$\delta\phi = i(a_0 h - \mathbf{a} \cdot \mathbf{p} - \boldsymbol{\theta} \cdot \mathbf{j} + \boldsymbol{\eta} \cdot \mathbf{k})\phi, \quad (2.7)$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are infinitesimal rotation and boost parameters, and the generators of the Poincaré group acting on fields are¹

$$h = i\partial_t, \quad (2.8a)$$

$$\mathbf{p} = -i\boldsymbol{\partial}, \quad (2.8b)$$

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + \boldsymbol{\Sigma}, \quad (2.8c)$$

$$\mathbf{k} = r h - t \mathbf{p} \pm i \boldsymbol{\Sigma}, \quad (2.8d)$$

with Σ^i the $(2s + 1)$ -dimensional matrix generators of the spin- s representation of rotations (e.g. for spin-1/2 Weyl fermions, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}/2$ with σ^i the Pauli matrices). Using (2.6) it is straightforward to construct Lorentz invariant actions, and correspondingly to prove Lorentz invariance of the S matrix. Let us briefly review this procedure.²

Recall the Poincaré algebra for generators of time translations H , space translations P^i , rotations J^i , and boosts K^i :

$$[H, P^i] = 0, \quad (2.9a)$$

$$[H, J^i] = 0, \quad (2.9b)$$

$$[P^i, P^j] = 0, \quad (2.9c)$$

$$[J^i, P^j] = i\epsilon^{ijk} P^k, \quad (2.9d)$$

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \quad (2.9e)$$

$$[J^i, K^j] = i\epsilon^{ijk} K^k, \quad (2.9f)$$

$$[H, K^i] = -iP^i. \quad (2.9g)$$

$$[P^i, K^j] = -iH\delta^{ij}, \quad (2.9h)$$

$$[K^i, K^j] = -i\epsilon^{ijk} J^k. \quad (2.9i)$$

Having built a Lagrangian that is invariant under (2.7), we may construct the corresponding conserved charges. Using (2.8), we find in canonical quantization that these charges obey the commutation relations (2.9).

Lorentz invariance of the S matrix demands that the free-particle charges, denoted by H_0 , $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{K}_0$, commute with the scattering operator, $S = \lim_{T \rightarrow \infty} \Omega(T)^\dagger \Omega(-T)$, where $\Omega(T) = e^{iHT} e^{-iH_0 T}$. We assume that momentum

¹In this chapter we use bold letters for Euclidean three-vectors, e.g. $\boldsymbol{\partial} = (\partial^i) = (\partial_i) = (\partial_x, \partial_y, \partial_z)$.

²For a pedagogical discussion, see [102].

and angular momentum operators for the interacting theory are unchanged from the free theory and furthermore demand translational and rotational invariance of the interaction

$$\mathbf{P} = \mathbf{P}_0, \quad \mathbf{J} = \mathbf{J}_0, \quad [H - H_0, \mathbf{P}_0] = [H - H_0, \mathbf{J}_0] = 0. \quad (2.10)$$

Then $[\mathbf{P}_0, S] = [\mathbf{J}_0, S] = 0$, and by the definition of S also $[H_0, S] = 0$. Finally, if one can show (2.9g) and that an asymptotic smoothness condition for $\Delta\mathbf{K} = \mathbf{K} - \mathbf{K}_0$ is obeyed, it follows that

$$\begin{aligned} [\mathbf{K}_0, S] &= \lim_{T \rightarrow \infty} [\mathbf{K}_0, \Omega(T)^\dagger \Omega(-T)] \\ &= \lim_{T \rightarrow \infty} \left\{ -[e^{iH_0 T} \Delta\mathbf{K} e^{-iH_0 T}] \Omega(T)^\dagger \Omega(-T) \right. \\ &\quad \left. + \Omega(T)^\dagger \Omega(-T) [e^{-iH_0 T} \Delta\mathbf{K} e^{iH_0 T}] \right\} = 0, \end{aligned} \quad (2.11)$$

completing the proof of the Lorentz invariance of the S -matrix. For later application, we note that of the commutation relations involving $\mathbf{K}$, it is only necessary to show the relation (2.9g); relations (2.9f), (2.9h) and (2.9i) are not required to complete the proof.³

2.2 Effective Field Theory and the Little Group

The field transformation law (2.6), based on finite dimensional representations of the Lorentz group, is not suitable for heavy particle effective field theories. For example, the associated irreducible representations of the Lorentz group are chiral, in conflict with the low-energy limit of a parity conserving theory such as QED or QCD. Let us consider instead the class of infinite dimensional induced representations. We first review their appearance in transformations of physical states, and then apply them as transformations acting on fields.

2.2.1 Little Group Formalism

Consider Lorentz transformations acting on the Hilbert space of physical states for a spin- s particle of mass M . These transformations are implemented by an induced representation [103]. In terms of a fixed timelike reference vector v^μ

³In fact, these relations *do* follow from the observation that having proven Lorentz invariance of the S matrix, it can be shown that $H, \mathbf{P}, \mathbf{J}$ and $\mathbf{K}$ are related to their free counterparts by the similarity transformation $\Omega(\pm\infty)$ [102].

(we assume $v^2 = 1$), define the associated “little group” as the subgroup of Lorentz transformations leaving v invariant, $\Lambda v = v$. The little group for massive particles is isomorphic to $SO(3)$, the group of rotations. Let $L(p)$ denote a standard Lorentz transformation taking Mv to p , yielding a (momentum-dependent) mapping of the Lorentz group into the little group,

$$\Lambda \rightarrow W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p). \quad (2.12)$$

We may define physical states to transform schematically as

$$|p, m\rangle \rightarrow U(\Lambda, p)|p, m\rangle = \sum_{m'=-s}^s D_{m'm}[W(\Lambda, p)]|\Lambda p, m'\rangle, \quad (2.13)$$

where $p^0 = \sqrt{M^2 + \mathbf{p}^2}$, and $D(W)$ is a spin- s representation matrix for rotations. A representation for the little group thus induces a representation for the full Lorentz group.

A convenient choice for the standard Lorentz transformation is $L(p) = \Lambda(p/M, v)$, where $\Lambda(w, v)$ denotes the generalized rotation in the plane of the unit vectors v and w such that $\Lambda(w, v)v = w$. This matrix is given by $\Lambda(w, v) = \exp[-i\theta \mathcal{J}_{\alpha\beta} w^\alpha v^\beta]$, with the Lorentz generators $\mathcal{J}_{\alpha\beta}$ defined in Eq. (2.65) and the angle θ chosen appropriately [84]. In the vector and spinor representations we have, respectively

$$\begin{aligned} \Lambda(w, v)^\mu{}_\nu &= g^\mu{}_\nu - \frac{1}{1 + v \cdot w} (w^\mu w_\nu + v^\mu v_\nu) \\ &\quad + w^\mu v_\nu - v^\mu w_\nu + \frac{v \cdot w}{1 + v \cdot w} (w^\mu v_\nu + v^\mu w_\nu), \end{aligned} \quad (2.14a)$$

$$\Lambda_{\frac{1}{2}}(w, v) = \frac{1 + \not{w}\not{v}}{\sqrt{2(1 + v \cdot w)}}. \quad (2.14b)$$

It is straightforward to verify that for elements of the little group, i.e. “rotations” with $\mathcal{R}v = v$, this choice of $L(p)$ implies

$$W(\mathcal{R}, p) = \mathcal{R}, \quad (2.15)$$

a property that greatly simplifies the construction of invariant Lagrangians, cf. Sects. 2.2.3, 2.3.1 and 2.3.2 below. Other choices of $L(p)$ do not share this property. For example, suppose that we introduce a spacelike vector s^μ with $s^2 = -1$. Then we may define $L'(p) = R(p)B(p)$, with $B(p)$ a boost taking Mv^μ to $MB(p)^\mu{}_\nu v^\nu = (v \cdot p)v^\mu + \sqrt{(v \cdot p)^2 - M^2}s^\mu$, and $R(p)$ a rotation taking $MB(p)^\mu{}_\nu v^\nu$ to p^μ . Such an $L'(p)$ provides a simple interpretation of $U[L(p)]|Mv, m\rangle$ in terms of helicity eigenstates (note that the spacelike vector is required to define a direction for helicity decomposition), but this consideration is secondary to the simplicity of (2.15) for our present purposes.

The remaining independent Lorentz generators represent “boosts” that shift v . They can be chosen as $\mathcal{B}(q) = \Lambda(v - q/M, v)$ with $(v - q/M)^2 = 1$. The appearance of the $1/M$ factor in $v - q/M$ will be explained in Sect. 2.2.3 below. For an infinitesimal momentum q , which obeys $v \cdot q = \mathcal{O}(q^2)$, these boosts are given by

$$\mathcal{B}(q)^\mu{}_\nu = g^\mu{}_\nu + \frac{v^\mu q_\nu - q^\mu v_\nu}{M} + \mathcal{O}(q^2), \quad (2.16a)$$

$$\mathcal{B}_{\frac{1}{2}}(q) = 1 - \frac{\not{q}\not{v}}{2M} + \mathcal{O}(q^2). \quad (2.16b)$$

For the transformation (2.13), we find

$$W(\mathcal{B}(q), p) = 1 - \frac{i}{2} \left[\frac{1}{M(M + v \cdot p)} (q^\alpha p_\perp^\beta - p_\perp^\alpha q^\beta) \right] \mathcal{J}_{\alpha\beta} + \mathcal{O}(q^2), \quad (2.17)$$

where for any four-vector k we define $k_\perp^\mu \equiv k^\mu - (v \cdot k)v^\mu$.

2.2.2 Field Transformation Law and Lorentz Invariance

In place of (2.6) let us postulate the transformation law for free massive fields,

$$\phi_a(x) \rightarrow D[W(\Lambda, i\partial)]_{ab} \phi_b(\Lambda^{-1}x). \quad (2.18)$$

For notational simplicity consider the special choice $v = (1, 0, 0, 0)$. Equation (2.18) together with Eq.(2.17) corresponds to replacing the boost generator (2.8d) by⁴

$$\mathbf{k} = \mathbf{r}h - \mathbf{t}p \pm i \frac{\boldsymbol{\Sigma} \times \boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}. \quad (2.19)$$

The generators (2.8a)–(2.8c) together with (2.19) will satisfy the Poincaré algebra when acting on fields satisfying

$$i\partial_t \phi = \pm \sqrt{M^2 - \boldsymbol{\partial}^2} \phi. \quad (2.20)$$

It follows that the conserved charges derived from a free field Lagrangian invariant under (2.18) will satisfy (2.9).

⁴For spin-1/2 particles, (2.19) may also be obtained by performing a Foldy-Wouthuysen transformation on Eq. (2.8d) [45].

In contrast to (2.6), transformation (2.18) acts on the field coordinates, spoiling gauge invariance. To include gauge interactions, we promote the partial derivatives in (2.18) to covariant derivatives $D_\mu = \partial_\mu - igA_\mu^A t^A \equiv \partial_\mu - igA_\mu$,

$$\phi_a(x) \rightarrow D[W(\Lambda, iD)]_{ab} \phi_b(\Lambda^{-1}x), \quad (2.21)$$

and correspondingly the infinitesimal generators become

$$h = i\partial_t, \quad (2.22a)$$

$$\mathbf{p} = -i\boldsymbol{\partial}, \quad (2.22b)$$

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + \boldsymbol{\Sigma}, \quad (2.22c)$$

$$\mathbf{k} = \mathbf{r}h - \mathbf{t}\mathbf{p} \pm i \frac{\boldsymbol{\Sigma} \times \mathbf{D}}{M + \sqrt{M^2 - \mathbf{D}^2}} + \mathcal{O}(g). \quad (2.22d)$$

In the expansion of $\mathbf{D}/(M + \sqrt{M^2 - \mathbf{D}^2})$ we assume a choice of ordering for the covariant derivatives. The $\mathcal{O}(g)$ terms in $\mathbf{k}$ denote field strength-dependent corrections that vanish for the non-interacting theory (i.e. $g \rightarrow 0$). Such $\mathcal{O}(g)$ terms can be introduced so that the resulting invariant Lagrangian is in “canonical form”, i.e. where the only time derivative acting on ϕ appears in the leading term,

$$\mathcal{L} = \bar{\phi}(iD_t + \dots)\phi. \quad (2.23)$$

The existence of suitable field strength-dependent terms, ensuring a boost generator $\mathbf{k}$ which yields a non-zero invariant Lagrangian, is implied by the all-orders construction in Sect. 2.3 and Appendix A. The explicit form of these corrections is not required for the following argument.

Although the field-dependent generators (2.22) do not obey simple commutation relations, we may nevertheless show that the S matrix derived from the resulting invariant action is Lorentz invariant (and hence that the conserved charges in the interacting theory satisfy the Poincare algebra). To see this, we assume as before the relations (2.10). Relation (2.9g) is satisfied if the explicit time dependence of the conserved charge $\mathbf{K}$ satisfies $\partial\mathbf{K}/\partial t = -\mathbf{P}$, so that

$$0 = \frac{d}{dt}\mathbf{K} = \frac{\partial}{\partial t}\mathbf{K} + i[H, \mathbf{K}] = -\mathbf{P} + i[H, \mathbf{K}]. \quad (2.24)$$

The fact that $\partial\mathbf{K}/\partial t = -\mathbf{P}$ follows from the assumed form of the infinitesimal generators (2.22). For the boost $\phi \rightarrow (1 + i\boldsymbol{\eta} \cdot \mathbf{k})\phi$, we find the conserved charge⁵

⁵The first ellipsis in (2.25) includes possible contributions from a surface term in $\delta\mathcal{L}$, which do not affect the term with explicit t dependence in (2.25).

$$\mathbf{K} = \sum_{\phi} i \int d^3x \frac{\delta \mathcal{L}}{\delta \dot{\phi}} \mathbf{k} \phi + \dots = \sum_{\phi} i \int d^3x \frac{\delta \mathcal{L}}{\delta \dot{\phi}} [-t\mathbf{p}] \phi + \dots = -t\mathbf{P} + \dots \quad (2.25)$$

Here the important point is that the remaining terms have no *explicit* time dependence, so that (2.24) follows.

Let us close this section with two comments. First, the choice $v = (1, 0, 0, 0)$ is not essential to the argument. The generators for arbitrary v can be obtained by a coordinate change using a boost which takes $(1, 0, 0, 0)$ to v . While the resulting explicit expressions for rotation and boost generators become more complicated, the demonstration of Lorentz invariance is not essentially changed. Second, having specified an ordering for covariant derivatives appearing in the boost generator $\mathbf{k}$, additional field strength-dependent corrections are determined at each order in $1/M$ by enforcing that the resulting invariant Lagrangian is in canonical form. We illustrate this with an explicit example in the following subsection. The existence of such a generator is implied by the analysis of Sect. 2.3 and Appendix A.

2.2.3 $1/M$ Expansion and Lagrangian Constraints

To enable the $1/M$ expansion we extract the rest mass by the field redefinition,

$$\phi(x) = e^{-iMt} \phi'(x). \quad (2.26)$$

In phenomenological applications it is also convenient to work with non-relativistic field normalization

$$\phi'(x) = \left(\frac{M^2}{M^2 - \mathbf{D}^2} \right)^{\frac{1}{4}} \phi''(x). \quad (2.27)$$

We enforce invariance under (2.22a), (2.22b) and (2.22c) by ensuring translational invariance (no explicit factors of x^μ) and rotational invariance. For the boost transformation (2.22d) we use $\boldsymbol{\eta} = -\mathbf{q}/M$ in (2.7) to preserve the power counting $D_t = \mathcal{O}(1/M)$ in (2.29). This explains the appearance of $1/M$ in (2.16). The resulting $1/M$ expansion becomes⁶

$$\phi'' \rightarrow e^{-iq \cdot x} \left\{ 1 + \frac{i\mathbf{q} \cdot \mathbf{D}}{2M^2} + \frac{i\mathbf{q} \cdot \mathbf{D}\mathbf{D}^2}{4M^4} - \frac{\boldsymbol{\Sigma} \times \mathbf{q} \cdot \mathbf{D}}{2M^2} \left[1 + \frac{\mathbf{D}^2}{4M^2} \right] + \mathcal{O}(g, 1/M^5) \right\} \phi''. \quad (2.28)$$

⁶For notational clarity we leave the coordinate change $x \rightarrow x' = \mathcal{B}^{-1}x$ implicit and suppress primes on coordinates and derivatives in (2.28) and (2.29).

Gauge fields are assumed to transform as usual, in the vector representation of the Lorentz group. Combined with derivatives acting on the transformed coordinate in (2.28), we have

$$D_t \rightarrow D_t + \frac{1}{M} \mathbf{q} \cdot \mathbf{D}, \quad \mathbf{D} \rightarrow \mathbf{D} + \frac{1}{M} \mathbf{q} D_t. \quad (2.29)$$

To illustrate the constraints, consider the canonical form of the Abelian gauged heavy spin-1/2 fermion effective Lagrangian (i.e., NRQED) through $\mathcal{O}(1/M^3)$. Identifying $\phi'' = \psi$ as a two-component spinor and setting $g = -e$ we obtain [77, 86]

$$\begin{aligned} \mathcal{L} = \psi^\dagger \bigg\{ & iD_t + c_2 \frac{\mathbf{D}^2}{2M} + c_4 \frac{\mathbf{D}^4}{8M^3} + c_F e \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2M} + c_D e \frac{[\boldsymbol{\partial} \cdot \mathbf{E}]}{8M^2} + i c_S e \frac{\boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8M^2} \\ & + c_{W1} e \frac{\{\mathbf{D}^2, \boldsymbol{\sigma} \cdot \mathbf{B}\}}{8M^3} - c_{W2} e \frac{D^i \boldsymbol{\sigma} \cdot \mathbf{B} D^i}{4M^3} + c_{p'p} e \frac{\boldsymbol{\sigma} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \mathbf{D}}{8M^3} \\ & + i c_M e \frac{\{\mathbf{D}^i, [\boldsymbol{\partial} \times \mathbf{B}]^i\}}{8M^3} + c_{A1} e^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8M^3} - c_{A2} e^2 \frac{\mathbf{E}^2}{16M^3} + \mathcal{O}(1/M^4) \bigg\} \psi. \end{aligned} \quad (2.30)$$

Here we have defined $E^i = (-i/e)[D_t, D^i]$, $\epsilon^{ijk} B^k \equiv (i/e)[D^i, D^j]$. Under (2.28), a straightforward computation yields

$$\delta \mathcal{L} = \frac{1}{M} \delta \mathcal{L}_1 + \frac{1}{M^2} \delta \mathcal{L}_2 + \frac{1}{M^3} \delta \mathcal{L}_3 + \dots, \quad (2.31)$$

where using $\boldsymbol{\Sigma} = \boldsymbol{\sigma}/2$ in (2.28),

$$\delta \mathcal{L}_1 = \psi^\dagger [(1 - c_2) i \mathbf{q} \cdot \mathbf{D}] \psi, \quad (2.32a)$$

$$\delta \mathcal{L}_2 = \psi^\dagger \left[-\frac{1}{2} (1 - c_2) \{ \mathbf{q} \cdot \mathbf{D}, D_t \} + \frac{e}{4} (1 - 2c_F + c_S) \boldsymbol{\sigma} \times \mathbf{q} \cdot \mathbf{E} \right] \psi, \quad (2.32b)$$

$$\delta \mathcal{L}_3 = \psi^\dagger \left[\frac{e}{8} c_D [D_t, \mathbf{q} \cdot \mathbf{E}] + \frac{e}{8} (c_F - c_D + 2c_M) \mathbf{q} \cdot [\boldsymbol{\partial} \times \mathbf{B}] + \frac{i}{4} (c_2 - c_4) \{ \mathbf{q} \cdot \mathbf{D}, \mathbf{D}^2 \} \right. \quad (2.32c)$$

$$\begin{aligned} & + \frac{ie}{8} c_S \{ D_t, \boldsymbol{\sigma} \times \mathbf{q} \cdot \mathbf{E} \} + \frac{ie}{8} (c_2 + 2c_F - c_S - 2c_{W1} + 2c_{W2}) \{ \mathbf{q} \cdot \mathbf{D}, \boldsymbol{\sigma} \cdot \mathbf{B} \} \\ & + \frac{ie}{8} (-c_2 + c_F - c_{p'p}) \{ \boldsymbol{\sigma} \cdot \mathbf{D}, \mathbf{q} \cdot \mathbf{B} \} \\ & \left. + \frac{ie}{8} (-c_F + c_S - c_{p'p}) \mathbf{q} \cdot \boldsymbol{\sigma} (\mathbf{D} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{D}) \right] \psi. \end{aligned}$$

From $\delta\mathcal{L}_1$ and $\delta\mathcal{L}_2$, we find

$$c_2 = 1, \quad c_S = 2c_F - 1. \quad (2.33)$$

The variation $\delta\mathcal{L}_3$ is equivalent to zero upon a field strength-dependent modification of the boost transformation (2.28),

$$\begin{aligned} \psi(x) \rightarrow e^{-iq \cdot x} \left\{ 1 + \frac{iq \cdot D}{2M^2} - \frac{\sigma \times q \cdot D}{4M^2} + \frac{ic_D}{8M^3} eq \cdot E \right. \\ \left. + \frac{c_S}{8M^3} eq \cdot \sigma \times E + \mathcal{O}\left(\frac{1}{M^4}\right) \right\} \psi(\mathcal{B}^{-1}x), \end{aligned} \quad (2.34)$$

and upon enforcing the constraints [58, 86]

$$c_4 = 1, \quad 2c_M = c_D - c_F, \quad c_{W2} = c_{W1} - 1, \quad c_{p'p} = c_F - 1. \quad (2.35)$$

The computation of the complete Lagrangian at $\mathcal{O}(1/M^4)$ is presented in Sect. 2.6.

2.3 Reparametrization Invariance and Invariant Operators

While in practice it may be convenient to enforce Lorentz invariance only after expanding the Lagrangian in a series of rotationally-invariant, but not Lorentz invariant, operators, it is interesting to consider formalism that permits an explicitly Lorentz-invariant construction. This formalism also addresses the question of existence of a suitable boost generator, extending (2.34) to arbitrary order in $1/M$.

This section begins by introducing covariant notation that can either be used in place of the $v = (1, 0, 0, 0)$ formalism above, or used to construct manifestly invariant operators. The relation between Lorentz invariance and reparameterization invariance is then demonstrated, and a general discussion of the invariant operator method is presented. In particular, we derive the necessary invariance equation (2.52) and present the solution to order $1/M^3$. A systematic, all-orders solution of the invariance equation is given in Appendix A.

2.3.1 Covariant Notation

The formalism of Sect. 2.4 allows us to straightforwardly extend the discussion to a general reference vector v and to arbitrary spin. Consider a term in the Lagrangian of the schematic form

$$\bar{\phi}_v \left\{ \dots v^\mu \dots D^\mu \dots \gamma^\mu \dots \right\} \phi_v, \quad (2.36)$$

where indices are contracted with $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. Invariance under generalized rotations of such a term in the action follows using the field transformation (2.15),

$$\phi_v(x) \rightarrow \mathcal{R}\phi_v(x'), \quad (2.37)$$

where $x' \equiv \mathcal{R}^{-1}x$. The transformation of the derivative and the gauge field are as usual,

$$\partial^\mu \rightarrow \partial^\mu = \mathcal{R}^\mu_\nu \partial'^\nu, \quad A^\mu \rightarrow \mathcal{R}^\mu_\nu A'^\nu(x'). \quad (2.38)$$

If the Lagrangian is already constructed such that all vector and spinor indices are contracted in (2.36), we can easily see that the Lagrangian is invariant under generalized rotations using the identities

$$v^\mu = \mathcal{R}^\mu_\nu v'^\nu, \quad \gamma^\mu = \mathcal{R}_{\frac{1}{2}} (\mathcal{R}^\mu_\nu \gamma'^\nu) \mathcal{R}_{\frac{1}{2}}^{-1}. \quad (2.39)$$

According to (2.17), the infinitesimal boosts are implemented by

$$\phi_v(x) \rightarrow W(\mathcal{B}, iD)\phi_v(x'), \quad (2.40)$$

where $x' \equiv \mathcal{B}^{-1}x$, together with the transformation of the derivative and gauge field,

$$\partial^\mu \rightarrow \partial^\mu = \mathcal{B}^\mu_\nu \partial'^\nu, \quad A^\mu(x) \rightarrow \mathcal{B}^\mu_\nu A'^\nu(x'). \quad (2.41)$$

We may proceed as in Sect. 2.2.3 above to construct invariant combinations of Lagrangian interactions of the form (2.36), order by order in $1/M$.

As an explicit example, let us focus presently on the phenomenologically important one-heavy particle sector of a spin-1/2 theory. To enable the $1/M$ expansion and convert to non-relativistic normalization, we introduce the field redefinition as in (2.26) and (2.27),

$$\psi_v(x) = e^{-iMv \cdot x} N(v, iD) \psi'_v(x), \quad N(v, iD) = \left(\frac{M^2}{M^2 + D_\perp^2} \right)^{\frac{1}{4}}. \quad (2.42)$$

The boost transformation (2.40) becomes

$$\psi'_v \rightarrow e^{iq \cdot x} \tilde{W}_{\frac{1}{2}}(\mathcal{B}, iD + Mv) \psi'_v, \quad (2.43)$$

where

$$\tilde{W}(\mathcal{B}, iD + Mv) = N(v + q/M, iD - q)^{-1} W(\mathcal{B}, iD + Mv) N(v, iD). \quad (2.44)$$

The $1/M$ expansion of this transformation is the extension to arbitrary v , for spin-1/2, of the previous (2.28):

$$\psi'_v \rightarrow e^{iq \cdot x} \left\{ 1 + \frac{iq \cdot D_\perp}{2M^2} - \frac{iq \cdot D_\perp D_\perp^2}{4M^4} + \frac{1}{4M^2} \sigma_{\alpha\beta} q^\alpha D_\perp^\beta \left[1 - \frac{D_\perp^2}{4M^2} \right] + \mathcal{O}(g, 1/M^5) \right\} \psi'_v. \quad (2.45)$$

Similarly, we find the extension to arbitrary v of the transformations (2.29)

$$v \cdot D \rightarrow v \cdot D + \frac{1}{M} q \cdot D_{\perp}, \quad D_{\perp}^{\mu} \rightarrow D_{\perp}^{\mu} - \frac{1}{M} q^{\mu} (v \cdot D). \quad (2.46)$$

Using these transformations one can build an invariant Lagrangian, which (in the Abelian case) is equivalent to the extension of the Lagrangian (2.30) to arbitrary v with the same constraints (2.33) and (2.35).

2.3.2 Reparametrization Invariance

We can reformulate the transformation law for generalized boosts by using the identities,

$$v^{\mu} = \mathcal{B}_{\nu}^{\mu} (\mathcal{B}^{-1})^{\nu}_{\rho} v^{\rho} \equiv \mathcal{B}_{\nu}^{\mu} w^{\nu}, \quad \gamma^{\mu} = \mathcal{B}_{\frac{1}{2}}^{\mu} (\mathcal{B}_{\nu}^{\mu} \gamma^{\nu}) \mathcal{B}_{\frac{1}{2}}^{-1}. \quad (2.47)$$

In place of (2.40) and (2.41) the transformation of any operator of the form (2.36) is identical to the transformation obtained by the substitutions

$$v \rightarrow w = v + q/M, \quad \phi_v \rightarrow \phi_w \equiv \mathcal{B}^{-1} W(\mathcal{B}, iD_{\mu}) \phi_v, \quad (2.48)$$

with no transformation of the coordinate and gauge field. The rules (2.48), with suitable choice for W , may be identified with the rules obtained by enforcing “reparameterization invariance” [84]. However, we emphasize that from the present perspective, we are not changing the reference vector v , but simply noticing the equivalence of (2.40) and (2.41) on the one hand, and (2.48) on the other hand, when acting on operators of the form (2.36).

2.3.3 Invariant Operator Method

It is not obvious that a non-zero Lagrangian, invariant under (2.40) and (2.41) to arbitrary order, will exist. For example, in (2.31) invariance relies on the possibility to enforce $\delta \mathcal{L}_n = 0$ by modifying the boost generator as in (2.34) and enforcing relations as in (2.33) and (2.35). It is not evident that this procedure can be extended to arbitrary order. We present here a method of constructing operators that are manifestly invariant under a particular choice of boost generator, to arbitrary order in $1/M$. The details of the construction are given in Appendix A.

The embedding of the little group into constrained representations of the full Lorentz group (cf. Sect. 2.4) provides a framework for constructing explicitly invariant operators. Suppose that we find an operator $\Gamma(v, iD)$ such that

$$\Gamma(\Lambda^{-1}v, iD) \Lambda^{-1} W(\Lambda, iD) = \Gamma(v, iD), \quad (2.49)$$

when acting on fields ϕ_v obeying the appropriate constraints, as given in Sect. 2.4 (e.g. $\not{\phi}\phi_v = \phi_v$ for spin-1/2). It follows from the rules (2.48) that the combination

$$\Phi_v \equiv \Gamma(v, iD)\phi_v \quad (2.50)$$

is invariant under the reparameterization implementation (2.48) of generalized boosts. Provided that invariance under generalized rotations (2.37)–(2.39) is maintained, we may build operators that are explicitly invariant. For example, in the spin-1/2 case

$$\bar{\Psi}_v i\not{D} \Psi_v, \quad \bar{\Psi}_v \Psi_v, \quad \bar{\Psi}_v i\sigma^{\mu\nu}[D_\mu, D_\nu]\Psi_v, \quad (2.51)$$

are invariant. Note that because of Eq. (2.15) the only constraints on $\Gamma(v, iD)$ from Eq. (2.49) come from boosts $\Lambda = \mathcal{B}$.

Applying field redefinitions as in (2.42), the condition (2.49) for Γ becomes

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}\tilde{W}(\mathcal{B}, iD + Mv) = \Gamma(v, iD). \quad (2.52)$$

We will refer to (2.52) as the “invariance equation”. Provided that such a $\Gamma(v, iD)$ can be found, the field

$$\Phi'_v(x) \equiv \Gamma(v, iD)\phi'_v(x) \quad (2.53)$$

obeys a simple transformation law under the reparameterization implementation of generalized boosts (2.48),

$$\Phi'_v \rightarrow \Phi'_w \equiv e^{iq \cdot x} \Phi'_v. \quad (2.54)$$

Noting that $e^{-iq \cdot x}(iD^\mu + Mw^\mu)e^{iq \cdot x} = iD^\mu + Mv^\mu$, invariant operators may thus be built from contractions of polynomials of γ^μ and $v^\mu + iD^\mu/M$, between $\bar{\Phi}'_v$ and Φ'_v . For example in the spin-1/2 case,

$$\bar{\Psi}'_v(i\not{D} + M\not{v})\Psi'_v, \quad \bar{\Psi}'_v\Psi'_v, \quad \bar{\Psi}'_v i\sigma^{\mu\nu}[D_\mu, D_\nu]\Psi'_v, \quad (2.55)$$

are invariant.

2.3.4 Solution for $\Gamma(v, iD)$

The key element of the invariant operator construction is a solution of the invariance equation (2.52). Without loss of generality, let us set $N(v, iD) = 1$; the solution for general N can then be obtained by $\Gamma(v, iD) \rightarrow \Gamma(v, iD)N(v, iD)^{-1}$. The method presented can be easily extended to arbitrary spin, but for illustration we focus on the one-heavy particle sector of a spin-1/2 theory.

In order to obtain a solution in closed form for the free theory, and to make contact with previous work, it is convenient to take the free theory limit for $W_{\frac{1}{2}}(\mathcal{B}, i\partial + Mv)$ of the form [84]

$$\begin{aligned} W_{\frac{1}{2}}(\mathcal{B}, i\partial + Mv) &= \mathcal{B}_{\frac{1}{2}} \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\text{free}}, v + q/M)^{-1} \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\text{free}}, v) \\ &= 1 + \frac{1}{4M^2} \sigma_{\perp}^{\mu\nu} q_{\mu} \partial_{\nu} \left[1 - \frac{iv \cdot \partial}{M} + \frac{1}{M^2} \left((iv \cdot \partial)^2 - \frac{1}{4} (i\partial_{\perp})^2 \right) \right] \\ &\quad + \mathcal{O}(1/M^5), \end{aligned} \quad (2.56)$$

where $\Lambda_{\frac{1}{2}}(u, v)$ was defined in (2.14), $\mathcal{V}_{\text{free}}^{\mu} \equiv v^{\mu} + i\partial^{\mu}/M$ and $\hat{\mathcal{V}}_{\text{free}}^{\mu} \equiv \mathcal{V}_{\text{free}}^{\mu}/|\mathcal{V}_{\text{free}}|$. We have also used that $/v\psi_v = \psi_v$. Inspection of (2.52) shows that an all-orders solution can be written for Γ in the non-interacting theory,

$$\begin{aligned} \Gamma(v, i\partial) &= \Lambda_{\frac{1}{2}}(\hat{\mathcal{V}}_{\text{free}}, v) = 1 + \frac{i\partial_{\perp}}{2M} + \frac{1}{M^2} \left[-\frac{1}{8} (i\partial_{\perp})^2 - \frac{1}{2} i\partial_{\perp} iv \cdot \partial \right] \\ &\quad + \frac{1}{M^3} \left[\frac{1}{4} (i\partial_{\perp})^2 iv \cdot \partial + \frac{i\partial_{\perp}}{2} \left(-\frac{3}{8} (i\partial_{\perp})^2 + (iv \cdot \partial)^2 \right) \right] + \mathcal{O}(1/M^4). \end{aligned} \quad (2.57)$$

In the interacting theory it turns out that one cannot simply replace ∂ by D in (2.57) to obtain a solution for $\Gamma(v, iD)$. It is instead necessary to add specific field strength dependent terms, first to W (as in (2.58) and (A.2a) below) in order to satisfy consistency conditions, and then to Γ in order to solve the invariance equation (2.52). The computations of Appendix A show that a solution for $\Gamma(v, iD)$ will exist if we specify

$$W_{\frac{1}{2}}(\mathcal{B}, iD + Mv) = 1 + \frac{1}{4M^2} \sigma_{\mu\nu}^{\perp} q^{\mu} D^{\nu} \left(1 - \frac{iv \cdot D}{M} \right) + \mathcal{O}(1/M^4), \quad (2.58)$$

with (2.58) reducing to (2.56) at $g = 0$. Let us proceed through $\mathcal{O}(1/M^3)$, writing

$$\Gamma = 1 + \frac{1}{M} \Gamma^{(1)} + \frac{1}{M^2} \Gamma^{(2)} + \frac{1}{M^3} \Gamma^{(3)} + \dots, \quad (2.59)$$

and deriving a solution to the invariance equation (2.52) order by order in $1/M$. In Appendix A we present a systematic construction that extends the solution to arbitrary order.

Modulo terms that vanish when acting on ψ_v with $\not{v}\psi_v = \psi_v$, we find

$$\Gamma^{(1)} = \frac{1}{2} i\not{D}_{\perp}. \quad (2.60a)$$

$$\Gamma^{(2)} = -\frac{1}{8} (iD_{\perp})^2 - \frac{1}{2} i\not{D}_{\perp} iv \cdot D + gA\sigma^{\mu\nu} G_{\mu\nu} + gB\gamma^{\mu} v^{\nu} G_{\mu\nu}. \quad (2.60b)$$

$$\begin{aligned}\Gamma^{(3)} = & \frac{1}{4}(iD_\perp)^2 iv \cdot D + \frac{i\cancel{D}_\perp}{2} \left[-\frac{3}{8}(iD_\perp)^2 + (iv \cdot D)^2 \right] \\ & - \frac{g}{8} G_{\mu\nu} v^\mu D_\perp^\nu - \frac{g}{16} \sigma_\perp^{\mu\nu} G_{\mu\nu} i\cancel{D}_\perp, \end{aligned} \quad (2.60c)$$

where we define $[iD_\mu, iD_\nu] = igG_{\mu\nu}$. Starting at order $1/M^2$ the solution is not unique. However, since we will consider arbitrary factors of $\mathcal{V}^\mu \equiv v^\mu + iD^\mu/M$ when constructing invariant operators, we can set $A = B = 0$ by considering instead of Γ , the operator Γ' given by

$$\Gamma(v, iD) = (1 - iA\sigma_{\mu\nu}[\mathcal{V}^\mu, \mathcal{V}^\nu] - iB\gamma_\mu \mathcal{V}_\nu[\mathcal{V}^\mu, \mathcal{V}^\nu] + \dots) \Gamma'(v, iD). \quad (2.61)$$

Similarly, we have absorbed additional $1/M^3$ terms in (2.60c). The remaining terms in (2.60) have free derivatives D_μ acting to the right, and cannot be removed as in (2.61).

A complete basis of bilinears required through order $1/M^3$ is

$$\begin{aligned}\mathcal{L} = \bar{\Psi}_v \Big\{ & M(\mathcal{V} - 1) - a_F g \frac{\sigma^{\mu\nu} G_{\mu\nu}}{4M} + ia_D g \frac{\{\mathcal{V}_\mu, [M\mathcal{V}_\nu, G^{\mu\nu}]\}}{16M^2} \\ & - a_{W1} g \frac{[M\mathcal{V}^\alpha, [M\mathcal{V}_\alpha, \sigma^{\mu\nu} G_{\mu\nu}]]}{16M^3} \\ & + a_{A1} g^2 \frac{G_{\mu\nu} G^{\mu\nu}}{16M^3} + a_{A2} g^2 \frac{\mathcal{V}_\alpha G^{\mu\alpha} G_{\mu\beta} \mathcal{V}^\beta}{16M^3} \Big\} \Psi_v. \end{aligned} \quad (2.62)$$

Performing field redefinitions to arrive at canonical form, we recover the result (2.30) with constraints (2.33) and (2.35). The computation at $\mathcal{O}(1/M^4)$ is presented in Sect. 2.6. We may perform a similar computation for heavy vector particles (or particles of arbitrary spin), and/or enforce constraints appropriate to self-conjugate fields (cf. Sect. 2.4).

The passage from (2.57) to (2.60) is not as simple as previously envisaged [84, 86], and careful attention must be paid to the interplay of Lorentz and gauge symmetry. The computations in Appendix A show that an arbitrary “covariantization” of (2.56) does *not* solve the invariance equation (2.52). The covariant little group element $W(\mathcal{B}, iD + Mv)$ must satisfy consistency conditions for a solution to exist, and specific field strength dependent terms, such as those appearing in (2.60c), are necessary in order that $\Gamma(v, iD)$ satisfy the resulting invariance equation (2.52). These considerations have previously been overlooked [84, 86]. For example, a naive covariantization of Eq. (2.57),

$$\begin{aligned}\Gamma^{\text{naive}}(v, iD) = & 1 + \frac{i\cancel{D}_\perp}{2M} + \frac{1}{M^2} \left[-\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\cancel{D}_\perp iv \cdot D \right] \\ & + \frac{1}{M^3} \left[\frac{1}{4}(iD_\perp)^2 iv \cdot D + \frac{i\cancel{D}_\perp}{2} \left(-\frac{3}{8}(iD_\perp)^2 + (iv \cdot D)^2 \right) \right] \\ & + \mathcal{O}(1/M^4), \end{aligned} \quad (2.63)$$

is not a solution to the invariance equation. The necessity for such additional field strength dependent terms can also be seen from the fact that the right hand side of (2.63) would imply a transformation $\psi_v \rightarrow \psi_w = \Gamma^{\text{naive}}(w, iD)^{-1} e^{iq \cdot x} \Gamma^{\text{naive}}(v, iD) \psi_v$ that takes ψ_v outside of the assumed representation space, with $\not{w} \psi_v = \psi_v$. In the heavy fermion Lagrangian, the effects of these field-strength dependent terms appear first at order $\mathcal{O}(1/M^4)$, where omission of the final term in (2.60c) would lead to incorrect $1/M^4$ Lagrangian coefficient relations (details are presented in later sections of this chapter).⁷

Before closing this section, let us summarize the value of the invariant operator method. Appendix A shows that we can find a suitable covariantization of $W(\mathcal{B}, i\partial + Mv)$ that allows solution of the invariance equation for $\Gamma(v, iD)$ to any order in $1/M$. Hence this method proves the existence of a covariantized boost operator and a non-zero, Lorentz invariant Lagrangian to arbitrary order. We may proceed in either of two ways to construct invariant Lagrangians. Firstly, we may proceed as in (2.62), where we construct manifestly invariant interactions through some fixed order in $1/M$; to achieve canonical form we must then perform field redefinitions. Alternatively, we may proceed as in (2.30) (or its generalization to arbitrary v), armed with the knowledge that a suitable boost generator as in (2.34) can be reconstructed order by order.

2.4 Higher-Spin and Self-conjugate Fields

Although the explicit results have so far focused on spin-1/2 fields transforming under an Abelian (i.e. complex) gauge group, the formalism extends straightforwardly to fields of arbitrary spin or to self-conjugate fields. Below we describe the formalism for embedding arbitrary spin representations within products of Dirac spinor and Lorentz vector representations of the Lorentz group. For a related discussion see e.g. [43]. We also describe constraints imposed on the effective theory deriving from self-conjugate fields, and the relation of such constraints to discrete symmetries C , P and T .

2.4.1 Higher Spin Representations

Irreducible higher spin representations can be built using products of the Dirac spinor and vector representations

$$\psi_v \rightarrow \Lambda_{\frac{1}{2}} \psi_v, \quad Z_v^\alpha \rightarrow \Lambda_{\beta}^{\alpha} Z_v^{\beta}, \quad (2.64)$$

⁷When building invariant fermion bilinears, the leading terms involve $iv \cdot D$ multiplying $1/M$ corrections appearing in $\Gamma(v, iD)$. Since such terms are eliminated in going to canonical form, nontrivial effects of the $1/M^3$ corrections to $\Gamma(v, iD)$ appear first at order $1/M^4$.

where $\Lambda = D(W)$ is a little group element as in Sect. 2.2.1, i.e., $\Lambda v = v$. The corresponding generators for these two representations are given by

$$\mathcal{J}_{\frac{1}{2}}^{\alpha\beta} = \frac{1}{2}\sigma^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta], \quad (\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(g_\mu^\alpha g_\nu^\beta - g_\mu^\beta g_\nu^\alpha). \quad (2.65)$$

We enforce a maximal set of constraints to isolate the appropriate irreducible representation.

Integer spin: For integer spin $s = n$, consider the totally symmetric and traceless tensor $Z_v^{\mu_1 \dots \mu_n}$, which has $(n+1)^2$ degrees of freedom. Imposing

$$v_{\mu_1} Z_v^{\mu_1 \dots \mu_n} = 0 \quad (2.66)$$

yields n^2 additional constraints, leaving us with $2n+1 = 2s+1$ degrees of freedom as desired. Under Lorentz transformations this field transforms as

$$Z_v^{\mu_1 \dots \mu_n} \rightarrow \Lambda_{\nu_1}^{\mu_1} \dots \Lambda_{\nu_n}^{\mu_n} Z_v^{\nu_1 \dots \nu_n}. \quad (2.67)$$

Using $\Lambda^T g \Lambda = g$ and $\Lambda v = v$, it is easy to see that symmetry, tracelessness and the constraint (2.66) are preserved by this transformation.

Half-integer spin: For half-integer spin $s = n + 1/2$, consider the spinor-tensor $\psi_v^{\mu_1, \mu_2, \dots, \mu_n}$, which is totally symmetric in the indices $\mu_1 \dots \mu_n$ and therefore has $2(n+1)(n+2)(n+3)/3$ degrees of freedom. We impose the constraints⁸

$$\not{v} \psi_v^{\mu_1 \dots \mu_n} = \psi_v^{\mu_1 \dots \mu_n}, \quad \gamma_{\mu_1} \psi_v^{\mu_1 \dots \mu_n} = 0. \quad (2.68)$$

The second constraint yields $n(n+1)(n+5)/3$ equations, while the first projects a four-component spinor onto a two-dimensional subspace, reducing the degrees of freedom by $1/2$. In total $2(n+1) = 2s+1$ degrees of freedom remain. Under Lorentz transformations this field transforms as

$$\psi_v^{\mu_1 \dots \mu_n} \rightarrow \Lambda_{\nu_1}^{\mu_1} \dots \Lambda_{\nu_n}^{\mu_n} \Lambda_{\frac{1}{2}} \psi_v^{\nu_1 \dots \nu_n}. \quad (2.69)$$

This is symmetric in $\mu_1 \dots \mu_n$. That Eq. (2.68) are preserved follows immediately from $\Lambda v = v$ and $\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda_\nu^\mu \gamma^\nu$.

The construction of heavy particle Lagrangians for higher-spin fields proceeds in a similar way to the spin-1/2 case described above, with extension of the spin matrices $(1, \sigma^i)$ to the appropriate set. It is possible to choose spin matrices of the appropriate dimension to construct the Lagrangian for a heavy particle of arbitrary spin valid to any order in the $1/M$ expansion.

⁸Note that the second constraint implies $g_{\mu\nu} \psi_v^{\mu\nu\mu_3 \dots \mu_n} = 0$ and, furthermore, is equivalent to imposing $v_{\mu_1} \psi_v^{\mu_1 \dots \mu_n} = 0$ and $\epsilon_{\nu\alpha\beta\mu_1} v^\nu \sigma^{\alpha\beta} \psi_v^{\mu_1 \dots \mu_n} = 0$.

2.4.2 Self-conjugate Parity and CPT

The self-conjugacy of $SU(2)$ implies that for any field $\phi(x)$ transforming as in (2.8) or (2.19) with the plus sign, the field

$$\phi^c(x) = S\phi^*(x), \quad (2.70)$$

transforms as in (2.8) or (2.19) with the minus sign. Here S is the $(2s+1) \times (2s+1)$ similarity transformation for the spin- s representation of $SU(2)$, such that $(-\Sigma^i)^* = S\Sigma^i S^{-1}$. In covariant language, this translates to the simultaneous transformations

$$\phi_v(x) \rightarrow \phi_v^c(x), \quad v^\mu \rightarrow -v^\mu. \quad (2.71)$$

In terms of the irreducible representations constructed in Sect. 2.4.1, the field transformation in (2.71) reads⁹

$$Z_v^{\mu_1 \dots \mu_s} \rightarrow (Z_v^c)^{\mu_1 \dots \mu_s} = (Z_v^{\mu_1 \dots \mu_s})^*, \quad \psi_v^{\mu_1 \dots \mu_s} \rightarrow (\psi_v^c)^{\mu_1 \dots \mu_s} = \mathcal{C}(\psi_v^{\mu_1 \dots \mu_s})^*, \quad (2.72)$$

for integer spin and half-integer spin fields, respectively. The charge conjugation matrix $\mathcal{C}$ acts on the spinor index of ψ_v . It is symmetric and unitary, and obeys $\mathcal{C}^\dagger \gamma^\mu \mathcal{C} = -\gamma^{\mu*}$. The parity (2.71) arises if the effective theory is describing a full theory of a self-conjugate field (necessarily transforming in a real representation of a gauge group). For example, the effective theory field for a real scalar $\varphi = \varphi^*$ can be obtained via

$$\varphi(x) = e^{-iMv \cdot x} \varphi_v(x) / \sqrt{M} = e^{iMv \cdot x} \varphi_v^*(x) / \sqrt{M} = \varphi^*(x). \quad (2.73)$$

Similarly, the effective theory for a Majorana fermion represented by a Dirac spinor $\psi_M = \psi_M^c$ can be obtained via

$$\psi_M = \sqrt{2} e^{-iMv \cdot x} (h_v + H_v) = \sqrt{2} e^{iMv \cdot x} (h_v^c + H_v^c) = \psi_M^c, \quad (2.74)$$

where $/v h_v = h_v$ and $/v H_v = -H_v$.

It follows from (2.71) that the allowed operators $\bar{\phi}_v \mathcal{O}(v) \phi_v$ in the Lagrangian representing a self-conjugate field can be chosen such that

$$\mathcal{O}(v) = \mathcal{C} \mathcal{O}(-v)^* \mathcal{C}^\dagger. \quad (2.75)$$

Since we are often interested in constructing the Lagrangian in canonical form, i.e., without higher $iv \cdot D$ derivatives acting on ϕ_v , it is important to ask whether this condition is preserved by the requisite field redefinitions. By a similar reasoning

⁹We here choose a basis such that $S = 1$ for vectors.

to above, operators of the form $\bar{\phi}_v[iv \cdot DX(v) + X^\dagger(v)iv \cdot D]\phi_v$ appearing in the Lagrangian must be such that $X(v) = CX(-v)^*\mathcal{C}^\dagger$. Hence field redefinitions of the form $\phi_v \rightarrow [1 - X(v)]\phi_v$ achieve canonical form of the Lagrangian while preserving (2.75).

The self-conjugate parity for heavy fields described above is equivalent to imposing a modified, heavy-particle version of CPT , where P and T act on a Dirac spinor χ_v in the usual way, but C acts as the identity:

$$C : \chi(t, \mathbf{x}) \rightarrow \chi(t, \mathbf{x}), \quad P : \chi(t, \mathbf{x}) \rightarrow \gamma_0 \chi(t, -\mathbf{x}), \quad T : \chi(t, \mathbf{x}) \rightarrow \gamma_1 \gamma_3 \chi(-t, \mathbf{x}). \quad (2.76)$$

In this formulation, the reference vector v^μ is unchanged while the field transformations under discrete symmetries C, P, T are implemented. Hence, it may be imposed in the case where the heavy particle effective theory is written in the rest frame where the reference vector has been fixed to $v^\mu = (1, 0)$.

2.5 NRQED Example: Lagrangian

Let us demonstrate the application of our formalism in the case of Nonrelativistic QED (NRQED) (i.e., the parity and time-reversal symmetric theory of a heavy spin-1/2 particle coupled to an Abelian gauge field) at $\mathcal{O}(1/M^4)$. We will also consider examples with multiple heavy particle fields, and other relativistic degrees of freedom beyond Abelian gauge fields.

NRQED is an effective field theory [21] describing the interactions of nonrelativistic fermions with electromagnetic fields. NRQED interactions at order $1/M^4$ have become relevant for describing radiative corrections to proton structure contributions in hydrogenic bound state spectroscopy [58, 93]. The NRQED Lagrangian, properly constrained by Lorentz invariance, trivializes the derivation of low-energy theorems of Compton scattering [95] and automatically incorporates the intricate singularity structure of scattering amplitudes [10, 99]. It can be used to rigorously compute radiative corrections to low-energy lepton-nucleon scattering, and it also provides a model-independent framework within which to analyze static properties of nucleons, such as polarizabilities and generalized electromagnetic moments [85].

Let us illustrate the formalism for constructing heavy particle Lagrangians, deriving a complete basis of operators and coefficient constraints through order $1/M^4$ for the effective theory of nonrelativistic nucleons and leptons interacting with photons.¹⁰ An important formal issue first arises at order $1/M^4$: as discussed in the previous sections, a “reparameterization invariance” ansatz for enforcing relativistic

¹⁰For definiteness we will often refer to the heavy fermion ψ as the “nucleon”, and to a second fermion χ in Sect. 2.8 as the “lepton”.

invariance breaks down at this order. We derive the correct implementation of Lorentz invariance constraints and the resulting Wilson coefficient relations (i.e., nonrenormalization theorems) through order $1/M^4$.

Let us begin by constructing the NRQED Lagrangian in the one-fermion sector through order $1/M^4$. Consider the Lagrangian for a heavy fermion coupled to an Abelian gauge field. We enforce hermiticity and invariance under parity, time-reversal and rotational symmetries. We also perform field redefinitions to eliminate time derivatives acting on the fermion field (apart from the leading term); we refer to this choice as the “canonical form” of the heavy particle Lagrangian. We thus find in the one-fermion sector,

$$\begin{aligned}
\mathcal{L} = & \psi^\dagger \left\{ iD_t + c_2 \frac{D^2}{2M} + c_4 \frac{D^4}{8M^3} + c_{Fg} \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{2M} + c_{Dg} \frac{[\boldsymbol{\partial} \cdot \mathbf{E}]}{8M^2} \right. \\
& + ic_S g \frac{\boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8M^2} \\
& + c_{W1} g \frac{\{D^2, \boldsymbol{\sigma} \cdot \mathbf{B}\}}{8M^3} - c_{W2} g \frac{D^i \boldsymbol{\sigma} \cdot \mathbf{B} D^i}{4M^3} + c_{p'p} g \frac{\boldsymbol{\sigma} \cdot \mathbf{D} \mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B} \boldsymbol{\sigma} \cdot \mathbf{D}}{8M^3} \\
& + ic_M g \frac{\{D^i, [\boldsymbol{\partial} \times \mathbf{B}]^i\}}{8M^3} + c_{A1} g^2 \frac{\mathbf{B}^2 - \mathbf{E}^2}{8M^3} - c_{A2} g^2 \frac{\mathbf{E}^2}{16M^3} \\
& + c_{X1} g \frac{[D^2, \mathbf{D} \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{D}]}{M^4} + c_{X2} g \frac{\{D^2, [\boldsymbol{\partial} \cdot \mathbf{E}]\}}{M^4} + c_{X3} g \frac{[\boldsymbol{\partial}^2 \boldsymbol{\partial} \cdot \mathbf{E}]}{M^4} \\
& + ic_{X4} g^2 \frac{\{D^i, [\mathbf{E} \times \mathbf{B}]^i\}}{M^4} + ic_{X5} g \frac{D^i \boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D}) D^i}{M^4} \\
& + ic_{X6} g \frac{\epsilon^{ijk} \sigma^i D^j [\boldsymbol{\partial} \cdot \mathbf{E}] D^k}{M^4} \\
& + c_{X7} g^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{B} [\boldsymbol{\partial} \cdot \mathbf{E}]}{M^4} + c_{X8} g^2 \frac{[\mathbf{E} \cdot \boldsymbol{\partial} \boldsymbol{\sigma} \cdot \mathbf{B}]}{M^4} + c_{X9} g^2 \frac{[\mathbf{B} \cdot \boldsymbol{\partial} \boldsymbol{\sigma} \cdot \mathbf{E}]}{M^4} \\
& + c_{X10} g^2 \frac{[E^i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} B^i]}{M^4} \\
& + c_{X11} g^2 \frac{[B^i \boldsymbol{\sigma} \cdot \boldsymbol{\partial} E^i]}{M^4} + c_{X12} g^2 \frac{\boldsymbol{\sigma} \cdot \mathbf{E} \times [\partial_t \mathbf{E} - \boldsymbol{\partial} \times \mathbf{B}]}{M^4} + \mathcal{O}(1/M^5) \Big\} \psi.
\end{aligned} \tag{2.77}$$

We have defined $D_t = \partial/\partial t + igZA^0$, $D^i = \partial/\partial x^i - igZA^i$, where $-gZ = -e$, $+e$ or 0 for an electron, proton or neutron, respectively. The operators up to $1/M^3$ were previously listed in [21, 77, 86]. We use the summation convention $X^i Y^i \equiv \sum_{i=1}^3 X^i Y^i$, and define $[X, Y] \equiv XY - YX$, $\{X, Y\} \equiv XY + YX$ to denote commutators and anticommutators as usual. Square brackets around quantities imply that derivatives act only within the bracket. Electric and magnetic fields are defined as usual by $\mathbf{E} = -[\partial_t \mathbf{A}] - [\boldsymbol{\partial} A^0]$ and $\mathbf{B} = [\boldsymbol{\partial} \times \mathbf{A}]$. By the definition of $\mathbf{E}$ and $\mathbf{B}$, $[\boldsymbol{\partial} \cdot \mathbf{B}] = 0$ and $[\partial_t \mathbf{B} + \boldsymbol{\partial} \times \mathbf{E}] = 0$.

The most general term in (2.77) is obtained by constructing all possible rotationally invariant, hermitian combinations of iD_t , D^i , E^i , iB^i , $i\sigma^i$, with parity requiring an even number of factors of D^i and E^i . Terms at $1/M^4$ with two field strength factors E^i or B^i are straightforward to tabulate; note that we have used $[\partial_t \mathbf{B}] = -[\partial \times \mathbf{E}]$ and the assumption of canonical form to eliminate time derivatives of the magnetic field. Remaining terms at $1/M^4$ involve one factor of electric field E^i and three spatial derivatives D^i . Spin-independent terms are straightforward to tabulate; the basis of operators parameterized by c_{X1} , c_{X2} , c_{X3} differs from other possible choices by terms involving commutators $[D^i, D^j]$, i.e., terms with two field strengths. For spin-dependent terms we use $[\partial \times \mathbf{E}] = -[\partial_t \mathbf{B}]$ and the assumption of canonical form to eliminate occurrences of $[\partial \times \mathbf{E}]$. The three-vector identity,

$$D^i(\mathbf{E} \times \boldsymbol{\sigma})^j + (\boldsymbol{\sigma} \times \mathbf{D})^j E^i + \sigma^i (\mathbf{D} \times \mathbf{E})^j = \mathbf{D} \cdot \mathbf{E} \times \boldsymbol{\sigma} \delta^{ij}, \quad (2.78)$$

applied to remaining terms of the form $\psi^\dagger D^i(\dots) D^j \psi$, leaves the basis of operators parameterized by c_{X5} , c_{X6} .

2.6 NRQED Example: Relativistic Invariance

The Lagrangian (2.77) is invariant, by construction, under rotations and spacetime translations. The remaining constraints of relativity are enforced by demanding invariance under boosts. Here we derive these additional constraints, first by a variational calculation in Sect. 2.6.1, and then by an equivalent invariant operator construction in Sect. 2.6.2.

