Byung Chan Eu

Kinetic Theory of Nonequilibrium Ensembles, Irreversible Thermodynamics, and Generalized Hydrodynamics

Volume 2. Relativistic Theories

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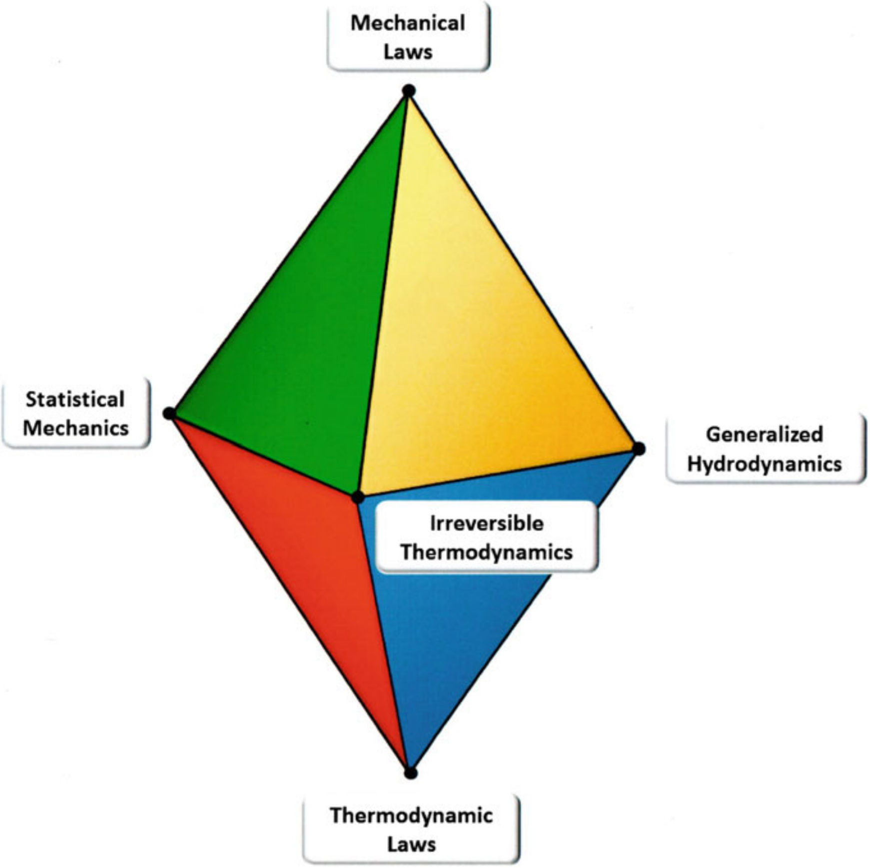

The tenet under which this work is carried out is figuratively summarized by the hexahedron displayed above: three vertices of subjects in the middle are constrained and in conformation to the two fundamental principles of physics, mechanical laws and laws of thermodynamics, at the two apices at the top and the bottom

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Volume 2. Relativistic Theories

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Preface

In Volume 2 of the monograph entitled *Kinetic Theory of Nonequilibrium Ensembles, Irreversible Thermodynamics, and Hydrodynamics*, relativistic theories are presented for dilute monatomic gases and photons by using covariant kinetic equations. Kinetic equations employed are covariant versions of Boltzmann kinetic equations and the Nordholm–Uehling–Uhlenbeck quantum kinetic equations applied to a mixture of material gases and photon gases. By using the aforementioned kinetic equations, we develop kinetic theories for relativistic irreversible thermodynamics and hydrodynamics in a parallel manner to the nonrelativistic theories we have presented in Volume 1 of the monograph. The materials were based on the manuscripts written in collaboration with Dr. Kefei Mao many years ago, which also made up a part of his Ph.D. dissertation, McGill University, Montreal, 1993. I would like to thank Dr. K. Mao for collaboration.

Montreal
March 2016

Byung Chan Eu

Contents

Chapter 1
Relativistic Kinetic Theory for Matter

In this chapter we make an important departure from the approach taken in Chaps. 3, 5, 6 and 7 of Volume 1 in which nonrelativistic kinetic equations have been discussed for gases and liquids. Now we consider relativistic kinetic equations for dilute uncorrelated particle systems. Specifically, our objective is to consider matter or a system of radiation and matter consisting of monatomic gas mixtures in the relativistic range of speed. In a previous paper [1] on radiation and matter we formulated a nonrelativistic kinetic theory of mixture of photon gas and matter in interaction with photons. But nonrelativistically treating photons presented awkward features theoretically. The principal motivation for the present relativistic kinetic theories is in our desire to remove the awkward features of treating radiation and matter in a nonrelativistic kinetic theory in our previous work [1]. There also are some important problems, particularly, in astrophysics and nuclear physics in which particles (e.g., electrons, neutrinos, protons, neutrons, and other elementary particles) move at high speed for which a relativistic treatment would be preferable to understand their transport properties in the early epoch of the universe [2]. Apart from this practical aspect of application of the theory we intend to formulate in the relativistic domain, there is also the desire to cover as widely as possible the range of applicability of the kinetic theory methodology we have pursued in the nonrelativistic domain of speed, especially, in connection with thermodynamics of irreversible processes and attendant hydrodynamics. Since it is simpler to first treat matter alone before considering radiation interacting with matter, we will separate the material gas from radiation and treat the subject relativistically by using the better known relativistic Boltzmann kinetic equation [3, 4] for dilute gases in Sect. 1.1 of this Volume. For readers less knowledgeable of the relativity theory and related mathematics Supplementary Notes on four-tensors and four-vectors are provided at the end of Sect. 1.1. Then in Chap. 2 a relativistic kinetic theory of radiation (photons) interacting with matter is formulated by using the covariant version of the Boltzmann–Nordholm–Uehling–Uhlenbeck (BNUU) kinetic equation [5, 6] for a mixture of quantum

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particles. In this chapter a generalized hydrodynamic theory is developed and a theory of radiative transport processes in the mixture od quantum particles including photons is formulated. Various radiative transport coefficients are derived for the mixture. The kinetic theory formulas so developed in Chap. 2 are explicitly applied to radiative transport processes in the mixture of material gases and photons. Especially, the transport coefficients are computed to compare with some experimental results in the third chapter. The author believes the last chapter may serve as an experimental validation of the theory developed in the second chapter of this volume.

1.1 Relativistic Boltzmann Equation for a Monatomic Gas Mixture

We consider an r-component mixture of relativistic monatomic gases in nonequilibrium state. The gas mixture is assumed to be sufficiently dilute so that particles are distributed uncorrelated to a good approximation in the phase space. The primary quantity of interest is then singlet distribution functions of particles assumed to describe a system of relativistic gases obeying the relativistic Boltzmann equations. In contrast to the nonrelativistic kinetic theory formalism, vectors and tensors in four-dimensional spacetime (simply, spacetime henceforth) must be used in relativistic kinetic theory. Before the kinetic equation is introduced, it is useful to fix the notation and convention for vectors and tensors appearing in the theory. For readers not well versed in algebras involving four-vectors and four-tensors a section on Supplementary Notes is appended at the end of this chapter, which the author hopes would help the readers to follow through the relativistic kinetic theory presented in this and following chapters.

1.1.1 Preparation

A point x in spacetime is denoted by a covariant four-vector

$$x := x^{\mu} = (ct, \mathbf{r}) \,, \tag{1.1}$$

where c is the speed of light, t is time, $\mathbf{r}$ is the three-vector for spatial position, and superscript index μ ranges 0, 1, 2, 3. Here 0 is for the time component and the arabic indices $(1, 2, 3)$ are for the space components. The conjugate four-momentum is denoted by

$$p_a := p_a^{\mu} = \left(p_a^0, \mathbf{p}_a\right) , \quad p_a^0 = \left(m_a^2 c^2 + \mathbf{p}_a^2\right)^{1/2} \tag{1.2}$$

with subscript a distinguishing the species. The covariant gradient operator ∂_{μ} is enumerated in spacetime by

$$\partial_\mu = (c^{-1}\partial_t, \nabla), \tag{1.3}$$

where the symbols ∂_t and ∇ are, respectively, defined by time derivative and spatial derivative operators

$$\partial_t = \partial/\partial t, \quad \nabla = \partial/\partial \mathbf{r}. \tag{1.4}$$

The following convention is adopted for the metric tensor $\boldsymbol{g}^{\mu\nu}$:

$$\boldsymbol{g}^{\mu\nu} = dia\boldsymbol{g}\,(1, -1, -1, -1)\,, \tag{1.5}$$

which abbreviates a diagonal 4 × 4 matrix.

The hydrodynamic velocity is denoted by $U^\mu(x)$, which will be more precisely defined later at an appropriate stage. Then associated with the metric tensor in (1.5) and hydrodynamic velocity U^μ is the projector[1] $\Delta^{\mu\nu}$:

$$\Delta^{\mu\nu}(x) = \boldsymbol{g}^{\mu\nu} - c^{-2}U^\mu(x)\,U^\nu(x)\,. \tag{1.6}$$

Here $U^\mu(x)$ is normalized by scaling it with speed of light c

$$c^{-2}U^\mu U_\mu = 1. \tag{1.7}$$

This projector tensor $\Delta^{\mu\nu}$ has the following properties:

$$\Delta^{\mu\nu} = \Delta^{\nu\mu}, \tag{1.8}$$

$$\Delta^{\mu\nu}\Delta_{\nu\sigma} = \Delta^\mu_\sigma, \tag{1.9}$$

$$\Delta^\mu_\mu = 3, \tag{1.10}$$

$$\Delta^{\mu\nu}U_\nu = 0. \tag{1.11}$$

These properties[2] will be frequently used when relativistic macroscopic evolution equations, such as hydrodynamic equations, are derived from the relativistic Boltzmann kinetic equation. Some identities derived from these properties and the hydrodynamic velocity four-vector are noted later in this chapter.

1.1.2 Relativistic Covariant Kinetic Equation

In relativistic kinetic theory, it is assumed that the covariant Boltzmann equation [3] is obeyed by the singlet distribution function $f_a(x, p_a)$ of species a for a relativistic monatomic dilute gas mixture of r components. It may be written in the form

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[1] The sign of the projector defined here is opposite to the projector used in Ref. [4].

[2] Property (1.10) in Ref. [4] is in error with regard to the sign.

$$p_a^\mu \partial_\mu f_a(x, p_a) = \sum_{b=1}^{r} C_{ab}(f_a, f_a) \quad (a = 1, 2, \ldots, r), \tag{1.12}$$

where the collision integral $C_{ab}(f_a, f_a)$ is given by the formula

$$\begin{aligned} C_{ab}(f_a, f_a) = G_{ab} \int\int\int & d^3\overline{\mathbf{p}}_b d^3\overline{\mathbf{p}}_a^* d^3\overline{\mathbf{p}}_b^* W_{ab}(p_a p_b | p_a^* p_b^*) \\ & \times \left[f_a^* \left(x, p_a^*\right) f_b^* \left(x, p_b^*\right) - f_a \left(x, p_a\right) f_b \left(x, p_b\right) \right] \end{aligned} \tag{1.13}$$

with the definitions of abbreviations

$$d^3\overline{\mathbf{p}}_b = \frac{d^3\mathbf{p}_b}{p_b^0}, \quad d^3\overline{\mathbf{p}}_b^* = \frac{d^3\mathbf{p}_b^*}{p_b^{0*}}, \text{ etc.}$$

and also with $W_{ab}(p_a p_b | p_a^* p_b^*)$ denoting the transition rate from the initial state (p_a, p_b) to the final state $\left(p_a^*, p_b^*\right)$ as a result of a collision between particles a and b. The factor G_{ab} defined by

$$G_{ab} = 1 - \delta_{ab}/2 \tag{1.14}$$

ensures that the final state is not counted twice. The asterisk denotes the post-collision value on completion of collision. The subscripts a and b are assigned to play a dual role of labeling a species and the particle state of that species. The transition rate is a scalar under Lorentz transformation [7] and obeys the microscopic reversibility (detailed balance)

$$W_{ab}(p_a p_b | p_a^* p_b^*) = W_{ba}(p_a^* p_b^* | p_a p_b) \tag{1.15}$$

as a consequence of time-reversal invariance of the dynamical equations of motion governing the collision dynamics. This symmetry property does not mean that the collision integral $C_{ab}(f_a, f_b)$ is symmetric with respect to the reversal of a collision process.

The symmetry property (1.15) is important for proving the H theorem as will be discussed later. It is important to note that the singlet distribution functions in the kinetic equation (1.12) are *coarse-grained in space over the intermolecular interaction range of particles*, so that the singlet distribution functions remain spatially unchanged over the collision volume in the course of a collision of particle pairs. Therefore their x dependence must be understood in the sense of the aforementioned coarse-graining. It is also important to recognize that the relativistic Boltzmann equation breaks the time reversal invariance and thus is irreversible, and this is a crucially important property of the relativistic Boltzmann kinetic equation postulated. It is remarked here that we do not pretend the kinetic equation is derived from the mechanical equations of motion of the particles comprising the gas. It is postulated and the results of *the postulate* is *a posteriori* verified to be true in the light of experiment.

1.1.3 Mean Values and Macroscopic Variables

Equipped with the distribution function it is possible to calculate statistical mechanical mean values for macroscopic observables including hydrodynamic variables. First of all, the distribution function is normalized to the number density $n_a(x)$ of species:

$$n_a(x) = \int d^3\mathbf{p}_a f_a\left(x, p_a\right). \tag{1.16}$$

This number density should not be confused with the hydrodynamic number density defined later. The particle flux of species a is given in the same manner as for $n_a(x)$ by the statistical mechanical formula

$$\mathbf{j}_a(x) := n_a\mathbf{u}_a = \int d^3\mathbf{p}_a\mathbf{v}_a f_a\left(x, p_a\right), \tag{1.17}$$

where $\mathbf{v}_a$, defined by

$$\mathbf{v}_a = \frac{c\mathbf{p}_a}{p^0}, \tag{1.18}$$

is the velocity of particle a. With the two quantities n_a and $\mathbf{u}_a$ a covariant vector, namely, particle four-flow, can be constructed and statistically calculated by using the formula

$$N_a^{\mu}\left(x\right) = n_a\left(c, \mathbf{u}_a\right) := \left\langle p_a^{\mu} f_a\left(x, p_a\right)\right\rangle. \tag{1.19}$$

Here the angular brackets are the abbreviation of the integral

$$\langle\cdots\rangle = c\int\frac{d^3\mathbf{p}_a}{p_a^0}\cdots := c\int d^3\overline{\mathbf{p}}_a\cdots. \tag{1.20}$$

This notation will be used throughout the present and following chapters. Similarly to $N_a^{\mu}\left(x\right)$, the covariant energy-momentum tensor $T_a^{\mu\nu}$ of species a is defined by the statistical mechanical formula

$$T_a^{\mu\nu} := \left\langle p_a^{\mu} p_a^{\nu} f_a\left(x, p_a\right)\right\rangle. \tag{1.21}$$

The mean values of different species are evidently additive over the species variable field in view of the absence of spatial correlation effects owing to the fact that the gas components are ideal. Therefore, the total number four-flow and total energy-momentum are, respectively, given by the sums

$$N^{\mu} = \sum_{a=1}^{r} N_a^{\mu} \tag{1.22}$$

and

$$T^{\mu\nu} = \sum_{a=1}^{r} T_a^{\mu\nu}. \tag{1.23}$$

The mean value of an observable $M_a(x, p_a)$ of species a will be expressed by the statistical mechanical formula

$$\langle M_a(x, p_a)\rangle = c\int d^3\overline{\mathbf{p}}_a M_a(x, p_a). \tag{1.24}$$

Before proceeding further, it would be useful to show that $f_a(x, p_a)$ is indeed a scalar under the Lorentz transformation. To show it, we first note

$$\delta\left(p^2 - m^2c^2\right) = \frac{1}{2p^0}\left[\delta\left(p^0 - \sqrt{m^2c^2 + \mathbf{p}^2}\right) + \delta\left(p^0 + \sqrt{m^2c^2 + \mathbf{p}^2}\right)\right].$$

Then, since

$$\theta\left(p^0\right)\delta\left(p^2 - m^2c^2\right) = \frac{1}{2p^0}\delta\left(p^0 - \sqrt{m^2c^2 + \mathbf{p}^2}\right) \tag{1.25}$$

for $p^0 - \sqrt{m^2c^2 + \mathbf{p}^2} \geq 0$, where $\theta\left(p^0\right)$ is the Heaviside step function defined by

$$\theta\left(p^0\right) = \begin{cases} 0 \text{ for } p^0 < 0 \\ 1 \text{ for } p^0 \geq 0 \end{cases},$$

the particle number four-flow in (1.19) can be written as

$$N_a^\mu(x) = 2\int d^4p_a\theta\left(p_a^0\right)\delta\left(p_a^2 - m_a^2c^2\right)p_a^\mu f_a(x, p_a). \tag{1.26}$$

The number four-flow $N_a^\mu(x)$ transforms like a four-velocity, provided that $f_a(x, p_a)$ is a scalar under the Lorentz transformation. So, the distribution functions will be treated as scalars.

1.1.4 Hydrodynamic Velocity

Unlike in the nonrelativistic kinetic theory in the chapters of Volume 1, the hydrodynamic velocity is not simply given by mean velocity $\mathbf{u}_a$ in (1.17) or its sum $\mathbf{u}$ over the species index in the case of a mixture.

There are two different definitions for hydrodynamic velocity in the literature. One is due to Eckart [8] and the other to Landau and Lifshitz [9]. In this work the Eckart definition will be adopted since it is simpler and more closely in line with the nonrelativistic counterpart. It is defined by

$$U^\mu = \frac{cN^\mu}{\sqrt{N^\nu N_\nu}}. \tag{1.27}$$

This definition clearly satisfies the normalization condition (1.7). With the definition of hydrodynamic velocity given by (1.27) we are naturally led to the hydrodynamic density ρ_a of species a. To see it, form the scalar product of U_μ with N^μ to obtain

$$c^{-2}N^\mu U_\mu = c^{-1}\sqrt{N^\nu N_\nu}. \tag{1.28}$$

The number four-flow may be enumerated in the notation of four-flow as a four-vector

$$N^\nu = n(c, \mathbf{u})^\nu, \tag{1.29}$$

where the superscript ν denotes the spacetime component of the four-vector and n is the total number density

$$n = \sum_{a=1}^{r} n_a \tag{1.30}$$

with n_a denoting the number density of a defined by (1.16), and $\mathbf{u}$ denoting the total particle velocity defined by

$$n\mathbf{u} = \sum_{a=1}^{r} n_a \mathbf{u}_a \tag{1.31}$$

with $\mathbf{u}_a$ given by (1.17). Accordingly, the total hydrodynamic number density ρ may be related to the mean velocity $\mathbf{u}$ obtained as follows, if the scalar product of N^ν with itself is taken:

$$c^{-1}\sqrt{N^\nu N_\nu} = n\sqrt{1 - \frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}. \tag{1.32}$$

This expression motivates to define hydrodynamic density ρ by the expression

$$\rho = n\sqrt{1 - \frac{\mathbf{u}\cdot\mathbf{u}}{c^2}}. \tag{1.33}$$

It clearly indicates that hydrodynamic density ρ is not the same as the number density n, to which the former tends as $u/c \to 0$, namely, the nonrelativistic limit. A clear distinction should be made between n and ρ to avoid possible confusion that might arise in connection with relativistic hydrodynamic equations and their nonrelativistic counterparts derived from the nonrelativistic Boltzmann equation in which ρ simply means the mass density. According to (1.33), the hydrodynamic density gets thinner than the number density n as the speed u increases toward c, vanishing[3] as $u \to c$.

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[3] This aspect seems to indicate that the vacuum may be described by hydrodynamic equations in the limit of $u/c \to 0$ in relativity.

From (1.27), (1.29), and (1.32) follows that the hydrodynamic four-velocity U^μ can be expressed in a more insightful form

$$U^\mu = \gamma \left(c, \mathbf{u}\right)^\mu , \tag{1.34}$$

where γ is given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.35}$$

This gives a lucid meaning to the hydrodynamic four-velocity in Eckart's definition. In contrast to the behavior of ρ, the hydrodynamic velocity increases in magnitude as u increases toward c.

Inserting ρ into (1.28) yields the hydrodynamic density expressed as

$$\rho = c^{-2} N^\mu U_\mu. \tag{1.36}$$

This formula motivates us to define species hydrodynamic density as

$$\rho_a = c^{-2} U_\mu N_a^\mu, \tag{1.37}$$

which on summing over all species yields the total hydrodynamic density ρ in (1.36) because

$$\rho = \sum_{a=1}^{r} \rho_a = c^{-2} U_\mu N^\mu. \tag{1.38}$$

Rearranging (1.27) and using (1.33) we obtain

$$N^\mu = \rho U^\mu, \tag{1.39}$$

which also implies

$$N_a^\mu = \rho_a U^\mu. \tag{1.40}$$

In fact, there should be generally another term, say, $\mathcal{V}^\mu$ in (1.39) such that $\mathcal{V}^\mu U_\mu = 0$. The term $\mathcal{V}^\mu$ is equal to zero in the Eckart definition of hydrodynamic velocity. The relations listed above will be found useful for deriving various evolution equations for macroscopic variables from the covariant kinetic equation.

1.1.5 Energy-Momentum Tensor and Related Observables

Various relevant macroscopic variables such as energy density, heat flux, and so forth can be defined in covariant form with the help of the hydrodynamic velocity and projectors by using the energy-momentum tensor $T_a^{\mu\nu}$. The scalar energy density $\mathcal{E}_a$ of species a is given by

$$E_a = \rho_a \mathcal{E}_a = c^{-2} U_\mu T_a^{\mu\nu} U_\nu, \tag{1.41}$$

where $\mathcal{E}_a$ is the energy density of species a, and the scalar (nonequilibrium) pressure $\mathfrak{p}_a$ by

$$\mathfrak{p}_a = \frac{1}{3} \Delta_{\mu\nu} T_a^{\mu\nu}. \tag{1.42}$$

That is, $\mathfrak{p}_a$ is the trace of the energy-momentum tensor divided by 3. The nonequilibrium enthalpy density may then be defined by the formula reminiscent of its equilibrium counterpart

$$\mathfrak{h}_a = \mathcal{E}_a + \mathfrak{p}_a / \rho_a = \mathcal{E}_a + \mathfrak{p}_a v_a, \tag{1.43}$$

where v_a is the specific volume defined by

$$v_a = 1/\rho_a. \tag{1.44}$$

We emphasize that both $\mathcal{E}_a$ and $\mathfrak{p}_a$ are of nonequilibrium and mechanical. As they stand, both $\mathcal{E}_a$ and $\mathfrak{p}_a$ are mechanical and hence so is $\mathfrak{h}_a$. Their thermal aspect is not apparent.

The heat flux (four-flow) Q_a^μ is also expressed in terms of a projection of energy-momentum tensor $T_a^{\nu\sigma}$ as follows:

$$Q_a^\mu = U_\nu T_a^{\nu\sigma} \Delta_\sigma^\mu. \tag{1.45}$$

It is useful to define the number diffusion flux (four-flow) J_a^μ by the expression

$$J_a^\mu = N_a^\mu - \mathfrak{c}_a N^\mu, \tag{1.46}$$

where $\mathfrak{c}_a$ is the number fraction defined by the hydrodynamic density ratio

$$\mathfrak{c}_a = \rho_a / \rho. \tag{1.47}$$

It should be remarked that in the relativistic kinetic theory the diffusion fluxes are defined in terms of the number fluxes instead of the mass fluxes generally used in the nonrelativistic kinetic theory, but J_a^μ is not a projection of $T_a^{\nu\sigma}$ unlike Q_a^μ or the pressure tensor $P_a^{\nu\sigma}$ discussed later. The reason is to distinguish diffusion fluxes from heat fluxes, or energy fluxes, because of the equivalence of mass and energy in relativity. However, it will be found convenient to use a modified heat flux defined by

$$Q_a'^\mu = Q_a^\mu - \mathfrak{J}_a^\mu, \tag{1.48}$$

where $\mathfrak{J}_a^\mu$ is a mechanical energy diffusion flux defined by

$$\mathfrak{J}_a^\mu = hJ_a^\mu. \tag{1.49}$$

Here h is the equilibrium enthalpy of the entire fluid; it is the equilibrium limit of $\mathfrak{h} = \sum_{a=1}^r \mathfrak{h}_a$. Therefore the nonequilibrium aspect of $\mathfrak{J}_a^\mu$ is borne by J_a^μ of species a. The definitions in (1.48) and (1.49) are motivated by the notion that heat in thermodynamics is not simply a form of mechanical energy flow, but an excess of energy carried by the material particle diffusing in matter; the latter part being represented by hJ_a^μ. Therefore $Q_a'^\mu$ is thought to inherently represent the "heat" as what we intuitively mean in our everyday life, it being a quantity intimately tied up with the second law of thermodynamics.

The nonequilibrium pressure $\mathfrak{p}_a$ is generally different from the *hydrostatic* (equilibrium) pressure of species a. It will be presently defined more precisely together with the pressure tensor and the related quantities. In the molecular theory of heat, what is just said is that which we mean by heat, if we want to identify it with the heat in macroscopic (thermodynamic) theory of heat; see the nonrelativistic theories part of this work in the previous volume. As will be seen later, it will be indeed also the case in the relativistic description of heat when irreversible thermodynamics is formulated later in this chapter.

Clearly, from the definitions of heat flux and diffusion flux it can be shown that there hold the following frequently used identities:

$$U_\mu Q_a^\mu = 0, \qquad U_\mu J_a^\mu = 0. \tag{1.50}$$

These identities can be easily verified by using the properties of projector and U^μ. Consequently, there also follows the identity[4]

$$U_\mu Q_a'^\mu = 0. \tag{1.51}$$

The total heat flux (four-flow) Q^μ is a sum of Q_a^μ over species:

$$Q^\mu = \sum_{a=1}^r Q_a^\mu \tag{1.52}$$

and similarly

$$Q'^\mu = \sum_{a=1}^r Q_a'^\mu. \tag{1.53}$$

However, owing to the definition of J_a^μ the diffusion fluxes are not all independent of each other since there holds the relation

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[4] Identities (1.50) and (1.51) imply that heat and diffusion fluxes are orthogonal to the hydrodynamic velocity U^μ.

$$\sum_{a=1}^{r} J_a^\mu = 0. \tag{1.54}$$

This implies that if the fluid is of a single component, there is no diffusion flux present.

The stress (pressure) tensor $P_a^{\mu\nu}$ is defined as a projection of the energy-momentum tensor onto spatial components

$$P_a^{\mu\nu} = \Delta_\sigma^\mu T_a^{\sigma\tau} \Delta_\tau^\nu. \tag{1.55}$$

In order to more clearly reveal its physical meaning we decompose $P_a^{\mu\nu}$ into the traceless symmetric part $\Pi_a^{\mu\nu}$, the excess trace (i.e., excess normal stress) part $\widetilde{\Delta}_a$, and the hydrostatic pressure p_a of species a into the form

$$P_a^{\mu\nu} = p_a \Delta^{\mu\nu} + \widetilde{\Delta}_a \Delta^{\mu\nu} + \Pi_a^{\mu\nu}, \tag{1.56}$$

where the hydrostatic pressure is defined in terms of the equilibrium energy-momentum tensor denoted $T_{\mathrm{e}a}^{\alpha\beta}$ by the formula

$$p_a = \frac{1}{3} \Delta_{\alpha\beta} T_{\mathrm{e}a}^{\alpha\beta}, \tag{1.57}$$

then the excess normal stress $\widetilde{\Delta}_a$ is defined by a fluctuation of $\mathfrak{p}_a$ from the hydrostatic pressure denoted by p_a

$$\widetilde{\Delta}_a = \frac{1}{3} \Delta_{\alpha\beta} T_a^{\alpha\beta} - p_a = \mathfrak{p}_a - p_a. \tag{1.58}$$

The equilibrium (more precisely, local equilibrium) energy-momentum tensor $T_{\mathrm{e}a}^{\alpha\beta}$ may be calculated with the equilibrium distribution function f_a^{e} with the statistical mechanical formula

$$T_{\mathrm{e}a}^{\mu\nu} = \left\langle p_a^\mu p_a^\nu f_a^{\mathbf{e}} \right\rangle. \tag{1.59}$$

The equilibrium distribution function $f_a^{\mathbf{e}}$ is the equilibrium solution of the covariant kinetic equation. The precise form of $f_a^{\mathbf{e}}$ will be obtained later at a more appropriate stage in the development of the theory under discussion.

The traceless symmetric part $\Pi_a^{\mu\nu}$ of $T_a^{\sigma\tau}$ is related to the viscous phenomena whereas the excess trace part $\widetilde{\Delta}_a$ is associated with dilatation/compression of the fluid (gas) over and above the hydrostatic value, that is, a fluctuation of pressure from the equilibrium value. The term $\frac{1}{3}\Delta_{\mu\nu}T_a^{\mu\nu}$ in (1.42) is the normal stress (trace part of the stress). Therefore $\widetilde{\Delta}_a$ may be regarded as the normal stress in excess of hydrostatic (equilibrium) pressure (force per unit area)—the equilibrium value of normal stress. We emphasize that p_a in (1.57) is an equilibrium property. Therefore $\widetilde{\Delta}_a$ is a purely nonequilibrium component of the normal stress.

The component stress tensors add up to the total stress tensor made up of the following three contributions, each composed of species components:

$$\Pi^{\mu\nu} = \sum_{a=1}^{r} \Pi_a^{\mu\nu}, \quad \widetilde{\Delta} = \sum_{a=1}^{r} \widetilde{\Delta}_a \tag{1.60}$$

and $\mathfrak{p}$, the total pressure, given by the formula

$$\mathfrak{p} = \sum_{a=1}^{r} \mathfrak{p}_a. \tag{1.61}$$

The energy-momentum tensor $T_a^{\mu\nu}$ can then be decomposed into the four major components

$$\begin{aligned} T_a^{\mu\nu} &= c^{-2} E_a U^\mu U^\nu + c^{-2} \left(Q_a^\mu U^\nu + Q_a^\nu U^\mu \right) \\ &\quad + c^{-2} \left(\mathfrak{J}_a^\mu U^\nu + \mathfrak{J}_a^\nu U^\mu \right) + P_a^{\mu\nu}. \end{aligned} \tag{1.62}$$

On summing this expression over species it follows

$$T^{\mu\nu} = c^{-2} E U^\mu U^\nu + c^{-2} \left(Q^\mu U^\nu + U^\mu Q^\nu \right) + P^{\mu\nu}, \tag{1.63}$$

for which (1.54) is made use of for the third term on the right in (1.62). We note that, energy E_a being written in terms of energy density $\mathcal{E}_a$,

$$E_a = \rho_a \mathcal{E}_a, \tag{1.64}$$

summing it over species gives the total energy E:

$$E := \rho \mathcal{E} = \sum_{a=1}^{r} \rho_a \mathcal{E}_a. \tag{1.65}$$

Therefore, with the definition of density fraction

$$\mathfrak{c}_a = \frac{\rho_a}{\rho} \tag{1.66}$$

the total energy density $\mathcal{E}$ is given by the sum of partial energy densities times the density fraction $\mathfrak{c}_a$:

$$\mathcal{E} = \sum_{a=1}^{r} \mathfrak{c}_a \mathcal{E}_a. \tag{1.67}$$

The hydrodynamic variables presented above obey the relativistic hydrodynamic equations obtained from the covariant Boltzmann equations that we have postulated for the relativistic gaseous mixture under consideration.

1.1.6 Relativistic Generalized Hydrodynamic Equations

Balance Equations for Conserved Variables

The particle number and the energy-momentum tensor are collision invariants of the relativistic Boltzmann collision integral, and their evolution equations turn out to be conservation laws of number density and energy-momentum. They are easily derived from the covariant Boltzmann equation. Upon operating contravariant derivative ∂_μ, namely, differentiating (1.19) for N^μ and (1.21) for $T^{\mu\nu}$ and using the covariant Boltzmann equation (1.12), the covariant balance equations for the particle number and energy-momentum are, respectively, obtained.

The particle number balance equation is given by

$$\partial_\mu N^\mu = 0, \tag{1.68}$$

and the energy-momentum balance equation by

$$\partial_\nu T^{\mu\nu} = 0. \tag{1.69}$$

Note that because N^μ and $T^{\mu\nu}$ are collision invariants of the covariant kinetic equation, there is no source term in the balance Eqs. (1.68) and (1.69) arising from the kinetic equation. Traditional fluid dynamic equations—conservation laws with appropriate constitutive relations—can be obtained from these balance equations, provided that the constitutive equations are supplied for the stress tensors, heat fluxes, and diffusion fluxes. For the purpose of obtaining them the covariant derivative ∂^μ is decomposed into time-like and space-like parts with the hydrodynamic velocity. Since

$$\partial^\mu = \boldsymbol{g}^{\mu\nu}\partial_\nu,$$

by eliminating the metric tensor with the help of the projector $\Delta^{\mu\nu}$ and defining the new symbols for derivatives

$$D = U^\nu \partial_\nu, \tag{1.70}$$

$$\nabla^\mu = \Delta^{\mu\nu}\partial_\nu, \tag{1.71}$$

the covariant derivative operator can be decomposed into two components in the form

$$\partial^\mu = c^{-2}U^\mu D + \nabla^\mu. \tag{1.72}$$

Similarly, the contravariant derivative can be decomposed as

$$\partial_\mu = c^{-2} U_\mu D + \nabla_\mu , \tag{1.73}$$

where

$$\nabla_\mu = \Delta_{\mu\nu} \partial^\nu . \tag{1.74}$$

From the property of the projectors $\Delta^{\mu\nu}$ in (1.9) follow the identities related to U^μ and ∇^μ:

$$U^\mu \nabla_\mu = 0; \;\; U_\mu \nabla^\mu = 0 \tag{1.75}$$

The operator D will be called *convective time derivative*, which is the relativistic analogue of the nonrelativistic substantial time derivative, and ∇^μ *covariant gradient operator*. In the local rest frame in which

$$U^\mu_{\mathrm{LR}} = (c, 0, 0, 0) \, , \tag{1.76}$$

D and ∇^μ are given by the usual time derivative and spatial derivative operators, respectively:

$$D_{\mathrm{LR}} = \partial/\partial t , \quad \nabla^0_{\mathrm{LR}} = 0 , \quad \nabla^k_{\mathrm{LR}} = -\nabla_{\mathrm{LR}\,k} = -\partial/\partial x^k \;\; (k = 1, 2, 3) \, . \tag{1.77}$$

By using the decomposition of ∂^μ and various definitions, such as hydrodynamic density, etc. presented earlier, it is possible to derive from the conservation laws (1.68) and (1.69) the balance equations of density, density fractions, momentum, and energy. To derive various balance and evolution equations it is convenient to collect some useful identities involving hydrodynamic velocity and projectors as well as D and ∇^μ. They are as follows:

$$\begin{aligned}
U^\mu \nabla_\mu &= 0, & U_\mu \nabla^\mu &= 0, & U^\mu \partial_\mu \rho &= -\rho \partial_\mu U^\mu , \\
U_\mu D U^\mu &= 0, & U_\mu \nabla_\sigma U^\mu &= 0, & U_\mu J^\mu_a &= 0, \\
U_\mu D J^\mu_a &= -J^\mu_a D U_\mu , & U_\mu D Q^\mu_a &= -Q^\mu_a D U_\mu , & U_\mu Q^\mu_a &= 0, \\
\Delta^\mu_\nu \nabla_\sigma U^\nu &= \nabla_\sigma U^\mu , & \Delta^\mu_\nu Q^\nu_a &= Q^\mu_a , \\
\Delta^\mu_\nu J^\nu_a &= J^\mu_a , & \Delta^\mu_\nu P^{\nu\omega}_a &= P^{\mu\omega}_a .
\end{aligned} \tag{1.78}$$

These identities are easy to prove, but handy to have collected to help derivations of various evolution equations. Since derivations of balance equations for conserved variables are fairly straightforward, we list only the results without the details of derivations.

Equation of Continuity

Substitution of the decomposition of ∂_μ in (1.73) into (1.68) yields the equation of continuity

$$D\rho = -\rho \nabla_\mu U^\mu . \tag{1.79}$$

One of identities in (1.78) is used for this equation.

Density Fraction Balance Equation

Upon applying convective derivative on (1.66) and using identities involving J_a^μ in (1.78) we obtain the density fraction balance equation

$$\rho D\mathfrak{c}_a = -\nabla_\mu J_a^\mu + c^{-2} J_a^\mu DU_\mu. \tag{1.80}$$

Equation (1.80) presumes that the material particles do not chemically react. The last term on the right is not present in the nonrelativistic counterpart; it indeed vanishes in the nonrelativistic limit as $u/c \to 0$.

Momentum Balance Equation

Operating the projector Δ_ν^μ on the energy-momentum balance equation yields the equation

$$\Delta_\nu^\mu \partial_\sigma T^{\nu\sigma} = 0. \tag{1.81}$$

Upon inserting the decomposition formula (1.63) for $T^{\nu\sigma}$ there then follows the momentum balance equation

$$\begin{aligned} c^{-2}\mathcal{E}\rho DU^\mu = {} & -\nabla^\mu p - \Delta_\omega^\mu \nabla_\nu \overline{P}^{\omega\nu} \\ & + c^{-2}\left(P^{\mu\nu} DU_\nu - \Delta_\omega^\mu DQ^\omega - Q^\mu \nabla_\nu U^\nu - Q^\nu \nabla_\nu U^\mu\right), \end{aligned} \tag{1.82}$$

where the pressure tensor $\overline{P}^{\omega\nu}$ may be further decomposed:

$$\overline{P}^{\omega\nu} = \sum_{a=1}^{r} \left(\widetilde{\Delta}_a \Delta^{\omega\nu} + \Pi_a^{\omega\nu}\right) = \widetilde{\Delta}\Delta^{\omega\nu} + \Pi^{\omega\nu}. \tag{1.83}$$

To derive the momentum balance equation we have used the identity

$$\Delta_\nu^\mu \nabla_\sigma U^\nu = \nabla_\sigma U^\mu$$

listed in (1.78). The momentum balance equation differs from its nonrelativistic counterpart in that there appears $c^{-2}\mathcal{E}$ multiplied to ρDU^μ on the left-hand side and on the right-hand side, in addition to the relativistic effect term (i.e., the last term), the divergence of the pressure tensor term is a little modified from the form reported by de Groot et al. [3] Nevertheless, as $u/c \to 0$ it reduces to the nonrelativistic momentum balance equation known in the classical hydrodynamics [9].

Energy Balance Equation

The energy balance equation also follows from the energy-momentum tensor balance equation (1.69) upon contracting it with $c^{-2}U_\mu$ and using some of the identities listed in (1.78). It is given by the equation

$$\rho D\mathcal{E} = -\nabla_\mu Q^\mu - \mathfrak{p}\nabla_\mu U^\mu + \overline{P}^{\mu\nu}\nabla_\nu U_\mu + 2c^{-2} Q^\mu DU_\mu, \tag{1.84}$$

which makes the energy balance equation look formally different from the nonrelativistic version, but the last term vanishes in the nonrelativistic limit as $u/c \to 0$, yielding the energy balance equation in the classical hydrodynamics [9].

The conservation laws derived earlier are found to contain flow four-tensors such as diffusion flux, heat flux, and diffusion flux. These flux tensors are nonconserved variables, which are basically different from the conserved variables such as density, energy, and momentum, because the latter are devoid of energy dissipation mechanisms, whereas the nonconserved variables dissipate energy in the course of their evolution. Their mechanical cause for energy dissipation lies in the fact that, while evolving, they do not conserve energy in the course of collision processes. Their evolution equations contain an energy dissipation term in contrast to the balance equations for the conserved variables as we have already noticed.

Evolution Equations for Nonconserved Variables

It is possible to derive the evolution equations of the aforementioned nonconserved variables, first defining the supermoment tensor four-flow by the formula [4]

$$\psi_a^{(q)\sigma\mu\cdots\nu} := \left\langle f_a(x, p_a) p_a^\sigma h_a^{(q)\mu\cdots\nu} \right\rangle \tag{1.85}$$

as a generic representative of nonconserved variables spanning the thermodynamic manifold, which we regard as the flux of moment $h_a^{(q)\mu\nu\cdots l}$. For want of terminology for this quantity we will henceforth call it supermoment. And then contracting the supermoment $\psi_a^{(q)\sigma\mu\cdots\nu}$ with hydrodynamic velocity—i.e., projecting it onto U_σ—we obtain the desired macroscopic flux tensors, such as shear stress tensors, heat flux four-vectors, etc., which are generically denoted by $\Phi_a^{(q)\sigma\cdots\nu}$:

$$\Phi_a^{(q)\mu\cdots\nu} := \rho_a \widehat{\Phi}_a^{(q)\mu\cdots\nu} = c^{-2} U_\sigma \psi_a^{(q)\sigma\mu\cdots\nu}. \tag{1.86}$$

Summing $\Phi_a^{(q)\mu\cdots\nu}$ over species components, we obtain the flux tensors for the entire mixture

$$\Phi^{(q)\mu\cdots\nu} := \rho \widehat{\Phi}^{(q)\mu\cdots\nu} = \sum_{a=1}^{r} \rho_a \widehat{\Phi}_a^{(q)\mu\cdots\nu} = \rho \sum_{a=1}^{r} \mathfrak{c}_a \widehat{\Phi}_a^{(q)\mu\cdots\nu}. \tag{1.87}$$

In the present theory, the set of $\Phi_a^{(q)\mu\cdots\nu}$ is ordered, for the leading elements, in the following sequence:

$$\Phi_a^{(1)\mu\nu} = \Pi_a^{\mu\nu}, \quad \Phi_a^{(2)} = \widetilde{\Delta}_a, \quad \Phi_a^{(3)\mu} = Q_a'^{\mu}, \quad \Phi_a^{(4)\mu} = J_a^{\mu}, \text{ etc.,} \tag{1.88}$$

which implies that the set for the moments $h_a^{(q)\mu\cdots\nu}$ should be correspondingly ordered. Here, as defined earlier, $\Pi_a^{\sigma\nu}$ denotes the shear stress tensor; $\widetilde{\Delta}_a$ the excess normal stress; $Q_a'^{\nu}$ the heat flux in excess of the heat carried by particle diffusion; and J_a^{ν} the number diffusion flux.

Leading Moments of the Hierarchy

The leading moments[5] corresponding to the nonconserved variables listed in (1.88) are as follows [4]:

$$h_a^{(1)\mu\nu} = c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta_{\sigma\tau}\Delta^{\mu\nu}\right) p_a^\sigma p_a^\tau, \tag{1.89}$$

$$h_a^{(2)} = c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \left[\frac{1}{3}\Delta_{\mu\nu} p_a^\mu p_a^\nu - c^{-2}\left(p_a/\rho_a\right)\left(U_\lambda p_a^\lambda\right)\right], \tag{1.90}$$

$$\begin{aligned} h_a^{(3)\mu} = {} & c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \left\{\Delta_\sigma^\mu p_a^\sigma p_a^\nu U_\nu \right. \\ & \left. - \left(\widehat{h}_a + m_a c^2\right)\left[p_a^\mu - c^{-2}\left(U_\lambda p_a^\lambda\right) U^\mu\right]\right\}, \end{aligned} \tag{1.91}$$

$$h_a^{(4)\mu} = c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \left[p_a^\mu - c^{-2}\left(p_a^\lambda U_\lambda\right) U^\mu\right], \quad \text{etc.}, \tag{1.92}$$

where $\widehat{h}_a$ is the density of h_a, namely, $\widehat{h}_a = h_a/\rho_a$. The parameter $\widehat{h}_a$ will turn out to be the equilibrium enthalpy density, when evaluated with the equilibrium distribution function. The choice made for (1.89)–(1.92) needs explanations, and, in particular, the appearance of the rest mass energy $m_a c^2$ in $h_a^{(3)\mu}$ will be elaborated on presently. The moments $h_a^{(q)\mu\cdots\nu}$ are chosen such that they emerge as the nonrelativistic moments for the nonconserved variables, such as shear stress, excess normal stress, etc. In this connection, subtracting the rest mass-energy flux represented by $m_a c^2 \left[p_a^\mu - c^{-2}\left(U_\lambda p_a^\lambda\right) U^\mu\right]$ is necessary to recover the nonrelativistic limit agreeing with the Boltzmann kinetic theory result.[6] In order to make this aspect of their correspondence with the nonrelativistic kinetic theory as apparent as possible, we would like to digress and introduce the notion of relativistic peculiar velocity, which would help to comprehend the moments chosen in this work.

Relativistic Peculiar Velocity

The moments chosen $h_a^{(q)\mu\cdots\nu}$ $(q = 1, 2, 3, 4\cdots)$ in (1.89)–(1.92) may be put into forms more readily recognizable in correspondence to the nonrelativistic kinetic theory, if we introduce the notion of *relativistic peculiar velocity* defined by the formula

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[5] It should be noted and kept in mind that the projector Δ_σ^μ, etc. are to be applied after the statistical mechanical averaging is performed on the molecular expression of the moment. Therefore, if one wishes to understand, for example, $h_a^{(1)\mu\nu}$ at the molecular level the factor

$$\left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right) p_a^\sigma p_a^\tau$$

should be taken as

$$p_a^\mu p_a^\nu - \frac{1}{3} \boldsymbol{g}^{\mu\nu}\left(p_a^\sigma p_{a\sigma}\right),$$

where $\left(p_a^\sigma p_{a\sigma}\right)$ is the trace of $p_a^\sigma p_a^\tau$, which is equal to $m_a c^2$. Here traceless symmetry generating operator $\left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right)$ is meant to be applied after statistical mechanical averaging is taken.

[6] According to the Boltzmann kinetic theory, i.e., nonrelativistic kinetic theory, $h_a^{(3)\mu} = \frac{1}{2} m_a \left(\mathbf{C}_a \cdot \mathbf{C}_a\right) C_{a\,\mu} - \widehat{h} J_{a\,\mu}$, where $\mathbf{C}_a$ is the peculiar velocity $\mathbf{C}_a = \mathbf{v}_a - \mathbf{u}$.

$$\mathfrak{C}_a^\mu = \frac{c^2}{\left(U_\lambda p_a^\lambda\right)} p_a^\mu - U^\mu. \tag{1.93}$$

It is a weighted momentum (in fact, a weighted four-velocity) relative to the hydrodynamic velocity U^μ. Since

$$\frac{c^2}{\left(U_\lambda p_a^\lambda\right)} = m_a^{-1} \left[\left(1 + \mathbf{p}_a \cdot \mathbf{p}_a / m_a^2 c^2\right)^{1/2} - \mathbf{p}_a \cdot \mathbf{u} / m_a c^2\right]^{-1},$$

$\mathfrak{C}_a^\mu$ tends to the nonrelativistic peculiar velocity as $u/c \to 0$:

$$\lim_{u/c \to 0} \mathfrak{C}_a^\mu = C_{a\,\mu} := v_{a\,\mu} - u_\mu, \tag{1.94}$$

where $v_{a\,\mu}$ and u_μ are, respectively, the μth spatial components of particle velocity of species a and the reference velocity—the fluid velocity. In connection with $\mathfrak{C}_a^\mu$ its definition is motivated if we examine the projection of particle momentum $\Delta_\sigma^\mu p_a^\sigma$, which can be written

$$\Delta_\sigma^\mu p_a^\sigma = p_a^\mu - c^{-2} \left(p_a^\lambda U_\lambda\right) U^\mu = c^{-2} \left(p_a^\lambda U_\lambda\right) \mathfrak{C}_a^\mu. \tag{1.95}$$

Therefore, from the mathematical standpoint the relativistic peculiar velocity $\mathfrak{C}_a^\mu$ is generated upon projecting p_a^σ onto the μ direction of hydrodynamic velocity.

The relativistic peculiar velocity $\mathfrak{C}_a^\mu$ has the properties:

$$\mathfrak{C}_a^\mu U_\sigma = 0 \tag{1.96}$$

and

$$\Delta_\sigma^\mu \mathfrak{C}_a^\sigma = \mathfrak{C}_a^\mu, \quad \Delta_{\mu\sigma} \mathfrak{C}_a^\sigma = \mathfrak{C}_{a\,\mu}, \text{ etc.} \tag{1.97}$$

The relativistic peculiar velocities are useful in deducing the nonrelativistic limits of the moments $h_a^{(q)\mu\cdots\nu}$ listed in (1.89)–(1.92). For the purpose $h_a^{(q)\mu\cdots\nu}$ can be written in forms more easily comparable to the nonrelativistic theory moments used in Chap. 3 of Volume 1 on the Boltzmann kinetic theory, if $h_a^{(q)\mu\cdots\nu}$ is cast in the form

$$h_a^{(q)\mu\cdots\nu} = \frac{c^2}{\left(U_\lambda p_a^\lambda\right)} M_a^{(q)\mu\cdots\nu}, \tag{1.98}$$

where the moments $M_a^{(q)\mu\cdots\nu}$ are expressed in terms of $\mathfrak{C}_a$ by the formulas

$$M_a^{(1)\mu\nu} = c^{-4} \left(U_\lambda p_a^\lambda\right)^2 \left[\mathfrak{C}_a \mathfrak{C}_a\right]^{(2)\mu\nu}, \tag{1.99}$$

$$M_a^{(2)} = \frac{1}{3} c^{-4} \left(U_\lambda p_a^\lambda\right)^2 \left(\mathfrak{C}_a^\mu \mathfrak{C}_{a\,\mu}\right) - c^{-2} \left(U_\lambda p_a^\lambda\right) \left(\frac{p_a}{n_a}\right), \tag{1.100}$$

$$M_a^{(3)\mu} = c^{-2}\left(U_\lambda p_a^\lambda\right)\left[\left(U_\lambda p_a^\lambda - m_a c^2\right) - \widehat{h}_a\right]\mathfrak{C}_a^\mu, \tag{1.101}$$

$$M_a^{(4)\mu} = c^{-2}\left(U_\lambda p_a^\lambda\right)\mathfrak{C}_a^\mu. \tag{1.102}$$

Here we have used the abbreviations for second rank tensors

$$[\mathfrak{C}_a\mathfrak{C}_a]^{(2)\mu\nu} = \Delta_\sigma^\mu \mathfrak{C}_a^\sigma \mathfrak{C}_a^\tau \Delta_\tau^\nu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\mathfrak{C}_a^\sigma\mathfrak{C}_a^\tau, \tag{1.103}$$

$$\mathrm{Tr}\mathfrak{C}_a\mathfrak{C}_a := \left(\mathfrak{C}_a^\mu \mathfrak{C}_{a\,\mu}\right) = \Delta_{\sigma\tau}\mathfrak{C}_a^\sigma\mathfrak{C}_a^\tau, \tag{1.104}$$

that is, $[\mathfrak{C}_a\mathfrak{C}_a]^{(2)\mu\nu}$ is the traceless symmetric part and $\left(\mathfrak{C}_a^\mu\mathfrak{C}_{a\,\mu}\right)$ the trace of the second rank tensor $\mathfrak{C}_a^\mu\mathfrak{C}_a^\nu$. In the nonrelativistic limit of $u/c \to 0$, where (1.94) holds, the moments listed in (1.99)–(1.102) are easily seen to tend to the nonrelativistic limits

$$\lim_{u/c\to 0} h_a^{(1)\mu\nu} = m_a\,[\mathbf{C}_a\mathbf{C}_a]_a^{(2)\mu\nu}\,, \tag{1.105}$$

$$\lim_{u/c\to 0} h_a^{(2)} = m_a\left[\frac{1}{3}\mathbf{C}_a\cdot\mathbf{C}_a - \frac{p_a}{n_a}\right], \tag{1.106}$$

$$\lim_{u/c\to 0} h_a^{(3)\mu} = \frac{1}{2}m_a\left(\mathbf{C}_a\cdot\mathbf{C}_a\right)C_a^\mu - \widehat{h}_a m_a C_{a\,\mu}, \tag{1.107}$$

$$\lim_{u/c\to 0} h_a^{(4)\mu} = m_a C_{a\,\mu}, \tag{1.108}$$

with which we derived the evolution equations for shear stress, excess normal stress, heat flux, and diffusion flux in the nonrelativistic kinetic theory. Especially, in the case of $h_a^{(3)\mu}$, if the rest mass energy flux $m_a c^2\mathfrak{C}_a^\mu$ were not subtracted from the energy flux (i.e. the first term on the right of (1.101), evidently the nonrelativistic limit of $h_a^{(3)\mu}$ in (1.107) would not have been obtained. This point was one of the principal motivations for inserting the $m_a c^2$ factor in (1.107). Physically, it also removes the contribution of the rest mass energy to heat flow.

As will be shown in a later section of this chapter, when averaged with the nonequilibrium distribution function f_a according to (1.85) and (1.86) they give rise to physically relevant macroscopic variables such as $\Pi_a^{\sigma\nu}$, Δ_a, $Q_a'^{\nu}$, and J_a^ν, which also tend, in the limit $u/c \to 0$, to the nonrelativistic nonconserved variables appearing in the nonrelativistic kinetic theory in Chap. 3 of Volume 1 of this work. The rest of the moment set can be suitably chosen in terms of higher-rank irreducible tensors constructed according to the Schmidt orthogonalization technique, beginning from the seed moments for density and internal energy; on this aspect see Chap. 3 mentioned. The higher-order moments are not listed here because they give rise to macroscopic variables rarely observed in experiments or in nature, and hence we are not going to use them in the present work. The moments listed here are irreducible Cartesian tensors and vectors and, in fact, those tending to the so-called first "13 moments" plus the moments for diffusion in the nonrelativistic kinetic theory in the limit of $u/c \to 0$ as will be shown later when we discuss the nonrelativistic limits of evolution equations obtained in the present theory.

In the following, we resume discussions with moments $h_a^{(q)\mu\cdots\nu}$ given in (1.89)–(1.92).

Generic Evolution Equation for Nonconserved Variables

To derive the evolution equation for $\widehat{\Phi}_a^{(q)\sigma\cdots\nu}$ we observe that the substantial time derivative $\mathfrak{D}$ in relativistic theory is defined by

$$\mathfrak{D} = N^\mu \partial_\mu. \tag{1.109}$$

It can be decomposed into the time and space components

$$\mathfrak{D} = \rho D + N^\mu \nabla_\mu. \tag{1.110}$$

However, since $N^\mu \nabla_\mu = \rho U^\mu \nabla_\mu = 0$ in Eckart's definition of hydrodynamic velocity, the substantial derivative $\mathfrak{D}$ simply becomes the convective time derivative D times ρ:

$$\mathfrak{D} = \rho D. \tag{1.111}$$

Now, the substantial time derivative of $\widehat{\Phi}_a^{(q)}$ can be recast into the form

$$\mathfrak{D}\widehat{\Phi}_a^{(q)\mu\cdots\nu} = N^\sigma \partial_\sigma \widehat{\Phi}_a^{(q)\mu\cdots\nu} = \partial_\sigma \left(N^\sigma \widehat{\Phi}_a^{(q)\mu\cdots\nu} \right) - \widehat{\Phi}_a^{(q)\mu\cdots\nu} \partial_\sigma N^\sigma. \tag{1.112}$$

Then, since $\partial_\mu N^\mu = 0$ according to (1.68), in the Eckart's definition of U^μ the substantial time derivative of flux density $\widehat{\Phi}_a^{(q)}$ on the left of (1.112) can be written as

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = \partial_\sigma \left(N^\sigma \widehat{\Phi}_a^{(q)\mu\cdots\nu} \right). \tag{1.113}$$

Rearrange the right hand side of this equation to the form

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\partial_\sigma \left(\psi_a^{(q)\sigma\mu\cdots\nu} - N^\sigma \widehat{\Phi}_a^{(q)\mu\cdots\nu} \right) + \partial_\sigma \psi_a^{(q)\sigma\mu\cdots\nu}. \tag{1.114}$$

Let us then define the "projection of supermoment tensor" $\Omega_a^{(q)\sigma\mu\cdots\nu}$ for the divergence term

$$\Omega_a^{(q)\sigma\mu\cdots\nu} = \psi_a^{(q)\sigma\mu\cdots\nu} - N^\sigma \widehat{\Phi}_a^{(q)\mu\cdots\nu} = \Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu}. \tag{1.115}$$

By using the covariant kinetic equation, $\partial_\sigma \psi^{(q)}$ is easily shown to yield the evolution equation

$$\partial_\sigma \psi_a^{(q)\sigma\mu\cdots\nu} = \left\langle f_a(x, p_a) p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} \right\rangle + \Lambda_a^{(q)\mu\cdots\nu}(x) \tag{1.116}$$

with the dissipation term $\Lambda_a^{(q)\mu\cdots\nu}$ defined by the formula

$$\Lambda_a^{(q)\mu\cdots\nu} = \sum_{b=1}^{r} \left\langle h_a^{(q)\mu\cdots\nu} C_{ab}(f_a, f_b) \right\rangle. \tag{1.117}$$

Thus, on substituting (1.116) into (1.114) we obtain the generic evolution equation for $\widehat{\Phi}_a^{(q)}$ in the form

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\partial_\sigma \Omega_a^{(q)\sigma\mu\cdots\nu} + Z_a^{(q)\mu\cdots\nu} + \Lambda_a^{(\alpha)\mu\cdots\nu}, \tag{1.118}$$

where the kinematic term $Z_a^{(q)\mu\cdots\nu}$ is now given by the expression

$$Z_a^{(q)\mu\cdots\nu} = \sum_{a=1}^{r} \left\langle f_a(x, p_a) p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} \right\rangle. \tag{1.119}$$

It can be shown that the kinematic term $Z_a^{(q)\mu\cdots\nu}$ tends to the nonrelativistic limit obtained from the nonrelativistic Boltzmann equation; for this purpose it will be found convenient to make use of the relativistic peculiar velocity introduced earlier. We also observe that in view of the nonrelativistic limit (1.94) the projection of supermoment tensor $\Omega_a^{(q)\sigma\mu\cdots\nu}$ tends to the nonrelativistic moment one-order higher than moment $M_a^{(q)\mu\cdots\nu}$:

$$\lim_{u/c\to 0} \Omega_a^{(q)\sigma\mu\cdots\nu} = \lim_{u/c\to 0} \left\langle f_a \mathfrak{C}_a^\sigma M_a^{(q)\mu\cdots\nu} \right\rangle.$$

The presence of this term in the generic evolution equation is clearly the principal reason that renders the set of evolution equations an open hierarchy.

On summing (1.118) over a, the evolution equation for $\widehat{\Phi}^{(q)\mu\nu\cdots l}$ of the entire mixture is obtained in the form

$$\rho D \widehat{\Phi}^{(q)\mu\cdots\nu} = -\partial_\sigma \Omega^{(q)\sigma\mu\cdots\nu} + Z^{(q)\mu\cdots\nu} + \Lambda^{(q)\mu\cdots\nu}, \tag{1.120}$$

with the definitions

$$\Omega^{(q)\sigma\mu\cdots\nu} = \sum_{a=1}^{r} \Omega_a^{(q)\sigma\mu\cdots\nu}, \tag{1.121}$$

$$Z^{(q)\mu\cdots\nu} = \sum_{a=1}^{r} Z_a^{(q)\mu\cdots\nu}. \tag{1.122}$$

The kinematic terms $Z_a^{(q)\mu\cdots\nu}$ consist of rather complicated terms, which include thermodynamic forces driving transport processes in the system as well as fluxes coupled to thermodynamic forces.

The divergence term $-\partial_\sigma \Omega^{(q)\sigma\mu\cdots\nu}$ also can be similarly evaluated. Some terms from it, in fact, should be combined with $Z_a^{(q)\mu\cdots\nu}$ to obtain a modified kinematic term $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$

$$\mathfrak{Z}_a^{(q)\mu\cdots\nu} = Z_a^{(q)\mu\cdots\nu} + K_a^{(q)\mu\cdots\nu}, \tag{1.123}$$

where

$$K_a^{(q)\mu\cdots\nu} = -c^{-2} U_\sigma D \left(\Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu}\right). \tag{1.124}$$

With the kinematic term so modified, the generic evolution equation (1.120) now reads

$$\rho D \widehat{\Phi}^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_a^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}^{(q)\mu\cdots\nu} + \Lambda^{(q)\mu\cdots\nu}, \tag{1.125}$$

where the supermoment flux tensor $\Theta_a^{(q)\sigma\mu\cdots\nu}$ making up the divergence term is defined by

$$\Theta_a^{(q)\sigma\mu\cdots\nu} = \Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu}. \tag{1.126}$$

This form (1.125) of relativistic generic evolution equation formally better resembles the nonrelativistic generic evolution equation for $\widehat{\Phi}^{(q)\mu\cdots\nu}$ appearing in Chap. 3, Volume 1 of this work. As a matter of fact, it can be easily shown that (1.125) tends to the nonrelativistic generic evolution equation in Chap. 3, Volume 1. Note that $\Theta_a^{(q)\sigma\mu\cdots\nu}$ is a tensor of rank one-order higher than the tensor $\mathfrak{Z}^{(q)\mu\cdots\nu}$, which physically may be regarded as the flux of macroscopic moment $\Phi^{(q)\mu\cdots\nu}$.

The generic kinematic term $Z_a^{(q)\mu\cdots\nu}$ given in (1.119) and the term $K_a^{(q)\mu\cdots\nu}$ are evaluated for the cases of $h_a^{(q)\mu\cdots\nu}$ $(q = 1, \ldots, 4)$ as defined by (1.89)–(1.92), and combined to obtain the modified kinematic term $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ in (1.123). For the results presented below we make use of the decomposition formula (1.62) of energy-momentum tensor $T_a^{\mu\nu}$ and various identities involving projectors listed in (1.78). The following system of abbreviations is employed to simplify the presentation of the complicated expressions involving tensor products:

$$[\mathbf{P} \cdot \nabla \mathbf{U}]^{(2)\mu\nu} = \frac{1}{2}\left(P^{\mu\sigma}\nabla_\sigma U^\nu + P^{\nu\sigma}\nabla_\sigma U^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\left(\Delta_{\omega\alpha} P^{\alpha\sigma}\nabla_\sigma U^\omega\right) \tag{1.127}$$

for the traceless symmetric part of tensor product $P^{\mu\sigma}\nabla_\sigma U^\nu$, a tensor of rank 2;

$$[\mathbf{P}]^{(2)\mu\nu} = \frac{1}{2}\left(P^{\mu\nu} + P^{\nu\mu}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\omega} P^{\omega\alpha}, \tag{1.128}$$

for the traceless symmetric part of tensor $P^{\mu\nu}$ of rank 2;

$$[\mathbf{P} \cdot D\mathbf{U}]^\mu = P^{\mu\nu} D U_\nu \tag{1.129}$$

for covariant vectors; and

$$[UU]^{(2)\mu\nu} = U^\mu U^\nu - \frac{1}{3}\Delta^{\mu\nu}(U^\sigma U_\sigma) = U^\mu U^\nu - \frac{1}{3}c^2\Delta^{\mu\nu}. \tag{1.130}$$

Modified Kinematic Terms $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$

In the following the modified kinematic terms $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ given in (1.123) are listed for $q = 1, 2, 3, 4$:

For the Case of Shear Stress

The modified kinematic term for $q = 1$ is given by the expression

$$\begin{aligned}\mathfrak{Z}_a^{(1)\mu\nu} &= -2\left[P_a \cdot \nabla U\right]^{(2)\mu\nu} - 2\left[Q_a \cdot DU\right]^{(2)\mu\nu} - 2\left[P_a U \cdot DU\right]^{(2)\mu\nu} \\ &\quad + c^{-2}\left(U^\mu P_a^{\nu\omega} DU_\omega + U^\nu P_a^{\mu\omega} DU_\omega\right) + c^{-2}\left[UU\right]^{(2)\mu\nu} P_a^{\tau\varepsilon} \nabla_\varepsilon U_\tau \\ &\quad - \frac{1}{3} c^{-2}\left(U^\mu Q_a^\varepsilon \nabla_\varepsilon U^\nu + U^\nu Q_a^\varepsilon \nabla_\varepsilon U^\mu\right) + c^{-2}\left[UU\right]^{(2)\mu\nu} Q_a^\omega DU_\omega \\ &\quad - c^{-4} U^\mu U^\nu Q_a^\omega DU_\omega + \frac{1}{3} m_a^2 c^2 \left\langle f_a \frac{p_a^\varepsilon}{\left(U_\lambda p_a^\lambda\right)} \right\rangle \left(U^\mu \nabla_\varepsilon U^\nu + U^\nu \nabla_\varepsilon U^\mu\right) \\ &\quad - c^2 \left\langle f_a \frac{p_a^\mu p_a^\nu p_a^\omega p_a^\varepsilon}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \nabla_\varepsilon U_\omega + \frac{1}{3} \Delta^{\mu\nu} m_a^2 c^4 \left\langle f_a \frac{p_a^\omega p_a^\varepsilon}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \nabla_\varepsilon U_\omega, \quad (1.131)\end{aligned}$$

where the second rank tensor $\mathbf{P}_a$ may be decomposed into traceless and trace parts as in (1.56).

For the Case of Excess Normal Stress

The modified kinematic term for $q = 2$ is given by the expression

$$\begin{aligned}\mathfrak{Z}_a^{(2)} &= -\rho_a D\left(p_a/\rho_a\right) - \frac{2}{3} P_a^{\mu\varepsilon} \nabla_\varepsilon U_\mu - \frac{2}{3} c^{-2} Q_a^\omega DU_\omega - J_a^\varepsilon \nabla_\varepsilon\left(p_a/\rho_a\right) \\ &\quad - c^{-2}\left(p_a/\rho_a\right) J_a^\omega DU_\omega + \frac{1}{3}\left[P_a^{\omega\varepsilon} - m_a^2 c^4 \left\langle f_a \frac{p_a^\omega p_a^\varepsilon}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle\right] \nabla_\varepsilon U_\omega. \quad (1.132)\end{aligned}$$

The pressure tensor in the third term on the right also can be decomposed as in (1.56).

For the Case of Heat Flux

The modified kinematic term $q = 3$ is given by the expression

$$\begin{aligned}\mathfrak{Z}_a^{(3)\mu} &= P_a^{\mu\omega} DU_\omega - Q_a^{\prime\sigma} \nabla_\sigma U^\mu - P_a^{\mu\sigma} \nabla_\sigma \widehat{h}_a^\prime - c^{-2} \mathfrak{J}_a^\sigma \nabla_\sigma U^\mu \\ &\quad + \widehat{h}_a^\prime J_a^\sigma \nabla_\sigma U^\mu + \rho_a\left(\widehat{h}_a^\prime - \mathcal{E}_a\right) DU^\mu - J_a^\mu D\widehat{h}_a^\prime + \varphi_a^{(3)\mu\omega\sigma} \nabla_\sigma U_\omega \\ &\quad - c^{-2} U^\mu\left(Q_a^\sigma DU_\sigma + c^{-2} \mathfrak{J}_a^\sigma DU_\sigma - \widehat{h}_a^\prime J_a^\omega DU_\omega\right) \\ &\quad - U^\mu\left(P_a^{\sigma\varepsilon} \nabla_\sigma U_\varepsilon - J_a^\sigma \nabla_\sigma \widehat{h}_a^\prime\right) \\ &\quad + \left[P_a^{\mu\sigma} - c^2 \left\langle f_a \frac{p_a^\mu p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)} \right\rangle\right] \nabla_\sigma \widehat{h}_a^\prime + c^2 \widehat{h}_a \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \nabla_\sigma U_\omega \\ &\quad + \left[m c^4 \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle - \varphi_a^{(3)\mu\omega\sigma}\right] \nabla_\sigma U_\omega, \quad (1.133)\end{aligned}$$

where $\varphi_a^{(3)\mu\omega\sigma}$ is defined by the rank 3 tensor

$$\varphi_a^{(3)\mu\omega\sigma} = \left\langle f_a p_a^\mu p_a^\omega p_a^\sigma \right\rangle .$$

The terms proportional to U^μ vanish in the local rest frame $U^\mu = (c, 0, 0, 0)$.

For the Case of Diffusion Flux

The modified kinematic term for $q = 4$ is given by the relatively simple form

$$\begin{aligned} \mathfrak{Z}_a^{(4)\mu} &= -\rho_a D U^\mu - J_a^\sigma \nabla_\sigma U^\mu - c^{-2} U^\mu J_a^\omega D U_\omega \\ &\quad - c^2 \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \nabla_\sigma U_\omega . \end{aligned} \tag{1.134}$$

Projection of Supermoments

Lastly, we examine the projection of supermoment $\psi_a^{(q)\sigma\mu\cdots\nu}$ making up the divergence term in the generic evolution equation (1.118), which on substitution of the moment $h_a^{(q)\mu\cdots\nu}$ is given by the integral

$$\Theta_a^{(q)\sigma\mu\cdots\nu} = \left\langle f_a \Delta_\omega^\sigma \frac{c^2 p_a^\omega}{\left(U_\lambda p_a^\lambda\right)} M_a^{(q)\mu\cdots\nu} \right\rangle . \tag{1.135}$$

The integral on the right of (1.135) cannot be generally completely reduced to a form given in terms of nonconserved variables within the framework of the "13" moments. The reason is that $\psi_a^{(q)\sigma\mu\cdots\nu}$ for $q = 1, 2, 3$ belong to the set outside the so-called first 13 moment subset. Nevertheless, it would be useful to examine their general characters as tensors. On substitution of the explicit forms for $M_a^{(q)\mu\cdots\nu}$ in the case of $q = 1, 2, 3, 4$ we can ascertain their characters as tensors case by case as shown below:

For the Case of Shear Stress

For $q = 1$(shear stress) $\Theta_a^{(1)\sigma\mu\nu}$ is given by the formula

$$\Theta_a^{(1)\sigma\mu\nu} = \left[\mathfrak{W}^{\sigma\cdot\cdot}\right]^{(2)\mu\nu} - U^\sigma \left[T_a\right]^{(2)\mu\nu} , \tag{1.136}$$

where

$$\left[\mathfrak{W}^{\sigma\cdot\cdot}\right]^{(2)\mu\nu} = \left\langle f_a \frac{c^2 p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)} \left(\Delta_\varepsilon^\mu \Delta_\tau^\nu - \frac{1}{3} \Delta_{\varepsilon\tau} \Delta^{\mu\nu} \right) p_a^\varepsilon p_a^\tau \right\rangle ,$$

which is a rank 3 tensor, traceless symmetric with respect to the last two indices. If the explicit expressions for projectors are made use of, the integrals may be partly expressed in terms of stress tensors or heat fluxes etc., but there remains an integral of a rank 3 tensor, which is traceless symmetric with respect to two indices out of three. Since it is rather bulky, but would not reveal its structure, we will leave it as it stands.

For the Case of Excess Normal Stress

For $q = 2$ (excess normal stress) $\Theta_a^{(2)\sigma}$ is given by the formula

$$\Theta_a^{(2)\sigma} = \frac{1}{3}c^2 \left\langle f_a \frac{p_a^\sigma \left(p_a^\mu p_{a\,\mu}\right)}{\left(U_\lambda p_a^\lambda\right)} \right\rangle - \frac{1}{3} Q_a^{\prime\sigma} - \frac{1}{3} c^{-2} \mathfrak{J}_a^\sigma \\ - \left(p_a/\rho_a\right) J_a^\sigma - U^\sigma \left(\mathfrak{p}_a - p_a\right), \tag{1.137}$$

which clearly shows it is a vector, the first term being vector but an integral of rank 3 tensor contracted to a vector with respect to two indices, whereas the remainder consists of a simple vectors.

For the Case of Heat Flux

For $q = 3$ (heat flux) $\Theta_a^{(3)\sigma\mu}$ is given by the expression

$$\Theta_a^{(3)\sigma\mu} = c^2 P_a^{\sigma\mu} + \rho_a \widehat{h}_a^\prime U^\sigma U^\mu - c^2 \widehat{h}_a^\prime \left\langle f_a \frac{p_a^\sigma p_a^\mu}{\left(U_\lambda p_a^\lambda\right)} \right\rangle, \tag{1.138}$$

which shows $\Theta_a^{(3)\sigma\mu}$ is a rank 2 tensor. The integral in the third term would not easily reduce to a well recognizable variable.

For the Case of Diffusion Flux

For $q = 4$ (diffusion flux) $\Theta_a^{(4)\sigma\mu}$ is given by the expression

$$\Theta_a^{(4)\sigma\mu} = c^2 \left\langle f_a \frac{p_a^\sigma p_a^\mu}{\left(U_\lambda p_a^\lambda\right)} \right\rangle - U^\sigma J_a^\mu - U^\mu J_a^\sigma - \rho_a U^\sigma U^\mu. \tag{1.139}$$

It is evidently a tensor of rank 2, but the integral is not well recognizable macroscopic variable unless the nonrelativistic limit is taken, in which case it becomes an energy-momentum tensor. Therefore it readily joins the rank of experimentally observable macroscopic observables in the present scheme of theory.

By this, we conclude examination of kinematic and divergence terms as well as the characters of $\Theta_a^{(q)\sigma\mu\cdots\nu}$ making up the divergence term in the generic evolution equations for nonconserved variables $\Phi_a^{(q)\mu\cdots\nu}$. Examination of the dissipation terms will have to wait until the thermodynamic theory is formulated, so that the thermodynamic consistency is properly imposed on them.

We emphasize that the evolution equations represented by the generic evolution Eq. (1.120) are coupled to the momentum, energy. and concentration balance equations presented earlier. As in the nonrelativistic kinetic theory this set of evolution equations is open and therefore must be suitably closed by means of a suitable closure relation. When the set is thus closed[7] and made consistent with the laws of thermodynamics, it provides relativistic generalized hydrodynamic equations consisting of (1.79)–(1.82), (1.84) and (1.118), if the number of moments is limited to a finite value by the closure. The generic evolution equation (1.118) can be easily shown to tend to the nonrelativistic counterparts in the limit of $u/c \rightarrow 0$. This will be shown in a later section.

The macroscopic moments $\{\Phi_a^{(q)} : q = 1, \ldots, 4, \ldots\}$ are defined in a significantly different manner from the conventional moment method [4]. As a consequence, their evolution equations are accordingly different. Furthermore, since the moment evolution equations will be subjected to the the laws of thermodynamics, the macroscopic theory on which the hydrodynamic theory is based remains thermodynamically consistent.

The covariant Boltzmann equation, as any kinetic equation, is a mathematical model for relativistic macroscopic fluids. The model is purported to provide us with a molecular theoretic picture and our way of comprehending macroscopic processes observed in nature and laboratory. The physical reality of such a model for molecular systems as a theoretical representation of macroscopic fluids should be subjected to the physical laws (e.g., the thermodynamic laws) governing macroscopic processes in fluids. The currently accepted paradigm for macroscopic physical phenomena is that all macroscopic phenomena are subject to the laws of thermodynamics. Therefore, the evolution equations derived from the covariant kinetic equations, including the balance equations for the conserved variables, must be subjected to the laws of thermodynamics before being judged physically reasonable and acceptable as a theory representing macroscopic processes in fluids [10–13]. In fact, this thermodynamic validation process makes the distribution functions with parameters therein acquire, as the candidates for approximate solutions of the covariant kinetic equation, the desired thermodynamic and microcopied theory meanings founded on thermodynamics of irreversible processes. Therefore various statistical mechanically calculated macroscopic observables consequently get endowed with physical meanings and anchored on the phenomenological thermodynamic and fluid dynamic observations and experiments. To achieve this aim we now introduce the Boltzmann entropy and the H theorem.

1.1.7 Boltzmann Entropy and the H *Theorem*

In the preceding sections the relativistic hydrodynamic variables have been presented as statistical mechanical averages computable with the help of the distribution function described by the covariant Boltzmann equation and their evolution equations have been derived therefrom. We have also posited that the hydrodynamic descrip-

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[7] The set $\left\{h_a^{(q)}\right\}$ must be closed such that the thermodynamic branch $f_a^{\mathrm{c}}(x, p_a)$ of the distribution function $f_a(x, p_a)$ is normalizable, that is, the set must be closed at the even order of $h_a^{(q)}$ so that the integral involved is convergent. See (1.210) below for $f_a^{\mathrm{c}}(x, p_a)$.

tion must remain consistent with the laws of thermodynamics, particularly, the second law of thermodynamics, so that the mathematical model theory formulated thereby can be firmly grounded in the physical reality as required of a physical theory. Since the variables describing macroscopic processes are generally varying in space–time, the thermodynamics must be necessarily in the realm of irreversible thermodynamics governing processes evolving in spacetime.

Boltzmann Entropy and Its Balance Equation

In order to formulate irreversible thermodynamics for the relativistic system it is necessary to introduce the Boltzmann entropy four-flow in a covariant form

$$S^{\mu}(x) = \sum_{a=1}^{r} S_a^{\mu}(x) = -k_B \sum_{a=1}^{r} \left\langle p_a^{\mu} f_a(x, p_a) \ln f_a(x, p_a) \right\rangle . \tag{1.140}$$

To distinguish it from the Clausius entropy used in thermodynamics, which was introduced for reversible cyclic processes [13–15], it should be more precisely called the Boltzmann entropy four-flow, but for brevity of terminology we will sometimes simply call it entropy four-flow. However, the distinction from the Clausius entropy of reversible thermodynamic processes should be kept in mind in order to prevent possible confusion that might arise when we use the term later in connection with irreversible thermodynamics of relativistic fluids. Incidentally, the statistical mechanical formula given here for $S^{\mu}(x)$ is without the (-1) factor on the right, which is inserted in the literature in kinetic theory since the entropy so defined appeared to be more convenient. It thus reads

$$S^{\mu}(x) = -k_B \sum_{a=1}^{r} \left\langle p_a^{\mu} f_a(x, p_a) \left[\ln f_a(x, p_a) - 1\right] \right\rangle .$$

See, for example, Ref. [3] for this definition. The factor was convenient to have for discussing equilibrium thermodynamics in statistical mechanics, but we find it superfluous and can be a source of confusion. It would be better if it were not inserted in the definition of $S^{\mu}(x)$.

With the four-vector $S^{\mu}(x)$ and hydrodynamic velocity U^{μ} the scalar Boltzmann entropy $S(x)$ can be constructed as follows:

$$S(x) = c^{-2} U_{\mu} S^{\mu}, \tag{1.141}$$

with which we may then define scalar Boltzmann entropy per particle $\mathcal{S}$:

$$\rho \mathcal{S}(x) := S(x) . \tag{1.142}$$

This is the quantity that plays an important role in determining the stability of the equilibrium solution because it may be regarded as a Lyapounov function [16] as it is done so for the nonrelativistic kinetic theory discussed in Chaps. 3, 5, 6, and 7 of Volume 1 of the present work.

With the definitions of the entropy four-flow and the scalar entropy density we can derive the (Boltzmann) entropy balance equation from the covariant relativistic Boltzmann equation. The covariant form of the Boltzmann entropy balance equation is given by

$$\partial_\mu S^\mu = \sigma_{\mathrm{ent}}\left(x\right), \tag{1.143}$$

where the Boltzmann entropy production $\sigma_{\mathrm{ent}}\left(x\right)$—the source term—is given in terms of the collision integral $C_{ab}(f_a, f_b)$ of the kinetic equation by the statistical mechanical formula

$$\sigma_{\mathrm{ent}}\left(x\right) = -k_B \sum_{a=1}^{r} \sum_{b=1}^{r} \left\langle \ln f_a(x, p_a) C_{ab}(f_a, f_a) \right\rangle . \tag{1.144}$$

To cast it into a more transparent form let us abbreviate the collision integral for brevity of notation as below:

$$\langle \cdots \rangle_{\mathrm{c}} = G_{ab} \int d^3\overline{p}_a \int d^3\overline{p}_a \int d^3\overline{p}_a^* \int d^3\overline{p}_b^* W_{ab}\left(p_a p_b | p_a^* p_b^*\right) \cdots . \tag{1.145}$$

By following the well-known procedure [3, 4] that makes use of the symmetry properties of the transition probability (1.15), it is easy to show[8] that the Boltzmann entropy production $\sigma_{\mathrm{ent}}\left(x\right)$ can be written as

$$\sigma_{\mathrm{ent}}\left(x\right) = -\frac{1}{4} k_B \sum_{a=1}^{r} \sum_{b=1}^{r} \left\langle \ln \left(\frac{f_a f_b}{f_a^* f_b^*} \right) \left(f_a^* f_b^* - f_a f_b \right) \right\rangle_{\mathrm{c}} . \tag{1.146}$$

Since $\ln\left(y/z\right)\left(y-z\right) \geq 0$ with the equality holding only if $y = z$, the right hand side of (1.146) is positive semidefinite:

$$\sigma_{\mathrm{ent}}\left(x\right) \geq 0 \tag{1.147}$$

with the equality holding only at equilibrium reached in long time. Hence at equilibrium

$$\left[\ln \left(\frac{f_a f_b}{f_a^* f_b^*} \right) \left(f_a^* f_b^* - f_a f_b \right) \right]_{f_a = f_a^{\mathrm{e}}, f_b = f_b^{\mathrm{e}}} = 0. \tag{1.148}$$

Equations (1.146)–(1.148) is the content of the H theorem in local form. Here f_k^{e} and $f_k^{*\mathrm{e}}$ denote the equilibrium solution of the covariant kinetic equation and the post-collision distribution function.

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[8] The proof follows the same procedure as for the nonrelativistic Boltzmann collision integral. For this reason it is not shown here to avoid repetition.

From the viewpoint of the stability theory of differential equation systems, which the covariant Boltzmann equations may be regarded as, the H theorem may be thought of as a Lyapounov stability theorem [16] for stability of the solution of the kinetic equation in question since the (Boltzmann) entropy can indeed serve as a Lyapounov function, and the equilibrium state characterized by f_a^{e} is stable in accord with the Lyapounov stability condition.

Although in the literature on kinetic theory and statistical mechanics the Boltzmann H theorem is generally regarded as a statistical mechanical representation of the second law of thermodynamics, we refrain from taking this viewpoint because there is a clear distinction that should be made between the Clausius entropy for a reversible process and the Boltzmann entropy of nonequilibrium dynamic processes as mentioned earlier in this chapter and in chapters on nonrelativistic kinetic theory in this work. In fact, the H theorem is a broader theorem that governs the stability of equilibrium state of the system in the phase space than the second law of thermodynamics that governs the stability of macroscopic processes at equilibrium, or reversible processes, in *the manifold of macroscopic variables.* For the latter is a projection of the phase space of a much higher dimension[9] onto a finite dimensional macroscopic variable manifold in which macroscopic irreversible thermodynamics is described. What we mean by projection will be elaborated on later when the theory is further developed.

The covariant Boltzmann entropy balance equation can be written in a more useful form in terms of the scalar entropy density

$$\rho D\mathcal{S} = -\partial_\mu J_{\mathrm{s}}^\mu + \sigma_{\mathrm{ent}}(x), \tag{1.149}$$

where the Boltzmann entropy flux is given by

$$J_{\mathrm{s}}^\mu = S^\mu - \mathcal{S}N^\mu = S^\mu - \rho\mathcal{S}U^\mu = \Delta_\nu^\mu S^\nu. \tag{1.150}$$

On inserting the statistical mechanical expression for the Boltzmann entropy four-flow S^ν, the statistical mechanical formula for J_{s}^μ now reads

$$J_{\mathrm{s}}^\mu = -k_B \sum_{a=1}^{r} \Delta_\nu^\mu \left\langle p_a^\mu f_a(x, p_a) \ln f_a(x, p_a) \right\rangle. \tag{1.151}$$

The derivation of (1.149) is made as follows: By using the same procedure as used for (1.112) and (1.113) it is easy to see that in Eckart's definition of U^μ the substantial derivative of the scalar Boltzmann entropy $\mathcal{S}$ can be written

$$\rho D\mathcal{S} = \partial_\mu (\rho\mathcal{S}N^\mu), \tag{1.152}$$

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[9] This point can be better comprehended if we recall that the distribution function $f_a(x, p_a)$ is a singlet distribution function of the many-particle distribution function representing the ensemble of representative systems.

which then can be rearranged to the form leading to (1.149):

$$\begin{aligned}\rho D\mathcal{S} &= -\partial_\mu \left(S^\mu - \rho \mathcal{S} N^\mu\right) + \partial_\mu S^\mu = -\partial_\mu \Delta^\mu_\nu S^\nu + \sigma_{\text{ent}}(x) \\ &= -\partial_\mu J^\mu_{\text{s}} + \sigma_{\text{ent}}(x) \,. \end{aligned} \tag{1.153}$$

As it stands, however, the H theorem is not equivalent to the second law of thermodynamics nor does it provide a clue as to whether and how the requirement of the second law of thermodynamics is fulfilled. To achieve the latter aim it is necessary to introduce another concept called *calortropy* as was done for nonrelativistic theories in the previous chapters of Volume 1, but it would first need the discussion of how we might determine the solution of the kinetic equation and the equilibrium solution before defining the calortropy.

Equilibrium Distribution Function

The Boltzmann H theorem makes it possible to obtain a unique equilibrium solution of the covariant kinetic equation since at equilibrium

$$\sigma_{\text{ent}} = 0, \tag{1.154}$$

and it implies that, if the equilibrium solution of the covariant kinetic equation is denoted by $f^{\text{e}}_a(p_a)$, then (1.154) necessarily means

$$p^\sigma_a \partial_\sigma f^{\text{e}}_a(p_a) = 0. \tag{1.155}$$

Equation (1.154), moreover, means that $\ln f^{\text{e}}_a(p_a)$ is a summation invariant of the covariant kinetic equation. Therefore there holds the condition

$$\ln f^{\text{e}}_a + \ln f^{\text{e}}_b = \ln f^{\text{e}*}_a + \ln f^{\text{e}*}_b, \tag{1.156}$$

which implies that the equilibrium distribution function must be a linear combination of basic collision invariants. Therefore $\ln f^{\text{e}}_k$ and $\ln f^{\text{e}*}_k$ $(k = a, b)$ may be generally written in the form

$$\ln f^{\text{e}}_k = a_k(x) + b_\mu(x)\, p^\mu_k \tag{1.157}$$

and similarly for $\ln f^{\text{e}*}_k$. Here a_k is a scalar and b_μ is a vector independent of p^μ_k. More precisely put, these parameters satisfy the Killing equation [3, 17] derived from the covariant kinetic equation

$$p^\mu_k \partial_\mu f_k(x, p_k) + m_k F^\mu(x, p_k) \frac{\partial}{\partial p^\mu} f_k(x, p_k) = \sum_{l=1}^{r} C_{kl}(f_k, f_l)\,, \tag{1.158}$$

where $F^\mu(x, p_k)$ is the Lorentz force.

$$F^\mu(x, p_k) = -\frac{q_k}{m_k c} F^{\mu\nu}(x) p_{k\nu} \tag{1.159}$$

with q_k denoting the charge of the particle k and $F^{\mu\nu}$ the electromagnetic field tensor. Here for generality we have inserted the Lorentz force [7, 18] in the kinetic equation. Insertion of (1.157) into (1.158) yields the equation

$$p_k^\mu \partial_\mu a_k(x) + p_k^\mu p_k^\nu \partial_\mu b_{k\nu}(x) + m_k b_{k\mu}(x) F^\mu(x, p_k) = 0. \tag{1.160}$$

The electromagnetic field tensor may be given in terms of the four-potential A^μ

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu. \tag{1.161}$$

On using this relation in (1.160) there follows the equation

$$p_{k\mu}\left[\partial^\mu a_k - c^{-1} q_k b_{k\nu} F^{\nu\mu}\right] + p_k^\mu p_k^\nu \partial_\mu b_{k\nu} = 0. \tag{1.162}$$

It is concluded from this equation that

$$\partial^\mu a_k - c^{-1} q_k b_{k\nu} F^{\nu\mu} = 0, \tag{1.163}$$

$$\partial^\mu b_k^\nu + \partial^\nu b_k^\mu = 0. \tag{1.164}$$

Equation (1.164) is called the Killing equation [19]. Its general solution can be written as

$$b_k^\mu(x) = b_k^\mu + \omega_k^{\mu\nu} x_\nu \tag{1.165}$$

with b_k^μ and $\omega_k^{\mu\nu} = -\omega_k^{\nu\mu}$ being independent of x. As shown later,

$$b_k^\mu = -U^\mu(x)/k_B T(x) \tag{1.166}$$

with $U^\mu(x)$ denoting the hydrodynamic velocity field, and $T(x)$ turns out to be the temperature field.[10] Therefore the most general equilibrium solution for the system obeying the covariant kinetic equation is a linear combination of translation and rigid rotation. If the rigid rotation is excluded, the hydrodynamic velocity field and hence the temperature field are independent of x. If we set

$$a_k(x) = \frac{\mu_k^{\mathrm{e}}(x)}{k_B T_{\mathrm{e}}}, \tag{1.167}$$

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[10] As a matter of fact, at this point $T(x)$ here is not known to be the temperature field. We are anticipating it to be the temperature of the equilibrium system when thermodynamic correspondence is made according to the thermodynamic theory of processes as will be shown later. Thus we may set $T_{\mathrm{e}} = T(x)$.

Equation (1.163) becomes

$$\partial^\mu \mu_k^{\mathrm{e}}(x) + c^{-1} q_k U_\nu F^{\nu\mu}(x) = 0 \tag{1.168}$$

and we obtain

$$\partial^\mu \partial^\nu \mu_k^{\mathrm{e}}(x) + c^{-1} q_k U_\sigma \partial^\mu F^{\sigma\nu}(x) = 0. \tag{1.169}$$

Take the antisymmetric part of this equation

$$U_\sigma \left[\partial^\mu F^{\sigma\nu}(x) - \partial^\nu F^{\sigma\mu}(x)\right] = 0. \tag{1.170}$$

Making use of the homogeneous Maxwell equations

$$\partial^\mu F^{\nu\sigma}(x) + \partial^\mu F^{\sigma\mu}(x) + \partial^\sigma F^{\mu\nu}(x) = 0, \tag{1.171}$$

and taking the convective time derivative $D = U^\nu \partial_\nu$ it is concluded that

$$DF^{\nu\mu}(x) = 0. \tag{1.172}$$

This implies that equilibrium is possible only if $F^{\nu\mu}(x)$ does not change in time, provided it remains constant in the rest frame with respect to U^μ. The condition suggests that

$$DA^\mu(x) = 0 \tag{1.173}$$

according to (1.161), that is, $F^{\nu\mu}(x) = A^\mu(x)$ within an arbitrary function of spacetime. And (1.168) now reads

$$\partial^\mu \left[\mu_a^{\mathrm{e}}(x) + c^{-1} q_a U_\nu A^\mu(x)\right] = 0. \tag{1.174}$$

Therefore

$$\mu_a^{\mathrm{e}}(x) = \mu_a^{\mathrm{e}} - c^{-1} q_a U_\nu A^\mu(x) \tag{1.175}$$

with μ_a^{e} denoting a constant. Combining these results into (1.157), we now conclude that the equilibrium distribution function is uniquely given by the form

$$f_a^{\mathrm{e}}(p_a) = \Gamma_a^{\mathrm{e}-1} \exp\left\{-\left(k_B T_{\mathrm{e}}\right)^{-1}\left[p_a^\mu + c^{-1} q_a A^\mu(x)\right] U_\mu + \mu_a^{\mathrm{e}}\left(k_B T_{\mathrm{e}}\right)^{-1}\right\}, \tag{1.176}$$

where Γ_a^{e} is the normalization factor defined by

$$\Gamma_a^{\mathrm{e}-1} = h^3 \left\langle \exp\left\{-\left(k_B T_{\mathrm{e}}\right)^{-1}\left[p_a^\mu + c^{-1} q_a A^\mu(x)\right] U_\mu + \mu_a^{\mathrm{e}}\left(k_B T_{\mathrm{e}}\right)^{-1}\right\}\right\rangle. \tag{1.177}$$

Here we have inserted h^{-3} ($h =$ Planck constant) to make the normalization of the distribution function dimensionless. If there is no external force field, then

$$f_a^{\mathrm{e}}\left(p_a\right)=\exp\left\{-\left(k_B T_{\mathrm{e}}\right)^{-1}\left[p_a^{\mu} U_{\mu}-\mu_a^{\mathrm{e}}\right]-\ln \Gamma_a^{\mathrm{e}}\right\}. \tag{1.178}$$

The uniqueness of $f_a^{\mathrm{e}}\left(p_a\right)$ owes it to the H theorem. In fact, as in the case of the nonrelativistic Boltzmann equation the H theorem may be regarded as the Lyapounov stability theorem [16] of equilibrium and, in particular, the equilibrium state defining $f_a^{\mathrm{e}}\left(p_a\right)$ living in the phase space.

Given the equilibrium distribution function with the parameters $a_a(x)$ and $b_a^{\mu}(x)$ as in (1.167) and (1.166) the equilibrium entropy

$$S^{\mathrm{e}}=-k_B \sum_{a=1}^{r}\left\langle f_a^{\mathrm{e}} c^{-2} U_{\mu} p_a^{\mu} \ln f_a^{\mathrm{e}}\right\rangle \tag{1.179}$$

is easily evaluated. It is convenient to introduce entropy density $\mathcal{S}^{\mathrm{e}}$ per particle by the relation

$$S^{\mathrm{e}}=\rho \mathcal{S}^{\mathrm{e}}. \tag{1.180}$$

Then we find

$$\rho \mathcal{S}^{\mathrm{e}}=\frac{\rho}{T^{\mathrm{e}}}\left(\mathcal{E}-\sum_{a=1}^{r} \mu_a^{\mathrm{e}} \mathfrak{c}_a\right)+k_B \ln \prod_{a=1}^{r}\left(\Gamma_a^{\mathrm{e}}\right)^{\rho_a}. \tag{1.181}$$

Setting

$$p^{\mathrm{e}} v_a=k_B T^{\mathrm{e}} \ln \prod_{a=1}^{r} \Gamma_a^{\mathrm{e} \rho_a}:=k_B T_{\mathrm{e}} \ln \Gamma^{\mathrm{e}}. \tag{1.182}$$

where we have set

$$p^{\mathrm{e}}:=\mathfrak{p} \text{ (hydrostatic pressure)} \tag{1.183}$$

to uniformize the notation on the equilibrium intensive variables, we obtain the equilibrium (Boltzmann) entropy density

$$\rho \mathcal{S}^{\mathrm{e}}=\frac{\rho}{T^{\mathrm{e}}}\left(\mathcal{E}-\sum_{a=1}^{r} \mu_a^{\mathrm{e}} \mathfrak{c}_a+p^{\mathrm{e}} v\right). \tag{1.184}$$

For (1.181) we have made use of the definitions of internal energy and density at equilibrium given in terms of the equilibrium energy-momentum tensor $T_a^{\mathrm{e}\mu\nu}$ and the equilibrium particle four-flow $N_a^{\mathrm{e}\mu}$, respectively. Identification of $k_B T_{\mathrm{e}} \ln \Gamma^{\mathrm{e}}$—equilibrium grand partition function of the uncorrelated gas—is, as a matter of fact, the necessary and sufficient conditions for $\mathcal{S}^{\mathrm{e}}$ to exist in a bilinear form in the equilibrium thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. We will shortly find $\mathcal{S}^{\mathrm{e}}$ to be an integral of the differential form for $\mathcal{S}^{\mathrm{e}}$ in $\mathfrak{P} \cup \mathfrak{T}$.

Now to endow the physical meanings to the quantities in (1.181) the statistically calculated S^{e} is corresponded to *the thermodynamic entropy for reversible processes*. It should be recalled that similar correspondences are made to the statistical mechanically calculated average values $\rho\mathcal{E}$, ρ, and ρ_a as well as p^{e} to the corresponding thermodynamic quantities in accordance with the spirit of the Gibbs ensemble theory [20].

According to thermodynamics [10–13, 21] the Clausius entropy $\mathcal{S}^{\mathrm{e}}$ obeys the differential form

$$D\mathcal{S}^{\mathrm{e}} = T^{\mathrm{e}-1}\left(D\mathcal{E} + p^{\mathrm{e}}Dv - \sum_{a=1}^{r} \mu_a^{\mathrm{e}} D\mathfrak{c}_a\right). \tag{1.185}$$

There also holds the Gibbs–Duhem relation,

$$\sum_{a=1}^{r} \mathfrak{c}_a D\left(\frac{\mu_a^{\mathrm{e}}}{T^{\mathrm{e}}}\right) = \mathcal{E}D\left(\frac{1}{T^{\mathrm{e}}}\right) + vD\left(\frac{p^{\mathrm{e}}}{T^{\mathrm{e}}}\right). \tag{1.186}$$

This equation may be regarded as necessary and sufficient conditions for the integrability of the differential form (1.185) in the manifold $\mathfrak{P} \cup \mathfrak{T}$, because if (1.186) is added to (1.185), then there follows an integral of (1.185) in a bilinear form of variables in $\mathfrak{P} \cup \mathfrak{T}$ for $\mathcal{S}^{\mathrm{e}}$ within a constant:

$$\mathcal{S}^{\mathrm{e}} = T^{\mathrm{e}-1}\left(\mathcal{E} + p^{\mathrm{e}}v - \sum_{a=1}^{r} \mu_a^{\mathrm{e}} \mathfrak{c}_a\right). \tag{1.187}$$

In (1.185) and (1.186) T^{e} and p^{e} stand for the equilibrium temperature and the hydrostatic pressure, which was introduced earlier. The Gibbs–Duhem relation (1.186) can be obtained if the normalization condition for f_a^{e} given in (1.178) is varied with respect to $T^{\mathrm{e}-1}$, μ_a^{e}, p^{e}. The parameter p^{e} defined by (1.182) is more explicitly found since Γ_a^{e} can be easily calculated from the expression

$$\Gamma_a^{\mathrm{e}} = \left\langle \exp\left(\frac{m_a\widehat{\mu}_a^{\mathrm{e}}}{k_B T^{\mathrm{e}}}\right) \exp\left(-\frac{m_a C_a^2}{2k_B T^{\mathrm{e}}}\right)\right\rangle^{-1}. \tag{1.188}$$

Therefore the parameter p^{e} thermodynamically identified with pressure is computable from the statistical formula (1.188). In this manner, (1.185) and (1.186) mean that the equilibrium thermodynamic entropy $\mathcal{S}^{\mathrm{e}}$ is given, within a constant, by Formula (1.187). In equilibrium thermodynamics the thermodynamic temperature can be given by the derivative of Clausius's thermodynamic entropy $\mathcal{S}^{\mathrm{e}}$: in the local rest frame

$$T^{\mathrm{e}-1} = \frac{\partial \mathcal{S}^{\mathrm{e}}}{\partial E}. \tag{1.189}$$

Making correspondence between the statistical mechanical equilibrium Boltzmann entropy and the thermodynamic Clausius entropy

$$\mathcal{S}^{\mathrm{e}}|_{\mathrm{st}} \Leftrightarrow \mathcal{S}^{\mathrm{e}}|_{\mathrm{th}} \tag{1.190}$$

and also between other quantities in (1.181) and (1.187) we are now able to identify the parameter b with the inverse temperature

$$b_a = -\frac{1}{k_B T^{\mathrm{e}}} := -\beta_{\mathrm{e}} \tag{1.191}$$

with k_B denoting the Boltzmann constant, and the parameters a_a with equilibrium chemical potentials

$$a_a = \frac{\mu_a^{\mathrm{e}}}{k_B T^{\mathrm{e}}}. \tag{1.192}$$

Finally, a_a and b_a having been so found, the equilibrium distribution functions f_a^{e} is fully determined with the help of the H theorem and the phenomenological equilibrium thermodynamics:

$$f_a^{\mathrm{e}} = \exp\left[-\beta_{\mathrm{e}}\left(U_\mu p_a^\mu - \mu_a^{\mathrm{e}}\right) - \ln \Gamma_a^{\mathrm{e}}\right], \tag{1.193}$$

which in the local rest frame may be written as

$$f_a^{\mathrm{e}} = \exp\left[-\beta_{\mathrm{e}}\left(c p_a^0 - \mu_a^{\mathrm{e}} + p_a^{\mathrm{e}} v_a\right)\right], \tag{1.194}$$

since $U_\mu = (c, 0, 0, 0)$ in the local rest frame. Incidentally, the equilibrium distribution function determined here[11] without $\exp\left(-\beta_{\mathrm{e}} p_a^{\mathrm{e}} v_a\right)$ is called the Jüttner function. It is a unique solution of the covariant Boltzmann equation, thanks to the H theorem. It will guide us to develop the nonequilibrium theory we have in mind. In the manner described above, for reversible processes the Boltzmann entropy at equilibrium is equal to the Clausius entropy according to the second law of thermodynamics.