2.6.1 Variational Method

As detailed in Sect. 2.2, under infinitesimal boosts, with infinitesimal boost parameter $\boldsymbol{\eta} = -\mathbf{q}/M$, we may choose the heavy fermion to transform as

$$\begin{aligned} \psi \rightarrow e^{-i\mathbf{q} \cdot \mathbf{x}} \left\{ 1 + \frac{i\mathbf{q} \cdot \mathbf{D}}{2M^2} + \frac{i\mathbf{q} \cdot \mathbf{D} \mathbf{D}^2}{4M^4} - \frac{\boldsymbol{\sigma} \times \mathbf{q} \cdot \mathbf{D}}{4M^2} \left[1 + \frac{\mathbf{D}^2}{4M^2} \right] \right. \\ \left. + \frac{ic_D g}{8M^3} \mathbf{q} \cdot \mathbf{E} + \frac{c_S g}{8M^3} \mathbf{q} \cdot \boldsymbol{\sigma} \times \mathbf{E} + \mathcal{O}(g/M^4, 1/M^6) + \dots \right\} \psi, \end{aligned} \quad (2.79)$$

while derivatives and gauge fields transform as Lorentz vectors:

$$\mathbf{B} \rightarrow \mathbf{B} - \frac{1}{M} \mathbf{q} \times \mathbf{E}, \quad \mathbf{E} \rightarrow \mathbf{E} + \frac{1}{M} \mathbf{q} \times \mathbf{B}, \quad \mathbf{D} \rightarrow \mathbf{D} + \frac{1}{M} \mathbf{q} D_t, \quad D_t \rightarrow D_t + \frac{1}{M} \mathbf{q} \cdot \mathbf{D}. \quad (2.80)$$

Field strength-dependent terms in (2.79) have been chosen to maintain canonical form. Since we are interested in the canonical Lagrangian through order $1/M^4$, we need not specify the explicit form of the order $1/M^4$ field strength-dependent terms, denoted by $\mathcal{O}(g/M^4)$. A straightforward computation yields

$$\delta\mathcal{L} = \frac{1}{M}\delta\mathcal{L}_1 + \frac{1}{M^2}\delta\mathcal{L}_2 + \frac{1}{M^3}\delta\mathcal{L}_3 + \frac{1}{M^4}\delta\mathcal{L}_4 + \dots, \quad (2.81)$$

where

$$\begin{aligned} \delta\mathcal{L}_1 &= \psi^\dagger [(1 - c_2) \mathbf{iq} \cdot \mathbf{D}] \psi, \\ \delta\mathcal{L}_2 &= \psi^\dagger \left[-\frac{1}{2}(1 - c_2) \{ \mathbf{q} \cdot \mathbf{D}, D_t \} + \frac{g}{4} (Z - 2c_F + c_S) \boldsymbol{\sigma} \times \mathbf{q} \cdot \mathbf{E} \right] \psi, \\ \delta\mathcal{L}_3 &= \psi^\dagger \left[\frac{g}{8} \mathbf{q} \cdot [\boldsymbol{\partial} \times \mathbf{B}] (c_F - c_D + 2c_M) + \frac{i}{4} \{ \mathbf{q} \cdot \mathbf{D}, \mathbf{D}^2 \} (c_2 - c_4) \right. \\ &\quad + \frac{ig}{8} \{ \mathbf{q} \cdot \mathbf{D}, \boldsymbol{\sigma} \cdot \mathbf{B} \} (c_2 Z + 2c_F - c_S - 2c_{W1} + 2c_{W2}) \\ &\quad + \frac{ig}{8} \{ \boldsymbol{\sigma} \cdot \mathbf{D}, \mathbf{q} \cdot \mathbf{B} \} (-c_2 Z + c_F - c_{p'p}) \\ &\quad \left. + \frac{ig}{8} \mathbf{q} \cdot \boldsymbol{\sigma} (\mathbf{D} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{D}) (-c_F + c_S - c_{p'p}) \right] \psi. \end{aligned} \quad (2.82)$$

From $\delta\mathcal{L}_1$, $\delta\mathcal{L}_2$ and $\delta\mathcal{L}_3$, we find [58, 86]¹¹

$$c_2 = 1, \quad c_S = 2c_F - Z, \quad c_4 = 1, \quad 2c_M = c_D - c_F, \quad c_{W2} = c_{W1} - Z, \quad c_{p'p} = c_F - Z. \quad (2.83)$$

Employing the above relations, the variation $\delta\mathcal{L}_4$ takes the form

$$\begin{aligned} \delta\mathcal{L}_4 &= \psi^\dagger \left[\frac{ig}{8} [\mathbf{D}^2, \mathbf{q} \cdot \mathbf{E}] \left(\frac{5Z}{4} - c_F + c_D - 32c_{X1} \right) \right. \\ &\quad + \frac{ig}{8} \{ \mathbf{q} \cdot \mathbf{D}, [\boldsymbol{\partial} \cdot \mathbf{E}] \} \left(-\frac{Z}{4} + c_F - 16c_{X2} \right) \\ &\quad + \frac{g^2}{8} \mathbf{q} \cdot \mathbf{E} \times \mathbf{B} \left(\frac{Z^2}{2} + 2c_F(Z - c_F) - 2Zc_D + c_{A2} + 16c_{X4} \right) \\ &\quad + \frac{g}{8} [\mathbf{q} \cdot \boldsymbol{\sigma} \times \boldsymbol{\partial} \boldsymbol{\partial} \cdot \mathbf{E}] \left(-Z + c_F - \frac{1}{4}c_D + c_{W1} + 8c_{X6} \right) \\ &\quad \left. + \frac{g}{8} D^i (q^i (\mathbf{E} \times \boldsymbol{\sigma})^j + (\mathbf{E} \times \boldsymbol{\sigma})^i q^j + \boldsymbol{\sigma} \times \mathbf{q} \cdot \mathbf{E} \delta^{ij}) D^j \left(\frac{Z}{2} - 2c_F + 16c_{X5} \right) \right] \psi, \end{aligned} \quad (2.84)$$

¹¹As noted in [58], we find the opposite sign in the relation for c_M in (2.83) compared to [86].

where we have suppressed terms that are removed by field strength-dependent modifications of the boost generator, denoted by $\mathcal{O}(g/M^4)$ in (2.79). We readily find,

$$\begin{aligned}
 32c_{X1} &= \frac{5Z}{4} - c_F + c_D, \\
 32c_{X2} &= -\frac{Z}{2} + 2c_F, \\
 32c_{X4} &= -Z^2 - 4c_F(Z - c_F) + 4Zc_D - 2c_{A2}, \\
 32c_{X5} &= -Z + 4c_F, \\
 32c_{X6} &= 4(Z - c_F) + c_D - 4c_{W1},
 \end{aligned} \tag{2.85}$$

while coefficients c_{X3} and $c_{X7\dots X12}$ are not constrained by Lorentz invariance. We thus find that seven new quantities are required at order $1/M^4$ to describe the proton's response to arbitrary background electromagnetic fields. The above relations following from relativistic symmetry are non-renormalizable.

2.6.2 Invariant Operators

An alternate method for enforcing Lorentz invariance is to construct the Lagrangian from explicitly invariant operators. We summarize here the main points; the details are presented in Sect. 2.3.

The basic building block in the construction is the field $\Psi_v = \Gamma(v, iD)\psi_v$, where ψ_v is a Dirac spinor field with $\not{v}\psi_v = \psi_v$. The matrix-valued operator $\Gamma(v, iD)$ is determined such that under an infinitesimal boost Λ , where $\Lambda^\mu_\nu v^\nu = v^\mu + q^\mu/M$, the field Ψ_v has a simple transformation law: $\Psi_v \rightarrow e^{iq \cdot x} \Psi_v$. Noting that $e^{-iq \cdot x}(iD^\mu + Mv^\mu + q^\mu)e^{iq \cdot x} = iD^\mu + Mv^\mu$, we may thus build invariant bilinears from contractions of polynomials of γ^μ and $\mathcal{V}^\mu \equiv v^\mu + iD^\mu/M$, between $\bar{\Psi}_v$ and Ψ_v .

The function $\Gamma(v, iD)$ is a solution to the invariance equation,

$$\Gamma(v + q/M, iD - q)\Lambda^{-1}W(\Lambda, iD + Mv) = \Gamma(v, iD), \tag{2.86}$$

where $W(\Lambda, p)$ is an element of the little group for timelike invariant vector v^μ , following from the theory of induced representations of the Lorentz group. Up to the relevant order for determining the $1/M^4$ Lagrangian we have [57]

$$\begin{aligned}
 \Gamma = 1 &+ \frac{i\not{D}_\perp}{2M} + \frac{1}{M^2} \left\{ -\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\not{D}_\perp iv \cdot D \right\} + \frac{1}{M^3} \left\{ \frac{1}{4}(iD_\perp)^2 iv \cdot D \right. \\
 &+ \frac{i\not{D}_\perp}{2} \left[-\frac{3}{8}(iD_\perp)^2 + (iv \cdot D)^2 \right] + \frac{gZ}{8} F_{\mu\nu} v^\mu D_\perp^\nu + \frac{gZ}{16} \sigma_\perp^{\mu\nu} F_{\mu\nu} i\not{D}_\perp \left. \right\} + \dots,
 \end{aligned} \tag{2.87}$$

where we have defined $D_\perp^\mu \equiv D^\mu - v^\mu v \cdot D$, and for Abelian gauge fields $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the last two terms of (2.87) are absent in the ansatz for reparameterization invariance given in [84], leading to incorrect Lorentz invariance constraints at $1/M^4$ and beyond. This subtlety is explained in Sects. 2.1–2.3 above.

A complete basis of invariant bilinears required through order $1/M^4$ is

$$\begin{aligned} \mathcal{L} = & \bar{\Psi}_v \left\{ M(\mathcal{V} - 1) - a_F g \frac{\sigma^{\mu\nu} F_{\mu\nu}}{4M} + ia_D g \frac{\{\mathcal{V}_\mu, [M\mathcal{V}_\nu, F^{\mu\nu}]\}}{16M^2} \right. \\ & - a_{W1} g \frac{[M\mathcal{V}^\alpha, [M\mathcal{V}_\alpha, \sigma^{\mu\nu} F_{\mu\nu}]]}{16M^3} \\ & \left. + a_{A1} g^2 \frac{F_{\mu\nu} F^{\mu\nu}}{16M^3} + a_{A2} g^2 \frac{\mathcal{V}_\alpha F^{\mu\alpha} F_{\mu\beta} \mathcal{V}^\beta}{16M^3} \right\} \Psi_v + a_{X3} \mathcal{B}_{X3} + \sum_{i=7}^{12} a_{Xi} \mathcal{B}_{Xi}. \end{aligned} \quad (2.88)$$

The bilinears $\mathcal{B}_{Xi}$ for $i = 3, 7 \dots 12$ are chosen to reduce to the respective operators multiplying c_{Xi} in (2.77) upon setting $v^\mu = (1, 0, 0, 0)$ and neglecting $1/M$ suppressed corrections. Since we are concerned only with the Lagrangian through order $1/M^4$ we do not specify an explicit choice for these $\mathcal{B}_{Xi}$. A computation shows that the field redefinition to recover canonical form is

$$\begin{aligned} \psi_v = & \left\{ 1 + \frac{1}{4M^2} (iD_\perp)^2 \left(1 - \frac{iv \cdot D}{M} \right) - \frac{gZ}{16M^2} \sigma_\perp^{\mu\nu} F_{\mu\nu} \right. \\ & - \frac{gZ}{4M^3} D_\perp^\mu v^\alpha F_{\alpha\mu} + \frac{igZ}{4M^3} \sigma_{\mu\nu} D_\perp^\mu v_\alpha F^{\alpha\nu} \\ & - \frac{gZ}{8M^3} v^\alpha F_{\alpha\mu} D_\perp^\mu + \frac{ga_F}{4M^3} [-D_\perp^\mu v^\alpha F_{\alpha\mu} + i\sigma_{\mu\nu} D_\perp^\mu v_\alpha F^{\alpha\nu}] - \frac{ga_D}{8M^3} v^\alpha F_{\alpha\mu} D_\perp^\mu \\ & \left. + \frac{iga_{W1}}{8M^3} \sigma_{\mu\nu} [D_\perp^\mu, v_\alpha F^{\alpha\nu}] \right\} \psi'_v. \end{aligned} \quad (2.89)$$

Upon setting $v^\mu = (1, 0, 0, 0)$, the resulting Lagrangian, expressed in terms of ψ'_v , is identical to the previous result (2.77) with constraints (2.83) and (2.85).

2.7 NRQED Example: One-Photon Matching

This section relates the matching conditions in the one-fermion sector to standard form factors of the nucleon. The coefficient relations of the previous section, derived from relativistic invariance, are verified explicitly. We focus here on operators contributing to the one-photon matrix element. Coefficient relations for operators contributing to the two-photon matrix element may be similarly verified.

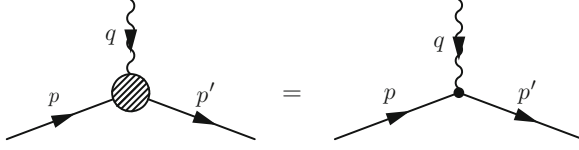


Fig. 2.1 Tree level matching of the one-photon amplitude in the full theory and NRQED. The *black dot* in the diagram on the right-hand side represents single-photon NRQED vertices

Consider first the operators contributing to the one-photon matrix element of the nucleon. The matching is performed in terms of standard invariant form factors,

$$\langle N(p') | J_\mu^{\text{em}} | N(p) \rangle = \bar{u}(p') \Gamma_\mu(q) u(p), \quad \Gamma_\mu(q) \equiv \gamma_\mu F_1^N(q^2) + \frac{i\sigma_{\mu\nu}}{2M_N} F_2^N(q^2) q^\nu, \quad (2.90)$$

where $q = p' - p$ and N denotes a proton or neutron; we suppress the superscript N in the following. Equating the effective theory with the full theory,¹² we find (cf. Fig. 2.1)

$$\begin{aligned} c_F &= \bar{F}_1 + \bar{F}_2 \equiv Z + a_N + \mathcal{O}(\alpha), \\ c_D &= \bar{F}_1 + 2\bar{F}_2 + 8\bar{F}'_1 \equiv Z + \frac{4}{3}M^2(r_E^N)^2 + \mathcal{O}(\alpha), \\ c_{W1} &= \bar{F}_1 + \frac{1}{2}\bar{F}_2 + 4\bar{F}'_1 + 4\bar{F}'_2, \\ c_{X3} &= \frac{1}{8}\bar{F}'_1 + \frac{1}{4}\bar{F}'_2 + \frac{1}{2}\bar{F}''_1, \end{aligned} \quad (2.91)$$

where Z denotes the electric charge, a_N is the anomalous magnetic moment of the nucleon, and r_E^N is the nucleon charge radius. We have introduced dimensionless barred quantities to denote derivatives with respect to q^2/M^2 at $q^2 = 0$: $\bar{F}_1 \equiv F_1(0) = Z$, $\bar{F}_2 \equiv F_2(0) = a_N$, $\bar{F}'_i \equiv M^2 F'_i(0)$, etc. The new quantity F'_1 appears at $1/M^4$. Expressions for other Wilson coefficients up to $1/M^3$ in terms of form factors can be found using (2.83). At $1/M^4$, we also find

$$\begin{aligned} c_{X1} &= \frac{5}{128}\bar{F}_1 + \frac{1}{32}\bar{F}_2 + \frac{1}{4}\bar{F}'_1, \\ c_{X2} &= \frac{3}{64}\bar{F}_1 + \frac{1}{16}\bar{F}_2, \\ c_{X5} &= \frac{3}{32}\bar{F}_1 + \frac{1}{8}\bar{F}_2, \\ c_{X6} &= -\frac{3}{32}\bar{F}_1 - \frac{1}{8}\bar{F}_2 - \frac{1}{4}\bar{F}'_1 - \frac{1}{2}\bar{F}'_2, \end{aligned} \quad (2.92)$$

¹²The nonrelativistic normalization of states in NRQED is obtained using $\bar{u}(p)u(p) = M/E_p$ in (2.90).

and it is readily verified that these expressions satisfy the constraints (2.85). In the presence of radiative corrections, the form factors on the right hand sides of (2.91) and (2.92) should be interpreted in an appropriate infrared regularization scheme; alternatively, the matching may be performed with infrared finite observables. The corresponding infrared subtractions and ultraviolet renormalizations must be performed to obtain the Wilson coefficients including radiative corrections.¹³

2.8 NRQED Example: Photon and Four-Fermion Sectors

So far our analysis has focused on the one-fermion sector. We have derived the form of the Lagrangian appropriate, e.g., to a proton in a background electromagnetic field. Let us consider the complete QED theory including dynamical photon, as well as a lepton (electron or muon) field. The case of a nonrelativistic lepton is appropriate to bound state hydrogen studies, or very low-energy lepton-nucleon (e.g. muon-proton) scattering, where $E \ll m_\ell, M$. We first consider this case, constructing the operator basis, deriving coefficient relations and identifying redundant operators. We then turn to a brief discussion of the case of a relativistic lepton, appropriate to e.g. low-energy electron-proton scattering with $m_\ell, E \ll M$.

2.8.1 Pure Photon Operators

The pure gauge sector for NRQED is the well known Euler-Heisenberg Lagrangian. Enforcing parity and time reversal symmetry and neglecting total derivatives we find

$$\begin{aligned} \mathcal{L}_\gamma = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_{V2}\frac{F_{\mu\nu}[\partial^2 F^{\mu\nu}]}{M^2} + c_{V4}\frac{F_{\mu\nu}[\partial^4 F^{\mu\nu}]}{M^4} \\ & + c_{E1}g^2\frac{(F_{\mu\nu}F^{\mu\nu})^2}{M^4} + c_{E2}g^2\frac{F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu}{M^4} + \dots \end{aligned} \quad (2.93)$$

The coefficients c_{V2} and c_{V4} may be set to zero through field redefinitions on A^μ , as discussed in Sect. 2.8.3 below.

¹³The expressions on the right hand side of (2.91) and (2.92) correspond to those referred to as c_i^{QED} in [77]. The renormalization procedure in dimensional regularization is described in [86].

2.8.2 Four-Fermion Operators

Consider four-fermion operators relevant for processes in the one-nucleon, one-lepton sector. We enforce hermiticity and invariance under parity, time-reversal and rotational symmetries. We use the notation $\overleftarrow{D}$ for a covariant derivative acting to the left, $X\overleftarrow{D}^i \equiv [\partial^i X] + igZX A^i$, and define $D_+ \equiv D + \overleftarrow{D}$, $D_- \equiv D - \overleftarrow{D}$. Having performed field redefinitions to eliminate operators with time derivatives acting on heavy fermions, the Lagrangian in this sector, through $1/M^4$, is

$$\begin{aligned}
\mathcal{L}_{\psi\chi} = & \frac{d_1}{M^2} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^i \chi + \frac{d_2}{M^2} \psi^\dagger \psi \chi^\dagger \chi + \frac{d_3}{M^4} \psi^\dagger D_+^i \psi \chi^\dagger D_+^i \chi \\
& + \frac{d_4}{M^4} \psi^\dagger D_-^i \psi \chi^\dagger D_-^i \chi \\
& + \frac{d_5}{M^4} \psi^\dagger (D^2 + \overleftarrow{D}^2) \psi \chi^\dagger \chi + \frac{d_6}{M^4} \psi^\dagger \psi \chi^\dagger (D^2 + \overleftarrow{D}^2) \chi \\
& + \frac{gd_7}{M^4} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B} \psi \chi^\dagger \chi + \frac{id_8}{M^4} \epsilon^{ijk} \psi^\dagger \sigma^i D_-^j \psi \chi^\dagger D_+^k \chi \\
& + \frac{id_9}{M^4} \epsilon^{ijk} \psi^\dagger \sigma^i D_+^j \psi \chi^\dagger D_-^k \chi \\
& + \frac{gd_{10}}{M^4} \psi^\dagger \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B} \chi + \frac{id_{11}}{M^4} \epsilon^{ijk} \psi^\dagger D_+^k \psi \chi^\dagger \sigma^i D_-^j \chi \\
& + \frac{id_{12}}{M^4} \epsilon^{ijk} \psi^\dagger D_-^k \psi \chi^\dagger \sigma^i D_+^j \chi \\
& + \frac{d_{13}}{M^4} \psi^\dagger \sigma^i D_+^j \psi \chi^\dagger \sigma^i D_+^j \chi + \frac{d_{14}}{M^4} \psi^\dagger \sigma^i D_-^j \psi \chi^\dagger \sigma^i D_-^j \chi \\
& + \frac{d_{15}}{M^4} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_+ \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_+ \chi \\
& + \frac{d_{16}}{M^4} \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_- \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{D}_- \chi + \frac{d_{17}}{M^4} \psi^\dagger \sigma^i D_-^j \psi \chi^\dagger \sigma^j D_-^i \chi \\
& + \frac{d_{18}}{M^4} \psi^\dagger \sigma^i (D^2 + \overleftarrow{D}^2) \psi \chi^\dagger \sigma^i \chi + \frac{d_{19}}{M^4} \psi^\dagger \sigma^i (D^i D^j + \overleftarrow{D}^j \overleftarrow{D}^i) \psi \chi^\dagger \sigma^j \chi \\
& + \frac{d_{20}}{M^4} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^i (D^2 + \overleftarrow{D}^2) \chi + \frac{d_{21}}{M^4} \psi^\dagger \sigma^i \psi \chi^\dagger \sigma^j (D^i D^j + \overleftarrow{D}^j \overleftarrow{D}^i) \chi.
\end{aligned} \tag{2.94}$$

Here χ is the nonrelativistic lepton field with mass M_χ and for notational simplicity we write all operators in terms of the common mass scale M .¹⁴ Covariant derivatives appearing within a fermion bilinear in (2.94) are understood to act only on fields

¹⁴Note that the coefficients $d_{1,2}$ in (2.94) are related to those of Caswell and Lepage [21] by a factor M_χ/M .

in that bilinear. The heavy field ψ transforms under boosts as in (2.79). Recalling that $\mathbf{q}$ in (2.79) is related to the mass-independent infinitesimal boost parameter by $\boldsymbol{\eta} = -\mathbf{q}/M$, the transformation law for χ is obtained by the replacement $M \rightarrow rM$ and $q \rightarrow rq$, where we define $r \equiv M_\chi/M$. We thus find

$$\begin{aligned}
\delta\mathcal{L}_{\psi\chi} = & \frac{1}{M^4} \left\{ \psi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \psi \chi^\dagger \chi \left[\frac{d_2}{2} - 2rd_4 - 2d_5 \right] \right. \\
& + \psi^\dagger \psi \chi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \chi \left[\frac{d_2}{2r} - 2d_4 - 2rd_6 \right] \\
& + \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \times \mathbf{D}_+ \psi \chi^\dagger \chi \left[-\frac{d_2}{4} + \frac{d_1}{4r} - 2d_8 - 2rd_9 \right] \\
& + \psi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \sigma^i \psi \chi^\dagger \sigma^i \chi \left[\frac{d_1}{2} - 2rd_{14} - 2d_{18} \right] \\
& + \psi^\dagger \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \times \mathbf{D}_+ \chi \left[-\frac{d_2}{4r} + \frac{d_1}{4} - 2rd_{11} - 2d_{12} \right] \\
& + \psi^\dagger \sigma^i \psi \chi^\dagger i\mathbf{q} \cdot \mathbf{D}_- \sigma^i \chi \left[\frac{d_1}{2r} - 2d_{14} - 2rd_{20} \right] \\
& + \psi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{D}_- \psi \chi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \chi \left[\frac{d_1}{4} - 2rd_{16} - d_{19} \right] \\
& + \psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{q} \psi \chi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{D}_- \chi \left[\frac{d_1}{4r} - 2d_{16} - rd_{21} \right] \\
& + \psi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{q} D_-^i \psi \chi^\dagger \sigma^i \chi \left[-\frac{d_1}{4} - 2rd_{17} - d_{19} \right] \\
& + \psi^\dagger \sigma^i \psi \chi^\dagger i\boldsymbol{\sigma} \cdot \mathbf{q} D_-^i \chi \left[-\frac{d_1}{4r} - 2d_{17} - rd_{21} \right] \Big\} \\
& + \mathcal{O}(1/M^5). \tag{2.95}
\end{aligned}$$

This enforces the relations

$$\begin{aligned}
rd_4 + d_5 &= \frac{d_2}{4}, \quad d_5 = r^2 d_6, \quad 8r(d_8 + rd_9) = -rd_2 + d_1, \\
8r(rd_{11} + d_{12}) &= -d_2 + rd_1, \\
rd_{14} + d_{18} &= \frac{d_1}{4}, \quad d_{18} = r^2 d_{20}, \quad 2rd_{16} + d_{19} = \frac{d_1}{4}, \\
r(d_{16} + d_{17}) + d_{19} &= 0, \quad d_{19} = r^2 d_{21}, \tag{2.96}
\end{aligned}$$

implying a total of 12 independent four-fermion operators through $1/M^4$, including two at order $1/M^2$. By performing field redefinitions on the gauge field A^μ , some of these four-fermion operators are found to mix with one-heavy particle sector operators, as discussed in Sect. 2.8.3 below.

Equation (2.94), with constraints (2.96), applies to the case of distinct heavy particles represented by ψ, χ , with arbitrary mass ratio M_χ/M . For certain applications, e.g. positronium or heavy quarkonium bound states, the fields ψ and χ can be taken to represent particle-antiparticle pairs with $r = M_\chi/M = 1$. Charge conjugation symmetry is then implemented by enforcing invariance under $\psi \leftrightarrow \chi$, thus reducing the basis of operators. This case has been investigated for QCD through $\mathcal{O}(1/M^4)$ by Brambilla et al. [17]. We find that our basis of four-fermion operators (2.94) and constraints (2.96) are equivalent to those found in [17] for this special case.¹⁵

2.8.3 Field Redefinitions and Redundant Operators

With a dynamical photon field, we may perform field redefinitions that maintain reality and gauge, parity, time reversal and rotational symmetries. In order to avoid upsetting the previously determined coefficient relations, we must also maintain the transformation law for A^μ as a four-vector under Lorentz transformations, i.e.,

$$A^0 \rightarrow A^0 - \frac{1}{M} \mathbf{q} \cdot \mathbf{A}, \quad \mathbf{A} \rightarrow \mathbf{A} - \frac{1}{M} \mathbf{q} A^0. \quad (2.97)$$

Let us write

$$A_\mu = A'_\mu + \Delta_\gamma A_\mu + \Delta_\psi A_\mu + \Delta_\chi A_\mu + \dots \quad (2.98)$$

For the pure gauge field terms the most general expression is

$$\Delta_\gamma A^\mu = a_{\gamma 1} \frac{\partial_\nu F^{\nu\mu}}{M^2} + a_{\gamma 2} \frac{\partial^2 \partial_\nu F^{\nu\mu}}{M^4} + \mathcal{O}(1/M^6). \quad (2.99)$$

Terms involving the heavy fermion ψ take the form

$$\begin{aligned} \frac{\Delta_\psi A^\mu}{g} &= \tilde{a}_{\psi 1} \frac{\bar{\Psi}_v \gamma^\mu \Psi_v}{M^2} + \tilde{a}_{\psi 2} \frac{\partial_\alpha (\bar{\Psi}_v \sigma^{\alpha\mu} \Psi_v)}{M^3} \\ &+ \tilde{a}_{\psi 3} g \frac{\bar{\Psi}_v \{\gamma^\mu, \sigma^{\alpha\beta} F_{\alpha\beta}\} \Psi_v}{M^4} + \tilde{a}_{\psi 4} \frac{\partial^2 (\bar{\Psi}_v \gamma^\mu \Psi_v)}{M^4} \\ &+ \tilde{a}_{\psi 5} g \frac{\bar{\Psi}_v \sigma^{\mu\alpha} \{\mathcal{V}^\beta, F_{\alpha\beta}\} \Psi_v}{M^4} + \mathcal{O}(1/M^5), \end{aligned} \quad (2.100)$$

¹⁵The difference between Abelian and nonAbelian gauge fields is trivial for four-fermion operators through this order.

where we have employed the invariant operator formalism of Sect. 2.6.2. In particular, $\Psi_v = \Gamma\psi_v$ with Γ from (2.87) and ψ_v from (2.89), expressed in terms of the field $\psi'_v \equiv \psi$ with canonical Lagrangian (2.77). As an alternative to the invariant operator formalism employed in (2.100) we may expand $\Delta_\psi A^0$ and $\Delta_\psi \mathbf{A}$ in a series of rotationally invariant operators with arbitrary coefficients, and subsequently constrain these coefficients using (2.97). The result is equivalent to (2.100), with five free parameters through $\mathcal{O}(1/M^4)$,

$$\begin{aligned} \frac{\Delta_\psi A^0}{g} &= a_{\psi 1} \frac{\psi^\dagger \psi}{M^2} + a_{\psi 2} \frac{\partial^2(\psi^\dagger \psi)}{M^4} - i \left(\frac{a_{\psi 1}}{4} - a_{\psi 4} \right) \frac{\psi^\dagger \boldsymbol{\sigma} \cdot \overleftarrow{\mathbf{D}} \times \mathbf{D} \psi}{M^4} \\ &\quad + a_{\psi 3} g \frac{\psi^\dagger \boldsymbol{\sigma} \cdot \mathbf{B} \psi}{M^4} + \mathcal{O}(1/M^5), \\ \frac{\Delta_\psi \mathbf{A}}{g} &= -a_{\psi 1} \frac{\psi^\dagger i \mathbf{D}_- \psi}{2M^3} + a_{\psi 4} \frac{\partial \times (\psi^\dagger \boldsymbol{\sigma} \psi)}{M^3} + a_{\psi 5} g \frac{\psi^\dagger \boldsymbol{\sigma} \times \mathbf{E} \psi}{M^4} + \mathcal{O}(1/M^5). \end{aligned} \quad (2.101)$$

The expansion of $\Delta_\chi A^\mu$ is obtained from (2.101) with the replacements $\psi \rightarrow \chi$, $M \rightarrow M_\chi$, $Z \rightarrow Z_\chi$ and $a_{\psi i} \rightarrow a_{\chi i}$. In terms of the field A'_μ in (2.98), we find in the pure photon sector,

$$\delta c_{V2} = -\frac{1}{2}a_{\gamma 1}, \quad \delta c_{V4} = -\frac{1}{2}a_{\gamma 2} - \frac{1}{4}a_{\gamma 1}^2 + 2a_{\gamma 1}c_{V2}, \quad (2.102)$$

while for the ψ sector,

$$\begin{aligned} \delta c_D &= -8Za_{\gamma 1} + 8a_{\psi 1}, \quad \delta c_{W1} = -4c_F a_{\gamma 1} + 8a_{\psi 4}, \\ \delta c_{A2} &= -16Z^2 a_{\gamma 1} + 16Za_{\psi 1}, \\ \delta c_{X3} &= -\frac{c_D a_{\gamma 1}}{8} + Za_{\gamma 2} - a_{\gamma 1}a_{\psi 1} + 4c_{V2}a_{\psi 1} + a_{\psi 2}, \quad \delta c_{X7} = -\frac{c_S Z a_{\gamma 1}}{4} + a_{\psi 3}, \\ \delta c_{X8} &= c_F Z a_{\gamma 1} - \frac{c_F a_{\psi 1}}{2} - Za_{\psi 4}, \quad \delta c_{X9} = -\frac{c_F^2 a_{\gamma 1}}{2} + c_F a_{\psi 4}, \\ \delta c_{X11} &= \frac{c_F^2 a_{\gamma 1}}{2} - c_F a_{\psi 4}, \quad \delta c_{X12} = \frac{c_S Z a_{\gamma 1}}{2} + a_{\psi 5}. \end{aligned} \quad (2.103)$$

Similar relations hold for the Wilson coefficients $c_i^{(\chi)}$ in the χ Lagrangian, defined as in (2.77), with $\psi \rightarrow \chi$, $Z \rightarrow Z_\chi$, $M \rightarrow M_\chi$, $c_i \rightarrow c_i^{(\chi)}$. Finally, for the four-fermion operator coefficients,

$$\begin{aligned} \frac{\delta d_2}{g^2} &= -Z_\chi a_{\psi 1} - \frac{Za_{\chi 1}}{r^2}, \quad \frac{\delta d_3}{g^2} = \frac{c_D^{(\chi)} a_{\psi 1}}{8r^2} + \frac{c_D a_{\chi 1}}{8r^2} + Z_\chi a_{\psi 2} + \frac{Za_{\chi 2}}{r^4}, \\ \frac{\delta d_4}{g^2} &= -\frac{Z_\chi a_{\psi 1}}{4r} - \frac{Za_{\chi 1}}{4r^3}, \quad \frac{\delta d_7}{g^2} = -\frac{ZZ_\chi}{4} (a_{\psi 1} - 4a_{\psi 4}) - Z_\chi a_{\psi 3}, \end{aligned}$$

$$\begin{aligned}
\frac{\delta d_8}{g^2} &= \frac{Z_\chi}{8} (a_{\psi 1} - 4a_{\psi 4}) - \frac{c_S a_{\chi 1}}{8r^2}, & \frac{\delta d_{10}}{g^2} &= -\frac{ZZ_\chi}{4r^4} (a_{\chi 1} - 4a_{\chi 4}) - \frac{Za_{\chi 3}}{r^4}, \\
\frac{\delta d_{11}}{g^2} &= -\frac{c_S^{(\chi)} a_{\psi 1}}{8r^2} + \frac{Z}{8r^4} (a_{\chi 1} - 4a_{\chi 4}), & \frac{\delta d_{13}}{g^2} &= -\frac{\delta d_{15}}{g^2} = \frac{c_F^{(\chi)} a_{\psi 4}}{2r} + \frac{c_F a_{\chi 4}}{2r^3}.
\end{aligned}
\tag{2.104}$$

The coefficient relations (2.83), (2.85) and (2.96) are preserved, since by construction the Lorentz transformation properties of A^μ are unchanged and hence the boost transformation rules (2.79) and (2.80) still apply.

We may use (2.102) to eliminate vacuum polarization terms c_{V2} and c_{V4} in favor of compensating terms in (2.103). Similarly, (2.103), together with the analogous relations for $c_i^{(\chi)}$, and (2.104), can be used to eliminate ten linear combinations of Wilson coefficients for two-fermion and four-fermion operators. Different applications may favor elimination of different operators.¹⁶

2.8.4 Relativistic Lepton

For applications such as lepton-nucleon scattering at energies $m_\ell, E \ll M$ (e.g., low-energy electron-proton scattering), the relevant effective theory involves a heavy fermion (e.g., the proton) interacting with an electromagnetically charged relativistic fermion (e.g., the electron). Let us briefly discuss this case. Enforcing parity, time-reversal, gauge, Lorentz as well as chiral symmetry at $m_\ell = 0$, we find the leptonic interactions with the photon,

$$\mathcal{L}_\ell = \bar{\ell} \left[i\not{D} - m_\ell + g c_F^{(\ell)} m_\ell \frac{\sigma^{\mu\nu} F_{\mu\nu}}{M^2} + g c_2^{(\ell)} m_\ell \frac{D^2}{M^2} + g c_D^{(\ell)} \frac{[\partial^\mu F_{\mu\nu}] \gamma^\nu}{M^2} + \mathcal{O}(1/M^4) \right] \ell,
\tag{2.105}$$

where we assume field redefinitions have been performed to remove power suppressed terms involving $(i\not{D} - m_\ell)\ell$.

Having performed field redefinitions to eliminate operators with time derivatives acting on fermion fields, the Lagrangian for the nucleon-relativistic lepton sector through $\mathcal{O}(1/M^3)$ is

$$\begin{aligned}
\mathcal{L}_{\psi\ell} &= \frac{b_1}{M^2} \psi^\dagger \psi \bar{\ell} \gamma^0 \ell + \frac{b_2}{M^2} \psi^\dagger \sigma^i \psi \bar{\ell} \gamma^i \gamma_5 \ell + \frac{b_3}{M^3} \psi^\dagger \psi m_\ell \bar{\ell} \ell + \frac{b_4}{M^3} \psi^\dagger i D_- \psi \bar{\ell} \gamma^i \ell \\
&\quad + \frac{b_5}{M^3} \psi^\dagger \psi \bar{\ell} i \gamma \cdot \mathbf{D}_- \ell + \frac{b_6}{M^3} \epsilon^{ijk} \psi^\dagger \sigma^i \psi m_\ell \bar{\ell} \sigma^{jk} \ell + \frac{b_7}{M^3} \epsilon^{ijk} \psi^\dagger \sigma^i \psi \bar{\ell} \gamma^j D_+^k \ell
\end{aligned}$$

¹⁶We have not specified gauge fixing and source terms, which are also affected by field redefinitions.

$$+ \frac{b_8}{M^3} \psi^\dagger \sigma^i \psi \bar{\ell} \gamma^0 \gamma_5 i D_-^i \ell + \frac{b_9}{M^3} \psi^\dagger \sigma^i \psi i D_-^i \psi \bar{\ell} \gamma^0 \gamma_5 \ell + \mathcal{O}(1/M^4), \quad (2.106)$$

where ℓ is the relativistic lepton field with mass m_ℓ and $\sigma^{ij} \equiv \frac{i}{2}[\gamma^i, \gamma^j]$. The heavy field ψ transforms under boosts as in (2.79), while ℓ transforms under finite dimensional representations of the Lorentz group in the usual way. Under Lorentz transformation, we thus find

$$\delta \mathcal{L}_{\psi\ell} = -\frac{1}{M^3} \psi^\dagger \psi \bar{\ell} q^i \gamma^i \ell (b_1 + 2b_4) - \frac{1}{M^3} \psi^\dagger \sigma^i q^i \psi \bar{\ell} \gamma^0 \gamma_5 \ell (b_2 + 2b_9) + \mathcal{O}(1/M^4). \quad (2.107)$$

This enforces the relations

$$b_4 = \frac{1}{2} b_1, \quad b_9 = -\frac{1}{2} b_2, \quad (2.108)$$

leaving seven operators in this sector through order $1/M^3$, including two at order $1/M^2$.

By performing field redefinitions on the gauge field A^μ , some of these four-fermion operators are found to mix with one-heavy particle operators. In addition to the contributions $\Delta_\gamma A^\mu$ and $\Delta_\psi A^\mu$ from (2.98) we may employ

$$\Delta_\ell A^\mu = g a_{\ell 1} \frac{\bar{\ell} \gamma^\mu \ell}{M^2} + \mathcal{O}(1/M^4). \quad (2.109)$$

We thus find the modified couplings in $\mathcal{L}_\ell$,

$$\delta c_D^{(\ell)} = -Z_\ell a_{\gamma 1} + a_{\ell 1}, \quad (2.110)$$

and for the four fermion operators in $\mathcal{L}_{\psi\ell}$,

$$\frac{\delta b_1}{g^2} = -Z a_{\ell 1} - Z_\ell a_{\psi 1}, \quad \frac{\delta b_7}{g^2} = -Z_\ell a_{\psi 4} - \frac{1}{2} c_F a_{\ell 1}, \quad (2.111)$$

with relation (2.108) remaining intact.

2.9 Discussion

The usual procedure of implementing Lorentz invariance via finite dimensional representations of the Lorentz group is insufficient for application to heavy particle effective theories. We have adapted the formalism of induced representations for application to heavy particle field transformation laws. Returning to the questions posed at the beginning of the chapter, we see that the parameter v enters as an

arbitrary reference vector in the effective theory construction. Rules identifiable with “reparameterization invariance” (2.48) are obtained by a rewriting of the transformation law for generalized boosts, and the class of reparameterization transformations consistent with Lorentz and gauge invariance is identified through a systematic solution of the invariance equation (2.52).

Let us compare our formalism to previous work. A naive ansatz for implementing Lorentz invariance via reparameterization invariance breaks down for $\Gamma(v, iD)$ starting at order $1/M^3$, corresponding to new effects at order $1/M^4$ in the canonical Lagrangian. The transformation law defined by $W(\Lambda, iD)$ is corrected at order $1/M^4$. These subtleties were not treated in the classic work of Luke and Manohar [84, 86], and the ansatz proposed there would lead to inconsistencies at the orders in $1/M$ specified above. Brambilla et al. [16] recognized that Wilson-coefficient dependent corrections to $W(\Lambda)$ must be included when deriving an invariant Lagrangian in canonical form. However, there the constraints of Lorentz invariance are derived (through order $1/M^2$) at the level of canonically quantized charges, a procedure that becomes increasingly cumbersome at high orders in the $1/M$ expansion. In Sect. 2.2 we have used general properties of commutators of the S matrix with conserved charges to derive constraints at the Lagrangian level that implement Lorentz invariance for heavy particle effective theories in canonical form. In Sect. 2.3 we have derived consistent reparameterization transformations that allow solution to the invariance equation (2.52), and hence the construction of manifestly invariant Lagrangians to arbitrary order.

At a practical level, the main results for building heavy fermion Lagrangians are contained in (2.34), or for the invariant operator method, in (2.59) and (2.60). We have illustrated the utility of these results by constructing the NRQED Lagrangian to order $1/M^4$. This provides the rigorous framework for a range of applications such as computing radiative corrections to low-energy lepton-nucleon scattering, and understanding a sharp discrepancy in proton charge radius measurements through scrutinizing proton structure effects in atomic bound states. In the next chapter, we will construct heavy particle Lagrangians for WIMPs interacting with Standard Model particles using the formalism developed here.

Chapter 3

Effective Theory at the Weak-Scale

Having established in the previous chapter ingredients for general construction of heavy particle Lagrangians, we may now proceed to use heavy-particle methods to efficiently describe the interactions of a WIMP, of mass M , with much lighter Standard Model degrees of freedom such as those of $n_f = 5$ flavor QCD (in the case $m_b \ll M$, where m_b is the bottom quark mass) or those of the electroweak sector (in the case $m_W \ll M$, where m_W is the $W^\pm$ boson mass).

Let us construct Lagrangians defined at the weak scale, $\mu_t \sim m_w \sim m_t$, for both theories with unbroken and broken electroweak symmetry. We take the theory symmetric under $SU(3)_c \times SU(2)_W \times U(1)_Y$ as our starting point in place of a specified UV theory for dark matter. Constraints from experimental searches at higher energies (e.g., DM annihilation for indirect searches and DM production for collider searches), as well as constraints from particular UV dynamics or from early universe cosmology, may be incorporated using renormalization group methods. The interplay of these high-energy constraints and the techniques for their robust correlation are interesting aspects of dark matter phenomenology we do not dwell on here.

We focus instead on direct nucleon scattering at low energies, and to investigate this process at energies below the weak scale, we construct the relevant effective theory symmetric under $SU(2)_W \times U(1)_Y$ describing interactions of the WIMP with quarks and gluons of $n_f = 5$ flavor QCD. The basis of operators below the weak scale, written in terms of the lightest, electrically neutral state, is independent of the WIMP electroweak quantum numbers.

We consider the leading $1/M$ interactions for a heavy WIMP of arbitrary spin, stabilized by a Z_2 symmetry, arising from SM extensions consisting of one electroweak multiplet (singlet, doublet and triplet), or two electroweak multiplets combined into admixtures (singlet-doublet and doublet-triplet) [6, 8, 9, 19, 22, 26–29, 33, 36, 42, 46, 56, 59, 65, 68, 72, 76, 78]. For the latter case, a consistent evaluation of amplitudes beyond tree level requires renormalization of WIMP couplings, and we define an extension of the onshell renormalization scheme for

the electroweak SM in the presence of nontrivial residual masses. The construction could be straightforwardly extended to include power corrections, and other light states within the context of specific UV completions. For illustration, the heavy scalar triplet Lagrangian is constructed up to order $1/M^3$, and a sample matching to a toy UV theory is performed. In the case of a singlet-doublet mixture, we also present an alternative derivation of the heavy particle Lagrangian starting from a relativistic theory.

This chapter is organized as follows. In Sect. 3.1 we consider the case of an electroweak singlet, listing the relevant operator building blocks for SM and WIMP degrees of freedom, and constructing the effective Lagrangians above and below the weak scale. For illustration, we also include relativistic contact operators for a scalar (real or complex) and fermion (Majorana or Dirac) electroweak singlet. In Sect. 3.2, we construct the effective theory for one or two heavy electroweak multiplets interacting with SM Higgs and electroweak gauge fields, accounting for masses induced by electroweak symmetry breaking (EWSB), and presenting the Lagrangian in terms of mass eigenstate fields from which the complete set of Feynman rules may be easily derived. In Sect. 3.3, we define an extension of the onshell renormalization scheme for the electroweak SM for a consistent loop-level evaluation of amplitudes. Section 3.4 specifies the low-energy operator basis for the interactions of a self-conjugate WIMP with quarks and gluons of $n_f = 5$ flavor QCD, relevant for spin-independent, low-velocity scattering with nucleons.

3.1 Singlet

The tabulation of operators at a given mass dimension involving SM fields [18, 55, 71] or involving both SM fields and a finite collection of DM fields of given SM quantum numbers, is a straightforward task, but requires some care, e.g., to construct a complete basis while avoiding redundant operators. Let us first consider the simplest case where the SM is extended at low energies by a single Lorentz scalar (real or complex) or fermion (Majorana or Dirac) SM gauge singlet, stabilized by a Z_2 symmetry. We consider separately the scenarios where the DM has mass comparable to or lighter than those of the SM degrees of freedom, and where the DM mass is much heavier. For the latter we use heavy-particle methods to efficiently describe the interactions between the heavy DM field and much lighter SM degrees of freedom.

3.1.1 Standard Model Building Blocks

We collect in Table 3.1 operator building blocks of SM fields invariant under the gauge symmetries of the unbroken and broken electroweak vacua. The set invariant under $SU(3)_c \times SU(2)_W \times U(1)_Y$ is appropriate for investigating processes at

Table 3.1 Gauge-invariant SM operator building blocks of indicated dimension

$SU(3)_c \times SU(2)_W \times U(1)_Y$ SM		$SU(3)_c \times U(1)_{\text{e.m.}}$ SM	
2	$H^\dagger H, B^{\mu\nu}$	2	$F_{\mu\nu}$
3	$\bar{Q}_L^i \gamma^\mu Q_L^j, \bar{u}_L^i \gamma^\mu u_R^j, \bar{d}_L^i \gamma^\mu d_R^j, H^\dagger i D^\mu_- H$	3	$\bar{q}^i [1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \gamma_5] q^j$
4	$\bar{Q}_L^i \gamma^\mu i D^\nu_- Q_L^j, \bar{u}_L^i \gamma^\mu i D^\nu_- u_R^j, \bar{d}_L^i \gamma^\mu i D^\nu_- d_R^j,$	4	$\bar{q}^i [1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \gamma_5] i D^{\rho\sigma}_+ q^j, G^A_{\mu\nu} G^A_{\rho\sigma}$
	$\bar{Q}_L^i [1, \sigma^{\mu\nu}] H d_R^j, \bar{Q}_L^i [1, \sigma^{\mu\nu}] H u_R^j,$	5	$\bar{q}^i [1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \gamma_5] D^{\rho\sigma}_+ q^j,$
	$G^A_{\mu\nu} G^A_{\rho\sigma}, W^B_{\mu\nu} W^B_{\rho\sigma}, B_{\mu\nu} B_{\rho\sigma}, H^\dagger \tau^a H W^a_{\mu\nu},$		$\bar{q}^i [1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \gamma_5] i D^{\rho\sigma}_- q^j,$
	$H^\dagger H B_{\mu\nu}, (H^\dagger H)^2, H^\dagger D^{\mu\nu}_+ H, H^\dagger i D^{\mu\nu}_- H$		$\bar{q}^i [1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \gamma_5] \tau^A q^j G^{A\rho\sigma}$

Here, $\bar{H} = i\tau^2 H^*$, and the hermitian conjugate of some building blocks are not listed. We collect within square brackets, $[\]$, different structures that may be applied to the same field bilinear. Total derivatives of building blocks are not listed above but should be considered in composing operators

energies comparable to or larger than electroweak scale particles. The set invariant under $SU(3)_c \times U(1)_{\text{e.m.}}$ is valid for the effective theory at energies below the electroweak scale having integrated out the massive gauge bosons $W^\pm, Z^0$, the top quark t , and the physical Higgs field h . For the low energy building blocks, we focus on operators describing interactions with quarks and gluons, and only consider interactions with the photon when it is leading such as in the case of a dipole moment. The construction of operator building blocks for interactions with leptons is similar to the quark operators [3, 7, 8, 22, 23].

In constructing operators, we perform Fierz rearrangements to the basis without spinor contractions between DM and SM fields, and use the shorthand $\partial_\pm^\mu \equiv \partial^\mu \pm \overleftarrow{\partial}^\mu$ and $\partial_\pm^{\mu\nu} \equiv (\partial^\mu \partial^\nu \pm \overleftarrow{\partial}^\nu \overleftarrow{\partial}^\mu)$, where $\overleftarrow{\partial}^\mu$ is a derivative acting to the left. We denote the Pauli matrices by τ^a , left- and right-handed fields with respective subscripts L and R , and flavor indices by $i, j = 1 \dots 3$.

A simple and phenomenologically motivated diagonal flavor structure may be obtained by imposing an ad hoc $U(3)_L \times U(3)_R^u \times U(3)_R^d$ symmetry (“minimal flavor violation”), under which the SM quarks and Yukawa matrices (promoted to background fields) transform as

$$\begin{aligned} Q_L &\rightarrow e^{i\epsilon_L} Q_L, & u_R &\rightarrow e^{i\epsilon_R^u} u_R, & d_R &\rightarrow e^{i\epsilon_R^d} d_R, \\ Y_u &\rightarrow e^{i\epsilon_R^u} Y_u e^{-i\epsilon_L}, & Y_d &\rightarrow e^{i\epsilon_R^d} Y_d e^{-i\epsilon_L}. \end{aligned} \quad (3.1)$$

This condition implies that, up to small Yukawa couplings of the b, c, s, d, u quarks and off-diagonal CKM elements, the low-energy building blocks are flavor-diagonal. This assumption, together with imposing a global chiral symmetry under $q_{L,R} \rightarrow e^{i\epsilon_{L,R}} q_{L,R}$ when quark masses vanish, allows us, e.g., to reduce the set of quark building blocks at low-energy to

$$\bar{q} [\gamma^\mu, \gamma^\mu \gamma_5] [1, iD_-^\rho, D_+^{\rho\sigma}, iD_-^{\rho\sigma}, t^A G^{A\rho\sigma}] q, \quad (3.2)$$

with (diagonal) flavor indices dropped. We have eliminated the building blocks

$$m_q \bar{q} [1, \gamma_5, \sigma^{\mu\nu}] [1, iD_-^\rho, \dots] q \quad (3.3)$$

by field redefinitions; consider e.g., the field redefinition of the quark field

$$q \rightarrow (1 + a_1 \mathcal{O}_1 + a_2 \mathcal{O}_2 \gamma_5 + a_3 \mathcal{O}_3^{\mu\nu} \sigma_{\mu\nu}) q, \quad (3.4)$$

where the a_n are arbitrary complex coefficients, and the $\mathcal{O}_n$ are hermitian, Lorentz- and gauge-invariant DM building blocks of mass dimension greater than or equal to two. Other redundancies in the low-energy basis and those appearing in the high-energy electroweak symmetric basis may be similarly accounted for by field redefinitions.

Table 3.2 Gauge-invariant DM operator building blocks of indicated dimension for a relativistic fermion and scalar, and a heavy-particle fermion

Fermion		Scalar	Heavy-particle		
3	$\bar{\psi}[1, i\gamma_5, \gamma^\mu\gamma_5, \{\gamma^\mu, \sigma^{\mu\nu}\}]\psi$	2	$ \phi ^2$	3	$\bar{\chi}_v[1, \{\sigma_\perp^{\mu\nu}\}]\chi_v$
4	$\bar{\psi}[\{1, i\gamma_5, \gamma^\mu\gamma_5\}, \gamma^\mu, \sigma^{\mu\nu}]i\partial^\rho\psi$	3	$\{\phi^*i\partial^\mu\phi\}$	4	$\bar{\chi}_v[\{1\}, \sigma_\perp^{\mu\nu}]i\partial_{\perp-}^\rho\chi_v$
5	$\bar{\psi}[1, i\gamma_5, \gamma^\mu\gamma_5, \{\gamma^\mu, \sigma^{\mu\nu}\}]\partial_+^{\rho\sigma}\psi$	4	$\phi^*\partial_+^{\rho\sigma}\phi$	5	$\bar{\chi}_v[1, \{\sigma_\perp^{\mu\nu}\}]\partial_{\perp+}^{\rho\sigma}\chi_v$
	$\bar{\psi}[\{1, i\gamma_5, \gamma^\mu\gamma_5\}, \gamma^\mu, \sigma^{\mu\nu}]i\partial_-^{\rho\sigma}\psi$		$\{\phi^*i\partial_-^{\rho\sigma}\phi\}$		$\bar{\chi}_v[\{1\}, \sigma_\perp^{\mu\nu}]i\partial_{\perp-}^{\rho\sigma}\chi_v$

For the relativistic case, building blocks within curly brackets, $\{ \}$, vanish for self-conjugate fields such as a Majorana fermion or a real scalar. For the heavy-particle case, building blocks within curly brackets, $\{ \}$, are odd under the parity in Eq. (2.71) and we must include an odd number of v^μ factors. The list for a heavy-particle scalar (of mass dimension 3/2) is obtained simply by omitting building blocks with the spin structure $\sigma_\perp^{\mu\nu}$.

3.1.2 Dark Matter Building Blocks

We collect in Table 3.2 hermitian DM bilinears invariant under the SM gauge symmetries and a stabilizing Z_2 symmetry. The relativistic set is appropriate for describing interactions in the regime where the DM has mass comparable to or less than the relevant SM degrees of freedom. We denote the scalar and fermion DM fields respectively by ϕ and a four-component spinor ψ , and consider both the case where there is an assumed $U(1)$ DM particle number, e.g., a Dirac fermion or complex scalar, and where the DM particle is self-conjugate, e.g., a Majorana fermion ($\psi = \psi^c$) or a real scalar ($\phi = \phi^*$).

In the regime where the DM particle is much heavier than the relevant SM degrees of freedom, we consider heavy-particle building blocks listed in Table 3.2. For integer spin we define $\bar{\chi}_v \equiv \chi_v^*$, while for half-integer spin χ_v carries spinor indices and we define $\bar{\chi}_v \equiv \chi_v^\dagger \gamma^0$. In writing the heavy-particle building blocks in Table 3.2 we assume field redefinitions that eliminate operators with $v \cdot \partial$ acting on χ_v , and hence only perpendicular components of derivatives, $\partial_\perp^\mu$, appear. We define perpendicular components using $g_\perp^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu$, e.g., $\partial_\perp^\mu \equiv \partial_\alpha g_\perp^{\alpha\mu} = \partial^\mu - v^\mu v \cdot \partial$ and $\sigma_\perp^{\mu\nu} \equiv \sigma_{\alpha\beta} g_\perp^{\alpha\mu} g_\perp^{\beta\nu}$. For heavy fields arising from self-conjugate relativistic fields such as a Majorana fermion or real scalar, we require invariance under the self-conjugate parity or CPT transformation described in Sect. 2.4.

3.1.3 High-Energy Basis

We may compose high-scale operators by combining the $SU(3)_c \times SU(2)_W \times U(1)_Y$ invariant SM building blocks in Table 3.1 with the DM building blocks of Table 3.2, considering both the case of relativistic DM and of heavy-particle DM. The Lorentz indices can be contracted with $g^{\mu\nu}$, $\epsilon^{\mu\nu\rho\sigma}$ and, in the case of heavy-particle DM, v^μ . We use the shorthand $\tilde{T}^{\mu\nu} = T_{\alpha\beta} \epsilon^{\alpha\beta\mu\nu}$, and in particular have $\tilde{T}_\perp^{\mu\nu} = T_{\perp\alpha\beta} \epsilon^{\alpha\beta\mu\nu}$,

where the perpendicular Lorentz indices of $T_{\perp\alpha\beta}$ require either μ or ν to be along the v direction. The quark flavor indices can be contracted with arbitrary functions of combinations of Yukawa couplings that preserve the symmetry in Eq. (3.1).

Our focus here is on leading interactions, but higher dimension operators are straightforward to construct. Redundant operators may be identified by performing the most general field redefinition for fields composing operators, and then eliminated by appropriately fixing the parameters of the field redefinition. This yields a linearly independent set having the minimal number of operators that completely parameterize the interactions. Field redefinitions also allow us to choose among linearly dependent operators the simplest or most convenient operator for describing a particular interaction.

Let us consider the regime where the DM mass, M , is comparable to or less than the mass of electroweak scale particles, i.e., $M \lesssim m_W \sim m_h \sim m_t$. Here we write higher dimension (contact) operators suppressed by Λ , the scale representing, e.g., the mass of a heavy mediator that has been integrated out. For the scalar case we have the following interactions,

$$\begin{aligned} \mathcal{L}'_{\phi, \text{SM}} = & -c'_{\phi 1} |\phi|^2 H^\dagger H + \frac{c'_{\phi 2}}{\Lambda^2} |\phi|^2 B_{\mu\nu} B^{\mu\nu} \\ & + \frac{c'_{\phi 3}}{\Lambda^2} |\phi|^2 G_{\mu\nu}^A G^{A\mu\nu} + \frac{c'_{\phi 4}}{\Lambda^2} |\phi|^2 (\bar{Q}_L H d_R + \text{h.c.}) + \dots, \end{aligned} \quad (3.5)$$

where the ellipsis denotes operators of dimension six and higher. There are no dimension five operators, but a long list of dimension six operators of which a few are included above for illustration. The interactions shown in Eq. (3.5) apply for either a real or complex scalar. For the fermion case we have the following interactions,

$$\begin{aligned} \mathcal{L}'_{\psi, \text{SM}} = & -\frac{c'_{\psi 1}}{\Lambda} \bar{\psi} \psi H^\dagger H - \frac{c'_{\psi 2}}{\Lambda} \bar{\psi} i \gamma_5 \psi H^\dagger H + \frac{c'_{\psi 3}}{\Lambda} \bar{\psi} \sigma_{\mu\nu} \psi B^{\mu\nu} + \frac{c'_{\psi 4}}{\Lambda} \bar{\psi} \tilde{\sigma}_{\mu\nu} \psi B^{\mu\nu} \\ & + \frac{c'_{\psi 5}}{\Lambda^2} \bar{\psi} \gamma^\mu i \partial_- \psi B_{\mu\nu} + \frac{c'_{\psi 6}}{\Lambda^2} \bar{\psi} \gamma^\mu i \partial_- \psi \tilde{B}_{\mu\nu} + \frac{c'_{\psi 7}}{\Lambda^2} \bar{\psi} \gamma^\mu \gamma_5 i \partial_- \psi B_{\mu\nu} \\ & + \frac{c'_{\psi 8}}{\Lambda^2} \bar{\psi} \gamma^\mu \gamma_5 i \partial_- \psi \tilde{B}_{\mu\nu} + \frac{c'_{\psi 9}}{\Lambda^2} \bar{\psi} \gamma^\mu \psi \bar{Q}_L f_L \gamma_\mu Q_L \\ & + \frac{c'_{\psi 10}}{\Lambda^2} \bar{\psi} \gamma^\mu \gamma_5 \psi \bar{Q}_L f_L \gamma_\mu Q_L + \frac{c'_{\psi 11}}{\Lambda^2} \bar{\psi} \gamma^\mu \psi \bar{u}_R f_R^\mu \gamma_\mu u_R \\ & + \frac{c'_{\psi 12}}{\Lambda^2} \bar{\psi} \gamma^\mu \gamma_5 \psi \bar{u}_R f_R^\mu \gamma_\mu u_R + \frac{c'_{\psi 13}}{\Lambda^2} \bar{\psi} \gamma^\mu \psi \bar{d}_R f_R^d \gamma_\mu d_R + \frac{c'_{\psi 14}}{\Lambda^2} \bar{\psi} \gamma^\mu \gamma_5 \psi \bar{d}_R f_R^d \gamma_\mu d_R \\ & + \frac{c'_{\psi 15}}{\Lambda^2} \bar{\psi} \gamma_\mu \psi H^\dagger i D_-^\mu H + \frac{c'_{\psi 16}}{\Lambda^2} \bar{\psi} \gamma_\mu \gamma_5 \psi H^\dagger i D_-^\mu H + \dots, \end{aligned} \quad (3.6)$$

where the ellipsis denotes higher dimension operators, and $f_L = f_L(Y_u^\dagger Y_u, Y_d^\dagger Y_d)$, $f_R^\mu = f_R^\mu(Y_u Y_u^\dagger)$ and $f_R^d = f_R^d(Y_d Y_d^\dagger)$ are arbitrary functions of the specified combinations of Yukawa couplings. When the fermion ψ is Majorana, the coefficients $c'_{\psi n}$ for $n = 3, 4, 7, 8, 9, 11, 13, 15$ vanish.