Before proceeding to the next stage of development of the theory, we would like to add a supplemental discussion on the question of temperature in relativistic theory, which we consider a rather important subject worth a further consideration in this chapter.

On the Question of Temperature in the Relativistic Theory

Before we deal with the question of temperature let us recap how f_a^{e} is determined. Based on the H theorem, the equilibrium solution f_a^{e} to the covariant Boltzmann equation is uniquely determined. The result is the well known Jüttner function [3] in which μ_a^{e} is the chemical potential, which also nominally serves the role of normalization factor for f_a^{e}, and β_{e} a parameter proportional to the inverse temperature: $\beta_{\mathrm{e}} := 1/k_B T^{\mathrm{e}}$. The physical meanings of the parameters are determined by comparing the thermodynamic entropy, pressure, and internal energy with the statistical

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[11] In this form the normalization factor for f_a^{e} is given by

$$n_a^{\mathrm{e}} \exp\left(-\beta_{\mathrm{e}} \mu_a^{\mathrm{e}}\right) / \left\langle \exp(-\beta_{\mathrm{e}} U_\nu p_a^\nu \right\rangle.$$

mechanical entropy, pressure, and internal energy calculated with the equilibrium solution of the kinetic equation by following the basic strategy of the Gibbs ensemble theory. If f_a^{e} in (1.178) is used in (1.179), the equilibrium entropy four-flow is found given by

$$S^{\mathrm{e}\mu} = k_B \beta_{\mathrm{e}} \sum_{a=1}^{r} \left(T_{\mathrm{e}a}^{\mu\nu} U_\nu + N_{\mathrm{e}a}^{\mu}/\beta_{\mathrm{e}} - \mu_a^{\mathrm{e}} N_{\mathrm{e}a}^{\mu} \right). \tag{1.195}$$

We thus find that the equilibrium scalar entropy is given by (1.187) and the thermodynamic temperature by (1.189). By making S^{e} in (1.195) correspond to the thermodynamic entropy, and other terms in it, to the energy, pressure and density, respectively, we are able to find the meaning of parameter T^{e}: it is thus identified with the thermodynamic absolute temperature: $T_{\mathrm{t}} = T^{\mathrm{e}}$. Therefore we have concluded

$$\beta_{\mathrm{e}} = 1/k_B T^{\mathrm{e}} = 1/k_B T_{\mathrm{t}}. \tag{1.196}$$

Because of this relation, T_{t} and T^{e} are henceforth interchangeably used. In fact, T^{e} means T_{t} unless stated otherwise. Since the chemical potential is defined in thermodynamics by

$$\mu_a^{\mathrm{e}}/T^{\mathrm{e}} = -\left(\frac{\partial S_{\mathrm{e}}}{\partial \rho_a} \right), \tag{1.197}$$

the normalization factor μ_a^{e} is indeed the chemical potential. The ideal gas equation of state is identified by

$$p^{\mathrm{e}} = \rho k_B T^{\mathrm{e}} \quad \left(v = \rho^{-1} \right). \tag{1.198}$$

It must be remembered that this T^{e} is the temperature of the equilibrium system.

The hydrodynamic pressure has been given by (1.57) in terms of the equilibrium energy-momentum tensor which we will work out more explicitly:

$$p^{\mathrm{e}} = \sum_{a=1}^{r} p_a^{\mathrm{e}} = \frac{1}{3} \sum_{a=1}^{r} \Delta_{\mu\nu} T_{\mathrm{e}a}^{\mu\nu} = \frac{1}{3} \sum_{a=1}^{r} \Delta_{\mu\nu} \left\langle p_a^{\mu} p_a^{\nu} f_a^{\mathrm{e}} \right\rangle. \tag{1.199}$$

Substituting the equilibrium distribution function (1.193) into it, we find

$$p^{\mathrm{e}} = \sum_{a=1}^{r} 4\pi m_a c^2 \beta_{\mathrm{e}}^{-2} K_2(m_a c^2 \beta_{\mathrm{e}}) e^{\beta_{\mathrm{e}} \mu_a^{\mathrm{e}}}, \tag{1.200}$$

where $K_2(z)$ is a modified Bessel function [22] of the second kind. The density ρ may be similarly calculated:

$$\rho = c^{-2} U_\mu N^\mu = \sum_{a=1}^{r} 4\pi m_a c^2 \beta_{\mathrm{e}}^{-1} K_2(m_a c^2 \beta_{\mathrm{e}}) e^{\beta_{\mathrm{e}} \mu_a^{\mathrm{e}}}. \tag{1.201}$$

This, together with (1.200), implies the equation of state (1.198). Therefore, the thermodynamically defined pressure through the equation of state (1.198) is consistent with the statistical mechanical definition of p^{e} in (1.200). The situation thus is seen to be the same as in the nonrelativistic kinetic theory in that the kinetically and thermodynamically defined pressure are identical. However, this is not the case for the internal energy.

In the nonrelativistic kinetic theory the temperature is defined such that the following relation holds true between the internal energy and the temperature [4, 23]

$$\mathcal{E} = \frac{3}{2} k_B T^{\mathrm{e}}. \tag{1.202}$$

The temperature defined in this manner is called the kinetic temperature and coincides with the thermodynamic temperature defined through the thermodynamic relation

$$T^{\mathrm{e}-1} = \frac{\partial S_{\mathrm{e}}}{\partial E}. \tag{1.203}$$

However, in the relativistic theory the situation is altered significantly since if the kinetic temperature were defined by (1.202), then it would not agree with the thermodynamic temperature defined by means of (1.203). Nevertheless, (1.203) is consistent with (1.198). One recovers the coincidence between (1.202) and (1.203) in the limit of $u/c \to 0$, *if the rest mass energy* mc^2 *is neglected;* see Ref. [23] for expression for $\mathcal{E}$ in the nonrelativistic theory. However, it would be preferable to have the thermodynamic temperature coincide with the kinetic temperature in the relativistic kinetic theory as well, since the kinetic theory should be a molecular representation of thermodynamics underlying macroscopic processes. Here we define the kinetic temperature for a system of particles with finite masses such that it precisely coincides with the thermodynamic temperature as follows:

$$\frac{3}{2} \rho k_B T^{\mathrm{e}} = \sum_{a=1}^{r} \left(2c^2\right)^{-1} \left\langle \left(p_a^\mu p_a^\nu U_\mu U_\nu - m_a^2 c^4\right) f_a^{\mathrm{e}}(x, p_a) \right\rangle. \tag{1.204}$$

This definition yields a correct nonrelativistic limit. It is rooted basically in the idea that the temperature is a measure of kinetic energy. Recall that a similar subtraction of rest mass energy was made when the moment $h_a^{(3)\mu}$ for heat flux was defined in the previous section.

Since the integrand in (1.204) is scalar and its value is independent of the frame of reference, it is convenient to choose the local rest frame, where $U_\mu = (c, 0, 0, 0)$, to evaluate the integral. Then,

$$f_a^{\mathrm{e}} = \exp\left[-\beta_{\mathrm{e}} \left(c p_a^0 - \mu_a^{\mathrm{e}}\right)\right], \tag{1.205}$$

where $\beta_{\mathrm{e}} = 1/k_B T_{\mathrm{t}}$ with the subscript t restored to mean the thermodynamic temperature. Thus we find

$$\frac{3}{2}\rho k_B T^{\mathrm{e}} = \sum_{i=1}^{r} \frac{1}{2} e^{\beta_{\mathrm{e}}\mu_a^{\mathrm{e}}} \int d^3\mathbf{p}_a \left(\varepsilon_a - m_a^2 c^4/\varepsilon_a\right) e^{-\beta_{\mathrm{e}} e_a}, \tag{1.206}$$

where

$$\varepsilon_a = c p_a^0 = \left(c^2 \mathbf{p}_a^2 + m_a^2 c^4\right)^{1/2}. \tag{1.207}$$

With the transformations

$$z_a = m_a c^2 \beta_{\mathrm{e}}, \quad \tau_a = \beta_{\mathrm{e}} e_a,$$

Equation (1.206) may be put in the form

$$\frac{3}{2}\rho k_B T^{\mathrm{e}} = \sum_{a=1}^{r} \frac{1}{2} e^{\beta_{\mathrm{e}}\mu_a^{\mathrm{e}}} 2\pi c^{-3} \beta_{\mathrm{e}}^{-4} \frac{4!}{2^2 2!} z_a^2 K_2(z_a). \tag{1.208}$$

In view of the fact that the chemical potential is given by

$$e^{-\beta_{\mathrm{e}}\mu_a^{\mathrm{e}}} = 4\pi m_a^2 c (\rho_a \beta_{\mathrm{e}})^{-1} K_2(z_a),$$

it follows from (1.208) that

$$T^{\mathrm{e}} = T_{\mathrm{t}}$$

since the right-hand side of (1.208) is equal to $\frac{3}{2}\rho k_B T_{\mathrm{t}}$. This proves the coincidence of the kinetic temperature with the thermodynamic temperature, if the kinetic temperature is defined as stated in (1.204). It must be noted that (1.204) is another way of expressing Tolman's equipartition theorem [24]. Therefore, the definition of temperature by (1.204) is seen to be based on the equipartition law. In other words, the relativistic kinetic temperature may be defined by means[12] of the equipartition theorem of Tolman, *which equally holds for both nonrelativistic and relativistic theories.* This interpretation of the definition of kinetic temperature provides us with *a universal way of defining temperature for both nonrelativistic and relativistic theories: the equipartition law of Tolman.*

The local rest frame version of the definition (1.204) of kinetic temperature in relativity was suggested by ter Haar and Wegeland [25], and it was later taken by Menon and Agrawal [26], but, as pointed out by Landsberg [27], their definition is preceded by Tolman [24]. On the other hand, in the Chapman-Enskog theory [23] the kinetic temperature is defined such that the equation of state in thermodynamics and in kinetic theory coincide with each other. Therefore, there appear to be two different modes of defining the temperature, but they are, in fact, identical at least in the case of dilute gases. Since

$$p_a^\mu p_a^\nu U_{\mathrm{e}\mu} U_{\mathrm{e}\nu} - m_a^2 c^4 = c^2 p_a^\mu p_a^\nu \Delta_{\nu\mu},$$

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[12] In the approach taken in the present work the thermodynamic correspondence is taken alternatively to Tolman's equipartition law.

substitution of this relation in (1.204) yields

$$\frac{3}{2}\rho k_B T_{\mathrm{e}} = \frac{1}{2}\Delta_{\sigma\mu} T_{\mathrm{e}}^{\mu\sigma} = \frac{3}{2} p^{\mathrm{e}},$$

which implies the equation of state for ideal gas at equilibrium. Henceforth p^{e} will be typeset p, as we have done in Chaps. 5–7 of Volume 1, without the superscript e for brevity of notation.

In summary of the discussion on temperature and related quantities in this section, we would like to point out the following: In order to maintain the continuity of nonrelativistic and relativistic kinetic theory and thermodynamics with regard to temperature and related quantities the rest mass energy contribution $m_a c^2$ must be subtracted from $p_a^{\mu} U_{\mu}$ appearing in the molecular expression in the kinetic energy or energy-momentum tensor. For example, see Sect. 1.1.6 where molecular moments $h_a^{(q)\mu\cdots\nu}$ are constructed for nonconserved macroscopic variables, especially, $h_a^{(3)\mu}$ for heat flux.

1.1.8 Functional Hypothesis

The covariant kinetic equations are first-order partial differential equations in the phase space. However, in relativistic kinetic theories as in nonrelativistic kinetic theories, they are not solved in the usual manner as we solve partial differential equations subject to initial and boundary conditions in the phase space. The reason is the same as for nonrelativistic kinetic equations, as discussed in Chaps. 3, 5, 6, and 7 of Volume 1, that the information needed for the purpose is overwhelming owing to the fact that there are too many particles to contend with. In the nonrelativistic kinetic theories discussed in the previous chapters of this work, we have seen approximate methods are used in a quite different approach under the aegis of a functional hypothesis. The details of the solution methods may vary depending on the approaches taken such as the Chapman–Enskog method [23, 28] or the moment methods [29], but the underlying feature is invariably a functional hypothesis, albeit not explicitly mentioned as such, because in this approach the distribution functions evolve as functionals of either the conserved variables or the variables of thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$.

Under a functional hypothesis, the solution of a kinetic equation, or more precisely put, the thermodynamic branch of the distribution function obeying the kinetic equation is looked for as a functional of macroscopic variables that the kinetic equation is purported to describe at the macroscopic level of description of the fluid. For example, in Enskog's method [28] and Chapman's method [23] of solving the Boltzmann equation we are looking for the distribution function as a functional of conserved macroscopic variables obeying the conservation laws, whereas in Grad's moment method [29] or in the modified moment method [4] the distribution function is assumed to be a functional of a complete set of macroscopic variables (moments) obeying their evolution equations. In practice, in the moment methods the set is truncated to a finite set as an approximation. To implement this idea we introduce the thermodynamic manifold, which is a union of manifold $\mathfrak{P}$ spanned by density, concentrations, velocity, internal energy, and fluxes and variables belonging to the manifold $\mathfrak{T}$ tangential to $\mathfrak{P}$; see Chap. 2 of Volume 1 for the precise meanings of manifolds $\mathfrak{P}$ and $\mathfrak{T}$. The union of these two manifolds will be also called the thermodynamic manifold as in the case of nonrelativistic theory.

In the case of relativistic kinetic theory we will naturally follow a similar approach and make the following functional hypothesis for the distribution function obeying the covariant Boltzmann equation:

Functional Hypothesis. *The nonequilibrium distribution function* f_a^{c} *evolves as a functional of macroscopic variables belonging to the manifold* $\mathfrak{P} \cup \mathfrak{T}$ and *obeying macroscopic evolution equations derived from the relativistic kinetic equation—namely, covariant (relativistic) Boltzmann equation:*

$$f_a^{\mathrm{c}}(x, p_a) = f_a(x, p_a | \mathfrak{P} \cup \mathfrak{T}). \tag{1.209}$$

The specific form of the thermodynamic branch may depend on whether the fluid mixture has a fixed composition or a variable composition, as the system exchanges particles with its surroundings. The form of thermodynamic branch of the distribution function in the case of a fixed composition, under the postulate of functional hypothesis, is assumed given by the nonequilibrium canonical form

$$f_a^{\mathrm{c}}(x, p_a) := \exp\left[-\beta\left(p_a^{\nu} U_{\nu} + H_a^{(1)} - \mu_a\right) - \ln \Gamma_a\right], \tag{1.210}$$

where

$$H_a^{(1)} = \sum_{q \geq 1} X_{a\nu\cdots\sigma}^{(q)} h_a^{(q)\sigma\cdots\nu} := \sum_{q \geq 1} X_a^{(q)} \odot h_a^{(q)} \tag{1.211}$$

and $\ln \Gamma_a$ is the normalization factor. If $f_a^{\mathrm{c}}(x, p_a)$ is normalized to a dimensionless number,[13] the normalization factor is given by the integral

$$\Gamma_a = n_a^{-1} \left\langle \exp\left[-\beta\left(p_a^{\nu} U_{\nu} + \sum_{q \geq 1} X_{a\nu\cdots\sigma}^{(q)} h_a^{(q)\sigma\cdots\nu} - \mu_a\right)\right]\right\rangle, \tag{1.212}$$

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[13] This means that the distribution function is given in the units of action cubed, e.g., h^3, where h is the Planck constant. This requires that the other averages defined earlier are in the units of h^3, the dependence of which can be appropriately made explicit later. Provided we remember this adjustment in dimension for them there is no harm done even if the formulation is carried out without the factor h^3 in the normalization factor.

where, with the factor h^{-3} inserted, the angular brackets $\langle A\rangle$ stand for

$$\langle A\rangle = \frac{c}{h^3}\int \frac{d^3 p_a}{p_a^0} A\left(p_a\right). \tag{1.213}$$

Here the Planck constant h cubed is inserted to make the integral dimensionless. Therefore the thermodynamic branch of distribution function $f_a^{\mathrm{c}}(x, p_a)$ may now be written as

$$f_a^{\mathrm{c}}(x, p_a) = \frac{1}{\Gamma_a}\exp\left[-\beta\left(p_a^{\nu}U_{\nu} + H_a^{(1)} - \mu_a\right)\right]. \tag{1.214}$$

The generalized potentials $X_{a\nu\cdots\sigma}^{(q)}$ do not depend on momentum p_a, but can be dependent on spacetime variable $x = (ct, \mathbf{x})$ on account of the fact that they are functionals of macroscopic variables, which are mean values of moments $h_a^{(q)}$ and, as such, depend on spacetime variable x. The canonical form $f_a^{\mathrm{c}}(x, p_a)$ given in (1.214) is a particular form admissible under the functional hypothesis that can, as will be shown later, produce a thermodynamic theory of irreversible macroscopic processes and hydrodynamics, both consistent with the laws of thermodynamics. For this reason $f_a^{\mathrm{c}}(x, p_a)$ given in (1.214) is called a thermodynamic branch of the distribution function obeying the relativistic Boltzmann equation postulated.

For the nonequilibrium contribution $H_a^{(1)}$ in f_a^{c} we have taken a linear combination of the tensorial moment set $\left\{h_a^{(q)\mu\cdots\nu}\right\}$ $(q \geq 1)$, because f_a^{c} is sought as a functional of macroscopic nonconserved variables $\left\{\Phi_a^{(q)\mu\cdots\nu}\right\}$ $(q \geq 1)$ among other macroscopic variables and, if the set $\left\{h_a^{(q)\mu\cdots\nu}\right\}$ $(q \geq 1)$ is averaged over the distribution, it gives rise to the set $\left\{\Phi_a^{(q)\mu\cdots\nu}\right\}$ $(q \geq 1)$. Henceforth, for notational brevity the superscripts $\mu\cdots\nu$ on the tensors will be omitted. In (1.211) $X_a^{(q)}$ are expansion coefficients which are conjugate tensors to the moment tensors $h_a^{(q)}$ and depend only on macroscopic variables belonging to the thermodynamic manifold $\mathfrak{P}\cup\mathfrak{T}$. These expansion coefficients, which we call the generalized potentials of irreversible transport processes, will have to be determined in terms of variables in the thermodynamic manifold in a thermodynamically consistent manner subjected to the laws of thermodynamics. The evolution of $f_a^{\mathrm{c}}(x, p_a)$ in spacetime is then determined as a functional of macroscopic variables through $X_a^{(\alpha)}$, β, and μ_a on which the local normalization factor Γ_a also depends. In the following we will often use the symbol $\odot$ for the notation of tensorial contraction for the sake of brevity of notation, whenever confusion or ambiguity would not arise thereby.

Since we are dealing with a mixture of non-interacting, uncorrelated molecules, the total distribution function of an N-molecule mixture is of interest. We may write it in the form

$$f_{\mathrm{c}}^{(N)}(x, p) = N!\prod_{a=1}^{r}\frac{\left(f_a^{\mathrm{c}}\right)^{N_a}}{N_a!}$$

$$= N!\prod_{a=1}^{r}\frac{1}{N_a!}\exp\left\{-\beta\sum_{ka=1}^{N_a}\left[p_{ka}^{\nu}U_{\nu} + H_{ka}^{(1)} - \mu_a + \frac{\ln\left(h^3\Gamma_a\right)}{\beta}\right]\right\}. \tag{1.215}$$

Here a composite subscript ka is introduced to distinguish different particles belonging to different species. The subscript ka means particle $k \in a$ (a = species). This form for $f_{\mathrm{c}}^{(N)}(x, p)$ may be regarded as the thermodynamic branch of the grand canonical distribution function for the mixture of r non-interacting components of ideal gas. Note that

$$N!/\prod_{a=1}^{r}N_a!$$

is the number of ways randomly distributing $N = \sum_{a=1}^{r} N_a$ particles into boxes of volume V, and f_a^{c} is the statistical weight of each assignment. Here we may set

$$\prod_{a=1}^{r}\prod_{ka=1}^{N_a}\exp\left(\ln\Gamma_a\right) = \exp\left(\ln\Gamma\right), \tag{1.216}$$

where Γ will now turn out to be the local form of nonequilibrium grand canonical partition function of the gas:

$$\beta^{-1}\ln\Xi = \int_V d\mathbf{r}\rho\beta^{-1}\ln\Gamma. \tag{1.217}$$

1.1.9 Calortropy

If we closely examine the original formulation of the concept of entropy by R. Clausius [14], we are unmistakably led to the conclusion that the Clausius entropy is for an equilibrium system that undergoes a reversible thermodynamic process. Nevertheless, in the kinetic theory literature the concept of entropy is indiscriminately applied whether the process in question is reversible or irreversible, especially, in most of works related to thermodynamics and irreversible thermodynamics. Although Clausius did not define the entropy for relativistic reversible processes, his reasoning can be applied to relativistic reversible processes and the concept of entropy can be seen to remain still valid, but for reversible processes only. Therefore, the Boltzmann entropy defined by (1.140) for arbitrary nonequilibrium processes cannot be simply identified with the equilibrium entropy, or Clausius entropy, if the process under consideration is away from equilibrium or irreversible. We have shown that the nonrelativistic kinetic theories could be made physically realistic by corresponding their thermodynamic formalism to the thermodynamically consistent phenomenological irreversible thermodynamics and the thermodynamic formalism requires the so-called calortropy that is definitely different from the Clausius entropy. Therefore, in the relativistic theory it would be also necessary to define the statistical mechanical calortropy (more precisely, calortropy four-flow in the relativistic theory) that cannot be necessarily the same as the Boltzmann entropy four-flow, if the macroscopic processes involved are of nonequilibrium or irreversible.

Here we define calortropy four-flow $\Psi^{\mu}(x)$ in terms of the nonequilibrium canonical form as follows:

$$\Psi^{\mu}(x) = -k_B \sum_{a=1}^{r} \left\langle p_a^{\mu} f_a(x, p_a) \ln f_a^{\mathrm{c}}(x, p_a) \right\rangle . \tag{1.218}$$

It is emphasized that $f_a^{\mathrm{c}}(x, p_a)$ is not the same as $f_a(x, p_a)$ that is the exact solution of the covariant kinetic equation in the phase space. Therefore, $\Psi^{\mu}(x)$ is not equal to $S^{\mu}(x)$. The scalar density $\widehat{\Psi}$ of calortropy can then be obtained by projecting Ψ^{μ} onto U^{μ}:

$$\Psi = \rho\widehat{\Psi} = c^{-2} U_{\mu} \Psi^{\mu}(x) . \tag{1.219}$$

Since the substantial time derivative of $\widehat{\Psi}$ can be cast into the form

$$\mathcal{D}\widehat{\Psi} = \rho D\widehat{\Psi} = \partial_{\mu}\left(\widehat{\Psi} N^{\mu}\right) \tag{1.220}$$

as is for the case of the Boltzmann entropy density, the evolution equation for the scalar calortropy density $\widehat{\Psi}$ is rearranged to the form

$$\rho D\widehat{\Psi} = -\partial_{\mu}\left(\Psi^{\mu} - \widehat{\Psi} N^{\mu}\right) + \partial_{\mu}\Psi^{\mu}. \tag{1.221}$$

Upon use of the definition for Ψ^{μ} in (1.218) and the covariant kinetic equation we then obtain the calortropy balance equation

$$\begin{aligned} \partial_{\mu}\Psi^{\mu} = & -k_B \sum_{a=1}^{r} \left\langle f_a p_a^{\mu} \partial_{\mu} \ln f_a^{\mathrm{c}}(x, p_a) \right\rangle \\ & - k_B \sum_{a=1}^{r} \sum_{b=1}^{r} \left\langle \ln f_a^{\mathrm{c}}(x, p_a) C_{ab}(f_a, f_b) \right\rangle . \end{aligned} \tag{1.222}$$

With the definitions of calortropy flux four-flow J_{c}^{μ} and calortropy production $\sigma_{\mathrm{c}}(x)$ by the formulas

$$J_{\mathrm{c}}^{\mu} = \Psi^{\mu} - \widehat{\Psi} N^{\mu}, \tag{1.223}$$

$$\sigma_{\mathrm{c}}(x) = -k_B \sum_{a=1}^{r} \sum_{b=1}^{r} \left\langle \ln f_a^{\mathrm{c}}(x, p_a) C_{ab}(f_a, f_b) \right\rangle , \tag{1.224}$$

the calortropy balance equation is obtained:

$$\rho D\widehat{\Psi} = -\partial_\mu J_{\mathrm{c}}^\mu - k_B \sum_{a=1}^{r} \left\langle f_a p_a^\mu \partial_\mu \ln f_a^{\mathrm{c}}(x, p_a) \right\rangle + \sigma_{\mathrm{c}}(x) \,. \tag{1.225}$$

Here the calortropy production $\sigma_{\mathrm{c}}(x)$ is definitely not the same as the Boltzmann entropy production $\sigma_{\mathrm{ent}}(x)$, nor is the calortropy four-flow J_{c}^μ the same as the Boltzmann entropy four-flow. If the projector Δ_ν^μ is made use of, the calortropy four-flow J_{c}^μ in (1.223) can be cast into a concise form as a projection of calortropy four-flow Ψ^ν onto hydrodynamic flow direction μ:

$$J_{\mathrm{c}}^\mu = \Delta_\nu^\mu \Psi^\nu. \tag{1.226}$$

Insertion of (1.218) into this expression yields the statistical mechanical expression for J_{c}^μ:

$$J_{\mathrm{c}}^\mu = -k_B \sum_{a=1}^{r} \Delta_\nu^\mu \left\langle p_a^\mu f_a(x, p_a) \ln f_a^{\mathrm{c}}(x, p_a) \right\rangle. \tag{1.227}$$

Therefore, J_{c}^μ is interpretable as a projection of the flux of calortropy Ψ^ν in the direction of hydrodynamic velocity U^μ.

The integrability of the calortropy balance equation in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ will be studied in detail and a thermodynamic theory of irreversible processes will be formulated later in this chapter. It should be pointed out that the presence of the second term on the right of (1.225)

$$-k_B \sum_{a=1}^{r} \left\langle f_a p_a^\mu \partial_\mu \ln f_a^{\mathrm{c}}(x, p_a) \right\rangle$$

is a crucial feature that makes the balance equation for the calortropy density integrable in $\mathfrak{P} \cup \mathfrak{T}$, different from the Boltzmann entropy balance equation (1.149), since the latter would not be integrable in $\mathfrak{P} \cup \mathfrak{T}$ owing to the absence of the aforementioned term as will be shown later. Consequently, it denies the relativistic Boltzmann entropy to acquire an exact differential form in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ and a theory of irreversible thermodynamics formulated in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. We have already seen that the same is true in the case of nonrelativistic theories, compelling us to introduce the notion of calortropy in lieu of entropy.

1.1.10 Relative Boltzmann Entropy

The calortropy is not the same as the Boltzmann entropy except at equilibrium or for reversible processes, and if there are irreversible processes present in the system they are, clearly, different. We, therefore, define the difference between them as the relative Boltzmann entropy (RBE):

$$S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right](x)=\Psi^{\mu}(x)-S^{\mu}(x)\,, \tag{1.228}$$

which, expressed in statistical mechanical form, is given by the formula

$$S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right](x)=k_B\sum_{a=1}^{r}\left\langle p_a^{\mu}\ln\left(\frac{f_a}{f_a^{\mathrm{c}}}\right)f_a\right\rangle. \tag{1.229}$$

The relative Boltzmann entropy can be shown positive semidefinite owing to the Klein inequality $x\ln x-x+1\geq 0$:

$$S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right](x)\geq 0, \tag{1.230}$$

the equality holding at equilibrium where $X_{a\sigma\cdots\nu}^{(q)}\to 0$ for all $q\geq 1$. Therefore, away from equilibrium the calortropy four-flow $\Psi^{\mu}(x)$ evidently is not the same as the Boltzmann entropy four-flow $S^{\mu}(x)$. We may also define the relative Boltzmann entropy density by the formula

$$\rho\mathcal{S}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=c^{-2}U_{\mu}S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right]. \tag{1.231}$$

The balance equation for the relative Boltzmann entropy density can then be formally given in the form

$$\rho D\mathcal{S}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=-\partial_{\mu}\left(\Delta_{\nu}^{\mu}S_{\mathrm{r}}^{\nu}\left[f|f^{\mathrm{c}}\right]\right)-k_B\sum_{a=1}^{r}\left\langle f_a p_a^{\mu}\partial_{\mu}\ln f_a^{\mathrm{c}}\right\rangle+\sigma_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right], \tag{1.232}$$

where the relative Boltzmann entropy production $\sigma_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]$ is given by the formula

$$\sigma_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=k_B\sum_{a=1}^{r}\sum_{b=1}^{r}\left\langle\ln\left(f_a/f_a^{\mathrm{c}}\right)C_{ab}(f_a,f_b)\right\rangle. \tag{1.233}$$

With the definition of the relative Boltzmann entropy flux four-flow

$$J_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=\Delta_{\nu}^{\mu}S_{\mathrm{r}}^{\nu}\left[f|f^{\mathrm{c}}\right], \tag{1.234}$$

Equation (1.232) can be put in a more cogent form

$$\rho D\mathcal{S}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=-\partial_{\mu}J_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]-k_B\sum_{a=1}^{r}\left\langle f_a p_a^{\mu}\partial_{\mu}\ln f_a^{\mathrm{c}}\right\rangle+\sigma_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]. \tag{1.235}$$

This relative Boltzmann entropy balance equation will be examined further, after the calortropy balance equation is integrated in the thermodynamic manifold.

1.2 Pfaffian Differential Form for Calortropy

Having defined the calortropy and obtained its balance equation, there arises the question of its dependence on the variables spanning the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ and its precise mathematical form. This question can be restated as the integrability of the calortropy balance equation in the thermodynamic manifold. It is an important question because, when integrated in the thermodynamic manifold, the calortropy and its differential form would provide us with the theory of thermodynamics of irreversible processes framed in the thermodynamic manifold. As a matter of fact, such a theory would be one of our important goals, as has been for the nonrelativistic theories studied in the kinetic theory chapters of Volume 1 in this work.

There are two routes of approach to this question. One is to take the statistical mechanical definition of calortropy and the nonequilibrium canonical form $f_a^{\mathrm{c}}(x, p_a)$ to calculate Ψ directly in terms of variables in manifold $\mathfrak{P} \cup \mathfrak{T}$. The other approach is to investigate the integrability of the calortropy balance equation by calculating the statistical mechanical formulas for quantities appearing on its right hand side that include calortropy four-flow, the calortropy production, and so on. Together with the result by the first approach, this second approach would provide the integrability proof for the calortropy differential form (i.e., extended Gibbs relation for calortropy), or equivalently the calortropy balance equation, and thereby a mathematical foundation of a thermodynamic theory of irreversible processes in dilute relativistic fluids. The extended Gibbs relation provides a working mathematical tool—a fundamental equation—for the thermodynamics of irreversible processes, as does the equilibrium Gibbs relation in the theory of equilibrium thermodynamics [10–13, 21].

1.2.1 Statistical Mechanical Method

The first method mentioned earlier uses the nonequilibrium canonical form $f_a^{\mathrm{c}}(x, p_a)$ in the statistical mechanical formula for the calortropy four-flow and calculate the calortropy density therewith. The calculation required is simple, and we obtain the expression for the scalar calortropy Ψ. Since

$$\Psi = \rho\widehat{\Psi} = c^{-2} U_\mu \Psi^\mu,$$

upon insertion of $f_a^{\mathrm{c}}(x, p_a)$ into the statistical mechanical formula for Ψ^μ it is easy to obtain the expression for calortropy density:

$$\widehat{\Psi} = T^{-1}\mathcal{E} + \sum_{a=1}^{r} \sum_{q\geq 1} \overline{X}^{(q)}_{a\nu\cdots\sigma} \widehat{\Phi}^{(q)\sigma\cdots\nu}_{a} - \sum_{a=1}^{r} \mathfrak{c}_a \overline{\mu}_a + k_B \rho^{-1} \ln \Gamma, \tag{1.236}$$

where

$$\overline{X}^{(q)}_{a\nu\cdots\sigma} = \frac{X^{(q)}_{a\nu\cdots\sigma}}{T}, \quad \overline{\mu}_a = \frac{\mu_a}{T} \tag{1.237}$$

and $\ln\Gamma$ is defined in terms of Γ_a for the nonequilibrium partition function of component a:

$$\ln\Gamma = \ln\prod_{a=1}^{r}\Gamma_a^{\rho_a}. \tag{1.238}$$

In anticipation of the later result obtained through thermodynamic correspondence of $\widehat{\Psi}$ with the phenomenological form[14] in Chap. 2 of Volume 1, we will further set

$$pv = k_B T \ln\Gamma. \tag{1.239}$$

Here, as it stands, p is an as-yet undetermined parameter. But its global form is given by

$$k_B T \ln\Xi = \int_V d\mathbf{r}\rho k_B T \ln\Gamma = \int_V d\mathbf{r} p = \langle p\rangle_V V, \tag{1.240}$$

where $\langle p\rangle_V$ is the volume average of parameter p:

$$\langle p\rangle_V = V^{-1}\int_V d\mathbf{r} p\left(\mathbf{r}\right). \tag{1.241}$$

In summary of the chain of equalities in (1.240), we obtain the global expression for $k_B T \ln\Xi$:

$$k_B T \ln\Xi = \langle p\rangle_V V. \tag{1.242}$$

Finally, provided (1.239) holds, (1.236) may be written as

$$\widehat{\Psi} = T^{-1}\mathcal{E} + \overline{p}v - \sum_{a=1}^{r}\overline{\mu}_a\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\overline{X}^{(q)}_{a\nu\cdots\sigma}\widehat{\Phi}_a^{(q)\sigma\cdots\nu}, \tag{1.243}$$

where

$$\overline{p} = \frac{p}{T}, \quad v = \rho^{-1} \quad (v = \text{specific volume}). \tag{1.244}$$

It should be noted that the parameters T, $X^{(q)}_{a\nu\cdots\sigma}$, μ_a, and p are not as yet clarified of their meanings, experimental or thermodynamical. Nevertheless, the expression for $\widehat{\Psi}$ in (1.237) evidently indicates that it is a bilinear form of variables in $\mathfrak{P}$ and their conjugate (in fact, tangent field) variables belonging to $\mathfrak{T}$, if we accept p as the

---

[14] Since the phenomenological theory of irreversible relativistic processes is not established, the theory formulated here is the reverse of that of the nonrelativistic theory taken in the nonrelativistic chapter. Here by extending the nonrelativistic theory into the relativistic domain it may be said that a relativistic theory of thermodynamics of irreversible processes is formulated.

tangent field to volume v. Note that we have made use of the statistical mechanical definitions of $\rho\mathcal{E}$, $\widehat{\Phi}_a^{(q)\sigma\cdots\nu}$, and N_a^μ to obtain (1.236). We remark that since $X_a^{(\alpha)} \odot \widehat{\Phi}_a^{(\alpha)} \neq 0$ and $\mu_a \neq \mu_a^{\mathrm{e}}$ if the system is removed from equilibrium, the calortropy density $\widehat{\Psi}$ in (1.237) evidently is not the same as the Clausius entropy for a reversible process in a system in equilibrium. The form for $\widehat{\Psi}$ in (1.237) extends $\mathcal{S}^{\mathrm{e}}$ in (1.187) into the domain of nonequilibrium, as will be evident upon the thermodynamic correspondence discussed below.

On applying the operator D to $\widehat{\Psi}$ in (1.243), we obtain

$$
\begin{aligned}
D\widehat{\Psi} = T^{-1}D\mathcal{E} + \overline{p}Dv - \sum_{a=1}^{r}\overline{\mu}_a D\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\overline{X}_{a\nu\cdots\sigma}^{(q)} D\widehat{\Phi}_a^{(q)\sigma\cdots\nu} \\
+ \mathcal{E}DT^{-1} + vD\overline{p} - \sum_{a=1}^{r}\mathfrak{c}_a D\overline{\mu}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\widehat{\Phi}_a^{(q)\sigma\cdots\nu} D\overline{X}_{a\nu\cdots\sigma}^{(q)}.
\end{aligned}
\tag{1.245}
$$

1.2.2 Method Using the Calortropy Balance Equation

The second method makes use of the calortropy balance equation and the statistical mechanical formulas for the terms on the right hand side of the equation. This approach, however, takes much more involved calculations making use of a number of identities, evolution equations for $\widehat{\Phi}_a^{(q)\sigma\cdots\nu}$, energy balance equation, number fraction balance equation, decomposition of the energy-momentum tensor. We list such identities and equations below: First we note

$$
\Delta_\nu^\mu N_a^\nu = 0 \tag{1.246}
$$

and using this identity we obtain

$$
\Delta_\nu^\mu \Psi^\nu = -\frac{Q^\mu}{T} + \sum_{a=1}^{r}\sum_{q\geq 1}\overline{X}_a^{(q)} \odot \psi_a^{(q)\nu}\Delta_\nu^\mu. \tag{1.247}
$$

The following are also useful:

$$
\psi_a^{(q)\mu} + \Delta_\nu^\mu \psi_a^{(q)\nu} = U^\mu \Phi_a^{(q)}, \tag{1.248}
$$

$$
T^{\mu\nu}U_\nu - Q^\mu = \rho\mathcal{E}U^\mu \quad (\because Q^\nu U_\nu = 0, \quad P^{\mu\nu}U_\nu = 0)\,, \tag{1.249}
$$

$$
\frac{1}{T}T^{\mu\nu}\partial_\mu U_\nu = c^{-2}\frac{1}{T}U^\mu Q^\nu \partial_\mu U_\nu + \frac{1}{T}P^{\mu\nu}\partial_\mu U_\nu. \tag{1.250}
$$

The dissipation terms $\Lambda_a^{(q)}$ are eliminated by using the generic evolution equation for $\widehat{\Phi}_a^{(q)}$, (1.118), which is again presented below for convenience

$$\rho D\widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\nabla_\sigma \Omega_a^{(q)\sigma\mu\cdots\nu} + Z_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}. \tag{1.251}$$

After some labor with the help of the identities involving projectors, equations (1.248)–(1.250), and the relations (1.238) and (1.239) between p and $\ln\Gamma$, we can cast the calortropy balance equation in (1.225) into the following differential form

$$\begin{aligned} D\widehat{\Psi} &= T^{-1}D\mathcal{E} + \overline{p}Dv - \sum_{a=1}^{r}\overline{\mu}_a D\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\overline{X}_a^{(q)} \odot D\widehat{\Phi}_a^{(q)} \\ &\quad + \mathcal{E}DT^{-1} + vD\overline{p} - \sum_{a=1}^{r}\mathfrak{c}_a D\overline{\mu}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\widehat{\Phi}_a^{(q)} \odot D\overline{X}_a^{(q)}. \end{aligned} \tag{1.252}$$

This differential form is identical with (1.245) derived directly from the statistical mechanical expression for $\widehat{\Psi}$ by operating operator D in (1.237). It should be emphasized that the relation between p and $\ln\Gamma$, therefore, is the necessary and sufficient condition for the bilinear form (1.252) in the thermodynamic manifold to be obtained from (1.225). The significant point of the second method, however, is that, especially in the light of the first method, the calortropy balance equation is exactly integrable in the thermodynamic manifold, thanks to the presence of the term

$$-k_B\sum_{a=1}^{r}\left\langle f_a p_a^\mu \partial_\mu \ln f_a^{\mathrm{c}}(x, p_a)\right\rangle$$

in the calortropy balance Eq. (1.225). It, unfortunately, would not be the case for the Boltzmann entropy because the aforementioned term is absent. In the case of the relative Boltzmann entropy balance equation, because the right hand side of (1.235), despite its formal similarity to the calortropy balance equation, defies integration in a bilinear form in the thermodynamic manifold $\mathfrak{P}\cup\mathfrak{T}$ owing to the ratio f_a/f_a^{c} present in $J_{\mathrm{r}}\left[f_a/f_a^{\mathrm{c}}\right]$. The mathematical reason for this aspect will be elaborated on later at a more appropriate point; see Sect. 1.12 on the relative Boltzmann entropy balance equation.

If the normalization condition for the nonequilibrium canonical form (1.215) is varied with respect to the parameters β, μ_a, $X_a^{(q)}$ and Γ_a there arises the equation

$$\mathcal{E}DT^{-1} + vD\overline{p} - \sum_{a=1}^{r}\mathfrak{c}_a D\overline{\mu}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\widehat{\Phi}_a^{(q)} \odot D\overline{X}_a^{(q)} = 0. \tag{1.253}$$

This is called the nonequilibrium Gibbs–Duhem (NGD) equation. If this is applied to both (1.245) and (1.252) then there follows the differential form for $\widehat{\Psi}$ in the thermodynamic manifold $\mathfrak{P}\cup\mathfrak{T}$:

$$D\widehat{\Psi} = T^{-1}D\mathcal{E} + \overline{p}Dv - \sum_{a=1}^{r}\overline{\mu}_a D\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1}\overline{X}_a^{(q)} \odot \widehat{\Phi}_a^{(q)}, \tag{1.254}$$

which is called the extended Gibbs relation (EGR) for $\widehat{\Psi}$. In view of the fact that (1.245) follows from the bilinear form (1.243) for $\widehat{\Psi}$, the EGR (1.254) is *an exact differential*, which evidently yields an integral (1.243) in $\mathfrak{P} \cup \mathfrak{T}$ apart from a constant that may be set equal to zero. The EGR will serve as the fundamental equation for the thermodynamics of irreversible processes described in this work. The NGD equation is, in fact, the necessary and sufficient condition for integrability of the Pfaffian differential form (1.254) in the thermodynamic manifold. For adding the NGD equation to the EGR side by side and integrating the resulting equation we obtain an integral of the latter in $\mathfrak{P} \cup \mathfrak{T}$, namely, (1.243) apart from a constant that may be set equal to zero.

This crucial feature of integrability is absent in the relativistic Boltzmann entropy balance equation because the distribution function f_a is not describable in the thermodynamic manifold, but it is a function living in the phase space of a gas consisting of an astronomical number of particles that would require an equally astronomical number of initial and boundary conditions to integrate. But that is not possible to achieve. A similar conclusion would be possible to draw for the relative Boltzmann entropy. It is interesting to note that the calortropy differential, the related differential forms, and the integrability of the calortropy differential were present in the nonrelativistic kinetic theory described in Volume 1 and they all survive intact in the relativistic kinetic theory based on irreversible covariant Boltzmann equation.

1.3 Thermodynamic Correspondence

In the section where the equilibrium distribution function is derived in this chapter, we have shown that parameters T, p, and μ_a are identified with the equilibrium thermodynamic intensive variables T_e, p_e, and μ_a^e as the system reaches equilibrium. The identifications are made by resorting to the device of thermodynamic correspondence of the statistical mechanical Boltzmann entropy $\mathcal{S}^\mathrm{e}$ with the equilibrium thermodynamic entropy (i.e., Clausius entropy). In the viewpoint of the functional hypothesis the equilibrium distribution function is attained as the generalized potentials vanish. Therefore it is reasonable to conjecture that there exists a phenomenological Pfaffian form for the calortropy density at a nonequilibrium state of the system, although such a form has not as yet been established phenomenologically in relativistic irreversible thermodynamics. Assuming that such a conjecture would be experimentally realizable, we postulate a phenomenological Pfaffian form [30] and make thermodynamic correspondences as was done for the case of the nonrelativistic theories. Thus if the correspondence is made between the thermodynamic and statistical mechanical (theoretical) variables spanning the manifold $\mathfrak{P}$ as below

$$\begin{aligned}&\widehat{\Psi}|_\mathrm{st} \Leftrightarrow \widehat{\Psi}|_\mathrm{th}; \quad \mathcal{E}|_\mathrm{st} \Leftrightarrow \mathcal{E}|_\mathrm{th}; \quad v|_\mathrm{st} \Leftrightarrow v|_\mathrm{th};\\ &\mathfrak{c}_a|_\mathrm{st} \Leftrightarrow \mathfrak{c}_a|_\mathrm{th}; \quad \widehat{\Phi}_a^{(q)}|_\mathrm{st} \Leftrightarrow \widehat{\Phi}_a^{(q)}|_\mathrm{th},\end{aligned} \tag{1.255}$$

then the conjugate intensive parameters spanning the tangent manifold $\mathfrak{T}$ must be identified with the corresponding thermodynamic parameters

$$T|_{\mathrm{st}} \Leftrightarrow T|_{\mathrm{th}}; \quad p|_{\mathrm{st}} \Leftrightarrow p|_{\mathrm{th}}; \quad \mu_a|_{\mathrm{st}} \Leftrightarrow \mu_a|_{\mathrm{th}}; \quad X_a^{(q)}|_{\mathrm{st}} \Leftrightarrow X_a^{(q)}|_{\mathrm{th}} \tag{1.256}$$

and the parameters T, p, μ_a, $X_a^{(q)}$ as yet undetermined are now endowed thermodynamic meanings of their correspondents. Here it must be empahsized that p is now identifiable with the nonequilibrium pressure $\mathfrak{p}$, which is defined by the trace part of energy-momentum tensor when the generalized hydrodynamic equations are derived in Sect. 1.1.6 of this Volume.

Equipped with these thermodynamic correspondences we are now able to carry out nonequilibrium statistical mechanical calculations for various thermodynamic quantities for the relativistic gas on the basis of the Pfaffian form (1.254) and a formulation of nonequilibrium statistical thermodynamics of relativistic irreversible processes is evidently possible. In other words, the Pfaffian form (1.254) for the calortropy density, so endowed with thermodynamic significance as in (1.256) on the one hand and the nonequilibrium partition function Γ given by

$$\Gamma = \prod_{a=1}^{r} \rho_a^{-1} \left\langle \exp\left[-\beta \sum_{j=1}^{N_a} \left(p_{ja}^{\nu} U_{\nu} + \sum_{q\geq 1} X_{ja\,\nu\cdots\sigma}^{(q)} h_{aj}^{(q)\sigma\cdots\nu} - \mu_a \right) \right] \right\rangle \tag{1.257}$$

on the other hand, lay mathematical foundations on which to erect nonequilibrium statistical thermodynamics of irreversible processes in dilute relativistic monatomic gases. The so-formulated nonequilibrium statistical thermodynamics is consistent with the laws of thermodynamics. This statement is delineated in the following.

1.4 Statistical Thermodynamics of Relativistic Gases

On the basis of the exact differential form (1.254) for $\widehat{\Psi}$ derived earlier we are now able to develop a relativistic theory of thermodynamics for irreversible processes in a form parallel to the nonrelativistic thermodynamics described in the chapters of Volume 1. Therefore we are not going to repeat the formalism here except pointing out that temperature is given by

$$T^{-1} = \left(\frac{D\widehat{\Psi}}{D\mathcal{E}} \right)_{v,\mathfrak{c},\widehat{\Phi}} . \tag{1.258}$$

Other parameters of manifold $\mathfrak{T}$ can be similarly obtained.

The following nonequilibrium statistical mechanical expressions can be also derived on the basis of (1.243) and (1.239), which, when combined with (1.254), gives rise to the differential form for thermodynamic potential pv

$$\begin{aligned} D\left(pv\right) &= \widehat{\Psi} DT + pDv + \sum_{a=1}^{r} \mathfrak{c}_a D\mu_a - \sum_{a=1}^{r} \sum_{q\geq 1} \widehat{\Phi}_a^{(q)\sigma\cdots\nu} DX_{a\,\sigma\cdots\nu}^{(q)} \\ &= D(k_B T \ln \Gamma). \end{aligned} \tag{1.259}$$

From this differential form we obtain the (local rest frame) relations between various macroscopic variables and the statistical mechanical formula for the thermodynamic potential given in terms of nonequilibrium partition function Γ:

$$\widehat{\Psi} = k_B \ln \Gamma + k_B \left(\frac{\partial \ln \Gamma}{\partial T} \right)_{v,\mu.X}, \tag{1.260}$$

$$p = k_B T \left(\frac{\partial \ln \Gamma}{\partial v} \right)_{T,\mu.X}, \tag{1.261}$$

$$\mathfrak{c}_a = k_B T \left(\frac{\partial \ln \Gamma}{\partial \mu_a} \right)_{T,\mu'.X}, \tag{1.262}$$

$$\widehat{\Phi}_a^{(q)\gamma\cdots\omega} = -k_B T \left(\frac{\partial \ln \Gamma}{\partial X_{a\,\gamma\cdots\omega}^{(q)}} \right)_{T,\mu.X'}. \tag{1.263}$$

With the help of the extended Gibbs relation and the nonequilibrium Gibbs–Duhem equation, the formal relations (1.260)–(1.263) can be made use of to calculate the whole gamut of macroscopic thermodynamic variables and their relations as well as numerous other thermodynamic quantities from the knowledge of the nonequilibrium partition function—namely, nonequilibrium statistical thermodynamics may be formulated for relativistic irreversible processes of dilute gases. To achieve this aim it would be necessary to explicitly compute Γ in the manifold $\mathfrak{P} \cup \mathfrak{T}$. This latter task will be in principle possible if the nonequilibrium partition function is calculated by some means either analytical or numerical and the hydrodynamic equations are also appropriately solved for transport processes. The content of this subsection provides the necessary theoretical framework and mathematical tools for such investigations. Such investigations are left to future work on nonequilibrium statistical thermodynamics of relativistic fluids.