We may also consider the regime where the DM mass, M , is much larger than the masses of electroweak scale particles, i.e., $M \gg m_W \sim m_h \sim m_t$. In this case operators are suppressed by the heavy DM mass M . For a heavy fermion we find the following interactions,

$$\mathcal{L}'_{\chi_v, \text{SM}} = \frac{c'_{\chi 1}}{M} \bar{\chi}_v \chi_v H^\dagger H + \frac{c'_{\chi 2}}{M} \bar{\chi}_v \sigma_{\mu\nu}^\perp \chi_v B^{\mu\nu} + \frac{c'_{\chi 3}}{M} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp \chi_v B^{\mu\nu} + \dots, \quad (3.7)$$

where the ellipsis denotes higher dimension operators. Working at leading order in $1/M$ for the DM-SM interactions, the implementation of Lorentz invariance on the heavy-particle interactions discussed in Chap. 2 imposes no constraints on the Wilson coefficients $c'_{\chi 1}$, $c'_{\chi 2}$ and $c'_{\chi 3}$. The magnetic and electric dipole moments, appearing with respective coefficients $c'_{\chi 2}$ and $c'_{\chi 3}$ above, carry the spin structure $\sigma_\perp$ and are odd under Eq. (2.71) or Eq. (2.76). Hence, both are omitted when considering a heavy scalar or a self-conjugate heavy particle. Higher dimension operators may be similarly constructed (e.g., see [59, 62, 86]), and an extension of the above basis for electroweak charged DM would be relevant for the leading $1/M$ corrections to the universal spin-independent cross section in the heavy WIMP limit [59–61].

3.1.4 Low-Energy Basis

The construction of low-scale operators using the $SU(3)_c \times U(1)_{\text{e.m.}}$ invariant SM building blocks in Table 3.1 and the various DM building blocks of Table 3.2 proceeds as in the previous case of high-scale operators. Since the low-scale interactions are obtained upon integrating out weak scale particles, $W^\pm, Z^0, h, t$, we write higher dimension operators suppressed by the weak scale, parametrically taken to be m_W . We again focus on leading interactions and obtained a complete but redundant-free basis by performing field redefinitions.

Let us first consider the regime where the DM mass is comparable to or smaller than the masses of active low-energy SM degrees of freedom, i.e., $M \lesssim m_p$. For the scalar case we have the following interactions,

$$\begin{aligned} \mathcal{L}_{\phi, \text{SM}} = \sum_{q=u,d,s,c,b} \left\{ \frac{c_{\phi 1}}{m_W^2} |\phi|^2 m_q \bar{q} q + \frac{c_{\phi 2}}{m_W^2} |\phi|^2 m_q \bar{q} i \gamma_5 q + \frac{c_{\phi 3}}{m_W^2} \phi^* i \partial^\mu \phi \bar{q} \gamma_\mu q \right. \\ \left. + \frac{c_{\phi 4}}{m_W^2} \phi^* i \partial^\mu \phi \bar{q} \gamma_\mu \gamma_5 q \right\} + \frac{c_{\phi 5}}{m_W^2} |\phi|^2 G_{\alpha\beta}^A G^{A\alpha\beta} + \frac{c_{\phi 6}}{m_W^2} |\phi|^2 G_{\alpha\beta}^A \tilde{G}^{A\alpha\beta} + \dots, \end{aligned} \quad (3.8)$$

where the ellipsis denotes higher dimension operators and the dependence of quark operator coefficients on the particular quark mass eigenstate q is suppressed. Upon inspection of the low-scale SM buildings blocks in Table 3.1 and the scalar building blocks in Table 3.2, we find that there are no higher dimension operators of odd dimension. For a real scalar the coefficients $c_{\phi n}$ vanish for $n = 3, 4, 8$. For the fermion case we have the following interactions,

$$\begin{aligned}
\mathcal{L}_{\psi, \text{SM}} = & \frac{c_{\psi 1}}{m_W} \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu} + \frac{c_{\psi 2}}{m_W} \bar{\psi} \tilde{\sigma}^{\mu\nu} \psi F_{\mu\nu} \\
& + \sum_{q=u,d,s,c,b} \left\{ \frac{c_{\psi 3}}{m_W^2} \bar{\psi} \gamma^\mu \psi \bar{q} \gamma_\mu q + \frac{c_{\psi 4}}{m_W^2} \bar{\psi} \gamma^\mu \psi \bar{q} \gamma_\mu \gamma_5 q \right. \\
& + \frac{c_{\psi 5}}{m_W^2} \bar{\psi} \gamma^\mu \gamma_5 \psi \bar{q} \gamma_\mu q + \frac{c_{\psi 6}}{m_W^2} \bar{\psi} \gamma^\mu \gamma_5 \psi \bar{q} \gamma_\mu \gamma_5 q + \frac{c_{\psi 7}}{m_W^3} \bar{\psi} \psi m_q \bar{q} q + \frac{c_{\psi 8}}{m_W^3} \bar{\psi} \psi m_q \bar{q} i \gamma_5 q \\
& + \frac{c_{\psi 9}}{m_W^3} \bar{\psi} i \gamma_5 \psi m_q \bar{q} q + \frac{c_{\psi 10}}{m_W^3} \bar{\psi} i \gamma_5 \psi m_q \bar{q} i \gamma_5 q + \frac{c_{\psi 11}}{m_W^3} \bar{\psi} \sigma_{\mu\nu} \psi m_q \bar{q} \sigma^{\mu\nu} q \\
& + \frac{c_{\psi 12}}{m_W^3} \bar{\psi} \tilde{\sigma}_{\mu\nu} \psi m_q \bar{q} \sigma^{\mu\nu} q + \frac{c_{\psi 13}}{m_W^3} \bar{\psi} i \partial_-^\mu \psi \bar{q} \gamma_\mu q + \frac{c_{\psi 14}}{m_W^3} \bar{\psi} i \partial_-^\mu \psi \bar{q} \gamma_\mu \gamma_5 q \\
& \left. + \frac{c_{\psi 15}}{m_W^3} \bar{\psi} \gamma_5 \partial_-^\mu \psi \bar{q} \gamma_\mu q + \frac{c_{\psi 16}}{m_W^3} \bar{\psi} \gamma_5 \partial_-^\mu \psi \bar{q} \gamma_\mu \gamma_5 q \right\} + \frac{c_{\psi 17}}{m_W^3} \bar{\psi} \psi G_{\alpha\beta}^A G^{A\alpha\beta} \\
& + \frac{c_{\psi 18}}{m_W^3} \bar{\psi} \psi G_{\alpha\beta}^A \tilde{G}^{A\alpha\beta} + \frac{c_{\psi 19}}{m_W^3} \bar{\psi} i \gamma_5 \psi G_{\alpha\beta}^A G^{A\alpha\beta} + \frac{c_{\psi 20}}{m_W^3} \bar{\psi} i \gamma_5 \psi G_{\alpha\beta}^A \tilde{G}^{A\alpha\beta} + \dots, \tag{3.9}
\end{aligned}$$

where the ellipsis denotes higher dimension quark and gluon operators as well as photon operators beyond the leading $\mathcal{O}(1/m_W)$ electromagnetic dipole interactions we have included above. The dependence of quark operator coefficients on the particular quark mass eigenstate q is again suppressed. For a Majorana fermion the coefficients $c_{\psi n}$ with $n = 1, 2, 3, 4, 11, 12, 13, 14, 15, 16$ vanish, leaving ten operators through $\mathcal{O}(1/m_W^3)$, as considered in [53]. Lepton operators may be treated similarly to the quark operators.

Let us consider the regime where the DM mass is much larger than the masses of active low-energy SM degrees of freedom, i.e., $M \gg m_b$. For a heavy fermion we find the following interactions,

$$\begin{aligned}
\mathcal{L}_{\chi_v, \text{SM}} = & \frac{c_{\chi 1}}{m_W} \bar{\chi}_v \sigma_{\mu\nu} \chi_v F^{\mu\nu} + \frac{c_{\chi 2}}{m_W} \bar{\chi}_v \tilde{\sigma}_{\mu\nu} \chi_v F^{\mu\nu} \\
& + \sum_{q=u,d,s,c,b} \left\{ \frac{c_{\chi 3}}{m_W^2} \bar{\chi}_v \chi_v \bar{q} \not{p} q + \frac{c_{\chi 4}}{m_W^2} \bar{\chi}_v \chi_v \bar{q} \not{p} \gamma_5 q \right. \\
& + \frac{c_{\chi 5}}{m_W^2} \bar{\chi}_v v^\mu \tilde{\sigma}_{\mu\nu}^\perp \chi_v \bar{q} \gamma^\nu q + \frac{c_{\chi 6}}{m_W^2} \bar{\chi}_v v^\mu \tilde{\sigma}_{\mu\nu}^\perp \chi_v \bar{q} \gamma^\nu \gamma_5 q + \frac{c_{\chi 7}}{m_W^3} \bar{\chi}_v \chi_v m_q \bar{q} q \\
& + \frac{c_{\chi 8}}{m_W^3} \bar{\chi}_v \chi_v m_q \bar{q} i \gamma_5 q + \frac{c_{\chi 9}}{m_W^3} \bar{\chi}_v i \partial_-^\mu \chi_v \bar{q} \gamma_\mu q + \frac{c_{\chi 10}}{m_W^3} \bar{\chi}_v i \partial_-^\mu \chi_v \bar{q} \gamma_\mu \gamma_5 q \\
& + \frac{c_{\chi 11}}{m_W^3} \bar{\chi}_v \chi_v \bar{q} \not{p} i v \cdot D_- q + \frac{c_{\chi 12}}{m_W^3} \bar{\chi}_v \chi_v \bar{q} \not{p} \gamma_5 i v \cdot D_- q + \frac{c_{\chi 13}}{m_W^3} \bar{\chi}_v \sigma_{\mu\nu}^\perp \chi_v m_q \bar{q} \sigma^{\mu\nu} q \\
& + \frac{c_{\chi 14}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp \chi_v m_q \bar{q} \sigma^{\mu\nu} q + \frac{c_{\chi 15}}{m_W^3} \bar{\chi}_v \sigma_{\mu\nu}^\perp \partial_+^{\perp\nu} \chi_v \bar{q} \gamma^\mu q + \frac{c_{\chi 16}}{m_W^3} \bar{\chi}_v \sigma_{\mu\nu}^\perp \partial_+^{\perp\nu} \chi_v \bar{q} \gamma^\mu \gamma_5 q \\
& + \frac{c_{\chi 17}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp \chi_v \bar{q} \gamma^\mu i D_-^{\perp\nu} q + \frac{c_{\chi 18}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp \chi_v \bar{q} \gamma^\mu \gamma_5 i D_-^{\perp\nu} q + \frac{c_{\chi 19}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp i \partial_-^{\perp\nu} \chi_v \bar{q} \gamma^\mu q \\
& \left. + \frac{c_{\chi 20}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp i \partial_-^{\perp\nu} \chi_v \bar{q} \gamma^\mu \gamma_5 q + \frac{c_{\chi 21}}{m_W^3} \bar{\chi}_v \sigma_{\mu\nu}^\perp i \partial_-^{\perp\nu} \chi_v \bar{q} \gamma^\mu q \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{c_{\chi 22}}{m_W^3} \bar{\chi}_v \sigma_{\mu\nu}^\perp i \partial_-^{\perp\nu} \chi_v \bar{q} \gamma^\mu \gamma_5 q + \frac{c_{\chi 23}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp \partial_+^{\perp\nu} \chi_v \bar{q} \gamma^\mu q \\
& + \frac{c_{\chi 24}}{m_W^3} \bar{\chi}_v \tilde{\sigma}_{\mu\nu}^\perp \partial_+^{\perp\nu} \chi_v \bar{q} \gamma^\mu \gamma_5 q \left. \vphantom{\frac{c_{\chi 24}}{m_W^3}} \right\} + \frac{c_{\chi 25}}{m_W^3} \bar{\chi}_v \chi_v G_{\alpha\beta}^A G^{A\alpha\beta} + \frac{c_{\chi 26}}{m_W^3} \bar{\chi}_v \chi_v G_{\alpha\beta}^A \tilde{G}^{A\alpha\beta} \\
& + \frac{c_{\chi 27}}{m_W^3} \bar{\chi}_v \chi_v v_\mu v_\nu G_\alpha^{A\mu} G^{A\alpha\nu} + \dots, \tag{3.10}
\end{aligned}$$

where the ellipsis denotes higher dimension operators. As discussed in Chap. 2, constraints on coefficients in Eq. (3.10) consistent with Lorentz symmetry may be derived by performing an infinitesimal boost under which the heavy field χ_v and a Lorentz vector A^μ transform as [57, 84],

$$\begin{aligned}
\chi_v & \rightarrow e^{iqx} \left[1 + \frac{iq \cdot D_\perp}{2m_W^2} + \dots \right] \chi_v, \quad A_\perp^\mu \rightarrow A_\perp^\mu - \frac{q^\mu v \cdot A}{m_W}, \\
v \cdot A & \rightarrow v \cdot A + \frac{q \cdot A_\perp}{m_W}. \tag{3.11}
\end{aligned}$$

Here the ellipsis denotes terms in the heavy-field transformation higher order in $1/m_W$, and the usual transformation for a Lorentz vector has been projected along v^μ and the perpendicular directions.

Working to leading order, the gluon and photon interactions in Eq. (3.10) are readily invariant under boosts. On the other hand, demanding that variations of quark operators under the transformations in Eq. (3.11) vanish, we find the constraints

$$(c_{\chi 3} - 2c_{\chi 9}) = (c_{\chi 4} - 2c_{\chi 10}) = (c_{\chi 5} + 2c_{\chi 19}) = (c_{\chi 6} + 2c_{\chi 20}) = c_{\chi 21} = c_{\chi 22} = 0. \tag{3.12}$$

This leaves 16 independent quark operators, and we may check that imposing parity and time-reversal symmetry yields the seven operators that describe nucleon-lepton interactions in NRQED [62]. The basis for a heavy scalar is obtained by omitting in Eq. (3.10) operators containing the spin structure $\sigma_{\mu\nu}^\perp$. The basis for a self-conjugate heavy-particle is obtained by imposing invariance under Eq. (2.71) or Eq. (2.76); in particular we find that for a self-conjugate heavy fermion the coefficients $c_{\chi n}$ vanish for $n = 1, 2, 3, 4, 9, 10, 13, 14, 15, 16, 17, 18, 23, 24$.

3.2 Multiplets and Mixtures

Let us use heavy particle effective theory to describe extensions of the SM consisting of one or two electroweak multiplets with masses large compared to the mass of electroweak-scale particles, $M, M' \gg m_W$. The extension to more than two multiplets is straightforward. We will construct the effective theory describing interactions of such heavy WIMPs with the SM in the regime $|M' - M|, m_W \ll M, M'$. In the case $|M' - M| \gg m_W$ the effects of the heavier multiplet appear as power corrections in the effective theory for the lighter multiplet. For notational clarity, below we omit the subscript v labeling a heavy-particle field.

Consider one or two multiplets of heavy-particle fields with arbitrary spin,¹ transforming under irreducible representations of electroweak $SU(2)_W \times U(1)_Y$. Let us collect the heavy fields in a column vector h , and their masses in a diagonal matrix M . The precise specification of M beyond tree level is described in Sect. 3.3. At leading order in the $1/M$ expansion, the most general gauge- and Lorentz-invariant Lagrangian, bilinear in h , and written in terms of the building-blocks h , v^μ , and SM fields, takes the form

$$\mathcal{L} = \bar{h} [iv \cdot D - \delta m - f(H)] h + \mathcal{O}(1/M), \quad (3.13)$$

where $iD_\mu = i\partial_\mu + g_1 Y B_\mu + g_2 W_\mu^a T^a$, and $f(H)$ is a linear matrix function of H (and H^*). For pure states gauge invariance implies $f(H) = \mathbb{0}$, while for mixed states $f(H)$ describes the mixing of the pure-state constituents through the Higgs field. In terms of a reference mass M_{ref} , the residual mass matrix is

$$\delta m = M - M_{\text{ref}} \mathbb{1}. \quad (3.14)$$

Note that if the masses composing M are degenerate, as for a single “pure” electroweak multiplet, we may choose M_{ref} appropriately to set $\delta m = \mathbb{0}$. In the case of two “mixed” electroweak multiplets M will have non-degenerate entries in general.

Upon accounting for EWSB we may write (3.13) as

$$\begin{aligned} \mathcal{L} = \bar{h} \bigg[& iv \cdot \partial + e Q v \cdot A + \frac{g_2}{c_W} v \cdot Z (T^3 - s_W^2 Q) \\ & + \frac{g_2}{\sqrt{2}} (v \cdot W^+ T^+ + v \cdot W^- T^-) - \delta M(v_{\text{wk}}) - f(\phi) \bigg] h + \mathcal{O}(1/M), \end{aligned} \quad (3.15)$$

where $T^\pm = T^1 \pm iT^2$, the charge matrix is $Q = T^3 + Y$ in units of the proton charge, and ϕ denotes the fluctuation of the Higgs field about $\langle H \rangle$,

$$H = \frac{v_{\text{wk}}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \phi_w^+ \\ \frac{1}{\sqrt{2}}(h + i\phi_Z) \end{pmatrix}. \quad (3.16)$$

The residual mass matrix now includes EWSB contributions,

$$\delta M(v_{\text{wk}}) = \delta m + f(\langle H \rangle), \quad (3.17)$$

and in the mass eigenstate basis for $\delta M(v_{\text{wk}})$, we will set the residual mass of the lightest, (assumed) electrically neutral WIMP, χ , to zero by appropriate choice of M_{ref} . Other states may have non-vanishing residual masses. In the following, we will suppress the subscript in v_{wk} ; the resulting v is not to be confused with the velocity v^μ .

The heavy-particle Lagrangian (3.13) can also be obtained at tree level from a manifestly relativistic Lagrangian by performing field redefinitions. We illustrate this for the singlet-doublet mixture in Sect. 3.2.5.² Let us now have a detailed look at extensions with one (pure states) or two (mixed states) electroweak multiplets.

¹For integer spin we define $\bar{h}_v \equiv h_v^\dagger$, while for half-integer spin h_v carries spinor indices and we define $\bar{h}_v \equiv h_v^\dagger \gamma^0$.

²We remark that the consistency of an effective description for the one-heavy particle sector for a self-conjugate field follows from the identification of lowest-lying states odd under a Z_2 symmetry.

3.2.1 Pure States

The pure-state heavy-particle Lagrangian is completely specified by electroweak quantum numbers since $\delta m = 0$ and $f(H) = 0$. We may proceed in generality, assuming a multiplet of fields in the isospin J representation of $SU(2)_W$ with hypercharge Y . The amplitudes for weak-scale matching in Sect. 4.2 will be given in terms of Y^2 and the Casimir $J(J+1)$. In particular, amplitudes with two $W^\pm$ bosons or two Z^0 bosons carry the respective factors

$$C_W = J(J+1) - Y^2, \quad C_Z = Y^2. \quad (3.18)$$

For extensions consisting of electroweak multiplets with non-zero hypercharge, we assume that higher-dimension operators cause the mass eigenstates after EWSB to be self-conjugate combinations. This forbids a phenomenologically disfavored tree-level vector coupling between the lightest, electrically neutral state, χ , and Z^0 .

As specific illustrations we consider the cases of an $SU(2)_W$ triplet ($J = 1$) with $Y = 0$, and a pair of $SU(2)_W$ doublets ($J = 1/2$) with opposite hypercharge $Y = \pm 1/2$. In supersymmetric extensions, these represent pure wino and pure higgsino states, respectively. Let us look at these cases in some detail.

Pure Triplet

Let the column vector $h_T = (h^1, h^2, h^3)$, with subscript T for triplet, be a heavy, self-conjugate, $SU(2)_W$ triplet with $Y = 0$. The heavy-particle Lagrangian for h_T is given by (3.13) with $(T^a)^{bc} = i\epsilon^{bac}$, $f(H) = 0$, and $\delta m = 0$. The electric charge eigenbasis is given by

$$\begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} \equiv \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_+ \\ h_- \end{pmatrix}. \quad (3.19)$$

In terms of the column vector $h = (h_0, h_+, h_-)$, where $h_0 \equiv \chi$, the Lagrangian is given by (3.15) with

$$Q = T^3 = \text{diag}(0, 1, -1), \quad T^+ = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}. \quad (3.20)$$

In contrast, the one-heavy particle sector for a heavy field carrying $U(1)$ global symmetry (e.g., heavy-quark number in a heavy quark effective theory) is identified by this quantum number.

Pure Doublet

Let h_ψ and h_{ψ^c} be heavy-particle doublets in the $(\mathbf{2}, 1/2)$ and $(\bar{\mathbf{2}}, -1/2)$ representations of $SU(2)_W \times U(1)_Y$.³ Anticipating perturbations that cause the mass eigenstates to be self-conjugate fields, let us introduce the linear combinations

$$h_{D_1} = \frac{h_\psi + h_{\psi^c}}{\sqrt{2}} = \begin{pmatrix} h_1 \\ h_0 \end{pmatrix}, \quad h_{D_2} = \frac{i(h_\psi - h_{\psi^c})}{\sqrt{2}} = \begin{pmatrix} h_2 \\ h'_0 \end{pmatrix}, \quad (3.21)$$

with subscript D for doublet. The heavy-particle Lagrangian for the column vector $h = (h_{D_1}, h_{D_2})$ is given by (1), with $f(H) = \mathbb{0}$, and gauge couplings

$$T^a = \begin{pmatrix} \frac{\tau^a - \tau^{aT}}{4} & \frac{-i(\tau^a + \tau^{aT})}{4} \\ \frac{i(\tau^a + \tau^{aT})}{4} & \frac{\tau^a - \tau^{aT}}{4} \end{pmatrix}, \quad Y = \frac{i}{2} \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad (3.22)$$

where τ^a are the Pauli isospin matrices. Neglecting the small mass perturbation mentioned above, the tree-level mass eigenstates are degenerate, and we may choose $\delta m = \mathbb{0}$. The charge eigenstates are given by

$$\begin{pmatrix} h_1 \\ h_0 \\ h_2 \\ h'_0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} h_0 \\ h'_0 \\ h_+ \\ h_- \end{pmatrix}. \quad (3.23)$$

In terms of the column vector $h = (h_0, h'_0, h_+, h_-)$, where $h_0 \equiv \chi$, the Lagrangian is given by (3.15) with $Q = \text{diag}(\mathbb{0}_2, 1, -1)$ and

$$\begin{aligned} T^3 &= \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad T^+ = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ T^- &= \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.24)$$

³This construction is analogous to that appearing in applications of heavy quark effective theory to processes where both a heavy quark and a heavy anti-quark are active degrees of freedom.

3.2.2 Higher-Order Example: Pure Triplet Scalar

As illustration let us investigate interactions for a scalar particle of mass M , charged under electroweak $SU(2)$, beyond the leading order required for the heavy WIMP limit. We work in terms of an effective heavy scalar field $\phi_v(x)$, in the isospin J representation of $SU(2)$. The covariant derivative is $D_\mu = \partial_\mu - ig_2 W_\mu^a t_J^a$ and $W_{\mu\nu} \equiv i[D_\mu, D_\nu]/g_2 \equiv W_{\mu\nu}^a t_J^a$ is the associated field strength. Through $\mathcal{O}(1/M^3)$, the scalar heavy particle effective theory in the one-heavy-particle sector takes the form,

$$\begin{aligned} \mathcal{L}_\phi = \phi_v^* \Big\{ & iv \cdot D - c_1 \frac{D_\perp^2}{2M} + c_2 \frac{D_\perp^4}{8M^3} + g_2 c_D \frac{v^\alpha [D_\perp^\beta, W_{\alpha\beta}]}{8M^2} \\ & + ig_2 c_M \frac{\{D_\perp^\alpha, [D_\perp^\beta, W_{\alpha\beta}]\}}{16M^3} + g_2^2 c_{A1} \frac{W^{\alpha\beta} W_{\alpha\beta}}{16M^3} + g_2^2 c_{A2} \frac{v_\alpha v^\beta W^{\mu\alpha} W_{\mu\beta}}{16M^3} \\ & + g_2^2 c_{A3} \frac{\text{Tr}(W^{\alpha\beta} W_{\alpha\beta})}{16M^3} + g_2^2 c_{A4} \frac{v_\alpha v^\beta \text{Tr}(W^{\mu\alpha} W_{\mu\beta})}{16M^3} \\ & + g_2^2 c'_{A1} \frac{\epsilon^{\mu\nu\rho\sigma} W_{\mu\nu} W_{\rho\sigma}}{16M^3} + g_2^2 c'_{A2} \frac{\epsilon^{\mu\nu\rho\sigma} v^\alpha v_\mu W_{\nu\alpha} W_{\rho\sigma}}{16M^3} \\ & + g_2^2 c'_{A3} \frac{\epsilon^{\mu\nu\rho\sigma} \text{Tr}(W_{\mu\nu} W_{\rho\sigma})}{16M^3} + g_2^2 c'_{A4} \frac{\epsilon^{\mu\nu\rho\sigma} v^\alpha v_\mu \text{Tr}(W_{\nu\alpha} W_{\rho\sigma})}{16M^3} + \dots \Big\} \phi_v, \end{aligned} \quad (3.25)$$

where we have employed appropriate field redefinitions to remove possible redundant operators involving factors of $v \cdot D$ acting on ϕ_v . Note that the operators with coefficients c'_{A1} through c'_{A4} violate parity and time reversal symmetries.⁴ For the effective theory describing a fundamental heavy scalar particle, we have $c_1 = c_2 = c_{A1} = 1$ and $c_D = c_M = c_{A2} = c_{A3} = c_{A4} = c'_{A1} = c'_{A2} = c'_{A3} = c'_{A4} = 0$ at tree level [66]. We find that Lorentz invariance implies the exact relations,

$$c_1 = c_2 = 1, \quad c_M = c_D. \quad (3.26)$$

Below, an explicit matching calculation provides a nontrivial illustration of the latter relation.

The complete Lagrangian including Standard Model particles and interactions can be written

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_{\text{SM}} + \mathcal{L}_{\phi, \text{SM}}. \quad (3.27)$$

Here $\mathcal{L}_{\text{SM}}$ is the usual Standard Model Lagrangian, and by convention we have included interactions with W_μ in $\mathcal{L}_\phi$. So far our discussion applies to a general irreducible $SU(2)$ representation for the heavy scalar field ϕ_v . Specializing to the case of a real scalar field, necessarily with integer isospin, the effective theory is invariant under Eq. (2.71). It is straightforward to verify that all interactions in $\mathcal{L}_\phi$ are invariant under this transformation.

⁴Additional CPT violating operators at $\mathcal{O}(1/M^2)$ and $\mathcal{O}(1/M^3)$ are constrained by Lorentz invariance to have vanishing coefficient.

In the one-heavy-particle sector, the remaining terms involving the Higgs field H , gauge fields, and fermions are ($\tilde{H} \equiv i\tau_2 H^*$)

$$\begin{aligned}
\mathcal{L}_{\phi, \text{SM}} = \phi_v^* \Big\{ & c_H \frac{H^\dagger H}{M} + \dots + c_Q \frac{t_J^a \bar{Q}_L \tau^a \not{p} Q_L}{M^2} + c_X \frac{i \bar{Q}_L \tau^a \gamma^\mu Q_L \{t_J^a, D_\mu\}}{2M^3} \\
& + c_{DQ} \frac{\bar{Q}_L \not{p} i v \cdot D Q_L}{M^3} + c_{Du} \frac{\bar{u}_R \not{p} i v \cdot D u_R}{M^3} + c_{Dd} \frac{\bar{d}_R \not{p} i v \cdot D d_R}{M^3} + c_{Hd} \frac{\bar{Q}_L H d_R + h.c.}{M^3} \\
& + c_{Hu} \frac{\bar{Q}_L \tilde{H} u_R + h.c.}{M^3} + g^2 c_{A1}^{(G)} \frac{G^A{}^{\alpha\beta} G_{\alpha\beta}^A}{16M^3} + g^2 c_{A2}^{(G)} \frac{v_\alpha v^\beta G^A{}^{\mu\alpha} G_{\mu\beta}^A}{16M^3} \\
& + g^2 c_{A1}^{(G)} \frac{\epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^A G_{\rho\sigma}^A}{16M^3} + g^2 c_{A2}^{(G)} \frac{\epsilon^{\mu\nu\rho\sigma} v^\alpha v_\mu G_{\nu\alpha}^A G_{\rho\sigma}^A}{16M^3} + \dots \Big\} \phi_v. \quad (3.28)
\end{aligned}$$

Terms odd under (2.71) have been omitted. Subleading terms containing only H , ϕ_v and their covariant derivatives are represented by the first ellipsis in (3.28). Terms bilinear in lepton fields, and terms bilinear in the hypercharge gauge field are also present in $\mathcal{L}_{\phi, \text{SM}}$ but have not been written explicitly. Repeated indices $a = 1\dots 3$ and $A = 1\dots 8$ imply a sum over gauge generators. Lorentz invariance implies

$$c_Q = c_X. \quad (3.29)$$

As an illustration of the construction and matching conditions for the heavy particle Lagrangian $\mathcal{L}_\phi$, consider the case of a fundamental scalar, ignoring scalar self interactions (i.e., ϕ^4 terms). For the matching of the terms containing a single gauge field, we consider the full theory result for the $W\phi\phi$ amputated three point function (cf. Fig. 2.1),

$$ig_2(p+p')^\mu F(q^2)(t_J^a)_{ji}, \quad (3.30)$$

where $q = p' - p$, and $F(q^2)$ is a model-dependent form factor. Setting $p^2 = p'^2 = M^2$, $v^\mu = (1, 0, 0, 0)$, the matching conditions for scalar scattering from a $\mu = 0$ or $\mu = i$ gauge field read

$$\begin{aligned}
F(0) - F'(0)q^2 + \dots &= 1 - c_D \frac{q^2}{8M^2} + \dots, \\
(p+p')^i \left[-F(0) \left(1 - \frac{p^2 + p'^2}{4M^2} \right) + F'(0)q^2 + \dots \right] \\
&= (p+p')^i \left[-1 + \frac{p^2 + p'^2}{4M^2} + c_M \frac{q^2}{8M^2} \right] + q^i \frac{p'^2 - p^2}{8M^2} (c_D - c_M) + \dots \quad (3.31)
\end{aligned}$$

An explicit computation of one-loop gauge boson corrections, employing dimensional regularization in $d = 4 - 2\epsilon$ dimensions, yields

$$\begin{aligned}
F(q^2) = 1 + \frac{g_2^2}{(4\pi)^2} \frac{q^2}{M^2} \Big\{ & C_2(r) \left[-\frac{2}{3\epsilon_{\text{IR}}} - 1 + \frac{4}{3} \log \frac{M}{\mu} \right] \\
& + C_2(G) \left[-\frac{1}{24\epsilon_{\text{IR}}} + \frac{3}{4} + \frac{1}{12} \log \frac{M}{\mu} \right] \Big\} + \dots \quad (3.32)
\end{aligned}$$

The quadratic Casimir coefficients for the isospin- J and adjoint representations of $SU(2)$ are $C_2(J) = J(J+1)$ and $C_2(G) = 2$. From (3.31) and (3.32), after effective theory subtractions the renormalized coefficients $c_D(\mu)$, $c_M(\mu)$ in the $\overline{\text{MS}}$ renormalization scheme are found to be

$$c_D(\mu) = c_M(\mu) = \frac{\alpha_2(\mu)}{4\pi} \left[-8J(J+1) + 12 + \left(\frac{32J(J+1)}{3} + \frac{4}{3} \right) \log \frac{M}{\mu} \right]. \quad (3.33)$$

Matching for a general ultraviolet completion model, and for other effective theory coefficients proceeds similarly.

Our focus will be on the limit $M \gg m_W$, where all nontrivial matching conditions at the scale $\mu \sim M$ become irrelevant.⁵ We leave a detailed investigation of the model-dependent form factor and subleading $1/M$ corrections to future work.

3.2.3 Admixtures

As an example of mixed states, let us consider in detail the singlet-doublet admixture. Results for the triplet-doublet admixture will also be given below.

Singlet-Doublet Admixture

Let h_S , with subscript S for singlet, be a heavy, self-conjugate, $SU(2)_W$ singlet with $Y = 0$ and mass M_S . Consider an admixture of h_S and the previously defined self-conjugate doublets h_{D_1} and h_{D_2} , with mass M_D . At leading order in the $1/M$ expansion, the gauge-invariant interactions of h_S , h_{D_1} and h_{D_2} involving the Higgs field are

$$\mathcal{L}_{H\bar{h}h} = -\bar{h}_S \left[y H^\dagger \frac{(h_{D_1} - i h_{D_2})}{\sqrt{2}} + y^* H^T \frac{(h_{D_1} + i h_{D_2})}{\sqrt{2}} \right] + \text{h.c.} = -\bar{h} f(H) h, \quad (3.34)$$

where we have imposed the invariance (2.71), and collected the heavy-particle fields in a column vector $h = (h_S, h_{D_1}, h_{D_2}) = (h_S, h_1, h_0, h_2, h'_0)$. The Higgs coupling matrix is given by

$$f(H) = \frac{a_1}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} + \frac{a_2}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \quad (3.35)$$

⁵In particular models with multiple mass scales, $1/M$ prefactors can be replaced by inverse powers of a smaller excitation energy. It is also of interest to investigate whether large anomalous dimensions could enhance the coefficients of particular subleading operators.

with real parameters $a_1 = \text{Re}(y)$ and $a_2 = \text{Im}(y)$. For comparison, the derivation in Sect. 3.2.5 obtains (3.34) at tree level starting from a manifestly relativistic Lagrangian. The residual mass matrix is $\delta m = \text{diag}(M_S, M_D \mathbb{1}_4) - M_{\text{ref}} \mathbb{1}_5$, and we define M_S and M_D to be real and positive.⁶ The gauge couplings are obtained by trivially extending (3.22) to include the singlet. This completely specifies the heavy-particle Lagrangian given in (3.13).

The mass induced by EWSB is accounted for at tree level by including contributions from (3.35),

$$\delta M(v) = \delta M + v \begin{pmatrix} 0 & 0 & a_1 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.36)$$

In the following, we use subscripts to denote the electric charge and bracketed superscripts to label the mass eigenstate. For neutral states we find the residual mass eigenvalues

$$\delta_0^{(0)} = M_D - M_{\text{ref}}, \quad \delta_0^{(\pm)} = \frac{M_D + M_S}{2} \pm \sqrt{\Delta^2 + (av)^2} - M_{\text{ref}}, \quad (3.37)$$

where we define

$$\Delta \equiv \frac{M_S - M_D}{2}, \quad a \equiv \sqrt{a_1^2 + a_2^2}. \quad (3.38)$$

By definition $a > 0$, and regardless of the sign of Δ , the smallest eigenvalue is $\delta_0^{(-)}$. Let us set this eigenvalue to zero by appropriately choosing the reference mass M_{ref} . The corresponding normalized eigenvectors in the (h_S, h_0, h'_0) basis of electrically neutral states are then

$$\begin{aligned} \vec{v}_0^{(0)} &= \frac{1}{a} \begin{pmatrix} 0 \\ a_2 \\ -a_1 \end{pmatrix}, \\ \vec{v}_0^{(\pm)} &= \frac{1}{\left[\left(\Delta \pm \sqrt{\Delta^2 + (av)^2} \right)^2 + (av)^2 \right]^{\frac{1}{2}}} \begin{pmatrix} \Delta \pm \sqrt{\Delta^2 + (av)^2} \\ a_1 v \\ a_2 v \end{pmatrix}, \end{aligned} \quad (3.39)$$

and we may construct the unitary matrix U_0 (on the three-dimensional neutral subspace) to translate to the mass eigenbasis,

$$U_0 = \begin{pmatrix} \vec{v}_0^{(0)} & \vec{v}_0^{(+)} & \vec{v}_0^{(-)} \end{pmatrix}, \quad \begin{pmatrix} h_S \\ h_0 \\ h'_0 \end{pmatrix} = U_0 \begin{pmatrix} h_0^{(0)} \\ h_0^{(+)} \\ h_0^{(-)} \end{pmatrix},$$

⁶An additional phase redefinition of h_ψ, h_{ψ^c} could be used to enforce the vanishing of a_1 or a_2 .

$$U_0^\dagger \delta M(v) U_0 = \text{diag} \left(\delta_0^{(0)}, \delta_0^{(+)}, \delta_0^{(-)} \right). \quad (3.40)$$

The tree-level masses for the electrically charged sector are unchanged by EWSB, given by $\delta_\pm^{(0)} = \delta_0^{(0)}$, and the corresponding charge eigenstates are given by

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} h_+^{(0)} \\ h_-^{(0)} \end{pmatrix}. \quad (3.41)$$

The basis of mass eigenstates is thus given by the column vector $h = \left(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(0)}, h_-^{(0)} \right)$, where $h_0^{(-)} \equiv \chi$, and the Lagrangian is given by (3.15) with

$$\begin{aligned} \delta M(v) &= \text{diag} \left(\delta_0^{(0)}, \delta_0^{(+)}, \delta_0^{(-)}, \delta_+^{(0)}, \delta_-^{(0)} \right) = av \text{diag} \left(t_\rho, 2s_\rho^{-1}, 0, t_\rho, t_\rho \right), \\ Q &= \text{diag}(0_3, 1, -1), \\ T^3 - s_W^2 Q &= \begin{pmatrix} 0 & \frac{i}{2}|s_\rho| & \frac{i}{2}|c_\rho| & 0 & 0 \\ -\frac{i}{2}|s_\rho| & 0 & 0 & 0 & 0 \\ -\frac{i}{2}|c_\rho| & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} - s_W^2 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} + s_W^2 \end{pmatrix}, \\ T^+ &= \frac{e^{-i\xi}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & -|s_\rho| \\ 0 & 0 & 0 & 0 & -|c_\rho| \\ i|s_\rho| & |c_\rho| & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^- = \frac{e^{+i\xi}}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & |s_\rho| & 0 \\ 0 & 0 & 0 & |c_\rho| & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i|s_\rho| & -|c_\rho| & 0 & 0 & 0 \end{pmatrix}, \\ f(\phi) &= a \begin{pmatrix} 0 & |c_\rho| \phi_Z & -|s_\rho| \phi_Z & 0 & 0 \\ |c_\rho| \phi_Z & s_\rho h & c_\rho h & |c_\rho| e^{+i\xi} \phi_W^- & |c_\rho| e^{-i\xi} \phi_W^+ \\ -|s_\rho| \phi_Z & c_\rho h & -s_\rho h & -|s_\rho| e^{+i\xi} \phi_W^- & -|s_\rho| e^{-i\xi} \phi_W^+ \\ 0 & |c_\rho| e^{-i\xi} \phi_W^+ & -|s_\rho| e^{-i\xi} \phi_W^+ & 0 & 0 \\ 0 & |c_\rho| e^{+i\xi} \phi_W^- & -|s_\rho| e^{+i\xi} \phi_W^- & 0 & 0 \end{pmatrix}, \end{aligned} \quad (3.42)$$

where we have introduced

$$\sin \rho \equiv \frac{av}{\sqrt{(av)^2 + \Delta^2}}, \quad \cos \rho \equiv \frac{\Delta}{\sqrt{(av)^2 + \Delta^2}}, \quad e^{\pm i\xi} \equiv \frac{(a_1 \pm ia_2)}{a}. \quad (3.43)$$

The shorthand notation $c_x \equiv \cos x$, $s_x \equiv \sin x$, and $t_x \equiv \tan x$ is used throughout this thesis. Note that s_ρ is positive, and that c_ρ can have either sign depending on the hierarchy between M_S and M_D . It is straightforward to extract Feynman rules from the Lagrangian (3.15) and the matrices (3.42). For example, the propagator for χ , and its coupling to the physical Higgs boson, h , are

$$\begin{aligned}
 \chi \xrightarrow{k} \chi &= \frac{i}{v \cdot k - \delta_0^{(-)} + i0}, & \begin{array}{c} \chi \quad \chi \\ \hline \hline | \\ h \end{array} &= ias_\rho.
 \end{aligned} \tag{3.44}$$

Triplet-Doublet Admixture

The construction for the triplet-doublet case follows closely that for the singlet-doublet case, with a heavy triplet h_T in place of the singlet h_S . Using $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ and $\bar{\boldsymbol{\tau}} = -(\tau^{1T}, \tau^{2T}, \tau^{3T})$, the gauge-invariant interactions of h_T , h_{D1} and h_{D2} involving the Higgs field can be written in the form $\mathcal{L}_{H\bar{h}h} = -\bar{h}f(H)h$, where we collect fields in a seven-component column vector $h = (h_T, h_{D1}, h_{D2})$, and the matrix $f(H)$ is given by

$$\begin{aligned}
 f(H) &= \frac{a_1}{\sqrt{2}} \begin{pmatrix} \mathbb{0}_3 & H^T \boldsymbol{\tau} - H^T \bar{\boldsymbol{\tau}} & i(-H^T \bar{\boldsymbol{\tau}} - H^T \boldsymbol{\tau}) \\ -\bar{\boldsymbol{\tau}} H^* + \boldsymbol{\tau} H & \mathbb{0}_2 & \mathbb{0}_2 \\ i(\boldsymbol{\tau} H + \bar{\boldsymbol{\tau}} H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \\
 &+ \frac{a_2}{\sqrt{2}} \begin{pmatrix} \mathbb{0}_3 & i(H^T \bar{\boldsymbol{\tau}} + H^T \boldsymbol{\tau}) & H^T \boldsymbol{\tau} - H^T \bar{\boldsymbol{\tau}} \\ i(-\boldsymbol{\tau} H - \bar{\boldsymbol{\tau}} H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ -\bar{\boldsymbol{\tau}} H^* + \boldsymbol{\tau} H & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \tag{3.45}
 \end{aligned}$$

with real parameters a_1 and a_2 . Upon accounting for mass contributions from EWSB, the basis of mass eigenstates is given by the column vector $h = (h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(+)}, h_+^{(-)}, h_-^{(+)}, h_-^{(-)})$, where $h_0^{(-)} \equiv \chi$, and the Lagrangian is given by (3.15) with

$$\begin{aligned}
 \delta M(v) &= \text{diag} \left(\delta_0^{(0)}, \delta_0^{(+)}, \delta_0^{(-)}, \delta_+^{(+)}, \delta_+^{(-)}, \delta_-^{(+)}, \delta_-^{(-)} \right) \\
 &= av \text{diag} \left(t_{\frac{\rho}{2}}, 2s_\rho^{-1}, 0, 2s_\rho^{-1}, 0, 2s_\rho^{-1}, 0 \right),
 \end{aligned}$$

$$Q = \text{diag}(0, 0, 0, 1, 1, -1, -1),$$

$$T^3 = \begin{pmatrix} 0 & \frac{i}{2}|s_{\frac{\rho}{2}}| & \frac{i}{2}|c_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 \\ -\frac{i}{2}|s_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{2}|c_{\frac{\rho}{2}}| & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - \frac{1}{2}s_{\frac{\rho}{2}}^2 & -\frac{1}{4}s_\rho & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4}s_\rho & 1 - \frac{1}{2}c_{\frac{\rho}{2}}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 + \frac{1}{2}s_{\frac{\rho}{2}}^2 & \frac{1}{4}s_\rho \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4}s_\rho & -1 + \frac{1}{2}c_{\frac{\rho}{2}}^2 \end{pmatrix},$$

$$\begin{aligned}
T^+ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i|s_{\frac{\rho}{2}}| & i|c_{\frac{\rho}{2}}| \\ 0 & 0 & 0 & 0 & 0 & 1 + c_{\frac{\rho}{2}}^2 & -\frac{1}{2}s_{\rho} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}s_{\rho} & 1 + s_{\frac{\rho}{2}}^2 \\ -i|s_{\frac{\rho}{2}}| & -1 - c_{\frac{\rho}{2}}^2 & \frac{1}{2}s_{\rho} & 0 & 0 & 0 & 0 \\ -i|c_{\frac{\rho}{2}}| & \frac{1}{2}s_{\rho} & -1 - s_{\frac{\rho}{2}}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
T^- &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i|s_{\frac{\rho}{2}}| & i|c_{\frac{\rho}{2}}| & 0 & 0 \\ 0 & 0 & 0 & -1 - c_{\frac{\rho}{2}}^2 & \frac{1}{2}s_{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_{\rho} & -1 - s_{\frac{\rho}{2}}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i|s_{\frac{\rho}{2}}| & 1 + c_{\frac{\rho}{2}}^2 & -\frac{1}{2}s_{\rho} & 0 & 0 & 0 & 0 \\ -i|c_{\frac{\rho}{2}}| & -\frac{1}{2}s_{\rho} & 1 + s_{\frac{\rho}{2}}^2 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
f(\phi) &= a \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}|\phi_Z & -|s_{\frac{\rho}{2}}|\phi_Z & -i|c_{\frac{\rho}{2}}|\phi_W^- & i|s_{\frac{\rho}{2}}|\phi_W^- & i|c_{\frac{\rho}{2}}|\phi_W^+ & -i|s_{\frac{\rho}{2}}|\phi_W^+ \\ |c_{\frac{\rho}{2}}|\phi_Z & s_{\rho}h & c_{\rho}h & 0 & \phi_W^- & 0 & \phi_W^+ \\ -|s_{\frac{\rho}{2}}|\phi_Z & c_{\rho}h & -s_{\rho}h & -\phi_W^- & 0 & -\phi_W^+ & 0 \\ i|c_{\frac{\rho}{2}}|\phi_W^+ & 0 & -\phi_W^+ & s_{\rho}h & c_{\rho}h - i\phi_Z & 0 & 0 \\ -i|s_{\frac{\rho}{2}}|\phi_W^+ & \phi_W^+ & 0 & c_{\rho}h + i\phi_Z & -s_{\rho}h & 0 & 0 \\ -i|c_{\frac{\rho}{2}}|\phi_W^- & 0 & -\phi_W^- & 0 & 0 & s_{\rho}h & c_{\rho}h + i\phi_Z \\ i|s_{\frac{\rho}{2}}|\phi_W^- & \phi_W^- & 0 & 0 & 0 & c_{\rho}h - i\phi_Z & -s_{\rho}h \end{pmatrix}, \tag{3.46}
\end{aligned}$$

where s_{ρ} and c_{ρ} are as defined in (3.43), with $a = \sqrt{a_1^2 + a_2^2}$ and $\Delta = (M_T - M_D)/2$. Again, s_{ρ} is positive and c_{ρ} can have either sign depending on the hierarchy between M_T and M_D .

3.2.4 Pure Case Limits

Appropriate parametric limits can be taken to decouple the pure state constituents of an admixture. This can be used to check the consistency of matching computations in Sect. 4.2. From the singlet-doublet admixture, we may recover the pure doublet (singlet) case by taking $a \rightarrow 0$ or $|\Delta| \rightarrow \infty$, with $\Delta > 0$ ($\Delta < 0$), or by taking $\rho \rightarrow 0$ ($\rho \rightarrow \pi$). Similarly, to recover the pure doublet (triplet) case from the triplet-doublet admixture, we decouple the triplet (doublet) component by taking $a \rightarrow 0$ or $|\Delta| \rightarrow \infty$, with $\Delta > 0$ ($\Delta < 0$), or by taking $\rho \rightarrow 0$ ($\rho \rightarrow \pi$).

3.2.5 Relativistic Example: Singlet-Doublet Mixture

The heavy-particle Lagrangians in Sect. 3.2 may be obtained from a manifestly relativistic Lagrangian by performing field redefinitions at tree level. Consider the case of a singlet-doublet mixture (see also [29]),

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{2} \bar{b}(i\partial\!\!\!/ - M_1)b + \bar{\psi}(i\partial\!\!\!/ - M_2)\psi - (y \bar{b} P_L H^\dagger \psi + y' \bar{b} P_L H^T \psi^c + \text{h.c.}), \quad (3.47)$$

where b is a gauge singlet (Majorana) fermion represented as a Dirac spinor with $b^c = b$, and ψ is a Dirac fermion in the $(\mathbf{2}, 1/2)$ representation of $SU(2)_W \times U(1)_Y$. In the above equation, $P_{R,L} = (1 \pm \gamma_5)/2$, and we have included all renormalizable gauge-invariant interactions involving the SM Higgs field. Expressing the result in terms of Majorana combinations,

$$\lambda_1 = \frac{1}{\sqrt{2}} (\psi + \psi^c), \quad \lambda_2 = \frac{i}{\sqrt{2}} (\psi - \psi^c), \quad (3.48)$$

and collecting the fermions in the column vector $\lambda = (b, \lambda_1, \lambda_2)$, we may write the interactions with the Higgs field as

$$\begin{aligned} \mathcal{L}_{H\bar{\lambda}\lambda} &= -\frac{1}{\sqrt{2}} \bar{b} \frac{1 - \gamma_5}{2} \left[(y H^\dagger + y' H^T) \lambda_1 - i(y H^\dagger - y' H^T) \lambda_2 \right] + \text{h.c.} \\ &\equiv -\frac{1}{2} \bar{\lambda} \left[f(H) + i\gamma_5 g(H) \right] \lambda, \end{aligned} \quad (3.49)$$

with

$$\begin{aligned} f(H) &= \frac{a_1}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \\ &\quad + \frac{a_2}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}, \\ g(H) &= \frac{b_1}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \\ &\quad + \frac{b_2}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix}. \end{aligned} \quad (3.50)$$

The real parameters a_i and b_i are given by

$$a_1 = \frac{1}{2} \text{Re}(y + y'), \quad a_2 = \frac{1}{2} \text{Im}(y - y'), \quad b_1 = \frac{1}{2} \text{Re}(y - y'), \quad b_2 = -\frac{1}{2} \text{Im}(y + y'). \quad (3.51)$$

We employ phase redefinitions of b , ψ_L and ψ_R to ensure that M_1 and M_2 are real and positive.⁷ The gauge generators will be those given in (3.22), extended trivially to include the singlet. Upon performing the tree-level field redefinition

$$\lambda = \sqrt{2}e^{-i(M-\delta M)v \cdot x}(h_v + H_v), \quad (3.52)$$

where the fields h_v and H_v obey $\not{v} h_v = h_v$ and $\not{v} H_v = -H_v$, we obtain the heavy-particle Lagrangian in (3.13). It follows from $\lambda^c = \lambda$ that the resulting Lagrangian is invariant under the simultaneous transformations in (2.71). Note that $f(H)$ is the only term surviving the projection from the condition $\not{v} h_v = h_v$. The remaining analysis follows that of Sect. 3.2.3.

3.3 Onshell Renormalization Scheme

A consistent evaluation of amplitudes beyond tree level demands renormalization of the Higgs-WIMP vertex, $h\bar{\chi}\chi$, that appears for admixtures. We define an extension of the onshell renormalization scheme for the electroweak SM (e.g., see [67]) by expressing the vertex amplitude in terms of physical masses in the SM and DM sectors. We begin by studying the singlet-doublet mixture, and will later quote the analogous results for the triplet-doublet mixture.

To avoid confusion with standard notation for counterterms, in this section (only) we denote a residual mass by μ , and a residual mass counterterm by $\delta\mu$. We keep the notation introduced in Sect. 3.2 for the residual mass eigenvalues, $\delta_0^{(0)}$, $\delta_0^{(\pm)}$, etc.

3.3.1 Singlet-Doublet Counterterm Lagrangian

Let us write the bare Lagrangian as the sum of renormalized and counterterm contributions

$$\begin{aligned} \mathcal{L} &= \bar{h}^{\text{bare}} \left[i v \cdot D - \mu^{\text{bare}} - f^{\text{bare}}(H^{\text{bare}}) \right] h^{\text{bare}} \\ &= \bar{h} \left[i v \cdot D + \delta Z_h i v \cdot D - \mu - \delta\mu - f(H^{\text{bare}}) - \delta f(H^{\text{bare}}) \right] h, \end{aligned} \quad (3.53)$$

where the bare quantities are given by

$$\begin{aligned} \mu^{\text{bare}} &= \text{diag}(\mu_S^{\text{bare}}, \mu_D^{\text{bare}}, \mu_D^{\text{bare}}, \mu_D^{\text{bare}}, \mu_D^{\text{bare}}), \\ f^{\text{bare}}(H) &= \frac{a_1^{\text{bare}}}{\sqrt{2}} \begin{pmatrix} 0 & H^\dagger + H^T & i(H^T - H^\dagger) \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \\ i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \end{aligned}$$

⁷An additional phase redefinition could be used to eliminate a_1 , a_2 , b_1 or b_2 .

$$\begin{aligned}
& + \frac{a_2^{\text{bare}}}{\sqrt{2}} \begin{pmatrix} 0 & -i(H^T - H^\dagger) & H^T + H^\dagger \\ -i(H - H^*) & \mathbb{0}_2 & \mathbb{0}_2 \\ H + H^* & \mathbb{0}_2 & \mathbb{0}_2 \end{pmatrix} \\
& \equiv a_1^{\text{bare}} f_1(H) + a_2^{\text{bare}} f_2(H), \tag{3.54}
\end{aligned}$$

and the expression for $f^{\text{bare}}(H)$ above is valid for arbitrary H (in particular, for H^{bare}). The gauge symmetry preserving counterterms are given by

$$\begin{aligned}
Z_h &= 1 + \delta Z_h = 1 + \text{diag}(\delta Z_S, \delta Z_D \mathbb{1}_4), \\
\mu + \delta\mu &= Z_h^{\frac{1}{2}} \mu^{\text{bare}} Z_h^{\frac{1}{2}} = \text{diag}(\mu_S + \delta\mu_S, (\mu_D + \delta\mu_D) \mathbb{1}_4), \\
f(H^{\text{bare}}) + \delta f(H^{\text{bare}}) &= Z_h^{\frac{1}{2}} f^{\text{bare}}(H^{\text{bare}}) Z_h^{\frac{1}{2}} = (a_1 + \delta a_1) f_1(H') + (a_2 + \delta a_2) f_2(H'). \tag{3.55}
\end{aligned}$$

We have introduced H' to absorb the renormalization of v :

$$H^{\text{bare}} = Z_H^{\frac{1}{2}} \begin{pmatrix} \phi_W^+ \\ \frac{1}{\sqrt{2}}(v - \delta v + h + i\phi_Z) \end{pmatrix} = Z_H^{\frac{1}{2}} \left(1 - \frac{\delta v}{v}\right) H'. \tag{3.56}$$

Note that the renormalization of v introduces a coupling $\sim \frac{\delta v}{v} h \bar{\chi} \chi$ through the $a_1 f_1(H') + a_2 f_2(H')$ term in (3.55). We will fix the counterterms by enforcing renormalization conditions on the residual mass matrix (two point functions). Three point functions involving the Higgs interaction will then be determined.

3.3.2 Propagator Corrections

Anticipating renormalization conditions that preserve the basis $h = (h_0^{(0)}, h_0^{(+)}, h_0^{(-)})$, $(h_+^{(0)}, h_-^{(0)})$ of mass eigenstates introduced in Sect. 3.2.3, let us express the counterterms in this basis,

$$\begin{aligned}
\delta\mu &= \delta\mu_D \mathbb{1}_5 + \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}| \frac{v}{a} (a_2 \delta a_1 - a_1 \delta a_2) & |s_{\frac{\rho}{2}}| \frac{v}{a} (a_1 \delta a_2 - a_2 \delta a_1) & 0 & 0 \\ \cdot & 2c_{\frac{\rho}{2}}^2 (\delta\Delta) + s_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) & -s_{\rho} (\delta\Delta) + c_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) & 0 & 0 \\ \cdot & \cdot & 2s_{\frac{\rho}{2}}^2 (\delta\Delta) - s_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \\
\delta Z_h &= \delta Z_D \mathbb{1}_5 + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \cdot & c_{\frac{\rho}{2}}^2 (\delta Z_S - \delta Z_D) - \frac{1}{2} s_{\rho} (\delta Z_S - \delta Z_D) & 0 & 0 & 0 \\ \cdot & \cdot & s_{\frac{\rho}{2}}^2 (\delta Z_S - \delta Z_D) & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \tag{3.57}
\end{aligned}$$

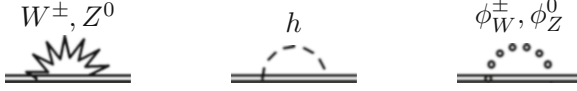


Fig. 3.1 One-loop corrections to two-point functions. *Double lines* denote heavy WIMPs, *zigzag lines* denote gauge bosons, $W^\pm$ or Z^0 , *dashed lines* denote the physical Higgs boson, h , and *dotted lines* denote Goldstone bosons, $\phi_W^\pm$ or ϕ_Z^0

where the above matrices are symmetric, and $(\delta\Delta) = (\delta\mu_S - \delta\mu_D)/2$. Due to the masslessness of the photon, the onshell renormalization factor for the electrically charged state, δZ_D , is infrared (IR) divergent. To avoid the associated complications, we may turn off δZ_D , corresponding to an additional overall renormalization of the fields with $\delta Z_S = \delta Z_D$. This overall renormalization will not impact the determination of physical masses or mass eigenstates. However, we will of course need to include additional wavefunction renormalization factors when computing physical amplitudes. In the following, we allow for arbitrary δZ_D .

We compute the one-loop corrections to the amputated two-point function, Σ_2 , from virtual Z^0 , $W^\pm$, h , ϕ_Z^0 and $\phi_W^\pm$ exchange, as illustrated in Fig. 3.1. In the following results, we set the external momentum to zero (i.e., we compute $\Sigma_2(0)$), and the first (second) subscript denotes the final (initial) state, with values $(1, 2, 3, 4, 5)$ corresponding to the mass eigenstates $(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(0)}, h_-^{(0)})$. Using Feynman-t'Hooft gauge, and expressing results in terms of the basic integral $I_3(\delta, m)$ of Appendix B.1, we find

$$\begin{aligned}
-i[\Sigma_2(0)]_{11} &= -\frac{g_2^2}{4c_W^2} c_\rho^2 I_3(\delta_0^{(-)}, m_Z) - \frac{g_2^2}{4c_W^2} s_\rho^2 I_3(\delta_0^{(+)}, m_Z) - \frac{g_2^2}{2} I_3(\delta_\pm^{(0)}, m_W) \\
&\quad + a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_Z) + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_Z), \\
-i[\Sigma_2(0)]_{22} &= -\frac{g_2^2}{4c_W^2} s_\rho^2 I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{2} s_\rho^2 I_3(\delta_\pm^{(0)}, m_W) + a^2 s_\rho^2 I_3(\delta_0^{(+)}, m_h) \\
&\quad + a^2 c_\rho^2 I_3(\delta_0^{(-)}, m_h) + a^2 c_\rho^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 c_\rho^2 I_3(\delta_\pm^{(0)}, m_W), \\
-i[\Sigma_2(0)]_{23} &= -i[\Sigma_2(0)]_{32} \\
&= -\frac{g_2^2}{8c_W^2} s_\rho I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{4} s_\rho I_3(\delta_\pm^{(0)}, m_W) + a^2 s_\rho c_\rho I_3(\delta_0^{(+)}, m_h) \\
&\quad - a^2 s_\rho c_\rho I_3(\delta_0^{(-)}, m_h) - \frac{a^2}{2} s_\rho I_3(\delta_0^{(0)}, m_Z) - a^2 s_\rho I_3(\delta_\pm^{(0)}, m_W), \\
-i[\Sigma_2(0)]_{33} &= -\frac{g_2^2}{4c_W^2} c_\rho^2 I_3(\delta_0^{(0)}, m_Z) - \frac{g_2^2}{2} c_\rho^2 I_3(\delta_\pm^{(0)}, m_W) + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_h) \\
&\quad + a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_h) + a^2 s_\rho^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 s_\rho^2 I_3(\delta_\pm^{(0)}, m_W), \\
-i[\Sigma_2(0)]_{44} &= -i[\Sigma_2(0)]_{55} \\
&= -e^2 I_3(\delta_\pm^{(0)}, \lambda) - \frac{g_2^2}{4c_W^2} (1 - 2s_W^2)^2 I_3(\delta_\pm^{(0)}, m_Z) - \frac{g_2^2}{4} I_3(\delta_0^{(0)}, m_W)
\end{aligned}$$

$$\begin{aligned}
& -\frac{g_2^2}{4}s_\rho^2 I_3(\delta_0^{(+)}, m_W) - \frac{g_2^2}{4}c_\rho^2 I_3(\delta_0^{(-)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_W) \\
& + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_W),
\end{aligned} \tag{3.58}$$

where λ is a fictitious photon mass, and the self-energy components not displayed above vanish. We may evaluate $\Sigma(v \cdot k)$ by the substitution $I_3(\delta, m) \rightarrow I_3(\delta - v \cdot k, m)$.

3.3.3 Renormalization Conditions

Let us fix the counterterms δa_1 , δa_2 , $\delta \mu_S$, $\delta \mu_D$ and δZ_S by enforcing that the physical residual masses of the neutral states are given by the renormalized parameters of the Lagrangian,

$$\begin{aligned}
& [\delta \mu]_{11} + \text{Re}[\Sigma_2(\delta_\pm^{(0)})]_{11} - \delta_0^{(0)}[\delta Z_h]_{11} = 0, \\
& [\delta \mu]_{22} + \text{Re}[\Sigma_2(\delta_0^{(+)})]_{22} - \delta_0^{(+)}[\delta Z_h]_{22} = 0, \\
& [\delta \mu]_{33} + \text{Re}[\Sigma_2(0)]_{33} = 0,
\end{aligned} \tag{3.59}$$

and that the lightest mass eigenstate is proportional to the vector $(0, 0, 1, 0, 0)$,

$$\begin{aligned}
& [\delta \mu]_{13} + \text{Re}[\Sigma_2(0)]_{13} = 0, \\
& [\delta \mu]_{23} + \text{Re}[\Sigma_2(0)]_{23} = 0.
\end{aligned} \tag{3.60}$$

This scheme defines renormalized values for a and t_ρ through the physical mass differences between neutral states,

$$\begin{aligned}
M_{h_0^{(+)}} - M_{h_0^{(-)}} &= 2avs_\rho^{-1}, \\
M_{h_0^{(0)}} - M_{h_0^{(-)}} &= avt_\rho,
\end{aligned} \tag{3.61}$$

where the mass of the neutral mass eigenstate $h_0^{(\cdot)}$ is denoted $M_{h_0^{(\cdot)}}$. Note also that the presence of $\delta Z_S \neq \delta Z_D$ is required to maintain the orientation of the lightest mass eigenstate under renormalization. Solving for the counterterms, we find from $[\delta \mu]_{13}$,

$$\frac{\delta a_1}{a_1} = \frac{\delta a_2}{a_2} \implies a_1 \delta a_1 + a_2 \delta a_2 = a^2 \frac{\delta a_1}{a_1}. \tag{3.62}$$

The remaining system of equations involving $[\delta \mu]_{23}$, $[\delta \mu]_{11}$, $[\delta \mu]_{22}$ and $[\delta \mu]_{33}$ then yields

$$\begin{aligned}
av \frac{\delta a_1}{a_1} &= -[\delta \mu]_{23} + t_\rho^{-1} ([\delta \mu]_{11} - [\delta \mu]_{33}) \\
&= [\Sigma_2(0)]_{23} + t_\rho^{-1} \left([\Sigma_2(0)]_{33} - [\Sigma_2(\delta_0^{(0)})]_{11} + \delta_0^{(0)}[\delta Z_h]_{11} \right),
\end{aligned}$$

$$\begin{aligned} \delta Z_S = \delta Z_D + \frac{1}{av} & \left\{ t_{\frac{\rho}{2}} [\Sigma_2(\delta_0^{(+)})]_{22} + 2[\Sigma_2(0)]_{23} \right. \\ & \left. + t_{\frac{\rho}{2}}^{-1} [\Sigma_2(0)]_{33} - 2s_{\rho}^{-1} [\Sigma_2(\delta_0^{(0)})]_{11} \right\}. \end{aligned} \quad (3.63)$$

We focus here on the counterterms δa_1 , δa_2 , and δZ_S which enter in the calculation of amplitudes relevant for WIMP-nucleon scattering. Explicit expressions for the remaining counterterms $\delta\mu_S$ and $\delta\mu_D$ may be similarly obtained. We note that the degeneracy between the mass of the $h_0^{(0)}$ state and the $h_{\pm}^{(0)}$ states is lifted by a finite amount, predicted in terms of renormalized parameters as

$$M_{h_{\pm}^{(0)}} - M_{h_0^{(0)}} = [\Sigma_2(\delta_{\pm}^{(0)})]_{44} - [\Sigma_2(\delta_0^{(0)})]_{11}, \quad (3.64)$$

where we have used that $[\delta\mu]_{11} = [\delta\mu]_{44}$, $[\delta Z_h]_{11} = [\delta Z_h]_{44}$ and $\delta_0^{(0)} = \delta_{\pm}^{(0)}$.

3.3.4 Extension to Triplet-Doublet

The extension to the triplet-doublet case is straightforward. The counterterms δa_1 , δa_2 , $\delta\mu_T$, $\delta\mu_D$, δZ_T and δZ_D are introduced in an analogous manner. In terms of the mass eigenbasis $h = (h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(+)}, h_+^{(-)}, h_-^{(+)}, h_-^{(-)})$ introduced in Sect. 3.2.3, the counterterms are given by the 7×7 matrices,

$$\delta\mu = \delta\mu_D \mathbb{1}_7 + \begin{pmatrix} \delta\mu_0 & 0 & 0 \\ 0 & \delta\mu_+ & 0 \\ 0 & 0 & \delta\mu_- \end{pmatrix}, \quad \delta Z_h = \delta Z_D \mathbb{1}_7 + \begin{pmatrix} \delta Z_0 & 0 & 0 \\ 0 & \delta Z_+ & 0 \\ 0 & 0 & \delta Z_- \end{pmatrix}, \quad (3.65)$$

where the submatrices for the neutral and charged sectors are specified by the following symmetric matrices,

$$\begin{aligned} \delta\mu_0 &= \begin{pmatrix} 0 & |c_{\frac{\rho}{2}}| \frac{v}{a} (a_2 \delta a_1 - a_1 \delta a_2) & |s_{\frac{\rho}{2}}| \frac{v}{a} (a_1 \delta a_2 - a_2 \delta a_1) \\ \cdot & 2c_{\frac{\rho}{2}}^2 (\delta\Delta) + s_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) & -s_{\rho} (\delta\Delta) + c_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) \\ \cdot & \cdot & 2s_{\frac{\rho}{2}}^2 (\delta\Delta) - s_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) \end{pmatrix}, \\ \delta\mu_{\pm} &= \begin{pmatrix} 2c_{\frac{\rho}{2}}^2 (\delta\Delta) + s_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) & -s_{\rho} (\delta\Delta) + c_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) \pm i \frac{v}{a} (a_1 \delta a_2 - a_2 \delta a_1) \\ \cdot & \cdot & 2s_{\frac{\rho}{2}}^2 (\delta\Delta) - s_{\rho} \frac{v}{a} (a_1 \delta a_1 + a_2 \delta a_2) \end{pmatrix}, \\ \delta Z_0 &= \begin{pmatrix} 0 & 0 & 0 \\ \cdot & c_{\frac{\rho}{2}}^2 (\delta Z_T - \delta Z_D) - \frac{1}{2} s_{\rho} (\delta Z_T - \delta Z_D) \\ \cdot & \cdot & s_{\frac{\rho}{2}}^2 (\delta Z_T - \delta Z_D) \end{pmatrix}, \\ \delta Z_{\pm} &= \begin{pmatrix} c_{\frac{\rho}{2}}^2 (\delta Z_T - \delta Z_D) - \frac{1}{2} s_{\rho} (\delta Z_T - \delta Z_D) \\ \cdot & \cdot & s_{\frac{\rho}{2}}^2 (\delta Z_T - \delta Z_D) \end{pmatrix}, \end{aligned} \quad (3.66)$$

with $(\delta\Delta) = (\delta\mu_T - \delta\mu_D)/2$. To fix counterterms, we impose the same renormalization conditions given in (3.59) and (3.60). We again require the one-loop corrections to the two-point function, Σ_2 . In the following results, the first (second) subscript denotes the final (initial) state, with values (1, 2, 3, 4, 5, 6, 7) corresponding to the mass eigenstates $(h_0^{(0)}, h_0^{(+)}, h_0^{(-)}, h_+^{(+)}, h_+^{(-)}, h_-^{(+)}, h_-^{(-)})$. Using Feynman-t'Hooft gauge and expressing results in terms of the basic integral $I_3(\delta, m)$ of Appendix B.1, we find

$$\begin{aligned}
-i[\Sigma_2(0)]_{11} &= -\frac{g_2^2}{4c_W^2}s_\rho^2 I_3(\delta_0^{(+)}, m_Z) - \frac{g_2^2}{4c_W^2}s_\rho^2 I_3(\delta_0^{(-)}, m_Z) - \frac{g_2^2}{2}s_\rho^2 I_3(\delta_\pm^{(+)}, m_W) \\
&\quad - \frac{g_2^2}{2}s_\rho^2 I_3(\delta_\pm^{(-)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_Z) + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_Z) \\
&\quad + 2a^2 c_\rho^2 I_3(\delta_\pm^{(+)}, m_W) + 2a^2 s_\rho^2 I_3(\delta_\pm^{(-)}, m_W), \\
-i[\Sigma_2(0)]_{22} &= -\frac{g_2^2}{4c_W^2}s_\rho^2 I_3(\delta_0^{(0)}, m_Z) \\
&\quad - \frac{g_2^2}{2}\left(1 + c_\rho^2\right)^2 I_3(\delta_\pm^{(+)}, m_W) - \frac{g_2^2}{8}s_\rho^2 I_3(\delta_\pm^{(-)}, m_W) \\
&\quad + a^2 c_\rho^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 I_3(\delta_\pm^{(-)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(-)}, m_h) + a^2 s_\rho^2 I_3(\delta_0^{(+)}, m_h), \\
-i[\Sigma_2(0)]_{23} &= -i[\Sigma_2(0)]_{32} = -\frac{g_2^2}{8c_W^2}s_\rho I_3(\delta_0^{(0)}, m_Z) \\
&\quad + \frac{g_2^2}{4}s_\rho\left(1 + c_\rho^2\right) I_3(\delta_\pm^{(+)}, m_W) + \frac{g_2^2}{4}s_\rho\left(1 + s_\rho^2\right) I_3(\delta_\pm^{(-)}, m_W) \\
&\quad - \frac{a^2}{2}s_\rho I_3(\delta_0^{(0)}, m_Z) + a^2 c_\rho s_\rho I_3(\delta_0^{(+)}, m_h) - a^2 c_\rho s_\rho I_3(\delta_0^{(-)}, m_h), \\
-i[\Sigma_2(0)]_{33} &= -\frac{g_2^2}{4c_W^2}s_\rho^2 I_3(\delta_0^{(0)}, m_Z) \\
&\quad - \frac{g_2^2}{2}\left(1 + s_\rho^2\right)^2 I_3(\delta_\pm^{(-)}, m_W) - \frac{g_2^2}{8}s_\rho^2 I_3(\delta_\pm^{(+)}, m_W) \\
&\quad + a^2 s_\rho^2 I_3(\delta_0^{(0)}, m_Z) + 2a^2 I_3(\delta_\pm^{(+)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(+)}, m_h) + a^2 s_\rho^2 I_3(\delta_0^{(-)}, m_h), \\
-i[\Sigma_2(0)]_{44} &= -i[\Sigma_2(0)]_{66} \\
&= -\frac{g_2^2}{c_W^2}\left(c_W^2 - \frac{1}{2}s_\rho^2\right)^2 I_3(\delta_\pm^{(+)}, m_Z) - \frac{g_2^2}{16c_W^2}s_\rho^2 I_3(\delta_\pm^{(-)}, m_Z) - \frac{g_2^2}{4}s_\rho^2 I_3(\delta_0^{(0)}, m_W) \\
&\quad - e^2 I_3(\delta_\pm^{(+)}, \lambda) - \frac{g_2^2}{4}\left(1 + c_\rho^2\right)^2 I_3(\delta_0^{(+)}, m_W) - \frac{g_2^2}{16}s_\rho^2 I_3(\delta_0^{(-)}, m_W) + a^2 I_3(\delta_\pm^{(-)}, m_Z) \\
&\quad + a^2 I_3(\delta_0^{(-)}, m_W) + a^2 c_\rho^2 I_3(\delta_0^{(0)}, m_W) + a^2 s_\rho^2 I_3(\delta_\pm^{(+)}, m_h) + a^2 c_\rho^2 I_3(\delta_\pm^{(-)}, m_h), \\
-i[\Sigma_2(0)]_{45} &= -i[\Sigma_2(0)]_{54} = -i[\Sigma_2(0)]_{67} = -i[\Sigma_2(0)]_{76} \\
&= \frac{g_2^2}{4c_W^2}s_\rho\left(c_W^2 - \frac{1}{2}s_\rho^2\right) I_3(\delta_\pm^{(+)}, m_Z) + \frac{g_2^2}{4c_W^2}s_\rho\left(c_W^2 - \frac{1}{2}s_\rho^2\right) I_3(\delta_\pm^{(-)}, m_Z)
\end{aligned}$$

$$\begin{aligned}
& -\frac{g_2^2}{8}s_\rho I_3(\delta_0^{(0)}, m_W) + \frac{g_2^2}{8}s_\rho \left(1 + c_\rho^2\right) I_3(\delta_0^{(+)}, m_W) + \frac{g_2^2}{8}s_\rho \left(1 + s_\rho^2\right) I_3(\delta_0^{(-)}, m_W) \\
& -\frac{a^2}{2}s_\rho I_3(\delta_0^{(0)}, m_W) + a^2 c_\rho s_\rho I_3(\delta_\pm^{(+)}, m_h) - a^2 c_\rho s_\rho I_3(\delta_\pm^{(-)}, m_h), \\
& -i[\Sigma_2(0)]_{55} = -i[\Sigma_2(0)]_{77} \\
& = -\frac{g_2^2}{16c_W^2}s_\rho^2 I_3(\delta_\pm^{(+)}, m_Z) - \frac{g_2^2}{c_W^2} \left(c_W^2 - \frac{1}{2}c_\rho^2\right)^2 I_3(\delta_\pm^{(-)}, m_Z) - \frac{g_2^2}{4}c_\rho^2 I_3(\delta_0^{(0)}, m_W) \\
& -e^2 I_3(\delta_\pm^{(-)}, \lambda) - \frac{g_2^2}{16}s_\rho^2 I_3(\delta_0^{(+)}, m_W) - \frac{g_2^2}{4} \left(1 + s_\rho^2\right)^2 I_3(\delta_0^{(-)}, m_W) + a^2 I_3(\delta_\pm^{(+)}, m_Z) \\
& + a^2 I_3(\delta_0^{(+)}, m_W) + a^2 s_\rho^2 I_3(\delta_0^{(0)}, m_W) + a^2 c_\rho^2 I_3(\delta_\pm^{(+)}, m_h) + a^2 s_\rho^2 I_3(\delta_\pm^{(-)}, m_h),
\end{aligned} \tag{3.67}$$

where λ is a fictitious photon mass, and the self-energy components not displayed above vanish. The remainder of the renormalization program proceeds as for the singlet-doublet system. In particular, the similarity of the neutral sectors implies relations similar to (3.63),

$$\begin{aligned}
av \frac{\delta a_1}{a_1} &= av \frac{\delta a_2}{a_2} = [\Sigma_2(0)]_{23} + t_\rho^{-1} \left([\Sigma_2(0)]_{33} - [\Sigma_2(\delta_0^{(0)})]_{11} + \delta_0^{(0)} [\delta Z_h]_{11} \right), \\
\delta Z_T &= \delta Z_D + \frac{1}{av} \left\{ t_\rho [\Sigma_2(\delta_0^{(+)})]_{22} + 2[\Sigma_2(0)]_{23} \right. \\
&\quad \left. + t_\rho^{-1} [\Sigma_2(0)]_{33} - 2s_\rho^{-1} [\Sigma_2(\delta_0^{(0)})]_{11} \right\},
\end{aligned} \tag{3.68}$$

where the self-energy components are those of the triplet-doublet system given in (3.67).

3.4 Low Energy Theory at the Weak Scale for Pure- and Mixed-State WIMPs

Let us construct the effective theory of DM with mass $M \gtrsim m_W$ interacting with $n_f = 5$ flavor QCD. The hierarchy of scales between the DM mass and the relevant low-energy degrees of freedom, $\Lambda_{\text{QCD}}, m_c, m_b \ll m_W$, allows us to use heavy particle effective theory to describe the DM field. The most general Lagrangian relevant for spin-independent, low-velocity scattering with nucleons, is then given at energies $E \ll m_W$ by,

$$\begin{aligned}
\mathcal{L}_{\chi v, \text{SM}} &= \bar{\chi}_v \chi_v \left\{ \sum_{q=u,d,s,c,b} \left[c_q^{(0)} O_q^{(0)} + c_q^{(2)} v_\mu v_\nu O_q^{(2)\mu\nu} \right] \right. \\
&\quad \left. + c_g^{(0)} O_g^{(0)} + c_g^{(2)} v_\mu v_\nu O_g^{(2)\mu\nu} \right\} + \dots,
\end{aligned} \tag{3.69}$$

where χ_v is the lightest, electrically neutral, self-conjugate WIMP of arbitrary spin, arising from the pure or mixed cases discussed in Sect. 3.2. The ellipsis in the above equation

includes higher-dimension operators suppressed by powers of $1/m_W$. The assumed self-conjugacy of χ_v implies that (3.69) is invariant under (2.71). The SM component of (3.69) is expressed in terms of quark and gluon fields as

$$\begin{aligned}
 O_q^{(0)} &= m_q \bar{q} q, \\
 O_g^{(0)} &= (G_{\mu\nu}^A)^2, \\
 O_q^{(2)\mu\nu} &= \frac{1}{2} \bar{q} \left(\gamma^{\{\mu} i D_-^{\nu\}} - \frac{1}{d} g^{\mu\nu} i \not{D}_- \right) q, \\
 O_g^{(2)\mu\nu} &= -G^{A\mu\lambda} G_{\lambda}^{A\nu} + \frac{1}{d} g^{\mu\nu} (G_{\alpha\beta}^A)^2.
 \end{aligned} \tag{3.70}$$

Here $D_- \equiv \vec{D} - \overleftarrow{D}$, and $A^{\{\mu} B^{\nu\}} \equiv (A^\mu B^\nu + A^\nu B^\mu)/2$ denotes symmetrization. The operators in (3.70) are expressed in terms of bare Lagrangian fields, where we employ dimensional regularization with $d = 4 - 2\epsilon$ spacetime dimensions. We use the background field method for gluons in the effective theory thus ignoring gauge-variant operators, and assume that appropriate field redefinitions are employed to eliminate operators that vanish by leading order equations of motion. We ignore flavor non-diagonal operators, whose nucleon matrix elements have an additional weak-scale suppression relative to those considered. We will not be concerned here with leptonic interactions.

For a self-conjugate WIMP, χ_v , with mass $M \gtrsim m_W$ and arbitrary spin, Eq. (3.69) represents the most general effective Lagrangian at leading order in $1/m_W$, relevant for spin-independent, low-velocity scattering with nucleons. The basis of operators below the weak scale is independent of the WIMP quantum numbers, and hence Eq. (3.69) agrees with the basis of operators in Eq. (3.10) for the singlet fermion case.

Details of the UV completion are encoded in the twelve matching coefficients $c_q^{(0)}$, $c_q^{(2)}$, $c_g^{(0)}$ and $c_g^{(2)}$. Matching onto the effective theory (3.69) is in general dependent on the specific SM extension. Although much of the formalism applies more generally, for definiteness we focus on the heavy WIMP limit, $M \gg m_W$, where universal features appear [59].

Chapter 4

Weak-Scale Matching

In the previous chapter we constructed the effective theory for WIMPs interacting with SM degrees of freedom in the electroweak symmetric phase (cf. Sect. 3.1.3), and for the lightest electrically neutral WIMP interacting with $n_f = 5$ QCD degrees of freedom (cf. Sects. 3.1.4 and 3.4). The few parameters of the high-scale electroweak symmetric theory imply definite predictions for WIMP-nucleon scattering at the low scale, but require matching between these two effective theories. In this chapter, we present formalism necessary to determine weak-scale matching coefficients in the computation of scattering cross sections for putative dark matter candidates interacting with the SM.

Even in many seemingly simple cases, determination of WIMP-nucleon cross sections demands an intricate analysis of competing amplitudes mediated by SM particles (see e.g., [28, 42, 48, 59, 64, 65, 78]). We set out the formalism for electroweak-scale matching computations for application to theories with a specified ultraviolet (UV) completion (e.g., supersymmetric models [73]), a basis of leading order contact interactions [12, 53], or to the heavy WIMP limit where theoretical control is maintained in the absence of a specified UV completion [59]. We review relevant aspects of techniques such as the background field method for matching to gluon operators [64, 90], and the treatment of effective theory subtractions.

An important simplification occurs when a scale separation exists between SM masses ($\sim m_W$) and the lightest new particle mass ($\sim M$), allowing an expansion in m_W/M . We consider in detail the limit $M \gg m_W$ where universal behavior appears, and present details of the first complete computation of the matching at leading order in perturbation theory onto the full basis of operators at the electroweak scale [59]. Within the heavy WIMP framework, we present a complete reduction of the required one- and two-loop amplitudes into a basis of heavy-particle loop integrals with nonzero residual mass.

This chapter is structured as follows. In Sect. 4.1 we perform matching for the electroweak singlet case, considering three mass regimes for the DM particle: much lighter than, comparable to, and much heavier than the weak-scale particles. Section 4.2 presents the details of the matching calculation for the case of pure and mixed electroweak multiplets, including the systematic reduction of heavy-particle integrals, and the implementation of background field techniques for gluon operators. The complete set of two-loop diagrams appearing in the matching to gluon operators are evaluated.

4.1 Singlet

Let us match the theory with $SU(3)_c \times SU(2)_W \times U(1)_Y$ symmetry, represented by the interactions in Eq. (3.5), (3.6) or (3.7), onto the theory with $SU(3)_c \times U(1)_{e.m.}$ symmetry, represented by the interactions in Eq. (3.8), (3.9) or (3.10), by integrating out the weak scale particles, $W^\pm, Z^0, h, t$, at energy scale $\mu_H \sim m_W \sim m_Z \sim m_h \sim m_t$. We illustrate this with tree-level matching for leading order operators and for the cases where the DM has mass $M \lesssim m_b \ll m_W, m_W \lesssim M$ and $m_W \ll M$. The matching for higher dimension operators and the treatment of power corrections and perturbative corrections can be analyzed similarly. In this section, we focus on matching for electroweak-singlet DM; weak scale matching for electroweak-charged WIMPs arising from extensions of the SM with one or two electroweak multiples require a more intricate analysis and will be presented in the next section.

4.1.1 Case I: $M \lesssim m_b \ll m_W$

In this case the matching is organized by a power counting employing a scale separation $M \ll m_W \ll \Lambda$, where Λ is the scale appearing in the expansion of Eqs. (3.5) and (3.6). Let us consider the Majorana fermion case with leading high-scale operators given by dimension five interactions,

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{2} \bar{\psi} (i \not{\partial} - M) \psi - \frac{1}{\Lambda} \bar{\psi} (c'_{\psi 1} + i c'_{\psi 2} \gamma_5) \psi H^\dagger H. \quad (4.1)$$

Upon integrating out the physical Higgs field h and performing the field redefinition,

$$\psi \rightarrow e^{-i\phi \gamma_5} \psi, \quad \tan 2\phi = \frac{c'_{\psi 2} v^2}{c'_{\psi 1} v^2 + M\Lambda}, \quad (4.2)$$

to retain a positive real mass convention for the DM field ψ , we obtain the low-scale effective lagrangian,

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_{\text{SM}} + \frac{1}{2} \bar{\psi} (i \not{\partial} - M') \psi + \frac{1}{m_W^3} \bar{\psi} (c_{\psi 7} + i c_{\psi 9} \gamma_5) \psi \sum_q m_q \bar{q} q \\
& + \frac{1}{m_W^3} \bar{\psi} (c_{\psi 17} + i c_{\psi 9} \gamma_5) \psi G_{\mu\nu}^A G^{A\mu\nu} + \dots, \quad (4.3)
\end{aligned}$$

where the sum runs over the active quark mass eigenstates $q = u, d, s, c, b$, and the ellipsis denotes higher-order perturbative corrections. There is a nontrivial relation between the DM mass and couplings of the high-scale effective lagrangian in Eq. (4.1) and those of the low-scale effective lagrangian in Eq. (4.3):

$$\begin{aligned}
M' &= \sqrt{\left(M + \frac{c'_{\psi 1} v^2}{\Lambda}\right)^2 + \left(\frac{c'_{\psi 2} v^2}{\Lambda}\right)^2}, \\
\{c_{\psi 7}, c_{\psi 9}\} &= \frac{m_W^3 M}{m_h^2 \Lambda M'} \left\{ c'_{\psi 1} + \frac{v^2}{M \Lambda} [c'^2_{\psi 1} + c'^2_{\psi 2}], c'_{\psi 2} \right\}, \\
\{c_{\psi 17}, c_{\psi 19}\} &= -\frac{\alpha_s(\mu_H)}{12\pi} \{c_{\psi 7}, c_{\psi 9}\}. \quad (4.4)
\end{aligned}$$

It is incorrect to assume that a vanishing $c'_{\psi 1}$ coefficient leads to a vanishing $c_{\psi 7}$ coefficient, and hence to a velocity-suppressed spin-independent cross section. The contribution to $c_{\psi 7}$ that goes as $\sim \frac{v^2}{M \Lambda} c'^2_{\psi 2}$ yields a spin-independent cross section that is not velocity-suppressed and may be large.

Weak scale matching for a real or complex scalar is similar to the Majorana case above. For example, we find for a real scalar,

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_{\text{SM}} + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} M^2 \phi^2 - c'_{\phi 1} \phi^2 H^\dagger H \\
\rightarrow & \mathcal{L}_{\text{SM}} + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} (M')^2 \phi^2 + \frac{c_{\phi 1}}{m_W^2} \phi^2 \sum_q m_q \bar{q} q + \frac{c_{\phi 6}}{m_W^2} \phi^2 G_{\mu\nu}^a G^{a\mu\nu} + \dots, \\
M' = & \sqrt{M^2 + c'^2_{\phi 1} v^2}, \quad c_{\phi 1} = \frac{m_W^2}{m_h^2} c'_{\phi 1}, \quad c_{\phi 6} = -\frac{\alpha_s(\mu_H)}{12\pi} c_{\phi 1}, \quad (4.5)
\end{aligned}$$

where the sum runs over the active quark mass eigenstates $q = u, d, s, c, b$, and the ellipsis denotes higher-order perturbative corrections.

For a Dirac fermion the leading high-scale operators are given by dimension five interactions including dipole operators. For matching to leading-order operators at the low scale, we find

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_{\text{SM}} + \bar{\psi} (i \not{\partial} - M) \psi - \frac{1}{\Lambda} \bar{\psi} (c'_{\psi 1} + i c'_{\psi 2} \gamma_5) \psi H^\dagger H + \frac{1}{\Lambda} \bar{\psi} (c'_{\psi 3} \sigma_{\mu\nu} + c'_{\psi 4} \tilde{\sigma}_{\mu\nu}) \psi B^{\mu\nu} \\
\rightarrow & \mathcal{L}_{\text{SM}} + \bar{\psi} (i \not{\partial} - M') \psi + \frac{1}{m_W} \bar{\psi} (c_{\psi 1} \sigma_{\mu\nu} + c_{\psi 2} \tilde{\sigma}_{\mu\nu}) \psi F^{\mu\nu} + \dots, \\
c_{\psi 1} = & \frac{m_W}{\Lambda} c_W c'_{\psi 3}, \quad c_{\psi 2} = \frac{m_W}{\Lambda} c_W c'_{\psi 4}, \quad (4.6)
\end{aligned}$$

with M' given in Eq.(4.4) and $c_W \equiv \cos \theta_W$. The ellipsis in Eq.(4.6) denotes higher-order perturbative corrections and contributions from integrating out Z^0 and h , suppressed by $1/m_W^2$ relative to the leading electric and magnetic dipole operators.

4.1.2 Case II: $m_W \lesssim M$

In this case the matching is organized by a power counting employing a scale separation $m_W \lesssim M \ll \Lambda$, and we may map onto the heavy-particle interactions in Eq.(3.10). Let us consider the Majorana fermion case starting with the effective lagrangian at the high-scale given in Eq.(4.1). For tree-level matching, we may pass to a heavy-particle description of the DM field by performing the field redefinition

$$\psi = \sqrt{2}e^{-iMv \cdot x}(\chi_v + X_v), \quad \not{p}\chi_v = \chi_v, \quad \not{p}X_v = -X_v. \quad (4.7)$$

Upon integrating out the heavy mode X_v and the physical Higgs field h , we obtain the low-scale heavy-particle lagrangian

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\text{SM}} + \bar{\chi}_v(iv \cdot \partial - \delta m)\chi_v + \frac{c_{\chi 7}}{m_W^3} \bar{\chi}_v \chi_v \sum_q m_q \bar{q}q + \frac{c_{\chi 25}}{m_W^3} \bar{\chi}_v \chi_v G_{\mu\nu}^A G^{A\mu\nu} + \dots \\ \delta m = \frac{c'_{\psi 1} v^2}{\Lambda}, \quad c_{\chi 7} = \frac{m_W^3}{m_h^2 \Lambda} c'_{\psi 1}, \quad c_{\chi 25} = -\frac{\alpha_s(\mu_H)}{12\pi} c_{\chi 7}, \end{aligned} \quad (4.8)$$

where the sum runs over the active quark mass eigenstates $q = u, d, s, c, b$ and the ellipsis denotes higher-order perturbative corrections. The residual mass δm is the leading contribution to the mass induced by electroweak symmetry breaking and may be eliminated by performing the field redefinition $\chi_v \rightarrow e^{-i\delta m v \cdot x} \chi_v$. The Majorana property of the DM particle, $\psi = \psi^c$, implies that the heavy-particle interactions in Eq.(4.8) are invariant under the transformations in Eq.(2.71) or (2.76). The operators with structure $\sim \bar{\psi} i \gamma_5 \psi$ do not contribute due to the projection property of the heavy field, $\not{p}\chi_v = \chi_v$.

Weak scale matching for a real or complex scalar is similar to the Majorana case above. Starting with the high-scale effective lagrangian in Eq.(4.5) for a real scalar, we may perform the field redefinition for tree-level matching,

$$\phi = e^{-iMv \cdot x} \frac{\chi_v}{\sqrt{M}}, \quad (4.9)$$

to pass to a heavy-particle description. Upon integrating out the physical Higgs field h , we then find the same low-scale heavy-particle lagrangian in Eq.(4.8), but with scalar χ_v , i.e., $\bar{\chi}_v = \chi_v^*$, and with residual mass and operator coefficients given by

$$\delta m = \frac{c'_{\phi 1} v^2}{2M}, \quad c_{\chi 7} = \frac{m_W^3}{m_h^2 M} c'_{\phi 1}, \quad c_{\chi 25} = -\frac{\alpha_s(\mu_H)}{12\pi} c_{\chi 7}. \quad (4.10)$$

For a Dirac fermion, we may pass to a heavy-particle description by performing the field redefinition in Eq. (4.7) on the high-scale effective lagrangian in Eq. (4.6). Upon integrating out the heavy mode X_v and weak scale particles, we obtain the low-scale effective lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \bar{\chi}_v (iv \cdot \partial - \delta m) \chi_v + \frac{1}{m_W} \bar{\chi}_v (c_{\chi 1} \sigma_{\perp \mu \nu} + c_{\chi 2} \tilde{\sigma}_{\perp \mu \nu}) \chi_v F^{\mu \nu} + \dots, \\ \delta m = \frac{c'_{\psi 1} v^2}{\Lambda}, \quad c_{\chi 1} = \frac{m_W}{\Lambda} c_W c'_{\psi 3}, \quad c_{\chi 2} = \frac{m_W}{\Lambda} c_W c'_{\psi 4}, \quad (4.11)$$

where the ellipsis denotes higher-order perturbative corrections and contributions from integrating out Z^0 and h , suppressed by $1/m_W^2$ relative to the leading electric and magnetic dipole operators. The Dirac fermion case leads to operators that are odd under the transformations in Eq. (2.71) or (2.76).

4.1.3 Case III: $m_W \ll M$

In this case the matching is organized by a power counting employing the scale separation $m_W \ll M$, and we may take the heavy-particle interactions in Eq. (3.7) as the effective theory at the high-scale. The structure of the effective theory at low energies is independent of whether the scale separation is $m_W \ll M$ or $m_W \lesssim M$, and hence upon matching we recover the results of the previous section but with different coefficient values. For example, in the case of a heavy particle derived from a Majorana or Dirac fermion, the result in Eq. (4.8) or (4.11) holds, with replacements

$$\Lambda \rightarrow M, \quad c'_{\psi 1} \rightarrow c'_{\chi 1}, \quad c'_{\psi 3} \rightarrow c'_{\chi 2}, \quad c'_{\psi 4} \rightarrow c'_{\chi 3}. \quad (4.12)$$

4.2 Multiplets and Mixtures

This section describes the matching of the effective theory described by (3.15) onto the effective theory described by (3.69), through integrating out weak-scale particles, $W^\pm$, Z^0 , h , ϕ_Z^0 , $\phi_W^\pm$, and t . The complete basis of twelve bare matching coefficients, $c_q^{(0)}$, $c_q^{(2)}$, $c_g^{(0)}$, and $c_g^{(2)}$, are determined at leading order in perturbation theory.

We may write the quark and gluon matching coefficients in terms of contributions from one-boson exchange (1BE) and two-boson exchange (2BE) diagrams,

$$c_q^{(0)} = c_q^{(0)}{}_{1\text{BE}} + c_q^{(0)}{}_{2\text{BE}} + \dots, \\ c_g^{(0)} = c_g^{(0)}{}_{1\text{BE}} + c_g^{(0)}{}_{2\text{BE}} + \dots,$$

$$\begin{aligned} c_q^{(2)} &= c_q^{(2)}{}_{2\text{BE}} + \dots, \\ c_g^{(2)} &= c_g^{(2)}{}_{2\text{BE}} + \dots, \end{aligned} \quad (4.13)$$

where the ellipses denote subleading contributions with more than two bosons exchanged. Note that spin-2 coefficients do not receive contributions from one-boson exchange amplitudes.

In the following analysis, we denote generic up- and down-type quarks by U and D , respectively, and an arbitrary quark flavor by q . We specify the contributions to the matching coefficients in terms of the constants

$$c_V^{(U)} = 1 - \frac{8}{3}s_W^2, \quad c_V^{(D)} = -1 + \frac{4}{3}s_W^2, \quad c_A^{(U)} = -1, \quad c_A^{(D)} = 1. \quad (4.14)$$

We systematically neglect subleading corrections involving light quark masses, and use CKM unitarity to simplify sums over quark flavors. Together with $|V_{tb}| \approx 1$ (and hence $|V_{td}| \approx |V_{ts}| \approx 0$), these assumptions lead to $c_u^{(S)} = c_c^{(S)}$ and $c_d^{(S)} = c_s^{(S)}$ for both $S = 0, 2$, reducing the number of independent matching coefficients to eight. When the interactions are isospin-conserving, e.g., as in the pure triplet case, we furthermore have $c_u^{(S)} = c_d^{(S)}$ and $c_c^{(S)} = c_s^{(S)}$ for both $S = 0, 2$, leaving only six independent coefficients. We use Feynman-t'Hooft gauge for the electroweak sector, and neglect higher-order corrections to the tree-level relations between residual masses, $\delta_0^{(0)} = \delta_\pm^{(0)}$ for the singlet-doublet system, and $\delta_0^{(\pm)} = \delta_\pm^{(\pm)}$ for the triplet-doublet system. In Sects. 4.2.1 and 4.2.3, we match to quark operators using onshell external quarks, and thus use the equivalence of $m_q u_q(p)$ and $\not{p} u_q(p)$.

4.2.1 Quark Matching: One-Boson Exchange

The matching condition for one-boson exchange is pictured in Fig. 4.1. The full-theory amplitude is given by

$$\begin{aligned} i\mathcal{M}_q &= i \left(\hat{\mathcal{M}}_{\text{tree}} + \hat{\mathcal{M}}_{\text{vertex},1} + \hat{\mathcal{M}}_{\text{vertex},2} + \hat{\mathcal{M}}_{\delta a_1} + \hat{\mathcal{M}}_{\delta Z} + \hat{\mathcal{M}}_{\delta v} \right) \\ &\quad - \frac{i}{-m_h^2} \frac{-ig_2 m_q}{2m_W} \bar{u}_q(p) u_q(p), \end{aligned} \quad (4.15)$$

where the $\hat{\mathcal{M}}_i$ are contributions to the $h\bar{\chi}\chi$ three-point function. These come from tree-level Higgs exchange ($\hat{\mathcal{M}}_{\text{tree}}$), one-loop diagrams with Higgs coupling to $W^\pm$ or Z^0 ($\hat{\mathcal{M}}_{\text{vertex},1}$), one-loop vertex corrections with Higgs coupling to the heavy particle ($\hat{\mathcal{M}}_{\text{vertex},2}$), the δa_1 counterterm ($\hat{\mathcal{M}}_{\delta a_1}$), wavefunction renormalization ($\hat{\mathcal{M}}_{\delta Z}$), and the renormalization of the Higgs vacuum expectation value ($\hat{\mathcal{M}}_{\delta v}$). Having included the counterterms, the sum of these contributions is finite. The one-boson exchange contribution to the spin-0 quark matching coefficient is thus

Singlet-Doublet Admixture

For the singlet-doublet case, we have the following contributions to the $h\bar{\chi}\chi$ three-point function,

$$\begin{aligned}
i\hat{\mathcal{M}}_{\text{tree}} &= ias_\rho, \quad i\hat{\mathcal{M}}_{\delta a_1} = ias_\rho \frac{\delta a_1}{a_1}, \quad i\hat{\mathcal{M}}_{\delta Z} = ias_\rho \delta Z_\chi, \quad i\hat{\mathcal{M}}_{\delta v} = ias_\rho \frac{\delta v}{v}, \\
i\hat{\mathcal{M}}_{\text{vertex},1} &= -\frac{g_2^3}{4c_W^3} c_\rho^2 m_Z I_1(\delta_0^{(0)}, m_Z) + \frac{g_2^2 a}{4c_W^2} s_\rho I_2(\delta_0^{(0)}, m_Z) \\
&\quad + \frac{g_2 a^2}{2} s_\rho^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(0)}, m_Z) \\
&\quad + \frac{3g_2 a^2}{2} \frac{m_h^2}{m_W} \left[s_\rho^2 I_1(\delta_0^{(-)}, m_h) + c_\rho^2 I_1(\delta_0^{(+)}, m_h) \right] \\
&\quad - \frac{g_2^3}{2} c_\rho^2 m_W I_1(\delta_0^{(0)}, m_W) + \frac{g_2^2 a}{2} s_\rho I_2(\delta_0^{(0)}, m_W) \\
&\quad + g_2 a^2 s_\rho^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(0)}, m_W), \\
i\hat{\mathcal{M}}_{\text{vertex},2} &= -a^3 s_\rho^3 I_4(\delta_0^{(-)}, \delta_0^{(-)}, m_h) + a^3 s_\rho c_\rho^2 I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_h) \\
&\quad - 2a^3 s_\rho c_\rho^2 I_4(\delta_0^{(-)}, \delta_0^{(+)}, m_h), \tag{4.19}
\end{aligned}$$

where δa_1 is given in (3.63), the onshell Z factor is given by

$$\begin{aligned}
Z_\chi^{-1} - 1 &= -\delta Z_\chi = [\delta Z_h]_{33} - \frac{\partial}{\partial v \cdot k} [\Sigma_2(v \cdot k)]_{33} = \delta Z_D - [\Sigma_2'(0)]_{33} \\
&\quad + \frac{1}{av} s_\rho^2 \left\{ -2s_\rho^{-1} [\Sigma_2(\delta_0^{(0)})]_{11} + t_\rho [\Sigma_2(\delta_0^{(+)})]_{22} \right. \\
&\quad \left. + 2[\Sigma_2(0)]_{23} + t_\rho^{-1} [\Sigma_2(0)]_{33} \right\}, \tag{4.20}
\end{aligned}$$

and δv is determined by the SM result [32],

$$\begin{aligned}
\frac{\delta v}{v} &= \frac{1}{2} \Sigma^{AA'}(0) - \frac{s_W}{c_W} \frac{\Sigma^{AZ}(0)}{m_Z^2} - \frac{c_W^2}{2s_W^2} \frac{\text{Re}[\Sigma^{ZZ}(m_Z^2)]}{m_Z^2} \\
&\quad + \frac{c_W^2 - s_W^2}{2s_W^2} \frac{\text{Re}[\Sigma^{WW}(m_W^2)]}{m_W^2} - \frac{1}{2} \text{Re}[\Sigma^{HH'}(m_h^2)]. \tag{4.21}
\end{aligned}$$

The two-point functions required in (4.21) are specified in (B.4) of Appendix B.1.¹ The one-boson exchange quark matching coefficient is obtained by collecting the above amplitudes into (4.16). Upon taking the pure-case limits described in Sect. 3.2.4, we recover the results (4.17) and (4.18) for a pure doublet. In the pure singlet limit, the one-boson exchange amplitudes vanish.

Triplet-Doublet Admixture

For the triplet-doublet case we have the following contributions to the $h\bar{\chi}\chi$ three-point function,

$$\begin{aligned}
i\hat{\mathcal{M}}_{\text{tree}} &= ias_\rho, \quad i\hat{\mathcal{M}}_{\delta a_1} = ias_\rho \frac{\delta a_1}{a_1}, \quad i\hat{\mathcal{M}}_{\delta Z} = ias_\rho \delta Z_\chi, \quad i\hat{\mathcal{M}}_{\delta v} = ias_\rho \frac{\delta v}{v}, \\
i\hat{\mathcal{M}}_{\text{vertex},1} &= -\frac{g_2^3}{4c_W^3} c_\rho^2 m_Z I_1(\delta_0^{(0)}, m_Z) + \frac{g_2^2 a}{4c_W^2} s_\rho I_2(\delta_0^{(0)}, m_Z) + \frac{g_2 a^2}{2} s_\rho^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(0)}, m_Z) \\
&\quad + \frac{3g_2 a^2}{2} \frac{m_h^2}{m_W} [s_\rho^2 I_1(\delta_0^{(-)}, m_h) + c_\rho^2 I_1(\delta_0^{(+)}, m_h)] - \frac{g_2^3}{8} s_\rho^2 m_W I_1(\delta_0^{(+)}, m_W) \\
&\quad - \frac{g_2^3}{2} (1 + s_\rho^2)^2 m_W I_1(\delta_0^{(-)}, m_W) + \frac{g_2^2 a}{2} s_\rho I_2(\delta_0^{(+)}, m_W) \\
&\quad + g_2 a^2 \frac{m_h^2}{m_W} I_1(\delta_0^{(+)}, m_W), \\
i\hat{\mathcal{M}}_{\text{vertex},2} &= -\frac{g_2^2 a}{8} s_\rho^3 I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_W) + \frac{g_2^2 a}{2} (1 + s_\rho^2) s_\rho c_\rho I_4(\delta_0^{(-)}, \delta_0^{(+)}, m_W) \\
&\quad + \frac{g_2^2 a}{2} (1 + s_\rho^2)^2 s_\rho I_4(\delta_0^{(-)}, \delta_0^{(-)}, m_W) + 2a^3 s_\rho I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_W) \\
&\quad + a^3 c_\rho^2 s_\rho I_4(\delta_0^{(+)}, \delta_0^{(+)}, m_h) - 2a^3 c_\rho^2 s_\rho I_4(\delta_0^{(-)}, \delta_0^{(+)}, m_h) \\
&\quad - a^3 s_\rho^3 I_4(\delta_0^{(-)}, \delta_0^{(-)}, m_h), \tag{4.22}
\end{aligned}$$

where δa_1 is specified in (3.68) and δv in (4.21). The onshell Z factor takes the same form as in (4.20), but uses the self-energy components for the triplet-doublet system given in (3.67). The one-boson exchange quark matching coefficient is obtained by collecting the above amplitudes into (4.16). Upon taking the pure case limits described in Sect. 3.2.4, we recover the results (4.17) and (4.18) for both pure triplet and pure doublet.

¹We are here neglecting contributions from states beyond the SM. Renormalization schemes relevant for WIMPs of mass $M \sim m_W$ are discussed in [38].

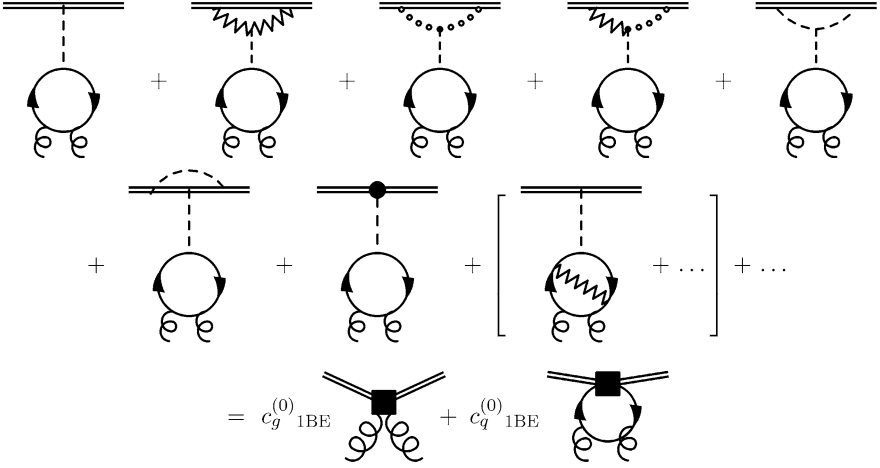


Fig. 4.2 Matching condition for one-boson exchange contributions to gluon operators. The notation for the different lines and vertices is as in Fig. 4.1. All active quark flavors, such as the top quark in the full theory, are included in the loops

4.2.2 Gluon Matching: One-Boson Exchange

One-boson exchange contributions to gluon matching are pictured in Fig. 4.2. The two-loop diagrams factorize into separate one-loop diagrams: the boson loop given by the amplitudes $\hat{\mathcal{M}}_i$ determined in the previous section, and the fermion loop familiar from, e.g., the top quark contribution to the effective $h(G_{\mu\nu}^A)^2$ vertex (e.g., see [25]). In terms of quark matching coefficients from one-boson exchange, $c_q^{(0)}$, the leading contribution to the bare gluon matching coefficient is thus

$$c_g^{(0)} = -\frac{g^2}{(4\pi)^2} \frac{1}{3} c_q^{(0)} + \mathcal{O}(\epsilon). \quad (4.23)$$

For the same reason discussed after Eq. (4.16), we neglect the one-boson exchange contributions containing $\mathcal{O}(\alpha_s^1)$ corrections to the effective $h(G_{\mu\nu}^A)^2$ coupling, shown within square brackets in Fig. 4.2. In the above result for $c_g^{(0)}$, the light quark contributions cancel between the full and effective theory amplitudes, leaving only contributions from the top quark. Further discussion of effective theory contributions can be found in Sect. 4.2.5.

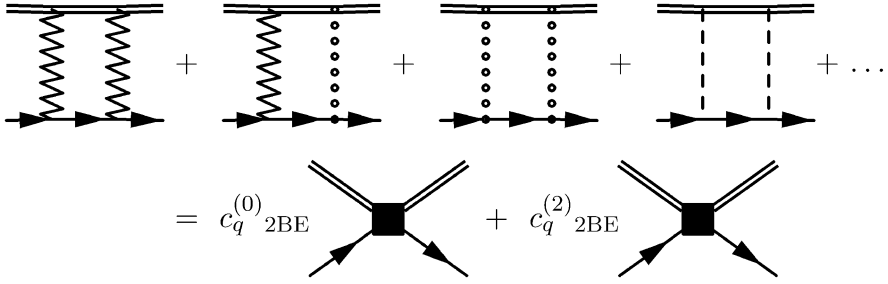


Fig. 4.3 Matching condition for two-boson exchange contributions to quark operators. The notation for the different lines and vertices is as in Fig. 4.1. The full theory diagrams illustrate the possible types of two-boson exchange. Crossed diagrams and time-reversed diagrams are not shown

4.2.3 Quark Matching: Two-Boson Exchange

Let us now consider quark matching from two-boson exchange, as displayed in Fig. 4.3. In covariant gauges, in particular Feynman-t'Hooft gauge employed here, the full theory contributions include diagrams with exchange of two gauge bosons ($W^\pm$ or Z^0), two Goldstone bosons (ϕ_Z^0 or $\phi_W^\pm$), one gauge and one Goldstone boson (Z^0 and ϕ_Z^0 , or $W^\pm$ and $\phi_W^\pm$), or two Higgs bosons. In terms of these contributions the total amplitude is

$$\mathcal{M}_q = \mathcal{M}_q^{ZZ} + \mathcal{M}_q^{WW} + \mathcal{M}_q^{W\phi_W} + \mathcal{M}_q^{Z\phi_Z} + \mathcal{M}_q^{\phi_W\phi_W} + \mathcal{M}_q^{\phi_Z\phi_Z} + \mathcal{M}_q^{hh}, \quad (4.24)$$

where the superscripts denote which bosons are exchanged, and the contributions from crossed diagrams and time-reversed diagrams are included in each amplitude. Upon expressing the amplitudes in terms of the integrals $J(m_V, M, \delta)$, $J^\mu(p, m_V, M, \delta)$, $J_-(p, m_V, M, \delta)$ and $J_-^\mu(m_V, M, \delta)$ defined in Appendix B.2, we may write each amplitude in the form

$$\mathcal{M}_q^{BB'} = \bar{u}_q(p) \left[m_q c_q^{(0)BB'} + \left(\not{p} \cdot v \cdot p - \frac{\not{p}}{d} \right) c_q^{(2)BB'} \right] u_q(p), \quad (4.25)$$

where the superscript BB' denotes the type of two-boson exchange. The contributions to spin-0 and spin-2 quark matching coefficients can then be read off as $c_q^{(0)BB'}$ and $c_q^{(2)BB'}$, respectively.

Pure States

For pure states the contributions come from diagrams with exchange of $W^\pm$ or Z^0 bosons. In terms of C_W and C_Z specified in (3.18), the amplitudes are

$$\begin{aligned}
i\mathcal{M}_q^{ZZ} &= \frac{g_2^4 \mathcal{C}_Z}{16c_W^4} \bar{u}_q(p) \left[[c_V^{(q)2} + c_A^{(q)2}] \not{p} [J(p, m_Z, 0, 0) + \not{p} J(m_Z, 0, 0)] \not{p} \right. \\
&\quad \left. + m_q [c_V^{(q)2} - c_A^{(q)2}] J(m_Z, 0, 0) \right] u_q(p), \\
i\mathcal{M}_U^{WW} &= \frac{g_2^4 \mathcal{C}_W}{8} \bar{u}_U(p) \not{p} [J(p, m_W, 0, 0) + \not{p} J(m_W, 0, 0)] \not{p} u_U(p), \\
i\mathcal{M}_D^{WW} &= \sum_U \frac{g_2^4 \mathcal{C}_W}{8} |V_{UD}|^2 \bar{u}_D(p) \not{p} [J(p, m_W, m_U, 0) + \not{p} J(m_W, m_U, 0)] \not{p} u_D(p).
\end{aligned} \tag{4.26}$$

Upon writing these amplitudes in the form of (4.25) and evaluating integrals, we find the contributions to the matching coefficients,

$$\begin{aligned}
c_{U\ 2\text{BE}}^{(0)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{8c_W^4} [c_V^{(U)2} - c_A^{(U)2}] + \mathcal{O}(\epsilon) \right\}, \\
c_D^{(0)\ 2\text{BE}} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \\
&\quad \left\{ \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{8c_W^4} [c_V^{(D)2} - c_A^{(D)2}] + \delta_{Db} \frac{m_W^{-3-2\epsilon}\mathcal{C}_W}{2} \left[-\frac{x_t}{4(x_t+1)^3} \right] + \mathcal{O}(\epsilon) \right\}, \\
c_U^{(2)\ 2\text{BE}} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \left[m_W^{-3-2\epsilon}\mathcal{C}_W + \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{2c_W^4} [c_V^{(U)2} + c_A^{(U)2}] \right] \right. \\
&\quad \left. \left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3} \log 2 \right) \epsilon \right] + \mathcal{O}(\epsilon^2) \right\}, \\
c_D^{(2)\ 2\text{BE}} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \left[m_W^{-3-2\epsilon}\mathcal{C}_W + \frac{m_Z^{-3-2\epsilon}\mathcal{C}_Z}{2c_W^4} [c_V^{(D)2} + c_A^{(D)2}] \right] \right. \\
&\quad \left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3} \log 2 \right) \epsilon \right] \\
&\quad \left. + \delta_{Db} \frac{m_W^{-3-2\epsilon}\mathcal{C}_W}{2} \left[\frac{(3x_t+2)}{3(x_t+1)^3} - \frac{2}{3} + \left(\frac{2x_t(7x_t^2-3)}{3(x_t^2-1)^3} \log x_t - \frac{2(3x_t+2)}{3(x_t+1)^3} \log 2 \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{2(25x_t^2-2x_t-11)}{9(x_t^2-1)^2(x_t+1)} - \frac{22}{9} + \frac{4}{3} \log 2 \right) \epsilon \right] + \mathcal{O}(\epsilon^2) \right\}, \tag{4.27}
\end{aligned}$$

where $x_t = m_t/m_W$, and the Kronecker delta, δ_{Db} , is equal to unity for $D = b$ and vanishes for $D = d, s$. We obtain the pure triplet (doublet) result upon setting $\mathcal{C}_W = 2$ and $\mathcal{C}_Z = 0$ ($\mathcal{C}_W = 1/2$ and $\mathcal{C}_Z = 1/4$) in (4.27).

Singlet-Doublet Admixture

For the singlet-doublet case the amplitudes for the different types of two-boson exchange are

$$\begin{aligned}
i\mathcal{M}_q^{ZZ} &= \frac{g_2^4}{64c_W^4} c_{\frac{p}{2}}^2 \bar{u}_q(p) \left[[c_V^{(q)2} + c_A^{(q)2}] \not{p} [J(p, m_Z, 0, \delta_0^{(0)}) + \not{p} J(m_Z, 0, \delta_0^{(0)})] \not{p} \right. \\
&\quad \left. + m_q [c_V^{(q)2} - c_A^{(q)2}] J(m_Z, 0, \delta_0^{(0)}) \right] u_q(p), \\
i\mathcal{M}_U^{WW} &= \frac{g_2^4}{16} c_{\frac{p}{2}}^2 \bar{u}_U(p) \not{p} [J(p, m_W, 0, \delta_0^{(0)}) + \not{p} J(m_W, 0, \delta_0^{(0)})] \not{p} u_U(p), \\
i\mathcal{M}_D^{WW} &= \sum_U \frac{g_2^4}{16} c_{\frac{p}{2}}^2 |V_{UD}|^2 \bar{u}_D(p) \not{p} [J(p, m_W, m_U, \delta_0^{(0)}) + \not{p} J(m_W, m_U, \delta_0^{(0)})] \not{p} u_D(p), \\
i\mathcal{M}_q^{Z\phi_Z} &= -\frac{g_2^3 a}{16c_W^2} s_\rho \frac{m_q}{m_W} \bar{u}_q(p) [v \cdot J_-(m_Z, 0, \delta_0^{(0)})] u_q(p), \\
i\mathcal{M}_U^{W\phi_W} &= -\frac{g_2^3 a}{8} s_\rho \frac{m_U}{m_W} \bar{u}_U(p) [v \cdot J_-(m_W, 0, \delta_0^{(0)})] u_U(p), \\
i\mathcal{M}_D^{W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \bar{u}_D(p) \\
&\quad \left[-\frac{m_D}{m_W} v \cdot J_-(m_W, m_U, \delta_0^{(0)}) + \not{p} \frac{m_U^2}{m_W} J_-(p, m_W, m_U, \delta_0^{(0)}) \right] u_D(p), \\
i\mathcal{M}_D^{\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} s_{\frac{p}{2}}^2 \frac{m_t^2}{m_W^2} |V_{tD}|^2 \bar{u}_D(p) [-m_D J(m_W, m_t, \delta_0^{(0)}) + \not{p} J(p, m_W, m_t, \delta_0^{(0)})] u_D(p), \\
i\mathcal{M}_U^{\phi_W\phi_W} &= 0, \quad i\mathcal{M}_q^{\phi_Z\phi_Z} = 0, \quad i\mathcal{M}_q^{hh} = 0.
\end{aligned} \tag{4.28}$$

The amplitudes $\mathcal{M}_q^{hh}$, $\mathcal{M}_q^{\phi_Z\phi_Z}$, and $\mathcal{M}_U^{\phi_W\phi_W}$ are suppressed by light quark masses. Comparing each amplitude above with (4.25), we find the contributions to spin-0 and spin-2 quark matching coefficients,

$$\begin{aligned}
c_q^{(0)ZZ} &= \frac{g_2^4}{64c_W^4} c_{\frac{p}{2}}^2 \left\{ [c_V^{(q)2} - c_A^{(q)2}] J(m_Z, 0, \delta_0^{(0)}) \right. \\
&\quad \left. + [c_V^{(q)2} + c_A^{(q)2}] \left[-J(m_Z, 0, \delta_0^{(0)}) - J_2(m_Z, 0, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_Z, 0, \delta_0^{(0)}) \right] \right\}, \\
c_q^{(2)ZZ} &= \frac{g_2^4}{64c_W^4} c_{\frac{p}{2}}^2 [c_V^{(q)2} + c_A^{(q)2}] \hat{J}(m_Z, 0, \delta_0^{(0)}), \\
c_U^{(0)WW} &= \frac{g_2^4}{16} c_{\frac{p}{2}}^2 \left[-J(m_W, 0, \delta_0^{(0)}) - J_2(m_W, 0, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_W, 0, \delta_0^{(0)}) \right],
\end{aligned}$$

$$\begin{aligned}
c_U^{(2)WW} &= \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 \hat{J}(m_W, 0, \delta_0^{(0)}), \\
c_D^{(0)WW} &= \sum_U \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 |V_{UD}|^2 \\
&\quad \left[-J(m_W, m_U, \delta_0^{(0)}) - J_2(m_W, m_U, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_W, m_U, \delta_0^{(0)}) \right], \\
c_D^{(2)WW} &= \sum_U \frac{g_2^4}{16} c_{\frac{\rho}{2}}^2 |V_{UD}|^2 \hat{J}(m_W, m_U, \delta_0^{(0)}), \\
c_q^{(0)Z\phi_Z} &= -\frac{g_2^3 a}{16 c_W^2} \frac{s_\rho}{m_W} J_{1-}(m_Z, 0, \delta_0^{(0)}), \\
c_q^{(2)Z\phi_Z} &= 0, \\
c_U^{(0)W\phi_W} &= -\frac{g_2^3 a}{8} \frac{s_\rho}{m_W} J_{1-}(m_W, 0, \delta_0^{(0)}), \\
c_U^{(2)W\phi_W} &= 0, \\
c_D^{(0)W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \left[-\frac{1}{m_W} J_{1-}(m_W, m_U, \delta_0^{(0)}) + \frac{1}{d} \frac{m_U^2}{m_W} J_{-}(m_W, m_U, \delta_0^{(0)}) \right], \\
c_D^{(2)W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \frac{m_U^2}{m_W} J_{-}(m_W, m_U, \delta_0^{(0)}), \\
c_D^{(0)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} s_{\frac{\rho}{2}}^2 \frac{m_t^2}{m_W^2} |V_{tD}|^2 \\
&\quad \left[J_2(m_W, m_t, \delta_0^{(0)}) - J(m_W, m_t, \delta_0^{(0)}) + \frac{1}{d} J_1(m_W, m_t, \delta_0^{(0)}) \right], \\
c_D^{(2)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} s_{\frac{\rho}{2}}^2 \frac{m_t^2}{m_W^2} |V_{tD}|^2 J_1(m_W, m_t, \delta_0^{(0)}), \tag{4.29}
\end{aligned}$$

where we have defined

$$\hat{J}(m_x, m_y, \delta_z) \equiv J_1(m_x, m_y, \delta_z) + 2J_2(m_x, m_y, \delta_z) + 2J(m_x, m_y, \delta_z). \tag{4.30}$$

The integrals $J(m_V, M, \delta)$, $J_1(m_V, M, \delta)$, $J_2(m_V, M, \delta)$, $J_{-}(m_V, M, \delta)$ and $J_{1-}(m_V, M, \delta)$ are given in Appendix B.2. The matching coefficients $c_q^{(0)2\text{BE}}$ and $c_q^{(2)2\text{BE}}$ for a given quark q are obtained by summing the nonvanishing contributions above,

$$\begin{aligned}
c_U^{(0)2\text{BE}} &= c_U^{(0)ZZ} + c_U^{(0)WW} + c_U^{(0)Z\phi_Z} + c_U^{(0)W\phi_W}, \\
c_D^{(0)2\text{BE}} &= c_D^{(0)ZZ} + c_D^{(0)WW} + c_D^{(0)Z\phi_Z} + c_D^{(0)W\phi_W} + c_D^{(0)\phi_W\phi_W},
\end{aligned}$$

$$\begin{aligned}
c_U^{(2)}{}_{2\text{BE}} &= c_U^{(2)\text{ZZ}} + c_U^{(2)\text{WW}} , \\
c_D^{(2)}{}_{2\text{BE}} &= c_D^{(2)\text{ZZ}} + c_D^{(2)\text{WW}} + c_D^{(2)W\phi_W} + c_D^{(2)\phi_W\phi_W} .
\end{aligned} \tag{4.31}$$

Upon taking the pure-case limits described in Sect. 3.2.4, we recover the results (4.26) and (4.27) for a pure doublet. In the pure singlet limit, the two-boson exchange amplitudes vanish.