1.5 Cumulant Expansion

1.5.1 Cumulant Expansion for Calortropy Production

To implement the present line of relativistic kinetic theory to investigate practical experimental problems the generalized hydrodynamic equations formally derived earlier in this chapter must be solved. To achieve this aim it is first necessary to calculate more explicitly a class of the most important components in the generalized hydrodynamic equations, that is, the dissipation terms, which describe the energy dissipation mechanisms involved in the irreversible processes in the gas.

Although we have no complete and exact solution of the covariant kinetic equation that we can make use of for such a purpose at this point of development in solution of the covariant kinetic equation, we are, nevertheless, able to formulate a thermodynamic theory of irreversible processes in a manner consistent with the laws of thermodynamics. For the purpose in mind it is reasonable to approximate the distribution function in the collision term $C_{ab}(f_a, f_b)$ by taking its thermodynamic branch $f_a^{\mathrm{c}}(x, p_a)$. Thus we consider the approximation in which deviation from $f_a^{\mathrm{c}}(x, p_a)$ is neglected:

$$\begin{aligned} f_a(x, p_a) &= f_a^{\mathrm{c}}(x, p_a) + \Delta f_a(x, p_a) \\ &\simeq f_a^{\mathrm{c}}(x, p_a), \end{aligned} \tag{1.264}$$

where $\Delta f_a(x, p_a)$ is a deviation[15] of $f_a(x, p_a)$ from the thermodynamic branch $f_a^{\mathrm{c}}(x, p_a)$:

$$\Delta f_a(x, p_a) = f_a(x, p_a) - f_a^{\mathrm{c}}(x, p_a). \tag{1.265}$$

The deviation $\Delta f_a(x, p_a)$ depends on fluctuations in temperature $\delta\overline{T}$, chemical potentials $\delta\overline{\mu}_a$, generalized potentials $\delta\overline{X}_a^{(q)}$, etc. of the solution of the kinetic equation from their thermodynamic branch. Recall that such fluctuations determine the relative Boltzmann entropy, which in a way determines and describes measures of deviation by the thermodynamic branch of distribution function $f_a^{\mathrm{c}}(x, p_a)$ from $f_a(x, p_a)$.

The dissipation terms $\Lambda_a^{(q)}$ and other terms in the evolution equations for the nonconserved variables of the generalized hydrodynamic equations will be evaluated with $f_a^{\mathrm{c}}(x, p_a)$. This would be an approximation for $\Lambda_a^{(q)}$ defined by the exact solution for the kinetic equation. However, it is a requisite approximation that gives rise to evolution equations for nonconserved variables satisfying the laws of thermodynamics and hence the thermodynamical consistency, because the exact solution $f_a(x, p_a)$ does not have an underlying nonequilibrium thermodynamics; in this connection recall that the Boltzmann entropy balance equation is not integrable in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$.

To achieve the desired aim we find it most expedient to employ a cumulant expansion method, which would be much facilitated if the nonequilibrium canonical form $f_a^{\mathrm{c}}(x, p_a)$ is written in a concise form. Thus we write

$$f_a^{\mathrm{c}}(x, p_a) = f_a^0 \exp(-\omega_a), \tag{1.266}$$

where

$$f_a^0 = \exp\left[-\beta\left(p_a^\mu U_\mu - \mu_a^{\mathrm{e}}\right) - \ln \Gamma_a\right], \tag{1.267}$$

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[15] It may be called the complement to the projection onto the manifold $\mathfrak{P} \cup \mathfrak{T}$ of $f_a(x, p_a)$.

$$\omega_a = \beta \left(\sum_{q\geq 1} X^{(q)}_{a\,\nu\cdots\sigma} h^{(q)\nu\cdots\sigma}_a - \Delta\mu_a \right), \tag{1.268}$$

$$\Delta\mu_a = \mu_a - \mu^{\mathrm{e}}_a \tag{1.269}$$

with μ^{e}_a denoting the equilibrium thermodynamic chemical potential of species a. Therefore the terms associated with ω_a will be attributable to nonequilibrium within the thermodynamic formalism based on the exact calortropy differential $D\widehat{\Psi}$ and related nonequilibrium statistical mechanics. To proceed further we need to define abbreviations and symbols for brevity of notation. Let

$$x_{ab} = \omega_a + \omega_b, \qquad y_{ab} = \omega^*_a + \omega^*_b, \tag{1.270}$$

where the asterisk denotes the post-collision values. We also define dimensionless quantities for various quantities involved in the calculation:

$$\begin{aligned} \pi^\mu_a &= cp^\mu_a \beta \qquad (\beta = 1/k_B T)\,, \\ \overline{U}^\mu &= U^\mu / c. \end{aligned} \tag{1.271}$$

With these dimensionless variables we find $\beta p^\mu_a U_\mu$ appearing in the Jüttner function can be made dimensionless as follows:

$$\beta p^\mu_a U_\mu = \pi^\mu_a \overline{U}_\mu, \tag{1.272}$$

It is also convenient to reduce the distribution function with the factor $\rho / (mk_B T)^{3/2}$ where m is the mean mass of the mixture. Therefore the distribution function is given by a dimensionless form

$$\overline{f}^{\mathrm{e}}_a = \rho^{-1} (mk_B T)^{3/2} f^{\mathrm{e}}_a. \tag{1.273}$$

To express the calortropy production $\sigma_{\mathrm{c}}(x)$ also in a dimensionless form the transition rate W_{ab} is scaled by the factor $\lambda^2 / mk_B T$ where λ is a parameter of dimension length, which we may take for the mean free path or the interaction range or the mean molecular size parameter. Thus the reduced (dimensionless) transition rate $\overline{W}_{ab}$ is defined by

$$\overline{W}_{ab} = \frac{mk_B T}{\lambda^2}. \tag{1.274}$$

With various reducing parameters introduced earlier, we find it useful to define an important scaling parameter $\boldsymbol{g}$

$$\boldsymbol{g} = \frac{1}{c\lambda^2 \rho} \left(\frac{mc^2}{k_B T} \right)^4. \tag{1.275}$$

It has a dimension of volume × time. This parameter is used to reduce the calortropy production to a dimensionless form $\overline{\sigma}_{\mathrm{c}}$ as follows:

$$\overline{\sigma}_{\mathrm{c}}(x) = (\boldsymbol{g}/k_B)\,\sigma_{\mathrm{c}}(x)\,. \tag{1.276}$$

Thus reduced to a dimensionless form, the calortropy production is given by the reduced collision integral

$$\overline{\sigma}_{\mathrm{c}}(x) = \frac{1}{4}\sum_{a=1}^{r}\sum_{b=1}^{r}\left\langle\!\left\langle (x_{ab} - y_{ab})\left(e^{-y_{ab}} - e^{-x_{ab}}\right)\right\rangle\!\right\rangle_{ab}, \tag{1.277}$$

where the double angular brackets $\langle\langle A\rangle\rangle_{ab}$ stand for the reduced (i.e., dimensionless) collision integral

$$\langle\langle A\rangle\rangle = G_{ab}\int d^3\overline{\pi}_a \int d^3\overline{\pi}_b \int d^3\overline{\pi}_a^* \int d^3\overline{\pi}_b^* \overline{f}_a^{\mathrm{e}}\overline{f}_b^{\mathrm{e}}\overline{W}_{ab}A \tag{1.278}$$

with the abbreviations $d^3\overline{\pi}_a$, etc. defined by

$$d^3\overline{\pi}_a = \frac{d^3\pi_a}{\pi_a^0},\ \text{etc.} \tag{1.279}$$

Now, the dimensionless calortropy production $\overline{\sigma}_{\mathrm{c}}(x)$ in (1.277) is expressible in the form

$$\overline{\sigma}_{\mathrm{c}}(x) = \frac{1}{4}\left[\mathcal{R}^{(+)}(\varsigma) - \mathcal{R}^{(-)}(\varsigma)\right]_{\varsigma=1}, \tag{1.280}$$

where, with the book-keeping parameter ς introduced for the purpose of ordering various expansion terms when the exponential function is expanded in power series of x_{ab} or y_{ab}, the functions $\mathcal{R}^{(\pm)}(\varsigma)$ are defined by the formulas:

$$\mathcal{R}^{(+)}(\varsigma) = \sum_{a=1}^{r}\sum_{b=1}^{r}\left\langle\!\left\langle (x_{ab} - y_{ab})\left(e^{-\varsigma y_{ab}} - 1\right)\right\rangle\!\right\rangle_{ab}, \tag{1.281}$$

$$\mathcal{R}^{(-)}(\varsigma) = \sum_{a=1}^{r}\sum_{b=1}^{r}\left\langle\!\left\langle (x_{ab} - y_{ab})\left(e^{-\varsigma x_{ab}} - 1\right)\right\rangle\!\right\rangle_{ab}. \tag{1.282}$$

Now we are ready to apply a cumulant expansion method [31, 32] to $\mathcal{R}^{(\pm)}(\varsigma)$. When the cumulant expansion is completed after expanding the exponential factors in $\mathcal{R}^{(\pm)}(\varsigma)$ in power series of ς, the book-keeping parameter ς will be set equal to 1.

Since we have already described the cumulant expansion method in detail, particularly, in Chap. 3 and also in other kinetic theory Chaps. 5, 6, and 7 for dense fluids of Volume 1, it is sufficient to list only the important results in the following. The first, second, and third-order cumulant are defined by expressions

$$\kappa := \kappa_1 = \frac{1}{2}\left\langle\!\!\left\langle \sum_{a=1}^{r}\sum_{a=1}^{r}\left(x_{ab}-y_{ab}\right)^2 \right\rangle\!\!\right\rangle_{ab}^{1/2}, \tag{1.283}$$

$$\kappa_2 := \frac{1}{4}\left\langle\!\!\left\langle \sum_{a=1}^{r}\sum_{b=1}^{r}\left(x_{ab}-y_{ab}\right)^2\left(x_{ab}+y_{ab}\right) \right\rangle\!\!\right\rangle_{ab}, \tag{1.284}$$

$$\kappa_3 := \frac{1}{4}\left\langle\!\!\left\langle \sum_{a=1}^{r}\sum_{b=1}^{r}\left(x_{ab}-y_{ab}\right)^2\left(x_{ab}^2+x_{ab}y_{ab}+y_{ab}^2\right) \right\rangle\!\!\right\rangle_{ab}, \text{ etc.} \tag{1.285}$$

Therefore, for example, to the first-order cumulant approximation the reduced calortropy production is given by the formula in terms of κ

$$\overline{\sigma}_{\mathrm{c}}(x) = \kappa \sinh \kappa \geq 0. \tag{1.286}$$

In this approximation, because of the first-order cumulant κ being positive semidefinite, the calortropy production $\overline{\sigma}_{\mathrm{c}}(x)$ always remains positive semidefinite regardless of approximations made to x_{ab} which depends on the generalized potentials $X^{(q)}_{a\,\nu\cdots\sigma}$. In other words, since $\overline{\sigma}_{\mathrm{c}}(x) \geq 0$ according to the second law of thermodynamics, in the first-order cumulant approximation the thermodynamic second law is guaranteed to be satisfied by the approximations made to $X^{(q)}_{a\,\nu\cdots\sigma}$ whether the approximations are with respect to the number of moments included (that is, q) or to their dependence on nonconserved variables.

We have found in the case of the nonrelativistic theory in the previous chapters that there is a wide range of phenomena to which the first-order cumulant approximation and thus $\overline{\sigma}_{\mathrm{c}}(x)$ in (1.286) can be applied in order to account for many nonlinear irreversible processes in nature [33–39]. In the case of relativistic theory the first-order cumulant approximation for $\overline{\sigma}_{\mathrm{c}}(x)$ is hopefully expected to yield similarly interesting comparisons with experiments. Higher-order cumulant approximations describe a non-monotonic dependence with respect to the generalized potentials or fluxes, and possess potentially intriguing features (e.g., self-organizing ability of the system beyond equilibrium), but they require rather laborious computations for the expansion coefficients. For this reason they have not as yet been much studied even in the nonrelativistic theories. See Ref. [40] for a third order cumulant approximation in the nonrelativistic Boltzmann kinetic theory.

The first-order cumulant κ is quadratic with respect to the generalized potentials $X^{(q)}_{i}$. It can be shown that κ is given by the formula

$$\kappa = \left[\sum_{a,b=1}^{r}\sum_{q,s\geq 1} X^{(q)}_{a\,\mu\cdots\sigma}\mathbb{R}^{(qs)\mu\cdots\sigma\nu\cdots\omega}_{ab} X^{(s)}_{b\,\nu\cdots\omega}\right]^{1/2}, \tag{1.287}$$

where $\mathbb{R}^{(qs)\mu\cdots\sigma\nu\cdots\omega}_{ab}$ is a collision bracket integral consisting of the collision integral of the covariant kinetic equation. They are given by

$$\mathbb{R}_{ab}^{(qs)\mu\cdots\sigma\nu\cdots\omega} = \frac{1}{4}\beta^2 \left[\Delta\left(h_a^{(q)} + h_{a'}^{(q)}\right)^{\mu\cdots\sigma} \Delta\left(h_a^{(s)} + h_{a'}^{(s)}\right)^{\nu\cdots\omega}\right]_{aa'} + \frac{1}{2}\beta^2 \sum_{b\neq a} \left[\Delta h_a^{(q)\mu\cdots\sigma} \Delta h_a^{(s)\nu\cdots\omega}\right]_{ab}, \tag{1.288}$$

$$\mathbb{R}_{aa}^{(qs)\mu\cdots\sigma\nu\cdots\omega} = \beta^2 \sum_{b\neq a} \left[\Delta h_{ab}^{(q)\mu\cdots\sigma} \Delta h_{ab}^{(s)\nu\cdots\omega}\right]_{ab}, \tag{1.289}$$

where the abbreviations on the right hand side are defined by the formulas

$$\begin{aligned}\Delta\left(h_a^{(q)} + h_{a'}^{(q)}\right)^{\mu\cdots\sigma} &= \left(h_a^{(q)} + h_{a'}^{(q)} - h_a^{(q)*} - h_{a'}^{(q)*}\right)^{\mu\cdots\sigma}, \\ \Delta h_a^{(q)\mu\cdots\sigma} &= \left(h_a^{(q)} - h_a^{(q)*}\right)^{\mu\cdots\sigma}, \\ \Delta h_{ab}^{(q)\mu\cdots\sigma} &= \left(h_a^{(q)} - h_b^{(q)*}\right)^{\mu\cdots\sigma}, \text{ etc.}\end{aligned} \tag{1.290}$$

and $[A^{\mu\cdots\sigma} B^{\nu\cdots\omega}]_{ab}$ is an abbreviation for the reduced collision bracket integral:

$$\left[A^{\mu\cdots\sigma} B^{\nu\cdots\omega}\right]_{ab} = G_{ab} \int d^3\overline{\pi}_a d^3\overline{\pi}_b d^3\overline{\pi}_a^* d^3\overline{\pi}_b^* f_a^{\mathrm{e}} f_b^{\mathrm{e}} \overline{W}_{ab} A^{\mu\cdots\sigma} B^{\nu\cdots\omega}. \tag{1.291}$$

The collision bracket integral $\mathbb{R}_{ab'}^{(qs)\mu\cdots\sigma\nu\cdots\omega}$ is an integral of tensor products, which can be expressed as a scalar collision bracket integral $\widehat{\mathbb{R}}_{ab'}^{(qs)}$ times an isotropic tensor for a spherically symmetric particle system. Thus for the case of $q = s = 1$ corresponding to the shear stress, that is, $\mathbb{R}_{ab'}^{(11)\mu\sigma\nu\omega}$ involving a rank 4 tensor, it is expressible in the form given in terms of isotropic irreducible tensors [4] of rank 4

$$\mathbb{R}_{ab}^{(11)\mu\sigma\,\nu\omega} = \widehat{\mathbb{R}}_{ab}^{(qs)} \left[\frac{1}{2}\left(\Delta^{\mu\nu}\Delta^{\sigma\omega} + \Delta^{\mu\omega}\Delta^{\nu\sigma}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\sigma\omega}\right], \tag{1.292}$$

$$\mathbb{R}_{aa}^{(11)\mu\cdots\sigma\,\nu\cdots\omega} = \widehat{\mathbb{R}}_{aa}^{(qs)} \left[\frac{1}{2}\left(\Delta^{\mu\nu}\Delta^{\sigma\omega} + \Delta^{\mu\omega}\Delta^{\nu\sigma}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\sigma\omega}\right], \tag{1.293}$$

where $\Delta^{\mu\nu}$, etc. are projectors. Here the scalar collision bracket integrals $\widehat{\mathbb{R}}_{ab}^{(qs)}$ are given by integrals involving a contraction to a scalar of two second-rank tensors:

$$\begin{aligned}\widehat{\mathbb{R}}_{ab}^{(qs)} &= \frac{1}{5}\mathbb{R}_{ab}^{(11)\mu\sigma\,\nu\omega} \left[\frac{1}{2}\left(\Delta^{\mu\nu}\Delta^{\sigma\omega} + \Delta^{\mu\omega}\Delta^{\nu\sigma}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\sigma\omega}\right] \\ &= \frac{1}{5}\left\{\frac{1}{4}\beta^2 \left[\Delta\left(h_a^{(q)} + h_{a'}^{(q)}\right) : \Delta\left(h_a^{(s)} + h_{a'}^{(s)}\right)\right]_{ab} \right. \\ &\qquad \left. + \frac{1}{2}\beta^2 \sum_{b\neq a} \left[\Delta h_a^{(q)} : \Delta h_a^{(s)}\right]_{ab}\right\},\end{aligned} \tag{1.294}$$

$$\widehat{\mathbb{R}}_{ab}^{(qs)} = \frac{1}{5}\beta^2 \sum_{b\neq a} \left[\Delta h_{ab}^{(q)} : \Delta h_{ab}^{(s)}\right]_{aa'} . \tag{1.295}$$

For the case of $q = s = 2$, there is no question of contraction because the integrand is already scalar. Therefore

$$\mathbb{R}_{ab}^{(22)} = \widehat{\mathbb{R}}_{ab}^{(22)} . \tag{1.296}$$

In the case of $q = s = 3$ or 4, that is, the collision bracket integrals for heat flux or diffusion flux—a vector

$$\mathbb{R}_{ab}^{(qq)\mu\nu} = \widehat{\mathbb{R}}_{ab}^{(qq)} \Delta^{\mu\nu} \quad (q = 3, 4) , \tag{1.297}$$

where

$$\begin{aligned}\widehat{\mathbb{R}}_{ab}^{(qq)} &= \frac{1}{3}\mathbb{R}_{ab}^{(qq)\mu\nu} \\ &= \frac{1}{6}\beta^2 \sum_{a\neq b} \left[\left[\Delta\left(h_a^{(q)} + h_{a'}^{(q)}\right) : \Delta\left(h_a^{(s)} + h_{a'}^{(s)}\right)\right]_{ab}\right] .\end{aligned} \tag{1.298}$$

The collision bracket integrals $\widehat{\mathbb{R}}_{aa'}^{(qs)}$ and $\widehat{\mathbb{R}}_{ab}^{(qs)}$ satisfy the Onsager reciprocal relations with respect to the interchange of subscripts a and b or superscripts q and s. Of course, they are symmetric tensors.

Finally, we note that κ^2 is a generalized form of the Rayleigh dissipation function [43] which is a quadratic function of velocities originally introduced by Rayleigh to account for energy dissipation in his theory of sound. With the linear approximation for $X_a^{(q)}$ the first-order cumulant κ is now expressible in terms of scalar $\widehat{\mathbb{R}}_{ab}^{(qs)}$ and $\widehat{\Phi}_a^{(q)}$ as in the expression

$$\kappa^2 = \sum_{a,b=1}^{r} \sum_{q,s\geq 1} \mathfrak{R}_{ab}^{(qs)} \Phi_a^{(q)\mu\cdots\sigma} \Phi_{a\,\mu\cdots\sigma}^{(q)} , \tag{1.299}$$

where we have defined the expansion coefficients $\mathfrak{R}_{ab}^{(qs)}$ by the formula

$$\mathfrak{R}_{ab}^{(qs)} = \boldsymbol{g}_a^{(q)} \widehat{\mathbb{R}}_{ab}^{(qs)} \boldsymbol{g}_b^{(s)} . \tag{1.300}$$

Interestingly, the calortropy production is related to a generalized form of the Rayleigh dissipation function. It is indeed the seat of energy dissipation arising from the irreversible processes progressing within the system.

1.5.2 Dissipation Terms in First-Order Cumulant Approximation

Since the calortropy production is related to the dissipation terms $\Lambda_a^{(q)\mu\cdots\nu}$ in (1.117) through the relation

$$\sigma_{\mathrm{c}}\left(x\right) = T^{-1} \sum_{a=1}^{r} \sum_{q \geq 1} X_{a\,\nu\cdots\mu}^{(q)} \Lambda_a^{(q)\mu\cdots\nu}, \tag{1.301}$$

$\Lambda_a^{(q)\mu\cdots\nu}$ can be also approximated by the first-order cumulant approximation. It is easily deduced that the first-order cumulant approximation for $\Lambda_a^{(q)\mu\cdots\nu}$ is given by the expression

$$\Lambda_a^{(q)\mu\cdots\nu}\left(x\right) = \left(k_B T/\boldsymbol{g}\right) \sum_{b=1}^{r} \sum_{s \geq 1} \widehat{\mathbb{R}}_{ab}^{(qs)} X_b^{(s)\mu\cdots\nu}\left(x\right) q_n(\kappa) \tag{1.302}$$

with $q_n(\kappa)$ denoting the first-order cumulant approximation for the nonlinear factor defined by

$$q_n(\kappa) = \frac{\sinh\left(\kappa\right)}{\kappa}. \tag{1.303}$$

The generalized potentials $X_b^{\mu\cdots\nu}\left(x\right)$ will be approximated to first-order in fluxes as will be shown in (1.313) below. Then with $\mathfrak{R}_{ab}^{(qs)}$ defined by (1.300) $\Lambda_a^{(q)\mu\cdots\nu}\left(x\right)$ can be written

$$\Lambda_a^{(q)\mu\cdots\nu}\left(x\right) = -\left(1/\beta \boldsymbol{g} g_a^{(q)}\right) \sum_{b=1}^{r} \sum_{s \geq 1} \mathfrak{R}_{ab}^{(qs)} \Phi_b^{(s)\mu\cdots\nu}\left(x\right) q_n(\kappa). \tag{1.304}$$

Before proceeding further, there is an important item to consider in connection with the dissipation terms of heat and diffusion four-flows, $\Phi_a^{(3)\mu}$ and $\Phi_a^{(4)\mu}$ of a mixture. In the case of a mixture, there holds the relation for diffusion fluxes of r species

$$\sum_{b=1}^{r} J_b^{\mu} = 0 \tag{1.305}$$

owing to the definition of diffusion four-flow; see (1.46). This means that one of the diffusion four-flows is dependent on the remaining $(r-1)$ diffusion four-flows. We will treat J_r^{μ} of the rth component as the dependent diffusion four-flow. Then the dependent diffusion four-flow must be removed by means of the condition (1.305). This task is easily achieved in the Rayleigh dissipation function κ. Upon elimination of J_r^{μ} of the rth species component the Rayleigh dissipation function now reads

$$\begin{aligned}\kappa^2 = &\sum_{a,b=1}^{r} \left(\mathfrak{R}_{ab}^{(11)} \Phi_a^{(1)\mu\nu} \Phi_{b\nu\mu}^{(1)} + \mathfrak{R}_{ab}^{(22)} \Phi_a^{(2)} \Phi_b^{(2)} + \mathfrak{R}_{ab}^{(33)} \Phi_a^{(3)\mu} \Phi_{b\mu}^{(3)} \right) \\ &+ \sum_{a=1}^{r} \sum_{b=1}^{r-1} \left(\mathfrak{R}_{ab}^{(34)} - \mathfrak{R}_{ar}^{(34)} \right) \Phi_a^{(3)\mu} \Phi_{b\mu}^{(4)} \\ &+ \sum_{a=1}^{r-1} \sum_{b=1}^{r} \left(\mathfrak{R}_{ab}^{(43)} - \mathfrak{R}_{rb}^{(43)} \right) \Phi_a^{(4)\mu} \Phi_{b\mu}^{(3)} \\ &+ \sum_{a=1}^{r-1} \sum_{b=1}^{r-1} \left(\mathfrak{R}_{ab}^{(44)} + \mathfrak{R}_{rr}^{(44)} - \mathfrak{R}_{ar}^{(44)} - \mathfrak{R}_{rb}^{(44)} \right) \Phi_a^{(4)\mu} \Phi_{b\mu}^{(4)}, \end{aligned} \tag{1.306}$$

for which we have used the linear approximation for the generalized potentials [see (1.299) and (1.300) as well as (1.313) below]. Upon use of this form (1.306) for the Rayleigh dissipation function κ, the dissipation terms $\Lambda_a^{(3)\mu}$ and $\Lambda_a^{(4)\mu}$ for the heat flux four-flow and diffusion flux four-flow are given by the formulas:

$$\Lambda_a^{(3)} = -\left(1/\boldsymbol{g}_a^{(3)}\right) \left[\sum_{b=1}^{r} \mathfrak{T}_{ab} \Phi_a^{(3)} + \sum_{b=1}^{r-1} \mathfrak{H}_{ab} \Phi_b^{(4)} \right], \tag{1.307}$$

$$\Lambda_a^{(4)} = -\left(1/\boldsymbol{g}_a^{(4)}\right) \left[\sum_{b=1}^{r} \mathfrak{K}_{ab} \Phi_b^{(3)} + \sum_{b=1}^{r-1} \mathfrak{D}_{ab} \Phi_b^{(4)} \right] \tag{1.308}$$

where various coefficients are defined below:

$$\begin{aligned} \mathfrak{T}_{ab} &= \frac{\mathfrak{R}_{ab}^{(33)}}{\beta_{\mathrm{e}} \boldsymbol{g}} \ (a, b = 1, \ldots, r,) ; \\ \mathfrak{H}_{ab} &= \frac{\mathfrak{R}_{ab}^{(34)} - \mathfrak{R}_{rb}^{(34)}}{\beta_{\mathrm{e}} \boldsymbol{g}} \ (a = 1, \ldots, r, b = 1, \ldots, r-1) ; \\ \mathfrak{K}_{ab} &= \frac{\mathfrak{R}_{ab}^{(43)} - \mathfrak{R}_{rb}^{(43)}}{\beta_{\mathrm{e}} \boldsymbol{g}} \ (a = 1, \ldots, r-1, b = 1, \ldots, r) ; \\ \mathfrak{D}_{ab} &= \frac{\mathfrak{R}_{ab}^{(44)} + \mathfrak{R}_{rr}^{(44)} - \mathfrak{R}_{ar}^{(44)} - \mathfrak{R}_{rb}^{(44)}}{\beta_{\mathrm{e}} \boldsymbol{g}} \ (a, b = 1, \ldots, r-1) . \end{aligned} \tag{1.309}$$

In addition to these coefficients, it is convenient to define the following coefficients related to shear stress and excess normal stress:

$$\mathfrak{B}_{ab} = \frac{\mathfrak{R}_{ab}^{(11)}}{\beta_{\mathrm{e}} \boldsymbol{g}}; \quad \mathfrak{V}_{ab} = \frac{\mathfrak{R}_{ab}^{(22)}}{\beta_{\mathrm{e}} \boldsymbol{g}} \ (a, b = 1, \ldots r,) . \tag{1.310}$$

It should be kept in mind that the nonlinear factor $q_n(\kappa)$ in (1.303) is calculated with κ given by (1.306), because the dissipation terms $\Lambda_a^{(3)}$ and $\Lambda_a^{(4)}$ are the consequences of κ^2 given by (1.306).

1.6 Generalized Potentials

In the evolution equations for nonconserved variables, especially, their dissipation terms there still remain the generalized potentials $X_{a\,\nu\cdots\sigma}^{(q)}$ to be determined in terms of variables spanning manifold $\mathfrak{P} \cup \mathfrak{T}$. In Chap. 3 of Volume 1, we have discussed a number of ways to calculate generalized potentials. Therefore anyone of the methods described in the chapter can be applied in covariant form to obtain approximate forms for the generalized potentials in the relativistic theory. To avoid repetition, we will present a summary of a simplest method of calculating them; namely, (1.263) is used to calculate $\Phi_a^{(q)\gamma\cdots\omega}$ by expanding $\ln\Gamma$ to first order of $X_{a\,\gamma\cdots\omega}^{(s)}$ in the power series expansion of f_a^{c} in $X_{a\,\gamma\cdots\omega}^{(s)}$. We thus obtain to the leading order in $X_{a\,\gamma\cdots\omega}^{(q)}$ the expression

$$\Phi_a^{(q)\mu\cdots\nu} = -\left(k_B T\right)^{-1} \sum_{s\geq 1} \left\langle c^{-2} U_\sigma p_a^\sigma h_a^{(q)\mu\cdots\nu} h_a^{(q)\beta\gamma\cdots\omega} f_a^{\mathrm{e}} \right\rangle X_{a\,\beta\gamma\cdots\omega} + O\left(X^2\right), \quad (1.311)$$

where the higher power terms of $X_{a\,\nu\cdots\mu}^{(s)}$ are omitted. Since moment tensors are constructed orthogonal to each other with respect to the indices q and s we obtain

$$\Phi_a^{(q)\mu\cdots\nu} = -\left(k_B T\right)^{-1} \left\langle c^{-2} U_\sigma p_a^\sigma h_a^{(q)\mu\cdots\nu} h_a^{(q)\beta\gamma\cdots\omega} f_a^{\mathrm{e}} \right\rangle X_{a\,\beta\gamma\cdots\omega}^{(q)}. \quad (1.312)$$

Therefore we see that $X_{a\,\beta\gamma\cdots\omega}^{(q)}$ is linearly proportional to $\Phi_a^{(q)\mu\cdots\nu}$ in the lowest order approximation. Since the tensorial rank remains invariant to the order of approximation in the expansion, we may generally write

$$X_a^{(q)\mu\cdots\nu} = -\boldsymbol{g}_a^{(q)} \Phi_a^{(q)\mu\cdots\nu}, \quad (1.313)$$

where the coefficient is, to first order, given by the statistical mechanical formula

$$1/\boldsymbol{g}_a^{(q)} = C_q \left\langle f_a^{\mathrm{e}} c^{-2} U_\alpha p_a^\alpha h_a^{(q)\mu\cdots\nu} h_{a\,\nu\cdots\mu}^{(q)} \right\rangle \quad (1.314)$$

with the values of C_q given by

$$C_q = \begin{cases} \frac{1}{5} & \text{for } q = 1 \\ 1 & \text{for } q = 2 \\ \frac{1}{3} & \text{for } q = 3, 4 \end{cases}. \quad (1.315)$$

In the case of the nonrelativistic theory of dilute monatomic gases it is found sufficient for many applications to take linear approximations similar to (1.313) for generalized potentials. It is believed also sufficient for the relativistic theory applications, especially, for the cases of not-too-far removed from equilibrium.

1.7 Linear Generalized Hydrodynamic Equations

We have now assembled all the necessary theoretical ingredients to calculate linear transport coefficients for irreversible processes—namely, transport processes—occurring in the vicinity of equilibrium. For this purpose we consider the linearized evolution equations—constitutive equations—for shear stress tensors, excess normal stress, heat fluxes, and diffusion fluxes for a dilute relativistic gas, since they appear in the hydrodynamic equations in the present kinetic theory. They will be simply listed here since more general constitutive equations will be examined in the next section where the quasilinear generalized hydrodynamic equations will be presented and examined together with their near-equilibrium limits. The linearized constitutive equations are as follows:

$$\rho D\widehat{\Phi}_a^{(1)\mu\nu} = -2\mathfrak{p}_a\,[\nabla U]^{(2)\mu\nu} - \frac{1}{\boldsymbol{g}_a^{(1)}}\sum_{b=1}^{r}\mathfrak{B}_{ab}\Phi_b^{(1)\mu\nu}, \tag{1.316}$$

$$\rho D\widehat{\Phi}_a^{(2)} = -\frac{5}{3}\mathfrak{p}_a\nabla_\mu U^\mu - \frac{1}{\boldsymbol{g}_a^{(2)}}\sum_{j=1}^{r}\mathfrak{V}_{ab}\Phi_b^{(2)}, \tag{1.317}$$

$$\rho D\widehat{\Phi}_a^{(3)\mu} = -\mathfrak{p}_a C_{pa} T\nabla^\mu \ln T - \frac{1}{\boldsymbol{g}_a^{(3)}}\left[\sum_{b=1}^{r}\mathfrak{T}_{ab}\Phi_a^{(3)\mu} + \sum_{b=1}^{r-1}\mathfrak{H}_{ab}\Phi_b^{(4)\mu}\right], \tag{1.318}$$

$$\rho D\widehat{\Phi}_a^{(4)\mu} = -\rho_a\nabla^\mu \ln \mathfrak{c}_a - \frac{1}{\boldsymbol{g}_a^{(4)}}\left[\sum_{b=1}^{r}\mathfrak{K}_{ab}\Phi_b^{(3)\mu} + \sum_{b=1}^{r-1}\mathfrak{D}_{ab}\Phi_b^{(4)\mu}\right], \tag{1.319}$$

for which we have taken the local equilibrium form for the temperature and pressure. Here recall $\mathfrak{p}_a = p_a^{\mathrm{e}}$. The symbol C_{pa} represents the heat capacity at constant pressure. More explicitly written out, the collision bracket integral $\mathfrak{R}_{ab}^{(q\gamma)}$ defining the coefficients $\mathfrak{B}_{ab}$, $\mathfrak{V}_{ab}$, $\mathfrak{T}_{ab}$, $\mathfrak{H}_{ab}$, $\mathfrak{K}_{ab}$, and $\mathfrak{D}_{ab}$ are as follows:

$$\begin{aligned}\mathfrak{R}_{aa}^{(qs)} = C_q\boldsymbol{g}_a^{(q)}\boldsymbol{g}_b^{(s)}\Bigg[&\frac{1}{4}\beta_{\mathrm{e}}^2\left\langle\!\!\left\langle\Delta\left(h_a^{(q)} + h_{a'}^{(q)}\right)^{\mu\cdots\sigma}\Delta\left(h_a^{(s)} + h_{a'}^{(s)}\right)_{\sigma\cdots\mu}\right\rangle\!\!\right\rangle_{aa'}\\ &+\frac{1}{2}\beta_{\mathrm{e}}^2\sum_{b\neq a}\langle\!\langle\Delta h_a^{(q)\mu\cdots\sigma}\Delta h_{a\,\sigma\cdots\mu}^{(s)}\rangle\!\rangle_{ab}\Bigg],\end{aligned} \tag{1.320}$$

$$\mathfrak{R}_{ab}^{(qs)} = C_q \boldsymbol{g}_a^{(q)} \boldsymbol{g}_b^{(s)} \beta_{\mathrm{e}}^2 \sum_{b\neq a} \left\langle\left\langle \Delta h_a^{(q)\mu\cdots\sigma} \Delta h_{b\,\sigma\cdots\mu}^{(s)} \right\rangle\right\rangle_{aa'} . \tag{1.321}$$

In (1.320) and (1.321) the subscript a' on the collision bracket integrals means that the integrals are performed for another particle of species a.

It should be noted that the collision bracket integrals are symmetric with respect to the interchange of species indices as well as indices for the coupled processes such as q and s, namely, in $\mathfrak{R}_{ab}^{(qs)}$. In other words, they satisfy the Onsager reciprocal relations [41, 42]:

$$\mathfrak{R}_{ab}^{(qs)} = \mathfrak{R}_{ba}^{(sq)} . \tag{1.322}$$

Together with the conserved variable equations, namely, conservation laws, (1.79), (1.80), (1.82), and (1.84), the linearized constitutive equations (1.316)–(1.319) constitute linearized generalized hydrodynamic equations, which generalize the classical hydrodynamic equations—namely, Navier–Stokes, Fourier, and Fick equations—into the relativistic regime, but linear and near equilibrium.

1.8 Linear Transport Coefficients

The linear transport coefficients are identified in terms of inverse matrices of collision bracket integrals $\mathfrak{R}_{aa}^{(qs)}$ and $\mathfrak{R}_{ab}^{(qs)}$, if the linear steady-state equations of (1.316)–(1.319) are algebraically solved for $\Phi_b^{(1)\mu\nu}$, etc. in terms of thermodynamic gradients:

$$-2\mathfrak{p}_a \left[\nabla U\right]^{(2)\mu\nu} - \frac{1}{\boldsymbol{g}_a^{(1)}} \sum_{b=1}^{r} \mathfrak{B}_{ab} \Phi_b^{(1)\mu\nu} = 0, \tag{1.323}$$

$$-\frac{5}{3}\mathfrak{p}_a \nabla_\mu U^\mu - \frac{1}{\boldsymbol{g}_a^{(2)}} \sum_{j=1}^{r} \mathfrak{V}_{ab} \Phi_b^{(2)} = 0, \tag{1.324}$$

$$-\mathfrak{p}_a C_{pa} T \nabla^\mu \ln T - \frac{1}{\boldsymbol{g}_a^{(3)}} \left[\sum_{b=1}^{r} \mathfrak{T}_{ab} \Phi_a^{(3)\mu} + \sum_{b=1}^{r-1} \mathfrak{H}_{ab} \Phi_b^{(4)\mu} \right] = 0, \tag{1.325}$$

$$-\rho_a \nabla^\mu \ln \mathfrak{c}_a - \frac{1}{\boldsymbol{g}_a^{(4)}} \left[\sum_{b=1}^{r} \mathfrak{K}_{ab} \Phi_b^{(3)} + \sum_{b=1}^{r-1} \mathfrak{D}_{ab} \Phi_b^{(4)} \right] = 0, \tag{1.326}$$

Since the first two equations (1.323) and (1.324) are not coupled to each other nor are they coupled to the rest of the equations in the set, they are separately solved. On the other hand, since the last two equations are coupled, they must be solved together. As a matter of fact, when solved algebraically for the fluxes, the set gives rise to linear constitutive relations—i.e., thermodynamic force–flux relations—in which linear transport coefficients [23], namely, the shear viscosity, bulk viscosity, heat conductivity, and diffusion coefficients, can be identified and expressed in terms of collision bracket integrals. To relate the linear constitutive equations to the conventional linear constitutive equations, recall that $\Phi_a^{(1)\mu\nu} = \Pi_a^{\mu\nu}$; $\Phi_a^{(2)} = \widetilde{\Delta}_a$; $\Phi_a^{(3)\mu} = Q_a'^{\mu}$; $\Phi_a^{(4)\mu} = J_a^{\mu}$.

The first two equations of the set (1.323)–(1.326) are solved in terms of thermodynamic forces (gradients) by simply inverting $r \times r$ matrices $\mathfrak{B} = (\mathfrak{B}_{ab})$ and $\mathfrak{V} = (\mathfrak{V}_{ab})$:

$$\Pi_a^{\mu\nu} = -\sum_{b=1}^{r} \left(\mathfrak{B}^{-1}\right)_{ab} [\nabla U]^{(2)\mu\nu} := -2\eta_a^0 [\nabla U]^{(2)\mu\nu}, \tag{1.327}$$

$$\widetilde{\Delta}_a = -\frac{5}{2}\sum_{b=1}^{r} \left(\mathfrak{V}^{-1}\right)_{ab} \nabla_\mu U^\mu := -\eta_{\mathrm{B}a}^0 \nabla_\mu U^\mu, \tag{1.328}$$

where η_a^0 and $\eta_{\mathrm{B}a}^0$ turn out to be the relativistic first-order Chapman–Enskog shear viscosity and bulk viscosity of species component a. We have thus derived the relativistic transport coefficients in terms of collision bracket integrals: The shear viscosity and the bulk viscosity are, respectively, identified with the statistical mechanical formulas in terms of collision bracket integrals making up the constitutive relations

$$\Pi_a^{\mu\nu} = -\sum_{b=1}^{r} \left(\mathfrak{B}^{-1}\right)_{ab} [\nabla U]^{(2)\mu\nu} = -2\eta_a^0 [\nabla U]^{(2)\mu\nu}, \tag{1.329}$$

$$\widetilde{\Delta}_a = -\sum_{b=1}^{r} \frac{5}{2} \left(\mathfrak{V}^{-1}\right)_{b} \nabla_\mu U^\mu = -\eta_{\mathrm{B}a}^0 \nabla_\mu U^\mu, \tag{1.330}$$

where the linear transport coefficients, namely, the shear viscosity and the bulk viscosity, are given by the statistical mechanical formulas

$$\eta_a^0 = \sum_{b=1}^{r} \left(\mathfrak{B}^{-1}\right)_{ab} = \sum_{b=1}^{r} \left[\left(\frac{2\mathfrak{R}_{ab}^{(11)}}{\beta_{\mathrm{e}}\boldsymbol{g}}\right)^{-1}\right]_{ab}, \tag{1.331}$$

$$\eta_{\mathrm{B}a}^0 = \sum_{b=1}^{r} \frac{5}{2} \left(\mathfrak{V}^{-1}\right)_{ab} = \sum_{b=1}^{r} \frac{5}{2} \left[\left(\frac{\mathfrak{R}_{ab}^{(22)}}{\beta_{\mathrm{e}}\boldsymbol{g}}\right)^{-1}\right]_{ab}. \tag{1.332}$$

To solve the last two coupled set for $Q_b'^{\mu}$ and J_b^{μ} we construct a $(2r-1)\times(2r-1)$ square matrix

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{T} & \mathfrak{H} \\ \mathfrak{K} & \mathfrak{D} \end{pmatrix}, \tag{1.333}$$

where submatrices are defined by square matrices $\mathfrak{T}$ and $\mathfrak{D}$ and rectangular matrices $\mathfrak{K}$ and $\mathfrak{H}$:

$$\mathfrak{T} = \begin{pmatrix} \mathfrak{T}_{11} \cdots \mathfrak{T}_{1r} \\ \vdots \ \ \ddots \ \ \vdots \\ \mathfrak{T}_{r1} \cdots \mathfrak{T}_{rr} \end{pmatrix}; \quad \mathfrak{D} = \begin{pmatrix} \mathfrak{D}_{11} & \cdots & \mathfrak{D}_{1,r-1} \\ \vdots & \ddots & \vdots \\ \mathfrak{D}_{r-1,1} & \cdots & \mathfrak{D}_{r-1,r-1} \end{pmatrix}; \tag{1.334}$$

$$\mathfrak{K} = \begin{pmatrix} \mathfrak{K}_{11}^{(JQ)} & \cdots & \mathfrak{K}_{1r}^{(JQ)} \\ \vdots & \ddots & \vdots \\ \mathfrak{K}_{r-1,1}^{(JQ)} & \cdots & \mathfrak{K}_{r-1,r}^{(JQ)} \end{pmatrix}; \quad \mathfrak{H} = \begin{pmatrix} \mathfrak{H}_{11}^{(QJ)} & \cdots & \mathfrak{H}_{1,r-1}^{(QJ)} \\ \vdots & \ddots & \vdots \\ \mathfrak{H}_{r1}^{(QJ)} & \cdots & \mathfrak{H}_{r-1,r-1}^{(QJ)} \end{pmatrix}; \tag{1.335}$$

and a $(2r-1)$-dimensional column vector composed of r-dimensional vector $Q_b^{\prime\mu}$ and $(r-1)$-dimensional vector J_b^μ:

$$\phi_b^\mu = \left(Q_b^{\prime\mu}, J_b^\mu\right)^t \tag{1.336}$$

and similarly a $(2r-1)$-dimensional column vector composed of the thermodynamic forces $T\nabla^\mu \ln T$ and $\nabla^\mu \ln \mathfrak{c}_a$:

$$\mathbf{F} = \left(T\nabla^\mu \ln T, \nabla^\mu \ln \mathfrak{c}_a\right)^t . \tag{1.337}$$

Thus the coupled subset is written as a single matrix equation

$$\sum_{j=1}^{2r-1} \mathfrak{S}_{ij} \phi_j^\mu = -\mathbf{F}_i^\mu . \tag{1.338}$$

The solution of this coupled set is given by

$$\phi_i^\mu = -\sum_{j=1}^{2r-1} \left(\mathfrak{S}^{-1}\right)_{ij} \mathbf{F}_j^\mu . \tag{1.339}$$

Explicitly expressed in terms of thermodynamic forces $T\nabla^\mu \ln T$ and $\rho_a \nabla^\mu \ln \mathfrak{c}_a$, we have the thermodynamic force–flux relations for the solutions:

$$\phi_a^\mu = -\sum_{j=1}^{r} \frac{|\mathfrak{S}|_{aj}}{\det|\mathfrak{S}|} T\nabla^\mu \ln T - \sum_{j=1}^{r-1} \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} \nabla^\mu \ln \mathfrak{c}_j \tag{1.340}$$

$$(a = 1, \ldots, r) ,$$

$$\phi_a^\mu = -\sum_{j=1}^{r} \frac{|\mathfrak{S}|_{aj}}{\det|\mathfrak{S}|} T\nabla^\mu \ln T - \sum_{j=1}^{r-1} \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} \nabla^\mu \ln \mathfrak{c}_j \tag{1.341}$$

$$(a = r+1, \ldots, 2r-1) .$$

Here the first equation is the thermodynamic force–flux relation for heat fluxes $Q_a^{\prime\mu} = \phi_i^\mu$ $(a = 1, \ldots, r)$ and the second equation is the thermodynamic force–flux relations for diffusion fluxes $J_a^\mu = \phi_a^\mu$, for which the index a runs from $r+1$ to $2r-1$. In these equations $|\mathfrak{S}|_{ij}$ denotes the (ij) cofactor of the $(2r-1) \times (2r-1)$ determinant $\det|\mathfrak{S}|$.

With the definitions of linear transport coefficients

$$\lambda_{ab}^0 = \frac{|\mathfrak{S}|_{ab}}{\det|\mathfrak{S}|} \quad (a, b = 1, \ldots, r)\,; \tag{1.342}$$

$$D_{aj}^{(dh)0} = \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} \quad (a = 1, \ldots, r;\ \ j = 1, \ldots, r-1)\,; \tag{1.343}$$

$$D_{aj}^{(td)0} = \frac{|\mathfrak{S}|_{aj}}{\det|\mathfrak{S}|} \quad (a = 1, \ldots, r-1;\ j = 1, \ldots, r)\,; \tag{1.344}$$

$$D_{aj}^0 = \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} \quad (a = 1, \ldots, r-1;\ j = 1, \ldots, r)\,, \tag{1.345}$$

the constitutive equations—the linear thermodynamic force–flux equations—are obtained. They are given by the equations

$$Q_a'^\mu = -\sum_{b=1}^{r} \lambda_{ab}^0 T \nabla^\mu \ln T - \sum_{j=1}^{r-1} D_{ab}^{(dh)0} \nabla^\mu \ln \mathfrak{c}_b \ \ (a = 1, \ldots, r)\,, \tag{1.346}$$

$$J_a^\mu = -\sum_{b=1}^{r} D_{ab}^{(td)0} T \nabla^\mu \ln T - \sum_{b=1}^{r-1} D_{ab}^0 \nabla^\mu \ln \mathfrak{c}_b \ \ (a = r+1, \ldots, 2r-1)\,. \tag{1.347}$$

The transport coefficients listed in (1.342)–(1.345) are: λ_{ab}^0 = thermal conductivity; D_{ab}^0 = diffusion coefficients; $D_{aj}^{(dh)0}$ = diffusive thermal conductivity; $D_{aj}^{(td)0}$ = thermal diffusivity.

In experiments, the partial transport coefficients such as η_a^0, etc. are usually not separately measured for different species. Therefore the observed coefficients are generally for the whole fluid except for diffusion coefficients. Therefore, to obtain them it would be necessary to sum the partial transport coefficients over species to obtain the transport coefficients measured for the fluid as a whole; that is, shear viscosity η^0, bulk viscosity η_{B}^0, and heat conductivity λ^0 of the entire fluid mixture:

$$\eta^0 = \sum_{a,b=1}^{r} \eta_{ab}^0, \quad \eta_{\mathrm{B}}^0 = \sum_{a,b=1}^{r} \eta_{\mathrm{B}ab}^0, \quad \lambda^0 = \sum_{a,b=1}^{r} \lambda_{ab}^0, \quad etc. \tag{1.348}$$

1.9 Generalized Hydrodynamic Equations

We have shown that the balance equations for the conserved variables, namely, the conservation laws, and evolution equations for the nonconserved variables derived from the covariant kinetic equation are constrained to satisfy the local positivity condition on the calortropy production, namely, the local second law of thermodynamics. The combined set of balance and evolution equations are called *the relativistic generalized hydrodynamic equations.* Their linearized version is already presented in the previous section. They reduce to the relativistic classical hydrodynamics as the system approaches equilibrium and also to the classical hydrodynamic equations of Navier, Stokes, Fourier, and Fick as $u/c \to 0$. Before ascertaining the limits mentioned, we first collect the generalized hydrodynamic equations in the linear approximation for the generalized potentials $X_a^{(q)\mu\nu\cdots\sigma}$ and in the first-order cumulant approximation for the dissipation terms $\Lambda_a^{(\alpha)\mu\nu\cdots\sigma}$.