Triplet-Doublet Admixture

We may similarly compute the two-boson exchange amplitudes for the triplet-doublet system, and upon comparing with (4.25), we find the following contributions to spin-0 and spin-2 quark matching coefficients,

$$\begin{aligned}
c_q^{(0)\text{ZZ}} &= \frac{g_2^4}{64c_W^4} c_{\frac{e}{2}}^2 \left\{ [c_V^{(q)2} - c_A^{(q)2}] J(m_Z, 0, \delta_0^{(0)}) \right. \\
&\quad \left. + [c_V^{(q)2} + c_A^{(q)2}] \left(-J(m_Z, 0, \delta_0^{(0)}) - J_2(m_Z, 0, \delta_0^{(0)}) + \frac{1}{d} \hat{J}(m_Z, 0, \delta_0^{(0)}) \right) \right\} , \\
c_q^{(2)\text{ZZ}} &= \frac{g_2^4}{64c_W^4} c_{\frac{e}{2}}^2 [c_V^{(q)2} + c_A^{(q)2}] \hat{J}(m_Z, 0, \delta_0^{(0)}) , \\
c_U^{(0)\text{WW}} &= \frac{g_2^4}{16} \left\{ (1 + s_{\frac{e}{2}}^2)^2 \left[-J(m_W, 0, \delta_0^{(-)}) - J_2(m_W, 0, \delta_0^{(-)}) + \frac{1}{d} \hat{J}(m_W, 0, \delta_0^{(-)}) \right] \right. \\
&\quad \left. + \frac{1}{4} s_\rho^2 \left[-J(m_W, 0, \delta_0^{(+)}) - J_2(m_W, 0, \delta_0^{(+)}) + \frac{1}{d} \hat{J}(m_W, 0, \delta_0^{(+)}) \right] \right\} , \\
c_U^{(2)\text{WW}} &= \frac{g_2^4}{16} \left[(1 + s_{\frac{e}{2}}^2)^2 \hat{J}(m_W, 0, \delta_0^{(-)}) + \frac{1}{4} s_\rho^2 \hat{J}(m_W, 0, \delta_0^{(+)}) \right] , \\
c_D^{(0)\text{WW}} &= \sum_U \frac{g_2^4}{16} |V_{UD}|^2 \\
&\quad \left\{ (1 + s_{\frac{e}{2}}^2)^2 \left[-J(m_W, m_U, \delta_0^{(-)}) - J_2(m_W, m_U, \delta_0^{(-)}) + \frac{1}{d} \hat{J}(m_W, m_U, \delta_0^{(-)}) \right] \right. \\
&\quad \left. + \frac{1}{4} s_\rho^2 \left[-J(m_W, m_U, \delta_0^{(+)}) - J_2(m_W, m_U, \delta_0^{(+)}) + \frac{1}{d} \hat{J}(m_W, m_U, \delta_0^{(+)}) \right] \right\} , \\
c_D^{(2)\text{WW}} &= \sum_U \frac{g_2^4}{16} |V_{UD}|^2 \left[(1 + s_{\frac{e}{2}}^2)^2 \hat{J}(m_W, m_U, \delta_0^{(-)}) + \frac{1}{4} s_\rho^2 \hat{J}(m_W, m_U, \delta_0^{(+)}) \right] , \\
c_q^{(0)Z\phi_Z} &= -\frac{g_2^3 a}{16c_W^2} \frac{s_\rho}{m_W} J_{1-}(m_Z, 0, \delta_0^{(0)}) ,
\end{aligned}$$

$$\begin{aligned}
c_q^{(2)Z\phi_Z} &= 0, \\
c_U^{(0)W\phi_W} &= -\frac{g_2^3 a}{8} \frac{s_\rho}{m_W} J_{1-}(m_W, 0, \delta_0^{(+)}) , \\
c_U^{(2)W\phi_W} &= 0, \\
c_D^{(0)W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \left[-\frac{1}{m_W} J_{1-}(m_W, m_U, \delta_0^{(+)}) + \frac{1}{d} \frac{m_U^2}{m_W} J_{-}(m_W, m_U, \delta_0^{(+)}) \right] , \\
c_D^{(2)W\phi_W} &= \sum_U \frac{g_2^3 a}{8} s_\rho |V_{UD}|^2 \frac{m_U^2}{m_W} J_{-}(m_W, m_U, \delta_0^{(+)}) , \\
c_D^{(0)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} \frac{m_t^2}{m_W^2} |V_{tD}|^2 \\
&\quad \left[J_2(m_W, m_t, \delta_0^{(+)}) - J(m_W, m_t, \delta_0^{(+)}) + \frac{1}{d} J_1(m_W, m_t, \delta_0^{(+)}) \right] , \\
c_D^{(2)\phi_W\phi_W} &= \frac{g_2^2 a^2}{4} \frac{m_t^2}{m_W^2} |V_{tD}|^2 J_1(m_W, m_t, \delta_0^{(+)}) , \tag{4.32}
\end{aligned}$$

where $\hat{J}(m_x, m_y, \delta_z)$ is given in (4.30), and the relevant integrals can be found in Appendix B.2. The total matching coefficients $c_q^{(0)}{}_{2\text{BE}}$ and $c_q^{(2)}{}_{2\text{BE}}$ are obtained by summing the contributions above as in (4.31). Upon taking the pure-case limits described in Sect. 3.2.4, we recover the results (4.26) and (4.27) for both pure triplet and pure doublet.

4.2.4 Gluon Matching: Two-Boson Exchange

The gluon matching condition for two-boson exchange is pictured in Fig. 4.4. If we consider the external gluons as a background field [90], we may express the full theory diagrams in terms of electroweak polarization tensors induced by a loop of quarks. For example, using the Feynman rules for the WIMP- Z^0 coupling from (3.15), the contributions from exchanging two Z^0 bosons may be written as

$$\mathcal{M}^{ZZ} \sim \int (dL) \frac{1}{-v \cdot L - \delta + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu v_\nu i\Pi_{(ZZ)}^{\mu\nu}(L) , \tag{4.33}$$

where $(dL) = d^d L / (2\pi)^d$ (this shorthand notation is used throughout this work), δ is a residual mass depending on the intermediate WIMP state, and $\Pi_{(ZZ)}^{\mu\nu}(L)$ is the two-gluon part of the Z^0 boson polarization tensor in a background gluon field. The amplitudes with exchange of one gauge and one Goldstone boson, two Goldstone bosons, or two Higgs bosons, have the same structure but with vector and scalar electroweak polarization tensors appearing. A similar analysis of gluon

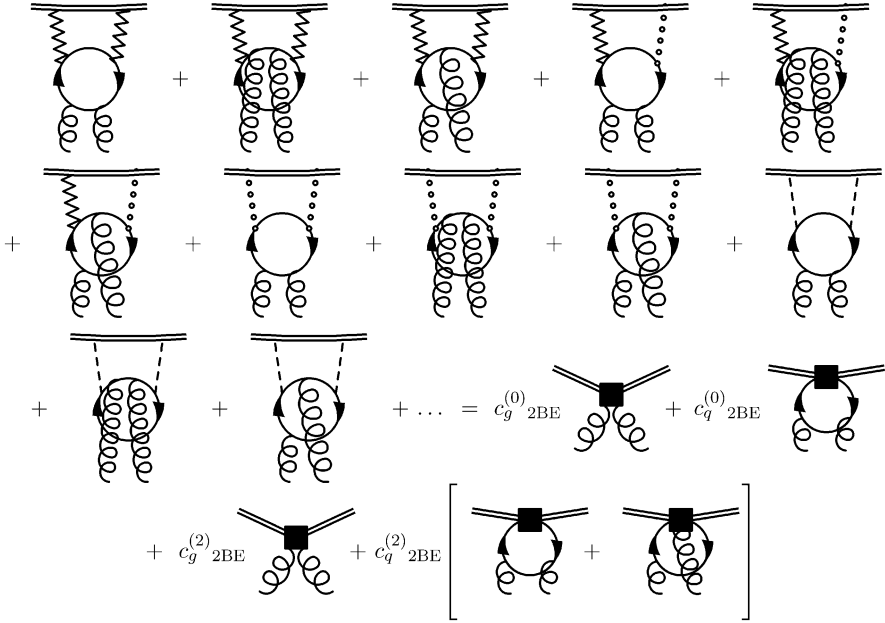


Fig. 4.4 Matching condition for two-boson exchange contributions to gluon operators. The notation for the different lines and vertices is as in Fig. 4.1. The diagrams with a quark loop are obtained by closing the external legs of the corresponding diagrams in Fig. 4.3, and considering the possible attachments of two external gluons. All active quark flavors, such as the top quark in the full theory, are included in the loops

contributions to DM-nucleon scattering in [64] focused on the spin-0 operator. Here we perform a complete matching for both spin-0 and spin-2 gluon operators, and consider new contributions appearing in the case of mixed states.

The background field method presents the following strategy for evaluating the two-loop diagrams of the full theory. First, we determine the two-gluon part of the relevant polarization tensors. These amplitudes depend only on SM parameters, and can be used for gluon matching in general DM scenarios; in particular, this part of the computation is independent of whether the heavy-particle expansion is employed. Second, we insert the polarization tensors into the boson loop and perform the remaining integrals. We illustrate this second part by identifying a basis of heavy-particle integrals to compute the universal heavy WIMP limit.

In our evaluation we neglect subleading corrections of $\mathcal{O}(m_q/m_W)$ for light quarks ($q \neq t$). The two-loop diagrams in the full theory (cf. Fig. 4.4) are UV finite, and may be evaluated in $d = 4$. However, we regulate the effective theory with dimensional regularization, and in performing the effective theory subtractions to determine Wilson coefficients it is convenient to also use dimensional regularization as IR regulator. Thus we choose to implement dimensional regularization as IR regulator also in the full theory. When considering only those terms contributing

We may expand the background field propagators at weak coupling,

$$\begin{aligned}
iS(p) &= \frac{i}{\not{p} - m} + g \int (dq) \frac{i}{\not{p} - m} i\mathcal{A}(q) \frac{i}{\not{p} - \not{q} - m} \\
&\quad + g^2 \int (dq_1)(dq_2) \frac{i}{\not{p} - m} i\mathcal{A}(q_1) \frac{i}{\not{p} - \not{q}_1 - m} i\mathcal{A}(q_2) \frac{i}{\not{p} - \not{q}_1 - \not{q}_2 - m} + \dots, \\
i\tilde{S}(p) &= \frac{i}{\not{p} - m} + g \int (dq) \frac{i}{\not{p} + \not{q} - m} i\mathcal{A}(q) \frac{i}{\not{p} - m} \\
&\quad + g^2 \int (dq_1)(dq_2) \frac{i}{\not{p} + \not{q}_1 + \not{q}_2 - m} i\mathcal{A}(q_1) \frac{i}{\not{p} + \not{q}_2 - m} i\mathcal{A}(q_2) \frac{i}{\not{p} - m} + \dots,
\end{aligned} \tag{4.40}$$

and upon insertion of these expressions into (4.37), the terms with two gluon fields are readily identified. Furthermore, in Fock-Schwinger gauge the gluon field can be written as

$$\begin{aligned}
\mathcal{A}(q) &= t^a \gamma^\alpha \int d^d x e^{iq \cdot x} A_\alpha^a(x) \\
&= t^a \gamma^\alpha \left[\frac{-i}{2} \frac{\partial}{\partial q_\rho} G_{\rho\alpha}^a(0) (2\pi)^d \delta^d(q) + \dots \right],
\end{aligned} \tag{4.41}$$

where the ellipsis denotes terms with derivatives acting on $G_{\mu\nu}^a$. Thus the amplitudes with gluon emission are given directly in terms of field-strengths, and intermediate steps involving gauge-variant combinations can be avoided.

In isolating the two-gluon amplitude, we may separately consider three cases depending on where the gluons are attached. Contributions with both gluons attached to the upper quark line in (4.34) are referred to as “*a*-type”, those with both gluons attached to the lower quark line in (4.34) are referred to as “*b*-type”, and those with one gluon attached to each of the upper and lower quark lines are referred to as “*c*-type”. We thus have

$$\tilde{\Pi}(L) = \tilde{\Pi}_a(L) + \tilde{\Pi}_b(L) + \tilde{\Pi}_c(L), \tag{4.42}$$

with

$$\begin{aligned}
i\tilde{\Pi}_a(L) &= \frac{-g^2}{4} \text{Tr}(t^a t^b) G_{\rho\alpha}^a(0) G_{\sigma\tau}^b(0) \int (dp) \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial q'_\sigma} \\
&\quad \text{Tr} \left[\Gamma \frac{1}{\not{p} - m_1} \gamma^\alpha \frac{1}{\not{p} - \not{q} - m_1} \gamma^\tau \frac{1}{\not{p} - \not{q} - \not{q}' - m_1} \Gamma' \frac{1}{\not{p} - \not{L} - m_2} \right]_{q=q'=0}, \\
i\tilde{\Pi}_b(L) &= \frac{-g^2}{4} \text{Tr}(t^a t^b) G_{\rho\alpha}^a(0) G_{\sigma\tau}^b(0) \int (dp) \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial q'_\sigma} \\
&\quad \text{Tr} \left[\Gamma \frac{1}{\not{p} - m_1} \Gamma' \frac{1}{\not{p} - \not{L} + \not{q} + \not{q}' - m_2} \gamma^\alpha \frac{1}{\not{p} - \not{L} + \not{q}' - m_2} \gamma^\tau \frac{1}{\not{p} - \not{L} - m_2} \right]_{q=q'=0},
\end{aligned}$$

$$i\tilde{\Pi}_c(L) = \frac{-g^2}{4} \text{Tr}(t^a t^b) G_{\rho\alpha}^a(0) G_{\sigma\tau}^b(0) \int (dp) \frac{\partial}{\partial q_\rho} \frac{\partial}{\partial q'_\sigma} \\ \text{Tr} \left[\Gamma \frac{1}{\not{p} - m_1} \gamma^\alpha \frac{1}{\not{p} - \not{q} - m_1} \Gamma' \frac{1}{\not{p} - \not{L} + \not{q}' - m_2} \gamma^\tau \frac{1}{\not{p} - \not{L} - m_2} \right]_{q=q'=0}, \quad (4.43)$$

where m_1 and m_2 are the masses of the quarks in the upper and lower lines in (4.34), respectively. To project these onto the spin-0 and spin-2 QCD gluon operators, $\mathcal{O}_g^{(0)}$ and $\mathcal{O}_g^{(2)}$ in (3.70), consider the four-index tensor $T_{\alpha\rho\gamma\delta} = G_{\alpha\rho}^A G_{\gamma\delta}^A$ with index symmetries $T_{\alpha\rho\gamma\delta} = T_{\gamma\delta\alpha\rho} = -T_{\rho\alpha\gamma\delta}$. We can decompose T into components $T = T^{(0)} + T^{(2)} + \Delta T$, where

$$T_{\alpha\rho\gamma\delta}^{(0)} = \frac{1}{d(d-1)} \mathcal{O}_g^{(0)} (g_{\alpha\gamma} g_{\rho\delta} - g_{\alpha\delta} g_{\rho\gamma}), \\ T_{\alpha\rho\gamma\delta}^{(2)} = \frac{1}{d-2} \left(-g_{\alpha\gamma} \mathcal{O}_{g\rho\delta}^{(2)} + g_{\alpha\delta} \mathcal{O}_{g\rho\gamma}^{(2)} - g_{\rho\delta} \mathcal{O}_{g\alpha\gamma}^{(2)} + g_{\rho\gamma} \mathcal{O}_{g\alpha\delta}^{(2)} \right), \quad (4.44)$$

and ΔT , satisfying

$$g^{\alpha\gamma} g^{\rho\delta} (\Delta T)_{\alpha\rho\gamma\delta} = v^\alpha v^\gamma g^{\rho\delta} (\Delta T)_{\alpha\rho\gamma\delta} = 0, \quad (4.45)$$

is not needed for the present analysis. The proportionality constants in $T^{(0)}$ and $T^{(2)}$ were obtained by contraction with $g^{\alpha\gamma} g^{\rho\delta}$ or $v^\alpha v^\gamma g^{\rho\delta}$. Upon applying the above decomposition to the expressions in (4.43), we obtain

$$i\tilde{\Pi}_k(L) \equiv \frac{-g^2}{8} \left[\frac{1}{d(d-1)} \mathcal{O}_g^{(0)} I_k^{(0)}(L) + \frac{1}{d-2} \mathcal{O}_g^{(2)\mu\nu} I_{k\mu\nu}^{(2)}(L) + \dots \right], \quad (4.46)$$

where $k = a, b, c$ and the ellipsis denotes irrelevant ΔT contributions.

Let us now determine $I_k^{(0)}(L)$ and $I_{k\mu\nu}^{(2)}(L)$ for the different cases of two-boson exchange. The trace and derivatives with respect to momenta q and q' in (4.43) are straightforward to evaluate, and the result is projected onto gluon operators of definite spin using (4.44). The quark-loop integral over momentum p is computed using standard methods, leaving an integral over a Feynman parameter, x , which will be evaluated after performing the boson-loop integral over momentum L . We may express the results in the form

$$I_k^{(S)}(L) \equiv \frac{i\Gamma[1+\epsilon]}{(4\pi)^{2-\epsilon}} \int_0^1 dx u_k(x) N_k^{(S)}(L), u_a(x) = \frac{(1-x)^3}{3!}, u_b(x) = \frac{x^3}{3!}, \\ u_c(x) = x(1-x), \quad (4.47)$$

where $S = 0, 2$ and for $S = 2$ the Lorentz indices are suppressed. Let us also introduce the parameters

$$z_n \equiv \frac{(-1)^n}{2^{3-n}} \frac{\Gamma[n + \epsilon]}{\Gamma[1 + \epsilon]}, \quad \Delta \equiv (1 - x)m_1^2 + xm_2^2 - x(1 - x)L^2 - i0, \quad (4.48)$$

which appear in the expressions for $N_k^{(S)}(L)$ given below.

For the operators of interest in (3.69), the relevant projections of the a - and b -type amplitudes in (4.43) and (4.44) are related by CP transformation. This condition can be stated in terms of $N_k^{(S)}(L)$ as

$$N_b^{(S)}(L) = N_a^{(S)}(L) \Big|_{x \leftrightarrow 1-x, m_1 \leftrightarrow m_2}. \quad (4.49)$$

In the case of flavor-diagonal currents (Z^0, ϕ_Z^0, h) where we set $m_1 = m_2 = m_q$ in $\tilde{\Pi}_k(L)$, the above relation implies $I_b^{(S)}(L) = I_a^{(S)}(L)$. For flavor-changing currents ($W^\pm, \phi_W^\pm$) we set the down-type quark mass $m_2 = m_D = 0$ but keep the up-type quark mass $m_1 = m_U \neq 0$ to accommodate the top quark. This asymmetry in treating the masses does not allow us to systematically recover $N_b^{(S)}(L)$ from $N_a^{(S)}(L)$ using the relation (4.49). Below we provide $N_b^{(S)}(L)$ explicitly for flavor-changing currents.

To illustrate the explicit implementation of this program, we again focus on the heavy WIMP limit, retaining the leading order (in $1/M$) WIMP-SM couplings as in (4.33). Anticipating the insertion of polarization tensors into the boson loop with leading order heavy-particle Feynman rules, we thus contract the free Lorentz indices of Γ and Γ' in (4.34), (4.36) with v_μ 's from the WIMP-vector boson vertices. It is straightforward to analyze the remaining components of $\Pi^{\mu\nu}(L)$ by the same methods. The following results are labelled by the bosons in the corresponding electroweak polarization tensor. For $N_k^{(0)}(L)$ we find,

$$\begin{aligned} N_a^{(0)}(W^+W^+) &= 64(3 - 2\epsilon)m_U^2 \left\{ 2(1 - \epsilon) \frac{z_2}{\Delta^{2+\epsilon}} + x(1 - x) (2(v \cdot L)^2 - L^2) \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\ N_b^{(0)}(W^+W^+) &= 0, \\ N_c^{(0)}(W^+W^+) &= 64(1 - \epsilon) \left\{ -2(1 + \epsilon)(3 - 2\epsilon) \frac{z_1}{\Delta^{1+\epsilon}} \right. \\ &\quad \left. + x(1 - x) [2(1 - 2\epsilon)(v \cdot L)^2 + (1 + 2\epsilon)L^2] \frac{z_2}{\Delta^{2+\epsilon}} \right\}, \\ N_a^{(0)}(ZZ) &= 32(3 - 2\epsilon)m_q^2 \left\{ [c_V^{(q)2} + c_A^{(q)2}] \right\} \end{aligned}$$

$$\begin{aligned}
& \left[2(1-\epsilon) \frac{z_2}{\Delta^{2+\epsilon}} + x(1-x)(2(v \cdot L)^2 - L^2) \frac{z_3}{\Delta^{3+\epsilon}} \right] \\
& - [c_V^{(q)2} - c_A^{(q)2}] \left[2(2-\epsilon) \frac{z_2}{\Delta^{2+\epsilon}} + x^2 L^2 \frac{z_3}{\Delta^{3+\epsilon}} \right] \Big\}, \\
N_c^{(0)}(ZZ) &= 32 \left\{ [c_V^{(q)2} + c_A^{(q)2}] (1-\epsilon) \left[-2(3-2\epsilon)(1+\epsilon) \frac{z_1}{\Delta^{1+\epsilon}} \right. \right. \\
& \quad \left. \left. + x(1-x) [2(1-2\epsilon)(v \cdot L)^2 \right. \right. \\
& \quad \left. \left. + (1+2\epsilon)L^2] \frac{z_2}{\Delta^{2+\epsilon}} \right] + [c_V^{(q)2} - c_A^{(q)2}] \epsilon(3-2\epsilon) m_q^2 \frac{z_2}{\Delta^{2+\epsilon}} \right\}, \\
N_a^{(0)}(W^+ \phi_W^+) &= -64(3-2\epsilon) m_U^2 v \cdot L \left[2[2-3x-\epsilon(1-x)] \frac{z_2}{\Delta^{2+\epsilon}} \right. \\
& \quad \left. + x^2(1-x) L^2 \frac{z_3}{\Delta^{3+\epsilon}} \right], \\
N_b^{(0)}(W^+ \phi_W^+) &= 0, \\
N_c^{(0)}(W^+ \phi_W^+) &= -64(3-2\epsilon)(1-\epsilon)(1-x) m_U^2 v \cdot L \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_a^{(0)}(Z\phi_Z) &= -32(3-2\epsilon) c_A^{(q)2} m_q v \cdot L \left[2[2-3x-\epsilon(1-x)] \frac{z_2}{\Delta^{2+\epsilon}} \right. \\
& \quad \left. + x[m_q^2 + x(1-x)L^2] \frac{z_3}{\Delta^{3+\epsilon}} \right], \\
N_c^{(0)}(Z\phi_Z) &= -32(3-2\epsilon)(1-\epsilon) c_A^{(q)2} m_q v \cdot L \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_a^{(0)}(\phi_W^+ \phi_W^+) &= 64(3-2\epsilon) m_U^4 \left[-2(2-\epsilon) \frac{z_2}{\Delta^{2+\epsilon}} \right. \\
& \quad \left. + x(1-x) L^2 \frac{z_3}{\Delta^{3+\epsilon}} \right], \\
N_b^{(0)}(\phi_W^+ \phi_W^+) &= 0, \\
N_c^{(0)}(\phi_W^+ \phi_W^+) &= 64(1-\epsilon)(3-2\epsilon) m_U^2 \left[-2(2-\epsilon) \frac{z_1}{\Delta^{1+\epsilon}} + x(1-x) L^2 \frac{z_2}{\Delta^{2+\epsilon}} \right], \\
N_a^{(0)}(\phi_Z \phi_Z) &= -32(3-2\epsilon) x m_q^2 L^2 \frac{z_3}{\Delta^{3+\epsilon}}, \\
N_c^{(0)}(\phi_Z \phi_Z) &= 32(3-2\epsilon) \left[2(1-\epsilon)(2-\epsilon) \frac{z_1}{\Delta^{1+\epsilon}} \right. \\
& \quad \left. - [(2-\epsilon) m_q^2 + (1-\epsilon)x(1-x)L^2] \frac{z_2}{\Delta^{2+\epsilon}} \right],
\end{aligned}$$

$$\begin{aligned}
N_a^{(0)}(hh) &= 32(3-2\epsilon)m_q^2 \left[-4(2-\epsilon)\frac{z_2}{\Delta^{2+\epsilon}} + x(1-2x)L^2\frac{z_3}{\Delta^{3+\epsilon}} \right], \\
N_c^{(0)}(hh) &= 32(3-2\epsilon) \left[-2(1-\epsilon)(2-\epsilon)\frac{z_1}{\Delta^{1+\epsilon}} \right. \\
&\quad \left. + [(1-\epsilon)x(1-x)L^2 - (2-\epsilon)m_q^2]\frac{z_2}{\Delta^{2+\epsilon}} \right]. \tag{4.50}
\end{aligned}$$

For $N_{k\mu\nu}^{(2)}(L)$ the open indices are to be contracted with $\mathcal{O}_g^{(2)\mu\nu}$, which is symmetric in μ and ν and satisfies $g_{\mu\nu}\mathcal{O}_g^{(2)\mu\nu} = 0$. The results are

$$\begin{aligned}
N_{a\mu\nu}^{(2)}(W+W+) &= 128(1-\epsilon) \left\{ -4(2-\epsilon)v_\mu v_\nu \frac{z_1}{\Delta^{1+\epsilon}} \right. \\
&\quad + 2 \left[(m_U^2 - x^2 L^2)v_\mu v_\nu + 2(2-\epsilon)x(1-x)v \cdot Lv_\mu L_\nu \right. \\
&\quad \left. - x(2-x-\epsilon)L_\mu L_\nu \right] \frac{z_2}{\Delta^{2+\epsilon}} \\
&\quad + x(1-x) \left[(m_U^2 - 2x^2(v \cdot L)^2)L_\mu L_\nu \right. \\
&\quad \left. - 2(m_U^2 - x^2 L^2)v \cdot Lv_\mu L_\nu \right] \frac{z_3}{\Delta^{3+\epsilon}} \Big\}, \\
N_{b\mu\nu}^{(2)}(W+W+) &= 128(1-\epsilon) \left\{ -4(2-\epsilon)v_\mu v_\nu \frac{z_1}{\Delta^{1+\epsilon}} \right. \\
&\quad - 2(1-x) \left[(1-x)L^2 v_\mu v_\nu - 2(2-\epsilon)xv \cdot Lv_\mu L_\nu \right. \\
&\quad \left. + (1+x-\epsilon)L_\mu L_\nu \right] \frac{z_2}{\Delta^{2+\epsilon}} \\
&\quad \left. + 2x(1-x)^3 \left[-(v \cdot L)^2 L_\mu L_\nu + L^2 v \cdot Lv_\mu L_\nu \right] \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{c\mu\nu}^{(2)}(W+W+) &= 128 \left\{ 2(1-\epsilon)(1-2\epsilon)v_\mu v_\nu \frac{z_1}{\Delta^{1+\epsilon}} \right. \\
&\quad \left. + x(1-x) \left[\epsilon L_\mu L_\nu + 2(1-2\epsilon)v \cdot Lv_\mu L_\nu - (1-2\epsilon)L^2 v_\mu v_\nu \right] \frac{z_2}{\Delta^{2+\epsilon}} \right\}, \\
N_{a\mu\nu}^{(2)}(ZZ) &= 64(1-\epsilon) \left\{ [c_V^{(q)2} - c_A^{(q)2}]x^2 m_q^2 L_\mu L_\nu \frac{z_3}{\Delta^{3+\epsilon}} \right. \\
&\quad + [c_V^{(q)2} + c_A^{(q)2}] \left[-4(2-\epsilon)v_\mu v_\nu \frac{z_1}{\Delta^{1+\epsilon}} \right. \\
&\quad \left. + 2[(m_q^2 - x^2 L^2)v_\mu v_\nu + 2(2-\epsilon)x(1-x)v \cdot Lv_\mu L_\nu \right.
\end{aligned}$$

$$\begin{aligned}
& +x(2-x-\epsilon)L_\mu L_\nu] \frac{z_2}{\Delta^{2+\epsilon}} \\
& +x(1-x) \left[(m_q^2 - 2x^2(v \cdot L)^2 L_\mu L_\nu - 2(m_q^2 - x^2 L^2)v \cdot L v_\mu L_\nu) \frac{z_3}{\Delta^{3+\epsilon}} \right] \Bigg\}, \\
N_{c\mu\nu}^{(2)}(ZZ) = & 64 \left\{ -[c_V^{(q)2} - c_A^{(q)2}] 2(1-\epsilon)m_q^2 v_\mu v_\nu \frac{z_2}{\Delta^{2+\epsilon}} \right. \\
& + [c_V^{(q)2} + c_A^{(q)2}] \left[2(1-\epsilon)(1-2\epsilon)v_\mu v_\nu \frac{z_1}{\Delta^{1+\epsilon}} \right. \\
& \left. \left. + x(1-x) [\epsilon L_\mu L_\nu + 2(1-2\epsilon)v \cdot L v_\mu L_\nu - (1-2\epsilon)L^2 v_\mu v_\nu] \frac{z_2}{\Delta^{2+\epsilon}} \right] \right\}, \\
N_{a\mu\nu}^{(2)}(W+\phi_W^+) = & -128(1-\epsilon)xm_U^2 \left\{ 2v_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}} - x(1-x)v \cdot L L_\mu L_\nu \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{b\mu\nu}^{(2)}(W+\phi_W^+) = & -128(1-\epsilon)(1-x)m_U^2 \left\{ 2(2-\epsilon)v_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}} \right. \\
& \left. + (1-x)^2 [L^2 v_\mu L_\nu - v \cdot L L_\mu L_\nu] \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{c\mu\nu}^{(2)}(W+\phi_W^+) = & -128(1-\epsilon)(1-x)m_U^2 v_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_{a\mu\nu}^{(2)}(Z\phi_Z) = & [c_A^{(q)2}] 64(1-\epsilon)xm_q \left\{ -2(3-\epsilon)v_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}} \right. \\
& \left. + [(m_q^2 - x^2 L^2)v_\mu L_\nu + xv \cdot L L_\mu L_\nu] \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{c\mu\nu}^{(2)}(Z\phi_Z) = & -[c_A^{(q)2}] 64(1-\epsilon)m_q v_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_{a\mu\nu}^{(2)}(\phi_W^+ \phi_W^+) = & 128(1-\epsilon)xm_U^2 L_\mu L_\nu \left\{ 2(2-\epsilon) \frac{z_2}{\Delta^{2+\epsilon}} - (1-x)m_U^2 \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{b\mu\nu}^{(2)}(\phi_W^+ \phi_W^+) = & 256(1-\epsilon)(2-\epsilon)(1-x)m_U^2 L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_{c\mu\nu}^{(2)}(\phi_W^+ \phi_W^+) = & 128(1-\epsilon)x(1-x)m_U^2 L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_{a\mu\nu}^{(2)}(\phi_Z \phi_Z) = & 64(1-\epsilon)xL_\mu L_\nu \left\{ -2(2-\epsilon) \frac{z_2}{\Delta^{2+\epsilon}} + m_q^2 \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{c\mu\nu}^{(2)}(\phi_Z \phi_Z) = & -64(1-\epsilon)x(1-x)L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}, \\
N_{a\mu\nu}^{(2)}(hh) = & 64(1-\epsilon)xL_\mu L_\nu \left\{ 2(2-\epsilon) \frac{z_2}{\Delta^{2+\epsilon}} - (1-2x)m_q^2 \frac{z_3}{\Delta^{3+\epsilon}} \right\}, \\
N_{c\mu\nu}^{(2)}(hh) = & 64(1-\epsilon)x(1-x)L_\mu L_\nu \frac{z_2}{\Delta^{2+\epsilon}}. \tag{4.51}
\end{aligned}$$

The results for $N_k^{(S)}(L)$ in (4.50) and (4.51) specify $I_a^{(S)}(L)$ through (4.47), and hence $\tilde{\Pi}_k(L)$ through (4.46), and $\tilde{\Pi}(L)$ through (4.42). This completes our determination of the polarization tensors in (4.34). The polarization tensors in (4.35) are obtained through the following relations

$$\begin{aligned}
 \Pi_{(W^-W^-)}^{\mu\nu}(L) &= \Pi_{(W^+W^+)}^{\mu\nu}(-L), \quad \Pi_{(\phi_Z Z)}^\mu(L) = \Pi_{(Z\phi_Z)}^\mu(-L), \\
 \Pi_{(\phi_W^- \phi_W^-)}(L) &= \Pi_{(\phi_W^+ \phi_W^+)}(-L), \\
 \Pi_{(\phi_W^- W^-)}^\mu(L) &= \Pi_{(W^+ \phi_W^+)}^\mu(-L), \quad \Pi_{(\phi_W^+ W^+)}^\mu(L) = \Pi_{(W^- \phi_W^-)}^\mu(-L), \\
 \Pi_{(W^- \phi_W^-)}^\mu(L) &= \Pi_{(W^+ \phi_W^+)}^\mu(-L).
 \end{aligned} \tag{4.52}$$

The identities in the first two lines are consequences of reversing the direction of momentum L in the diagrams in (4.34). The last relation follows from Hermitian conjugation and the identification $\overline{S(p)} \equiv \gamma^0 S(p)^\dagger \gamma^0 = \tilde{S}(p)$. We note that polarization tensors with one gauge and one Goldstone boson are odd in L , while all others are even in L . This property also holds for the corresponding $N_k^{(S)}(L)$, and we use it in the next section to systematically reduce the boson loop integrals into a convenient basis.

Basis Reduction of the Full Theory Boson Loop

Having determined the generalized polarization tensors, we now proceed with the reduction of the remaining boson loop integrals. Upon insertion of the polarization tensors into the boson loop, we find the required set of basic loop integrals

$$\begin{aligned}
 \int (dL) \left[\frac{1}{v \cdot L - \delta + i0} + \frac{1}{-v \cdot L - \delta + i0} \right] &= \frac{1}{(L^2 - m_V^2 + i0)^2} N_k^{(S)}(L) \\
 &\equiv \mathcal{I}_{\text{even}}(\delta, m_V) N_k^{(S)}(L), \\
 \int (dL) \left[\frac{1}{v \cdot L - \delta + i0} - \frac{1}{-v \cdot L - \delta + i0} \right] &= \frac{1}{(L^2 - m_V^2 + i0)^2} N_k^{(S)}(L) \\
 &\equiv \mathcal{I}_{\text{odd}}(\delta, m_V) N_k^{(S)}(L), \quad (4.53)
 \end{aligned}$$

where δ is the residual mass of the intermediate WIMP state, and m_V is the mass of the exchanged bosons. We suppress the arguments, (δ, m_V) , of these integral operators when making generic statements below. The integral operator $\mathcal{I}_{\text{even}}$

requires that $N_k^{(S)}(L)$ be even in L as in those for polarization tensors with a single type of boson, while the integral operator $\mathcal{I}_{\text{odd}}$ requires that $N_k^{(S)}(L)$ be odd in L as in those for polarization tensors with one gauge and one Goldstone boson. Let us denote even $N_k^{(S)}(L)$ by $N_{k\text{even}}^{(S)}(L)$ and odd $N_k^{(S)}(L)$ by $N_{k\text{odd}}^{(S)}(L)$. The subscripts even and odd may be dropped if we mean either type, or if the exchanged bosons are specified.

To reduce (4.53) to a set of basis integrals for evaluation, we begin by replacing factors of L^2 in $N_k^{(S)}(L)$ with

$$L^2 = -\frac{\Delta}{x(1-x)} + \frac{m_1^2}{x} + \frac{m_2^2}{(1-x)}, \quad (4.54)$$

which follows from the definition of Δ in (4.48). The $N_k^{(S)}(L)$ of (4.50) and (4.51) may then be written in terms of Δ and the vectors v_μ and L_μ . In $N_{k\text{even}}^{(S)}(L)$ each term must have two or zero v_μ 's, while in $N_{k\text{odd}}^{(S)}(L)$ each term must have one v_μ . Organizing the result in powers of $(v \cdot L)$, we obtain

$$\begin{aligned} N_{k\text{even}}^{(0)}(L) &= (v \cdot L)^0 \sum_n a_n^{(1)} \Delta^{-n-\epsilon} + (v \cdot L)^2 \sum_n a_n^{(2)} \Delta^{-n-\epsilon}, \\ N_{k\text{odd}}^{(0)}(L) &= (v \cdot L)^1 \sum_n a_n^{(3)} \Delta^{-n-\epsilon}, \\ N_{k\text{even}}^{(2)\mu\nu}(L) &= (v \cdot L)^0 \sum_n \left[v^\mu v^\nu a_n^{(4)} \Delta^{-n-\epsilon} \right. \\ &\quad \left. + L^\mu L^\nu a_n^{(5)} \Delta^{-n-\epsilon} \right] + (v \cdot L)^1 \sum_n v^\mu L^\nu a_n^{(6)} \Delta^{-n-\epsilon} \\ &\quad + (v \cdot L)^2 \sum_n L^\mu L^\nu a_n^{(7)} \Delta^{-n-\epsilon}, \\ N_{k\text{odd}}^{(2)\mu\nu}(L) &= (v \cdot L)^0 \sum_n v^\mu L^\nu a_n^{(8)} \Delta^{-n-\epsilon} + (v \cdot L)^1 \sum_n L^\mu L^\nu a_n^{(9)} \Delta^{-n-\epsilon}, \end{aligned} \quad (4.55)$$

where the sums run over $n = 1, 2, \dots$, and the coefficients $a_n^{(i)}$ are functions of x and ϵ . The above $N_k^{(S)}(L)$ structures require the set of integrals

$$\begin{aligned} H(n) &= \mathcal{I}_{\text{even}} \Delta^{-n-\epsilon}, \quad H^\mu(n) = \mathcal{I}_{\text{odd}} \Delta^{-n-\epsilon} L^\mu, \quad H^{\mu\nu}(n) = \mathcal{I}_{\text{even}} \Delta^{-n-\epsilon} L^\mu L^\nu, \\ F(n) &= \int (dL) \frac{1}{(L^2 - m_V^2 + i0)^2} \Delta^{-n-\epsilon}. \end{aligned} \quad (4.56)$$

The integrals H^μ and $H^{\mu\nu}$ may be expressed in terms of $H(n)$ and $F(n)$ through standard reduction methods and the relation

$$\begin{aligned} & \left[\frac{1}{v \cdot L - \delta + i0} \pm \frac{1}{-v \cdot L - \delta + i0} \right] v \cdot L \\ &= \delta \left[\frac{1}{v \cdot L - \delta + i0} \mp \frac{1}{-v \cdot L - \delta + i0} \right] + 1 \mp 1. \end{aligned} \quad (4.57)$$

Furthermore, recursion relations in n may be derived by taking derivatives of parameters. A detailed discussion of these relations, as well as the evaluation of the above integrals, can be found in Appendix B.3. Note that the $(v \cdot L)^2$ term in $N_{k \text{ even}}^{(2) \mu\nu}(L)$ also requires the integral

$$\int (dL) \frac{1}{(L^2 - m_V^2 + i0)^2} \Delta^{-n-\epsilon} L^\mu L^\nu \sim g^{\mu\nu}, \quad (4.58)$$

however this does not contribute since it vanishes upon contraction with the traceless spin-2 gluon operator, $O_g^{(2) \mu\nu}$. Upon feeding the general expressions for $N_k^{(S)}(L)$ in (4.55) into the integrals in (4.53), we find the following decomposition in terms of basis integrals,

$$\begin{aligned} \mathcal{I}_{\text{even}} N_{k \text{ even}}^{(0)}(L) &= \sum_n \left[a_n^{(1)} H(n) + a_n^{(2)} [\delta^2 H(n) + 2\delta F(n)] \right], \\ \mathcal{I}_{\text{odd}} N_{k \text{ odd}}^{(0)}(L) &= \sum_n a_n^{(3)} [\delta H(n) + 2F(n)], \\ \mathcal{I}_{\text{even}} N_{k \text{ even}}^{(2) \mu\nu}(L) &= v^\mu v^\nu \sum_n \left[a_n^{(4)} H(n) + a_n^{(5)} H_1(n) \right. \\ &\quad \left. + a_n^{(6)} [\delta^2 H(n) + 2\delta F(n)] + a_n^{(7)} \delta^2 H_1(n) \right], \\ \mathcal{I}_{\text{odd}} N_{k \text{ odd}}^{(2) \mu\nu}(L) &= v^\mu v^\nu \sum_n \left[a_n^{(8)} [\delta H(n) + 2F(n)] + a_n^{(9)} \delta H_1(n) \right], \end{aligned} \quad (4.59)$$

where

$$H_1(n) = \frac{1}{3-2\epsilon} \left\{ (4-2\epsilon) [\delta^2 H(n) + 2\delta F(n)] + \frac{H(n-1)}{x(1-x)} - \left[\frac{m_1^2}{x} + \frac{m_2^2}{1-x} \right] H(n) \right\}. \quad (4.60)$$

The above results apply generally to both pure and mixed states. Comparing with the explicit expressions for $N_k^{(S)}(L)$ in (4.50) and (4.51), we find that $H(n)$ for $n = 1, 2, 3$ and $F(n)$ for $n = 2, 3$ are required.

For pure states there is no residual mass, and $\mathcal{I}_{\text{odd}}$ is irrelevant since the only contributions are from exchanges of $W^\pm$ and Z^0 , involving $N_{k\text{even}}^{(S)}(L)$. The vanishing of certain contributions in $\mathcal{I}_{\text{even}}N_{k\text{even}}^{(S)}(L)$ at $\delta = 0$ can be traced to the identity in (4.57).² Setting $\delta = 0$ in $\mathcal{I}_{\text{even}}N_{k\text{even}}^{(S)}(L)$ above and using the explicit expressions for $N_k^{(S)}(L)$ in (4.50) and (4.51), we find pure-state results that depend on $H(n)$ only,

$$\begin{aligned}
\mathcal{I}(0, m_W)N_a^{(0)}(W^+W^+) &= 64(1+\epsilon)(3-2\epsilon)m_U^2 \\
&\quad \left\{ (2+\epsilon)(1-x)m_U^2H(3) - (1+2\epsilon)H(2) \right\}, \\
\mathcal{I}(0, m_W)N_c^{(0)}(W^+W^+) &= 32(1-\epsilon^2)\left\{ (1+2\epsilon)(1-x)m_U^2H(2) + 2(1-2\epsilon)H(1) \right\}, \\
\mathcal{I}(0, m_Z)N_a^{(0)}(ZZ) &= \frac{32(1+\epsilon)(3-2\epsilon)m_q^2}{1-x} \\
&\quad \left\{ [c_V^{(q)2} + c_A^{(q)2}](1-x)[(2+\epsilon)m_q^2H(3) - (1+2\epsilon)H(2)] \right. \\
&\quad \left. + [c_V^{(q)2} - c_A^{(q)2}][(2+\epsilon)xm_q^2H(3) - (2-\epsilon+2\epsilon x)H(2)] \right\}, \\
\mathcal{I}(0, m_Z)N_c^{(0)}(ZZ) &= 16(1+\epsilon)\left\{ [c_V^{(q)2} + c_A^{(q)2}](1-\epsilon) \right. \\
&\quad \left. [(1+2\epsilon)m_q^2H(2) + 2(1-2\epsilon)H(1)] \right. \\
&\quad \left. + [c_V^{(q)2} - c_A^{(q)2}]\epsilon(3-2\epsilon)m_q^2H(2) \right\}, \\
\mathcal{I}(0, m_W)N_{a\mu\nu}^{(2)}(W^+W^+) &= \frac{128(1-\epsilon)v_\mu v_\nu}{(3-2\epsilon)(1-x)}\left\{ (2-\epsilon)(2-x-3\epsilon+4\epsilon x)H(1) \right. \\
&\quad \left. + (1+\epsilon)(1-x)m_U^2 \right. \\
&\quad \left. [(2+\epsilon)(1-x)m_U^2H(3) + (3-4x-4\epsilon+2\epsilon x)H(2)] \right\}, \\
\mathcal{I}(0, m_W)N_{c\mu\nu}^{(2)}(W^+W^+) &= \frac{64(1-\epsilon)v_\mu v_\nu}{(3-2\epsilon)} \\
&\quad \left\{ -(3-\epsilon-4\epsilon^2)(1-x)m_U^2H(2) + \epsilon(7-8\epsilon)H(1) \right\}, \\
\mathcal{I}(0, m_Z)N_{a\mu\nu}^{(2)}(ZZ) &= \frac{64(1-\epsilon)v_\mu v_\nu}{(3-2\epsilon)(1-x)}
\end{aligned}$$

²In particular, this can be used to demonstrate gauge invariance for the electroweak part of the amplitudes since in a general R_ξ gauge the ξ -dependent terms carry a factor of $(v \cdot L)$.

$$\begin{aligned}
& \left\{ [c_V^{(q)2} + c_A^{(q)2}] \left[(2 - \epsilon)(2 - x - 3\epsilon + 4\epsilon x)H(1) \right. \right. \\
& \quad + m_q^2(1 + \epsilon) \left[(2 + \epsilon)(1 - x)m_q^2 H(3) \right. \\
& \quad \left. \left. + (3 - 5x - 4\epsilon + 5\epsilon x)H(2) \right] \right] \\
& \quad + [c_V^{(q)2} - c_A^{(q)2}] (1 + \epsilon)(2 + \epsilon)xm_q^2 [m_q^2 H(3) - H(2)] \left. \right\}, \\
\mathcal{I}(0, m_Z) N_{c\mu\nu}^{(2)}(ZZ) &= \frac{32(1 - \epsilon)v_\mu v_\nu}{(3 - 2\epsilon)} \left\{ [c_V^{(q)2} + c_A^{(q)2}] \right. \\
& \quad \left[- (1 + \epsilon)(3 - 4\epsilon)m_q^2 H(2) + \epsilon(7 - 8\epsilon)H(1) \right] \\
& \quad \left. - [c_V^{(q)2} - c_A^{(q)2}] 2(1 + \epsilon)(3 - 2\epsilon)m_q^2 H(2) \right\}, \tag{4.61}
\end{aligned}$$

where the subscript on $\mathcal{I}_{\text{even}}$ has been suppressed. The reduction for admixtures, where there are nonzero residual masses and the integral $\mathcal{I}_{\text{odd}}$ is relevant, is also straightforward to obtain.

We collect in Appendix B.3 useful results for the remaining task of integrating over Feynman parameters. The singularity structure and evaluation of integrals can be classified into three cases corresponding to zero, one, or two heavy fermions contributing to the electroweak polarization tensor. The case of zero heavy fermions is for polarization tensors with no top quark in the loop. With subleading powers of light quark masses neglected, only polarization tensors of $W^\pm$ and Z^0 bosons are relevant in this case. The case of one heavy fermion is for polarization tensors of flavor-changing currents with one top quark and one down-type quark. The case of two heavy fermions is for polarization tensors of flavor-diagonal currents with a top quark loop.

Full Theory Contributions and Matching Coefficients for Pure States

Let us now determine the full theory contributions to the matching using the generalized electroweak polarization tensors and the reduction method for the boson loop integral. For pure states, the total amplitude receives two-boson exchange contributions from $W^\pm$ and Z^0 bosons,

$$\mathcal{M} = \mathcal{M}^{WW} + \mathcal{M}^{ZZ}, \tag{4.62}$$

which may be written in terms of electroweak polarization tensors in a background field as

$$\begin{aligned}
 i\mathcal{M}^{WW} &= \frac{ig_2^2 \mathcal{C}_W}{2} \int (dL) \frac{1}{-v \cdot L + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} v_\mu v_\nu \left[i\Pi_{(W^+ W^+)}^{\mu\nu}(L) + i\Pi_{(W^- W^-)}^{\mu\nu}(L) \right], \\
 i\mathcal{M}^{ZZ} &= \frac{ig_2^2 \mathcal{C}_Z}{c_W^2} \int (dL) \frac{1}{-v \cdot L + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu v_\nu i\Pi_{(ZZ)}^{\mu\nu}(L), \quad (4.63)
 \end{aligned}$$

with $\mathcal{C}_W$ and $\mathcal{C}_Z$ given in (3.18). The parity of the polarization tensors under $L \rightarrow -L$ and the identities in (4.52) allow us to write the above amplitudes in terms of the integrals defined in (4.53),

$$\begin{aligned}
 i\mathcal{M}^{WW} &= \frac{ig_2^2 \mathcal{C}_W}{2} \mathcal{I}_{\text{even}}(0, m_W) v_\mu v_\nu i\Pi_{(W^+ W^+)}^{\mu\nu}(L), \\
 i\mathcal{M}^{ZZ} &= \frac{ig_2^2 \mathcal{C}_Z}{2c_W^2} \mathcal{I}_{\text{even}}(0, m_Z) v_\mu v_\nu i\Pi_{(ZZ)}^{\mu\nu}(L). \quad (4.64)
 \end{aligned}$$

Upon inserting the explicit polarization tensors from (4.34) into the expressions above, we may employ the reduction of integrals given in (4.61) and write each contribution in terms of the gluon operators of definite spin,

$$\mathcal{M}^{BB'} = \mathcal{M}^{BB'(0)} \mathcal{O}_g^{(0)} + \mathcal{M}^{BB'(2)} v_\mu v_\nu \mathcal{O}_g^{(2)\mu\nu}, \quad (4.65)$$

where the superscript BB' denotes the different types of two-boson exchange. From the expression in (4.65), we readily identify the contribution of each amplitude to $c_g^{(0)}{}_{2\text{BE}}$ and $c_g^{(2)}{}_{2\text{BE}}$ as $\mathcal{M}^{BB'(0)}$ and $\mathcal{M}^{BB'(2)}$, respectively. Let us decompose $\mathcal{M}^{WW(S)}$, for $S = 0, 2$, into contributions from each up-type quark flavor, and the a -, b -, and c -type gluon attachments,

$$\mathcal{M}^{WW(S)} = -\frac{[\Gamma(1 + \epsilon)]^2}{(4\pi)^d} \frac{\pi g^2 g_2^4 \mathcal{C}_W}{m_W^{3+4\epsilon}} \frac{1}{16} \sum_{U=u,c,t} \sum_{k=a,b,c} \mathcal{M}_{U,k}^{WW(S)}. \quad (4.66)$$

Similarly, we decompose $\mathcal{M}^{ZZ(S)}$ into contributions from each quark flavor, and the a -, b -, and c -type gluon attachments,

$$\mathcal{M}^{ZZ(S)} = -\frac{[\Gamma(1 + \epsilon)]^2}{(4\pi)^d} \frac{\pi g^2 g_2^4}{m_Z^{3+4\epsilon}} \frac{\mathcal{C}_Z}{64c_W^4} \sum_{q=u,c,t,d,s,b} \sum_{k=a,b,c} \mathcal{M}_{q,k}^{ZZ(S)}. \quad (4.67)$$

The results for $W^\pm$ exchange are as follows. The amplitudes with one top quark are

$$\begin{aligned}
\mathcal{M}_{t,a}^{WW(0)} &= 4x_t^2 \log \frac{x_t + 1}{x_t} - \frac{2x_t(6x_t^2 + 9x_t + 2)}{3(x_t + 1)^2}, \\
\mathcal{M}_{t,b}^{WW(0)} &= 0, \\
\mathcal{M}_{t,c}^{WW(0)} &= -4x_t^2 \log \frac{x_t + 1}{x_t} + \frac{2(6x_t^3 + 9x_t^2 + 2x_t - 2)}{3(x_t + 1)^2}, \\
\mathcal{M}_{t,a}^{WW(2)} &= \frac{16(30x_t^4 - 3x_t^2 - 4)}{9} \log \frac{x_t + 1}{x_t} \\
&\quad - \frac{8(60x_t^5 + 90x_t^4 + 14x_t^3 - 14x_t^2 - 8x_t - 9)}{9(x_t + 1)^2}, \\
\mathcal{M}_{t,b}^{WW(2)} &= \frac{16(3x_t + 2)}{9(x_t + 1)^3} \frac{1}{\epsilon} + \frac{32x_t(15x_t^9 - 48x_t^7 + 52x_t^5 - 15x_t^3 + 14x_t^2 - 6)}{9(x_t^2 - 1)^3} \log x_t \\
&\quad - \frac{32(15x_t^{10} - 48x_t^8 + 52x_t^6 - 12x_t^4 + 3x_t^2 - 2)}{9(x_t^2 - 1)^3} \log(x_t + 1) \\
&\quad - \frac{32(3x_t^4 - 21x_t^3 + 3x_t^2 + 9x_t - 2)}{9(x_t^2 - 1)^3} \log 2 \\
&\quad + \frac{8(180x_t^8 + 90x_t^7 - 426x_t^6 - 183x_t^5 + 285x_t^4 + 111x_t^3 - 220x_t^2 - 4x_t + 71)}{27(x_t^2 - 1)^2(x_t + 1)}, \\
\mathcal{M}_{t,c}^{WW(2)} &= -48x_t^2 \log \frac{x_t + 1}{x_t} + \frac{8x_t(6x_t^2 + 9x_t + 2)}{(x_t + 1)^2}, \tag{4.68}
\end{aligned}$$

where $x_t = m_t/m_W$. The amplitudes with only light quarks are

$$\begin{aligned}
\mathcal{M}_{U,a}^{WW(0)} &= \mathcal{M}_{U,b}^{WW(0)} = 0, \quad \mathcal{M}_{U,c}^{WW(0)} = -\frac{4}{3}, \\
\mathcal{M}_{U,a}^{WW(2)} &= \mathcal{M}_{U,b}^{WW(2)} = \frac{32}{9\epsilon} + \frac{568}{27} - \frac{64}{9} \log 2, \quad \mathcal{M}_{U,c}^{WW(2)} = 0, \tag{4.69}
\end{aligned}$$

for $U = u, c$. The results for Z^0 exchange with a top quark loop are

$$\begin{aligned}
\mathcal{M}_{t,a}^{ZZ(0)} &= \mathcal{M}_{t,b}^{ZZ(0)} = [c_V^{(t)2} + c_A^{(t)2}] \\
&\quad \left[\frac{4y_t^2(32y_t^6 - 28y_t^4 + 14y_t^2 - 1)}{(4y_t^2 - 1)^{7/2}} \arctan(\sqrt{4y_t^2 - 1}) - \frac{\pi y_t}{2} \right. \\
&\quad \left. + \frac{4y_t^2(y_t^2 - 1)(24y_t^2 - 1)}{3(4y_t^2 - 1)^3} \right] + [c_V^{(t)2} - c_A^{(t)2}]
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{16y_t^4(24y_t^4 - 21y_t^2 + 5)}{(4y_t^2 - 1)^{7/2}} \arctan(\sqrt{4y_t^2 - 1}) \right. \\
& \left. + \frac{2(144y_t^6 - 70y_t^4 + 9y_t^2 - 2)}{3(4y_t^2 - 1)^3} - \frac{3\pi y_t}{2} \right], \\
\mathcal{M}_{t,c}^{ZZ(0)} &= [c_V^{(t)2} + c_A^{(t)2}] \\
& \left[-\frac{8y_t^2(8y_t^2 - 1)(2y_t^2 - 1)}{(4y_t^2 - 1)^{5/2}} \right. \\
& \left. \arctan(\sqrt{4y_t^2 - 1}) - \frac{4(24y_t^4 - 7y_t^2 + 1)}{3(4y_t^2 - 1)^2} + 2\pi y_t \right], \\
\mathcal{M}_{t,a}^{ZZ(2)} &= \mathcal{M}_{t,b}^{ZZ(2)} = [c_V^{(t)2} + c_A^{(t)2}] \\
& \left[\frac{16(480y_t^8 - 420y_t^6 + 214y_t^4 - 47y_t^2 + 4)}{9(4y_t^2 - 1)^{7/2}} \arctan(\sqrt{4y_t^2 - 1}) \right. \\
& \left. + \frac{8(240y_t^6 - 314y_t^4 + 92y_t^2 - 9)}{9(4y_t^2 - 1)^3} - \frac{10\pi y_t}{3} \right] \\
& + [c_V^{(t)2} - c_A^{(t)2}] \left[-\frac{8y_t^2(48y_t^4 - 34y_t^2 + 13)}{9(4y_t^2 - 1)^3} \right. \\
& \left. - \frac{32y_t^2(16y_t^6 - 14y_t^4 + 4y_t^2 - 1)}{3(4y_t^2 - 1)^{7/2}} \arctan(\sqrt{4y_t^2 - 1}) + \frac{2\pi y_t}{3} \right], \\
\mathcal{M}_{t,c}^{ZZ(2)} &= \{ [c_V^{(t)2} + c_A^{(t)2}] + 2[c_V^{(t)2} - c_A^{(t)2}] \} \\
& \left[-\frac{32y_t^2(16y_t^4 - 10y_t^2 + 3)}{(4y_t^2 - 1)^{5/2}} \arctan(\sqrt{4y_t^2 - 1}) \right. \\
& \left. - \frac{16y_t^2(8y_t^2 - 5)}{(4y_t^2 - 1)^2} + 8\pi y_t \right], \tag{4.70}
\end{aligned}$$

where $y_t = m_t/m_Z$. The amplitudes for Z^0 exchange with a light quark loop are

$$\begin{aligned}
\mathcal{M}_{q,a}^{ZZ(0)} &= \mathcal{M}_{q,b}^{ZZ(0)} = 0, \quad \mathcal{M}_{q,c}^{ZZ(0)} = [c_V^{(q)2} + c_A^{(q)2}] \left[-\frac{4}{3} \right], \\
\mathcal{M}_{q,a}^{ZZ(2)} &= \mathcal{M}_{q,b}^{ZZ(2)} = [c_V^{(q)2} + c_A^{(q)2}] \left[\frac{32}{9\epsilon} + \frac{568}{27} - \frac{64}{9} \log 2 \right], \quad \mathcal{M}_{q,c}^{ZZ(2)} = 0, \tag{4.71}
\end{aligned}$$

where $q = u, d, s, c, b$. The $\frac{1}{\epsilon}$ pieces in the above amplitudes are IR divergences that cancel upon subtraction of the effective theory contributions, $\mathcal{M}_{\text{EFT}}^{(S)}$, discussed in Sect. 4.2.5. The bare coefficients are then given by

$$c_g^{(S)}{}_{2\text{BE}} = \mathcal{M}^{WW(S)} + \mathcal{M}^{ZZ(S)} - \mathcal{M}_{\text{EFT}}^{(S)}, \tag{4.72}$$

where the remaining $\frac{1}{\epsilon}$ pieces are UV divergences.