1.9.1 Nonlinear Generalized Hydrodynamic Equations

The set of evolution equations thus obtained constitutes a thermodynamically consistent relativistic hydrodynamics for nonlinear transport processes. They are collected below:

Equation of Continuity

The equation of continuity for the entire mixture is

$$D\rho = -\rho\nabla_\mu U^\mu \text{ or } \rho D v = \nabla_\mu U^\mu, \tag{1.349}$$

where v denotes the specific volume defined by $v = 1/\rho$.

Number Fraction Balance Equation

The number fraction (or concentration) balance equation is given by

$$\rho D\mathfrak{c}_a = -\nabla_\mu J_a^\mu + c^{-2} J_a^\mu D U_\mu. \tag{1.350}$$

Unlike its nonrelativistic counter part it has a source term arising from the relativistic effect—the second term on the right.

Momentum Balance Equation

The momentum balance equation is also in a more complicated form than the nonrelativistic version because of the relativistic effects.

$$\begin{aligned} c^{-2}\mathcal{E}\rho D U^\omega &= -\nabla^\omega p - \Delta\nabla_\nu \overline{P}^{\mu\nu} \\ &\quad + c^{-2}\left(P^{\omega\nu} D U_\nu - \Delta_\mu^\omega D Q^\mu - Q^\omega \nabla_\nu U^\nu - Q^\nu \nabla_\nu U^\omega\right). \end{aligned} \tag{1.351}$$

The last group of terms on the right represents the relativistic effect.

Energy Balance Equation

The energy balance equation is given by

$$\rho D\mathcal{E} = -\nabla_\mu Q^\mu + P^{\mu\nu}\nabla_\nu U_\mu + 2c^{-2}Q^\mu DU_\mu. \tag{1.352}$$

It also has a source term related to heat four-flow, which is a relativistic effect.

The nonconserved variables such as shear stress, etc. have terms reminiscent of the kinematic terms of the nonrelativistic evolution equations, but although their evolution equations are more complex than the nonrelativistic versions, the dissipation terms are found to be in the formally same forms as for the nonrelativistic versions. They are presented in the first cumulant approximation. Therefore they are highly nonlinear.

Shear Stress Evolution Equation

The shear stress evolution equation for $\widehat{\Pi}_a^{\mu\nu} = \Pi_a^{\mu\nu}/\rho$ is given by

$$\begin{aligned}
\rho D\widehat{\Pi}_a^{\mu\nu} = & -\partial_\sigma \Omega_a^{(1)\sigma\mu\nu} - 2\left[\mathbf{P}_a\cdot\nabla\mathbf{U}\right]^{(2)\mu\nu} - 2c^{-2}\left[\mathfrak{J}_a D\mathbf{U}\right]^{(2)\mu\nu} \\
& - \sum_{b=1}^{r}\left(\rho\beta_{\mathrm{e}}\boldsymbol{g}\boldsymbol{g}_b^{(1)}\right)^{-1}\mathfrak{R}_{ab}^{(11)}\Phi_b^{(1)\mu\nu}\left(\sinh\kappa/\kappa\right) \\
& - 2c^{-2}\left[\mathbf{Q}_a D\mathbf{U}\right]^{(2)\mu\nu} - c^{-2}\left[\mathbf{U}\mathbf{P}_a\cdot\mathbf{DU}\right]^{(2)\mu\nu} \\
& - \left\langle f_a \frac{p_a^\sigma p_a^\tau p^\omega}{\left(U_\lambda p_a^\lambda\right)}\right\rangle \left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right) DU_\omega \\
& - c^2\left\langle f_a \frac{p_a^\omega p_a^\sigma p_a^\varepsilon p_a^\tau}{\left(U_\lambda p_a^\lambda\right)^2}\right\rangle \left(\Delta_\varepsilon^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\varepsilon\tau}\right)\nabla_\sigma U_\omega \\
& - \left\langle f_a \frac{p_a^\omega p_a^\tau p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)}\right\rangle \left(U^\nu\Delta_\omega^\mu + U^\mu\Delta_\omega^\nu\right)\nabla_\sigma U_\tau \\
& + \frac{1}{3}\Delta_{\omega\tau}\left\langle f_a \frac{p_a^\omega p_a^\tau p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)}\right\rangle \nabla_\sigma\left(U^\mu U^\nu\right),
\end{aligned} \tag{1.353}$$

where the terms except for the first line and the dissipation term in the second line are relativistic effects. The second term containing the pressure tensor $\mathbf{P}_a$ in the first line may be further decomposed into traceless and trace parts according to (1.56).

Excess Normal Stress Evolution Equation

With the definition $\widehat{\Delta}_a := \widetilde{\Delta}_a/\rho$, the excess normal stress evolution equation is given by

$$\rho D\widehat{\Delta}_a = -\partial_\sigma \Omega_a^{(2)\sigma} - \frac{2}{3}P_a^{\alpha\sigma}\nabla_\sigma U_\alpha - \frac{1}{3}\left[c^{-2}\mathfrak{J}_a^\alpha + \left\langle f_a \frac{m_a^2 c^2 p_a^\alpha}{\left(U_\lambda p_a^\lambda\right)}\right\rangle\right] DU_\alpha$$

$$
\begin{aligned}
&- \frac{1}{3} c^{-2} Q_a^{\alpha} D U_\alpha + \frac{1}{3} \left[P_a^{\sigma\omega} - \frac{1}{3} \left\langle f_a \frac{m_a^2 c^4 p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \right] \nabla_\sigma U_\omega \\
&- \sum_{b=1}^{r} \left(\rho \beta_{\mathrm{e}} \boldsymbol{g}\boldsymbol{g}_b^{(2)} \right)^{-1} \mathfrak{R}_{ab}^{(22)} \Phi_b^{(2)} \left(\sinh \kappa / \kappa \right), \qquad (1.354)
\end{aligned}
$$

where the third term in the square brackets gives rise to (2/3) $J_a^\alpha d_t u_\alpha$ in the limit $u/c \to 0$, whereas the term in the square brackets in the second line vanishes in the nonrelativistic limit. The pressure tensor can be also decomposed into traceless and trace parts.

Heat Flux Evolution Equation

The heat four-flow evolution equation is given by

$$
\begin{aligned}
\rho D \widehat{Q}_a'^{\mu} = {} & -\partial_\sigma \Omega_a^{(3)\sigma\mu} - Q_a'^{\sigma} \nabla_\sigma U^\mu - J_a^\mu D \widehat{h}_a - P_a^{\mu\sigma} \nabla_\sigma \widehat{h}_a + \varphi_a^{(3)\mu\sigma\omega} \nabla_\omega U_\sigma \\
& - \sum_{b=1}^{r} \sum_{s=3}^{4} \left(\rho \beta_{\mathrm{e}} \boldsymbol{g}\boldsymbol{g}_b^{(3)} \right)^{-1} \mathfrak{R}_{ab}^{(3s)} \Phi_b^{(s)\mu} \left(\sinh \kappa / \kappa \right) \\
& + \left[P_a^{\mu\sigma} - c^2 \left\langle f_a \frac{p_a^\mu p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)} \right\rangle \right] \nabla_\sigma \widehat{h}_a \\
& + \left[m_a c^4 \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle - \varphi_a^{(3)\mu\sigma\omega} \right] \nabla_\omega U_\sigma \\
& - U^\mu P_a^{\tau\sigma} \nabla_\sigma U_\tau + U^\mu J_a^\sigma \nabla_\sigma \widehat{h}_a - c^{-2} U^\mu Q_a^\sigma D U_\sigma \\
& - c^{-2} U^\mu \mathfrak{J}_a^\sigma D U_\sigma - \rho_a U^\mu D \widehat{h}_a + m_a c^2 \left\langle f_a \frac{p_a^\omega p_a^\mu}{\left(U_\lambda p_a^\lambda\right)} \right\rangle D U_\omega \\
& + \widehat{h}_a \left\langle f_a \frac{p_a^\omega p_a^\mu}{\left(U_\lambda p_a^\lambda\right)} \right\rangle D U_\omega + c^2 \widehat{h}_a \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \nabla_\sigma U_\omega . \qquad (1.355)
\end{aligned}
$$

The second line represents the dissipation term in the first-order cumulant approximation and

$$
\varphi_a^{(3)\mu\sigma\omega} = \left\langle f_a p_a^\mu p_a^\sigma p_a^\omega \right\rangle . \qquad (1.356)
$$

Diffusion Flux Evolution Equation

The mass four-flow evolution equation is given by

$$
\begin{aligned}
\rho D \widehat{J}_a^{\mu} = {} & -\partial_\sigma \Omega_a^{(4)\sigma\mu} - \rho_a D U^\mu - J_a^\sigma \nabla_\sigma U^\mu \\
& - \sum_{b=1}^{r} \sum_{s=3}^{4} \left(\beta_{\mathrm{e}} \boldsymbol{g}\boldsymbol{g}_b^{(4)} \right)^{-1} \mathfrak{R}_{ab}^{(4s)} \Phi_b^{(s)\mu} \left(\sinh \kappa / \kappa \right)
\end{aligned}
$$

$$-\left\langle f_a \frac{p_a^\mu p_a^\omega}{\left(U_\lambda p_a^\lambda\right)} \right\rangle DU_\omega - c^2 \left\langle f_a \frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2} \right\rangle \nabla_\sigma U_\omega, \tag{1.357}$$

where the last line represents the relativistic effects and the second line is the dissipation term in the first-order cumulant approximation.

Recall that $\Phi_a^{(1)\mu\nu} = \Pi_a^{\mu\nu}$; $\Phi_a^{(2)} = \widetilde{\Delta}_a$; $\Phi_a^{(3)\mu} = Q_a'^{\mu}$; $\Phi_a^{(4)\mu} = J_a^\mu$, and the nonlinear evolution equations presented are coupled to each other and to the conservation laws presented earlier. The evolution equations for the conserved and nonconserved variables presented above constitute the relativistic generalized hydrodynamic equations for the relativistic dilute gas derived from the covariant Boltzmann equation.

When the dependent diffusion flux is chosen—we choose J_r^μ as the dependent one—the dissipation terms for $s = 3, 4$ are modified as well as the Rayleigh dissipation function κ; see (1.307) and (1.308) for $\Lambda_a^{(3)}$ and $\Lambda_a^{(4)}$, respectively, and (1.306) for κ. Therefore, the dissipation terms in (1.355) and (1.357) must be replaced by the modified expressions as follows:

$$\begin{aligned}
&-\sum_{b=1}^{r}\sum_{s=3}^{4}\left(\beta_{\mathrm{e}}\boldsymbol{g}\boldsymbol{g}_b^{(3)}\right)^{-1}\mathfrak{R}_{ab}^{(3s)}\Phi_b^{(s)\mu}\left(\sinh\kappa/\kappa\right)\Rightarrow\\
&-\frac{1}{\boldsymbol{g}_b^{(3)}}\left[\sum_{b=1}^{r}\mathfrak{T}_{ab}\Phi_a^{(3)\mu}+\sum_{b=1}^{r-1}\mathfrak{H}_{ab}\Phi_b^{(4)\mu}\right]\left(\sinh\kappa/\kappa\right),
\end{aligned}\tag{1.358}$$

$$\begin{aligned}
&-\sum_{b=1}^{r}\sum_{s=3}^{4}\left(\beta_{\mathrm{e}}\boldsymbol{g}\boldsymbol{g}_b^{(4)}\right)^{-1}\mathfrak{R}_{ab}^{(4s)}\Phi_b^{(s)\mu}\left(\sinh\kappa/\kappa\right)\Rightarrow\\
&-\frac{1}{\boldsymbol{g}_a^{(4)}}\left[\sum_{b=1}^{r}\mathfrak{K}_{ab}\Phi_b^{(3)\mu}+\sum_{b=1}^{r-1}\mathfrak{D}_{ab}\Phi_b^{(4)\mu}\right]\left(\sinh\kappa/\kappa\right),
\end{aligned}\tag{1.359}$$

where κ must be replaced by the quadratic form κ in (1.306) in the nonlinear factor $(\sinh\kappa/\kappa)$ on the right of (1.358) and (1.359).

1.9.2 Quasilinear Generalized Hydrodynamic Equations

If we keep the dissipation terms in the first-order cumulant approximation in (1.353)–(1.357) while linearizing the kinematic terms and the divergence terms as in the linearized evolution equations for nonconserved variables (1.316)–(1.319) then the quasilinear evolution equations are obtained as below:

$$\rho D\widehat{\Phi}_a^{(1)\mu\nu} = -2\mathfrak{p}_a\left[\nabla U\right]^{(2)\mu\nu} - \frac{1}{\boldsymbol{g}_a^{(1)}}\sum_{b=1}^{r}\mathfrak{B}_{ab}\Phi_b^{(1)\mu\nu}q_n\left(\kappa\right), \tag{1.360}$$

$$\rho D\widehat{\Phi}_a^{(2)} = -\frac{5}{3}\mathfrak{p}_a \nabla_\mu U^\mu - \frac{1}{g_a^{(2)}} \sum_{j=1}^{r} \mathfrak{V}_{ab} \Phi_b^{(2)} q_n(\kappa), \tag{1.361}$$

$$\begin{aligned} \rho D\widehat{\Phi}_a^{(3)\mu} &= -\mathfrak{p}_a \widehat{C}_{pa} T \nabla^\mu \ln T \\ &\quad - \frac{1}{g_a^{(3)}} \left[\sum_{b=1}^{r} \mathfrak{T}_{ab} \Phi_a^{(3)\mu} + \sum_{b=1}^{r-1} \mathfrak{H}_{ab} \Phi_b^{(4)\mu} \right] q_n(\kappa), \end{aligned} \tag{1.362}$$

$$\begin{aligned} \rho D\widehat{\Phi}_a^{(4)\mu} &= -\rho_a \nabla^\mu \ln \mathfrak{c}_a \\ &\quad - \frac{1}{g_a^{(4)}} \left[\sum_{b=1}^{r} \mathfrak{K}_{ab} \Phi_b^{(3)} + \sum_{b=1}^{r-1} \mathfrak{D}_{ab} \Phi_b^{(4)} \right] q_n(\kappa). \end{aligned} \tag{1.363}$$

These are the simplest model of nonlinear evolution equations for nonconserved variables we can devise within the framework of relativistic generalized hydrodynamics consistent with the laws of thermodynamics. Together with the balance equations (1.349)–(1.352)—i.e., the conservation laws—they are called *the relativistic quasilinear generalized hydrodynamic equations*. This set would be completely parallel to the nonrelativistic version we have examined and applied in Chap.3 of Volume 1 where the nonrelativistic hydrodynamic theory is formulated. The quasilinear generalized hydrodynamic equations are also in complete conformation to the laws of thermodynamics as is the full version of the evolution equations considered earlier. The theory of transport processes based on them, therefore, would be thermodynamically consistent, yet highly linear with respect to the nonconserved variables $\Phi_a^{(q)\mu\cdots\nu}$ because of the nonlinear factor $q_n(\kappa)$.

1.9.3 Quasilinear Transport Processes

The time scale of evolution is much shorter for the nonconserved variables $\widehat{\Phi}_a^{(q)\mu\cdots\nu}$ than the conserved variables. Therefore (1.360)–(1.363) generally reach a steady state much faster than the conservations laws (1.349)–(1.352) do the steady state. Therefore at the steady state of the transport processes reached in a much shorter time regime than conserved variables we have

$$D\widehat{\Phi}_a^{(q)\mu\cdots\nu} = 0 \tag{1.364}$$

and obtain the steady-state nonlinear constitutive equations

$$-2\mathfrak{p}_a [\nabla U]^{(2)\mu\nu} - \frac{1}{g_a^{(1)}} \sum_{b=1}^{r} \mathfrak{B}_{ab} \Phi_b^{(1)\mu\nu} q_n(\kappa) = 0, \tag{1.365}$$

$$-\frac{5}{3}\mathfrak{p}_a \nabla_\mu U^\mu - \frac{1}{g_a^{(2)}} \sum_{j=1}^{r} \mathfrak{V}_{ab} \Phi_b^{(2)} q_n(\kappa) = 0, \tag{1.366}$$

$$-\mathfrak{p}_a C_{pa} T \nabla^\mu \ln T - \frac{1}{\boldsymbol{g}_a^{(3)}} \left[\sum_{b=1}^{r} \mathfrak{T}_{ab} \Phi_a^{(3)\mu} + \sum_{b=1}^{r-1} \mathfrak{H}_{ab} \Phi_b^{(4)\mu} \right] q_n(\kappa) = 0, \tag{1.367}$$

$$-\rho_a \nabla^\mu \ln \mathfrak{c}_a - \frac{1}{\boldsymbol{g}_a^{(4)}} \left[\sum_{b=1}^{r} \mathfrak{K}_{ab} \Phi_b^{(3)} + \sum_{b=1}^{r-1} \mathfrak{D}_{ab} \Phi_b^{(4)} \right] q_n(\kappa) = 0. \tag{1.368}$$

It can be easily shown that the solutions of these steady-state equations are given by the nonlinear thermodynamic force–flux relations much similar to the linear counterparts given earlier in (1.329), (1.330), (1.346), and (1.347) except for the presence of the nonlinear factor q_L, which can be expressed by the formula

$$q_L := q_n^{-1}(\kappa) = \frac{\sinh^{-1} \kappa_L}{\kappa_L}. \tag{1.369}$$

Here the new dissipation function κ_L is given by a quadratic form of the thermodynamic gradients $[\nabla U]^{(2)\mu\nu}$, $\nabla_\mu U^\mu$, $T \nabla^\mu \ln T$, and $\nabla^\mu \ln \mathfrak{c}_a$:

$$\begin{aligned} \kappa_L = \Bigg\{ \sum_{a,b=1}^{r} \Big[& \mathfrak{C}_{ab}^{(11)} [\nabla U]^{(2)\mu\nu} [\nabla^\mu U]_{\nu\mu}^{(2)} + \mathfrak{C}_{ab}^{(22)} \left(\nabla_\mu U^\mu\right)^2 \\ & + \sum_{q,s=3,4} \mathfrak{C}_{ab}^{(qs)} \mathfrak{F}^{(q)\mu} \mathfrak{F}_\mu^{(s)} \Big] \Bigg\}^{1/2}, \end{aligned} \tag{1.370}$$

where the coefficients $\mathfrak{C}_{ab}^{(qs)}$ are scalars consisting of the coefficients $\mathfrak{B}_{ab}$, $\mathfrak{V}_{ab}$, $\mathfrak{T}_{ab}, \ldots, \mathfrak{D}_{ab}$, which in turn are given in terms of the collision bracket integrals. The coefficients $\mathfrak{C}_{ab}^{(11)}$, etc. are calculated by solving the quasilinear constitutive equations (1.365)–(1.368) and inverting the nonlinear factor $q_n(\kappa)$. It takes a lengthy algebraic process to show it, but the procedure is straightforward and completely parallel to the one discussed for nonrelativistic theory in Sect. 9.1.5, Chap. 9 of Volume 1 to which the reader is referred for details. Therefore we will not prove (1.369) here, leaving it as an exercise for the reader.

1.9.4 Quasilinear Transport Coefficients

The quasilinear constitutive equations (1.365)–(1.368) enable us to obtain the statistical mechanical formulas for quasilinear transport coefficients for the associated nonlinear transport processes. They are given below:

$$\eta_a = \sum_{b=1}^{r} \left[\left(\frac{2\mathfrak{R}_{ab}^{(11)}}{\beta_e \boldsymbol{g}} \right)^{-1} \right]_{ab} q_L := \eta_a^0 q_L; \tag{1.371}$$

$$\eta_{\mathrm{B}a} = q_L \sum_{b=1}^{r} \frac{5}{2} \left[\left(\frac{\mathfrak{R}_{ab}^{(22)}}{\beta_{\mathrm{e}} \boldsymbol{g}} \right)^{-1} \right]_{ab} q_L := \eta_{\mathrm{B}a}^{0} q_L ; \tag{1.372}$$

$$\lambda_{ab} = \frac{|\mathfrak{S}|_{ab}}{\det |\mathfrak{S}|} q_L := \lambda_{ab}^{0} q_L \quad (a, b = 1, \ldots, r) ; \tag{1.373}$$

$$D_{aj}^{(dh)} = \frac{|\mathfrak{S}|_{a,r+j}}{\det |\mathfrak{S}|} q_L := D_{aj}^{(dh)0} q_L \; (a = 1, \ldots, r; \;\; j = 1, \ldots, r-1) ; \tag{1.374}$$

$$D_{aj}^{(td)} = \frac{|\mathfrak{S}|_{aj}}{\det |\mathfrak{S}|} q_L := D_{aj}^{(td)0} q_L \;\; (a = 1, \ldots, r-1; \; j = 1, \ldots, r) ; \tag{1.375}$$

$$D_{aj} = \frac{|\mathfrak{S}|_{a,r+j}}{\det |\mathfrak{S}|} q_L := D_{aj}^{0} q_L \;\; (a = 1, \ldots, r-1; \; j = 1, \ldots, r) . \tag{1.376}$$

Since q_L is dependent on a quadratic form of the thermodynamic gradients, the shear viscosity η_a is non-Newtonian and vanishes as $|\kappa_L|$ increases; to be mathematically more precise, as $|\kappa_L| \to \infty$ the non-Newtonian shear viscosity shows the asymptotic behavior as below:

$$\eta_a \to |\kappa_L|^{-1+\varepsilon} \, (\varepsilon > 0) . \tag{1.377}$$

That is, there is a shear thinning. Other nonlinear transport coefficients behave similarly. Notice also that the nonlinear transport coefficients presented above break the Curies principle. In other words, for example, the presence of a temperature gradient causes the non-Newtonian shear viscosity η_a diminishes in magnitude—a phenomenon not seen in linear transport processes occurring near equilibrium, because two processes are linearly independent of each other in the linear regime of thermodynamic forces.

1.10 Model Quasilinear Generalized Hydrodynamic Equations

If the nonlinear factor q_L replaces the nonlinear factor $q_n(\kappa)$ in (1.360)–(1.363) a model quasilinear generalized hydrodynamic equations are obtained when they are combined with the conservation laws (1.349)–(1.352):

$$\rho D \widehat{\Phi}_a^{(1)\mu\nu} = -2 \mathfrak{p}_a \left[\nabla U \right]^{(2)\mu\nu} - \frac{1}{\boldsymbol{g}_a^{(1)}} \sum_{b=1}^{r} \mathfrak{B}_{ab} \Phi_b^{(1)\mu\nu} q_L \left(\Gamma_\kappa \right) , \tag{1.378}$$

$$\rho D \widehat{\Phi}_a^{(2)} = -\frac{5}{3} \mathfrak{p}_a \nabla_\mu U^\mu - \frac{1}{\boldsymbol{g}_a^{(2)}} \sum_{j=1}^{r} \mathfrak{V}_{ab} \Phi_b^{(2)} q_L \left(\Gamma_\kappa \right) , \tag{1.379}$$

$$\rho D \widehat{\Phi}_a^{(3)\mu} = -\mathfrak{p}_a C_{pa} T \nabla^\mu \ln T$$

$$- \frac{1}{\boldsymbol{g}_a^{(3)}} \left[\sum_{b=1}^{r} \mathfrak{T}_{ab} \Phi_a^{(3)\mu} + \sum_{b=1}^{r-1} \mathfrak{H}_{ab} \Phi_b^{(4)\mu} \right] q_L \left(\Gamma_\kappa \right), \tag{1.380}$$

$$\rho D \widehat{\Phi}_a^{(4)\mu} = -\rho_a \nabla^\mu \ln \mathfrak{c}_a - \frac{1}{\boldsymbol{g}_a^{(4)}} \left[\sum_{b=1}^{r} \mathfrak{K}_{ab} \Phi_b^{(3)} + \sum_{b=1}^{r-1} \mathfrak{D}_{ab} \Phi_b^{(4)} \right] q_L \left(\Gamma_\kappa \right). \tag{1.381}$$

This model produces a mathematically less steep differential equations, but still may be applicable to flow phenomena far removed from equilibrium where the thermodynamic gradients can be steep. From the numerical computational standpoint this feature would be a great advantage. We note that this model naturally arises when the quasilinear evolution equation (1.360)–(1.363) are analyzed in the neighborhood of their steady state. For details, see Chap. 9 of Volume 1, where nonrelativistic quasilinear evolution equations are examined at their steady state.

It should be also noted that the coefficients in this set of equations are all determined in terms of the collision bracket integrals or linear transport coefficients. Instead of calculating the linear transport coefficients from the collision bracket integrals, one may take a semiempirical approach in which empirical linear transport coefficients or model transport coefficients may be made use of.

In Chap. 9 of Volume 1, we have discussed applications of nonrelativistic quasilinear generalized hydrodynamic equations to a number of rarefied gas dynamic flow problems in which the thermodynamic gradients are generally steep, but the accuracy of their predictions is excellent. In view of such successful applications the quasilinear relativistic generalized hydrodynamic equations presented here may provide equally accurate results for flow problems in the relativistic domain, particularly because the relativistic quasilinear generalized hydrodynamic equations tend to the nonrelativistic version as $u/c \to 0$. It would be interesting to see if the quasilinear relativistic generalized hydrodynamic equations can be equally successful for relativistic flow phenomena.

1.11 Nonrelativistic Limits of Evolution Equations

The important motivation of studying nonrelativistic limits lies in ascertaining the continuity of the relativistic irreversible thermodynamic formalism and generalized hydrodynamics with their nonrelativistic counterparts we have formulated in various chapters in Volume 1 of this work. This way, we would be assured that the relativistic formulations have not been strayed, but provide appropriate relativistic extensions of the theories mentioned. To investigate nonrelativistic limits of generalized hydrodynamic equations, we first examine limits of operators, moments, and energy-momentum tensors before we consider various evolution equations derived earlier.

1.11.1 Nonrelativistic Limits of Operators and Related Quantities

Since the hydrodynamic velocity may be enumerated and written in the form

$$U^{\mu} = c\gamma\left(1, \mathbf{u}/c\right), \quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}, \tag{1.382}$$

its nonrelativistic limit is enumerated as given below

$$\lim_{u/c \to 0} U^{\mu} = \left(c, u_{\mu}\right), \tag{1.383}$$

where u_{μ} denotes the μth component of three-dimensional velocity vector $\mathbf{u}$. The covariant derivative is enumerated as below:

$$\partial_{\mu} = (c^{-1}\partial_t, \partial/\partial x^{\mu}). \tag{1.384}$$

Since the convective derivative operator D is then enumerated

$$D = U^{\mu}\partial_{\mu} = \gamma\left(c, \mathbf{u}\right)^{\mu}\left(c^{-1}\partial_t, \partial/\partial x^{\mu}\right) = \gamma\left(\partial_t + \mathbf{u}\cdot\nabla\right),$$

it tends to the nonrelativistic limit

$$D = d_t + O\left(u^2/c^2\right), \tag{1.385}$$

where d_t is the nonrelativistic substantial time derivative

$$d_t = \partial_t + \mathbf{u}\cdot\nabla. \tag{1.386}$$

For the space component ∇^{μ} of ∂^{μ} we may first decompose it in the form

$$\nabla^{\mu} = \Delta^{\mu\nu}\partial_{\nu} = \Delta^{\mu 0}\partial_0 + \Delta^{\mu k}\partial_k \tag{1.387}$$

and recalling $\partial_{\mu} = \left(c^{-1}\partial_t, \partial/\partial x^{\mu}\right)$ and making use of the limits of projectors

$$\Delta^{\mu 0} = -\delta_{\mu 0} - \gamma^2\frac{\mathbf{u}_{\mu}}{c}, \tag{1.388}$$

$$\Delta^{\mu k} = -\delta_{\mu k} - \gamma^2\frac{\mathbf{u}_{\mu}\mathbf{u}_k}{c^2} \tag{1.389}$$

it can be shown that

$$\nabla^{\mu} = \partial/x_{\mu} + O\left(\frac{u^2}{c^2}\right) = \nabla_{\mu} + O\left(\frac{u^2}{c^2}\right) \quad (k = 1, 2, 3), \tag{1.390}$$

where $\nabla_\mu = \partial/x_\mu$ is the three-dimensional gradient operator. Similarly, for $\bigtriangledown_\mu$ we find

$$\bigtriangledown_\mu = \nabla_\mu + O\left(\frac{u^2}{c^2}\right) \quad (k = 1, 2, 3)\,. \tag{1.391}$$

Therefore the relativistic operators D and $\bigtriangledown^\mu$ or $\bigtriangledown_\mu$ can be replaced, respectively, with the nonrelativistic substantial time derivative d_t and the gradient operator $\nabla_k = \partial/\partial x_k$ in the limit of $u/c \to 0$. This means that, for example,

$$\bigtriangledown_\mu U^\nu = \partial u_\nu/\partial x_\mu; \qquad \bigtriangledown^\mu U_\nu = -\partial u_\nu/\partial x_\mu. \tag{1.392}$$

The limiting behaviors of the operators listed above would make it easy to deduce the nonrelativistic limits of macroscopic evolution equations. We note the following limiting behaviors:

$$\begin{aligned} c^{-2}\mathcal{E}\rho D U^k &= nc^{-1}\mathcal{E}\left(d_t + O\left(\frac{u}{c}\right)\right)(1, \mathbf{u}/c)^\mu \\ &= n\frac{\mathcal{E}}{c^2} d_t \mathbf{u}_k \\ &= \rho\left[d_t + O\left(c^{-2}\right)\right], \end{aligned} \tag{1.393}$$

where ρ is now the mass density. We also note the nonrelativistic limits of the following quantities:

$$\begin{aligned} p_i^0 &= \left(m_a^2 c^2 + \mathbf{p}\cdot\mathbf{p}\right)^{1/2} = m_a c + \frac{\mathbf{p}\cdot\mathbf{p}}{2m_a c} + O\left(u^2/c^2\right), \\ p_a^\mu &= \left(p_a^0, \mathbf{p}\right) = m_a c\left(1 + \frac{\mathbf{p}\cdot\mathbf{p}}{2m_a^2 c^2} + O\left(c^{-3}\right), \frac{\mathbf{p}}{m_a c}\right), \\ p_a^\mu U_\mu &= \gamma(c p_a^0 - \mathbf{p}_a\cdot\mathbf{u}) = m_a c^2\left[1 + \frac{1}{2} m_a \mathbf{C}_a\cdot\mathbf{C}_a + O(c^{-4})\right] \\ \Delta_\mu^\nu p_a^\mu &= c^{-2}\left(p_a^\lambda U_\lambda\right)\mathfrak{C}_a^\mu = m_a \mathbf{C}_a + O(c^{-2}), \end{aligned} \tag{1.394}$$

where $\mathbf{C}_a$ is the (nonrelativistic) peculiar velocity $\mathbf{C}_a = \mathbf{v}_a - \mathbf{u}$. With the nonrelativistic limits presented above, it is now possible to examine various quantities appearing in relativistic equations and formulas to obtain their nonrelativistic limits.

1.11.2 Nonrelativistic Limits of Energy-Momentum Tensor

The statistical mechanical expressions of energy-momentum tensor components have the following nonrelativistic limit expressions:

$$
\begin{aligned}
T^{00} &= c\sum_{a=1}^{r}\int \frac{d^3p_a}{p_a^0}p_a^0p_a^0f_a = nmc^2 + E_k + O(c^{-1}),\\
T^{0k} &= T^{k0} = c\sum_{a=1}^{r}\int \frac{d^3p_a}{p_a^0}p_a^0p_a^kf_a = \sum_{a=1}^{r}\int d^3p_ap_a^kf_a \quad (k=1,2,3),\\
T^{kj} &= c\sum_{a=1}^{r}\int \frac{d^3p_a}{p_a^0}p_a^kp_a^jf_a = \sum_{a=1}^{r}\int d^3p_ap_a^kp_a^jf_a \quad (k,j=1,2,3).
\end{aligned} \tag{1.395}
$$

Taking projections of $T^{\mu\nu}$, we obtain statistical mechanical expressions of various projected components and their statistical formulas as well as the limits:

$$
\begin{aligned}
E &= c^{-2}U_\mu T^{\mu\nu}U_\nu\\
&= \sum_{a=1}^{r}\int d^3p_af_a\frac{\left(U_\mu p_a^\mu\right)}{cp_a^0}\left(p_a^\nu U_\nu\right),
\end{aligned} \tag{1.396}
$$

which, as $u/c \to 0$, tends to the limit :

$$
E \to \sum_{a=1}^{r}n_am_ac^2 + \sum_{a=1}^{r}\int d^3p_af_a\frac{1}{2}m_a\left(\mathbf{C}_a\cdot\mathbf{C}_a\right) + O\left(c^{-2}\right). \tag{1.397}
$$

Here the integral on the right is the statistical mean value of nonrelativistic internal energy; see Chap. 3 of Volume 1. There is a rest mass energy additionally contributing to E, which should be subtracted from E. Similarly, since we may write the partial heat four-flow

$$
\begin{aligned}
Q_a^\mu &= c^{-2}\Delta_\omega^\mu T_a^{\omega\tau}U_\tau\\
&= \int d^3p_af_a\frac{\left(p_a^\omega U_\omega\right)}{cp_a^0}\left(p_a^\tau U_\tau - m_ac^2\right)\mathfrak{C}_a^\mu + \int d^3p_af_a\frac{\left(p_a^\omega U_\omega\right)}{cp_a^0}m_ac^2\mathfrak{C}_a^\mu
\end{aligned} \tag{1.398}
$$

and since $\lim_{u/c\to 0}\mathfrak{C}_a^\mu = C_{a\mu}$, we obtain in the limit of $u/c \to 0$ the nonrelativistic limit

$$
\lim_{u/c\to 0} Q_a^\mu = \int d^3p_af_a\frac{1}{2}m_a\left(\mathbf{C}_a\cdot\mathbf{C}_a\right)C_{a\mu} + \int d^3p_af_am_ac^2C_{a\mu} + O(c^{-2}), \tag{1.399}
$$

to which, as for the internal energy, the rest mass energy contributes an energy flux, but it should be subtracted from Q^μ. Thus we may define $Q^{'\mu}$

$$
Q^{'\mu} = Q_a^\mu - m_ac^2N_a^\mu, \tag{1.400}
$$

so that the limit

$$\lim_{u/c\to 0} Q'^{\mu} = \int d^3 p_a f_a \frac{1}{2} m_a \left(\mathbf{C}_a \cdot \mathbf{C}_a\right) C_{a\,\mu} + O(c^{-2}) \tag{1.401}$$

represents an internal energy (heat) flux as in the nonrelativistic kinetic theory. Recall that a similar subtraction was found necessary when the molecular moment for heat flux $h_a^{(3)\mu}$ was defined. In any case, Q'^{μ} defined in (1.400) now coincides with the heat flux obtained from $h_a^{(3)\mu}$ in the nonrelativistic limit.

As for the pressure (stress) tensor $P_a^{\mu\nu}$, since it is given by the projection of $T_a^{\mu\nu}$

$$P_a^{\mu\nu} = \Delta_{\omega}^{\mu} T_a^{\omega\tau} \Delta_{\tau}^{\nu},$$

its statistical mechanical formula is

$$P_a^{\mu\nu} = c \int \frac{d^3 p_a}{p_a^0} \Delta_{\omega}^{\mu} p_a^{\omega} p_a^{\tau} \Delta_{\tau}^{\nu} f_a, \tag{1.402}$$

which may be expressed in terms of relativistic peculiar velocities by the formula

$$P_a^{\mu\nu} = \int d^3 p_a f_a \frac{\left(p_a^{\omega} U_{\omega}\right)^2}{c^3 p_a^0} \mathfrak{C}_a^{\mu} \mathfrak{C}_a^{\nu}. \tag{1.403}$$

This statistical mechanical representation for $P_a^{\mu\nu}$ tends to the nonrelativistic limit as below:

$$\lim_{u/c\to 0} P_a^{\mu\nu} = \int d^3 p_a f_a m_a C_a^{\mu} C_a^{\nu} + O(u^2/c^2). \tag{1.404}$$

We have already discussed the nonrelativistic limits of $h_a^{(q)\mu\cdots\nu}$, expressed in terms of relativistic peculiar moments $\mathfrak{C}_a^{\mu}$ in an earlier section of this chapter; see (1.105)–(1.108). Since they are in complete agreement with the nonrelativistic Boltzmann kinetic theory results, we are now assured that various nonconserved variables $\Phi_a^{(q)\mu\cdots\nu}$ obeying the generic evolution equations have correct values in the limit of $u/c \to 0$. To find the nonrelativistic limit of the relativistic generic evolution equation we now examine the nonrelativistic limits (1.405) of kinematic terms $Z_a^{(q)\mu\cdots\nu}$ and its modified form $Z_a^{(q)\mu\cdots\nu}$ and $K_a^{(q)\mu\cdots\nu}$.

1.11.3 Nonrelativistic Limit of the Kinematic Terms

We find it convenient to make use of the relativistic peculiar velocity $\mathfrak{C}_a^{\sigma}$ in order to investigate the nonrelativistic limit of the kinematic term $Z_a^{(q)\mu\cdots\nu}$ and the related. The kinematic term $Z_a^{(q)\mu\cdots\nu}$ can be expressed in terms of $\mathfrak{C}_a^{\sigma}$

$$\begin{aligned} Z_a^{(q)\mu\cdots\nu} &= \left\langle f_a p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} \right\rangle \\ &= \left\langle f_a \partial_\sigma \left[\left(\mathfrak{C}_a^\sigma + U^\sigma \right) M_a^{(q)\mu\cdots\nu} \left(\mathfrak{C}_a \right) \right] \right\rangle, \end{aligned} \tag{1.405}$$

For the relation of $h_a^{(q)\mu\cdots\nu}$ to $M_a^{(q)\mu\cdots\nu}$ we refer to (1.98) and for the definitions of $M_a^{(q)\mu\cdots\nu}$ in terms of $\mathfrak{C}_a^\sigma$ see (1.99)–(1.102). The right-hand side of (1.405) then can be expressed as follows:

$$\begin{aligned} Z_a^{(q)\mu\cdots\nu} &= \left\langle f_a \left(D + \mathfrak{C}_a^\sigma \nabla_\sigma \right) M_a^{(q)\mu\cdots\nu} \right\rangle \\ &\quad + c^{-2} U_\sigma \left\langle f_a M_a^{(q)\mu\cdots\nu} D \mathfrak{C}_a^\sigma \right\rangle + \left\langle f_a M_a^{(q)\mu\cdots\nu} \left(\nabla_\sigma \mathfrak{C}_a^\sigma + \nabla_\sigma U^\sigma \right) \right\rangle. \end{aligned} \tag{1.406}$$

To arrive at this expression we have made use of the identities:

$$\mathfrak{C}_a^\sigma U_\sigma = 0, \quad U^\sigma \nabla_\sigma = 0, \quad U_\sigma D U^\sigma = 0, \quad U_\sigma \nabla_\sigma U^\sigma = 0. \tag{1.407}$$

See identities given below (1.97) in Sect. 1.1.6. The first term on the right hand side of (1.406) tends to the nonrelativistic limit

$$\lim_{u/c\to 0} \left\langle f_a \left(D + \mathfrak{C}_a^\sigma \nabla_\sigma \right) M_a^{(q)\mu\cdots\nu} \right\rangle = \left\langle f_a \mathcal{D}_t h_{a\,\mathrm{nr}}^{(q)\mu\cdots\nu} \right\rangle$$

where

$$\mathcal{D}_t = d_t + \mathbf{C}_a \cdot \nabla \tag{1.408}$$

$$h_{a\,\mathrm{nr}}^{(q)\mu\cdots\nu} = \lim_{u/c\to 0} M_a^{(q)\mu\cdots\nu} \tag{1.409}$$

with $h_{a\,\mathrm{nr}}^{(q)\mu\cdots\nu}$ denoting the nonrelativistic limits of the relativistic moments $h_a^{(q)\mu\cdots\nu}$ given in (1.105)–(1.108). The remaining terms on the right of (1.406) contribute terms of $O(u^2/c^2)$, namely, relativistic effects.

1.11.4 Nonrelativistic Limits of $\Omega_a^{(q)\sigma\mu\cdots\nu}$ and $K_a^{(q)\mu\cdots\nu}$

Using the various limits of $\mathfrak{C}_a^\sigma$ and moments $h_a^{(q)\mu\cdots\nu}$ and observing $\Omega_a^{(q)\sigma\mu\cdots\nu}$ is expressible in the form

$$\Omega_a^{(q)\sigma\mu\cdots\nu} = \left\langle f_a \mathfrak{C}_a^\sigma M_a^{(q)\mu\cdots\nu} \right\rangle, \tag{1.410}$$

we obtain the nonrelativistic limit

$$\lim_{u/c\to 0} \Omega_a^{(q)\sigma\mu\cdots\nu} = \left\langle f_a C_{a\,\sigma} h_{a\,\mathrm{nr}}^{(q)\mu\cdots\nu} \right\rangle. \tag{1.411}$$

This is the nonrelativistic limit of supermoment $\psi_{a\,\mathrm{nr}}^{(q)\sigma\mu\cdots\nu}$, which is one order higher than the moment $h_{a\,\mathrm{nr}}^{(q)\mu\cdots\nu}$.

As for $K_a^{(q)\mu\cdots\nu}$, since it is expressible

$$K_a^{(q)\mu\cdots\nu} = c^{-2} \left\langle f_a \left(\mathfrak{C}_a^{\sigma} + U^{\sigma} \right) M_a^{(q)\mu\cdots\nu} \left(\mathfrak{C}_a \right) \right\rangle DU_{\omega}, \tag{1.412}$$

it is seen to be $O(u^2/c^2)$ as $u/c \to 0$ and hence represents a relativistic effect. Therefore the modified kinematic term $\mathfrak{z}_a^{(q)\mu\cdots\nu}$ simply tends to the nonrelativistic limit of the kinematic term $Z_a^{(q)\mu\cdots\nu}$:

$$\lim_{u/c\to 0} \mathfrak{z}_a^{(q)\mu\cdots\nu} = \left\langle f_a \mathcal{D}_t h_{a\,\mathrm{nr}}^{(q)\mu\cdots\nu} \right\rangle + O(u^2/c^2). \tag{1.413}$$

Therefore, we conclude that the relativistic generic evolution equation tends to the nonrelativistic limit, namely, nonrelativistic generic evolution equation derived from the nonrelativistic Boltzmann equation. Therefore, this nonrelativistic limit of $Z_a^{(q)\mu\cdots\nu}$ would be recovered from the relativistic kinematic terms presented earlier. The limiting form used for the phase space integrals in (1.395) and various projections of them also apply to the collision integral and collision bracket integrals. Therefore it is expected that together with the nonrelativistic limits of various observables, operators, and integrals appearing in $Z_a^{(q)\mu\cdots\nu}$ we have presented earlier, the generalized hydrodynamic equations can be verified to tend to those in the nonrelativistic kinetic theory presented in Chap. 3 of Volume 1. To avoid repetition we leave the verification to the reader. In this regard, we note that

$$c^{-2} \left[\mathfrak{J}_a D\mathbf{U} \right]^{(2)\mu\nu} = \frac{h}{c^2} \left[\mathbf{J}_a D\mathbf{U} \right]^{(2)\mu\nu} \to - \left[\mathbf{J}_a d_t \mathbf{u} \right]^{(2)\mu\nu},$$

since $U_\nu \to -u_\nu$ in the limit of $u/c \to 0$.

Having verified that the relativistic generalized hydrodynamic equations tend to the nonrelativistic generalized hydrodynamic equations we are now assured of an appropriate extension of the nonrelativistic hydrodynamics to the relativistic domain of flow phenomena for a gaseous mixture described by the covariant kinetic equation postulated. As we have already discussed earlier, if the dissipation terms $\Lambda_a^{(q)\mu\cdots\sigma}$ $(q = 1, 2, 3, 4)$ are calculated with the nonequilibrium canonical form under the functional hypothesis and in the first-order cumulant approximation, which assures the thermodynamic consistency; and if the generalized potentials $X_a^{(q)\mu\cdots\sigma}$ $(q = 1, 2, 3, 4)$ are calculated to the first order in $\Phi_a^{(q)\mu\cdots\sigma}$; and lastly if a suitable closure is taken for the macroscopic observables so that the set of generalized hydrodynamic equations is closed, we may then investigate flow processes in the system subject to suitable initial and boundary conditions in a manner completely consistent with the laws of thermodynamics. Especially, the quasilinear model for generalized hydrodynamic equations enable us to investigate nonlinear phenomena such as non-Newstonian flow processes consistent with laws of thermodynamics. In view of the fact that nonrelativistic quasilinear model for generalized hydrodynamic equations have shown their capability for highly nonlinear flow phenomena we might speculate the relativistic quasilinear generalized hydrodynamic equations may yield equally interesting results. Lastly, we emphasize that the generalized hydrodynamic equations are thermodynamically consistent and support the thermodynamics of irreversible processes described by the thermodynamic theory based on the calortropy and calortropy differential form presented earlier, because the generalized hydrodynamic equations are field equations for macroscopic variables spanning the manifold $\mathfrak{P} \cup \mathfrak{T}$.

1.12 Relative Boltzmann Entropy Balance Equation

Since the calortropy Ψ is not equal to the Boltzmann entropy S, we have earlier defined their difference as the relative Boltzmann entropy and have shown its balance equation; see (1.228) and subsequent equations. Its balance equation can be a little more explicitly worked out to help us a little better comprehend the nature of the relative Boltzmann entropy. We discuss about it in the following.

1.12.1 A Form for ln $\left(f_a/f_a^{\mathrm{c}}\right)$

To make progress in the calculation of the relative Boltzmann entropy we now look for ln $\left(f_a/f_a^{\mathrm{c}}\right)$, as for the Boltzmann gas discussed in Chap. 3 of Volume 1, in a form whose leading terms are isomorphic to f_a^{c}:

$$k_B \ln\left(f_a/f_a^{\mathrm{c}}\right) = -p_a^{\nu} U_{\nu} \delta\overline{T} - \sum_{q \geq 1} \delta\overline{X}_{a\,\nu\cdots\sigma}^{(q)} h_a^{(q)\sigma\cdots\nu} + \delta\overline{\mu}_a - k_B \ln\left(\Upsilon_a / \Gamma_a\right). \tag{1.414}$$

Here Υ_a not only includes the normalization factor for f_a but also the correction factor to the leading term displayed. The symbols $\delta\overline{T}$, $\delta\overline{\mu}_a$ and $\delta\overline{X}_{a\,\nu\cdots\sigma}^{(q)}$ are defined by

$$\delta\overline{T} = \frac{1}{T} - \frac{1}{T^t};\ \delta\overline{\mu}_a = \frac{\mu_a}{T} - \frac{\mu_a^t}{T^t};\ \delta\overline{X}_a^{(q)} = \frac{X_a^{(q)}}{T} - \frac{X_a^{(q)t}}{T^t} \tag{1.415}$$

with the parameters superscripted with t denoting the irreversible thermodynamic parameters of manifold $\mathfrak{T}$, which are determined through the extended Gibbs relation for calortropy, (1.254). In other words, T^t, μ_a^t, and $X_a^{(q)t}$ are deterministic thermodynamic parameters belonging to manifold $\mathfrak{T}$, whereas the parameters T, μ_a, and $X_a^{(q)}$ are not. The latter are phase space functions that should be determined as parts of the solution for the covariant Boltzmann equation in the phase space, subject to the initial and boundary conditions, although from the irreversible thermodynamic standpoint they may be expressed as tangents to the manifold $\mathfrak{P}$. Therefore,

by looking for the solution of the covariant Boltzmann equation in the form as in (1.414) the entire phase space information on f_a beyond that of f_a^{c} is vested in the parameters T, μ_a, and $X_a^{(q)}$ and thus in the fluctuations $\delta\overline{I}$, $\delta\overline{\mu}_a$, and $\delta\overline{X}_a^{(q)}$.

If we express f_a in the form

$$f_a = \exp\left[-\left(k_B T\right)^{-1}\left(\mathcal{H}_a + \Theta_a\right)\right]\left(1 + \Delta f_a\right), \tag{1.416}$$

where

$$\mathcal{H}_a := p_a^{\nu} U_{\nu} + \sum_{q\geq 1} X_{a\,\nu\cdots\sigma}^{(q)} h_a^{(q)\sigma\cdots\nu} - \mu_a \tag{1.417}$$

and Δf_a is a correction to the nonequilibrium canonical form, then Υ_a is identified as follows:

$$\Upsilon_a\left(x, p_a\right) = \frac{\left\langle \exp\left[-\left(k_B T\right)^{-1}\mathcal{H}_a\right]\right\rangle}{n_a}\frac{\left\langle 1 + \exp\left[-\left(k_B T\right)^{-1}\mathcal{H}_a\right]\Delta f_a\right\rangle}{\left(1 + \Delta f_a\right)}. \tag{1.418}$$

The factor $\Upsilon_a\left(x, p_a\right)$ may be regarded as the normalization factor for f_a. We emphasize that, when they appear in the solution of the Boltzmann equation and the Boltzmann entropy is under consideration, the parameters T and μ_a are not necessarily the same as the absolute temperature and chemical potentials in the absence of the thermodynamic structure associated with the calortropy. They are just functions of phases (x, p_a) satisfying the covariant Boltzmann kinetic equation.

With so prepared $\ln\left(f_a/f_a^{\mathrm{c}}\right)$ and the nature of Υ_a clarified, we can now calculate the relative Boltzmann entropy balance equation, but it is not going to be integrable in the thermodynamic manifold because of Υ_a as will be shown.

The relative Boltzmann entropy four-flow may then be written in the form

$$S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right](x) = -\sum_{a=1}^{r}\left[T_a^{\mu\nu} U_{\nu}\delta\overline{I} + \sum_{q\geq 1}\delta\overline{X}_{a\,\nu\cdots\sigma}^{(q)}\psi_a^{(q)\sigma\cdots\nu\mu} - N_a^{\mu}\delta\overline{\mu}_a + k_B\left\langle f_a p_a^{\mu}\delta\overline{\ln\Gamma_a}\right\rangle\right], \tag{1.419}$$

where we have defined the symbol Γ_a by the formula

$$\delta\overline{\ln\Gamma_a} = \ln\left(\Gamma_a/\Upsilon_a\right) = \ln\Gamma_a - \ln\Upsilon_a. \tag{1.420}$$

Recall that $T_a^{\mu\nu}$ is the energy-momentum tensor, supermoment four-flow $\psi_a^{(q)\sigma\cdots\nu\mu}$, and particle four-flow N_a^{μ} defined by

$$T_a^{\mu\nu} = \left\langle p_a^{\mu} p_a^{\nu} f_a\right\rangle, \quad \psi_a^{(q)\sigma\cdots\nu\mu} = \left\langle p_a^{\mu} h_a^{(q)\sigma\cdots\nu} f_a\right\rangle, \quad N_a^{\mu} = \left\langle p_a^{\mu} f_a\right\rangle.$$

Let us define the scalar relative Boltzmann entropy density $\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]$ by the formula

$$\rho\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=c^{-2}U_{\mu}S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right] \tag{1.421}$$

as we have for other scalar densities such as the Boltzmann entropy, calortropy etc., and calculate the balance equation for $\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]$. Formally, it is given by

$$\begin{aligned}\rho D\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=\partial_{\mu}\left(\Delta_{\nu}^{\mu}S_{\mathrm{r}}^{\nu}\left[f|f^{\mathrm{c}}\right]\right)+k_{B}\sum_{a=1}^{r}\left\langle f_{a}p_{a}^{\mu}\partial_{\mu}\ln\left(f_{a}/f_{a}^{\mathrm{c}}\right)\right\rangle \\ +k_{B}\sum_{a=1}^{r}\sum_{b=1}^{r}\left\langle\partial_{\mu}\ln\left(f_{a}/f_{a}^{\mathrm{c}}\right)C_{ab}\left(f_{a}f_{b}\right)\right\rangle.\end{aligned} \tag{1.422}$$

The three components on the right can be more explicitly calculated by using (1.414).

1.12.2 *The Final Form of the Relative Boltzmann Entropy Balance Equation*

Putting the terms calculated earlier into the relative Boltzmann entropy balance equation, we obtain $\rho D\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]$ in the form

$$\begin{aligned}\rho D\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=Q^{\mu}\partial_{\mu}\delta\overline{I}+\delta\overline{I}\partial_{\mu}Q^{\mu}-T^{\mu\nu}\partial_{\mu}\left(U_{\nu}\delta\overline{I}\right)+\sum_{a=1}^{r}N_{a}^{\mu}\partial_{\mu}\delta\overline{\xi}_{a} \\ -\rho\sum_{a=1}^{r}\sum_{q\geq1}\widehat{\Phi}_{a}^{(q)\sigma\cdots\nu}D\delta\overline{X}_{a\,\alpha\cdots\sigma}^{(q)} \\ -\sum_{a=1}^{r}\sum_{q\geq1}\delta\overline{X}_{a\,\alpha\cdots\sigma}^{(q)}\rho D\widehat{\Phi}_{a}^{(q)\sigma\cdots\alpha\nu}-\rho\sum_{a=1}^{r}k_{B}D\left\langle\delta\overline{\ln\Gamma_{a}}\right\rangle,\end{aligned} \tag{1.423}$$

where we may use the evolution equation for $\left\langle\delta\overline{\ln\Gamma_{a}}\right\rangle$ expressed as follows:

$$\begin{aligned}\rho D\left\langle\delta\overline{\ln\Gamma_{a}}\right\rangle=\partial_{\mu}\left(\Delta_{\nu}^{\mu}\left\langle f_{a}p_{a}^{\nu}\delta\ln\Gamma_{a}\right\rangle\right)+\left\langle f_{a}p_{a}^{\mu}\partial_{\mu}\delta\ln\Gamma_{a}\right\rangle \\ +\sum_{b=1}^{r}\left\langle\delta\ln\Gamma_{a}C_{ab}\left(f_{a}f_{b}\right)\right\rangle.\end{aligned} \tag{1.424}$$

This evolution equation is obtained by making use of the generic form for evolution equations for scalar quantities; see, for example, evolution equation for $\widehat{\Phi}_{a}^{(q)\alpha\cdots\sigma}$, (1.118). Therefore $\left\langle\delta\overline{\ln\Gamma_{a}}\right\rangle$ is looked upon as a macroscopic quantity. This viewpoint is sufficient for our purpose here. If the decomposition of energy-momentum tensor, various identities involving covariant derivatives, the hydrodynamic velocity U^{μ}, and the products of the latter with Q^{μ} and $P^{\mu\nu}$, the following can be shown:

$$\begin{aligned}\mathrm{Term1} &:= Q^{\mu}\partial_{\mu}\delta\overline{I} + \delta\overline{I}\partial_{\mu}Q^{\mu} - T^{\mu\nu}\partial_{\mu}U_{\nu}\delta\overline{I} \\ &= -\delta\overline{I}\rho D\mathcal{E} - \rho\mathcal{E}D\delta\overline{I}.\end{aligned} \tag{1.425}$$

For this we have used the energy balance equation (1.84). With this result we finally obtain the relative Boltzmann entropy balance equation in a relatively simple form

$$\begin{aligned}D\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right] = &-\delta\overline{I}D\mathcal{E} + \sum_{a=1}^{r}\delta\overline{\xi}_{a}D\mathfrak{c}_{a} - \sum_{a=1}^{r}\sum_{q\geq 1}\delta\overline{X}^{(q)}_{a\,\alpha\cdots\sigma}D\widehat{\Phi}^{(q)\sigma\cdots\alpha\nu}_{a} \\ &-\mathcal{E}D\delta\overline{I} + \sum_{a=1}^{r}\mathfrak{c}_{a}D\delta\overline{\mu}_{a} - \sum_{a=1}^{r}\sum_{q\geq 1}\widehat{\Phi}^{(q)\sigma\cdots\alpha\nu}_{a}D\delta\overline{X}^{(q)}_{a\,\alpha\cdots\sigma} \\ &-k_B\sum_{a=1}^{r}D\left\langle\delta\overline{\ln\Gamma_{a}}\right\rangle.\end{aligned} \tag{1.426}$$

We note that we can obtain this form directly from the statistical mechanical expression (1.419) and (1.421), if the relative Boltzmann entropy density is differentiated with the convective derivative D. As a matter of fact, this procedure serves as a check for the final result.

The differential form (1.426), however, is not integrable in the thermodynamic manifold because the last term $D\left\langle\delta\overline{\ln\Gamma_{a}}\right\rangle$ cannot be expressed in a bilinear form of variables belonging to manifold $\mathfrak{P}\cup\mathfrak{T}$. For it is given in terms of the solution of the covariant Boltzmann equation, subject to the initial and boundary conditions, in the phase space. Therefore $D\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]$ is not an exact differential in the thermodynamic manifold $\mathfrak{P}\cup\mathfrak{T}$.

Following the procedure described in Chap. 3, it is possible to develop a system of evolution equations $\delta\overline{I}$, $\delta\overline{\mu}_{a}$, $\delta\overline{X}^{(q)}_{a\,\alpha\cdots\sigma}$, etc. on use of the covariant kinetic equation. We do not wish to repeat it here, especially since studying such a set of equations is beyond the scope of this work, which is to investigate the kinetic theory foundations of thermodynamic theories of irreversible processes and accompanying hydrodynamics. We will leave it to the future development on the subject of fluctuation theory.

1.13 Concluding Remarks

In this chapter, we have applied the modified moment method to the covariant Boltzmann equation for a relativistic gas mixture to study the thermodynamics of irreversible processes and hydrodynamics associated therewith. As in the nonrelativistic theories, the method modifies the conventional moment method of H. Grad [29] in such a way as to make the hydrodynamic theory of irreversible processes strictly conform to the demand of the laws of thermodynamics. It thus affords us a rigorous conclusion concerning the calortropy differential and foundations of irreversible thermodynamics for the system away from equilibrium. It is found that the statistical mechanically defined calortropy admits a mathematical structure for a theory of irreversible thermodynamics consistent with the *H* theorem and the laws of thermodynamics and, in particular, the second law of thermodynamics. Therefore it has been possible to show that there exists a theory of thermodynamics of irreversible processes satisfactorily describable on the basis of a mathematical theory of macroscopic irreversible processes derived from the covariant kinetic equation in a manner consistent with the thermodynamic principles.

In the literature on the molecular theory approach to irreversible thermodynamics the important notion and properties of the Clausius entropy have been uncritically extended to nonequilibrium transport processes. The present work shows that such an uncritical extension is not warranted for both nonrelativistic and relativistic kinetic theory. In this chapter, by using the covariant Boltzmann equation for gas mixture we have shown how one may develop a thermodynamically consistent theory of irreversible processes, irreversible thermodynamics, and attendant hydrodynamics on the basis of calortropy. We have also pointed out that the Boltzmann *H* theorem is a stability theorem for the equilibrium solution for the covariant Boltzmann equation, but it is not a statistical mechanical representation of *the second law of thermodynamics*. The *H* theorem is a broader stability theorem for the distribution function in the phase space than it is a statistical mechanical representation of the second law of thermodynamics that governs the macroscopic processes, reversible or irreversible, in the thermodynamic manifold, which is much smaller in dimension than the phase space of a gas consisting of molecular particles. In fact, for nonequilibrium systems the Boltzmann entropy is a quantity depending on the path of evolution taken by the system in the nonequilibrium thermodynamic manifold, when the system makes transition from a state to another in the manifold. The notion of equilibrium entropy, which is a state function giving rise to an exact differential in the thermodynamic manifold, does not apply to the nonequilibrium Boltzmann entropy, if it is defined by the statistical formula associated with the *H* theorem for the covariant kinetic equation.

The present study also demonstrates for the case of relativistic kinetic theory that the calortropy differential produces a thermodynamic theory of nonequilibrium systems in a way rather parallel to equilibrium thermodynamics. Nevertheless, such a formalism still requires integration of the relativistic generalized hydrodynamic equations in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ as an integral part of the theory. Since the latter are not trivial to solve exactly in analytic form except for a few special cases, there remains a great deal of work to be done on the subject matter of hydrodynamic flows in the future, unquestionably and necessarily, in the numerical computational approach. Here we now have acquired a thermodynamically consistent coherent mathematical theory of nonequilibrium statistical thermodynamics, irreversible processes, and hydrodynamics far removed from equilibrium at arbitrary degree.

The relativistic kinetic theory presented in essence puts the statistical mechanics of nonequilibrium thermodynamics on par with the Gibbs ensemble theory of equilibrium statistical thermodynamics in the sense that all thermodynamic functions and evolution equations and relations are expressed in terms of nonequilibrium partition function which must be computed for each and every irreversible process in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ to achieve the desired goal of a molecular theory of matter.

All the macroscopic evolution equations are presented in such forms that they consist of terms easily identifiable with the corresponding terms in their nonrelativistic versions and purely relativistic terms which vanish in the nonrelativistic limit. Thus, the relativistic corrections to the classical generalized hydrodynamic equations are clearly exposed and made easy to identify and examine their effects. Since the full relativistic generalized hydrodynamic equations are much more complicated to solve than their already difficult nonrelativistic counterparts, the first-order relativistic corrections to the latter are probably all we can hope for in practice at present. The quasilinear relativistic generalized hydrodynamic equations presented here should be useful for studies in macroscopic flow processes and transport processes in a thermodynamically consistent manner. The fact that their nonrelativistic counterparts have been found useful raises a hope that the relativistic versions should be useful for studying relativistic irreversible processes in gases.

1.14 Supplementary Notes on Four-Tensors and Vectors

1.14.1 Four-Vectors, Four-Tensors and Related Algebra

In the present Supplementary Notes, we briefly review some topics on four vectors and tensors [44] associated with the relativity theory for the reader not familiar with the subject matter.

A point x in spacetime denotes a four-vector

$$x = x^\mu = (ct, \mathbf{r}) = (ct, x_1, x_2, x_3) \tag{1.427}$$

and the conjugate four-momentum is

$$p^\mu = (p^0, \mathbf{p}), \tag{1.428}$$

where cp^0 is energy with abbreviation

$$p^0 = \sqrt{m^2c^2 + \mathbf{p}^2}. \tag{1.429}$$

The length of time-like vectors is positive. Let the length squared of four-momentum p^μ be denoted

$$p^2 := p^\mu p_\mu. \tag{1.430}$$

Then since we employ the metric tensor

$$g^{\mu\nu} = \operatorname{diag}(1, -1, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.431}$$

we find

$$p^2 = p^\mu p_\mu = \left(p^0\right)^2 - \mathbf{p} \cdot \mathbf{p} = (mc)^2. \tag{1.432}$$

Note that p^μ therefore is time-like since $(mc)^2 > 0$.

Let the distribution function be $f(x, p)$. The particle number density $n_i(\mathbf{x}, t)$ may be defined as

$$n(\mathbf{x}, t) = \int d^3 p f(x, p) \tag{1.433}$$

and particle flow as

$$\mathbf{j}(\mathbf{x}, t) = \int d^3 p \mathbf{v} f(x, p), \tag{1.434}$$

where

$$\mathbf{v} = \frac{c\mathbf{p}}{p^0} \tag{1.435}$$

is the velocity of a relativistic particle of momentum $\mathbf{p}$. The particle number four-flow $N^\mu(x) = (cn(\mathbf{x}, t), \mathbf{j}(\mathbf{x}, t))$ can be written as

$$N^\mu = c \int \frac{d^3 p}{p^0} p^\mu f(x, p). \tag{1.436}$$

As is demonstrated by de Groot et al. [3] $f(x, p)$ is a scalar under Lorentz transformation.

1.14.2 The Scalar Product of Four-Vectors

The scalar product of four-vectors is given by

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3. \tag{1.437}$$

This four-scalar is invariant under rotations of the four dimensional coordinate system. The component A^0 is called the time component, and A^1, A^2, A^3 the space component of the four-vector. Depending on the sign of the scalar product, A^i and B_i are classified as follows:

$$\begin{aligned}&\text{if } A^i B_i > 0;\ \ A^i \text{ and } B_i \text{ are time-like,}\\&\text{if } A^i B_i < 0;\ \ A^i \text{ and } B_i \text{ are space-like,}\\&\text{if } A^i B_i = 0;\ \ A^i \text{ and } B_i \text{ are null-vectors.}\end{aligned}$$

Under purely spatial rotations (transformations not affecting the time axis) the three space components of the four vector A^i form a three-dimensional vector $\mathbf{A}$. The time-component of A^i is a three-dimensional scalar with respect to rotations.

The components of the four-vector may be enumerated as

$$A^i = (A^0, \mathbf{A}). \tag{1.438}$$

The covariant component of the same four-vector is given by

$$A_i = (A^0, -\mathbf{A}). \tag{1.439}$$

Therefore the scalar product $A^i A_i$ is given by

$$A^i A_i = \left(A^0\right)^2 - \mathbf{A}^2. \tag{1.440}$$

For example, for four-vector x^i

$$x^i = (ct, \mathbf{r})\,, \quad x_i = (ct, -\mathbf{r})\,, \quad x^i x_i = c^2 t^2 - \mathbf{r}^2 \tag{1.441}$$

Thus

$$dx^i dx_i = c^2\,(dt)^2 - (d\mathbf{r})^2\,. \tag{1.442}$$

1.14.3 Four-Tensor of Rank Two

For a four-tensor of rank 2, there is a set of 16 components A^{ik} which under coordinate transformations transform like the products of components of two four-vectors. The components of a second-rank tensor can be written in three forms: contravariant A^{ik}, covariant A_{ik}, and mixed ${A^i}_k$ and ${A_i}^k$. For symmetric tensors ${A^i}_k = {A_i}^k$, which then may be written as A^i_k. Raising or lowering the a space index changes the sign of the component, whereas raising or lowering the time index (0) does not. Thus

$$\begin{aligned} &A_{00} = A^{00}, \quad A_0{}^0 = A^{00}, \quad A_0{}^1 = A^{01}, \\ &A_{01} = -A^{01}, \quad A^0{}_1 = -A^{01}, \quad A_1{}^1 = -A^{11}, \quad \text{etc.} \end{aligned} \tag{1.443}$$

Under purely spatial transformations, the 9 components A^{11}, A^{12}, etc. (e.g., direct products of three-dimensional vectors) form a three-tensor whereas A^{01}, A^{02}, etc. constitute three-dimensional vectors, while A^{00} is a three-dimensional scalar (e.g., a scalar product of two three-dimensional vectors).

Contraction of a tensor A_i^i is the trace of the tensor:

$$\begin{aligned} A_i^i &= A_0^0 + A_1^1 + A_2^2 + A_3^3 \\ &:= \mathrm{Tr} A \end{aligned} \tag{1.444}$$

Unit four-tensor δ_i^k is defined by

$$\begin{aligned} \delta_i^k &= 1, \quad \text{if } i = k \\ &= 0, \quad \text{if } i \neq k. \end{aligned} \tag{1.445}$$

Thus

$$\delta_i^k A^i = A^k. \tag{1.446}$$

Raising or lowering index is achieved by a metric tensor $\boldsymbol{g}^{ik}$ or $\boldsymbol{g}_{ik}$:

$$\begin{aligned} \boldsymbol{g}^{ik} A_k &= A^i, \\ \boldsymbol{g}_{ik} A^k &= A_i. \end{aligned} \tag{1.447}$$

In matrix form, they therefore must have the form

$$\left(\boldsymbol{g}^{ik}\right) = (\boldsymbol{g}_{ik}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.448}$$

Then the scalar product of two four-vectors can be written as

$$A^i A_i = \boldsymbol{g}^{ik} A_i A_k = \boldsymbol{g}_{ik} A^i A^k. \tag{1.449}$$

1.14.4 Completely Antisymmetric Unit Tensors of Rank Four e^{iklm}

As δ^i_k, g_{ik}, and g^{ik} are, four-tensors e^{iklm} are the same in all coordinates. This is the tensor whose components change sign under interchange of any pair of indices, and whose nonzero components are ± 1. Therefore, if, for example, two indices are the same, then $e^{iilm} = 0$ owing to the antisymmetry. This means that only

$$e^{0123} = 1, \tag{1.450}$$

and, if so, then

$$e_{0123} = -1. \tag{1.451}$$

To see this, we observe

$$e_{0123} = g_{00}g_{11}g_{22}g_{33}e^{0123} = 1(-1)^3 1 = -1. \tag{1.452}$$

With respect to rotations of the coordinate system, e^{iklm} behaves like a tensor. But even if the sign of one or three of the coordinates is changed, the sign of e^{iklm} do not change because e^{iklm} is the same in all coordinate systems. Therefore e^{iklm} is not a tensor, but a pseudotensor. Pseudotensors of any rank, in particular pseudoscalar, behave like tensors under all transformations except to reflection, which is not reducible to a rotation.

1.14.5 Terminology Definition

It is convenient to list the well known definitions of terms used in connection with tensor algebra.

Polar vector: under a reflection of the coordinate system, i.e., under a change in sign of all coordinates, the components of ordinary vector also change the sign. Such vectors are said to be polar.

Axial vector: On the other hand, the components of a vector that can be written as a cross product of two polar vectors do not change sign under inversion. Such vectors are said to be axial.

Pseudoscalar: The scalar product of a polar vector with an axial vector changes its sign under reflection. Such a scalar is called the pseudoscalar. Note that scalar products of two polar vectors or two axial vectors do not change the sign under reflection.

Pseudovector: Axial vectors (e.g., $\mathbf{C} = \mathbf{A} \times \mathbf{B}$) are pseudovectors dual to some antisymmetric tensor. Thus it can be written as

$$C_{\beta\gamma} = A_\beta B_\gamma - A_\gamma B_\beta. \tag{1.453}$$

Then the pseudovector C_α can be defined

$$C_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} C^{\beta\gamma}. \tag{1.454}$$

This vector is dual to the antisymmetric tensor $C^{\beta\gamma}$.

Antisymmetric four-tensor: The space components $(i, k = 1, 2, 3)$ of antisymmetric tensor A^{ik} form a three-dimensional antisymmetric tensor with respect to purely spatial transformations; its components can be expressed in terms of the components of a three-dimensional axial vector. With respect to the same transformations the components A^{01}, A^{02}, A^{03} form a three-dimensional polar vector. Thus the components of an antisymmetric four-tensors can be written as a matrix

$$\left(A^{ik}\right) = \begin{pmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_x & a_y \\ -p_y & a_x & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{pmatrix}, \tag{1.455}$$

where, with respect to spatial transformations, **p** and **a** are polar and axial vectors, respectively. This four-tensor will be enumerated as

$$A^{ik} = (\mathbf{p}, \mathbf{a}). \tag{1.456}$$

Then the covariant components of the same tensor are enumerated as

$$A_{ik} = (-\mathbf{p}, \mathbf{a}). \tag{1.457}$$

This should be understood by the fact that **p** is a polar vector that changes sign on reflection while the sign of the axial vector **a** remains unchanged.

Four-gradient of scalar ϕ: The covariant component of the four-vector (∂_i) is enumerated as

$$\frac{\partial\phi}{\partial x^i} = \left(\frac{1}{c}\frac{\partial\phi}{\partial t}, \nabla\phi\right) \tag{1.458}$$

In fact, the differential of scalar

$$d\phi = \frac{\partial\phi}{\partial x^i} dx^i \tag{1.459}$$

is also a scalar. In general, the operators of differential with respect to the coordinates x^i, $\partial/\partial x^i$, should be regarded as a covariant component of the operator four-vector. Therefore, for example, the divergence of a four-vector A^i

$$\frac{\partial A^i}{\partial x^i} := \partial_i A^i \tag{1.460}$$

is a scalar.

1.14.6 Four Types of Integrations

1. **Integral over a curve in four-space**. The element of integration is the line element dx^i
$$\int dx^i f. \tag{1.461}$$
2. **Integral over a (two-dimensional) surface in four-space**. In three space the projections of the area of the parallelogram formed from $d\mathbf{r}$ and $d\mathbf{r}'$ on the coordinate plane $x_\alpha x_\beta$ are
$$A = dx_\alpha dx'_\beta - dx_\beta dx'_\alpha. \tag{1.462}$$
Similarly, in four-space the infinitesimal element df^{ik} is given by
$$df^{ik} = dx^i dx'^k - dx^k dx'^i. \tag{1.463}$$
These are projections of the element of area on the coordinate planes. In three-dimensional space one uses as surface element in place of the tensor $df_{\alpha\beta}$ the vector dual to the tensor $df_{\alpha\beta}$:
$$df_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} df^{\beta\gamma}. \tag{1.464}$$
This is what we understand as the normal vector to the surface in three-dimensional vector analysis, the magnitude of which is equal to the area of the element. In four-space we construct the tensor df^{*ik} dual to the tensor df^{ik},
$$df^{*ik} = \frac{1}{2} e^{iklm} df_{lm}. \tag{1.465}$$
Geometrically, it describes an element of surface equal to and "normal" to the element of surface df^{ik}. All elements lying in it are orthogonal to all segments in the element df^{ik}. Obviously, it follows that
$$df^{ik} df^*_{ik} = 0. \tag{1.466}$$
3. **Integral over a hypersurface**, i.e., over a three-dimensional manifold. In three-dimensional space the volume of the parallelepiped spanned by three vectors is equal to the determinant of the third rank formed from the components of the vectors:

$$dS^{ikl} = \begin{vmatrix} dx^i & dx'^i & dx''^i \\ dx^k & dx'^k & dx''^k \\ dx^l & dx'^l & dx''^l \end{vmatrix}, \tag{1.467}$$

which form a tensor of rank 3, antisymmetric in all three indices. As element of integration over the hypersurface, it is more convenient to use the four-vector dS^i, dual to dS^{ikl}

$$dS^i = -\frac{1}{6}e^{iklm}dS_{klm}, \tag{1.468}$$

$$dS_{klm} = e_{nklm}dS^n. \tag{1.469}$$

Here

$$dS^0 = dS^{123}, \quad dS^1 = dS^{023}, \ldots. \tag{1.470}$$

Geometrically, dS^i is a four-vector equal in magnitude to the "areas" of the hypersurface element, and normal to this element (i.e., perpendicular to all lines drawn in the hypersurface element). In particular, $dS^0 = dxdydz$, i.e., the three-dimensional volume element, the projection of the hypersurface element on the hyperplane $x^0 =$ constant.

4. **Integral over a four-dimensional volume**: The element of integration is scalar

$$d\Omega = dx^0dx^1dx^2dx^3 = cdtdV \tag{1.471}$$

Analogs of Gauss and Stokes theorems:

$$\oint A^i dS_i = \int \frac{\partial A^i}{\partial x^i}d\Omega \qquad \text{(Gauss theorem)}, \tag{1.472}$$

$$\begin{aligned} \oint A_i dx^i &= \int \frac{\partial A^i}{\partial x^k}df^{ki} \qquad \text{(Stokes theorem)} \\ &= \frac{1}{2}\int df^{ki}\left(\frac{\partial A^k}{\partial x^i} - \frac{\partial A^i}{\partial x^k}\right). \end{aligned} \tag{1.473}$$

1.14.7 Four-Dimensional Velocity

Four-velocity of a particle is defined by

$$u^i = \frac{dx^i}{ds}. \tag{1.474}$$

To find its components, since

$$ds = cdt\sqrt{1 - \frac{v^2}{c^2}}, \tag{1.475}$$

where v is the ordinary three-dimensional velocity of the particle,

$$u^1 = \frac{v_x}{c\sqrt{1 - \frac{v^2}{c^2}}} \text{ etc.}, \tag{1.476}$$

the four-velocity is given by

$$u^i = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{v_x}{c\sqrt{1 - \frac{v^2}{c^2}}} \right). \tag{1.477}$$

Since

$$dx_i dx^i = ds^2$$

we obtain

$$u^i u_i = 1 \tag{1.478}$$

Similarly, the second derivative is given by

$$w^i := \frac{d^2 x^i}{ds^2} = \frac{du^i}{ds}. \tag{1.479}$$

Therefore

$$w^i u_i = 0. \tag{1.480}$$

The algebra surveyed in this appendix is made use for various evolution equations in the relativistic kinetic theory.

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Chapter 2
Relativistic Kinetic Theory of Matter and Radiation

The relativistic kinetic theory of irreversible processes in a system of matter presented in Chap. 1 of this volume is generalized to include radiation in this chapter.[1] The generalization enables us to remove the awkwardness inherent to the nonrelativistic theory of radiation and matter reported in the paper [1] by Eu and Mao and provides kinetic theory foundations for the relativistic irreversible thermodynamics and radiation hydrodynamics for a system of radiation and matter. In this chapter, the relativistic Boltzmann equations—more precisely, a relativistic theory (covariant) form of the Boltzmann–Nordholm–Uehling–Uhlenbeck equations [2–4]—is employed to formulate a theory of transport processes in a system of radiation interacting with matter in a manner consistent with the laws of thermodynamics.

The motivations to have a covariant kinetic theory are following: Since photons are inherently relativistic, the relativity principle requires that governing equations of the system must be Lorentz covariant. In Ref. [1] the nonrelativistic kinetic theory was used to study irreversible processes in a system of radiation interacting with matter and, in particular, to study Doppler shift corrections for photon frequencies, but the nonrelativistic kinetic theory treatment of the subject matter was found often cumbersome and awkward. It was also difficult to make sure that the definitions of statistical mechanical formulas for macroscopic variables and the evolution equations indeed had correct nonrelativistic limits. These weaknesses can be assuredly removed if a covariant theory is formulated. Secondly, there are some problems, especially, in the study of the early epoch of the universe and also in the nuclear physics of high energy heavy ion collisions, for which a relativistic formalism is required since post-collision particles move at high speed. It will be shown that the present covariant kinetic theory recovers all the nonrelativistic evolution equations

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[1] This chapter is a substantially revised version of the unpublished paper by the author with K. Mao, which was also part of the PhD thesis of K. Mao, McGill University, Montreal, 1993, under supervision of B.C. Eu. The revision is concerned with the theory of irreversible thermodynamics making use of the calortropy and the evolution equations for nonconserved variables. There are other aspects which are significantly revised.

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Irreversible Thermodynamics, and Generalized Hydrodynamics,
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in correct forms with proper relativistic connections and thus verifies the previous nonrelativistic formulation [1]. In any case, in the present chapter we formulate a relativistic kinetic theory for a monatomic gas mixture interacting with photons treated relativistically and quantum mechanically. In the kinetic equations for molecular particles and photons, the collision terms are so modified as to make them more suitable for quantum particles. Since the collision terms in the kinetic equations used in the present chapter are quantum mechanical, they naturally affect the dissipation terms of nonconserved variables in the evolution equations, but we are, nevertheless, able to make use of the approach taken in Chap. 1 of this Volume except for some details related to the quantum collision terms. As a matter of fact, the affected parts of the theory turn out to be all related to transport coefficients originating from the collision terms in the kinetic equations. Therefore, most of evolution equations involved in the theory remain the same as those in Chap. 1 of this Volume except for the dissipation terms and quantities related to them. For this reason we will dispense with the details of derivations of evolution equations, wherever possible without sacrificing the comprehension, but will present the results only. The distribution functions used in the present chapter should be regarded as the Wigner representations of the density matrices. For this part of discussion on the kinetic equation and distribution functions employed for quantum particles, see Chap. 9, Sects. 9.1–9.3, of Ref. [5], where a nonrelativistic theory is used to discuss about Wigner distribution functions.

First, covariant Boltzmann equations are briefly presented for a system consisting of photons and material particles with internal degrees of freedom. The material gas molecules can make transitions between various internal states in interaction with photons by absorbing, emitting, or scattering the latter. These dynamical processes at the particulate level are described by the Boltzmann collision terms in the covariant Boltzmann equations. Consequently, the generalized hydrodynamic equations are derived from the covariant kinetic equations for radiation and matter and made fully consistent with the thermodynamic laws at any degree of removal from equilibrium.

The consistency between the relativistic Boltzmann distribution function and the Planck distribution function has been a point of controversy in the past [6–8], and the present covariant kinetic theory makes it possible to examine the question. As a matter of fact, we show that both distribution functions are consistent with each other within the framework of relativity and that their mutual consistency is intimately related to the H theorem and therefore deeply rooted in the second law of thermodynamics framed in terms of calortropy. In this connection, it is interesting to recall that the original derivation by Planck of the radiation distribution function itself was based on thermodynamics [9].

2.1 Covariant Boltzmann Equations for Matter and Radiation

We assume that the system consists of a gas with internal degrees of freedom (e.g., atoms or diatomic molecules, etc.) which interact with a radiation field (photons) not necessarily in equilibrium. The system is assumed to be free from external fields.

The internal quantum states of material particles will be denoted by i and the particles in different internal quantum states a will be regarded as different species of particles. Therefore, the material gas is considered a mixture of particles with different "colors" distinguished by the value of i. For example, if the particle species is the hydrogen atom, then the index i represents the $1s$, $2p$, $\cdots$, states of hydrogen suitably arranged. The mass of the particle species a will be denoted by m_a and its momentum by $\mathbf{p}_a$, the kinetic energy by cp_a^0 with c denoting the speed of light.

It then is convenient, as in Chap. 1 of the present Volume, to enumerate the four-momentum of species a by the formula

$$p_a^\mu = \left(p_a^0, \mathbf{p}_a\right), \tag{2.1}$$

where the time component p_a^0 is given by the expression

$$p_a^0 = \sqrt{\mathbf{p}_a \cdot \mathbf{p}_a + m_a^2 c^2}. \tag{2.2}$$

This formula looks like the one holding for structureless particles considered in Chap. 1 of the present Volume, but different masses m_a are assigned to the particles in different internal states. This point can be illustrated in the following way: According to relativistic quantum mechanics, the Dirac equation for the hydrogen atom, for example, gives the energy eigenvalues [10] depending on the two quantum numbers, the principal quantum number n and the angular momentum quantum number J,

$$E_{nJ} = m_e c^2 \left[1 + Z_e^2 \alpha^2 / \left(n - \varepsilon_J\right)^2\right]^{-1/2}, \tag{2.3}$$

where ε_J is defined by the formula

$$\varepsilon_J = J + \frac{1}{2} - \left[\left(J + \frac{1}{2}\right)^2 - Z_e^2 \alpha^2\right]^{1/2}; \tag{2.4}$$

α is the fine structure constant

$$\alpha = \frac{e^2}{\hbar} \approx \frac{1}{137};$$

Z_e is the electron number in atom; and m_e is the rest mass of the electron. If we set $E_a^{nJ} = m_a c^2$ where a stands for the set (n, J), (2.3) suggests that the mass of the excited particles is not the same as the rest mass of the ground-state particles. Therefore, m_a contains the information on the internal state of the particle. It is easy to show by expansion that approximately

$$m_a = m_e + c^{-2} E_a, \tag{2.5}$$

where E_a is the energy eigenvalue:

$$E_a = -\frac{Z_e^2 \alpha^2}{2a^2} \quad (a = n - \varepsilon_J) . \tag{2.6}$$

The time component p_a^0 in the four-momentum in (2.1) is understood in the sense of (2.5) or its precursor (2.3) and will be used accordingly in the calculations made in connection with the two-body collision dynamics in this chapter. The singlet distribution function of species a will be denoted by $f_a(x, p_a)$, the italic subscript r being reserved for radiation.

We consider the following microscopic collision processes:

$$a + b \rightleftharpoons a^* + b^*, \tag{M1}$$

$$a + b \rightleftharpoons k + l, \tag{M2}$$

$$a + \hbar\omega \rightleftharpoons a^* + \hbar\omega^*, \tag{M3}$$

$$a + \hbar\omega \rightleftharpoons b + \hbar\omega^*, \tag{M4}$$

where a, b, k and l stand for the matter species and the asterisk denotes the post-collision quantity. Therefore, for example, $\hbar\omega^*$ means a photon of radiation frequency ω^* after collision. Let subscript a stand for the species of particles including photons, which will be designated the rth species.

In contrast to the four-momentum of matter species a given in (2.1), the photon four-momentum is defined as follows: Since photon momentum $\mathbf{p}_r$ is related to the wave vector $\mathbf{k}_r$ of photon propagation, it can be written

$$\mathbf{p}_r = \hbar \mathbf{k}_r := \hbar k_r \widehat{\mathbf{k}}_r, \tag{2.7}$$

where $\widehat{\mathbf{k}}_r$ is the dimensionless unit vector of the photon wave vector $\mathbf{k}_r$:

$$\widehat{\mathbf{k}}_r = \mathbf{k}_r / |\mathbf{k}_r| = \mathbf{k}_r / k_r . \tag{2.8}$$

If we set

$$k_r = \frac{\omega}{c}, \tag{2.9}$$

where ω is the photon frequency, then the photon momentum $\mathbf{p}_r$ may be written

$$\mathbf{p}_r = \frac{\hbar\omega}{c}\widehat{\mathbf{k}}_r. \tag{2.10}$$

We now observe that the photon is massless.[2] Consistent with this fact, the photon four-momentum p_r^μ is enumerated by

$$p_r^\mu = \hbar\left(k_r, \mathbf{k}_r\right)^\mu = \hbar\left(\frac{\omega}{c}, \mathbf{k}_r\right)^\mu = \frac{\hbar\omega}{c}\left(1, \widehat{\mathbf{k}}_r\right)^\mu. \tag{2.11}$$

Therefore we find its scalar product is equal to zero:

$$p_r^\mu p_{r\,\mu} = \left(\frac{\hbar\omega}{c}\right)^2\left(1 - \widehat{\mathbf{k}}_r \cdot \widehat{\mathbf{k}}_r\right) = 0. \tag{2.12}$$

This is in contrast to $p_a^\mu p_{a\,\mu} = m_a^2 c^2$ for the case of matter species a.

It is postulated that the singlet distribution function $f_a(x, p_a)$ for species a obeys the covariant Boltzmann equation[3]

$$p_a^\nu \partial_\nu f_a(x, p_a) = \mathfrak{R}_a\left[f_a\right], \tag{2.13}$$

where collision term $\mathfrak{R}_a\left[f_a\right]$, which is decomposable into component collision terms

$$\mathfrak{R}_a\left[f_a\right] = \sum_{b=1}^{r} \mathfrak{R}_{ab}\left[f_a f_b\right], \tag{2.14}$$

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[2] Later, this statement is shown to be true from the thermodynamic consideration.

[3] This is a covariant kinetic equation for particles free from an external force field F^μ. This restriction, however, can be relaxed for the case of the Lorentz force, provided that the external force leaves the rest mass of the particle unaltered at the end of the particle collision

$$p^\mu F_\mu = 0.$$

Furthermore, it is assumed that the particles having momenta in the range $\Delta^3 p$ at the beginning of the proper time interval $\Delta\tau$ occupy an equal momentum range at the end of the interval. This would true if the force has the property

$$\frac{\partial}{\partial p^\mu} F^\mu(x, p) = 0.$$

These properties are met by the Lorentz force. For such an external force the covariant kinetic equation may be modified to the form

$$p_a^\nu \partial_\nu f_a(x, p_a) + m F_a^\mu f_a(x, p_a) = \mathfrak{R}_a\left[f_a\right],$$

where F_a^μ is external force on particle a.

is given by the Wang Chang–Uhlenbeck collision term [2] with the quantum effects put in as in the Boltzmann–Nordholm–Uehling–Uhlenbeck (BNUU) equation [3, 4]

$$\mathfrak{R}_{ab}\left[f_a f_b\right] = \left[C(f_a f_b) + \sum_{k=i}^{r}\sum_{l=i}^{r} C_r\left(f_a f_b | f_k^* f_l^*\right)\right]. \tag{2.15}$$

Here the subscript b, k, l runs over all species including matter particles and photons; that is, $b = 1, 2, \ldots, m, r$, where m is the number of material species. The index r is reserved for photons. We refer the reader to Ref. [5] where a nonrelativistic theory of irreversible processes is discussed, using the BNUU equation for a dilute quantum gas mixture, since the present covariant kinetic equation is a relativistic generalization of the BNUU equation. The Boltzmann collision integrals—more precisely, relativistic BNUU collision integrals—$C(f_a f_b)$ and $C_r\left(f_a f_b | f_k^* f_l^*\right)$ are given by the expressions for the matter part

$$\begin{aligned}
C(f_a f_b) = G_a &\int d^3\overline{p}_a^* \int d^3\overline{p}_b^* \int d^3\overline{p}_b W^{(e)}\left(p_a p_b | p_a^* p_b^*\right) \\
&\times \left[f_a^* f_b^* \left(1 + \epsilon_a f_a\right)\left(1 + \epsilon_b f_b\right) - f_a f_b \left(1 + \epsilon_a f_a^*\right)\left(1 + \epsilon_b f_b^*\right)\right] \\
&+ \sum_{k=a}^{r}\sum_{l=a}^{r} G_a \int d^3\overline{p}_k^* \int d^3\overline{p}_l^* \int d^3\overline{p}_b W^{(in)}\left(p_a p_b | p_k^* p_l^*\right) \\
&\times \left[f_k^* f_l^* \left(1 + \epsilon_a f_a\right)\left(1 + \epsilon_b f_b\right) - f_a f_b \left(1 + \epsilon_k f_k^*\right)\left(1 + \epsilon_l f_l^*\right)\right]
\end{aligned} \tag{2.16}$$

and for the radiation part

$$\begin{aligned}
C_r\left(f_a f_b | f_k^* f_l^*\right) = G_r &\int d^3\overline{p}_k^* \int d^3\overline{p}_l^* \int d^3\overline{p}_b W^{(r)}\left(p_a p_b | p_k^* p_l^*\right) \times \\
&\times \left[f_k^* f_l^* \left(1 + \epsilon_a f_a\right)\left(1 + \epsilon_b f_b\right) - f_a f_b \left(1 + \epsilon_k f_k^*\right)\left(1 + \epsilon_l f_l^*\right)\right] \\
&+ \sum_{k=i}^{r}\sum_{l=i}^{r} G_a \int d^3\overline{p}_k^* \int d^3\overline{p}_l^* \int d^3\overline{p}_b W^{(in)}\left(p_a p_b | p_k^* p_l^*\right) \\
&\times \left[f_k^* f_l^* \left(1 + \epsilon_a f_a\right)\left(1 + \epsilon_b f_b\right) - f_a f_b \left(1 + \epsilon_k f_k^*\right)\left(1 + \epsilon_l f_l^*\right)\right].
\end{aligned} \tag{2.17}$$

Here $d^3\overline{p}_a^* = d^3\mathbf{p}_a / p_a^0$, etc. and

$$\epsilon_a, \epsilon_k = \begin{cases} +1 & \text{for fermions} \\ -1 & \text{for bosons} \\ \;\;0 & \text{Boltzmann particles} \end{cases} .$$

The symbols G_a and G_r stand for the statistical weights for the material particles and photons. The $W^{(e)}$, $W^{(in)}$, and $W^{(r)}$ denote the transition probabilities for elastic, inelastic and "reactive" collisions, respectively. Since the photons do not interact with each other the transition probabilities for photon-photon interactions are equal to zero: $W^{(e)}\left(p_r p_r | p_r^* p_r^*\right) = 0$, $W^{(in)}\left(p_r p_r | p_r^* p_r^*\right) = 0$, and $W^{(r)}\left(p_r p_r | p_r^* p_r^*\right) = 0$. These transition probabilities can be calculated from quantum mechanics for a given dynamic mechanism. We assume that they are known through few-body mechanics. In kinetic theory, such few-body dynamical information is assumed known. Writing the kinetic equations for material particles and photons as in (2.13), we have assumed both the material gas and photon gas act as a dilute gas mixture.

We close this section by adding that the transition probabilities have the following symmetry properties

$$W^{(s)}\left(p_a p_b | p_k^* p_l^*\right) = W^{(s)}\left(p_k^* p_l^* | p_a p_b\right), \text{etc.} \quad (s = e, in, r)\,. \tag{2.18}$$

These relations originate from the microscopic reversibility of microscopic dynamic events and are crucially important for proving the H theorem.

2.2 Boltzmann Entropy and the *H* Theorem

The covariant kinetic equation postulated here satisfies the H theorem, which enables us to determine uniquely the equilibrium solution of the kinetic equation as in the case of the covariant kinetic equation discussed in Chap. 1 of this volume. The H theorem also guides us to construct a thermodynamic theory of irreversible processes occurring in the system as will be shown. To state the H theorem explicitly we first must introduce the Boltzmann entropy four-flow $S^\mu(x)$ for the system of radiation and matter by the formula

$$S^\mu(x) = -k_B \sum_{a=1}^{r} \left\langle p_a^\mu \left[f_a \ln f_a - \epsilon_a \left(1 + \epsilon_a f_a\right) \ln\left(1 + \epsilon_a f_a\right)\right]\right\rangle, \tag{2.19}$$

where $f_a = f_a(x, p_a)$. The angular brackets $\langle \cdots \rangle$ abbreviate the integration over momentum space:

$$\langle A_a(x, p_a)\rangle = G_a c \int d^3\overline{p}_a A_a(x, p_a) \quad (a = 1, \cdots, r) \tag{2.20}$$

with the abbreviation

$$d^3\overline{p}_a = d^3 p_a / p_a^0.$$

Recall that the subscript a is for either matter or photons, but the subscript r stands for radiation (photon). By differentiating $S^\mu(x)$ and using the covariant kinetic equation, we obtain the balance equation for the Boltzmann entropy four-flow

$$\partial_\mu S^\mu = \sigma_{\text{ent}}(x), \tag{2.21}$$

where the Boltzmann entropy production $\sigma_{\text{ent}}(x)$ is given by the statistical mechanical formula

$$\sigma_{\text{ent}}(x) = -k_B \sum_{a=1}^{r} \langle \{\ln f_a(x, p_a) - \ln [1 + \epsilon_a f_a(x, p_a)]\} \mathfrak{R}_a [f_a] \rangle . \tag{2.22}$$

It is easy to show that $\sigma_{\text{ent}}(x)$ is positive semidefinite and equal to zero, only if the system is at equilibrium:

$$\begin{aligned}\sigma_{\text{ent}}(x) = \frac{1}{4} k_B \sum_{a,b=1}^{r} \Big\langle \ln \Big\{ & f_a^* f_b^* (1 + \epsilon_a f_a)(1 + \epsilon_b f_b) \\ & / \ f_a f_b \left(1 + \epsilon_a f_a^*\right)\left(1 + \epsilon_b f_b^*\right) \Big\} \mathfrak{R}_{ab} [f_a f_b] \Big\rangle \\ \geq 0. & \end{aligned} \tag{2.23}$$

This is the local form of the H theorem for the covariant kinetic equation postulated, (2.13). The equality holds at equilibrium reached in long time. As in the case of a nonrelativistic gas and a relativistic single-component gas discussed in the previous chapter of this volume, the H theorem may be regarded as a stability theorem of Lyapounov [11], which concerns the stability of the equilibrium solution and the state of equilibrium.

The scalar density $\widehat{\mathcal{S}}$ of Boltzmann entropy four-flow S^μ is defined by contracting S^μ with the hydrodynamic velocity U^μ:

$$\mathcal{S}(x) := \rho \widehat{\mathcal{S}} = c^{-2} U_\mu S^\mu (x) . \tag{2.24}$$

Here again we take Eckart's definition of hydrodynamic velocity; see (1.27), Chap. 1 of this Volume. The balance equation of $\widehat{\mathcal{S}}$ is then obtained by operating the convective time derivative operator D on this expression and using the same procedure as used for the balance equation for $\widehat{\mathcal{S}}$ in (1.149)–(1.153), Chap. 1, Vol. 2:

$$\rho D \widehat{\mathcal{S}} = -\partial_\mu J_{\text{s}}^\mu + \sigma_{\text{ent}}(x), \tag{2.25}$$

where Boltzmann entropy flux J_{s}^μ is given by

$$J_{\text{s}}^\mu = \Delta_\nu^\mu S^\nu \tag{2.26}$$

with Δ_ν^μ denoting a projector.[4]

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[4] Here we would like to emphasize that the hydrodynamic velocity is defined according to Eckart, but for the mixture of matter and photons:

Owing to the H theorem, the equilibrium distribution functions satisfying the covariant kinetic equation can be uniquely determined with the help of the Clausius entropy (i.e., the entropy of reversible processes) and the second law of thermodynamics. To implement this step we need to develop the theory a little further. We will presently return to this task to construct the equilibrium solutions more explicitly later.

2.2.1 Equilibrium Solution of the Covariant Kinetic Equation

One of the most important results provided by the H theorem is that the equilibrium solution of the covariant kinetic equation is uniquely determined by the vanishing Boltzmann entropy production:

$$\begin{aligned}\sigma_{\mathrm{ent}}(x)\left.\right|_{\mathrm{eqil}} &= \frac{k_B}{4}\sum_{a,b=1}^{r}\left\langle \ln\left[\frac{\left(\epsilon_a + f_a^{\mathrm{e}-1}\right)\left(\epsilon_b + f_b^{\mathrm{e}-1}\right)}{\left(\epsilon_a + f_a^{\mathrm{e}*-1}\right)\left(\epsilon_b + f_b^{\mathrm{e}*-1}\right)}\right]\mathfrak{R}_{ab}\left[f_a^{\mathrm{e}}, f_b^{\mathrm{e}}\right]\right\rangle \\ &= 0, \end{aligned} \qquad (2.27)$$

which implies the logarithmic factor in it is a collision invariant. The superscript e denotes equilibrium. That is, $\ln\left(\epsilon_a + f_a^{\mathrm{e}-1}\right)$ must be a summation invariant:

$$\ln\left(\epsilon_a + f_a^{\mathrm{e}-1}\right) + \ln\left(\epsilon_b + f_b^{\mathrm{e}-1}\right) = \ln\left(\epsilon_a + f_a^{\mathrm{e}*-1}\right) + \ln\left(\epsilon_b + f_b^{\mathrm{e}*-1}\right). \qquad (2.28)$$

This in turn means that the logarithmic functions $\ln\left(\epsilon_a + f_a^{\mathrm{e}-1}\right)$ must be also a linear combination of basic collision invariants. Therefore, the equilibrium distribution function may be written in the form[5]

$$\ln\left(\epsilon_a + f_a^{\mathrm{e}-1}\right) = \beta_{\mathrm{e}}\left(p_a^{\nu}U_{\nu} - \mu_a^{\mathrm{e}}\right). \qquad (2.29)$$

Rearranging this expression we find the equilibrium distribution function in the form

$$f_a^{\mathrm{e}} = \frac{1}{e^{\beta_{\mathrm{e}}\left(p_a^{\nu}U_{\nu} - \mu_a^{\mathrm{e}}\right)} - \epsilon_a} \quad (a = 1, 2, \ldots, m, r). \qquad (2.30)$$

(Footnote 4 continued)

$$U^{\mu} = \frac{cN^{\mu}}{\sqrt{N^{\nu}N_{\nu}}},$$

where N^{μ} is the sum of N_i^{μ} with the index i running for entire species including photons according to our model.

[5] This form is without a constant factor that has to do with the normalization of f_a^{e}. See Chap. 1 where the case for classical particles is described in connection with equilibrium distribution functions.