Full Theory Contributions and Matching Coefficients for Admixtures

For admixtures, the total amplitude receives contributions from other types of two-boson exchange beyond WW and ZZ ,

$$\mathcal{M} = \mathcal{M}^{WW} + \mathcal{M}^{ZZ} + \mathcal{M}^{\phi_W \phi_W} + \mathcal{M}^{\phi_Z \phi_Z} + \mathcal{M}^{hh} + \mathcal{M}^{Z\phi_Z} + \mathcal{M}^{W\phi_W}. \quad (4.73)$$

Let us first consider the singlet-doublet case. In terms of the electroweak polarization tensors, we find integrals involving nonzero residual masses,

$$\begin{aligned} i\mathcal{M}^{WW} &= \frac{ig_2^2}{4} c_{\frac{\rho}{2}}^2 \int (dL) \frac{1}{-v \cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} v_\mu v_\nu \\ &\quad \left[i\Pi_{(W^+W^+)}^{\mu\nu}(L) + i\Pi_{(W^-W^-)}^{\mu\nu}(L) \right], \\ i\mathcal{M}^{ZZ} &= \frac{ig_2^2}{4c_W^2} c_{\frac{\rho}{2}}^2 \int (dL) \frac{1}{-v \cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu v_\nu i\Pi_{(ZZ)}^{\mu\nu}(L), \\ i\mathcal{M}^{hh} &= ia^2 \int (dL) \left[\frac{s_\rho^2}{-v \cdot L - \delta_0^{(-)} + i0} + \frac{c_\rho^2}{-v \cdot L - \delta_0^{(+)} + i0} \right] \\ &\quad \frac{1}{(L^2 - m_h^2 + i0)^2} i\Pi_{(hh)}(L), \\ i\mathcal{M}^{\phi_Z \phi_Z} &= ia^2 s_{\frac{\rho}{2}}^2 \int (dL) \frac{1}{-v \cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} i\Pi_{(\phi_Z \phi_Z)}(L), \\ i\mathcal{M}^{\phi_W \phi_W} &= ia^2 s_{\frac{\rho}{2}}^2 \int (dL) \frac{1}{-v \cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} \\ &\quad \left[i\Pi_{(\phi_W^+ \phi_W^+)}(L) + i\Pi_{(\phi_W^- \phi_W^-)}(L) \right], \\ i\mathcal{M}^{Z\phi_Z} &= \frac{g_2 a}{4c_W} s_\rho \int (dL) \frac{1}{-v \cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_Z^2 + i0)^2} v_\mu \\ &\quad \left[i\Pi_{(Z\phi_Z)}^\mu(L) - i\Pi_{(\phi_Z Z)}^\mu(L) \right], \\ i\mathcal{M}^{W\phi_W} &= \frac{ig_2 a}{4} s_\rho \int (dL) \frac{1}{-v \cdot L - \delta_0^{(0)} + i0} \frac{1}{(L^2 - m_W^2 + i0)^2} v_\mu \\ &\quad \left[i\Pi_{(W^+ \phi_W^+)}^\mu(L) - i\Pi_{(W^- \phi_W^-)}^\mu(L) \right. \\ &\quad \left. + i\Pi_{(\phi_W^+ W^+)}^\mu(L) - i\Pi_{(\phi_W^- W^-)}^\mu(L) \right]. \end{aligned} \quad (4.74)$$

Using the behavior of the polarization tensors under $L \rightarrow -L$ and the identities in (4.52), we may write these amplitudes in terms of the integrals defined in (4.53),

$$\begin{aligned}
i\mathcal{M}^{WW} &= \frac{ig_2^2}{4} c_{\frac{\rho}{2}}^2 \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_W) v_\mu v_\nu i\Pi_{(W^+W^+)}^{\mu\nu}(L), \\
i\mathcal{M}^{ZZ} &= \frac{ig_2^2}{8c_W^2} c_{\frac{\rho}{2}}^2 \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_Z) v_\mu v_\nu i\Pi_{(ZZ)}^{\mu\nu}(L), \\
i\mathcal{M}^{hh} &= \frac{ia^2}{2} s_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(-)}, m_h) i\Pi_{(hh)}(L) + \frac{ia^2}{2} c_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(+)}, m_h) i\Pi_{(hh)}(L), \\
i\mathcal{M}^{\phi_Z\phi_Z} &= \frac{ia^2}{2} s_{\frac{\rho}{2}}^2 \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_Z) i\Pi_{(\phi_Z\phi_Z)}(L), \\
i\mathcal{M}^{\phi_W\phi_W} &= ia^2 s_{\frac{\rho}{2}}^2 \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_W) i\Pi_{(\phi_W^+\phi_W^+)}(L), \\
i\mathcal{M}^{Z\phi_Z} &= \frac{g_2 a}{4c_W} s_\rho \mathcal{I}_{\text{odd}}(\delta_0^{(0)}, m_Z) v_\mu i\Pi_{(Z\phi_Z)}^\mu(L), \\
i\mathcal{M}^{W\phi_W} &= \frac{ig_2 a}{2} s_\rho \mathcal{I}_{\text{odd}}(\delta_0^{(0)}, m_W) v_\mu i\Pi_{(W^+\phi_W^+)}^\mu(L). \tag{4.75}
\end{aligned}$$

The required polarization tensors are specified in (4.34), and, in particular, the complete set of functions $N_k^{(S)}(L)$ are explicitly given in (4.50) and (4.51). Thus, the general result in (4.59) for reducing these integrals may be applied. Each amplitude may be written in the form of (4.65), i.e., in terms of its contributions to the gluon operators of definite spin. The bare coefficients are then given by

$$\begin{aligned}
c_g^{(S)}{}_{2\text{BE}} &= \mathcal{M}^{(S)WW} + \mathcal{M}^{(S)ZZ} + \mathcal{M}^{(S)hh} + \mathcal{M}^{(S)\phi_Z\phi_Z} \\
&\quad + \mathcal{M}^{(S)\phi_W\phi_W} + \mathcal{M}^{(S)Z\phi_Z} + \mathcal{M}^{(S)W\phi_W} - \mathcal{M}_{\text{EFT}}^{(S)}, \tag{4.76}
\end{aligned}$$

where the remaining $\frac{1}{\epsilon}$ pieces are UV divergences. We may again organize each contribution in the previous equation in terms of the quark flavors in the loop, and the a -, b -, and c -type gluon attachments, as we have done in (4.66) and (4.67).

For the triplet-doublet case we find,

$$\begin{aligned}
i\mathcal{M}^{WW} &= \frac{ig_2^2}{16} s_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(+)}, m_W) v_\mu v_\nu i\Pi_{(W^+W^+)}^{\mu\nu}(L) \\
&\quad + \frac{ig_2^2}{4} (1 + s_{\frac{\rho}{2}}^2)^2 \mathcal{I}_{\text{even}}(\delta_0^{(-)}, m_W) v_\mu v_\nu i\Pi_{(W^+W^+)}^{\mu\nu}(L), \\
i\mathcal{M}^{ZZ} &= \frac{ig_2^2}{8c_W^2} c_{\frac{\rho}{2}}^2 \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_Z) v_\mu v_\nu i\Pi_{(ZZ)}^{\mu\nu}(L), \\
i\mathcal{M}^{hh} &= \frac{ia^2}{2} s_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(-)}, m_h) i\Pi_{(hh)}(L) + \frac{ia^2}{2} c_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(+)}, m_h) i\Pi_{(hh)}(L),
\end{aligned}$$

$$\begin{aligned}
i\mathcal{M}^{\phi_Z\phi_Z} &= \frac{ia^2}{2}s_\rho^2 \mathcal{I}_{\text{even}}(\delta_0^{(0)}, m_Z) i\Pi_{(\phi_Z\phi_Z)}(L), \\
i\mathcal{M}^{\phi_W\phi_W} &= ia^2 \mathcal{I}_{\text{even}}(\delta_0^{(+)}, m_W) i\Pi_{(\phi_W^+\phi_W^+)}(L), \\
i\mathcal{M}^{Z\phi_Z} &= \frac{g_2 a}{4c_W} s_\rho \mathcal{I}_{\text{odd}}(\delta_0^{(0)}, m_Z) v_\mu i\Pi_{(Z\phi_Z)}^\mu(L), \\
i\mathcal{M}^{W\phi_W} &= \frac{ig_2 a}{2} s_\rho \mathcal{I}_{\text{odd}}(\delta_0^{(+)}, m_W) v_\mu i\Pi_{(W^+\phi_W^+)}^\mu(L). \tag{4.77}
\end{aligned}$$

The rest of the analysis proceeds as above, using the same polarization tensors and integral reduction method. We check for both types of admixtures that the expected results are recovered upon taking the pure-case limits described in Sect. 3.2.4.

4.2.5 Effective Theory Amplitudes and Infrared Regulator

In the computation of both pure- and mixed-case amplitudes above, we have neglected subleading corrections of $\mathcal{O}(m_q/m_W)$ by Taylor expanding integrands about vanishing light quark masses.³ This requires a regulator to control IR divergences (the full theory diagrams in Figs. 4.2 and 4.4 are UV finite but the projection onto the spin-2 operator $\mathcal{O}_g^{(2)}$ is IR divergent).

It is technically simplest to compute the full and effective theory amplitudes using dimensional regularization as IR regulator. Effective theory loop diagrams on the right hand sides of Figs. 4.2 and 4.4 then result in dimensionfull but scaleless integrals that are required to vanish. Upon subtracting the effective theory amplitude, remaining $1/\epsilon$ pieces in matching coefficients are identified as UV divergences.

We have obtained identical renormalized matching coefficients by retaining light quark masses, $m_q \neq 0$, as an alternative IR regulator. In this scheme, the effective theory loop diagrams on the right-hand side of Figs. 4.2 and 4.4 yield nonvanishing contributions. The full theory diagrams on the left-hand side are correspondingly modified so that, upon subtracting the effective theory amplitude, consistent results are obtained.

³For matching onto quark operators, we of course include the leading m_q factor appearing in $\mathcal{O}_q^{(0)}$ and $\mathcal{O}_q^{(2)}$. For matching onto gluon operators we may neglect light quark masses.

4.2.6 Extended Higgs Sector for Pure Case

Additional structure in the Higgs sector may impact cross-section predictions for spin-independent WIMP-nucleon scattering. For the pure case, we may easily modify the one-boson exchange matching for quark and gluon operators to include a second CP-even Higgs of mass $m_H > m_h$, arising in the context of the type-II two-Higgs-doublet model. Two-boson exchange amplitudes are not modified.

Parameterization

Let us consider the light CP-even Higgs to be the “SM-like” Higgs with mass $m_h = 126$ GeV. For direct detection, the inclusion of a second Higgs doublet introduces the parameters $m_H \approx m_A, \alpha, \beta$ or equivalently, $m_H, t_\beta, \beta - \alpha \equiv X \in (0, \pi)$, where m_H is the mass of the heavy CP-even Higgs and t_β is the ratio of VEVs. The relevant couplings of the CP-even Higgses, h and H , to vector bosons and up- and down-type quarks are given in Eqs. (24)–(26) of [20] which we reproduce here,

$$\begin{aligned} g_{hVV} &= s_X, & g_{HVV} &= c_X, \\ g_{hdd} &= s_X - t_\beta c_X, & g_{Hdd} &= c_X + t_\beta s_X, \\ g_{huu} &= s_X + t_\beta^{-1} c_X, & g_{Hu u} &= c_X - t_\beta^{-1} s_X. \end{aligned} \quad (4.78)$$

To consider small deviations about the alignment limit, we may further use the parameterization $\eta \equiv t_\beta c_X, s_X \equiv \sqrt{1 - c_X^2}$, and thus work with the set of parameters m_H, t_β, η . The couplings are then

$$\begin{aligned} g_{hVV} &= \sqrt{1 - t_\beta^{-2} \eta^2}, & g_{HVV} &= t_\beta^{-1} \eta, \\ g_{hdd} &= \sqrt{1 - t_\beta^{-2} \eta^2} - \eta, & g_{Hdd} &= t_\beta^{-1} \eta + t_\beta \sqrt{1 - t_\beta^{-2} \eta^2}, \\ g_{huu} &= \sqrt{1 - t_\beta^{-2} \eta^2} + t_\beta^{-2} \eta, & g_{Hu u} &= t_\beta^{-1} \eta - t_\beta^{-1} \sqrt{1 - t_\beta^{-2} \eta^2}. \end{aligned} \quad (4.79)$$

Expanding about $\eta = 0$ recovers the results in Eqs. (44)–(46) of [20].

A plot of the couplings of the light CP-even Higgs in units of the SM values is shown in Fig. 4.5 for $t_\beta = 5$. The couplings to vector bosons and up-type quarks are close to 1 in the whole range, with deviations suppressed by t_β^{-2} . The coupling to down-type quarks is given by

$$g_{hdd} = (1 - \eta) + \mathcal{O}(t_\beta^{-2}), \quad (4.80)$$

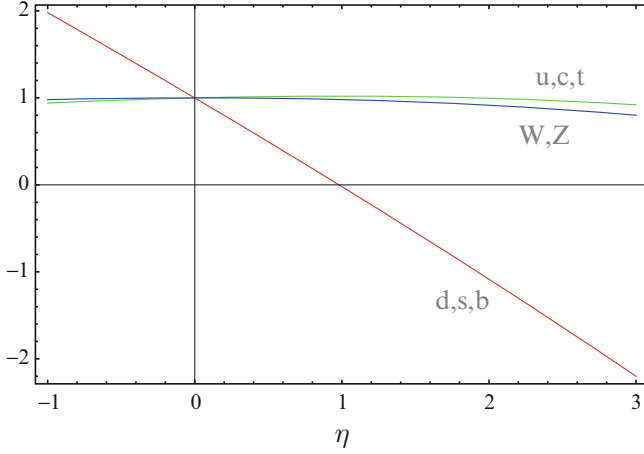


Fig. 4.5 Couplings of the light CP-even Higgs to W, Z (blue), up-type quarks (green) and down-type quarks (red) in units of the SM values, using $t_\beta = 5$. For $g_{hdd} = (1 - \eta) + \mathcal{O}(t_\beta^{-2})$, the SM magnitude is obtained for $\eta = 0, 2$, but with an opposite sign for $\eta = 2$. For up to 50 % deviation about the “alignment limit” one may consider the regions $-0.5 < \eta < 0.5$ and $1.5 < \eta < 2.5$

in units of the SM value. Hence, about $\eta = 0$ and $\eta = 2$ the magnitude of the coupling is near that of the SM, but with opposite sign for the latter. The couplings are not sensitive to t_β for $t_\beta \gg 1$ since the subleading terms are $\mathcal{O}(1/t_\beta^2)$.

Modification of Weak-Scale Matching Amplitudes

In the type-II two-Higgs-doublet model, only the CP-even Higgs boson, H and h , couple to SM vector bosons at tree-level, and thus A , $H^\pm$, and the Goldstones do not appear in the weak-scale matching computation at leading order. The heavy Higgs appears in the same diagrams as the light Higgs but with modified couplings to vector bosons and quarks. The couplings of the light Higgs are also modified. We may thus account for these effects by making the following replacement in the coefficients $c_q^{(0)}$,

$$x_h^{-2} \rightarrow x_h^{-2} \left(F_h^q + \frac{m_h^2}{m_H^2} F_H^q \right), \quad F_h^q = g_{hVV} g_{hqq}, \quad F_H^q = g_{HVV} g_{Hqq}. \quad (4.81)$$

4.2.7 Bare Matching Coefficients

We may now collect the results of the preceding analysis of quark and gluon matching to present the bare coefficients of the effective theory at the weak scale. We have analyzed the Wilson coefficients of the effective theory described by (3.69)

in terms of contributions from exchanges of one or two electroweak bosons, as expressed in (4.13). The results for one-boson exchange matching to quark and gluon operators are given by (4.16) and (4.23), respectively. The results for two-boson exchange matching to quark and gluon operators are given by summing contributions of the form (4.25) and (4.65), respectively.

For pure cases, the results for the bare matching coefficients are as follows,

$$\begin{aligned}
c_U^{(0)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ -\frac{m_W^{-3-2\epsilon}}{2x_h^2} \left[C_W + \frac{C_Z}{c_W^3} \right] + \frac{m_Z^{-3-2\epsilon}C_Z}{8c_W^4} [c_V^{(U)2} - c_A^{(U)2}] + \mathcal{O}(\epsilon) \right\}, \\
c_D^{(0)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ -\frac{m_W^{-3-2\epsilon}}{2x_h^2} \left[C_W + \frac{C_Z}{c_W^3} \right] + \frac{m_Z^{-3-2\epsilon}C_Z}{8c_W^4} [c_V^{(D)2} - c_A^{(D)2}] \right. \\
&\quad \left. -\delta_{Db} m_W^{-3-2\epsilon} C_W \frac{x_t}{8(x_t+1)^3} + \mathcal{O}(\epsilon) \right\}, \\
c_g^{(0)} &= \frac{\pi[\Gamma(1+\epsilon)]^2 g_2^4 g^2}{(4\pi)^{4-2\epsilon}} \left\{ \frac{m_W^{-3-4\epsilon}}{2} \left[\frac{1}{3x_h^2} \left[C_W + \frac{C_Z}{c_W^3} \right] + C_W \left[\frac{1}{3} + \frac{1}{6(x_t+1)^2} \right] \right] \right. \\
&\quad + \frac{m_Z^{-3-4\epsilon}C_Z}{64c_W^4} \left[4[c_V^{(D)2} + c_A^{(D)2}] + [c_V^{(U)2} + c_A^{(U)2}] \right. \\
&\quad \left[\frac{8}{3} + \frac{32y_t^6(8y_t^2-7)}{(4y_t^2-1)^{7/2}} \arctan(\sqrt{4y_t^2-1}) \right. \\
&\quad \left. \left. -\pi y_t + \frac{4(48y_t^6-2y_t^4+9y_t^2-1)}{3(4y_t^2-1)^3} \right] + [c_V^{(U)2} - c_A^{(U)2}] \right. \\
&\quad \left[3\pi y_t - \frac{4(144y_t^6-70y_t^4+9y_t^2-2)}{3(4y_t^2-1)^3} \right. \\
&\quad \left. \left. \left. -\frac{32y_t^4(24y_t^4-21y_t^2+5)}{(4y_t^2-1)^{7/2}} \arctan(\sqrt{4y_t^2-1}) \right] \right] + \mathcal{O}(\epsilon) \right\}, \\
c_U^{(2)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \left[m_W^{-3-2\epsilon} C_W + \frac{m_Z^{-3-2\epsilon}C_Z}{2c_W^4} [c_V^{(U)2} + c_A^{(U)2}] \right] \right. \\
&\quad \left. \left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3} \log 2 \right) \epsilon \right] + \mathcal{O}(\epsilon^2) \right\}, \\
c_D^{(2)} &= \frac{\pi\Gamma(1+\epsilon)g_2^4}{(4\pi)^{2-\epsilon}} \left\{ \left[m_W^{-3-2\epsilon} C_W + \frac{m_Z^{-3-2\epsilon}C_Z}{2c_W^4} [c_V^{(D)2} + c_A^{(D)2}] \right] \right. \\
&\quad \left. \left[\frac{1}{3} + \left(\frac{11}{9} - \frac{2}{3} \log 2 \right) \epsilon \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \delta_{Db} \frac{m_W^{-3-2\epsilon} \mathcal{C}_W}{2} \left[\frac{(3x_t + 2)}{3(x_t + 1)^3} - \frac{2}{3} + \left(\frac{2x_t(7x_t^2 - 3)}{3(x_t^2 - 1)^3} \log x_t - \frac{2(3x_t + 2)}{3(x_t + 1)^3} \log 2 \right. \right. \\
& \left. \left. - \frac{2(25x_t^2 - 2x_t - 11)}{9(x_t^2 - 1)^2(x_t + 1)} - \frac{22}{9} + \frac{4}{3} \log 2 \right) \epsilon \right] + \mathcal{O}(\epsilon^2) \Big\}, \\
c_g^{(2)} = & \frac{\pi[\Gamma(1 + \epsilon)]^2 g_s^4 g^2}{(4\pi)^{4-2\epsilon}} \left\{ \frac{m_W^{-3-4\epsilon} \mathcal{C}_W}{2} \left[-\frac{16}{9\epsilon} - \frac{284}{27} + \frac{32}{9} \log 2 - \frac{2(3x_t + 2)}{9(x_t + 1)^3} \frac{1}{\epsilon} \right. \right. \\
& + \frac{8(6x_t^8 - 18x_t^6 + 21x_t^4 - 3x_t^2 - 2)}{9(x_t^2 - 1)^3} \log(x_t + 1) \\
& + \frac{4(3x_t^4 - 21x_t^3 + 3x_t^2 + 9x_t - 2)}{9(x_t^2 - 1)^3} \log 2 \\
& - \frac{4(12x_t^8 - 36x_t^6 + 39x_t^4 + 14x_t^3 - 9x_t^2 - 6x_t - 2)}{9(x_t^2 - 1)^3} \log x_t \\
& \left. - \frac{144x_t^6 + 72x_t^5 - 312x_t^4 - 105x_t^3 - 40x_t^2 + 47x_t + 98}{27(x_t^2 - 1)^2(x_t + 1)} \right] \\
& + \frac{m_Z^{-3-4\epsilon} \mathcal{C}_Z}{64c_W^4} \left[\left[8[c_V^{(U)2} + c_A^{(U)2}] + 12[c_V^{(D)2} + c_A^{(D)2}] \right] \right. \\
& \left[-\frac{16}{9\epsilon} - \frac{284}{27} + \frac{32}{9} \log 2 \right] \\
& + [c_V^{(U)2} + c_A^{(U)2}] \left[\frac{128(24y_t^8 - 21y_t^6 - 4y_t^4 + 5y_t^2 - 1)}{9(4y_t^2 - 1)^{7/2}} \right. \\
& \arctan(\sqrt{4y_t^2 - 1}) - \frac{4\pi y_t}{3} \\
& \left. + \frac{16(48y_t^6 + 62y_t^4 - 47y_t^2 + 9)}{9(4y_t^2 - 1)^3} \right] + [c_V^{(U)2} - c_A^{(U)2}] \\
& \left[\frac{16y_t^2(624y_t^4 - 538y_t^2 + 103)}{9(4y_t^2 - 1)^3} - \frac{52\pi y_t}{3} \right. \\
& \left. \left. + \frac{128y_t^2(104y_t^6 - 91y_t^4 + 35y_t^2 - 5)}{3(4y_t^2 - 1)^{7/2}} \arctan(\sqrt{4y_t^2 - 1}) \right] \right] + \mathcal{O}(\epsilon) \Big\}, \quad (4.82)
\end{aligned}$$

where, as before, $x_t = m_t/m_W$ and $y_t = m_t/m_Z$. Above, the Kronecker delta, δ_{Db} , is equal to unity for $D = b$, and vanishes for $D = d, s$. The pure triplet (doublet) results are given by setting $\mathcal{C}_W = 2$ and $\mathcal{C}_Z = 0$ ($\mathcal{C}_W = 1/2$ and $\mathcal{C}_Z = 1/4$). The renormalization of the theory involving these bare coefficients will be detailed in the next chapter. In particular, the relation between the bare coefficient $c_g^{(2)}$ given above and the renormalized coefficient $c_g^{(2)}(\mu)$ involves a nontrivial subtraction requiring the $\mathcal{O}(\epsilon)$ part of $c_q^{(2)}$ which we have retained.

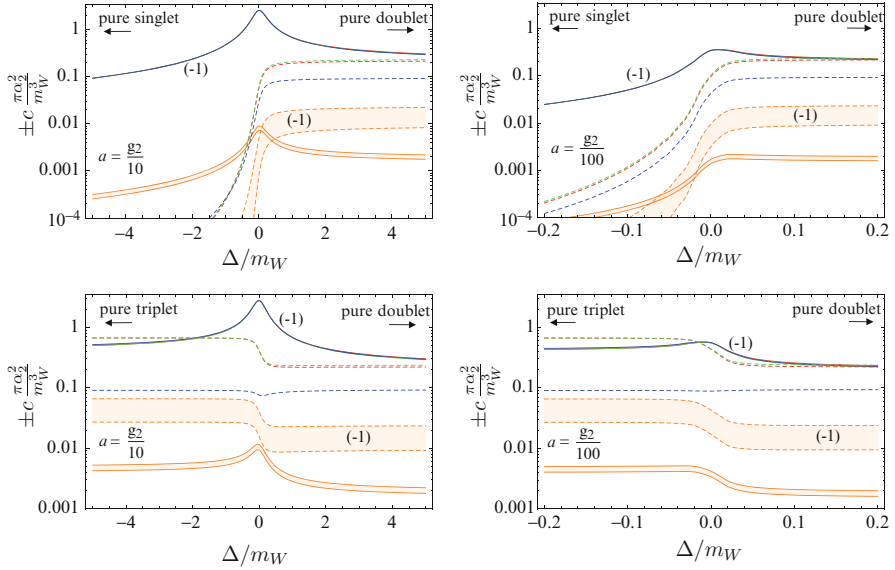


Fig. 4.6 Renormalized coefficients (with $\pi\alpha_2^2/m_W^3$ extracted) for the singlet-doublet (*upper panels*) and triplet-doublet (*lower panels*) mixtures as a function of the respective mass splittings $\Delta = (M_S - M_D)/2$ and $\Delta = (M_T - M_D)/2$, in units of m_W . The panels on the left (right) use $a = g_2/10$ ($a = g_2/100$). The negative coefficients $c_q^{(0)}$ and $c_g^{(2)}$ are presented with opposite sign, as indicated by (-1) . The *solid red, green, and blue lines* are respectively for $-c_{U=u,c}^{(0)}$, $-c_{D=d,s}^{(0)}$, and $-c_b^{(0)}$. The *dashed red, green, and blue lines* are respectively for $c_{U=u,c}^{(2)}$, $c_{D=d,s}^{(2)}$, and $c_b^{(2)}$. Some quark matching coefficients appear degenerate. The *orange band with solid borders* is $c_g^{(0)}$, and the *orange band with dashed borders* is $-c_g^{(2)}$. The *band thickness* represents renormalization scale variation, taking $m_W^2/2 < \mu_t^2 < 2m_t^2$ [59]. We indicate the pure-case limits at large $|\Delta|$

The results for admixtures are similarly obtained by collecting contributions to the coefficients specified in (4.13). For example, the amplitudes in (4.19) for a singlet-doublet admixture, combined with the integrals defined in Appendix B.1, specify $c_q^{(0)}{}_{1\text{BE}}$ through (4.16), and $c_g^{(0)}{}_{1\text{BE}}$ through (4.23). The coefficients $c_q^{(S)}{}_{2\text{BE}}$ are specified in (4.31) in terms of the results in (4.29), which require the integrals in Appendix B.2. Finally, $c_g^{(S)}{}_{2\text{BE}}$ is specified in (4.76) in terms of the amplitudes in (4.75) which require the polarization tensors in (4.34), the basis reduction in (4.59), and the integrals in Appendix B.3.

The matching coefficients for admixtures are functions of the mass splitting Δ and the coupling a , as defined in (3.38) for the singlet-doublet mixture. We illustrate numerical values in Fig. 4.6 for both the singlet-doublet and triplet-doublet mixtures. Numerical inputs are collected in Table B.1 of Appendix B.4. Depending on the value of a , the $\mathcal{O}(\alpha_2^1)$ tree-level Higgs exchange contribution to the spin-0 coefficients may dominate near $\Delta = 0$. When m_W/Δ suppression is significant,

the $\mathcal{O}(\alpha_s^2)$ loop contributions dominate. The curves approach the correct pure-case values upon taking the limits described in Sect. 3.2.4. In particular, the coefficients vanish in the pure singlet limit.

The contributions of these coefficients to scattering cross sections depend on the detailed mapping onto the low-energy $n_f = 3$ flavor theory through renormalization group running and heavy quark threshold matching, and on the evaluation of nucleon matrix elements at a low scale, $\mu \sim 1$ GeV. These effects enhance the contribution from certain coefficients, upsetting the α_s counting reflected in the relative magnitudes of the high-scale coefficients. One example is the enhancement of the spin-0 gluon contribution due both to a large anomalous dimension in the RG running, and to the large nucleon matrix element of the scalar gluon operator [97]. Another example is the enhanced impact of numerically subleading contributions due to a partial cancellation at leading order. The relative signs between high-scale coefficients in Fig. 4.6, combined with details of the mapping onto low-energy coefficients and evaluation of matrix elements, lead to a cancellation between the spin-0 and spin-2 amplitude contributions [59, 65]. Therefore, a robust determination of DM-nucleon scattering cross sections demands a careful analysis of the complete set of leading operators in (3.70).

The coefficient $c_g^{(2)}$ has been omitted in previous works [64, 65]. Due to a cancellation between spin-0 and spin-2 amplitude contributions to cross sections, the effect of neglecting $c_g^{(2)}$ ranges from a factor of a few to an order of magnitude difference in cross sections. For the pure-doublet and pure-triplet states, neglecting $c_g^{(2)}$ leads to an $\mathcal{O}(10\text{--}20\%)$ shift in the spin-2 amplitude, depending on the choice of renormalization scale, and an underestimation of its perturbative uncertainty by $\mathcal{O}(70\%)$. For comparison, neglecting $c_q^{(2)}$ for $q = b, c, s, d, u$ shifts the spin-2 amplitude by $\mathcal{O}(1\%)$, $\mathcal{O}(10\%)$, $\mathcal{O}(10\%)$, $\mathcal{O}(30\%)$, and $\mathcal{O}(50\%)$, respectively.

Although we find that cancellations are generic, their severity depends on SM parameters and on properties of DM such as its electroweak quantum numbers. The presence of additional low-lying states could also have impact, and the formalism for weak-scale matching presented here can be readily extended to investigate such scenarios. For example, including a second Higgs doublet in the pure-state analysis simply requires modification of the vertices in the amplitudes computed in Figs. 4.1 and 4.2. An extra Higgs boson modifies the spin-0 amplitude, and could potentially weaken the cancellation between spin-0 and spin-2 amplitudes. The case where the second Higgs-like doublet itself plays the role of DM (e.g., “inert Higgs DM” [78]) is related to the pure-doublet case in the heavy WIMP limit by heavy particle universality.

While we have focused here on the case of a heavy, self-conjugate WIMP, deriving from one or two electroweak multiplets, much of the formalism applies more generally. The generalized electroweak polarization tensors obtained through background field techniques depend only on SM parameters, and hence can be applied for gluon operator matching in general DM scenarios. Within the context of heavy particle effective theories, the new integral basis evaluated here may be applied to other processes such as low-energy lepton-nucleon scattering [62].

Chapter 5

QCD Analysis and Hadronic Matrix Elements

Estimating the correct order of magnitude of scattering cross sections in many simple and motivated models of dark matter requires careful treatment of competing Standard Model contributions in weak-scale matching, and of QCD corrections when passing from a theory renormalized at the electroweak scale to a low-energy theory of quarks and gluons. In the previous chapter, we focused on weak-scale matching conditions necessary for robustly computing WIMP-nucleon interactions, both in specified UV completions involving electroweak-charged DM, and in the model-independent heavy WIMP limit. In this chapter, we tackle the remaining analysis below the weak scale, applicable to a broad class of theories, independent of assuming a particular UV model or the heavy WIMP limit. Although we focus here on the analysis for the effective theory given in Eq. (3.69), it is straightforward to consider an extended basis of operators relevant for other scenarios, e.g., χ that is not self-conjugate.

The formalism to systematically map high-scale matching coefficients onto the low-energy theory, where matrix elements are evaluated, requires several ingredients. A careful analysis of operator renormalization is necessary to determine physical amplitudes from bare coefficients obtained from weak-scale matching. In particular, a nontrivial subtraction appears in the analysis of quark and gluon tensor operators. Renormalization group evolution of coefficients between heavy quark thresholds, and matching of $(n_f + 1)$ -flavor and n_f -flavor QCD at the bottom ($\mu_b \sim m_b$) and charm ($\mu_c \sim m_c$) heavy quark thresholds allow for precise connection between high- and low-scale coefficients. These tools account for operator mixing and large logarithms $\sim \log \frac{m_t}{m_c}$, and provide an estimate of uncertainty from higher order perturbative QCD corrections. Analysis of the necessary hadronic matrix elements at the low-scale identifies sources of uncertainty and areas for potential improvement in determination of inputs from lattice studies.

This chapter is organized as follows. In Sect. 5.1 we present the framework for renormalization of the bare lagrangian obtained from matching at the weak scale, and explicit results for the renormalized coefficients for pure states. Sections 5.2

and 5.3 perform the analysis for renormalization group evolution of coefficients and matching conditions at heavy quark thresholds, respectively. In Sect. 5.4 we provide an overview of hadronic matrix elements necessary for spin-independent WIMP-nucleon scattering.

5.1 Operator Renormalization

Having determined the bare matching coefficients of the low-energy theory at the weak scale, we may proceed to map onto a theory valid at lower energy scales. The lower energy scale, μ , may refer to a light dark matter mass scale, $\mu \sim M \sim \text{few GeV}$, for computing annihilation processes, or to hadronic scales, $\Lambda_{\text{QCD}} \lesssim \mu \lesssim m_c$, where hadronic matrix elements for scattering cross sections are computed in three-flavor QCD.

The first task is to renormalize the bare lagrangian obtained from matching at the weak scale. Inspection of the low-energy SM building blocks in Table 3.1 and Eq.(3.2) shows that, up to field redefinitions,¹ the strong interaction matrix elements relevant for renormalization of low-energy operators through dimension seven involve the vector and axial-vector current of dimension three,

$$V_\mu^{(q)} = \bar{q}\gamma_\mu q, \quad A_\mu^{(q)} = \bar{q}\gamma_\mu\gamma_5 q, \quad (5.1)$$

the scalar (spin-0) and tensor (spin-2) operators of dimension four in Eq.(3.69),

$$\begin{aligned} O_q^{(0)} &= m_q \bar{q}q, \\ O_g^{(0)} &= G_{\mu\nu}^A G^{A\mu\nu}, \\ O_q^{(2)\mu\nu} &= \frac{1}{2} \bar{q} \left(\gamma^{\{\mu} i D_-^{\nu\}} - \frac{g^{\mu\nu}}{d} i \not{D} - \right) q, \\ O_g^{(2)\mu\nu} &= -G^{A\mu\lambda} G_{\lambda}^{A\nu} + \frac{g^{\mu\nu}}{d} (G_{\alpha\beta}^A)^2, \end{aligned} \quad (5.2)$$

and the pseudoscalar (spin-0) and pseudotensor (spin-2) operators of dimension four,

$$\begin{aligned} O_{5q}^{(0)} &= m_q \bar{q} i \gamma_5 q, \\ O_{5g}^{(0)} &= G_{\mu\nu}^A \tilde{G}^{A\mu\nu}, \end{aligned}$$

¹For example, the tensor $m_q \bar{q} \sigma^{\mu\nu} q$ appearing in the low-energy bases of fermion operators in Eqs. (3.9) and (3.10) may be rewritten in terms of the vector and tensor currents in Eqs. (5.1)–(5.3) by performing the field redefinition $q \rightarrow (1 + a \mathcal{O}_{\text{DM}}^{\mu\nu} \sigma_{\mu\nu}) q$ with appropriate choice of parameter a and DM bilinear $\mathcal{O}_{\text{DM}}$.

$$\begin{aligned}
O_{5q}^{(2)\mu\nu} &= \frac{i}{2}\bar{q} \left(\gamma^{\{\mu} iD_-^{\nu\}} - \frac{g^{\mu\nu}}{d} i\not{D} - \right) \gamma_5 q, \\
O_{5g}^{(2)\mu\nu} &= -G^{A\mu\lambda} \tilde{G}^{A\nu}_{\lambda} + \frac{g^{\mu\nu}}{d} G_{\alpha\beta}^A \tilde{G}^{A\alpha\beta}.
\end{aligned} \tag{5.3}$$

Here $A^{\{\mu}B^{\nu\}} \equiv (A^\mu B^\nu + A^\nu B^\mu)/2$ denotes symmetrization. We consider operators $O_q^{(S)}$ and $O_{5q}^{(S)}$ for each active quark and each spin $S = 0, 2$, allowing for independent coefficients but not including flavor-violating operators. The basis of operators in Eqs. (5.2) and (5.3) are separately closed under renormalization, and contribute to spin-independent and spin-dependent scattering on the nucleon, respectively.

5.1.1 Renormalization Constants

We define the operator renormalization constants Z_{ij} in the $\overline{\text{MS}}$ scheme, except for the axial-vector and pseudoscalar operators where we consider an additional finite renormalization to retain a conventional anomaly condition. The relation between renormalized and bare operators, and the corresponding coefficients are given by

$$O_i^{\text{bare}} = Z_{ij}(\mu) O_j^{\text{ren}}(\mu), \quad c_i^{\text{ren}}(\mu) = Z_{ji}(\mu) c_j^{\text{bare}}, \tag{5.4}$$

with an implicit sum over repeated indices. Let us proceed to collect the renormalization constants of operators arising in the low-energy effective theory for the DM particle. The vector currents, representing conserved quark number, evolve trivially under QCD renormalization:

$$\partial_\mu V^{(q)\mu} = 0, \quad Z_V = 1. \tag{5.5}$$

The axial anomaly in the presence of $SU(3)_c$ gauge fields implies non-conservation of the (flavor singlet) axial currents. In a renormalization prescription that retains the one-loop anomaly condition, we have

$$\partial_\mu A^{(q)\mu} = \frac{1}{2} g^2 \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a, \tag{5.6}$$

and the renormalization coefficient through two loop is [82]

$$\begin{aligned}
Z_A &= (Z_{\text{MS}}^s)^{-1} (Z_5^s)^{-1}, \\
Z_{\text{MS}}^s &= 1 + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[\frac{22}{3\epsilon} C_F C_A + \frac{5}{3\epsilon} C_F n_f \right] + \mathcal{O}(\alpha_s^3), \\
Z_5^s &= 1 + \frac{\alpha_s}{4\pi} [-4C_F] + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[22C_F^2 - \frac{107}{9} C_F C_A + \frac{31}{18} C_F n_f \right] + \mathcal{O}(\alpha_s^3).
\end{aligned} \tag{5.7}$$

Three three-loop corrections are also available [82], but are presently beyond phenomenological relevance.

For the scalar and tensor operators in Eq. (5.2), the operator renormalization in the $\overline{\text{MS}}$ scheme is independent of quark flavor [54, 100]. In the basis $(u, d, s, \dots, g)$ and through the first order pole in $1/\epsilon$, we have

$$\begin{aligned}\hat{Z}^{(0)} &= \mathbb{1} + \frac{1}{\epsilon} \left(\begin{array}{ccc|c} 0 & & & 0 \\ & \ddots & & \vdots \\ & & 0 & 0 \\ \hline 2\gamma_m & \dots & 2\gamma_m & \tilde{\beta} \end{array} \right) = \mathbb{1} + \frac{1}{\epsilon} \frac{\alpha_s}{4\pi} \left(\begin{array}{ccc|c} 0 & & & 0 \\ & \ddots & & \vdots \\ & & 0 & 0 \\ \hline -16 & \dots & -16 & \beta_0 \end{array} \right) + \dots, \\ \hat{Z}^{(2)} &= \mathbb{1} + \frac{1}{\epsilon} \frac{\alpha_s}{4\pi} \left(\begin{array}{ccc|c} -\frac{32}{9} & & & \frac{2}{3} \\ & \ddots & & \vdots \\ & & -\frac{32}{9} & \frac{2}{3} \\ \hline \frac{32}{9} & \dots & \frac{32}{9} & -\frac{2n_f}{3} \end{array} \right) + \dots, \quad (5.8)\end{aligned}$$

where n_f is the number of light quark flavors, $\beta_0 = 11 - 2n_f/3$ and the ellipses denote terms higher order in α_s . The all-orders expression for $\hat{Z}^{(0)}$ is specified in terms of functions

$$\tilde{\beta} = \beta/g, \quad \beta = \frac{dg}{d \log \mu}, \quad \gamma_m = \frac{d \log m_q}{d \log \mu}. \quad (5.9)$$

For completeness, we give explicit expressions for β and γ_m in Appendix C. We presently require $\hat{Z}^{(0)}$ and $\hat{Z}^{(2)}$ through $\mathcal{O}(\alpha_s)$ in order to derive the relation between bare and renormalized coefficients at first non-vanishing order. From the definition in Eq. (5.4), the renormalized coefficients for the scalar operators are

$$\begin{aligned}c_q^{(0)}(\mu) &= \sum_{q'} Z_{q'q}^{(0)}(\mu) c_{q'}^{(0)\text{bare}} + Z_{gq}^{(0)}(\mu) c_g^{(0)\text{bare}} = c_q^{(0)\text{bare}} + \mathcal{O}(\alpha_s^2), \\ c_g^{(0)}(\mu) &= \sum_q Z_{qg}^{(0)}(\mu) c_q^{(0)\text{bare}} + Z_{gg}^{(0)}(\mu) c_g^{(0)\text{bare}} = c_g^{(0)\text{bare}} + \mathcal{O}(\alpha_s^2), \quad (5.10)\end{aligned}$$

while for the tensor operators, we find

$$\begin{aligned}c_q^{(2)}(\mu) &= \sum_{q'} Z_{q'q}^{(2)}(\mu) c_{q'}^{(2)\text{bare}} + Z_{gq}^{(2)}(\mu) c_g^{(2)\text{bare}} = c_q^{(2)\text{bare}} + \mathcal{O}(\alpha_s), \\ c_g^{(2)}(\mu) &= \sum_q Z_{qg}^{(2)}(\mu) c_q^{(2)\text{bare}} + Z_{gg}^{(2)}(\mu) c_g^{(2)\text{bare}} = \sum_q \frac{1}{\epsilon} \frac{\alpha_s}{6\pi} c_q^{(2)\text{bare}} \\ &\quad + c_g^{(2)\text{bare}} + \mathcal{O}(\alpha_s^2). \quad (5.11)\end{aligned}$$

In particular, a nontrivial subtraction, requiring the $\mathcal{O}(\epsilon)$ part of the coefficients $c_q^{(2)\text{bare}}$, is necessary to obtain the renormalized coefficient $c_g^{(2)}(\mu)$.

Finally, let us consider the pseudoscalar and pseudotensor operators. These arise at subleading order in many WIMP models, but for completeness and potential future applications we summarize their leading renormalization properties here. The leading operator renormalization factors are

$$\begin{aligned}\hat{Z}_5^{(0)} &= \mathbb{1} + \frac{\alpha_s}{4\pi} \left[\frac{1}{\epsilon} \left(\begin{array}{ccc|c} 0 & & & 0 \\ & \ddots & & \vdots \\ & & 0 & 0 \\ \hline -16 & \cdots & -16 & \beta_0 \end{array} \right) + \left(\begin{array}{ccc|c} \frac{16}{3} & & & 0 \\ & \ddots & & \vdots \\ & & \frac{16}{3} & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array} \right) \right], \\ \hat{Z}_5^{(2)} &= \mathbb{1} + \frac{1}{\epsilon} \frac{\alpha_s}{4\pi} \left(\begin{array}{ccc|c} -\frac{32}{9} & & & \frac{2}{3} \\ & \ddots & & \vdots \\ & & -\frac{32}{9} & \frac{2}{3} \\ \hline \frac{32}{9} & \cdots & \frac{32}{9} & -\frac{2n_f}{3} \end{array} \right). \quad (5.12)\end{aligned}$$

Note that an extra finite renormalization beyond $\overline{\text{MS}}$ is required to retain one-loop exactness of the axial anomaly.

5.1.2 Renormalized Matching Coefficients for Pure States

In Sect. (4.1), we illustrated the framework for matching effective theories at the weak scale with explicit $\mathcal{O}(\alpha_2^1)$ matching for leading operators in the case of electroweak singlet DM. The bare coefficients for the scalar quark and gluon operators obtained in those cases are trivially related to renormalized coefficients.

To illustrate operator renormalization, let us turn to the case of electroweak-charged DM where the required amplitudes for matching are $\mathcal{O}(\alpha_2^2)$ one- and two-loop diagrams. The relevant bare effective lagrangian at the weak-scale is given by Eq. (3.69). In the heavy WIMP limit, the bare matching coefficients for pure-states are given in Eq. (4.82). We have retained the $\mathcal{O}(\epsilon)$ part of coefficients $c_q^{(2)}$ in Eq. (4.82), as required by the nontrivial subtraction in Eq. (5.11). Employing Eqs. (5.10) and (5.11), we find the (finite) renormalized coefficients,

$$\begin{aligned}c_U^{(0)}(\mu) &= \frac{\pi\alpha_2^2}{m_W^3} \left\{ -\frac{1}{2x_h^2} \left[C_W + \frac{C_Z}{c_W^3} \right] + \frac{C_Z}{8c_W} [c_V^{(U)2} - c_A^{(U)2}] \right\}, \\ c_D^{(0)}(\mu) &= \frac{\pi\alpha_2^2}{m_W^3} \left\{ -\frac{1}{2x_h^2} \left[C_W + \frac{C_Z}{c_W^3} \right] + \frac{C_Z}{8c_W} [c_V^{(D)2} - c_A^{(D)2}] - \delta_{Db} C_W \frac{x_t}{8(x_t + 1)^3} \right\}, \\ c_s^{(0)}(\mu) &= \frac{\pi\alpha_2^2}{m_W^3} \frac{\alpha_s(\mu)}{4\pi} \left\{ \frac{1}{2} \left[\frac{1}{3x_h^2} \left[C_W + \frac{C_Z}{c_W^3} \right] + C_W \left[\frac{1}{3} + \frac{1}{6(x_t + 1)^2} \right] \right] \right\}\end{aligned}$$

$$\begin{aligned}
& + \frac{\mathcal{C}_Z}{64c_W} \left[4[c_V^{(D)2} + c_A^{(D)2}] + [c_V^{(U)2} + c_A^{(U)2}] \left[\frac{8}{3} + \frac{32y_t^6(8y_t^2 - 7)}{(4y_t^2 - 1)^{7/2}} \right. \right. \\
& \left. \left. \arctan(\sqrt{4y_t^2 - 1}) - \pi y_t + \frac{4(48y_t^6 - 2y_t^4 + 9y_t^2 - 1)}{3(4y_t^2 - 1)^3} \right] \right. \\
& \left. + [c_V^{(U)2} - c_A^{(U)2}] \left[3\pi y_t - \frac{4(144y_t^6 - 70y_t^4 + 9y_t^2 - 2)}{3(4y_t^2 - 1)^3} \right. \right. \\
& \left. \left. - \frac{32y_t^4(24y_t^4 - 21y_t^2 + 5)}{(4y_t^2 - 1)^{7/2}} \arctan(\sqrt{4y_t^2 - 1}) \right] \right] \Bigg\}, \\
c_U^{(2)}(\mu) &= \frac{\pi\alpha_2^2}{m_W^3} \left\{ \frac{\mathcal{C}_W}{3} + \frac{\mathcal{C}_Z}{6c_W} [c_V^{(U)2} + c_A^{(U)2}] \right\}, \\
c_D^{(2)}(\mu) &= \frac{\pi\alpha_2^2}{m_W^3} \left\{ \frac{\mathcal{C}_W}{3} + \frac{\mathcal{C}_Z}{6c_W} [c_V^{(D)2} + c_A^{(D)2}] + \delta_{Db} \frac{\mathcal{C}_W}{2} \left[\frac{(3x_t + 2)}{3(x_t + 1)^3} - \frac{2}{3} \right] \right\}, \\
c_s^{(2)}(\mu) &= \frac{\pi\alpha_2^2}{m_W^3} \frac{\alpha_s(\mu)}{4\pi} \left\{ \mathcal{C}_W \left[- \frac{2(8x_t^3 + 24x_t^2 + 27x_t + 10)}{9(x_t + 1)^3} \log \frac{\mu}{m_W} \right. \right. \\
& - \frac{4x_t(7x_t^2 - 3)}{9(x_t^2 - 1)^3} \log 2 - \frac{2(12x_t^5 - 36x_t^4 + 36x_t^3 - 12x_t^2 + 3x_t - 2)}{9(x_t - 1)^3} \log x_t \\
& + \frac{4(6x_t^8 - 18x_t^6 + 21x_t^4 - 3x_t^2 - 2)}{9(x_t^2 - 1)^3} \log(x_t + 1) \\
& \left. - \frac{48x_t^6 + 60x_t^5 - 68x_t^4 - 107x_t^3 - 52x_t^2 + 49x_t + 54}{18(x_t^2 - 1)^2(x_t + 1)} \right] \\
& + \frac{\mathcal{C}_Z}{c_W} \left[\left[2[c_V^{(U)2} + c_A^{(U)2}] + 3[c_V^{(D)2} + c_A^{(D)2}] \right] \left[-\frac{1}{4} - \frac{2}{9} \log \frac{\mu}{m_Z} \right] \right. \\
& + [c_V^{(U)2} + c_A^{(U)2}] \left[-\frac{\pi y_t}{48} + \frac{2(24y_t^8 - 21y_t^6 - 4y_t^4 + 5y_t^2 - 1)}{9(4y_t^2 - 1)^{7/2}} \right. \\
& \left. \left. \arctan(\sqrt{4y_t^2 - 1}) + \frac{48y_t^6 + 62y_t^4 - 47y_t^2 + 9}{36(4y_t^2 - 1)^3} \right] \right. \\
& + [c_V^{(U)2} - c_A^{(U)2}] \left[-\frac{13\pi y_t}{48} + \frac{2y_t^2(104y_t^6 - 91y_t^4 + 35y_t^2 - 5)}{3(4y_t^2 - 1)^{7/2}} \right. \\
& \left. \left. \arctan(\sqrt{4y_t^2 - 1}) + \frac{y_t^2(624y_t^4 - 538y_t^2 + 103)}{36(4y_t^2 - 1)^3} \right] \right] \Bigg\}. \tag{5.13}
\end{aligned}$$

In the next chapter, we study the renormalization group evolution of these coefficients down to low-energies, including matching effects at heavy quark thresholds.

5.2 Renormalization Group Evolution

After matching onto the QCD theory with $n_f = 5$ flavors and determining renormalized coefficients, two ingredients remain for mapping onto a theory valid at lower energy scales: renormalization group evolution of coefficients between heavy quark thresholds, and matching of $(n_f + 1)$ -flavor and n_f -flavor QCD at the bottom ($\mu_b \sim m_b$) and charm ($\mu_c \sim m_c$) heavy quark thresholds. We perform the necessary analysis for the running of coefficients in this section, and tackle threshold matching in the next.

From the relations between bare and renormalized operators and coefficients in Eq. (5.4), we obtain the scale evolution of renormalized operators and coefficients,

$$\frac{d}{d \log \mu} O_i = -\gamma_{ij} O_j, \quad \frac{d}{d \log \mu} c_i = \gamma_{ji} c_j, \quad \gamma_{ij} \equiv Z_{ik}^{-1} \frac{d}{d \log \mu} Z_{kj}, \quad (5.14)$$

where the superscript “ren” on renormalized quantities has been dropped, and we have defined the anomalous dimension matrix γ_{ij} . In the $\overline{\text{MS}}$ scheme the anomalous dimension reduces to

$$\gamma_{ij} = -g \frac{\partial}{\partial g} Z_{(1)ij}, \quad (5.15)$$

where $Z_{(1)ij}$ is the coefficient of $1/\epsilon$ in the expansion

$$Z_{ij} = \delta_{ij} + \sum_{n=1}^{\infty} \frac{Z_{(n)ij}}{\epsilon^n}. \quad (5.16)$$

Let us now proceed to solve the evolution of coefficients from scale μ_h down to scale μ , for the vector, axial-vector, scalar and tensor operators in Eqs. (5.1)–(5.3).

For the vector current, the renormalization constant in Eq. (5.5) leads to a vanishing anomalous dimension, and hence a trivial scale evolution:

$$c_V^{(q)}(\mu) = c_V^{(q)}(\mu_h). \quad (5.17)$$

For the axial-vector current non-trivial renormalization begins at two loop order. Recall that the renormalization constant in Eq. (5.7) includes a finite contribution beyond $\overline{\text{MS}}$. From the definition in Eq. (5.14), we find a flavor diagonal anomalous dimension with leading behavior

$$\gamma_A = \left(\frac{\alpha_s}{4\pi} \right)^2 \gamma_{A,2} + \mathcal{O}(\alpha_s^3), \quad \gamma_{A,2} = 12 C_F n_f. \quad (5.18)$$

The corresponding solution for the coefficient running is

$$c_A^{(q)}(\mu) = \exp \left\{ -\frac{\gamma_{A,2}}{8\pi\beta_0} [\alpha_s(\mu) - \alpha_s(\mu_h)] + \mathcal{O}(\alpha_s^2) \right\} c_A^{(q)}(\mu_h). \quad (5.19)$$

The phenomenological impact of this correction is moderate ($\sim 10\%$ corrections from running between $\mu_h \sim m_W$ and $\mu \sim 1$ GeV), but could become relevant if a WIMP signal is detected.

For the scalar and tensor operators relevant for spin-independent WIMP-nucleon scattering, given in Eq. (5.2), the anomalous dimension matrices follow from the renormalization constants in Eq. (5.8) and the relation in Eq. (5.15). In the basis $(u, d, s, \dots, g)$, we have

$$\begin{aligned} \hat{\gamma}^{(0)} &= \left(\begin{array}{ccc|c} 0 & & & 0 \\ & \ddots & & \vdots \\ & & 0 & 0 \\ \hline -2\gamma'_m & \dots & -2\gamma'_m & \tilde{\beta}' \end{array} \right) = \frac{\alpha_s}{4\pi} \left(\begin{array}{ccc|c} 0 & & & 0 \\ & \ddots & & \vdots \\ & & 0 & 0 \\ \hline 32 & \dots & 32 & -2\beta_0 \end{array} \right) + \dots, \\ \hat{\gamma}^{(2)} &= \frac{\alpha_s}{4\pi} \left(\begin{array}{ccc|c} \frac{64}{9} & & & -\frac{4}{3} \\ & \ddots & & \vdots \\ & & \frac{64}{9} & -\frac{4}{3} \\ \hline -\frac{64}{9} & \dots & -\frac{64}{9} & \frac{4n_f}{3} \end{array} \right) + \dots, \end{aligned} \quad (5.20)$$

where the renormalization of the scalar operators is determined by derivatives of the QCD beta function β , and the quark mass anomalous dimension γ_m as

$$\tilde{\beta}' \equiv g \frac{\partial}{\partial g} \tilde{\beta}, \quad \gamma'_m = g \frac{\partial}{\partial g} \gamma_m, \quad (5.21)$$

with $\tilde{\beta} \equiv \beta/g$ and explicit expressions for β, γ_m collected in Appendix C.

Using the all-orders expression for $\hat{\gamma}^{(0)}$, we find the following solution to the coefficient evolution for the scalar case,

$$c_q^{(0)}(\mu) = c_q^{(0)}(\mu_h) + \frac{2[\gamma_m(\mu_h) - \gamma_m(\mu)]}{\tilde{\beta}(\mu_h)} c_g^{(0)}(\mu_h), \quad c_g^{(0)}(\mu) = \frac{\tilde{\beta}(\mu)}{\tilde{\beta}(\mu_h)} c_g^{(0)}(\mu_h). \quad (5.22)$$

In the tensor case, the leading behavior of the anomalous dimension $\hat{\gamma}^{(2)}$ in Eq. (5.20) yields the following solution at leading logarithmic order,

$$\begin{aligned}
c_q(\mu) &= \left\{ \frac{3}{16 + 3n_f} \left[\frac{16}{3n_f} r(n_f) + 1 \right] + \frac{n_f - 1}{n_f} r(0) \right\} c_q(\mu_h) \\
&\quad + \left\{ \frac{3}{16 + 3n_f} \left[\frac{16}{3n_f} r(n_f) + 1 \right] - \frac{1}{n_f} r(0) \right\} \sum_{q' \neq q} c_{q'}(\mu_h) \\
&\quad - \frac{16}{16 + 3n_f} [r(n_f) - 1] c_g(\mu_h), \\
c_g(\mu) &= \frac{3}{16 + 3n_f} \left[n_f r(n_f) + \frac{16}{3} \right] c_g(\mu_h) + \frac{3}{16 + 3n_f} [1 - r(n_f)] \sum_q c_q(\mu_h),
\end{aligned} \tag{5.23}$$

where the factor $r(n_f)$ is given by

$$r(n) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_h)} \right)^{-\frac{2}{3\beta_0} \left(\frac{16}{3} + n \right)}. \tag{5.24}$$

For most phenomenological applications we may simply evaluate the tensor matrix elements in terms of known parton distribution functions (PDFs) at the weak scale $\mu \sim m_W$. This avoids the need for renormalization group analysis (apart from matching to a convenient scale to evaluate matrix elements) and heavy-quark threshold matching conditions. Nonetheless, we have included the above results for completeness and for future analyses which may require an evaluation of tensor matrix elements at low scales, e.g., in considering multi-nucleon contributions to matrix elements.