The equilibrium distribution function f_a^{e} contains as-yet undetermined parameters β_{e} and μ_a^{e} which must be determined upon corresponding the statistical mechanical formula for the equilibrium Boltzmann entropy to their phenomenological counterpart, namely, the Clausius entropy. This can be achieved if the equilibrium Boltzmann entropy density $\widehat{\mathcal{S}}^{\mathrm{e}}$, which is identical with the Clausius entropy for reversible processes in the equilibrium system, is evaluated with the equilibrium solution f_a^{e}. Since $\mathcal{S}^{\mathrm{e}} = c^{-2} U_\mu S^{\mathrm{e}\mu}(x)$ is given by the statistical mechanical formula

$$\mathcal{S}^{\mathrm{e}} = k_B \sum_{a=1}^{r} c^{-2} U_\mu \left\langle p_a^\mu \left[f_a^{\mathrm{e}} \ln\left(\epsilon_a + f_a^{\mathrm{e}-1}\right) + \epsilon_a \ln\left(1 + \epsilon_a f_a^{\mathrm{e}}\right)\right]\right\rangle, \tag{2.31}$$

inserting f_a^{e} we obtain

$$\mathcal{S}^{\mathrm{e}} = k_B \sum_{a=1}^{r} c^{-2} U_\mu \left\langle p_a^\mu \left[f_a^{\mathrm{e}} \beta_{\mathrm{e}} \left(p_a^\nu U_\nu - \mu_a^{\mathrm{e}}\right) + \epsilon_a \ln\left(1 + \epsilon_a f_a^{\mathrm{e}}\right)\right]\right\rangle. \tag{2.32}$$

This expression upon use of the statistical mechanical formulas for the internal energy—energy-momentum tensor—and density yields

$$\mathcal{S}^{\mathrm{e}} = \beta_{\mathrm{e}} k_B \rho \mathcal{E}^{\mathrm{e}} - \beta_{\mathrm{e}} k_B \rho \sum_{a=1}^{r} \mu_a^{\mathrm{e}} \mathfrak{c}_a + k_B \sum_{a=1}^{r} \epsilon_a \left\langle c^{-2} U_\mu p_a^\mu \ln\left(1 + \epsilon_a f_a^{\mathrm{e}}\right)\right\rangle. \tag{2.33}$$

Here the internal energy density $\rho\mathcal{E}$ is calculated from the projection of the energy-momentum tensor

$$\rho \mathcal{E}^{\mathrm{e}} = \sum_{a=1}^{r} c^{-2} U_\mu T^{\mathrm{e}\mu\nu} U_\nu = \sum_{a=1}^{r} c^{-2} U_\mu \left\langle f_a^{\mathrm{e}} p_a^\mu p_a^\nu \right\rangle U_\nu \tag{2.34}$$

and the density ρ_a of species a with the statistical mechanical formula

$$\rho = \sum_{a=1}^{r} \rho_a = c^{-2} \sum_{a=1}^{r} U_\mu N_a^\mu = c^{-2} \sum_{a=1}^{r} U_\mu \left\langle f_a^{\mathrm{e}} p_a^\mu \right\rangle. \tag{2.35}$$

The ratio of densities ρ_a and ρ provides the statistical mechanical formula for the density fraction $\mathfrak{c}_a = \rho_a/\rho$—concentration of a. The last term on the right of (2.33) is identified[6] with the statistical mechanical expression for relativistic hydrostatic pressure

$$p^{\mathrm{e}} v_a = \beta_{\mathrm{e}}^{-1} k_B \epsilon_a \left\langle c^{-2} U_\mu p_a^\mu \ln\left(1 + \epsilon_a f_a^{\mathrm{e}}\right)\right\rangle, \tag{2.36}$$

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[6] It should be noted that $\mathfrak{p}^{\mathrm{e}} v_a$ on the left of (2.36) defines macroscopic parameter $\mathfrak{p}^{\mathrm{e}}$ by the statistical mechanical formula on the right, which will turn out to be hydrostatic (i.e., equilibrium) pressure, when $\mathcal{S}^{\mathrm{e}}$ calculated therewith is identified with the equilibrium entropy on correspondence with the phenomenological Clausius entropy deduced from the second law of thermodynamics.

where $v_a = 1/\rho_a$ is the specific volume. This formula is the equivalent of the nonrelativistic formula for hydrostatic pressure in the grand canonical theory to which it reduces since $c^{-2}U_\mu p_a^\mu \to m_a$ in the limit of $u/c \to 0$. Formula (2.36) is the necessary and sufficient condition for $\mathcal{S}^{\mathrm{e}}$ to become a bilinear form in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. In this connection it should be noted that if equilibrium grand partition function Ξ^{e} were constructed for the mixture of matter and photons such that the equation of state is given by the formula

$$p^{\mathrm{e}}V = \beta_{\mathrm{e}}^{-1} \ln \Xi^{\mathrm{e}}, \tag{2.37}$$

then there would be the correspondence

$$\Xi^{\mathrm{e}} \Leftrightarrow k_B \epsilon_a \int_V d\mathbf{r} \left\langle c^{-2}U_\mu p_a^\mu \ln\left(1 + \epsilon_a f_a^{\mathrm{e}}\right)\right\rangle. \tag{2.38}$$

In summary, the equilibrium Boltzmann entropy density is given by the formula

$$\mathcal{S}^{\mathrm{e}} = \beta_{\mathrm{e}} k_B \rho \mathcal{E}^{\mathrm{e}} - \beta_{\mathrm{e}} k_B \rho \sum_{a=1}^{r} \mu_a^{\mathrm{e}} \mathfrak{c}_a + \beta_{\mathrm{e}} k_B \rho p^{\mathrm{e}} v. \tag{2.39}$$

On the other hand, if we calculate the equilibrium calortropy density differential from the balance equation (2.25), the following equation is obtained at equilibrium

$$\begin{aligned} \rho D\widehat{\mathcal{S}}^{\mathrm{e}} &= k_B \sum_{a=1}^{r} \partial_\mu \beta_{\mathrm{e}} \Delta_\nu^\mu \left\langle f_a^{\mathrm{e}} p_a^\mu \left(p_a^\nu U_\nu - \mu_a^{\mathrm{e}}\right)\right\rangle \\ &\quad - k_B \sum_{a=1}^{r} \left\langle \left(1 + \epsilon_a f_a^{\mathrm{e}}\right)^{-1} \Delta_\nu^\mu p_a^\mu \partial_\mu f_a^{\mathrm{e}} \right\rangle \\ &= \beta_{\mathrm{e}} k_B \sum_{a=1}^{r} \left(-\partial_\mu Q_{\mathrm{e}}^\mu - \rho \mu_a^{\mathrm{e}} D\mathfrak{c}_a\right) \end{aligned} \tag{2.40}$$

for the following reasons: Firstly, at equilibrium

$$p_a^\mu \partial_\mu f_a^{\mathrm{e}} = 0; \tag{2.41}$$

secondly, Q_{e}^μ is the heat four-flow defined by projection of the energy-momentum tensor $T_{\mathrm{e}}^{\mu\nu}$

$$Q_{\mathrm{e}}^\mu = -\Delta_\nu^\mu T_{\mathrm{e}}^{\mu\nu} U_\nu = -\Delta_\nu^\mu \left\langle f_a^{\mathrm{e}} p_a^\mu p_a^\nu \right\rangle U_\nu; \tag{2.42}$$

and thirdly, the following identification is possible:

$$\Delta_\nu^\mu N_a^\nu = \Delta_\nu^\mu \left\langle f_a^{\mathrm{e}} p_a^\mu \right\rangle,$$

which may be rearranged to the form

$$\begin{aligned}\Delta^{\mu}_{\nu} N^{\nu}_{a} &= \left(g^{\mu}_{\nu} - c^{-2} U^{\mu} U_{\nu}\right) N^{\nu}_{a} = N^{\mu}_{a} - c^{-2} U^{\mu} U_{\nu} N^{\nu}_{a} \\ &= N^{\mu}_{a} - \rho_a U^{\mu}. \end{aligned} \tag{2.43}$$

The symbol $\mathfrak{c}_a$ is the number fraction defined earlier: $\mathfrak{c}_a = \rho_a/\rho$. Therefore, $\Delta^{\mu}_{\nu} N^{\nu}_{a}$ may be regarded as the flux J^{μ}_{a} of N^{μ}_{a} relative to $\rho_a U^{\mu}$ in the number fraction balance equation

$$\rho D \mathfrak{c}_a = -\partial_{\mu} J^{\mu}_{a} = -\partial_{\mu} \Delta^{\mu}_{\nu} N^{\nu}_{a}. \tag{2.44}$$

Using the energy balance equation for an equilibrium process, we then obtain

$$\begin{aligned}\rho D \mathcal{E}^{\mathrm{e}} &= -\partial_{\mu} Q^{\mu}_{\mathrm{e}} - p^{\mathrm{e}} \nabla_{\mu} U^{\mu} = -\partial_{\mu} Q^{\mu}_{\mathrm{e}} + p^{\mathrm{e}} \rho^{-1} D\rho \\ &= -\partial_{\mu} Q^{\mu}_{\mathrm{e}} - \rho p^{\mathrm{e}} D v, \end{aligned} \tag{2.45}$$

where $v = 1/\rho$. Therefore, we finally obtain from (2.40) the equilibrium Boltzmann entropy differential

$$D\widehat{\mathcal{S}}^{\mathrm{e}} = \beta_{\mathrm{e}} k_B \left(D\mathcal{E} + p^{\mathrm{e}} D v - \sum_{a \neq r} \mu^{\mathrm{e}}_{a} D \mathfrak{c}_a \right). \tag{2.46}$$

Here the photon is excluded from the sum over species because for photons $\mu^{\mathrm{e}}_{r} = 0$ as will be shown shortly when we discuss the Boltzmann entropy of photons. This differential form in comparison with (2.39) implies that the integrability condition of (2.46) is

$$\mathcal{E} D \left(\beta_{\mathrm{e}} k_B\right) - \sum_{a=1}^{r} \mathfrak{c}_a D \left(\beta_{\mathrm{e}} k_B \mu^{\mathrm{e}}_{a}\right) + v D \left(\beta_{\mathrm{e}} k_B \mathfrak{p}^{\mathrm{e}}\right) = 0, \tag{2.47}$$

which is the equilibrium Gibbs–Duhem equation, provided

$$\beta_{\mathrm{e}} = \frac{1}{k_B T^{\mathrm{e}}}. \tag{2.48}$$

That is, making correspondence between the statistical mechanical expression (2.39) with the corresponding thermodynamic (Clausius) entropy for a reversible process within a constant, what is the same thing, comparing (2.47) with the thermodynamic (phenomenological) equilibrium Gibbs–Duhem equation

$$\mathcal{E} D T^{\mathrm{e}-1} - \sum_{a=1}^{r} \mathfrak{c}_a D \left(\frac{\mu^{\mathrm{e}}_{a}}{T^{\mathrm{e}}}\right) + v D \left(\frac{p^{\mathrm{e}}}{T^{\mathrm{e}}}\right) = 0, \tag{2.49}$$

we find the parameter β_{e} as in (2.48) in terms of the absolute temperature T^{e} of the equilibrium system and the parameter μ^{e}_{a} as the equilibrium chemical potential of species a. This way, the parameters β_{e} (or T^{e}), p^{e}, and μ_a^{e} are fully identified with their thermodynamic counterparts in (statistical mechanical) Boltzmann entropy $\widehat{\mathcal{S}}^{\mathrm{e}}$ and the equilibrium distribution function f_a^{e}. This of course means that the Boltzmann entropy at equilibrium is identical with the Clausius entropy of reversible processes.

Now we would like to show the equilibrium distribution function of radiation is given by the expression

$$f_r^{\mathrm{e}} = \left(e^{\beta_{\mathrm{e}} p_r^\nu U_\nu} - 1\right)^{-1}, \tag{2.50}$$

if the photon number is variable [1]. Since f_r^{e} is a Lorentz scalar, which can be demonstrated by the same method as shown in Chap. 1 of volume 2, it is permissible to use the local rest frame formula for U_ν:

$$U_\nu = (c, 0, 0, 0)\,. \tag{2.51}$$

Then, if the wave vector of radiation is denoted $\mathbf{k}_r = k_r \widehat{\mathbf{k}}_r$ where $k_r = |\mathbf{k}_r| = \omega/c$ ($\omega =$ frequency) and $\widehat{\mathbf{k}}_r$ is the unit vector in the direction of propagation, the covariant form of the equilibrium photon distribution function is given by a familiar looking expression

$$f_r^{\mathrm{e}} = \left(e^{\beta_{\mathrm{e}} \hbar\omega - \mu_r^{\mathrm{e}}} - 1\right)^{-1}. \tag{2.52}$$

We now show that $\mu_r^{\mathrm{e}} = 0$ if the photon number is variable. The equilibrium number density is given by the formula

$$\begin{aligned} \rho_r &= N_r/V = \left\langle f_r^{\mathrm{e}}(\omega) \right\rangle \\ &= \pi^{-2} c^{-3} \int_0^\infty d\omega \omega^2 \left\{ \exp\left[\beta_{\mathrm{e}} \left(\hbar\omega - \mu_r^{\mathrm{e}}\right)\right] - 1 \right\}^{-1}. \end{aligned} \tag{2.53}$$

Since the local grand canonical partition function for photons Γ_r^{e} is given by

$$\Gamma_r^{\mathrm{e}} = -\left(V/\pi^2 c^3 \beta_{\mathrm{e}}\right) \int_0^\infty d\omega \omega^2 \ln\left\{1 - \exp\left[\beta_{\mathrm{e}}\left(\mu_r^{\mathrm{e}} - \hbar\omega\right)\right]\right\}, \tag{2.54}$$

we find

$$\rho_r = -\left(\frac{\partial \ln \Gamma_r^{\mathrm{e}}}{\partial \mu}\right)_{T^{\mathrm{e}}, V}, \tag{2.55}$$

where μ is the chemical potential. If we set $\mu = \mu_r^{\mathrm{e}}$ and differentiate (2.54) with μ_r^{e}, there follows (2.53) exactly. Thus we have identified μ_r^{e} with the photon chemical potential. Now since

$$\mu_r^{\mathrm{e}} = -T^{\mathrm{e}} \left(\frac{\partial \mathcal{S}_{\mathrm{r}}^{\mathrm{e}}}{\partial \rho_r}\right)_{T^{\mathrm{e}},, V} = \left(\frac{\partial \mathcal{A}_{\mathrm{r}}^{\mathrm{e}}}{\partial \rho_r}\right)_{T^{\mathrm{e}}, V}, \tag{2.56}$$

where $\mathcal{S}_{\mathrm{r}}^{\mathrm{e}}$ and $\mathcal{A}_{\mathrm{r}}^{\mathrm{e}}$ are, respectively, the equilibrium Boltzmann entropy density and work function for photon. If $\mathcal{A}_{\mathrm{r}}^{\mathrm{e}}$ is required to be at a minimum at equilibrium as it is varied with respect to ρ_r, it follows that

$$\left(\frac{\partial \mathcal{A}_{\mathrm{r}}^{\mathrm{e}}}{\partial \rho_r}\right)_{T^{\mathrm{e}},V} = 0$$

and hence the equilibrium radiation has a vanishing chemical potential:

$$\mu_r^{\mathrm{e}} = 0. \tag{2.57}$$

Therefore the equilibrium radiation distribution function is given by

$$f_r^{\mathrm{e}} = \left(e^{\beta_{\mathrm{e}}\hbar\omega} - 1\right)^{-1}. \tag{2.58}$$

The meaning of β_{e} is the same as (2.48) because according to the Stefan–Boltzmann law [12] the radiation energy density is given by

$$\mathcal{E}_r^{\mathrm{e}} = 4\frac{a_{\mathrm{SB}} T^{\mathrm{e}4}}{c}, \tag{2.59}$$

where a_{SB} is the Stefan–Boltzmann constant. If $\mathcal{E}_r$ is calculated by using f_r^{e} in (2.58) and equated with the phenomenological $\mathcal{E}_r$ in (2.59) we conclude

$$(\beta_{\mathrm{e}} T^{\mathrm{e}})^{-4} = \frac{60\hbar c^3}{\pi^2} a_{\mathrm{SB}}.$$

The left hand side, $(\beta_{\mathrm{e}} T^{\mathrm{e}})^{-4}$, must be a universal constant since the Stefan–Boltzmann constant is universal as verified by experiments, i.e., the Kirchhoff law [9, 13]. The universal constant turns out to be the Boltzmann constant raised to the fourth power. Hence we find that the parameter β_{e} in f_r^{e} is given by the same formula as (2.48). We therefore conclude that both radiation and matter distribution functions share a single temperature parameter characterized by the black body radiation in thermal equilibrium with matter. Furthermore, they are mutually consistent, and their mutual consistency is demanded by the H theorem.

It is interesting to note that Einstein derived (2.52) from the equilibrium condition between radiation and material particles in his famous paper of 1917 [14]. The present approach elucidates the kinetic theory and dynamic bases which are absent in his theory. In the recent past, Boyer [6] claimed that the Boltzmann distribution functions for relativistic material particles and the Planck distribution function are not consistent within the framework of quantum theory, and this has been a point of controversy [7, 8]. The derivation of (2.30) and (2.52) presented earlier shows that they are completely consistent with each other, provided that the irreversible covariant kinetic equation (2.13) satisfying the H theorem is postulated. Moreover, it shows that the Planck distribution law is deeply rooted in the H theorem and thus the second law of thermodynamics for radiation in equilibrium with matter. It also points out that it is not possible to think of a radiation distribution function without taking into account the corresponding material particle distribution function and the interaction between radiation and matter. It therefore may be said that this is the first practical and useful result of the present covariant kinetic theory formulation of the theory of radiation and matter.

2.2.2 Equilibrium Energy-Momentum Tensors of Radiation and Matter

We define the energy-momentum tensor of matter or radiation in equilibrium by the statistical mechanical formula

$$T_{\mathrm{e}i}^{\mu\nu} = \left\langle p_i^{\mu} p_i^{\nu} f_i^{\mathrm{e}}(p_i) \right\rangle, \quad (i = a, r). \tag{2.60}$$

With the help of the projector defined by

$$\Delta^{\mu\nu} = g^{\mu\nu} - c^{-2} U^{\mu} U^{\nu} \tag{2.61}$$

the equilibrium energy-momentum tensor can be decomposed into component as follows:

$$T_{\mathrm{e}i}^{\mu\nu} = \mathfrak{p}_i^{\mathrm{e}} \Delta^{\mu\nu} + c^{-2} \rho_i \mathcal{E}_i^{\mathrm{e}} U^{\mu} U^{\nu}, \tag{2.62}$$

where the energy density $\mathcal{E}_i^{\mathrm{e}}$ is defined by

$$\rho_i \mathcal{E}_i^{\mathrm{e}} = c^{-2} U_{\mu} T_{\mathrm{e}i}^{\mu\nu} U_{\nu} = c^{-2} \left\langle \left(p_i^{\mu} U_{\mu}\right) \left(p_i^{\nu} U_{\nu}\right) f_i^{\mathrm{e}}(p_i) \right\rangle \tag{2.63}$$

and the hydrostatic pressure $\mathfrak{p}_i^{\mathrm{e}}$ is given by

$$\mathfrak{p}_i^{\mathrm{e}} = \frac{1}{3} \Delta_{\mu\nu} T_{\mathrm{e}i}^{\mu\nu} = \frac{1}{3} \Delta_{\mu\nu} \left\langle p_i^{\mu} p_i^{\nu} f_i^{\mathrm{e}}(p_i) \right\rangle. \tag{2.64}$$

Therefore, the total material internal energy and total material pressure are given, respectively, by the formulas

$$\rho_{\mathrm{m}} \mathcal{E}_{\mathrm{m}}^{\mathrm{e}} = \sum_{a \neq r} \rho_a \mathcal{E}_a^{\mathrm{e}}, \quad \mathfrak{p}_{\mathrm{m}}^{\mathrm{e}} = \sum_{a \neq r} \mathfrak{p}_a^{\mathrm{e}} = \rho_{\mathrm{e}} k_B T^{\mathrm{e}}. \tag{2.65}$$

Note that the summation sign here excludes photons. The last equality is easy to show in the local rest frame [15]. The second equation of (2.65) is a well known result for gases consisting of structureless particles. It also holds for dilute gases consisting of particles with an internal structure. By using the definition of the projector $\Delta_{\mu\nu}$ and the identity

$$p_r^{\mu} p_{r\mu} = 0$$

for photons, we can show from (2.63) and (2.64)

$$\mathfrak{p}_r^{\mathrm{e}} = \frac{1}{3} \rho_r \mathcal{E}_r^{\mathrm{e}}. \tag{2.66}$$

In fact, this relation is indeed verifiable by using the local rest frame. By using the energy-momentum tensor for radiation, we can also derive the Stefan–Boltzmann law [12],

$$\rho_r \mathcal{E}_r^{\mathrm{e}} = a_{SB} T^{\mathrm{e}4}, \tag{2.67}$$

where a_{SB} is the Stefan–Boltzmann constant:

$$a_{\mathrm{SB}} = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3}. \tag{2.68}$$

In connection with this result, we recall that the parameter β_{e} in the radiation distribution function is determined such that the equilibrium radiation energy is given by (2.67). The β_{e} so determined exactly coincides with the one given in (2.48). This coincidence is a result of the demand made by the H theorem that the material gas and radiation be in equilibrium. Other thermodynamic quantities for radiation and matter can be calculated with the equilibrium distribution functions determined here.

2.3 Number and Energy-Momentum Tensor Balance Equations

The covariant kinetic equation postulated for matter and radiation gives rise to the particle number and energy-momentum tensor balance equations as does the relativistic kinetic equation considered in the previous chapter of this volume. They will be derived for matter and radiation in the following, but if derived separately for matter and radiation treating them as different species, they give rise to additional source terms because of interactions between molecules and photons, which may be regarded as "chemical reactions" in the present model.

Let us define the number four-flow of species i by the statistical representation

$$N_i^{\mu} = \left\langle p_i^{\mu} f_i(x, p_i) \right\rangle \tag{2.69}$$

and the energy-momentum tensor of species i by

$$T_i^{\mu\nu} = \left\langle p_i^{\mu} p_i^{\nu} f_i(x, p_i) \right\rangle, \tag{2.70}$$

where index i runs over matter species $(1, \ldots, m)$ and similar expressions for radiation specifically designated by index r. Therefore the total energy-momentum tensor of matter as a whole is given by

$$T_{\mathrm{m}}^{\mu\nu} = \sum_{i=1}^{m} T_i^{\mu\nu} = \sum_{i=1}^{m} \left\langle p_i^{\mu} p_i^{\nu} f_i(x, p_i) \right\rangle, \tag{2.71}$$

and the energy-momentum tensor for radiation by

$$T_r^{\mu\nu} = \left\langle p_r^{\mu} p_r^{\nu} f_r(x, p_r) \right\rangle. \tag{2.72}$$

The number four-flow for the entire matter species is then given by

$$N_{\mathrm{m}}^{\mu} = \sum_{i=1}^{m} N_i^{\mu} = \sum_{i=1}^{m} \left\langle p_i^{\mu} f_i(x, p_i) \right\rangle \tag{2.73}$$

and for radiation by the statistical mechanical formula

$$N_r^{\mu} = \left\langle p_r^{\mu} f_r(x, p_r) \right\rangle. \tag{2.74}$$

By using the kinetic equation, the balance equation for N_a^{μ} for matter species a can be shown given by the equation

$$\partial_{\mu} N_a^{\mu} = \Lambda_a^{(n)}, \tag{2.75}$$

where

$$\Lambda_a^{(n)} = \langle \mathfrak{R}_a [f_a] \rangle . \tag{2.76}$$

This source term $\Lambda_a^{(n)}$ is not equal to zero because of the interactions of matter particles with photons and making transition to other matter species as assumed in the present model. However, when summed over all matter species, that is, for N_{m}^{μ}

$$\partial_{\mu} N_{\mathrm{m}}^{\mu} = 0, \tag{2.77}$$

because the overall matter is conserved and hence

$$\sum_{i=1}^{m} \Lambda_i^{(n)} = 0. \tag{2.78}$$

Similarly to the balance equation for N_a^{μ}. Hence the energy-momentum balance equation acquires a source term:

$$\partial_{\nu} T_{\mathrm{m}}^{\mu\nu} = \Lambda_{\mathrm{m}}^{\mu}, \tag{2.79}$$

where the source term is given by

$$\Lambda_{\mathrm{m}}^{\mu} = \sum_{i=1}^{m} \left\langle p_i^{\mu} \mathfrak{R}_i \left[f_i \right] \right\rangle . \tag{2.80}$$

Therefore, the energy-momentum tensor for matter alone is not a conserved quantity, because it is not a collision invariant. For radiation we obtain the number balance equation

$$\partial_{\mu} N_r^{\mu} = \Lambda_r^{(n)}, \tag{2.81}$$

and the energy momentum balance equation

$$\partial_{\nu} T_r^{\mu\nu} = \Lambda_r^{\mu} \tag{2.82}$$

with the source terms

$$\Lambda_r^{(n)} = \langle \mathfrak{R}_r \left[f_r \right] \rangle \tag{2.83}$$

and

$$\Lambda_r^{\mu} = \left\langle p_r^{\mu} \mathfrak{R}_r \left[f_r \right] \right\rangle , \tag{2.84}$$

respectively. For the system of matter and radiation as a whole the number four-flow is then given by the sum

$$N^{\mu} = N_{\mathrm{m}}^{\mu} + N_r^{\mu} . \tag{2.85}$$

With the total number four-flow N^{μ} we define hydrodynamic four-velocity U^{μ} by the Eckart convention

$$U^{\mu} = \frac{cN^{\mu}}{\sqrt{N^{\nu} N_{\nu}}} . \tag{2.86}$$

Hydrodynamic number density ρ_i of species i is then given by

$$\rho_i = c^{-2} U_{\mu} N_i^{\mu} \quad (i = 1, 2, \ldots, m, r) . \tag{2.87}$$

The total number density therefore is given by the sum

$$\rho = \rho_{\mathrm{m}} + \rho_r = \sum_{i=1}^{\{m,r\}} \rho_i = c^{-2} U_{\mu} N^{\mu} , \tag{2.88}$$

where the upper limit $\{m, r\}$ of the sum means that the sum is over all matter particle species $(1, \ldots, m)$ and radiation r.

The scalar energy density $\mathcal{E}_i$ of species i (either matter or radiation) is defined by a projection of energy-momentum tensor of the species:

$$E_i = \rho_i \mathcal{E}_i = c^{-2} U_{\mu} T_i^{\mu\nu} U_{\nu} . \tag{2.89}$$

Other projections of the energy-momentum tensor are the heat flux (four-flow) Q_i^μ and the stress tensor $P_i^{\mu\nu}$, respectively, defined by

$$Q_i^\mu = U_\nu T_i^{\nu\sigma} \Delta_\sigma^\mu, \tag{2.90}$$

$$P_i^{\mu\nu} = \Delta_\sigma^\mu T_i^{\sigma\tau} \Delta_\tau^\nu, \tag{2.91}$$

where Δ_σ^μ is a projection operator as defined in the previous chapter. Therefore the energy-momentum tensor of a species is decomposed into four components according to the projections given above:

$$\begin{aligned} T_i^{\mu\nu} &= c^{-2}\rho_i \mathcal{E}_i U^\mu U^\nu + c^{-2}\left(Q_i'^\mu U^\nu + U^\mu Q_i'^\nu\right) \\ &\quad + c^{-2}\left(\mathfrak{J}_i^\mu U^\nu + U^\mu \mathfrak{J}_i^\nu\right) + P_i^{\mu\nu}, \end{aligned} \tag{2.92}$$

where $Q_i'^\mu$ is the net heat flux above and beyond energy carried by matter

$$Q_i'^\mu = Q_i^\mu - \widehat{h} J_i^\mu \tag{2.93}$$

and $\mathfrak{J}_i^\mu$ is the diffusion of energy attributable to particle diffusion:

$$\mathfrak{J}_i^\mu = \widehat{h} J_i^\mu \tag{2.94}$$

with $\widehat{h}$ denoting the enthalpy density and J_i^μ the number diffusion four-flow defined by

$$J_i^\mu = N_i^\mu - \mathfrak{c}_a N^\mu. \tag{2.95}$$

The index i runs for all matter species and radiation since the decomposition holds both for matter and radiation. From the number and energy-momentum balance equations derived earlier follow various balance equations for conserved variables—number, momentum, energy—when various projections of the energy-momentum tensor balance equations and the number balance equations are taken. The procedure is described in the previous chapter of this volume. Since it basically remains the same, we will simply present the results only, but separately for matter and radiation because we are interested in hydrodynamics and transport processes of radiation and matter separately.

For the purpose in mind we recall that because $N^\mu \nabla_\mu = 0$ in Eckart's definition of hydrodynamic velocity the substantial time derivative $\mathfrak{D} = \rho D + N^\mu \nabla_\mu$ is simply given by

$$\mathfrak{D} = \rho D. \tag{2.96}$$

Therefore the substantial time derivative of macroscopic quantity (density) ψ may be written as

$$\rho D\psi = \partial_\mu \left(\psi N^\mu\right). \tag{2.97}$$

Using this identity various evolution equations for macroscopic properties follow straightforwardly with the help of their statistical mechanical definitions.

2.3.1 Equation of Continuity

From the total number density balance equation follows the equation of continuity for the system

$$D\rho = -\rho \nabla_\mu U^\mu. \tag{2.98}$$

It should be noted that this is for the totality of material particles and photons in the system. With the definition of specific volume $v = 1/\rho$, (2.98) is equivalently written

$$\rho D v = \nabla_\mu U^\mu. \tag{2.99}$$

2.3.2 Balance Equations for Conserved Variables of Matter

Various balance equations for conserved variables follow for matter species similarly to (2.98).

Density Fraction Balance Equation

The density fraction (concentration) balance equation follows from (2.75)

$$\rho D \mathfrak{c}_a = -\partial_\mu J_a^\mu + c^{-2} J_a^\mu D U_\mu + \Lambda_a^{(n)}, \tag{2.100}$$

where diffusion four-flow J_a^μ is defined by (2.95). The term $\Lambda_a^{(n)}$ appears because the number of particle species a is not conserved owing to its transition to another material species upon interaction with photons; see the collision processes (M1)–(M4) assumed for the model for collisions.

Momentum Balance Equation

Operating the projector Δ_ν^μ on the energy-momentum balance equation for matter we obtain the momentum balance equation

$$\begin{aligned} c^{-2}\mathcal{E}_{\mathrm{m}} \rho D U^\mu = & -\nabla^\mu p_{\mathrm{m}} - \Delta_\omega^\mu \nabla_\nu \overline{P}_{\mathrm{m}}^{\omega\nu} \\ & + c^{-2}\left(P_{\mathrm{m}}^{\mu\nu} D U_\nu - \Delta_\omega^\mu D Q_{\mathrm{m}}^\omega \right. \\ & \left. - Q_{\mathrm{m}}^\mu \nabla_\nu U^\nu - Q_{\mathrm{m}}^\nu \nabla_\nu U^\mu\right) + \Delta_\nu^\mu \Lambda_{\mathrm{m}}^{(n)\nu}, \end{aligned} \tag{2.101}$$

for which we have made use of the decomposition of energy-momentum tensor $T_{\mathrm{m}}^{\mu\nu}$:

$$T_{\mathrm{m}}^{\mu\nu} = c^{-2}\rho_{\mathrm{m}}\mathcal{E}_{\mathrm{m}} U^\mu U^\nu + c^{-2}\left(Q_{\mathrm{m}}^\mu U^\nu + U^\mu Q_{\mathrm{m}}^\nu\right) + P_{\mathrm{m}}^{\mu\nu}. \tag{2.102}$$

This decomposition of $T_i^{\mu\nu}$ follows from summing components (2.92) over matter species. Consequently, the total energy of matter (the total internal energy of material gas) is given by

$$\rho_{\mathrm{m}}\mathcal{E}_{\mathrm{m}} = \sum_{a\neq r}^{m} \rho_a \mathcal{E}_a \tag{2.103}$$

with

$$\rho_{\mathrm{m}} = \sum_{a=1}^{m} \rho_a; \tag{2.104}$$

the heat flux four-flow Q_{m}^{μ} by

$$Q_{\mathrm{m}}^{\mu} = U_\mu T_{\mathrm{m}}^{\sigma\tau} \Delta_\tau^{\mu}; \tag{2.105}$$

and the pressure tensor $P_{\mathrm{m}}^{\mu\nu}$ by

$$P_{\mathrm{m}}^{\mu\nu} = \sum_{a\neq r} P_a^{\mu\nu} = \Delta_\sigma^{\mu} T_{\mathrm{m}}^{\sigma\tau} \Delta_\tau^{\nu}. \tag{2.106}$$

In (2.101)

$$p_{\mathrm{m}} := \mathfrak{p}_{\mathrm{m}}^{\mathrm{e}} = \frac{1}{3}\Delta_{\mu\nu} T_{\mathrm{em}}^{\nu\mu}. \tag{2.107}$$

The pressure tensor $\overline{P}_{\mathrm{m}}^{\sigma\nu}$ is further decomposable into excess trace $(\widetilde{\Delta}_{\mathrm{m}})$ and traceless symmetric $(\Pi_{\mathrm{m}}^{\sigma\nu})$ parts:

$$\overline{P}_{\mathrm{m}}^{\sigma\nu} = \left(\frac{1}{3} T_{\mathrm{m}\mu}^{\mu} - p_{\mathrm{m}}\right) \Delta^{\sigma\nu} + \Pi_{\mathrm{m}}^{\sigma\nu} := p_{\mathrm{m}}\Delta^{\sigma\nu} + \widetilde{\Delta}_{\mathrm{m}}\Delta^{\sigma\nu} + \Pi_{\mathrm{m}}^{\sigma\nu}, \tag{2.108}$$

where

$$\widetilde{\Delta}_{\mathrm{m}} = \frac{1}{3}\Delta_{\mu\nu} P_{\mathrm{m}}^{\mu\nu} - \mathfrak{p}_{\mathrm{m}}^{\mathrm{e}} := \mathfrak{p} - p_{\mathrm{m}}^{\mathrm{e}}, \tag{2.109}$$

$$\Pi_{\mathrm{m}}^{\mu\nu} = P_{\mathrm{m}}^{\mu\nu} - \frac{1}{3}\Delta_{\sigma\tau} T_{\mathrm{m}}^{\sigma\tau} \Delta_{\mu\nu}. \tag{2.110}$$

Therefore, $\widetilde{\Delta}_{\mathrm{m}}$ is the excess normal stress above and beyond the hydrostatic pressure p_{m}. It consequently vanishes at equilibrium. It should be noted that we have used the identity $\Delta_\nu^{\mu} \nabla_\sigma U^\nu = \nabla_\sigma U^\mu$ for the derivation of (2.101).

Energy Balance Equation

The energy balance equation also follows upon contracting the energy-momentum balance equation (2.79) with $c^{-2}U_\mu$ and using some of the identities listed in (1.78). We thereby obtain the energy balance equation

$$\rho D\mathcal{E}_\mathrm{m} = -\partial_\mu Q^\mu_\mathrm{m} + P^{\mu\nu}_\mathrm{m}\nabla_\nu U_\mu + c^{-2}Q^\mu_\mathrm{m} DU_\mu + c^{-2}U_\nu\Lambda^\nu_\mathrm{m}. \tag{2.111}$$

The last term in this equation appears because of the interaction of particle species with photons making transition to different particle species—or acquiring a spectrum of "colors", figuratively speaking. The term $2c^{-2}Q^\mu_\mathrm{m} DU_\mu$ vanishes as $u/c \to 0$ in the nonrelativistic limit.

2.3.3 Balance Equations for Conserved Variables of Radiation

The balance equations for conserved variables of radiation can be derived from (2.81) and (2.82) in the same manner as for material balance equations.

Radiation Density Fraction Balance Equation

With the definition of radiation (photon number) fraction as ratio of ρ_r to ρ

$$\mathfrak{c}_r = \frac{\rho_r}{\rho}, \tag{2.112}$$

we obtain, by making use of identities (1.78), the radiation fraction balance equation

$$\rho D\mathfrak{c}_r = -\partial_\nu J^\nu_r + c^{-2}J^\nu_r DU_\nu + \Lambda^{(n)\mu}_r. \tag{2.113}$$

The term $c^{-2}J^\nu_r DU_\nu$ vanishes in the nonrelativistic limit as $u/c \to 0$.

Radiation Energy Balance Equation

Contracting the energy-momentum balance equation for radiation (2.82) with $c^{-2}U_\mu$ the radiation energy balance equation is obtained:

$$\rho D\mathcal{E}_r = -\partial_\nu Q^\nu_r + P^{\mu\nu}_r\nabla_\nu U_\mu + c^{-2}Q^\nu_r DU_\nu + c^{-2}U_\mu\Lambda^\mu_r. \tag{2.114}$$

Here $\mathcal{E}_r$ is the radiation energy density and the equation obtained above describes the evolution of the radiation energy density. We have not listed the radiation momentum balance equation here, since it is related to the radiation heat flux evolution equation, which will be included in the flux evolution equations for nonconserved variables given below.

2.4 Evolution Equations for Nonconserved Variables

We have already seen that there appear macroscopic variables such as J^μ_i, Q^μ_i, and $P^{\mu\nu}_i$ in the balance equations presented for conserved variables, which do not as yet have their own evolution equations. They are, as a matter of fact, examples for what we call nonconserved variables. The variables J_i^μ, Q_i^μ, and $P_i^{\mu\nu}$ are the leading elements of a hierarchy of macroscopic nonconserved moments, which can be generated from the kinetic equation. We have seen such examples in nonrelativistic kinetic theories and also in the relativistic kinetic theory of pure gases discussed in the previous chapter. We derive them from the covariant kinetic equation by using the moments denoted by $h_i^{(q)\sigma\cdots\alpha}$ for the generic molecular symbol for nonconserved macroscopic variables. The subscript i of moment $h_i^{(q)\sigma\cdots\alpha}$ denotes the species $i \in (1, \ldots, m, r)$ and the superscript q denotes the qth element of the moment set: $q = 1, 2, 3, 4, \ldots$. The set of moments is constructed by using the conserved moments as seed moments, which have appeared in the balance equations of conserved variables. The higher-order moments following the seed moments are hierarchically constructed by using the Schmidt orthogonalization method employed for constructing an orthogonal set of moments. This method was used in the nonrelativistic kinetic theory of the present work and also in the previous chapter on the relativistic kinetic theory of matter of this volume. The same procedure is employed for the present covariant kinetic equation. The reader is referred to the sections for the Schmidt orthogonalization method of constructing an orthogonal moment set in the previous chapters of this work. We present the matter and radiation parts separately.

2.4.1 Moment Set for Matter

The leading elements of the moment set for matter species are represented by the tensor polynomials $h_a^{(q)\alpha\cdots\omega}$, which may be chosen to be orthogonal tensor polynomials of momentum p_a^σ and ordered as follows:

$$h_a^{(1)\mu\nu} = c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta_{\sigma\tau}\Delta^{\mu\nu}\right) p_a^\sigma p_a^\tau, \tag{2.115}$$

$$h_a^{(2)} = \frac{1}{3}c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \Delta_{\mu\nu} p_a^\mu p_a^\nu - p_a^{\mathrm{e}}/\rho_a + \left(\vartheta^{-1} - 1\right)\left(\overline{E} - E_a\right), \tag{2.116}$$

$$\begin{aligned} h_a^{(3)\mu} = c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \{&\Delta_\sigma^\mu p_a^\nu p_a^\sigma U_\nu \\ &- \left(\widehat{h}_a + m_a c^2\right)\left[p_a^\mu - c^{-2}\left(U_\lambda p_a^\lambda\right) U^\mu\right]\}, \end{aligned} \tag{2.117}$$

$$h_a^{(4)\mu} = c^2 \left(U_\lambda p_a^\lambda\right)^{-1} \left[p_a^\mu - c^{-2}\left(p_a^\lambda U_\lambda\right) U^\mu\right], \quad etc., \tag{2.118}$$

where ρ_a and $\widehat{h}_a$ are the density and the enthalpy density of “color” species a, respectively; E_a is the energy of the internal degrees of freedom of particle a;

$$\overline{E} = \sum_{a \neq r} E_a, \tag{2.119}$$

$$\vartheta^{-1} = C_{\mathrm{int}}/C_{\mathrm{v}} \qquad (C_{\mathrm{v}} = C_{\mathrm{tr}} + C_{\mathrm{int}}) \tag{2.120}$$

with C_{tr} and C_{int} denoting the translational and the internal specific heat per molecule at constant volume, respectively; and $p_a := p_a^{\text{e}}$ for notational brevity. The set of moments $\left\{h_a^{(q)\alpha\cdots\omega}\right\}$ is constructed in the same spirit as taken for it in Chap. 1 such that its elements reduce to the same nonrelativistic limits as for the nonrelativistic theory counterparts in the limit of $u/c \to 0$. In the case of gases without an internal structure the last term in (2.116) for $h_a^{(2)}$ vanishes and the expression for $h_a^{(2)}$ of a monatomic gas used in the previous chapter is recovered. Note that the quantum aspect does not appear explicitly in the kinematic terms even though particles behave quantum mechanically, at least in their collision processes, in the present theory. The set is ordered as presented, because the present ordering is most suitable for the theory presented below in this work. We also note that the moment set also can be expressible in terms of relativistic peculiar velocity $\mathfrak{C}_a^{\sigma}$ as was done in Chap. 1 of this Volume. Such a representation would be most convenient when their nonrelativistic limits are discussed. The physical meanings of the tensor moments $h_a^{(q)\alpha\cdots\omega}$ will become evident as the theory is formulated in the following.

To begin the derivation of evolution equations, the macroscopic four-flow tensor (i.e., supermoment) $\psi_a^{(q)\sigma\mu\cdots\nu}$ is defined by the statistical mechanical average of flow $p_a^{\sigma} h_a^{(q)\mu\cdots\nu}$ of molecular tensor moment $h_a^{(q)\mu\cdots\nu}$:

$$\psi_a^{(q)\sigma\mu\cdots\nu} = \left\langle p_a^{\sigma} h_a^{(q)\mu\cdots\nu} f_a(x, p_a)\right\rangle . \tag{2.121}$$

We then define macroscopic flux tensor $\Phi_a^{(q)\mu\cdots\nu}$ for nonconserved fluxes by taking contraction of supermoment $\psi_a^{(q)\sigma\mu\cdots\nu}$ with hydrodynamic velocity U_σ

$$\Phi_a^{(q)\mu\cdots\nu} = c^{-2} U_\sigma \psi_a^{(q)\sigma\mu\cdots\nu} = c^{-2} U_\sigma \left\langle p_a^{\sigma} h_a^{(q)\mu\cdots\nu} f_a(x, t)\right\rangle . \tag{2.122}$$

Therefore summing it over all matter species we obtain the macroscopic moment for matter as a whole

$$\Phi_{\text{m}}^{(q)\mu\cdots\nu} := \rho \widehat{\Phi}_{\text{m}}^{(q)\mu\cdots\nu} = \sum_{a=1(a\neq r)}^{m} \Phi_a^{(q)\mu\ldots\nu} \quad (q = 1, 2, \cdots, 4) . \tag{2.123}$$

The macroscopic moment tensor $\Phi_a^{(q)\mu\cdots\nu}$ will be sometimes referred to as the macroscopic flux of order q. Here the subscript m stands for matter in order to distinguish the material part from the radiation part of the flux. The leading examples for macroscopic tensor moments $\Phi_a^{(q)\mu\cdots\nu}$ are stress tensors, heat fluxes, diffusion fluxes.

2.4.2 Evolution Equations for Nonconserved Moments for Matter

By using (2.121), (2.123), and the substantial time derivative[7] $\mathfrak{D} = \rho D$, where $D := U^\nu \partial_\nu$, we easily obtain the evolution equation for $\widehat{\Phi}_a^{(q)\mu\cdots\nu}$ of matter. The procedure is illustrated below as a generic example.

Operating ρD on $\widehat{\Phi}_a^{(q)\mu\cdots\nu} := \Phi_a^{(q)\mu\cdots\nu}/\rho$, we obtain

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = \partial_\sigma \left(\widehat{\Phi}_a^{(q)\mu\cdots\nu} N^\sigma \right), \tag{2.124}$$

which can be rearranged to the form

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\partial_\sigma \left(\psi_a^{(q)\sigma\mu\cdots\nu} - \widehat{\Phi}_a^{(q)\mu\cdots\nu} N^\sigma \right) + \partial_\sigma \psi_a^{(q)\sigma\mu\cdots\nu}. \tag{2.125}$$

The flux of $\widehat{\Phi}_a^{(q)\mu\cdots\nu}$ may be defined by the formula

$$\Delta_\gamma^\mu \psi_a^{(q)\sigma\gamma\cdots\nu} = \left(\psi_a^{(q)\sigma\mu\cdots\nu} - \widehat{\Phi}_a^{(q)\mu\cdots\nu} N^\sigma \right). \tag{2.126}$$

Upon use of the covariant kinetic equation, we obtain the evolution equation for supermoment $\psi_a^{(q)\sigma\mu\cdots\nu}$

$$\partial_\sigma \psi_a^{(q)\sigma\mu\cdots\nu} = \left\langle f_a(x, p_a) p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} \right\rangle + \Lambda_a^{(q)\mu\cdots\nu}(x), \tag{2.127}$$

where the dissipation term $\Lambda_a^{(q)\mu\cdots\nu}(x)$ is defined by the kinetic theory formula

$$\Lambda_a^{(q)\mu\cdots\nu}(x) = \left\langle h_a^{(q)\mu\cdots\nu} \mathfrak{R}_a \left[f_a(x, p_a) \right] \right\rangle. \tag{2.128}$$

Therefore,we are finally able to write the generic evolution equation for $\Phi_a^{(q)\mu\cdots\nu}$ in the form

$$\rho D \widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\partial_\sigma \left(\Delta_\gamma^\mu \psi_a^{(q)\sigma\gamma\cdots\nu} \right) + Z_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}(x). \tag{2.129}$$

In this equation the term $Z_a^{(q)\mu\cdots\nu}$ is defined by the formula

$$Z_a^{(q)\mu\cdots\nu} = \left\langle p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} f_a(x, p_a) \right\rangle, \tag{2.130}$$

which is called the kinematic term of the evolution equation for $\Phi_a^{(q)\mu\cdots\nu}$. Equation (2.129) is the generic form of evolution equation for all species and orders ($q = 1, 2, \cdots$). As in the case of generic evolution equation for matter alone, the kinematic term is modified by adding to it the contribution from the divergence term, and the

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[7] If we choose the Eckart convention [16] for hydrodynamic velocity, the convective time derivative D is identical with the substantial time derivative. For this, see Chap. 1 of this volume.

generic evolution equation reads

$$\rho D\widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_a^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}(x)\,, \tag{2.131}$$

where $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ and $\Theta_a^{(q)\sigma\mu\cdots\nu}$, similarly to (1.123) and (1.126), are given respectively by the formulas

$$\mathfrak{Z}_a^{(q)\mu\cdots\nu} = Z_a^{(q)\mu\cdots\nu} - c^{-2}U_\sigma D\left(\Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu}\right), \tag{2.132}$$

$$\Theta_a^{(q)\sigma\mu\cdots\nu} = \Delta_\omega^\sigma \psi_a^{(q)\omega\mu\cdots\nu}. \tag{2.133}$$

This equation has formally a similar mathematical structure to the evolution equation for nonrelativistic nonconserved flux $\Phi_a^{(q)}$ appearing in nonrelativistic kinetic theories and also to those in the relativistic theory of matter alone. We have repeated about it to set it apart from the relativistic evolution equations for radiation, which are found different from those for matter.

Upon summing (2.129) over all matter species we obtain the generic evolution equation for $\Phi_{\mathrm{m}}^{(q)\mu\cdots\nu}$:

$$\rho D\widehat{\Phi}_{\mathrm{m}}^{(q)\mu\cdots\nu} = -\partial_\sigma\left(\Delta_\gamma^\mu \psi_{\mathrm{m}}^{(q)\sigma\gamma\cdots\nu}\right) + Z_{\mathrm{m}}^{(q)\mu\cdots\nu} + \Lambda_{\mathrm{m}}^{(q)\mu\cdots\nu}(x)\,, \tag{2.134}$$

where

$$\Lambda_{\mathrm{m}}^{(q)\mu\cdots\nu}(x) = \sum_{a\neq r}\left\langle h_a^{(q)\mu\cdots\nu}\mathfrak{R}_a\left[f_a(x,p_a)\right]\right\rangle, \tag{2.135}$$

$$Z_{\mathrm{m}}^{(q)\mu\cdots\nu} = \sum_{a\neq r}\left\langle p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} f_a(x,p_a)\right\rangle. \tag{2.136}$$

Let us recall the decomposition of $T_{\mathrm{m}}^{\mu\nu}$ into the traceless symmetric part $\Pi_{\mathrm{m}}^{\mu\nu}$ of the stress tensor, the excess trace part $\widetilde{\Delta}_{\mathrm{m}}$ of the stress tensor, and the heat flux Q_{m}^μ as well as diffusion flux $\mathfrak{J}_{\mathrm{m}}^\mu$ and also the definition of diffusion four-flow

$$J_a^\mu = N_a^\mu - \mathfrak{c}_a N_{\mathrm{m}}^\mu, \tag{2.137}$$

where the statistical mechanical formula for number four-flow is defined by

$$N_{\mathrm{m}}^\mu = \sum_{a\neq r} N_a^\mu = \sum_{a\neq r}\left\langle p_a^\mu f_a(x,p_a)\right\rangle. \tag{2.138}$$

Then, the leading members of the macroscopic moment set just defined can be identified with shear stress tensor, excess normal stress, heat flux, and diffusion flux as follows:

$$\Phi_{\mathrm{m}}^{(1)\mu\nu} = \Pi_{\mathrm{m}}^{\mu\nu}; \quad \Phi_{\mathrm{m}}^{(2)} = \widetilde{\Delta}_{\mathrm{m}} + \left[\frac{2}{3}\left(\vartheta^{-1} - 1\right)\right]\Delta E;$$
$$\Phi_{\mathrm{m}}^{(3)\mu} = Q_{\mathrm{m}}^{\mu} - \mathfrak{a}_{\mathrm{m}} h_{\mathrm{m}} J_{\mathrm{m}}^{\mu} := Q_{\mathrm{m}}^{\prime\mu}; \quad \Phi_{\mathrm{m}}^{(4)\mu} = J_{\mathrm{m}}^{\mu}, \quad \text{etc.} \tag{2.139}$$

Here

$$\Delta E = \sum_{a \neq r}^{m} \left(\overline{E} - E_a\right). \tag{2.140}$$

It is important to note that the second member $\Phi_{\mathrm{m}}^{(2)}$ of the macroscopic moment set in (2.139) has an additional term, which is absent in the moment $h_a^{(2)}$ for monatomic gases [1] since the latter does not have the term arising from the internal energy fluctuation, namely, $\left[\frac{2}{3}\left(\vartheta^{-1} - 1\right)\right]\Delta E$.