Our main focus in the phenomenological analysis will be on the spin-independent cross section for heavy self-conjugate dark matter, for which the scalar and tensor operators of dimension four are dominant. We will see that subleading orders in the coefficient running in Eq. (5.22) can shift physical cross sections by large factors. It is straightforward to include higher-order corrections in the solution to the tensor coefficient running [88, 101], and to extend the analysis to treat the pseudoscalar and pseudotensor operators of Eq. (5.3) [41].

5.3 Threshold Matching and Low Energy Coefficients

Nontrivial matching conditions relate theories with different numbers of active quark flavors. Due to the lightness of the charm mass, and correspondingly poorly convergent $\alpha_s(m_c)$ expansion, WIMP-nucleon cross sections can depend sensitively on higher-order corrections. The vector and axial-vector operators have trivial matching conditions, while, as previously stated, the matrix element of tensor operators may be evaluated at the weak scale. Pseudoscalar and pseudotensor operators have velocity-suppressed matrix elements in physical WIMP-nucleon processes

at small relative velocity. We thus focus on threshold matching for the scalar operators relevant for spin-independent WIMP-nucleon scattering. We summarize the knowledge of these threshold corrections, and present final expressions for low energy coefficients, accounting for both renormalization group evolution and threshold matching.

5.3.1 Heavy Quark Threshold Matching Conditions

After evolving to the scale $\mu_b \sim m_b$, we integrate out the bottom quark, and enforce the following matching condition for physical matrix elements, up to $1/m_b$ power corrections of the operators in $n_f = 5$ and $n_f = 4$ theories,

$$\begin{aligned} c_g^{(S)}(\mu) O_g^{(S)}(\mu) + \sum_{q=u,d,s,c,b} c_q^{(S)}(\mu) O_q^{(S)}(\mu) \\ = c_g^{(S)'}(\mu) O_g^{(S)'}(\mu) + \sum_{q=u,d,s,c} c_q^{(S)'}(\mu) O_q^{(S)'}(\mu). \end{aligned} \quad (5.25)$$

The primed and un-primed coefficients in the above equation are in the $n_f = 4$ and $n_f = 5$ theories, respectively. For the scalar operators the leading contributions are [69, 91]

$$\begin{aligned} c_g^{(0)'}(\mu_b) &= \left[1 + \frac{\alpha_s(\mu_b)}{4\pi} \left[-\frac{4}{3} \log \frac{\mu}{m_b} \right] + \dots \right] c_g^{(0)}(\mu_b) \\ &\quad + \frac{\alpha_s(\mu_b)}{4\pi} \left[-\frac{1}{3} \right] \left[1 + \frac{\alpha_s(\mu_b)}{4\pi} \left[11 - \frac{4}{3} \log \frac{\mu}{m_b} \right] + \dots \right] c_b^{(0)}(\mu_b), \\ c_q^{(0)'}(\mu_b) &= c_q^{(0)}(\mu_b) + \dots, \end{aligned} \quad (5.26)$$

where the ellipses denote higher-order $\alpha_s(\mu_b)$ corrections. The contributions from $c_b^{(0)}(\mu_b)$, both to $c_g^{(0)'}(\mu_b)$ and $c_q^{(0)'}(\mu_b)$, are available through $\mathcal{O}(\alpha_s^3)$ in [25], yielding corrections to $c_q^{(0)'}(\mu_b)$ at $\mathcal{O}(\alpha_s^2)$. We are unable to find results for higher-order contributions from $c_g^{(0)}(\mu_b)$ and $c_q^{(0)}(\mu_b)$ in the literature. The impact of the available higher-order contributions are investigated in the next chapter. For completeness we also include here the leading threshold matching condition for tensor operators,

$$\begin{aligned} c_g^{(2)'}(\mu_b) &= c_g^{(2)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} \left[\frac{4}{3} \log \frac{\mu_b}{m_b} \right] c_b^{(2)}(\mu_b), \\ c_q^{(2)'}(\mu_b) &= c_q^{(2)}(\mu_b). \end{aligned} \quad (5.27)$$

Similar conditions hold at the scale $\mu_c \sim m_c$ after integrating out the charm quark.

5.3.2 Low Energy Coefficients

Let us put together the ingredients of renormalization group evolution and threshold matching to complete the mapping of coefficients from a high scale to a low scale. The operator coefficients $c_i^{(S)}(\mu)$, for the n_f -flavor theory at scale μ , may be collected in the column $(n_f + 1)$ -vector $\vec{c}_{(n_f)}^{(S)}(\mu) = \{c_u^{(S)}(\mu), c_d^{(S)}(\mu), c_s^{(S)}(\mu), \dots, c_g^{(S)}(\mu)\}$. The coefficient running from a high scale, μ_h , to a low scale, μ , with n_f active quark flavors, and the matching at scale μ between the n_f -flavor and $(n_f - 1)$ -flavor theories may be represented as matrices $R_{(n_f)}^{(S)}(\mu, \mu_h)$ and $M_{(n_f-1, n_f)}^{(S)}(\mu)$, respectively, as

$$\vec{c}_{(n_f)}^{(S)}(\mu) = R_{(n_f)}^{(S)}(\mu, \mu_h) \vec{c}_{(n_f)}^{(S)}(\mu_h), \quad \vec{c}_{(n_f-1)}^{(S)}(\mu) = M_{(n_f-1, n_f)}^{(S)}(\mu) \vec{c}_{(n_f)}^{(S)}(\mu). \quad (5.28)$$

Thus, composing the two effects, the mapping of coefficients in the $n_f = 5$ theory at the high scale μ_t onto coefficients in the $n_f = 3$ theory at the low scale μ_0 is given by

$$\vec{c}_{(3)}^{(S)}(\mu_0) = R_{(3)}^{(S)}(\mu_0, \mu_c) M_{(3,4)}^{(S)}(\mu_c) R_{(4)}^{(S)}(\mu_c, \mu_b) M_{(4,5)}^{(S)}(\mu_b) R_{(5)}^{(S)}(\mu_b, \mu_t) \vec{c}_{(5)}^{(S)}(\mu_t). \quad (5.29)$$

Collecting the results in Eqs. (5.22) and (5.26) for the scalar operators, the $(n_f + 1) \times (n_f + 1)$ matrix $R_{(n_f)}^{(0)}(\mu, \mu_h)$ is given by

$$R_{(n_f)}^{(0)}(\mu, \mu_h) = \left(\begin{array}{c|c} 1 & 2[\gamma_m(\mu) - \gamma_m(\mu_h)]/\tilde{\beta}(\mu_h) \\ \vdots & \vdots \\ 1 & 2[\gamma_m(\mu) - \gamma_m(\mu_h)]/\tilde{\beta}(\mu_h) \\ \hline 0 \cdots 0 & \tilde{\beta}(\mu)/\tilde{\beta}(\mu_h) \end{array} \right), \quad (5.30)$$

while the $n_f \times (n_f + 1)$ matrix $M_{(n_f-1, n_f)}^{(0)}(\mu_q)$ is given by

$$M_{(n_f-1, n_f)}^{(0)}(\mu_q) = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ \hline 0 \cdots 0 & -\frac{\alpha_s(\mu_q)}{12\pi} \left[1 + \frac{\alpha_s(\mu_q)}{4\pi} \left[11 - \frac{4}{3} \log \frac{\mu_q}{m_q} \right] \right] & 1 - \frac{\alpha_s(\mu_q)}{3\pi} \log \frac{\mu_q}{m_q} \end{array} \right), \quad (5.31)$$

with m_q the mass of the heavy quark associated with the threshold at μ_q . We may similarly collect the results in Eqs. (5.23) and (5.27) for the tensor operators. The $(n_f + 1) \times (n_f + 1)$ matrix $R_{(n_f)}^{(2)}(\mu, \mu_h)$ is given by

$$R_{(n_f)}^{(2)}(\mu, \mu_h) = \left(\begin{array}{c|c} r(0)\mathbb{1} + \frac{1}{n_f} \left[\frac{16r(n_f)+3n_f}{16+3n_f} - r(0) \right] \mathbb{J} & \begin{array}{c} \frac{16[1-r(n_f)]}{16+3n_f} \\ \vdots \\ \frac{16[1-r(n_f)]}{16+3n_f} \end{array} \\ \hline \frac{3[1-r(n_f)]}{16+3n_f} & \dots \frac{3[1-r(n_f)]}{16+3n_f} \frac{16+3n_f r(n_f)}{16+3n_f} \end{array} \right), \quad (5.32)$$

where the $n_f \times n_f$ matrices $\mathbb{1}$ and $\mathbb{J}$ are respectively the identity and the matrix with all elements 1, and we have used the factor

$$r(n) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_h)} \right)^{-\frac{2}{3\beta_0} \left(\frac{16}{3} + n \right)}. \quad (5.33)$$

The $n_f \times (n_f + 1)$ matrix $M_{(n_f-1, n_f)}^{(2)}(\mu_q)$ is given by

$$M_{(n_f-1, n_f)}^{(2)}(\mu_q) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ & \vdots & \vdots \\ & 1 & 0 \\ \hline 0 \dots 0 & \frac{\alpha_s(\mu_q)}{3\pi} \log \frac{\mu}{m_q} & 1 \end{array} \right). \quad (5.34)$$

In the next chapter, we investigate the numerical impact of these corrections.

5.4 Hadronic Matrix Elements

Having determined the effective theory in terms of quark and gluon degrees of freedom in $n_f = 3$ (or $n_f = 4$) flavor QCD, we may evaluate the resulting nucleon matrix elements. This analysis is sufficient to determine low-velocity WIMP-nucleon scattering cross sections. We focus here on ingredients necessary for scalar and tensor operators relevant to spin-independent scattering; the extension for vector and axial-vector currents, and for pseudoscalar and pseudotensor operators is straightforward. The operators are to be evaluated at a renormalization scale $\mu \sim 1\text{--}2$ GeV appropriate to $n_f = 3$ or $n_f = 4$ flavor QCD.

5.4.1 Scalar Matrix Elements

For the dimension four scalar and tensor operators, we restrict attention to forward matrix elements, neglecting $1/m_N$ suppressed corrections. Let us define

$$\begin{aligned} \langle N(k) | O_q^{(0)} | N(k) \rangle &\equiv m_N f_{q,N}^{(0)}, \\ \frac{-9\alpha_s(\mu)}{8\pi} \langle N(k) | O_g^{(0)}(\mu) | N(k) \rangle &\equiv m_N f_{G,N}^{(0)}(\mu), \end{aligned} \quad (5.35)$$

where the appearance of the numerical factor involving $\alpha_s(\mu)$ is purely conventional. The operator matrix elements are not independent, being linked by the relation

$$\begin{aligned} m_N \bar{u}(k) u(k) &= \langle N(k) | \theta_\mu^\mu | N(k) \rangle \\ &= (1 + \gamma_m) \sum_q \langle N(k) | m_q \bar{q} q | N(k) \rangle + \frac{\beta}{2g} \langle N(k) | (G_{\mu\nu}^a)^2 | N(k) \rangle, \end{aligned} \quad (5.36)$$

where $N = p$ or n , m_N is the nucleon mass, and we have ignored power corrections in the above equation. The QCD beta function β and quark mass anomalous dimension γ_m are given in Appendix C. We may solve

$$f_{G,N}^{(0)}(\mu) = -\frac{9\alpha_s}{4\pi} \tilde{\beta}(\mu)^{-1} \left[1 - [1 + \gamma_m(\mu)] \sum_{q=u,d,s} f_{q,N}^{(0)} \right] = 1 - \sum_{q=u,d,s} f_{q,N}^{(0)} + \dots, \quad (5.37)$$

where the second equality is obtained by neglecting γ_m and $\mathcal{O}(\alpha_s^2)$ contributions to β . We will see that corrections to the leading relation are important for the robust cross section predictions. For the quark matrix elements, we define the RG invariant combinations,

$$\begin{aligned} \Sigma_{\pi N} &= \frac{m_u + m_d}{2} \langle p | (\bar{u}u + \bar{d}d) | p \rangle, \\ \Sigma_- &= (m_d - m_u) \langle p | (\bar{u}u - \bar{d}d) | p \rangle, \\ \Sigma_s &= m_s \langle p | \bar{s}s | p \rangle. \end{aligned} \quad (5.38)$$

We use updated lattice results for the pion-nucleon sigma term, $\Sigma_{\pi N} = 39_{-8}^{+18}$ MeV [35], and the strange scalar matrix element, $\Sigma_s = 40 \pm 20$ MeV [74]. For Σ_s we assume a 50 % uncertainty (cf. the estimated 25 % in [74]). We use $\Sigma_- = 2(2)$ MeV [49], and the quark mass ratios adopted from PDG values (symmetrizing errors)

$$\frac{m_u}{m_d} = 0.49 \pm 0.13, \quad \frac{m_s}{m_d} = 19.5 \pm 2.5. \quad (5.39)$$

The independent quark matrix elements for the proton and neutron become

$$\begin{aligned} f_{u,p}^{(0)} &= 0.018(8), \quad f_{d,p}^{(0)} = 0.030(15), \quad f_{s,p}^{(0)} = 0.043(21), \\ f_{u,n}^{(0)} &= 0.015(7), \quad f_{d,n}^{(0)} = 0.034(14), \quad f_{s,n}^{(0)} = 0.043(21). \end{aligned} \quad (5.40)$$

For the neutron, we neglect higher order corrections proportional to $m_u - m_d$ and α . In both cases, the gluon matrix element $f_{G,N}^{(0)}$ is obtained from the quark matrix elements via (5.36). For example, in the case of the proton, we find the gluon matrix elements at scale $\mu = 1.2 \text{ GeV}$,

$$\begin{aligned} f_{G,p}^{(0)\text{LO}} &= 0.910(45), \\ f_{G,p}^{(0)\text{NLO}} &= 0.714(46)(24), \\ f_{G,p}^{(0)\text{NNLO}} &= 0.661(48)(37), \\ f_{G,p}^{(0)\text{NNNLO}} &= 0.633(48)(48), \end{aligned} \quad (5.41)$$

where LO labels the leading $\alpha_s(\mu)$ result, corresponding to the expression in the rightmost side of Eq. (5.37), and N^kLO are results including corrections from higher-order α_s terms in β and γ_m . The first uncertainty is from the quark matrix elements; cf. Eq. (5.40). For illustration, we also show the sensitivity of the higher order results to scale variation using the range $1 < \mu < 1.4$. This scale dependence is only from the operators, and will be cancelled by scale dependence in the coefficients, coming renormalization group evolution, as we will see in the next chapter.

A determination of matrix elements in $n_f = 4$ QCD would avoid uncertainties associated with the low scale $\mu \sim 1 \text{ GeV}$ and the charm scale $\sim m_c$, such as power corrections and perturbative corrections at the charm threshold. In the following chapter we consider such an evaluation of the spin-independent cross section, assuming the same values for the light quark matrix elements and obtaining the gluon matrix element from Eq. (5.36) with an active charm quark. Recent lattice measurements of the charm matrix element [47, 52] span a large range of values, $0.031 < f_{c,N}^{(0)} < 0.133$, consistent with estimates from perturbative QCD [74]. Improved measurement of the charm matrix element is a target for maturing lattice studies, with important implications for WIMP-nucleon scattering cross sections.

5.4.2 Tensor Matrix Elements

The forward matrix elements of the tensor operators are parameterized as

$$\begin{aligned} \langle N(k) | O_q^{(2)\mu\nu}(\mu) | N(k) \rangle &\equiv \frac{1}{m_N} \left(k^\mu k^\nu - \frac{g^{\mu\nu}}{4} m_N^2 \right) f_{q,N}^{(2)}(\mu), \\ \langle N(k) | O_g^{(2)\mu\nu}(\mu) | N(k) \rangle &\equiv \frac{1}{m_N} \left(k^\mu k^\nu - \frac{g^{\mu\nu}}{4} m_N^2 \right) f_{G,N}^{(2)}(\mu). \end{aligned} \quad (5.42)$$

Table 5.1 Operator coefficients derived from MSTW PDF analysis at different values of μ

μ	$f_{u,p}^{(2)}(\mu)$	$f_{d,p}^{(2)}(\mu)$	$f_{s,p}^{(2)}(\mu)$	$f_{G,p}^{(2)}(\mu)$
1.0	0.404(6)	0.217(4)	0.024(3)	0.36(1)
1.2	0.383(6)	0.208(4)	0.027(2)	0.38(1)
1.4	0.370(5)	0.202(4)	0.030(2)	0.40(1)

The matrix elements of spin-two operators define moments of parton distribution functions. Under this identification,

$$f_{q,p}^{(2)}(\mu) = \int_0^1 dx x [q(x, \mu) + \bar{q}(x, \mu)] , \quad (5.43)$$

where $q(x, \mu)$ is the parton distribution function evaluated at scale μ . Neglecting power corrections, the sum of spin two operators is identified as the traceless part of the QCD energy momentum tensor, hence independent of scale,

$$\sum_{q=u,d,s} f_{q,p}^{(2)}(\mu) + f_{G,p}^{(2)}(\mu) = 1 . \quad (5.44)$$

Similar to the spin-zero case, we use approximate isospin symmetry to set

$$f_{u,n}^{(2)} = f_{d,p}^{(2)} , \quad f_{d,n}^{(2)} = f_{u,p}^{(2)} , \quad f_{s,n}^{(2)} = f_{s,p}^{(2)} . \quad (5.45)$$

Here the operators are scale dependent and we must choose a renormalization scale to evaluate them. Table 5.1 lists coefficient values for renormalization scales $\mu = 1 \text{ GeV}$, $\mu = 1.2 \text{ GeV}$ and $\mu = 1.4 \text{ GeV}$ using the parameterization and analysis of Martin et al. [87].

Chapter 6

Heavy WIMP-Nucleon Scattering Cross Sections

The formalism collected in the previous chapters describe a complete set of tools for making robust cross section predictions for heavy WIMPs scattering with nucleons. The formalism for heavy particle lagrangians in Chap. 2 is used in Chap. 3 to construct effective theories at the weak scale describing electroweak symmetric interactions of the WIMP with SM Higgs and gauge bosons. In Chap. 4 we perform the matching between the electroweak symmetric theory and the low-energy theory of quarks and gluons, determining weak-scale coefficients in terms of a few WIMP parameters such as its electroweak charges. The renormalization of the scalar and tensor coefficients, and their mapping to lower energies using renormalization group and effective theory methods are detailed in the previous chapter. In this chapter, we assemble these pieces into a framework for careful analysis of cross section predictions and their perturbative and input uncertainties.

The SM exhibits a surprising transparency of nucleons to WIMP scattering, due to a cancellation between spin-0 and spin-2 amplitude contributions [59, 65]. Robust cross section predictions demand a complete treatment of both perturbative and hadronic uncertainties, including resummation of large logarithms in perturbative QCD (pQCD). This chapter presents a careful study of the various contributions and their uncertainties. We trace the evolution of coefficients from the high to low scale, paying close attention to the effects of renormalization, heavy quark thresholds and to the sizes of matrix elements. We identify the dominant uncertainties, and determine strategies for evaluation of perturbative contributions and for scrutinizing the final cross sections. We also identify Standard Model inputs whose improved precision would impact our knowledge of WIMP-nucleon cross sections. Robust predictions for the cross sections of the pure triplet, pure doublet, singlet-doublet admixture, and triplet-doublet admixture are given.

Although we find that cancellations are generic, their severity depends on SM parameters and on properties of DM such as its electroweak quantum numbers. The presence of additional low-lying states could also have impact, and we investigate such scenarios. For example, an extra Higgs boson modifies the spin-0

amplitude, and could potentially weaken the cancellation between spin-0 and spin-2 amplitudes. We also consider WIMPs in larger $SU(2)_W$ representations.

This chapter is organized as follows. In Sect. 6.1 we trace the numerical evolution of scalar and tensor coefficients from high to low scales, illustrating the impact of renormalization group running, heavy quark threshold contributions, and nucleon matrix elements. In Sect. 6.2, we survey the uncertainties from perturbative corrections and hadronic inputs. Section 6.3 presents cross section predictions for spin-independent low-velocity scattering on a proton. The sensitivity of cross sections and their fractional uncertainty are illustrated, and several cross checks are performed.

6.1 Cross Section Assembly Line

We may now put together the ingredients for mapping weak scale parameters onto coefficients of operators in $n_f = 3$ or $n_f = 4$ flavor QCD. Let us illustrate the numerical coefficients at different stages of the low-energy effective theory. For studying the effects of renormalization group running, threshold matching and the sizes of the nucleon matrix elements, we focus on default “central” values. Analysis of perturbative and hadronic uncertainties is necessary for robust determination of cross section predictions, and will be discussed in the following section.

For definiteness, let us illustrate with the following evaluation. The renormalization group running and heavy quark matching for spin-2 operators are evaluated at LO. The RG running from μ_c to μ_0 from (5.20) is evaluated with NNNLO corrections, including contributions to β/g through $\mathcal{O}(\alpha_s^4)$ and γ_m through $\mathcal{O}(\alpha_s^4)$. Accordingly, the spin-0 gluonic matrix element from (5.37) is also evaluated at NNNLO, including contributions to β/g through $\mathcal{O}(\alpha_s^4)$ and γ_m through $\mathcal{O}(\alpha_s^3)$. We perform the RG running and heavy quark matching from μ_t to μ_c at NLO. The motivation for these choices will be explained in the following section. Tables 6.1 and 6.2 show the results for the scalar and tensor coefficients of the pure triplet case,

Table 6.1 Scalar coefficients at each stage of the effective theory, labelled by the scale μ and the number of active quark flavors n_f , with overall factors $\pi\alpha_2^2/m_W^3$ extracted

	u	d	s	c	b	g
$c^{(0)}(\mu_t, 5)$	−0.407	−0.407	−0.407	−0.407	−0.424	0.004
$c^{(0)}(\mu_b, 5)$	−0.418	−0.418	−0.418	−0.418	−0.436	0.009
$c^{(0)}(\mu_b, 4)$	−0.418	−0.418	−0.418	−0.418	—	0.012
$c^{(0)}(\mu_c, 4)$	−0.443	−0.443	−0.443	−0.443	—	0.022
$c^{(0)}(\mu_c, 3)$	−0.443	−0.443	−0.443	—	—	0.028
$c^{(0)}(\mu_0, 3)$	−0.458	−0.458	−0.458	—	—	0.033
$\langle N c^{(0)}(\mu_0, 3) O^{(0)} N \rangle$ (MeV)	−8	−13	−18	—	—	−128

The final line shows the proton matrix elements (including coefficient) for each contribution to the total scalar amplitude

with overall factors $\pi\alpha_2^2/m_W^3$ extracted. The coefficients are labelled by the scale μ and number of active quark flavors n_f . We take $\mu_t = 126$, $\mu_b = m_b$, $\mu_c = m_c$ and $\mu_0 = 1.2$ as default values. Coefficients at the weak scale are obtained from renormalization of the bare matching coefficients (first line). Renormalization group evolution is performed down to the bottom threshold (second line). The bottom quark is integrated out (third line). Renormalization group evolution is performed down to the charm threshold (fourth line). The charm quark is integrated out (fifth line). Renormalization group evolution is performed down to the low scale (sixth line). The final line shows an evaluation of the amplitude, i.e., the operator matrix element including the coefficient.

For the scalar case, the total amplitude is,

$$\mathcal{M}_N^{(0)} = m_W^{-3} \langle N | \left(\sum_{q=u,d,s} \left[c_q^{(0)} O_q^{(0)} \right] + c_g^{(0)} O_g^{(0)} \right) | N \rangle = -167 \text{ MeV}, \quad (6.1)$$

where the overall factors $\pi\alpha_2^2/m_W^3$ have been extracted in the right hand side. The gluon contribution is dominant, gaining an order of magnitude from the coefficient running and heavy quark contributions, and having a large matrix element reflecting the gluon content of the nucleon.

For the tensor case, the gluon coefficient flips in sign at scale $\mu \sim 34 \text{ GeV}$, during the running from μ_t to μ_b . Two evaluations of the total amplitude,

$$\mathcal{M}_N^{(2)} = m_W^{-3} \langle N | \left(\sum_q \left[c_q^{(2)} v_\mu v_\nu O_q^{(2)\mu\nu} \right] + c_g^{(2)} v_\mu v_\nu O_g^{(2)\mu\nu} \right) | N \rangle, \quad (6.2)$$

are considered in Table 6.2: one at a high-scale $\mu \sim 70 \text{ GeV}$ using input from PDFs for five active quark flavors and the gluon, and another at a low-scale using input

Table 6.2 Tensor coefficients at each stage of the effective theory, labelled by the scale μ and the number of active quark flavors n_f , with overall factors $\pi\alpha_2^2/m_W^3$ extracted

	u	d	s	c	b	g
$c^{(2)}(\mu_t, 5)$	0.667	0.667	0.667	0.667	0.091	-0.050
$c^{(2)}(\mu_b, 5)$	0.498	0.498	0.498	0.498	0.073	0.080
$c^{(2)}(\mu_b, 4)$	0.498	0.498	0.498	0.498	—	0.080
$c^{(2)}(\mu_c, 4)$	0.418	0.418	0.418	0.418	—	0.140
$c^{(2)}(\mu_c, 3)$	0.418	0.418	0.418	—	—	0.140
$c^{(2)}(\mu_0, 3)$	0.405	0.405	0.405	—	—	0.147
$\langle N c^{(2)}(70, 5) O^{(2)} N \rangle \text{ (MeV)}$	116	71	24	17	1	-9
$\langle N c^{(2)}(\mu_0, 3) O^{(2)} N \rangle \text{ (MeV)}$	109	59	8	—	—	40

The final lines show the proton matrix elements (including coefficient) for each contribution to the total scalar amplitude for a high- and low-scale evaluation

from PDFs for three active quark flavors and the gluon. The dominant contribution is from light quarks, but the gluon contribution is not negligible. The high- and low-scale evaluations (with overall factors $\pi\alpha_2^2/m_W^3$ extracted) yield 220 MeV and 216 MeV, respectively, showing agreement up to higher order perturbative corrections and power corrections.

6.2 Survey of Uncertainties

Note that the scalar and tensor amplitude contributions in Tables 6.1 and 6.2 have similar magnitudes but opposite sign. This leads to destructive interference and anomalously small cross sections. In this situation the prediction becomes sensitive even to subdominant effects, and a robust conclusion requires careful study of each contribution, and of uncertainties from perturbative and power corrections, and from Standard Model inputs such as nucleon matrix elements. In this section, we consider uncertainties from scale variation and hadronic inputs, taking for definiteness the case of a pure triplet scattering on a proton.

For scale variations, we consider the ranges: $m_W^2/2 \leq \mu_t^2 \leq 2m_t^2$, $m_b^2/2 \leq \mu_b^2 \leq 2m_b^2$, $m_c^2/2 \leq \mu_c^2 \leq 2m_c^2$. Inspection of the renormalization group evolution for scalar operators, given in Eq. (5.30), and the scalar gluon matrix element, given in Eq. (5.37), shows that the gluon contribution to the total scalar amplitude is given by

$$c_g^{(0)}(\mu_c, 3) \frac{\tilde{\beta}(\mu_0)}{\tilde{\beta}(\mu_c)} \frac{1}{\tilde{\beta}(\mu_0)} 2 \left[1 - [1 + \gamma_m(\mu_0)] \sum_{q=u,d,s} f_{q,N}^{(0)} \right], \quad (6.3)$$

and that the μ_0 -dependent part of the contribution from light quarks is given by

$$c_g^{(0)}(\mu_c, 3) \frac{2[\gamma_m(\mu_0) - \gamma_m(\mu_c)]}{\tilde{\beta}(\mu_c)} \sum_{q=u,d,s} f_{q,N}^{(0)}, \quad (6.4)$$

where $c_g^{(0)}(\mu_c, 3)$ is the scalar gluon coefficient at the charm scale μ_c with $n_f = 3$ flavors. The sum of the above contributions is independent of the low scale μ_0 , and hence we may fix μ_0 in our evaluation, taking $\mu_0 = m_c = 1.4$ GeV for definiteness. We find a large residual uncertainty at LO from μ_c and μ_b scale variation. The RG running from μ_c to μ_0 from (5.20) is thus evaluated with NNNLO corrections, including contributions to β/g through $\mathcal{O}(\alpha_s^4)$ and γ_m through $\mathcal{O}(\alpha_s^4)$. Accordingly, the scalar gluon matrix element from (5.37) is also evaluated at NNNLO, including contributions to β/g through $\mathcal{O}(\alpha_s^4)$ and γ_m through $\mathcal{O}(\alpha_s^3)$. The impact of these higher order corrections will be illustrated for the pure triplet in the following section. We perform the RG running and heavy quark matching from μ_t to μ_c at NLO, and ignore power corrections appearing at relative order $\alpha_s(m_c)\Lambda_{\text{QCD}}^2/m_c^2$;

typical numerical prefactors appearing in the coefficients of the corresponding power-suppressed operators [97] suggest that these effects are small. For the scalar amplitude (with overall factors $\pi\alpha_2^2/m_W^3$ extracted), we find

$$\mathcal{M}_p^{(0)} = -167 \begin{pmatrix} +1 \\ -1 \end{pmatrix} \begin{pmatrix} +0 \\ -1 \end{pmatrix} \begin{pmatrix} +5 \\ -14 \end{pmatrix} (2)(3)(5) \text{ MeV} , \quad (6.5)$$

where the first three uncertainties are from the scale variation of μ_t , μ_b and μ_c , respectively, and the last three are respectively from the up, down and strange matrix elements in Eq. (5.40). The residual perturbative uncertainty is dominated by the charm scale. In the next section, we will investigate the impact of higher order $\alpha(\mu_c)$ corrections in the running from μ_b to μ_c and in the matching condition at the charm threshold. We will also consider an alternative evaluation of matrix elements in $n_f = 4$ -flavor QCD.

For the tensor operators, the renormalization group running and heavy quark matching are evaluated at LO. We find the tensor amplitude (with overall factors $\pi\alpha_2^2/m_W^3$ extracted)

$$\mathcal{M}_p^{(2)} = 216 \begin{pmatrix} +11 \\ -7 \end{pmatrix} (2)(2)(1)(2) \text{ MeV} , \quad (6.6)$$

where the first uncertainty is from scale variation of μ_t , and we have neglected other scale uncertainties being much smaller. The sensitivity to μ_t is from log terms, $\sim \log \mu_t$, in the high-scale matching coefficients, and its reduction would require a NLO weak-scale matching computation. The remaining uncertainties in Eq. (6.8) are from PDF inputs for up, down, strange, and glue, respectively. Up to power corrections and subleading $\mathcal{O}(\alpha_s)$ corrections, our evaluation is equivalent to an evaluation in either the $n_f = 4$ or $n_f = 5$ flavors theories, taking the c - and b -quark momentum fractions of the proton as input. We have verified this with matrix elements taken from [87]; e.g., the numerical difference between the result in Eq. (6.8) and the high-scale evaluation in Table 6.2 is within our error budget.

For the combined, scalar and tensor amplitudes (with overall factors $\pi\alpha_2^2/m_W^3$ extracted), we find

$$\mathcal{M}_p^{(2)} + \mathcal{M}_p^{(0)} = 49 \begin{pmatrix} +19 \\ -10 \end{pmatrix} (7) \text{ MeV} , \quad (6.7)$$

where the first and second uncertainties represent the total (combined in quadrature) scale and hadronic uncertainties, respectively. The cancellation between scalar and tensor contributions leads to a small total amplitude (and hence cross section), with large fractional uncertainties. For the pure doublet case, the cancellation is stronger, giving

$$\mathcal{M}_p^{(2)} + \mathcal{M}_p^{(0)} = 1.5 \begin{pmatrix} +7 \\ -4 \end{pmatrix} (3) \text{ MeV} . \quad (6.8)$$

In the next section we translate these amplitudes into cross section predictions with estimated uncertainties.

6.3 Cross Section Predictions and Consistency Checks

The low-velocity, spin-independent, cross section for WIMP scattering on a nucleus of mass number A and charge Z may be written

$$\sigma_{A,Z} = \frac{m_r^2}{\pi} |Z\mathcal{M}_p + (A-Z)\mathcal{M}_n|^2 \approx \frac{m_r^2 A^2}{\pi} |\mathcal{M}_p|^2, \quad (6.9)$$

where $\mathcal{M}_p$ and $\mathcal{M}_n$ are the matrix elements for scattering on a proton or neutron respectively and $m_r = Mm_N/(M+m_N)$ denotes the reduced mass of the dark-matter nucleus system. These matrix elements are the sum of the scalar and tensor matrix elements, $\mathcal{M}_N^{(0)}$ and $\mathcal{M}_N^{(2)}$, studied in the previous section. Explicitly, the matrix elements are

$$\begin{aligned} \mathcal{M}_N = m_W^{-3} \langle N | & \left(\sum_{q=u,d,s} \left[c_q^{(0)} O_q^{(0)} + c_q^{(2)} v_\mu v_\nu O_q^{(2)\mu\nu} \right] \right. \\ & \left. + c_g^{(0)} O_g^{(0)} + c_g^{(2)} v_\mu v_\nu O_g^{(2)\mu\nu} \right) | N \rangle. \end{aligned} \quad (6.10)$$

In the $M \gg m_N$ limit, the cross section scales as A^4 . At finite velocity, a nuclear form factor modifies this behavior [73]. Cross sections for scattering on the neutron and proton are numerically similar; we present results for the latter.

Since the pure-state weak-scale matching coefficients only depend electroweak charges and the mapping of the high-scale theory onto the low-scale theory only on known Standard Model inputs, upon specifying gauge representations for the WIMP, we obtain parameter-free cross section predictions. The triplet cross section is

$$\sigma_{\text{SI}}^T = 1.3_{-0.5}^{+1.2+0.4} \times 10^{-47} \text{ cm}^2, \quad (6.11)$$

where the first (second) error represents 1σ uncertainty from pQCD (hadronic inputs). Subleading corrections in ratios m_b/m_W and Λ_{QCD}/m_c are expected to be within this error budget. Stronger cancellation between spin-0 and spin-2 amplitudes in the doublet case implies a smaller cross section,

$$\sigma_{\text{SI}}^D \lesssim 10^{-48} \text{ cm}^2 \quad (95\% \text{ C.L.}). \quad (6.12)$$

The cancellation between scalar and tensor contributions leads to small cross sections and large fractional error that depend sensitively on subleading perturbative corrections and parametric inputs. As illustrations, sensitivity to the Higgs mass, m_h , is shown in Fig. 6.1, the impact of higher order pQCD corrections are shown in

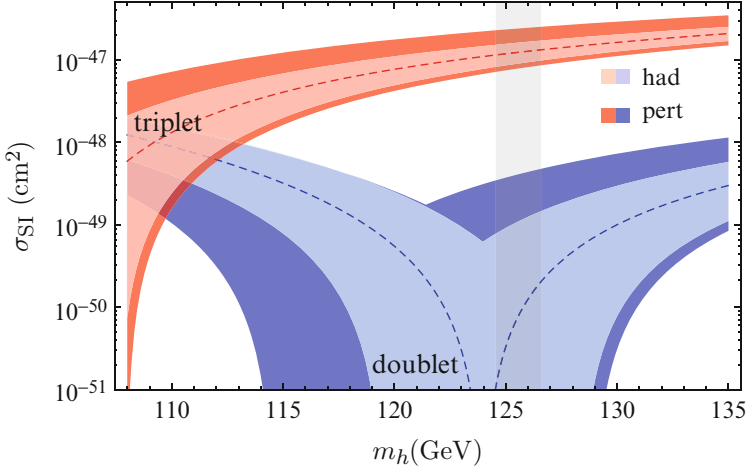


Fig. 6.1 SI cross sections for low-velocity scattering on the proton as a function of m_h , for the pure cases indicated. Here and in the plots below, *dark (light) bands* represent 1σ uncertainty from pQCD (hadronic inputs). The *vertical band* indicates the physical value of m_h

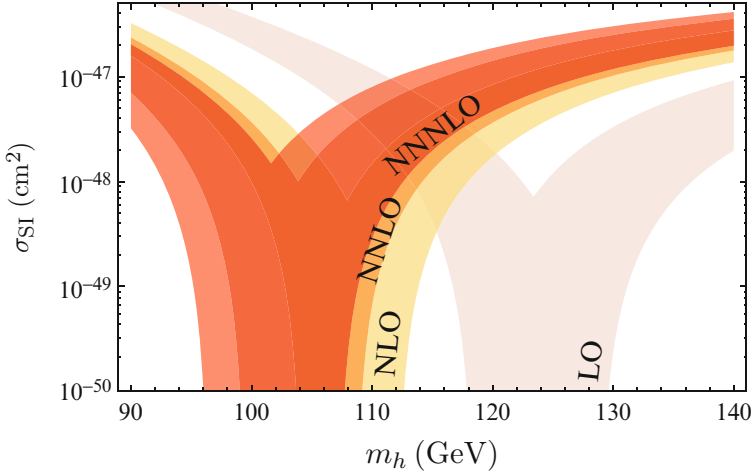


Fig. 6.2 SI cross section for low-velocity scattering on the proton as a function of m_h , for the pure-triplet case. Labels refer to inclusion of LO, NLO, NNLO and NNNLO corrections in the RG running from μ_c to μ_0 and in the spin-0 gluon matrix element. *Bands* represent 1σ uncertainty from neglected higher order pQCD corrections

Fig. 6.2, and the sensitivity to the strange quark content of the nucleon are shown in Figs. 6.3 and 6.4.

Robust cross section predictions demand a complete treatment of both perturbative and hadronic uncertainties. We may consider the impact of higher order $\alpha_s(\mu_c)$ corrections in the running between μ_b and μ_c as well as in the

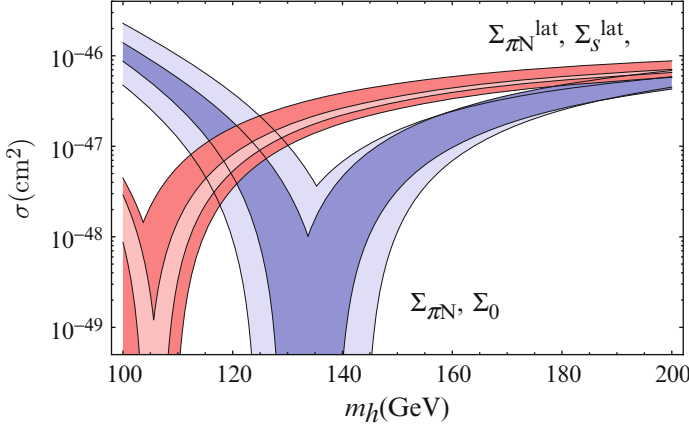


Fig. 6.3 Cross section for low-velocity scattering on a nucleon for a pure triplet. The *dark shaded region* represents the 1σ uncertainty from perturbative QCD, estimated by varying factorization scales. The *light shaded region* represents the 1σ uncertainty from hadronic inputs. We consider here “traditional” inputs $\Sigma_{\pi N}$ and Σ_0 [15, 92], as well as recent lattice determinations of $\Sigma_{\pi N}^{\text{lat}}$ and Σ_s^{lat} for the strange content of the nucleon

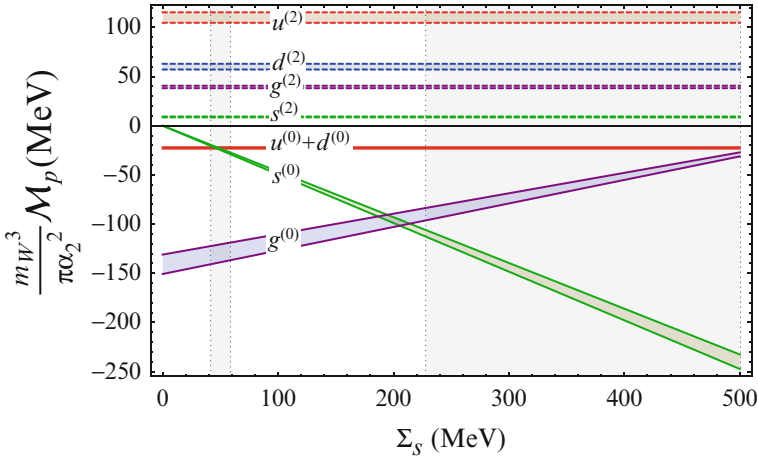


Fig. 6.4 Breakdown of contributions to the matrix element $\mathcal{M}_p$ using the representative values $m_h = 120$ GeV and $\Sigma_{\pi N}^{\text{lat}} = 47(9)$ MeV. The labels $u^{(S)}$, $d^{(S)}$, $s^{(S)}$ and $g^{(S)}$ refer to spin-S up, down, strange and gluon operator contributions, respectively. The *thickness* represents the 1σ uncertainty from perturbative QCD. The *left-hand vertical band* corresponds to the lattice value $\Sigma_s^{\text{lat}} = 50(8)$ MeV and the *right-hand vertical band* corresponds to the range $\Sigma_s = 366(142)$ MeV

threshold matching at μ_c . The cross sections shown in Fig. 6.5 are evaluated with NNNLO corrections in the running from μ_b to μ_c and with the NNNLO threshold matching relations available from [25]. We check that the scheme dependence of the matching conditions have minimal impact. The scale variation of μ_c would

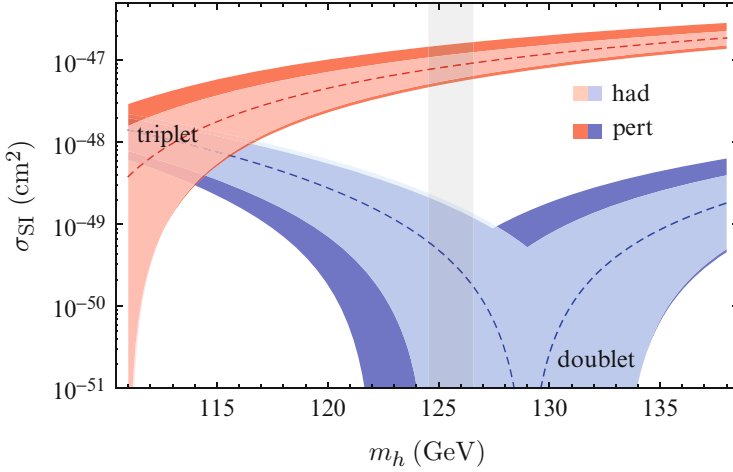


Fig. 6.5 Impact of NNNLO perturbative QCD corrections in the running from μ_b to μ_c and in (part of) threshold matching at μ_c on the spin-independent cross sections for low-velocity scattering on the proton

be further minimized given complete results for heavy quark decoupling. Using the physical Higgs mass, we find that the conclusion for the pure doublet cross section given in Eq. (6.12) remains the same, while the triplet prediction becomes $\sigma_{\text{SI}}^T = 0.8^{+0.7+0.3}_{-0.3-0.3} \times 10^{-47} \text{ cm}^2$, consistent with Eq. (6.11) given the uncertainties.

To avoid large perturbative uncertainties associated with the charm scale, we may also consider evaluating matrix elements in the $n_f = 4$ flavor theory. Figure 6.6 shows the results as a function of the charm scalar matrix element. Cancellation for the doublet is strongest near matrix element values estimated from pQCD. Direct determination of this matrix element could make the difference between a prediction and an upper bound for this (albeit small) cross section.

Previous computations of WIMP-nucleon scattering have focused on a different mass regime where other degrees of freedom are relevant [34], or have neglected the contribution $c_2^{(2)}$ from spin-2 gluon operators [65]. For pure states, this would lead to an $\mathcal{O}(20\%)$ shift in the spin-2 amplitude, with an underestimation of the perturbative uncertainty by $\mathcal{O}(70\%)$. For comparison, neglecting the spin-2 quark contribution from (b, c, s, d, u) shifts the spin-2 amplitude by $\mathcal{O}(1\%, 10\%, 10\%, 30\%, 50\%)$. Due to amplitude cancellations, the resulting effect on the cross sections in Fig. 6.1 ranges from a factor of a few to an order of magnitude.

For mixed states, we obtain the results pictured in Fig. 6.7. For weakly coupled WIMPs, we consider $\kappa \lesssim 1$. The presence of a scale separation $M, M' \gg m_W$, implies that the partner state contributes at leading order when $|\Delta| \lesssim m_W$, or more precisely $|\Delta| \lesssim m_W(4\pi\kappa)^2$. Within this regime, the purely spin-0 contributions from tree-level Higgs exchange can dominate (cf. [27]). However, when m_W/Δ

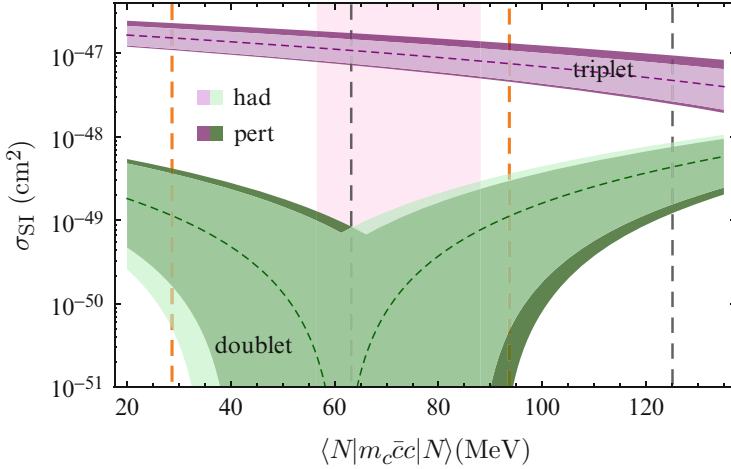


Fig. 6.6 SI cross sections for low-velocity scattering on the proton, evaluated in the $n_f = 4$ flavor theory as a function of the charm scalar matrix element, for the pure cases indicated. The pink region corresponds to charm content estimated from pQCD [74]. The region between orange (black) dashed lines correspond to direct lattice determinations in [47, 52]

suppression is significant, loop-induced contributions become relevant, and the opposite signs of spin-0 and spin-2 amplitudes lead to cancellations in the κ - Δ plane. In the decoupling limit of SUSY, κ depends on t_β and the sign of μ , taking values $\kappa \leq \frac{1}{2} \tan \theta_W$ ($\kappa \leq \frac{1}{2}$) for a bino-higgsino (wino-higgsino) mixture.

A simple dimensional estimate of the pure-state cross section yields¹ $\sigma_{\text{SI}} \sim \alpha_2^4 m_N^4 / m_W^6 \sim 10^{-45} \text{ cm}^2$. However, destructive interference between spin-0 and spin-2 amplitudes leads to anomalously small cross sections. The degree of cancellation depends on SM parameters such as m_h in Fig. 6.1, and on the choice of WIMP quantum numbers. Extending our computation to pure states of arbitrary isospin, J , and hypercharge, Y , the resulting cross section is minimum for $(J, Y) = (\frac{1}{2}, \frac{1}{2})$ corresponding to the doublet, and increases for larger J at fixed Y ; e.g., the result for $Y = 0$ is $\sigma_{\text{SI}}^{(J,0)} = [J(J+1)/2]^2 \sigma_{\text{SI}}^T$.

Additional structure in the Higgs sector may also have impact. We illustrate this with a second CP-even Higgs of mass $m_H > m_h = 126 \text{ GeV}$, arising in the context of the type-II two-Higgs-doublet model. Upon including diagrams with both Higgses, we obtain pure-state cross sections in terms of m_H , $t_\beta \equiv \tan \beta$ and $\eta \equiv t_\beta \cos(\beta - \alpha)$ (following the parameterization in [20] for departures from the “alignment limit”). For $t_\beta \gg 1$ and $|\eta| \leq \mathcal{O}(1)$, the couplings of the SM-like Higgs to $W^\pm, Z^0, u, c, t$ are given by $1 + \mathcal{O}(1/t_\beta^2)$, while those to d, s, b are given

¹Results consistent with this estimate were obtained in previous works missing the cancellation [28, 42].

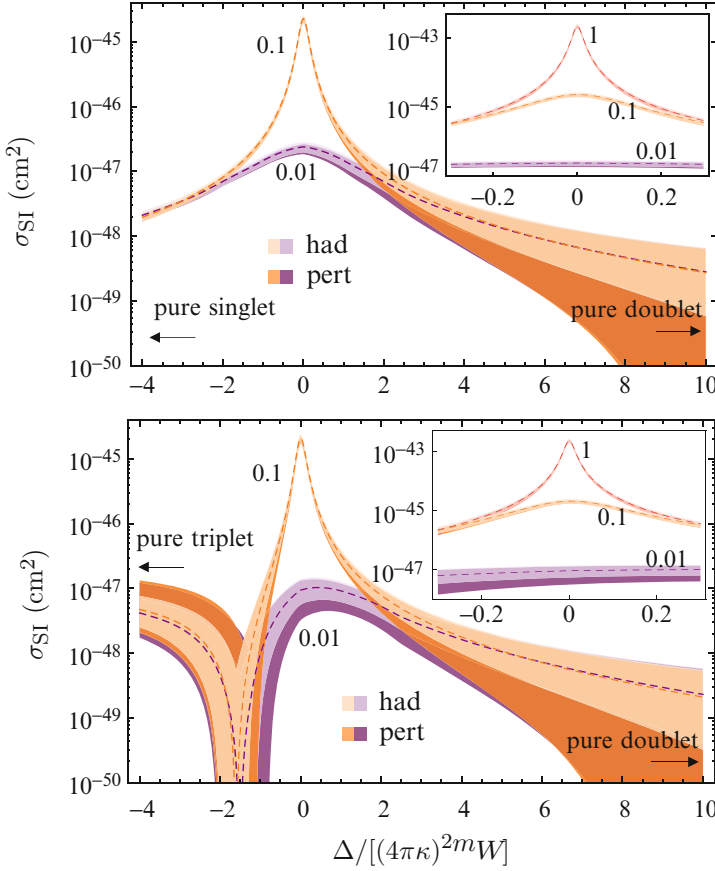


Fig. 6.7 SI cross sections for low-velocity scattering on the proton for the singlet-doublet and doublet-triplet admixtures, as a function of the mass splitting between pure-state constituents, $\Delta/[(4\pi\kappa)^2 m_W]$ (in conveniently chosen units such that interesting features of the curves with different κ may be displayed on the same scale). We indicate pure case limits and label each curve with the κ value used. *Inset* plots use the same units

by $(1 - \eta) + \mathcal{O}(1/t_\beta^2)$, measured relative to SM values. Existing phenomenological constraints are not sensitive to the sign of the latter, allowing for both $\eta \approx 0, 2$ where the magnitude is near the SM value [37]. Figure 6.8 shows cross section predictions for pure states with quantum numbers (J, Y) indicated, including $(2, 0)$, the smallest representation for which WIMP decay by dimension five operators is forbidden by gauge invariance [28].

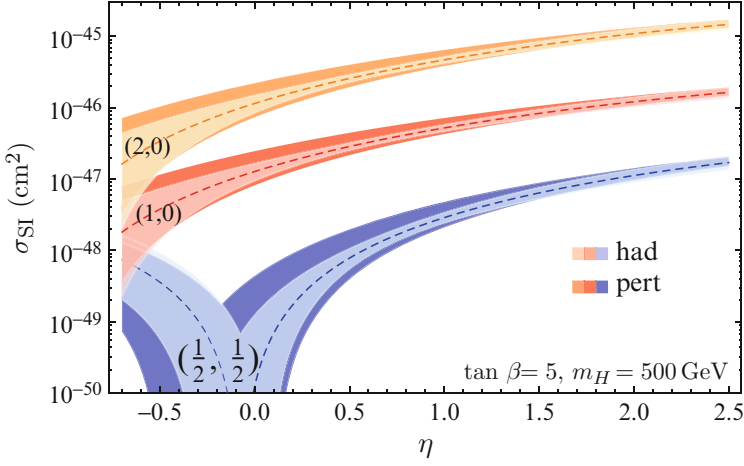


Fig. 6.8 SI cross sections for low-velocity scattering on the proton as a function of $\eta \equiv t_\beta \cos(\beta - \alpha)$, for pure states with quantum numbers (J, Y) . The values $|\eta|, |\eta - 2| \lesssim 0.5$ are phenomenologically allowed [37]. Cross sections assuming only a SM-like Higgs are at $\eta = 0$

Chapter 7

Conclusions

Data from a range of astrophysical and cosmological observations, such as the brightness of supernovae at various redshifts [79, 81], the spectra of cosmic microwave background anisotropies [2, 63], and the primeval abundance of light elements [96], are sensitive to the matter (Ω_M) and baryon (Ω_B) densities in the universe, leading, e.g., to the precise measurements [2]:

$$\Omega_M h^2 = 0.1423 \pm 0.0029, \quad \Omega_B h^2 = 0.02207 \pm 0.00033. \quad (7.1)$$

Evidence for the existence of dark matter may be summarized by the inequality of Ω_M and Ω_B , motivating both experimental and theoretical studies to elucidate the nature of particle dark matter.

Presently, this research program faces several challenges. First, there are no definitive signals from dark matter search experiments thus far, only a few anomalies considered tenuous due to large astrophysical backgrounds and contradictory results among various searches. Second, while theoretical visions offer some guidance to experimental searches, it is often difficult to extract definite predictions from the multi-dimensional parameter space of models. Third, relating experimental signals and theoretical models require a robust analysis of model-independent uncertainties, e.g., from astrophysical or Standard Model inputs.

In this thesis, we developed effective theory tools aimed at controlling the uncertainties inherent to a vast landscape of theoretical models and to our knowledge of Standard Model physics relevant for WIMP-nucleon scattering. We employed two scale separations, $M \gg m_W$ above the weak scale and $m_W \gg m_b$ below the weak scale, to identify universal components underlying the interactions of WIMPs with the Standard Model.

Above the electroweak scale, we developed heavy WIMP effective theory (HWET) for systematically describing the interactions of heavy WIMPs ($M \gg m_W$) with Standard Model gauge and Higgs bosons. In the absence of a specified UV completion, HWET provides a universal parametrization of these interactions for

WIMPs arising in models known (e.g., SUSY), unknown, or in models where predictions require a nonperturbative description (e.g., dark baryons). The universality sharpens in the heavy WIMP limit, where cross section predictions for WIMPs of given electroweak quantum numbers depend on Standard Model inputs only.

Below the electroweak scale, we constructed the basis of operators describing the interactions of the lightest electrically neutral WIMP state χ with the quarks and gluons of 5-flavor QCD. This effective theory at low energies is independent of the heavy WIMP assumption, and applies to a broad class of underlying models for the WIMP. We focused on the basis for self-conjugate χ with mass $M \gtrsim m_W$, providing the necessary tools to relate the theory above the electroweak scale to the theory at low-energies, such as weak-scale matching techniques, operator renormalization, renormalization group evolution and heavy quark threshold matching.

Employing this effective theory framework, we presented the first complete solution of the twelve operator coefficients describing interactions below the weak scale, in the heavy WIMP limit. We emphasized the importance of the complete set of matching contributions; in particular, we found that scalar gluon contributions dominate, and that tensor gluon contributions are sizable and represent the largest source of perturbative uncertainty in the tensor amplitude. We presented a careful treatment of both perturbative and hadronic input uncertainties and of the resummation of large logarithms $\sim \log m_t/m_c$. We also identified aspects of Standard Model physics whose further study would improve our knowledge of WIMP-nucleon scattering, such as a complete set of threshold matching corrections at the charm scale, and the nucleon scalar matrix elements evaluated in $n_f = 4$ QCD. The present analysis is systematically improvable in small ratios m_W/M , m_b/m_W , Λ_{QCD}/m_c and in powers of α_s at various renormalization scales.

There is a surprising transparency of WIMPs to nucleons due to a cancellation between scalar and tensor amplitudes. In this situation, cross section predictions and its fractional uncertainty become sensitive to perturbative contributions and parametric inputs. We presented absolute predictions for WIMPs transforming under irreducible representations of $SU(2)_W \times U(1)_Y$ (Fig. 6.1), and investigated the impact of perturbative QCD corrections (Figs. 6.2 and 6.5), tensor gluon contributions neglected in previous works, and our knowledge of nucleon scalar matrix elements (Figs. 6.3, 6.4 and 6.6). We also considered the impact of model-dependent extensions such as additional WIMPs (Fig. 6.7) and an extended Higgs sector (Fig. 4.5). With obvious modifications, our results can be extended to include additional low-energy field content beyond those considered here, or to investigate power corrections to the heavy WIMP limit. It is also interesting to investigate the impact of nuclear effects on the cancellation between scalar and tensor contributions [94].

The analysis of dark matter observables is a new application of heavy particle effective theory. We have developed several nontrivial aspects, including a clear statement of Lorentz invariance for bottom-up construction of heavy particle lagrangians, the introduction of a consistent renormalization scheme for heavy WIMP-SM vertices, and the computation of new heavy-particle loop integrals with nonzero residual masses. The new integral basis evaluated here may be applied

to other processes such as low-energy lepton-nucleon scattering [62]. Beyond the application to direct scattering developed in this thesis, an extension of HWET with hard-collinear modes provides the framework for computing the annihilation of WIMPs to electroweak gauge bosons, including the resummation of large logarithms, $\sim \log \frac{M}{m_W}$, relevant for indirect detection experiments [11].

Heavy WIMPs are plausible dark matter particle candidates, compatible with the absence of experimental signals thus far (e.g., at the LHC and LUX facilities) and with relic abundance estimates. Prospects for detecting heavy WIMPs in underground direct detection search experiments are challenged by large backgrounds from coherent neutrino scattering. On the other hand, it is in this regime where definite predictions, precisely constrained by heavy particle universality, can be made. If such backgrounds can be controlled, then the tools developed in this thesis, together with improvements in our knowledge of Standard Model contributions (e.g., of perturbative QCD and nucleon matrix elements contributions), provide precise cross section targets with minimal model-dependence, allowing us to test the WIMP hypothesis robustly. If neutrino-scattering backgrounds prove insurmountable, then we might refocus efforts on alternative experimental strategies such as production at future colliders and indirect detection at future telescopes.

Appendix A

Solution to the Invariance Equation

Section 2.3.4 describes the solution of the invariance equation (2.52) for the function $\Gamma(v, iD)$ in the free theory. The solution in the interacting theory is not simply obtained from the free one by replacing ∂ with D . Here we present a method of solution that is valid to any order in $1/M$. Since we use $\Gamma(v, iD)$ to construct the invariant Lagrangian, the existence of a solution for $\Gamma(v, iD)$ proves that a non-zero Lagrangian exists at any order in $1/M$. First, we will construct the general solution in Sect. A.1 and then explicitly apply this construction to the spin 1/2 theory up to order $1/M^3$ in Sect. A.2.

A.1 Series Solution for Γ

Recall Eq. (2.52) for Γ required to build explicitly invariant operators,

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}W(\mathcal{B}, iD + Mv) = \Gamma(v, iD), \quad (\text{A.1})$$

where to first order in q we have $\mathcal{B}^{-1}v = v + q/M$. Let us expand in orders of $1/M$ and define

$$X \equiv \mathcal{B}^{-1}W = 1 + q^\mu X_\mu = 1 + q^\mu \left[\frac{1}{M} X_\mu^{(1)} + \frac{1}{M^2} X_\mu^{(2)} + \dots \right], \quad (\text{A.2a})$$

$$\Gamma = 1 + \frac{1}{M} \Gamma^{(1)} + \frac{1}{M^2} \Gamma^{(2)} + \dots \quad (\text{A.2b})$$

We note that the variation in Γ arises from the variations in v and in iD ,

$$\delta\Gamma = \Gamma(v + q/M, iD - q) - \Gamma(v, iD) = q^\mu \left(-\frac{\partial}{\partial iD^\mu} \Gamma + \frac{1}{M} \frac{\partial}{\partial v^\mu} \Gamma \right). \quad (\text{A.3})$$

Equating orders in $1/M$, we find

$$\frac{\partial}{\partial iD^\mu} \Gamma^{(n)} = \frac{\partial}{\partial v^\mu} \Gamma^{(n-1)} + \Gamma^{(n-1)} X_\mu^{(1)} + \Gamma^{(n-2)} X_\mu^{(2)} + \dots + \Gamma^{(0)} X_\mu^{(n)} \equiv Y_\mu^{(n)}, \quad (\text{A.4})$$

where we define $\Gamma^{(0)} = 1$. Note that Eq. (A.4) is understood to be contracted with q^μ so that pieces proportional to v^μ should be dropped. We can solve this equation for $\Gamma^{(n)}$ obtaining

$$\begin{aligned} \Gamma^{(n)} &= \sum_{m=1}^n \frac{(-1)^{m-1}}{m!} iD_\perp^{\mu_1} iD_\perp^{\mu_2} \dots iD_\perp^{\mu_m} \frac{\partial}{\partial iD^{\mu_1}} \frac{\partial}{\partial iD^{\mu_2}} \dots \frac{\partial}{\partial iD^{\mu_{m-1}}} Y_{\mu_m}^{(n)} \\ &= iD_\perp^\mu Y_\mu^{(n)} - \frac{1}{2!} iD_\perp^\mu iD_\perp^\nu \frac{\partial}{\partial iD^\mu} Y_\nu^{(n)} + \dots, \end{aligned} \quad (\text{A.5})$$

provided that at each order, the $Y^{(n)}$ derived from the already determined $\Gamma^{(1)}, \dots, \Gamma^{(n-1)}$ satisfy¹

$$\frac{\partial}{\partial iD^{[\nu}} Y_{\mu]}^{(n)} = 0, \quad (\text{A.6})$$

where $A^{[\mu} B^{\nu]} = (A^\mu B^\nu - A^\nu B^\mu)/2$ denotes antisymmetrization. Using the definition of $Y^{(n)}$ we can show that this imposes constraints on $X^{(n)}$, for $n \geq 2$,

$$\frac{\partial}{\partial iD^{[\nu}} X_{\mu]}^{(n)} = -\frac{\partial}{\partial v^{[\mu}} X_{\nu]}^{(n-1)} + X_{[\mu}^{(n-1)} X_{\nu]}^{(1)} + X_{[\mu}^{(n-2)} X_{\nu]}^{(2)} + \dots + X_{[\mu}^{(1)} X_{\nu]}^{(n-1)} \equiv Z_{\mu\nu}^{(n)}. \quad (\text{A.7})$$

For Eq. (A.7) to have a solution, a consistency condition on $Z_{\mu\nu}^{(n)}$ requires that²

$$0 = v_\sigma \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial iD^\rho} Z_{\mu\nu}^{(n)}. \quad (\text{A.8})$$

We can show by induction that Eq. (A.7) can be solved at each order. Since $X^{(1)}$ is dimensionless, it cannot depend on iD ; hence $Z^{(2)}$ from (A.7) is also independent of iD and solves (A.8). Now assume that we have constructed solutions $X^{(n)}$ to Eq. (A.7) for $n = 1, \dots, N-1$ (necessarily obeying the constraint (A.8)). Application of the Jacobi identity shows that the constraint (A.8) is then obeyed for $n = N$ and a solution to Eq. (A.7) can be found for $n = N$.

Let us find a solution to Eq. (A.7) that reduces to a given X_{free} for the non-interacting theory (e.g., $X_{\text{free}} = \mathcal{B}^{-1}W$ from (2.56)). First, note that the existence

¹This is the analog of $\vec{\nabla} \times \vec{E} = \vec{0}$ for the existence of a solution ϕ of $\vec{\nabla}\phi = \vec{E}$ in electrostatics.

²This is the analog of $\vec{\nabla} \cdot \vec{B} = 0$ for the existence of a solution $\vec{A}$ of $\vec{\nabla} \times \vec{A} = \vec{B}$ in magnetostatics.

of the free case solution given in (2.57) implies that the $X^{(n)}$ defined in the free case from (2.56) must obey the constraint (A.7). Let us define naively covariantized quantities $\hat{X}^{(n)} = X_{\text{free}}^{(n)} \Big|_{\partial \rightarrow D}$, with a definite ordering prescription, e.g. as in (2.63), and define $\hat{Z}^{(n)}$ by

$$\hat{Z}_{\mu\nu}^{(n)} \equiv \frac{\partial}{\partial iD^{[\nu}} \hat{X}_{\mu]}^{(n)}. \quad (\text{A.9})$$

A straightforward calculation then shows that (A.7) is solved by

$$X_{\mu}^{(n)} = \hat{X}_{\mu}^{(n)} + 2 \sum_{m=1}^{n-1} \frac{(-1)^m}{(m+1)!} iD_{\perp}^{\nu_1} \cdots iD_{\perp}^{\nu_m} \frac{\partial}{\partial iD^{\nu_1}} \cdots \frac{\partial}{\partial iD^{\nu_{m-1}}} \left(Z_{\nu_m \mu}^{(n)} - \hat{Z}_{\nu_m \mu}^{(n)} \right). \quad (\text{A.10})$$

In the free case we have $Z^{(n)} = \hat{Z}^{(n)}$ and $X^{(n)}$ reduces to the free case solution. Having found a suitable $X^{(n)}$ satisfying (A.7) we may then proceed to build $\Gamma^{(n)}$ satisfying (A.4), and hence Γ satisfying (2.52).

Note that $Z^{(n)}$ has mass dimension $n - 2$ so that $n = 4$ is the first order at which field strength dependent terms can cause $Z^{(n)} \neq \hat{Z}^{(n)}$. Correspondingly, our choice (A.10) ensures that field-strength dependent corrections to $X^{(n)} - \hat{X}^{(n)}$ can first appear at order $n = 4$. This can be explicitly seen in the solution for the spin 1/2 theory in the next section.

A.2 Explicit Solution for Γ in the Spin 1/2 Theory

To illustrate, let us calculate Γ for the spin 1/2 theory. Consider the free solution (2.56),

$$X_{\mu}(v, i\partial) = \frac{1}{2M} \gamma_{\mu}^{\perp} + \frac{1}{4M^2} \sigma_{\mu\nu}^{\perp} \partial^{\nu} \left[1 - \frac{iv \cdot \partial}{M} + \frac{1}{M^2} \left((iv \cdot \partial)^2 - \frac{1}{4} (i\partial_{\perp})^2 \right) + \dots \right], \quad (\text{A.11})$$

and the arbitrary covariantization,

$$\hat{X}_{\mu}(v, iD) = \frac{1}{2M} \gamma_{\mu}^{\perp} + \frac{1}{4M^2} \sigma_{\mu\nu}^{\perp} D^{\nu} \left[1 - \frac{iv \cdot D}{M} + \frac{1}{M^2} \left((iv \cdot D)^2 - \frac{1}{4} (iD_{\perp})^2 \right) + \dots \right]. \quad (\text{A.12})$$

A corresponding solution for Γ in the free theory is displayed in (2.57). Now let us follow the construction of the previous section order by order.

Order $1/M$: First, we determine,

$$Y_\mu^{(1)} = X_\mu^{(1)} = \hat{X}_\mu^{(1)} = \frac{\gamma_\mu^\perp}{2}. \quad (\text{A.13})$$

This function clearly satisfies Eq. (A.6) so that we may solve for

$$\Gamma^{(1)} = \frac{1}{2} i \not{D}_\perp. \quad (\text{A.14})$$

Order $1/M^2$: Continuing to the next order, we evaluate

$$Z_{\mu\nu}^{(2)} = -\frac{i}{4} \sigma_{\mu\nu}^\perp = \hat{Z}_{\mu\nu}^{(2)}, \quad (\text{A.15a})$$

$$X_\mu^{(2)} = \frac{1}{4} \sigma_{\mu\nu}^\perp D^\nu = \hat{X}_\mu^{(2)}, \quad (\text{A.15b})$$

$$Y_\mu^{(2)} = -\frac{1}{2} \gamma_\mu^\perp i v \cdot D - \frac{1}{4} i D_\mu^\perp. \quad (\text{A.15c})$$

Solving for $\Gamma^{(2)}$ yields

$$\Gamma^{(2)} = -\frac{1}{8} (i D_\perp)^2 - \frac{1}{2} i \not{D}_\perp i v \cdot D. \quad (\text{A.16})$$

Order $1/M^3$: At the next order, we find

$$Z_{\mu\nu}^{(3)} = \frac{i}{4} \sigma_{\mu\nu}^\perp i v \cdot D = \hat{Z}_{\mu\nu}^{(3)}, \quad (\text{A.17a})$$

$$X_\mu^{(3)} = -\frac{1}{4} \sigma_{\mu\nu}^\perp D^\nu i v \cdot D = \hat{X}_\mu^{(3)}, \quad (\text{A.17b})$$

$$\begin{aligned} Y_\mu^{(3)} = & \frac{1}{2} \gamma_\mu^\perp (i v \cdot D)^2 + \frac{3}{8} i D_\mu^\perp i v \cdot D + \frac{1}{8} i v \cdot D i D_\mu^\perp - \frac{1}{2} i \not{D}_\perp i D_\mu^\perp \\ & - \frac{1}{16} (i D_\perp)^2 \gamma_\mu^\perp + \frac{1}{8} i \not{D}_\perp \sigma_{\mu\nu}^\perp D^\nu. \end{aligned} \quad (\text{A.17c})$$

After some manipulations, the resulting $\Gamma^{(3)}$ is

$$\begin{aligned} \Gamma^{(3)} = & \frac{1}{4} (i D_\perp)^2 i v \cdot D + \frac{i \not{D}_\perp}{2} \left[-\frac{3}{8} i \not{D}_\perp (i D_\perp)^2 + (i v \cdot D)^2 \right] - \frac{g}{8} v^\alpha G_{\alpha\beta} D_\perp^\beta \\ & - \frac{g}{16} \sigma_{\alpha\beta}^\perp G^{\alpha\beta} i \not{D}_\perp + \frac{g}{8} \left[i \gamma_\perp^\beta \sigma_\perp^{\mu\alpha} [D_\mu, G_{\beta\alpha}] \right. \\ & \left. - v^\alpha [D_\perp^\mu, G_{\alpha\mu}] - [D_\perp^\mu, G_{\mu\beta}^\perp] \gamma_\perp^\beta \right]. \end{aligned} \quad (\text{A.18})$$

Order $1/M^4$: Continuing to higher order we find

$$Z_{\mu\nu}^{(4)} = \hat{Z}_{\mu\nu}^{(4)} + \frac{g}{32} \left(-iG_{\mu\nu}^{\perp} + \sigma_{\mu\sigma}^{\perp} G_{\nu}^{\perp\sigma} - \sigma_{\nu\sigma}^{\perp} G_{\mu}^{\perp\sigma} \right), \quad (\text{A.19a})$$

$$X_{\mu}^{(4)} = \sigma_{\mu\nu}^{\perp} D^{\nu} \left[\frac{1}{4} (iv \cdot D)^2 - \frac{1}{16} (iD_{\perp})^2 \right] + \frac{g}{32} iD_{\perp}^{\nu} \left(-iG_{\mu\nu}^{\perp} + \sigma_{\mu\sigma}^{\perp} G_{\nu}^{\perp\sigma} - \sigma_{\nu\sigma}^{\perp} G_{\mu}^{\perp\sigma} \right). \quad (\text{A.19b})$$

Note that $X_{\mu}^{(4)}$ differs from the trial solution $\hat{X}_{\mu}^{(4)}$. We may continue in this manner to construct $Y_{\mu}^{(4)}$ and $\Gamma^{(4)}$.

Appendix B

Integrals and Inputs for Weak Scale Matching

B.1 Self Energy Integrals and Standard Model Two-Point Functions

Here and in the following sections we use the notation

$$[c_\epsilon] = \frac{i\Gamma(1+\epsilon)}{(4\pi)^{2-\epsilon}}, \quad (dL) = \frac{d^d L}{(2\pi)^d}. \quad (\text{B.1})$$

The self-energies in Sect. 3.3 and the $h\bar{\chi}\chi$ three-point functions in Sect. 4.2.1 require the following integrals,

$$\begin{aligned} I_1(\delta, m) &= \int (dL) \frac{1}{v \cdot L - \delta + i0} \frac{1}{(L^2 - m^2 + i0)^2} \\ &= \frac{\partial}{\partial m^2} I_3(\delta, m) \\ &= [c_\epsilon] m^{-2\epsilon} \left\{ \frac{2}{\sqrt{m^2 - \delta^2 - i0}} \left[\arctan \left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] \right. \\ &\quad \left. + \mathcal{O}(\epsilon) \right\}, \\ I_2(\delta, m) &= \int (dL) v \cdot L \frac{1}{v \cdot L - \delta + i0} \frac{1}{(L^2 - m^2 + i0)^2} \\ &= \delta I_1(\delta, m) + \frac{i}{(4\pi)^2} B_0(0, m, m) \end{aligned}$$

$$\begin{aligned}
&= [c_\epsilon] m^{-2\epsilon} \left\{ \frac{1}{\epsilon} + \frac{2\delta}{\sqrt{m^2 - \delta^2 - i0}} \left[\arctan \left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] \right. \\
&\quad \left. + \mathcal{O}(\epsilon) \right\}, \\
I_3(\delta, m) &= \int (dL) \frac{1}{v \cdot L - \delta + i0} \frac{1}{L^2 - m^2 + i0} \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2\delta}{\epsilon} + 4\sqrt{m^2 - \delta^2 - i0} \left[\arctan \left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) \right. \right. \\
&\quad \left. \left. - \frac{\pi}{2} \right] - 4\delta + \mathcal{O}(\epsilon) \right\}, \\
I_4(\delta_1, \delta_2, m) &= \int (dL) \frac{1}{v \cdot L - \delta_1 + i0} \frac{1}{v \cdot L - \delta_2 + i0} \frac{1}{L^2 - m^2 + i0}. \tag{B.2}
\end{aligned}$$

For $I_4(\delta_1, \delta_2, m)$, let us specialize to $\delta_2 = 0$ or $\delta_1 = \delta_2$,

$$\begin{aligned}
I_4(\delta, 0, m) &= \frac{1}{\delta} [I_3(\delta, m) - I_3(0, m)] \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2}{\epsilon} + \frac{4\sqrt{m^2 - \delta^2 - i0}}{\delta} \left[\arctan \left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] \right. \\
&\quad \left. - 4 + \frac{2\pi m}{\delta} + \mathcal{O}(\epsilon) \right\}, \\
I_4(\delta, \delta, m) &= \frac{\partial}{\partial \delta} I_3(\delta, m) \\
&= [c_\epsilon] m^{-2\epsilon} \left\{ -\frac{2}{\epsilon} - \frac{4\delta}{\sqrt{m^2 - \delta^2 - i0}} \left[\arctan \left(\frac{\delta}{\sqrt{m^2 - \delta^2 - i0}} \right) - \frac{\pi}{2} \right] \right. \\
&\quad \left. + \mathcal{O}(\epsilon) \right\}. \tag{B.3}
\end{aligned}$$

The two-point functions for the electroweak SM bosons appearing in (4.21) are obtained by summing the fermionic and bosonic contributions given below. Following Denner [32], we have

$$\begin{aligned}
\Sigma^{AA'}(0) &= -\frac{\alpha}{4\pi} \left\{ 3B_0(0, m_W, m_W) + 4m_W^2 B'_0(0, m_W, m_W) \right. \\
&\quad \left. - \frac{4}{3} \sum_{f,i} [N_c^f Q_f^2 B_0(0, m_{f,i}, m_{f,i})] \right\}, \\
\frac{\Sigma^{AZ}(0)}{m_Z^2} &= -\frac{\alpha}{4\pi} \left\{ -\frac{2c_W}{s_W} B_0(0, m_W, m_W) \right\},
\end{aligned}$$

$$\begin{aligned}
\frac{\Sigma^{ZZ}(m_Z^2)_{\text{fermion}}}{m_Z^2} &= -\frac{\alpha}{4\pi} \left\{ \frac{2}{3} \left[-B_0(m_Z, 0, 0) + \frac{1}{3} \right] \sum_{f,i} N_c^f [(g_f^+)^2 + (g_f^-)^2] \right. \\
&\quad + \frac{2}{3} N_c^t \left[[(g_t^+)^2 + (g_t^-)^2] \left[- \left(1 + \frac{2m_t^2}{m_Z^2} \right) B_0(m_Z, m_t, m_t) \right. \right. \\
&\quad \left. \left. + B_0(m_Z, 0, 0) + \frac{2m_t^2}{m_Z^2} B_0(0, m_t, m_t) \right] \right. \\
&\quad \left. \left. + \frac{3}{4s_W^2 c_W^2} \frac{m_t^2}{m_Z^2} B_0(m_Z, m_t, m_t) \right] \right\}, \\
\frac{\Sigma^{ZZ}(m_Z^2)_{\text{boson}}}{m_Z^2} &= -\frac{\alpha}{4\pi} \frac{1}{s_W^2 c_W^2} \left\{ \frac{1}{12} (4c_W^2 - 1)(12c_W^4 + 20c_W^2 + 1) B_0(m_Z, m_W, m_W) \right. \\
&\quad - \frac{1}{3} c_W^2 (12c_W^4 - 4c_W^2 + 1) B_0(0, m_W, m_W) - \frac{1}{6} B_0(0, m_Z, m_Z) \\
&\quad - \frac{1}{12} \left(\frac{m_h^4}{m_Z^4} - 4 \frac{m_h^2}{m_Z^2} + 12 \right) B_0(m_Z, m_Z, m_h) - \frac{1}{6} \frac{m_h^2}{m_Z^2} B_0(0, m_h, m_h) \\
&\quad \left. + \frac{1}{12} \left(1 - \frac{m_h^2}{m_Z^2} \right)^2 B_0(0, m_Z, m_h) - \frac{1}{9} (1 - 2c_W^2) \right\}, \\
\frac{\Sigma^{WW}(m_W^2)_{\text{fermion}}}{m_W^2} &= -\frac{\alpha}{4\pi} \frac{1}{2s_W^2} \left\{ \frac{2}{3} \left[\frac{1}{3} - B_0(m_W, 0, 0) \right] \sum_{f,i} \frac{N_c^f}{2} \right. \\
&\quad + \frac{2}{3} N_c^t \left[\frac{1}{2} \left(\frac{m_t^4}{m_W^4} + \frac{m_t^2}{m_W^2} - 2 \right) B_0(m_W, m_t, 0) + B_0(m_W, 0, 0) \right. \\
&\quad \left. \left. + \frac{m_t^2}{m_W^2} B_0(0, m_t, m_t) - \frac{m_t^4}{2m_W^4} B_0(0, m_t, 0) \right] \right\}, \\
\frac{\Sigma^{WW}(m_W^2)_{\text{boson}}}{m_W^2} &= -\frac{\alpha}{4\pi} \left\{ 4B_0(m_W, m_W, \lambda) - \frac{4}{3} B_0(0, m_W, m_W) + \frac{2}{3} B_0(0, m_W, \lambda) \right. \\
&\quad + \frac{2}{9} + \frac{1}{12s_W^2} \left[\frac{1}{c_W^4} (4c_W^2 - 1)(12c_W^4 + 20c_W^2 + 1) B_0(m_W, m_W, m_Z) \right. \\
&\quad - 2(8c_W^2 + 1) B_0(0, m_W, m_W) - \frac{2}{c_W^2} (8c_W^2 + 1) B_0(0, m_Z, m_Z) \\
&\quad \left. + \frac{s_W^4}{c_W^4} (8c_W^2 + 1) B_0(0, m_W, m_Z) - \frac{2}{3} (1 - 4c_W^2) \right] \\
&\quad + \frac{1}{12s_W^2} \left[- \left(\frac{m_h^4}{m_W^4} - 4 \frac{m_h^2}{m_W^2} + 12 \right) B_0(m_W, m_W, m_h) \right. \\
&\quad \left. - 2B_0(0, m_W, m_W) - 2 \frac{m_h^2}{m_W^2} B_0(0, m_h, m_h) \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(1 - \frac{m_h^2}{m_W^2} \right)^2 B_0(0, m_W, m_h) - \frac{2}{3} \Big] \Big\} , \\
\Sigma^{HH'}(m_h^2)_{\text{fermion}} &= -\frac{\alpha}{4\pi} \frac{3m_t^2}{2s_W^2 m_W^2} \left[(4m_t^2 - m_h^2) B'_0(m_h, m_t, m_t) - B_0(m_h, m_t, m_t) \right] , \\
\Sigma^{HH'}(m_h^2)_{\text{boson}} &= -\frac{\alpha}{4\pi} \left\{ -\frac{1}{2s_W^2} \left[\left(6m_W^2 - 2m_h^2 + \frac{m_h^4}{2m_W^2} \right) B'_0(m_h, m_W, m_W) \right. \right. \\
&\quad \left. \left. - 2B_0(m_h, m_W, m_W) \right] - \frac{1}{4s_W^2 c_W^2} \left[\left(6m_Z^2 - 2m_h^2 + \frac{m_h^4}{2m_Z^2} \right) \right. \right. \\
&\quad \left. \left. B'_0(m_h, m_Z, m_Z) - 2B_0(m_h, m_Z, m_Z) \right] - \frac{9m_h^4}{8s_W^2 m_W^2} B'_0(m_h, m_h, m_h) \right\} , \\
&\hspace{15cm} \text{(B.4)}
\end{aligned}$$

where the sums over indices f and i are for SM fermion flavors and generations, respectively. Above, N_c^f and Q_f respectively denote the number of colors and the electric charge of fermion f . We have also used

$$\alpha = \frac{g_2^2 s_W^2}{4\pi} , \quad g_f^+ = \frac{1}{8s_W^2 c_W^2} [c_V^{(f)2} + c_A^{(f)2}] , \quad g_f^- = \frac{1}{8s_W^2 c_W^2} [c_V^{(f)2} - c_A^{(f)2}] , \quad \text{(B.5)}$$

where

$$c_V^{(\ell)} = -1 + 4s_W^2 , \quad c_A^{(\ell)} = 1 , \quad c_V^{(\nu)} = 1 , \quad c_A^{(\nu)} = -1 , \quad \text{(B.6)}$$

with ℓ and ν denoting charged lepton and neutrino, respectively. The coefficients $c_V^{(f)}$ and $c_A^{(f)}$ for quarks can be found in (4.14). The basic integral appearing above is

$$\begin{aligned}
\frac{i}{(4\pi)^2} B_0(M, m_0, m_1) &= \int (dL) \frac{1}{L^2 - m_0^2 + i0} \frac{1}{(L+p)^2 - m_1^2 + i0} \\
&= [c_\epsilon] \left[\frac{1}{\epsilon} + 2 - \log(m_0 m_1) + \frac{m_0^2 - m_1^2}{M^2} \log \frac{m_1}{m_0} \right. \\
&\quad \left. - \frac{m_0 m_1}{M^2} \left(\frac{1}{r} - r \right) \log r + \mathcal{O}(\epsilon) \right] , \quad \text{(B.7)}
\end{aligned}$$

where $p^2 = M^2$ and

$$r = X + \sqrt{X^2 - 1} , \quad \frac{1}{r} = X - \sqrt{X^2 - 1} , \quad X = \frac{m_0^2 + m_1^2 - M^2 - i0}{2m_0 m_1} . \quad \text{(B.8)}$$

We find the following limits,

$$B_0(0, m, m) = (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{\epsilon} - 2 \log m + \mathcal{O}(\epsilon) \right] ,$$

$$\begin{aligned}
B_0(0, m, 0) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{\epsilon} - 2 \log m + 1 + \mathcal{O}(\epsilon) \right], \\
B_0(0, m_0, m_1) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{\epsilon} - \frac{m_0^2}{m_0^2 - m_1^2} \log m_0^2 \right. \\
&\quad \left. + \frac{m_1^2}{m_0^2 - m_1^2} \log m_1^2 + 1 + \mathcal{O}(\epsilon) \right], \\
B_0(M, m, 0) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{\epsilon} + 2 - \frac{m^2}{M^2} \log m^2 \right. \\
&\quad \left. + \frac{m^2 - M^2}{M^2} \log(m^2 - M^2 - i0) + \mathcal{O}(\epsilon) \right], \\
B_0(M, 0, 0) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{\epsilon} + 2 - \log(-M^2 - i0) + \mathcal{O}(\epsilon) \right], \\
\lim_{\lambda \rightarrow 0} B_0(m, m, \lambda) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{\epsilon} + 2 - \log m^2 + \mathcal{O}(\epsilon) \right]. \tag{B.9}
\end{aligned}$$

In the present application, only the real parts of the integrals are relevant. For the derivative of the integral we have,

$$\begin{aligned}
B'_0(M, m, m) &\equiv \frac{\partial}{\partial p^2} B_0(M, m, m) \\
&= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{m^2}{M^4} \left(\frac{1}{r} - r \right) \log r - \frac{1}{M^2} \left(1 + \frac{r^2 + 1}{r^2 - 1} \log r \right) + \mathcal{O}(\epsilon) \right], \tag{B.10}
\end{aligned}$$

which has the following limits,

$$\begin{aligned}
B'_0(0, m, m) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[\frac{1}{6m^2} + \mathcal{O}(\epsilon) \right], \\
B'_0(M, 0, 0) &= (4\pi)^\epsilon \Gamma(1 + \epsilon) \left[-\frac{1}{M^2} + \mathcal{O}(\epsilon) \right]. \tag{B.11}
\end{aligned}$$

B.2 Box Integrals

The integrals required for the two-boson exchange amplitudes in Sect. 4.2.3 may be written in terms of the integral operators $\mathcal{I}_{\text{even}}$ and $\mathcal{I}_{\text{odd}}$ defined in (4.53) as

$$\begin{aligned}
J(m_V, M, \delta) &= \mathcal{I}_{\text{even}}(\delta, m_V) \frac{1}{L^2 - M^2 + i0}, \\
J^\mu(p, m_V, M, \delta) &= \mathcal{I}_{\text{even}}(\delta, m_V) \frac{1}{L^2 + 2L \cdot p - M^2 + i0} L^\mu \\
&= v \cdot p v^\mu J_1(m_V, M, \delta) + p^\mu J_2(m_V, M, \delta) + \mathcal{O}(p^3),
\end{aligned}$$

$$\begin{aligned}
J_-(p, m_V, M, \delta) &= -\mathcal{I}_{\text{odd}}(\delta, m_V) \frac{1}{L^2 + 2L \cdot p - M^2 + i0} \\
&= v \cdot p J_-(m_V, M, \delta) + \mathcal{O}(p^3), \\
J_-^\mu(m_V, M, \delta) &= -\mathcal{I}_{\text{odd}}(\delta, m_V) \frac{1}{L^2 - M^2 + i0} L^\mu = v^\mu J_{1-}(m_V, M, \delta). \tag{B.12}
\end{aligned}$$

Note that $J^\mu(p, m_V, M, \delta)$ and $J_-(p, m_V, M, \delta)$ vanish when p^μ vanishes since the integrands are then odd in L^μ . By standard manipulations, we may express the integrals J_1, J_2, J , and J_- , as

$$\begin{aligned}
J_1(m_V, M, \delta) &= -8[c_\epsilon](1 + \epsilon) \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx \rho^2 (1 - x) \\
&\quad [xm_V^2 + (1 - x)M^2 + \rho^2 + 2\rho\delta - i0]^{-2-\epsilon}, \\
J_2(m_V, M, \delta) &= 4[c_\epsilon] \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx (1 - x) \\
&\quad [xm_V^2 + (1 - x)M^2 + \rho^2 + 2\rho\delta - i0]^{-1-\epsilon}, \\
J(m_V, M, \delta) &= -4[c_\epsilon] \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx \\
&\quad [xm_V^2 + (1 - x)M^2 + \rho^2 + 2\rho\delta - i0]^{-1-\epsilon}, \\
J_-(m_V, M, \delta) &= 4[c_\epsilon] \frac{\partial}{\partial \delta} \frac{\partial}{\partial m_V^2} \int_0^\infty d\rho \int_0^1 dx (1 - x) \\
&\quad [xm_V^2 + (1 - x)M^2 + \rho^2 + 2\rho\delta - i0]^{-1-\epsilon}. \tag{B.13}
\end{aligned}$$

Let us introduce the integral

$$\hat{J}(m_V, M, \delta) = [c_\epsilon] \int_0^\infty d\rho \int_0^1 dx (1 - x) [xm_V^2 + (1 - x)M^2 + \rho^2 + 2\rho\delta - i0]^{-1-\epsilon}, \tag{B.14}$$

and write the above integrals in terms of $\hat{J}(m_V, M, \delta)$ as

$$\begin{aligned}
J_2(m_V, M, \delta) &= 4 \frac{\partial}{\partial m_V^2} \hat{J}(m_V, M, \delta), \\
J_-(m_V, M, \delta) &= 4 \frac{\partial}{\partial \delta} \frac{\partial}{\partial m_V^2} \hat{J}(m_V, M, \delta), \\
J(m_V, M, \delta) &= -4 \frac{\partial}{\partial m_V^2} [\hat{J}(m_V, M, \delta) + \hat{J}(M, m_V, \delta)], \\
J_1(m_V, M, \delta) &= 4 \frac{\partial}{\partial m_V^2} \left[-\hat{J}(m_V, M, \delta) + \frac{\partial}{\partial A} \hat{J}(m_V, M, \delta/A) \Big|_{A=1} \right]. \tag{B.15}
\end{aligned}$$

For J_{1-} , we may use the identity (4.57) to write

$$\begin{aligned}
 J_{1-}(m_V, M, \delta) &= -2 \int (dL) \frac{1}{(L^2 - m_V^2 + i0)^2} \frac{1}{L^2 - M^2 + i0} - \delta J(m_V, M, \delta) \\
 &= \frac{2[c_\epsilon] m_V^{-2-2\epsilon}}{\epsilon(1-\epsilon)} \left(1 - \frac{M^2}{m_V^2}\right)^{-2} \left[\epsilon + \frac{M^2}{m_V^2} \left(1 - \epsilon - \frac{m_V^{2\epsilon}}{M^{2\epsilon}}\right) \right] \\
 &\quad - \delta J(m_V, M, \delta).
 \end{aligned} \tag{B.16}$$

Having determined the above integrals in terms of $\hat{J}(m_V, M, \delta)$, it remains to compute this function. Let us write

$$\begin{aligned}
 \hat{J}(m_V, M, \delta) &= -\frac{[c_\epsilon]}{\epsilon} \frac{\partial}{\partial M^2} \int_0^\infty d\rho \int_0^1 dx [x m_V^2 + (1-x) M^2 + \rho^2 + 2\rho\delta - i0]^{-\epsilon} \\
 &= -\frac{[c_\epsilon]}{\epsilon} \frac{\partial}{\partial M^2} \int_0^\infty d\rho \frac{1}{m_V^2 - M^2} \frac{1}{1-\epsilon} \\
 &\quad \left\{ [m_V^2 + \rho^2 + 2\rho\delta - i0]^{1-\epsilon} - [M^2 + \rho^2 + 2\rho\delta - i0]^{1-\epsilon} \right\} \\
 &= -\frac{[c_\epsilon]}{\epsilon} \frac{\partial}{\partial M^2} \frac{1}{m_V^2 - M^2} \frac{1}{1-\epsilon} \left\{ m_V^{3-2\epsilon} f_1(\delta/m_V, 1-\epsilon) \right. \\
 &\quad \left. - M^{3-2\epsilon} f_1(\delta/M, 1-\epsilon) \right\},
 \end{aligned} \tag{B.17}$$

where

$$\begin{aligned}
 f_1(\delta, a) &= \int_0^\infty d\rho (1 + \rho^2 + 2\rho\delta - i0)^a \\
 &= (1 - \delta^2 - i0)^{a+\frac{1}{2}} \frac{\sqrt{\pi}}{2} \frac{\Gamma(-a - \frac{1}{2})}{\Gamma(-a)} - \delta^{2a+1} \int_0^1 dx [\delta^{-2} - 1 + x^2 - i0]^a.
 \end{aligned} \tag{B.18}$$

Although for the present application we require only $\delta > 0$, the expression is for general sign of δ . We presently need $f_1(\delta, a)$ for $a = 1 - \epsilon$, and hence consider

$$\begin{aligned}
 \frac{\sqrt{\pi}}{2} \frac{\Gamma(-\frac{3}{2} + \epsilon)}{\Gamma(-1 + \epsilon)} &= -\frac{2\pi}{3} \epsilon + \frac{2\pi}{9} (6 \log 2 - 5) \epsilon^2 + \mathcal{O}(\epsilon^3), \\
 \int_0^1 dx [\delta^{-2} - 1 + x^2 - i0]^{1-\epsilon} &= B^2 + \frac{1}{3} + \epsilon \left\{ \frac{2}{9} + \frac{4}{3} B^2 - \frac{4}{3} B^3 \operatorname{arccot} B \right. \\
 &\quad \left. - \left(B^2 + \frac{1}{3} \right) \log(B^2 + 1) \right\} + \epsilon^2 \left\{ \frac{4}{27} + \frac{20}{9} B^2 + \frac{4}{9} B^3 (6 \log 2B - 5) \operatorname{arccot} B \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{4}{3} B^3 i \left[\text{Li}_2 \left(\frac{1+iB}{1-iB} \right) - \text{arccot}^2 B + \frac{\pi^2}{12} \right] + \frac{1}{2} \left(B^2 + \frac{1}{3} \right) \log^2(B^2 + 1) \\
& - \left(\frac{4}{3} B^2 + \frac{2}{9} \right) \log(B^2 + 1) \Big\} + \mathcal{O}(\epsilon^3), \tag{B.19}
\end{aligned}$$

where $B^2 = 1/\delta^2 - 1 - i0$. For $B^2 > 0$, the bracket involving dilogarithm may be written

$$\begin{aligned}
i \left[\text{Li}_2 \left(\frac{1+iB}{1-iB} \right) - \text{arccot}^2 B + \frac{\pi^2}{12} \right] &= -\text{Im Li}_2 \left(\frac{1+iB}{1-iB} \right) \\
&= -\text{Cl}_2 \left[\arccos \left(\frac{1-B^2}{1+B^2} \right) \right], \tag{B.20}
\end{aligned}$$

where Cl_2 is the Clausen function of order two. The general expression is required for continuing to arbitrary mass parameters. Having determined $f_1(\delta, 1-\epsilon)$, we may proceed to compute $\hat{J}(m_V, M, \delta)$ using (B.17), and then $J_2(m_V, M, \delta)$, $J(m_V, M, \delta)$, $J_-(m_V, M, \delta)$ and $J_1(m_V, M, \delta)$ using (B.15), and $J_{1-}(m_V, M, \delta)$ using (B.16).

For $M = 0$, the expressions in (B.13), the expressions for $J_2(m_V, M, \delta)$, $J_1(m_V, M, \delta)$, and $J_-(m_V, M, \delta)$ in (B.15), and the expression for $J_{1-}(m_V, M, \delta)$ in (B.16), remain valid. The integral $J(m_V, 0, \delta)$ is now given by

$$J(m_V, 0, \delta) = -4[c_\epsilon] \frac{\partial}{\partial m_V^2} \left\{ -\frac{1}{\epsilon} m_V^{-1-2\epsilon} \left[f_1(\delta/m_V, -\epsilon) - f_0(\delta/m_V, -\epsilon) \right] \right\}, \tag{B.21}$$

and the integral $\hat{J}(m_V, 0, \delta)$ by

$$\begin{aligned}
\hat{J}(m_V, 0, \delta) &= \frac{[c_\epsilon] m_V^{-2}}{\epsilon} \int_0^\infty d\rho \left\{ (\rho^2 + 2\rho\delta - i0)^{-\epsilon} \right. \\
&\quad \left. - \frac{m_V^{-2}}{1-\epsilon} \left[(m_V^2 + \rho^2 + 2\rho\delta - i0)^{1-\epsilon} - (\rho^2 + 2\rho\delta - i0)^{1-\epsilon} \right] \right\} \\
&= \frac{[c_\epsilon] m_V^{-1-2\epsilon}}{\epsilon} \left\{ f_0(\delta/m_V, -\epsilon) - \frac{1}{(1-\epsilon)} [f_1(\delta/m_V, 1-\epsilon) \right. \\
&\quad \left. - f_0(\delta/m_V, 1-\epsilon)] \right\}, \tag{B.22}
\end{aligned}$$

where $f_1(\delta, a)$ is given by (B.18) and

$$f_0(\delta, a) = \int_0^\infty d\rho (\rho^2 + 2\rho\delta - i0)^a = \frac{\delta^{1+2a} \Gamma(1+a) \Gamma(-a - \frac{1}{2})}{2\sqrt{\pi}}. \tag{B.23}$$

We also need $f_1(\delta/m_V, a)$ for $a = -\epsilon$, which we may write as

$$f_1(\delta/m_V, -\epsilon) = \frac{1}{1-\epsilon} m_V^{-1+2\epsilon} \frac{\partial}{\partial m_V^2} \left[m_V^{3-2\epsilon} f_1(\delta/m_V, 1-\epsilon) \right]. \tag{B.24}$$

At vanishing residual mass, $\delta = 0$, only the integrals $J(m_V, M, 0)$, $J_1(m_V, M, 0)$ and $J_2(m_V, M, 0)$ are required, and from (B.13) they can be easily represented in closed form,

$$\begin{aligned}
 J(m_V, M, 0) &= [c_\epsilon] \frac{2\sqrt{\pi}}{(1-2\epsilon)} \frac{\Gamma(\frac{1}{2} + \epsilon)}{\Gamma(1 + \epsilon)} \frac{m_V^{1-2\epsilon}}{(M^2 - m_V^2)^2} \\
 &\quad \left[1 + 2\epsilon - 2 \left(\frac{M}{m_V} \right)^{1-2\epsilon} + (1-2\epsilon) \left(\frac{M}{m_V} \right)^2 \right], \\
 J_2(m_V, M, 0) &= -J_1(m_V, M, 0) = [c_\epsilon] \frac{4\sqrt{\pi}}{(3-2\epsilon)(1-2\epsilon)} \frac{\Gamma(\frac{1}{2} + \epsilon)}{\Gamma(1 + \epsilon)} \frac{m_V^{3-2\epsilon}}{(M^2 - m_V^2)^3} \\
 &\quad \left[1 + 2\epsilon - (3-2\epsilon) \left(\frac{M}{m_V} \right)^{1-2\epsilon} + (3-2\epsilon) \left(\frac{M}{m_V} \right)^2 \right. \\
 &\quad \left. - (1+2\epsilon) \left(\frac{M}{m_V} \right)^{3-2\epsilon} \right]. \tag{B.25}
 \end{aligned}$$

The result $J_2(m_V, M, 0) = -J_1(m_V, M, 0)$ follows from the observation that when $\delta = 0$ the identity in (4.57) implies $v_\mu J^\mu(p, m_V, M, 0) = 0$. The case $\delta = M = 0$ is simply obtained by substitution in (B.25).

B.3 Heavy Particle Integrals with Electroweak Polarization Tensor Insertion

The two-boson exchange amplitudes for gluon matching require the integrals $H(n)$, $F(n)$, $H^{\mu\nu}(n)$, and $H^\mu(n)$ defined in (4.56). Let us parameterize the last two as

$$H^{\mu\nu}(n) = H_1(n) v^\mu v^\nu + H_2(n) g^{\mu\nu}, \quad H^\mu(n) = H_3(n) v^\mu. \tag{B.26}$$

Upon contracting the above expressions with v_μ and $g_{\mu\nu}$, we may solve for the relations

$$\begin{aligned}
 H_1(n) &= \frac{1}{3-2\epsilon} [(4-2\epsilon) v_\mu v_\nu H^{\mu\nu}(n) - H^\mu_\mu(n)], \\
 H_2(n) &= \frac{1}{3-2\epsilon} [H^\mu_\mu(n) - v_\mu v_\nu H^{\mu\nu}(n)], \\
 H_3(n) &= v_\mu H^\mu(n). \tag{B.27}
 \end{aligned}$$

Using the identities in (4.54) and (4.57), we further obtain

$$\begin{aligned}
 v_\mu H^\mu(n) &= \delta H(n) + 2F(n), \\
 v_\mu v_\nu H^{\mu\nu}(n) &= \delta^2 H(n) + 2\delta F(n), \\
 H^\mu_\mu(n) &= \left[\frac{m_1^2}{x} + \frac{m_2^2}{(1-x)} \right] H(n) - \frac{H(n-1)}{x(1-x)}, \tag{B.28}
 \end{aligned}$$

and hence the boson loops are completely specified by $H(n)$ and $F(n)$. In evaluating these functions it may be advantageous to relate to more basic integrals by means of derivatives. Let us write,

$$\begin{aligned} H(n) &= 2 \frac{\partial}{\partial m_V^2} \int (dL) \frac{1}{v \cdot L - \delta + i0} \frac{1}{L^2 - m_V^2 + i0} \Delta^{-n-\epsilon}, \\ F(n) &= \frac{\partial}{\partial m_V^2} \int (dL) \frac{1}{L^2 - m_V^2 + i0} \Delta^{-n-\epsilon}, \end{aligned} \quad (\text{B.29})$$

with Δ as defined in (4.48). The singularity structure and evaluation of the above integrals can be classified into three cases, corresponding to zero, one, or two heavy fermions contributing to the electroweak polarization tensor. For pure states we obtain analytic expressions for all integrals, while for mixed states we encounter several integrals that require numerical evaluation of one Feynman parameter integral.

B.3.1 Case of Zero Heavy Fermions

Upon setting $m_1 = m_2 = 0$ in Δ and performing the integration in $d = 4 - 2\epsilon$ dimensions, we obtain

$$\begin{aligned} F(n) &= [c_\epsilon] \frac{\Gamma(2-n-2\epsilon)\Gamma(n+2\epsilon)}{\Gamma(2-\epsilon)\Gamma(1+\epsilon)} [x(1-x)]^{-n-\epsilon} m_V^{-2n-4\epsilon}, \\ H(n) &= [c_\epsilon] \frac{4\Gamma(n+2\epsilon)}{\Gamma(n+\epsilon)\Gamma(1+\epsilon)} [x(1-x)]^{-n-\epsilon} \frac{\partial}{\partial m_V^2} I(n), \end{aligned} \quad (\text{B.30})$$

where

$$I(n) = \int_0^1 dy (1-y)^{n-1+\epsilon} \int_0^\infty d\rho (\rho^2 + 2\rho\delta + ym_V^2 - i0)^{-n-2\epsilon}. \quad (\text{B.31})$$

We may reduce to the case of $I(1)$ by noticing that

$$\begin{aligned} I(n+1) &= -\frac{m_V^{-2}}{n+2\epsilon} \int_0^1 dy (1-y)^{n+\epsilon} \frac{d}{dy} \int_0^\infty d\rho (\rho^2 + 2\rho\delta + ym_V^2 - i0)^{-n-2\epsilon} \\ &= \frac{m_V^{-2}}{n+2\epsilon} \left[\int_0^\infty d\rho (\rho^2 + 2\rho\delta - i0)^{-n-2\epsilon} + (n+\epsilon)I(n) \right] \\ &= \frac{m_V^{-2}}{n+2\epsilon} \left[\delta^{1-2n-4\epsilon} \frac{\Gamma(1-n-2\epsilon)\Gamma(n-\frac{1}{2}+2\epsilon)}{2\sqrt{\pi}} + (n+\epsilon)I(n) \right]. \end{aligned} \quad (\text{B.32})$$

Finally, for $I(1)$ we require

$$I(1) = \delta^{-1-4\epsilon} \int_0^1 dy (1 + \epsilon \log(1-y) + \dots) \int_1^\infty d\rho (\rho^2 + \alpha^2)^{-1} \\ \left(1 - 2\epsilon \log(\rho^2 + \alpha^2) + \dots \right), \quad (\text{B.33})$$

where $\alpha = (ym_V^2/\delta^2 - 1 - i0)^{\frac{1}{2}}$. The relevant integrals are

$$\int_1^\infty d\rho \frac{1}{\rho^2 + \alpha^2} = \frac{1}{\alpha} \arctan \alpha, \\ \int_1^\infty d\rho \frac{\log(\rho^2 + \alpha^2)}{\rho^2 + \alpha^2} = \frac{1}{\alpha} \left[2 \log(2\alpha) \arctan \alpha \right. \\ \left. - \frac{1}{2i} \left(\text{Li}_2 \left(\frac{1-i\alpha}{1+i\alpha} \right) - \text{Li}_2 \left(\frac{1+i\alpha}{1-i\alpha} \right) \right) \right]. \quad (\text{B.34})$$

We perform the remaining integral over Feynman parameter y numerically.

B.3.2 Case of One Heavy Fermion

Let us set $m_1 = M$ (not to be confused with heavy WIMP mass M used elsewhere in the thesis) and $m_2 = 0$ in Δ , and consider separately the finite integrals for a - and c -type contributions, and the IR divergent integrals for b -type contributions.

Finite Integrals for a - and c -Type Contributions

For the finite a - and c -type contributions we may take $d = 4$. Let us evaluate the required integrals $F(2)$ and $H(1)$, and obtain the remaining integrals by differentiating with respect to M . We find

$$F(2) = \frac{i}{(4\pi)^2} \frac{\partial}{\partial m_V^2} \left\{ \left[x(1-x)m_V \left(1 - \frac{M^2}{xm_V^2} \right) \right]^{-2} \left[-\log \frac{M^2}{xm_V^2} + \frac{M^2}{xm_V^2} - 1 \right] \right\}, \\ H(1) = \frac{i}{(4\pi)^2} \frac{\partial}{\partial m_V^2} \left\{ 8 \left[x(1-x)m_V^2 \left(1 - \frac{M^2}{xm_V^2} \right) \right]^{-1} \left[\sqrt{m_V^2 - \delta^2} \arctan \right. \right. \\ \left. \left. \left(\sqrt{\frac{m_V^2}{\delta^2} - 1} \right) - \sqrt{\frac{M^2}{x} - \delta^2} \arctan \left(\sqrt{\frac{M^2}{x\delta^2} - 1} \right) - \frac{\delta}{2} \log \frac{xm_V^2}{M^2} \right] \right\}. \quad (\text{B.35})$$

The integrals have been obtained by breaking an integration region into pieces, e.g.,

$$\begin{aligned}
& \int_{\delta}^{\infty} d\rho \left[\log(\rho^2 + m_V^2 - \delta^2) - \log\left(\rho^2 + \frac{M^2}{x} - \delta^2\right) \right] \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\delta}^{\infty} d\rho \left[\log(\rho^2 + m_V^2 - \delta^2 - i\varepsilon) - \log\left(\rho^2 + \frac{M^2}{x} - \delta^2 - i\varepsilon\right) \right] \\
&= \delta \lim_{\varepsilon \rightarrow 0} \int_1^{\infty} d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right) \right] \\
&= \delta \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^{\infty} d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right) \right] \right. \\
&\quad \left. - \int_0^1 d\rho \left[\log\left(\rho^2 + \frac{m_V^2}{\delta^2} - 1 - i\varepsilon\right) - \log\left(\rho^2 + \frac{M^2}{x\delta^2} - 1 - i\varepsilon\right) \right] \right\}. \quad (\text{B.36})
\end{aligned}$$

Since the original integral is independent of ε , either choice of $\text{sgn}(\varepsilon)$ is correct provided it is used consistently in both terms. The continuation away from $\delta \rightarrow 0$ is thus obtained above by taking, e.g., $\delta \rightarrow \delta + i\varepsilon$ everywhere. For the evaluation of integrals over x involving $H(1)$, let us write

$$H(1) \equiv 2 \frac{\partial}{\partial m_V^2} K(1) \equiv 2 \frac{\partial}{\partial m_V^2} \left\{ \frac{M^2}{xm_V^2 - M^2} k(1) \right\}. \quad (\text{B.37})$$

We then have

$$x^n K(1) = \left(\frac{M^2}{m_V^2} \right)^n K(1) + \frac{\left(\frac{M^2}{m_V^2} \right)^n - x^n}{\frac{M^2}{m_V^2} - x} \frac{M^2}{m_V^2} k(1), \quad (\text{B.38})$$

so that all powers $x^n K(1)$ can be reduced to the case $n = 0$, in addition to the remaining straightforward integral involving a polynomial in x times $k(1)$, which in practice is evaluated numerically. The remaining integrals involving $F(2)$ are similarly straightforward to evaluate.

Infrared Divergent Integrals for b -Type Contributions

Let us now turn to the integrals for b -type contributions, where we work in $d = 4 - 2\varepsilon$ spacetime dimensions to account for singular behavior at the endpoints of the x integration. We find,

$$\begin{aligned}
F(1) = & [c_\epsilon][x(1-x)]^{-1-\epsilon} \frac{\Gamma(1+2\epsilon)}{[\Gamma(1+\epsilon)]^2} \left\{ m_V^{-2-4\epsilon} \left[(r^2-1)^{-2} \left(r^2 \log r^2 - r^2 + 1 \right) \right. \right. \\
& + \epsilon (r^2-1)^{-2} \left(2r^2 \log r^2 - r^2 \log^2 r^2 - r^2 + 1 + r^2 \text{Li}_2(1-r^2) \right) \Big] \\
& + m_V^{-2} \left[\left(\frac{r^2}{x} - 1 \right)^{-2} \left(\frac{r^2}{x} \log \frac{r^2}{x} - \frac{r^2}{x} + 1 \right) - (r^2-1)^{-2} \right. \\
& \left. \left. \left(r^2 \log r^2 - r^2 + 1 \right) \right] \right\}, \tag{B.39}
\end{aligned}$$

where $r \equiv M/m_V$. The first term in curly braces is obtained by taking $x = 1$ inside the $\int dy$ integral, and the second term is the remainder having no singularity in the final $\int dx$ integral at $x = 1$.

Similarly we find,

$$\begin{aligned}
H(1) = & [c_\epsilon][x(1-x)]^{-1-\epsilon} \frac{4\Gamma(1+2\epsilon)}{[\Gamma(1+\epsilon)]^2} \frac{\partial}{\partial m_V^2} \left\{ \delta^{-1-4\epsilon} \left[Y_0(1) + \epsilon (Y_1 + Y_2) \right] \right. \\
& \left. + \delta^{-1} \left[Y_0(x) - Y_0(1) \right] \right\}, \tag{B.40}
\end{aligned}$$

where

$$\begin{aligned}
Y_0(x) = & \frac{2}{r_V^2 - \frac{r_M^2}{x}} \left\{ \sqrt{r_V^2 - 1} \arctan \left(\sqrt{r_V^2 - 1} \right) - \sqrt{\frac{r_M^2}{x} - 1} \arctan \left(\sqrt{\frac{r_M^2}{x} - 1} \right) \right. \\
& \left. - \frac{1}{2} \log \frac{xm_V^2}{M^2} \right\}, \tag{B.41}
\end{aligned}$$

with $r_V \equiv m_V/\delta$ and $r_M \equiv M/\delta$. As in the discussion after (B.36), continuation away from $\delta = 0$ is given by taking $\delta \rightarrow \delta + i\varepsilon$ with arbitrary choice of $\text{sgn}(\varepsilon)$. The remaining terms Y_1 and Y_2 are given by

$$\begin{aligned}
Y_1 = & \int_0^1 dy \int_0^\infty d\beta (r_M^2 - r_V^2)^{-1} \frac{d}{d\beta} \log^2 \left[\beta^2 + 2\beta + yr_V^2 + (1-y)r_M^2 \right] \\
= & (r_M^2 - r_V^2)^{-1} \left\{ -4\pi \sqrt{r_V^2 - 1} \left[1 - \log \left(2\sqrt{r_V^2 - 1} \right) \right] \right. \\
& \left. + 4\pi \sqrt{r_M^2 - 1} \left[1 - \log \left(2\sqrt{r_M^2 - 1} \right) \right] - y_1 \left(\sqrt{r_V^2 - 1} \right) + y_1 \left(\sqrt{r_M^2 - 1} \right) \right\}, \\
Y_2 = & \int_0^1 dy \log(1-y) \left(yr_V^2 + (1-y)r_M^2 - 1 \right)^{-1} \arctan \left(\sqrt{yr_V^2 + (1-y)r_M^2 - 1} \right), \tag{B.42}
\end{aligned}$$

where

$$y_1(A) \equiv \int_0^1 dx \log^2(x^2 + A^2). \quad (\text{B.43})$$

For Y_2 , we evaluate the remaining integral over Feynman parameter y numerically.

B.3.3 Case of Two Heavy Fermions

Let us set $m_1 = m_2 = M$ (not to be confused with heavy WIMP mass M used elsewhere in the thesis) in Δ , and work in $d = 4$ dimensions. Naming $x(1-x) \equiv z$, we find,

$$\begin{aligned} F(1) &= \frac{i}{(4\pi)^2} \left[zm_V^2 \left(1 - \frac{M^2}{zm_V^2} \right)^2 \right]^{-1} \left[\frac{M^2}{zm_V^2} \log \frac{M^2}{zm_V^2} - \frac{M^2}{zm_V^2} + 1 \right], \\ H(1) &= \frac{i}{(4\pi)^2} \frac{\partial}{\partial m_V^2} \left\{ 8 \left[zm_V^2 \left(1 - \frac{M^2}{zm_V^2} \right) \right]^{-1} \left[\sqrt{m_V^2 - \delta^2} \arctan \left(\sqrt{\frac{m_V^2}{\delta^2} - 1} \right) \right. \right. \\ &\quad \left. \left. - \sqrt{\frac{M^2}{z} - \delta^2} \arctan \left(\sqrt{\frac{M^2}{z\delta^2} - 1} \right) - \frac{\delta}{2} \log \frac{zm_V^2}{M^2} \right] \right\}. \end{aligned} \quad (\text{B.44})$$

The remaining integrals can be obtained by differentiating the above results with respect to M . In practice, we evaluate the remaining integral over Feynman parameter x (or z) numerically.

B.4 Numerical Inputs

Table B.1 Inputs to the numerical analysis

Parameter	Value	Reference	Parameter	Value	Reference
$ V_{td} , V_{ts} $	~ 0	—	m_t	172 GeV	[87]
$ V_{tb} $	~ 1	—	m_b	4.75 GeV	[87]
m_e	0.511 MeV	[13]	m_c	1.4 GeV	[87]
m_μ	106 MeV	[13]	m_s	93.5 MeV	[13]
m_τ	1.78 GeV	[13]	m_d	4.70 MeV	[13]
m_h	126 GeV	[1, 24]	m_u	2.15 MeV	[13]
m_W	80.4 GeV	[13]	c_W	m_W/m_Z	—
m_Z	91.188 GeV	[13]	$\alpha_s(m_Z)$	0.118	[13]

We use the inputs of Table B.1 in the numerical analysis of coefficients appearing in Fig. 4.6. Light fermion masses enter the analysis indirectly via the onshell renormalization scheme. The matching in (4.21) requires a limit of the photon two-point function which receives contributions from momentum regions of light (u , d and s) quark loops that are outside the domain of validity of QCD perturbation theory. A complete nonperturbative treatment of this function is not numerically relevant to the present analysis; for definiteness, we model these contributions using $\overline{\text{MS}}$ light quark masses (cf. Table B.1) in the one-loop evaluation of the two-point function. Varying these mass inputs by an order of magnitude in either direction does not appreciably change the numerical matching coefficients of Fig. 4.6.

Appendix C

Inputs for Analysis of QCD Effects and Hadronic Matrix Elements

C.1 QCD Functions

The QCD beta function β , and the quark mass anomalous dimension γ_m , are defined as

$$\begin{aligned}\frac{\beta}{g} &= \frac{d \log g}{d \log \mu} = -\beta_0 \left(\frac{\alpha_s}{4\pi} \right) - \beta_1 \left(\frac{\alpha_s}{4\pi} \right)^2 - \beta_2 \left(\frac{\alpha_s}{4\pi} \right)^3 - \beta_3 \left(\frac{\alpha_s}{4\pi} \right)^4 + \dots, \\ \gamma_m &= \frac{d \log m_q}{d \log \mu} = \gamma_0 \left(\frac{\alpha_s}{4\pi} \right) + \gamma_1 \left(\frac{\alpha_s}{4\pi} \right)^2 + \gamma_2 \left(\frac{\alpha_s}{4\pi} \right)^3 + \gamma_3 \left(\frac{\alpha_s}{4\pi} \right)^4 \dots, \end{aligned} \quad (\text{C.1})$$

where the ellipses denote terms higher order in α_s , and the required functions are

$$\begin{aligned}\beta_0 &= 11 - 0.66667n_f \\ \beta_1 &= 102 - 12.6667n_f \\ \beta_2 &= 1428.50 - 279.611n_f + 6.01852n_f^2 \\ \beta_3 &= 29243 - 6946.30n_f + 405.089n_f^2 + 1.49931n_f^3 \end{aligned} \quad (\text{C.2})$$

and

$$\begin{aligned}\gamma_0 &= 8 \\ \gamma_1 &= 134.667 - 4.44445n_f \\ \gamma_2 &= 2498 - 292.367n_f - 3.45679n_f^2 \\ \gamma_3 &= 50659 - 9783.04n_f + 141.395n_f^2 + 2.96613n_f^3. \end{aligned} \quad (\text{C.3})$$

The strong coupling α_s is given as a function of scale as

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 t} \left\{ 1 - \frac{\beta_1 \log t}{\beta_0^2 t} + \frac{\beta_1^2}{\beta_0^4 t^2} \left[\left[\log t - \frac{1}{2} \right]^2 - \frac{5}{4} + \frac{\beta_0 \beta_2}{\beta_1^2} \right] - \frac{1}{\beta_0^6 t^3} \left[\beta_1^3 \left[\log^3 t - \frac{5}{2} \log^2 t - 2 \log t + \frac{1}{2} \right] + 3\beta_0 \beta_1 \beta_2 - \frac{1}{2} \beta_0^2 \beta_3 \right] \right\}, \quad (\text{C.4})$$

with

$$t = \log \left[\frac{\mu^2}{\Lambda(n_f)^2} \right], \quad \Lambda(5) = 0.213066, \quad \Lambda(4) = 0.297608, \quad \Lambda(3) = 0.339872. \quad (\text{C.5})$$

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