Since the generalized hydrodynamic theory arising from the covariant kinetic equation for the matter part is already presented in Chap. 1 of Volume 2, here attention will be more closely paid to the radiation part.

2.4.3 Evolution Equations for Nonconserved Moments for Radiation

We have formulated the radiation part of the kinetic equation with the radiation momenta on equal footing with the particle momenta. However, whereas the material particles obey the relativistic equations of motion, the radiation (photon) obeys a wave equation. According to (2.10) the photon momentum $\mathbf{p}_r$ is related to the wave vector $\mathbf{k}_r$ of radiation by the formula

$$\mathbf{p}_r = \hbar \mathbf{k}_r = \frac{\hbar\omega}{c}\widehat{\mathbf{k}}_r. \tag{2.141}$$

The photon momentum is rendered dimensionless if it is multiplied by $c\beta$:

$$\mathbf{q}_r = \beta\hbar\omega\widehat{\mathbf{k}}_r. \tag{2.142}$$

With this dimensionless unit vector for the space component, we form the dimensionless four-vector q_r^{ν} for photon

$$q_r^{\mu} = \beta\hbar\omega\left(1, \widehat{\mathbf{k}}_r\right) = c\beta p_r^{\mu}. \tag{2.143}$$

The trace of this four-vector is then seen to be null, reflecting the fact that the photon has a zero rest mass:

$$\left(q_r^{\mu} q_{r\,\mu}\right) = 0. \tag{2.144}$$

In addition to this, we find

$$p_r^\mu U_\mu = \hbar\omega\gamma\left(1 - \frac{\mathbf{n}\cdot\mathbf{u}}{c}\right), \tag{2.145}$$

where $\mathbf{n}$ is the unit vector in the direction of photon propagation. With this consideration indicating the distinctive nature of photons and photon momentum $p_r^\mu = q_r^\mu / c\beta$, which we have already alluded to in the early part of this chapter, we construct a set of moments for radiation, which is comparable to the moment set of matter (2.115)–(2.118). The following is the leading elements of the set for radiation:

$$h_r^{(1)\mu\nu} = c^2\left(U_\lambda p_r^\lambda\right)^{-1}\left(\Delta_\sigma^\mu \Delta_\tau^\nu - \frac{1}{3}\Delta_{\sigma\tau}\Delta^{\mu\nu}\right) p_r^\sigma p_r^\tau, \tag{2.146}$$

$$h_r^{(2)} = \frac{1}{3}c^2\left(U_\lambda p_r^\lambda\right)^{-1}\Delta_{\mu\nu} p_r^\mu p_r^\nu - \mathfrak{p}_r/\rho_r, \tag{2.147}$$

$$h_r^{(3)\mu} = c^2\left(U_\lambda p_r^\lambda\right)^{-1}\left\{\Delta_\sigma^\mu p_r^\nu p_r^\sigma U_\nu - h_r\left[p_r^\mu - c^{-2}\left(U_\lambda p_r^\lambda\right)U^\mu\right]\right\}, \tag{2.148}$$

$$h_r^{(4)\mu} = c^2\left(U_\lambda p_r^\lambda\right)^{-1}\left[p_r^\mu - c^{-2}\left(p_r^\lambda U_\lambda\right)U^\mu\right]. \tag{2.149}$$

Here it must be noted that $\left(p_r^\nu p_{r\,\nu}\right) = 0$ which makes it unnecessary to use h_r unmodified in contrast to the case of matter; (2.117). The moment set presented here consists of orthogonal tensor polynomials $h_r^{(\delta)}(q_r)$, which are constructed by means of the Schmidt orthogonalization method similarly to $h_m^{(\delta)}$ presented in (2.115)–(2.118). They can be expressible in terms of isotropic tensors of unit dimensionless four-vectors q_r^ν.

With the hydrodynamic four-velocity defined for matter and radiation together, we define the relativistic peculiar velocity $\mathfrak{C}_a^\sigma$ in the same manner as for case of matter alone. Therefore the index now covers both matter and radiation. With so defined $\mathfrak{C}_a^\sigma$, the moment set for both matter and radiation can be expressed in terms of relativistic peculiar velocities. Such moment sets can be used to examine the nonrelativistic limits of the relativistic generic evolution equations for both matter and radiation in the same line of approach as taken for the case of matter alone in the previous chapter.

As is the case for the matter parts of the macroscopic fluxes, if the radiation part of the macroscopic supermoment $\psi_r^{(q)\sigma\mu\cdots\nu}$ is defined by the statistical mechanical expression

$$\psi_r^{(q)\sigma\mu\cdots\nu} = \left\langle f_r(x, p_r) p_r^\sigma h_r^{(q)\mu\cdots\nu}\right\rangle. \tag{2.150}$$

Then the macroscopic fluxes $\Phi_r^{(q)\mu\cdots\nu}$ are obtained by contracting $\psi_r^{(q)\sigma\mu\cdots\nu}$ with U^μ—projecting $\psi_r^{(q)\sigma\mu\cdots\nu}$ onto hydrodynamic velocity U^σ:

$$\Phi_r^{(q)\mu\cdots\nu} = c^{-2} U_\sigma \psi_r^{(q)\sigma\mu\cdots\nu}. \tag{2.151}$$

As is for material particles, the leading macroscopic moments (i.e. fluxes) $\Phi_r^{(q)\mu\cdots\nu}$ for radiation (photon) are identified with the radiation shear stress tensor $\Pi_r^{\mu\nu}$; the excess trace part of the radiation stress tensor $\widetilde{\Delta}_r$; the heat flux Q_r^{μ}; the number flux J_r^{μ}. Then their statistical definitions can be obtained if the energy-momentum tensor of radiation is made use of

$$T_r^{\mu\nu} = \left\langle p_r^{\mu} p_r^{\nu} f_r(x, p_r) \right\rangle . \tag{2.152}$$

Recall that since $T_r^{\mu\nu}$ is decomposable into $\mathcal{E}_r$, $Q_r^{\prime\mu}$, $\mathfrak{J}_r^{\mu} := hJ_a^{\mu}$, and $P_r^{\mu\nu}$ with the help of the projection operators. It is then straightforward to find the desired statistical mechanical formulas for the decomposed components. Note also that as is for matter the radiation stress tensor is decomposable as

$$P_r^{\mu\nu} = p_r^{\mathrm{e}} \Delta^{\mu\nu} + \widetilde{\Delta}_r \Delta^{\mu\nu} + \Pi_r^{\mu\nu}, \tag{2.153}$$

where p_r^{e} is the equilibrium radiation pressure and

$$\widetilde{\Delta}_r = \frac{1}{3}\Delta_{\mu\nu} P_r^{\mu\nu} - p_r^{\mathrm{e}} = \mathfrak{p}_r - p_r^{\mathrm{e}}, \tag{2.154}$$

$$\Pi_r^{\mu\nu} = P_r^{\mu\nu} - \frac{1}{3}\Delta_{\sigma\tau} T_r^{\sigma\tau} \Delta^{\mu\nu} \tag{2.155}$$

with $\mathfrak{p}_r$ denoting the nonequilibrium radiation pressure—the trace part of the radiation pressure tensor $P_r^{\mu\nu}$. The excess normal stress of radiation $\widetilde{\Delta}_r$ defined by (2.154) presumes the possibility of radiation pressure $\mathfrak{p}_r$ fluctuating from p_r^{e} if the radiation is in nonequilibrium.

Therefore, as for the case of matter, we can easily find the physical meanings of $\Phi_r^{(q)\mu\cdots\nu}$ $(q = 1, 2, \ldots, 4)$ as follows:

$$\begin{aligned} \Phi_r^{(1)\mu\nu} &= \Pi_r^{\mu\nu}; \quad \Phi_r^{(2)} = \widetilde{\Delta}_r; \\ \Phi_r^{(3)\mu} &= Q_r^{\mu} - hJ_r^{\mu} := Q_r^{\prime\mu}; \quad \Phi_r^{(4)\mu} = J_r^{\mu};\ \text{etc.} \end{aligned} \tag{2.156}$$

The evolution equations for $\Phi_r^{(q)}$ can also be derived from the covariant Boltzmann equation (2.13) by using the same method as for the matter parts. They may be written in the form

$$\rho D\widehat{\Phi}_r^{(q)\mu\cdots\nu} = -\partial_\sigma \left(\Delta_\gamma^{\mu} \psi_r^{(q)\sigma\gamma\cdots\nu} \right) + Z_r^{(q)\mu\cdots\nu} + \Lambda_r^{(q)\mu\cdots\nu}, \tag{2.157}$$

where

$$Z_r^{(q)\mu\cdots\nu} = \left\langle p_r^{\sigma} \partial_\sigma h_r^{(q)\mu\cdots\nu} f_r(x, p_r) \right\rangle , \tag{2.158}$$

$$\Lambda_r^{(q)\mu\cdots\nu} = \left\langle h_r^{(q)\mu\cdots\nu} \mathfrak{R}_r \left[f_r(x, p_r) \right] \right\rangle \qquad (q = 1, 2, \cdots). \tag{2.159}$$

Equation (2.157) can be also put into a modified form

$$\rho D\widehat{\Phi}_r^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_r^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_r^{(q)\mu\cdots\nu} + \Lambda_r^{(q)\mu\cdots\nu} \tag{2.160}$$

together with the modified kinematic term $\mathfrak{Z}_r^{(q)\mu\cdots\nu}$ and the higher order term $\Theta_r^{(q)\sigma\mu\cdots\nu}$ in the divergence term similar to (2.132) and (2.133):

$$\mathfrak{Z}_r^{(q)\mu\cdots\nu} = Z_r^{(q)\mu\cdots\nu} - c^{-2} U_\sigma D \left(\Delta_\omega^\sigma \psi_r^{(q)\omega\mu\cdots\nu} \right), \tag{2.161}$$

$$\Theta_r^{(q)\sigma\mu\cdots\nu} = \Delta_\omega^\sigma \psi_r^{(q)\omega\mu\cdots\nu}. \tag{2.162}$$

The evolution equations for conserved variables and nonconserved variables for both matter and radiation are what we immediately need to formulate a theory of irreversible macroscopic processes in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$ for the system of matter and radiation under consideration. The procedure used for evaluation of modified kinematic terms $\mathfrak{Z}_a^{(q)\mu\cdots\nu}$ $(a = 1, 2, \ldots, m, r; q = 1, 2, \cdots)$ is the same as for the generic evolution equation studied in Chap. 1 of this volume. We collect and summarize them in the following before proceeding to investigate them with regard to irreversible thermodynamics undergoing in the system.

2.5 Summary of Macroscopic Evolution Equations

Summarized in subsections below are the balance equation for conserved and nonconserved variables derived from the covariant kinetic equations for both matter and radiation.

2.5.1 Equation of Continuity

The equation of continuity holds for the combined system of matter and radiation

$$D\rho = -\rho \nabla_\mu U^\mu. \tag{2.163}$$

This equation in essence is the evolution equation for density.

2.5.2 Conservation Laws for Matter

The evolution equations for the concentration fraction of matter, the hydrodynamic velocity, and the energy of matter are listed in the order as below:

$$\begin{aligned} \rho D\mathfrak{c}_a &= -\partial_\mu J_a^\mu + c^{-2} J_a^\mu DU_\mu + \Lambda_a^{(n)}, &(2.164)\\ c^{-2}\mathcal{E}_{\mathrm{m}}\rho DU^\mu &= -\nabla^\mu p_{\mathrm{m}} - \Delta_\sigma^\mu \partial_\nu \left(\Pi_{\mathrm{m}}^{\sigma\nu} + \widetilde{\Delta}_{\mathrm{m}}\Delta^{\sigma\nu}\right) \\ &\quad + c^{-2}\left(P_{\mathrm{m}}^{\mu\nu} DU_\nu - \Delta_\omega^\mu D Q_{\mathrm{m}}^\omega - Q_{\mathrm{m}}^\mu \nabla_\nu U^\nu - Q_{\mathrm{m}}^\nu \nabla_\nu U^\mu\right) \\ &\quad + \Delta_\nu^\mu \Lambda_{\mathrm{m}}^{(n)\nu}, &(2.165)\\ \rho D\mathcal{E}_{\mathrm{m}} &= -\partial_\mu Q_{\mathrm{m}}^\mu + P_{\mathrm{m}}^{\mu\nu}\nabla_\nu U_\mu + c^{-2} Q_{\mathrm{m}}^\mu DU_\mu + c^{-2} U_\nu \Lambda_{\mathrm{m}}^\nu. &(2.166) \end{aligned}$$

2.5.3 Conservation Laws for Radiation

The evolution equations are listed for the concentration fraction for radiation and the radiation energy in the following order:

$$\begin{aligned} \rho D\mathfrak{c}_r &= -\partial_\nu J_r^\nu + c^{-2} J_r^\nu DU_\nu + \Lambda_r^{(n)\mu}, &(2.167)\\ \rho D\mathcal{E}_r &= -\partial_\nu Q_r^\nu + P_r^{\mu\nu}\ \nabla_{\nu\mu} + c^{-2} Q_r^\nu DU_\nu + c^{-2} U_\mu \Lambda_r^\mu. &(2.168) \end{aligned}$$

2.5.4 Evolution Equations for Nonconserved Variables for Matter

The generic evolution equations for nonconserved variables for matter species a can be given in generic form

$$\begin{aligned} \rho D\widehat{\Phi}_a^{(q)\mu\cdots\nu} &= -\nabla_\sigma \Theta_a^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_a^{(q)\mu\cdots\nu} + \Lambda_a^{(q)\mu\cdots\nu}(x)\,, &(2.169)\\ \Theta_a^{(q)\sigma\mu\cdots\nu} &= \Delta_\gamma^\mu \psi_a^{(q)\gamma\sigma\cdots\nu}, &(2.170) \end{aligned}$$

where $\Theta_a^{(q)\sigma\mu\cdots\nu}$ is the flux of supermoment defined earlier, which may be regarded as the flux of macroscopic moment $\Phi_a^{(q)\mu\cdots\nu}$. It is a moment one-order higher than $\Phi_a^{(q)\mu\cdots\nu}$ in the language of moment methods. Because of $\Theta_a^{(q)\sigma\mu\cdots\nu}$ the evolution equation (2.169) represents an open hierarchy of evolution equations for macroscopic moments. The index a runs for all matter species in the system, and $q = 1, 2, 3, 4$, etc. For the entire matter as a whole the generic evolution equation reads

$$\begin{aligned} \rho D\widehat{\Phi}_{\mathrm{m}}^{(q)\mu\cdots\nu} &= -\nabla_\sigma \Theta^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_{\mathrm{m}}^{(q)\mu\cdots\nu} + \Lambda_{\mathrm{m}}^{(q)\mu\cdots\nu}(x)\,, &(2.171)\\ \Theta_{\mathrm{m}}^{(q)\sigma\mu\cdots\nu} &= \Delta_\gamma^\mu \psi_{\mathrm{m}}^{(q)\gamma\sigma\cdots\nu}. &(2.172) \end{aligned}$$

The kinematic terms and $\Theta_a^{(q)\sigma\mu\cdots\nu}$ can be evaluated in terms of variables belonging to the manifold $\mathfrak{P}$ by using exactly the same procedure and method and the same expressions as described in the previous chapter. The dissipation terms are calculated with the collision integral $\mathfrak{R}\left[f_a\right]$ of the covariant kinetic equation (2.13). Although the details of $\mathfrak{R}[f_a]$ is different from the kinetic equation used in the previous chapter, the formal structure of the dissipation terms remains the same as those in the previous chapter with regard to the dependence of generalized potentials and ultimately macroscopic moments $\Phi_a^{(q)\mu\cdots\nu}$. Therefore, the results of evaluation of the kinematic terms of the material part are not presented here to avoid repetition.

2.5.5 Evolution Equations for Nonconserved Variables for Radiation

The generic evolution equation for nonconserved variables for radiation is given by formally the same equation as the material part (2.171):

$$\rho D\widehat{\Phi}_r^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_r^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_r^{(q)\mu\cdots\nu} + \Lambda_r^{(q)\mu\cdots\nu}(x)\,, \tag{2.173}$$

$$\Theta_r^{(q)\sigma\mu\cdots\nu} = \Delta_\gamma^\mu \psi_r^{(q)\gamma\sigma\cdots\nu}, \tag{2.174}$$

but the kinematic terms $\mathfrak{Z}_r^{(q)\mu\cdots\nu}$ will be worked out since the details of the moments $h_r^{(q)\mu\nu}$ in (2.146)–(2.149) are slightly different from their matter counter parts $h_a^{(q)\mu\nu}$ given in (2.115)–(2.118). They constitute counterparts to the kinematic terms for relativistic evolution equations of matter given in Chap. 1 of this volume. They will be explicitly presented in Sect. 2.12.2 of this chapter. For more explicit formulas for dissipation terms $\Lambda_a^{(q)\mu\cdots\nu}$ and $\Lambda_r^{(q)\mu\cdots\nu}$ some suitable approximations will be required, so that they remain consistent with the laws of thermodynamics. Since a thermodynamic theory of irreversible processes is required to discuss the thermodynamic consistency of the evolution equations formulated here we would like turn our attention to the formulation of theory of irreversible thermodynamics.

2.6 Calortropy and Its Balance Equation

To make use of the laws of thermodynamics in developing a kinetic theory of irreversible transport processes it is necessary to link up the kinetic theory with the laws of thermodynamics by some means. For this purpose the conventional trail of thought followed in kinetic theory is to regard the Boltzmann entropy and, particularly, the H theorem as the statistical mechanical representations for the Clausius entropy and the second law of thermodynamics, respectively. We have avoided this line of thinking for the reason clarified in the previous chapter. We will follow the same line of approach as for the previous chapters of this work.

Since the second law of thermodynamics is entirely phrased in terms of macroscopic observables for the system of interest, which are described by a system of Pfaffian differential forms, the Boltzmann entropy must seamlessly fit in such mathematical structures in the appropriate macroscopic variable manifold, if we are to reach the desired goal. To be more specific, the differential of Boltzmann entropy, in particular, must be an exact Pfaffian differential in the manifold of macroscopic variables appropriate for thermodynamic description of irreversible processes in the system. We have seen in the previous chapters of this work that this, unfortunately, is not the case for the Boltzmann entropy and the H function. This particular feature would not change for the system of matter and radiation, if the system is away from equilibrium. We, therefore, must look for an alternative to the Boltzmann entropy that will provide us a desired theory of irreversible thermodynamics also for matter and radiation. The mathematical framework for the desired theory is again provided by a quantity called *calortropy*, if the system is away from equilibrium, as we have seen in the previous chapters on the nonrelativistic and relativistic kinetic equations for dilute gases.

We introduce the calortropy four-flow $\Psi^{\mu}(x)$ by the statistical mechanical formula

$$\Psi^{\mu}(x) = -k_B \sum_{a=1}^{r} \left\langle p_a^{\mu} \left[f_a \ln f_a^{\mathrm{c}} - \epsilon_a (1 + \epsilon_a f_a) \ln \left(1 + \epsilon_a f_a^{\mathrm{c}} \right) \right] \right\rangle, \tag{2.175}$$

where $f_a^{\mathrm{c}}(x, p_a)$ is called the nonequilibrium canonical form, which is not necessarily equal to $f_a(x, p_a)$ at all states of the system except at equilibrium. A more concrete form of nonequilibrium canonical form is given under the functional hypothesis for the distribution function as will be discussed presently. For the moment it will be sufficient to know there exists a nonequilibrium canonical form $f_a^{\mathrm{c}}(x, p_a)$, which is generally not equal to $f_a(x, p_a)$ except at equilibrium. The scalar calortropy density is then obtained if the calortropy four-flow $\Psi^{\mu}(x)$ is contracted with U^{μ}:

$$\Psi(x) := \rho \widehat{\Psi}(x) = c^{-2} U_{\mu} \Psi^{\mu}(x). \tag{2.176}$$

Geometrically, this is a projection of Ψ^{μ} onto the direction of U^{μ}. The balance equation for the calortropy density is obtained from the formal expression (2.175) by following the procedure used to obtain the balance equation (2.25) for $\widehat{\mathcal{S}}$.

According to the aforementioned procedure, we begin with the substantial derivative of $\widehat{\Psi}$ expressed in the form

$$\rho D \widehat{\Psi} = -\partial_{\mu} \left(\Psi^{\mu} - \widehat{\Psi} N^{\mu} \right) + \partial_{\mu} \Psi^{\mu}, \tag{2.177}$$

which follows upon operating on $\widehat{\Psi}$ the substantial time derivative operator $\mathfrak{D} = N^{\mu} \partial_{\mu} = \rho D + N^{\mu} \nabla_{\mu} = \rho D$. We then define the calortropy flux four-flow J_{c}^{μ} by the formula

$$J_{\mathrm{c}}^{\mu} = \Psi^{\mu} - \widehat{\Psi} N^{\mu}. \tag{2.178}$$

If the definitions of number four-flow N^{μ} and the projector $\Delta^{\mu\nu}$ are made use of, the calortropy flux four-flow J_{c}^{μ} can be written as a projection of Ψ^{ν} in the direction perpendicular to U^{ν}:

$$J_{\mathrm{c}}^{\mu} = \Delta_{\nu}^{\mu} \Psi^{\nu}. \tag{2.179}$$

Substituting (2.175) into this expression, we obtain the statistical mechanical formula for the calortropy flux four-flow

$$J_{\mathrm{c}}^{\mu} = -k_B \sum_{a=1}^{r} \Delta_{\nu}^{\mu} \left\langle p_a^{\nu} \left[f_a \ln f_a^{\mathrm{c}} - \epsilon_a \left(1 + \epsilon_a f_a\right) \ln \left(1 + \epsilon_a f_a^{\mathrm{c}}\right)\right]\right\rangle . \tag{2.180}$$

For the balance equation for Ψ^{μ}, upon operating ∂_{μ} on (2.175) we obtain the calortropy balance equation for calortropy four-flow Ψ^{μ}

$$\partial_{\mu} \Psi^{\mu}(x) = \Sigma_{\mathrm{c}} + \sigma_{\mathrm{c}}, \tag{2.181}$$

where σ_{c} is the calortropy production defined by the statistical mechanical formula

$$\sigma_{\mathrm{c}}(x) = k_B \sum_{a=1}^{r} \left\langle \ln \left(\epsilon_a + f_a^{\mathrm{c}-1}\right) \mathfrak{R}_a \left[f_a\right]\right\rangle , \tag{2.182}$$

and Σ_{c} a kinematic term defined by the formula

$$\Sigma_{\mathrm{c}} = k_B \sum_{a=1}^{r} \left\langle p_a^{\mu} \left[f_a \partial_{\mu} \ln \left(1 + \epsilon_a f_a^{\mathrm{c}-1}\right) + \epsilon_a \partial_{\mu} \ln \left(f_a^{\mathrm{c}} + \epsilon_a\right)\right]\right\rangle . \tag{2.183}$$

Substituting (2.179), (2.182), and (2.183) into (2.177) we obtain the calortropy density balance equation

$$\rho D \widehat{\Psi} = -\partial_{\mu} J_{\mathrm{c}}^{\mu} + \Sigma_{\mathrm{c}} + \sigma_{\mathrm{c}}(x) . \tag{2.184}$$

Although reminiscent of the Boltzmann entropy balance equation (2.25), this balance equation is clearly different from it in its mathematical structure, as we have seen in the previous chapter for the case of matter alone. We, in fact, notice that the calortropy balance equation is in the ranks of nonconserved variables such as $\Phi_a^{(q)}$ or $\Phi_r^{(q)}$—see (2.129) or (2.157) for the evolution equation for macroscopic moment $\Phi_a^{(q)}$—because the term Σ_{c} may be regarded as the kinematic term in the evolution equation for $\widehat{\Psi}$, and $\sigma_{\mathrm{c}}(x)$ its dissipation term. Furthermore, $f_a^{\mathrm{c}}(x, p_a)$ is, evidently, not an exact solution of the covariant kinetic equation in the sense that it will be found to be a projection of $f_a(x, p_a)$ onto the thermodynamic manifold characterizing the system at the level of thermodynamic description of the irreversible processes. In fact, f_a^{c} may be qualified for an approximate solution of the kinetic equation, which may be sought as closely to the exact solution f_a as possible. Equation (2.184), together with (2.182) and (2.183), forms the basis of formulating a thermodynamic theory of irreversible processes as will be shown presently.

2.7 Relative Boltzmann Entropy and Its Balance Equation

Since the calortropy four-flow is obviously not the same as the Boltzmann entropy four-flow we may examine their difference

$$S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right](x)=\Psi^{\mu}(x)-S^{\mu}(x), \tag{2.185}$$

which we call the relative Boltzmann entropy. It contains the portion of molecular theoretic information lost by the nonequilibrium canonical form f_a^{c} and will tell us of the extent of information shedding incurred by replacing f_a with f_a^{c} in $\ln f_a$ and $\ln(1+\epsilon_a f_a)$ in the expression for $S^{\mu}(x)$.

On inserting the statistical mechanical formulas for S^{μ} and Ψ^{μ} into (2.185) the statistical mechanical formula for the relative Boltzmann entropy is obtained:

$$S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right](x)=k_B\sum_{a=1}^{r}\left\langle p_a^{\mu}\left[f_a\ln\left(\frac{f_a}{f_a^{\mathrm{c}}}\right)-\epsilon_a(1+\epsilon_a f_a)\ln\left(\frac{1+\epsilon_a f_a}{1+\epsilon_a f_a^{\mathrm{c}}}\right)\right]\right\rangle. \tag{2.186}$$

The scalar relative Boltzmann entropy density is obtained by contracting four-flow $S_{\mathrm{r}}^{\mu}[f|f^{\mathrm{c}}]$ with U_{μ}:

$$\mathcal{S}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=\rho\widehat{\mathcal{S}}_{\mathrm{r}}\left[f|f^{\mathrm{c}}\right]=c^{-2}U_{\mu}S_{\mathrm{r}}^{\mu}\left[f|f^{\mathrm{c}}\right]. \tag{2.187}$$

Its balance equation can be easily found to have the form

$$\rho D\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}]=-\partial_{\mu}J_{\mathrm{r}}^{\mu}[f|f^{\mathrm{c}}]+\Sigma_{\mathrm{c}}(x)+\left[\sigma_{\mathrm{c}}(x)-\sigma_{\mathrm{ent}}(x)\right]. \tag{2.188}$$

with the relative Boltzmann entropy flux $J_{\mathrm{r}}^{\mu}[f|f^{\mathrm{c}}]$ given by the expression

$$\begin{aligned}J_{\mathrm{r}}^{\mu}[f|f^{\mathrm{c}}]&=\Delta_{\nu}^{\mu}S_{\mathrm{r}}^{\nu}[f|f^{\mathrm{c}}]=\Delta_{\nu}^{\mu}\left(\Psi^{\nu}-S^{\nu}\right)\\&=k_B\sum_{a=1}^{r}\Delta_{\nu}^{\mu}\left\langle p_a^{\nu}\left[f_a\ln\left(\frac{f_a}{f_a^{\mathrm{c}}}\right)-\epsilon_a\left(1+\epsilon_a f_a\right)\ln\left(\frac{1+\epsilon_a f_a}{1+\epsilon_a f_a^{\mathrm{c}}}\right)\right]\right\rangle.\end{aligned} \tag{2.189}$$

We will return to examine and further elucidate the formal balance equations for $\widehat{\Psi}$ and $\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}]$ after the nonequilibrium canonical form f_a^{c} and its attendant theory of macroscopic transport processes are more explicitly developed.

2.8 Functional Hypothesis and Nonequilibrium Canonical Form

We have formally derived a complete set of macroscopic evolution equations from the covariant kinetic equation. The formal evolution equations include the conservation laws as well as the evolution equations for nonconserved variables. By making the macroscopic evolution equations to conform to the principles of thermodynamics, it would be possible to formulate a thermodynamically consistent theory of transport processes and hydrodynamics out of the aforementioned evolution equations, if we adopt the procedure taken in the cases of kinetic equations considered in the previous chapters in this work. Thus, we now would like accordingly to formulate the thermodynamics of irreversible processes and generalized hydrodynamics for a system of radiation and matter. This section is devoted to the preparation to achieve the stated aim.

Since we are interested in thermodynamic description of macroscopic transport processes in the system, any solution of the covariant kinetic equation (2.13), regardless of whether exact or approximate, should be formulated to yield a thermodynamic theory of irreversible processes that is consistent with the laws of thermodynamics. In the absence of, and the unlikelihood of acquiring, an exact solution known to produce such a theory, we must be content to explore an approximate solution.[8] We would like to look for such an approximate solution under the functional hypothesis as in the case of nonrelativistic kinetic equations and the covariant kinetic equation for a single-component gas discussed in the previous chapters. At the risk of being repetitive, we state the functional hypothesis for the distribution function below:

Functional Hypothesis: *The distribution function obeying the covariant kinetic equation* (2.13) *evolves as a functional of macroscopic observables spanning the thermodynamic manifold for the system, which obey their own evolution equations descending from the kinetic equation.*

In the functional hypothesis, the aforementioned evolution equations of macroscopic variables serve as field equations through which the evolution of the distribution function is described, since they provide the spacetime dependence of the distribution function. This distribution function denoted by $f_a^{\mathrm{c}}(x, p_a)$, which has already symbolically appeared in calortropy and relative Boltzmann entropy, is called nonequilibrium canonical form. It may be, in fact, regarded as a thermodynamic branch of the distribution $f_a(x, p_a)$ obeying the covariant kinetic equation, since it affords us with thermodynamics of irreversible processes. It should be emphasized that it is not the same as $f_a(x, p_a)$ in general: $f_a^{\mathrm{c}}(x, p_a) \neq f_a(x, p_a)$, except at equilibrium to which the system approaches over a sufficiently long period of time. However, we will require both $f_a(x, p_a)$ and $f_a^{\mathrm{c}}(x, p_a)$ to have the same normalization to the number density. The consequence of this requirement is that

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[8] The solution is approximate in the sense that it is given by a projection of the phase space distribution function onto the thermodynamic manifold of macroscopic variables whose dimension is much smaller than the full phase space of the system.

$$\rho_i = c^{-2}U_\mu \left\langle p_i^\mu f_i \right\rangle = c^{-2}U_\mu \left\langle p_i^\mu f_i^{\mathrm{c}} \right\rangle$$

for both matter species and radiation as was done in the case of matter alone in the previous chapter. This requirement is physically sensible and reasonable.

In the modified moment method [5, 17, 18] used in the present work, $f_a^{\mathrm{c}}(x, p_a)$ is constructed with the nonconserved moments, denoted by tensors $h_a^{(q)\mu\cdots\nu}$, such that the evolution equations of the moments are consistent with the second law of thermodynamics. The relativistic extension of this method for a material particle mixture is presented in Chap. 1 of this Volume. In the present case of radiation and matter the nonequilibrium canonical form assumes distinctive forms for radiation (r) and matter $(a = 1, \ldots, m)$: $f_i^{\mathrm{c}}(x, p_i)$ $(i = 1, \ldots, m, r)$.

2.8.1 Nonequilibrium Canonical Form for Material Particles

The nonequilibrium canonical form of distribution may be taken for the material particles in the form

$$f_a^{\mathrm{c}}(x, p_a) = \left(e^{\beta\mathcal{H}_a(p_a)} - \epsilon_a\right)^{-1} \qquad (a = 1, 2, \ldots, m), \tag{2.190}$$

where $\mathcal{H}_a(p_a)$ is given by the formula

$$\mathcal{H}_a(p_a) = p_a^\mu U_\mu + \sum_{q\geq 1} X_{a\,\alpha\cdots\omega}^{(q)} h_a^{(q)\sigma\cdots\omega} - \mu_a. \tag{2.191}$$

In this expression $\beta := 1/k_BT$, μ_a, $X_{a\,\alpha\cdots\omega}^{(q)}$, are as-yet-undetermined parameters depending on macroscopic variables and spacetime position x. Their precise meanings and x dependence will be determined as the nonequilibrium (irreversible thermodynamic) theory of macroscopic variables is developed under the functional hypothesis. The parameters β, μ_a, $X_{a\,\alpha\cdots\omega}^{(q)}$ must be determined such that the second law of thermodynamics is satisfied, and $h_a^{(q)\sigma\cdots\omega}$ are the molecular moments already introduced, which yield macroscopic moments (fluxes), when averaged over the ensemble with the statistical weight $f_a(x, p_a)$. The set of the macroscopic moments so obtained spans the manifold of macroscopic variables necessary to describe macroscopic transport processes in the system. Such macroscopic variables obey the evolution equations of nonconserved variables presented earlier. Henceforth, whenever convenient, the symbol $\odot$ will be used to abbreviate the scalar product of tensors for the sake of notational brevity. Thus with this symbol we write, for example,

$$X_{a\,\alpha\cdots\omega}^{(q)} h_a^{(q)\sigma\cdots\omega} = X_a^{(q)} \odot h_a^{(q)}$$

for the contraction of tensors.

2.8.2 Nonequilibrium Canonical Form for Radiation

The nonequilibrium canonical form of radiation distribution function is sought after in a form similar to the Planck distribution function—an equilibrium distribution function of radiation. Thus, we take it in the form

$$f_r^{\mathrm{c}}(x, p_r) = \left(e^{\beta \overline{W}_r(p_r)} - 1\right)^{-1}. \tag{2.192}$$

The function $\overline{W}_r(p_r)$ in the exponent, as yet unspecified, will be also sought after in such a way that it is thermodynamically consistent as is the distribution function for matter $f_a^{\mathrm{c}}(x, p_a)$. This function is sought in a dimensionless form.

We now look for $\overline{W}_r$ in terms of q_r^ν, the dimensionless momentum four-vector defined earlier; see (2.143). It is convenient to define

$$W_r = \beta \overline{W}_r\left(q_r\right). \tag{2.193}$$

In view of the well known equilibrium radiation (Planck) distribution function, which has been obtained as an equilibrium solution of the covariant kinetic equation, and the exponential form for $f_a\left(x, p_a\right)$ in (2.190), it is sensible to take $\overline{W}_r\left(q_r\right)$ in the form

$$\overline{W}_r\left(q_r\right) = p_r^\mu U_\mu + H_r^{(1)}\left(q_r\right) - \mu_r, \tag{2.194}$$

where μ_r is found to be the chemical potential and $H_r^{(1)}\left(q_r\right)$ is the nonequilibrium part that can be determined in a way analogous to the one taken for the matter distribution function f_a^{c} in (2.190). That is, it is made up of nonequilibrium contributions in the form

$$H_r^{(1)}\left(q_r\right) = \sum_{q \geq 1} X_r^{(q)} \odot h_r^{(q)}\left(q_r\right), \tag{2.195}$$

where $h_r^{(q)}$ are orthogonal tensor polynomials of $p_r^\nu = q_r^\nu / \left(c\beta\right)$, which are also orthogonal to each other and to the conserved moments for radiation. The generalized potentials $X_r^{(q)}$ are the functions of macroscopic variables such as the radiation shear stress, the excess trace part of the radiation stress tensor, radiation heat flux, etc. as well as β and μ_r. The nonequilibrium contribution $\beta H_r^{(1)}$ thus can be expressed as a dimensionless function of the reduced four-vector q_r^ν and dimensionless variables.

In the case of radiation and matter, since the photons are put on equal footing with matter as far as the kinetic processes are concerned and the whole system is considered a mixture of photons and material particles, it is reasonable to define a single temperature for the mixture as has been found to be the case for the equilibrium system considered earlier. The fundamental reason for this point of view is that the photons do not interact with each other and hence cannot come to equilibrium on their own without a help from matter—e.g., material gas. This is the basic reason why the same parameter β, which will turn out to be related to the temperature, is used for both material gases and radiation in the nonequilibrium canonical forms. Consistent with this viewpoint, we may define the temperature by the statistical mechanical formula for the system[9]

$$\frac{3}{2}\rho_t k_B T = \sum_{a=i}^{r} \frac{1}{2} \left\langle \left[\left(p_a^\nu U_\nu \right)^2 - m_a^2 c^4 \right] f_a \left(x, p_a \right) \right\rangle, \tag{2.196}$$

where $m_r = 0$ in the case of photon species and

$$\rho_t = \rho + \frac{2a_{\mathrm{SB}} c^2 T^3}{k_B} \quad (a_{\mathrm{SB}} = \text{Stefan–Boltzmann constant})\,. \tag{2.197}$$

This definition of temperature is reasonable, especially, for the present dilute gaseous system. It is basically rooted in equipartition law of energy by Tolman [19], which was later discussed by Landsberg [20] and by ter Haar and Wergeland [21]. The second term on the right-hand side (2.196) arises from the recognition that the photon number depends on temperature, but also is not conserved. Writing the nonequilibrium photon number density in such a form means that the temperature is determined such that (2.197) holds for nonequilibrium. Equation (2.196) can be recast into the form

$$\frac{3}{2}\rho_t k_B T = \sum_{a=1}^{r} \Delta_{\mu\nu} \left\langle p_a^\nu p_a^\mu f_a(x, p_a) \right\rangle = \sum_{a=1}^{r} \Delta_{\mu\nu} T^{\nu\mu}. \tag{2.198}$$

The nonequilibrium canonical forms (2.190) and (2.192) involve parameters β, ρ, ρ_r, and $\mathfrak{p}$, which have been defined statistically, but their operational meanings as thermodynamic variables are not as yet fixed. Their phenomenological meanings can be gained only through corresponding the statistically derived macroscopic variables and their relations to the phenomenological thermodynamic counterparts. This procedure is facilitated by deriving the statistical mechanical relations of macroscopic variables, which are consistent with the laws of thermodynamics. Therefore it is now necessary to formulate a theory of thermodynamics of irreversible processes in the system of radiation and matter. To this end we now closely examine the mathematical aspects of calortropy and the related in the light of the laws of thermodynamics for irreversible processes in the system. The approach to this goal is along the same line as for Chap. 1 of Volume 2 and also for the nonrelativistic kinetic theory discussed in Volume 1 of this work.

2.8.3 Calortropy of Matter and Radiation

The calortropy density $\Psi(x)$ of radiation and matter defined by (2.176) can be rearranged to the formula

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[9] This is also for the case of matter consisting of a gas.

$$\begin{aligned}\Psi(x) &:= \rho\widehat{\Psi}(x)\\ &= k_B\sum_{a=1}^{r} c^{-2}U_\mu\left\langle p_a^\mu\left[f_a\ln\left(\epsilon_a+f_a^{\mathrm{c}-1}\right)+\epsilon_a\ln\left(1+\epsilon_a f_a^{\mathrm{c}}\right)\right]\right\rangle.\end{aligned} \tag{2.199}$$

From this formula we obtain the statistical mechanical expressions for calortropy density of matter and radiation separately

$$\Psi_{\mathrm{m}}(x) = k_B\sum_{a=1}^{m} c^{-2}U_\mu\left\langle p_a^\mu\left[f_a\ln\left(\epsilon_a+f_a^{\mathrm{c}-1}\right)+\epsilon_a\ln\left(1+\epsilon_a f_a^{\mathrm{c}}\right)\right]\right\rangle, \tag{2.200}$$

$$\Psi_r(x) = k_B c^{-2}U_\mu\left\langle p_r^\mu\left[f_a\ln\left(\epsilon_r+f_r^{\mathrm{c}-1}\right)+\epsilon_r\ln\left(1+\epsilon_r f_r^{\mathrm{c}}\right)\right]\right\rangle. \tag{2.201}$$

Upon inserting f_a^{c} and f_r^{c} in (2.190) and (2.192), respectively, and making use of definitions of energy densities of matter and radiation, we obtain Ψ_{m} and Ψ_r:

$$T\widehat{\Psi}_{\mathrm{m}}(x) = \mathcal{E}_{\mathrm{m}} + \mathfrak{p}_{\mathrm{m}}v_{\mathrm{m}} - \sum_{a=1}^{m}\mu_a\mathfrak{c}_a + \sum_{a=1}^{m}\sum_{q\geq 1} X_{a\,\mu\nu\cdots\sigma}^{(q)}\widehat{\Phi}_a^{(q)\sigma\cdots\mu\nu}, \tag{2.202}$$

$$T\widehat{\Psi}_r(x) = \mathcal{E}_r + \mathfrak{p}_r v_r - \mu_r + \sum_{q\geq 1} X_{r\,\mu\nu\cdots\sigma}^{(q)}\widehat{\Phi}_r^{(q)\sigma\cdots\mu\nu}, \tag{2.203}$$

where pressures are defined for matter and radiation, respectively, by the statistical mechanical formulas

$$\beta\mathfrak{p}_{\mathrm{m}} = \sum_{a=1}^{m}\epsilon_a\left\langle c^{-2}U_\mu p_a^\mu\ln\left(1+\epsilon_a f_a^{\mathrm{c}}\right)\right\rangle, \tag{2.204}$$

$$\beta\mathfrak{p}_r = \epsilon_r\left\langle c^{-2}U_\mu p_r^\mu\ln\left(1+\epsilon_r f_r^{\mathrm{c}}\right)\right\rangle, \tag{2.205}$$

and $v_{\mathrm{m}} = 1/\rho_{\mathrm{m}}$, $v_r = 1/\rho_r$, and we have set

$$\beta = 1/k_B T. \tag{2.206}$$

It should be noted that (2.204) and (2.205) are necessary and sufficient conditions for $\widehat{\Psi}_{\mathrm{m}}$ and $\widehat{\Psi}_r$ to be bilinear forms of macroscopic variables in manifold $\mathfrak{P}\cup\mathfrak{T}$. Thus combining (2.202) and (2.203) we obtain the bilinear form for calortropy density of the system:

$$T\widehat{\Psi}(x) = \mathcal{E} + \mathfrak{p}v - \sum_{a=1}^{r}\mu_a\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1} X_{a\,\nu\cdots\mu\sigma}^{(q)}\widehat{\Phi}_a^{(q)\sigma\mu\cdots\nu} \tag{2.207}$$

with

$$\rho\mathcal{E} = \rho_{\mathrm{m}}\mathcal{E}_{\mathrm{m}} + \rho_r\mathcal{E}_r = \sum_{a=1}^{r}\rho_a\mathcal{E}_a \quad or \quad \mathcal{E} = \sum_{a=1}^{r}\mathfrak{c}_a\mathcal{E}_a. \tag{2.208}$$

The physical meanings of parameters T, $\mathfrak{p}$, μ_a, and $X_a^{(q)}$ will be presently elucidated by making thermodynamic correspondence of statistical mechanical result for $\widehat{\Psi}$ with the phenomenological (i.e., thermodynamic) calortropy density.

2.9 Calortropy Differential

The conserved variable balance equations and the flux (moment) evolution equations for radiation and matter derived in the previous sections must be subjected to the demands of the second law of thermodynamics, so that the theory of irreversible processes based them becomes consistent with the second law of thermodynamics. To achieve this aim the calortropy balance equation must be shown integrable in the thermodynamic manifold $\mathfrak{P}\cup\mathfrak{T}$.

2.9.1 Pfaffian Form for Calortropy

To prove the integrability of the calortropy density balance equation it is necessary to show the right hand side of (2.184) must be expressible in an integrable differential form in the thermodynamic manifold. If f_a^{c} is substituted into the $\ln f_a^{\mathrm{c}}$ term in (2.180), the divergence term in question is given by the expression

$$\begin{aligned}
\partial_\mu\left(\Psi^\nu\Delta_\nu^\mu\right) = & -k_B\sum_{a=1}^{r}\beta\partial_\mu Q_a^\mu - k_B\sum_{a=1}^{r}Q_a^\mu\partial_\mu\beta \\
& + k_B\sum_{a=1}^{r}\beta X_{a\,\alpha\cdots\sigma}^{(q)}\partial_\mu\left(\Delta_\nu^\mu\psi_a^{(q)\sigma\cdots\alpha\nu}\right) \\
& + k_B\sum_{a=1}^{r}\Delta_\nu^\mu\psi_a^{(q)\sigma\cdots\alpha\nu}\partial_\mu\left(\beta X_{a\,\alpha\cdots\sigma}^{(q)}\right) \\
& + k_B\sum_{a=1}^{r}\partial_\mu\Delta_\nu^\mu\left\langle\epsilon_a p_a^\nu\ln\left(1+\epsilon_a f_a^{\mathrm{c}}\right)\right\rangle.
\end{aligned} \tag{2.209}$$

On the other hand, the kinematic term Σ_{c} of the calortropy balance equation is expressible by the form

$$\begin{aligned}\Sigma_{\mathrm{c}} = {} & T^{\mu\nu}U_{\nu}\partial_{\mu}T^{-1} + \frac{1}{T}T^{\mu\nu}\partial_{\mu}U_{\nu} + \sum_{a=1}^{r}\sum_{q\geq 1}\psi_{a}^{(q)\mu}\odot\partial_{\mu}\left(\frac{X_{a}^{(q)}}{T}\right) \\ & + \sum_{a=1}^{r}\sum_{q\geq 1}\frac{X_{a}^{(q)}}{T}\odot\left\langle f_{a}\partial_{\mu}p_{a}^{\mu}h_{a}^{(q)}\right\rangle - \sum_{a=1}^{r}\partial_{\mu}\left(\frac{\mu_{a}}{T}N_{a}^{\mu}\right) \\ & + k_{B}\sum_{a=1}^{r}\epsilon_{a}\left\langle p_{a}^{\mu}\partial_{\mu}\ln\left(1+\epsilon_{a}f_{a}^{\mathrm{c}}\right)\right\rangle \end{aligned} \tag{2.210}$$

and the calortropy production σ_{c} by a linear combination of dissipation terms $\Lambda_{a}^{(q)}$ of the nonconserved variable evolution equations:

$$\sigma_{\mathrm{c}} = T^{-1}\sum_{a=1}^{r}\sum_{q\geq 1}X_{a}^{(q)}\odot\Lambda_{a}^{(q)}. \tag{2.211}$$

Upon substituting the three components (2.209)–(2.211) into the calortropy density balance equation, we obtain the calortropy density differential

$$\begin{aligned} D\widehat{\Psi} = {} & T^{-1}\left(D\mathcal{E} + \mathfrak{p}Dv - \sum_{a=1}^{r}\mu_{a}D\mathfrak{c}_{a} + \sum_{a=1}^{r}\sum_{q\geq 1}X_{a}^{(q)}\odot D\widehat{\Phi}_{a}^{(q)}\right) \\ & + \mathcal{E}DT^{-1} + vD\left(\mathfrak{p}T^{-1}\right) - \sum_{a=1}^{r}\mathfrak{c}_{a}D\left(\frac{\mu_{a}}{T}\right) + \sum_{a=1}^{r}\sum_{q\geq 1}\widehat{\Phi}_{a}^{(q)}\odot D\left(\frac{X_{a}^{(q)}}{T}\right), \end{aligned} \tag{2.212}$$

if

$$\mathfrak{p} = \mathfrak{p}_{\mathrm{m}} + \mathfrak{p}_{r} \tag{2.213}$$

and $\mathfrak{p}_{\mathrm{m}}$ and $\mathfrak{p}_{r}$ are defined by (2.204) and (2.205), respectively. Therefore, the definitions of macroscopic parameters $\mathfrak{p}_{\mathrm{m}}$ and $\mathfrak{p}_{r}$ by the formulas (2.204) and (2.205) are necessary and sufficient conditions for (2.212) to exist. The physical meanings of parameters $\mathfrak{p}_{\mathrm{m}}$ and $\mathfrak{p}_{r}$ remain as yet not elucidated except that they are macroscopic parameters. Similarly, the meanings of parameter T, μ_{a}, and $X_{a}^{(q)}$ also remain undetermined as macroscopic variables. In any case, to obtain this Pfaffian differential form [22] (2.212) we have made use of the energy balance equations for matter and radiation, and the identities listed below:

$$\begin{aligned}\partial_{\mu}\left(\frac{\mu_{a}}{T}N_{a}^{\mu}\right) &= \partial_{\mu}\left(\frac{\mu_{a}}{T}\mathfrak{c}_{a}\rho U^{\mu}\right) \\ &= \rho U^{\mu}\partial_{\mu}\left(\frac{\mu_{a}}{T}\mathfrak{c}_{a}\right) + \frac{\mu_{a}}{T}\mathfrak{c}_{a}\partial_{\mu}\left(\rho U^{\mu}\right) \\ &= \rho D\left(\beta\mu_{a}\mathfrak{c}_{a}\right), \end{aligned} \tag{2.214}$$

$$\partial_\mu \left(\rho U^\mu\right) = D\rho + \rho \nabla_\mu U^\mu = 0, \tag{2.215}$$

and

$$\begin{aligned}
\partial_\mu \left(\frac{\mathfrak{p}v}{T}\rho U^\mu\right) &= \frac{\mathfrak{p}v}{T}\partial_\mu \left(\rho U^\mu\right) + \rho U^\mu \partial_\mu \left(\frac{\mathfrak{p}v}{T}\right) \\
&= \rho U^\mu \partial_\mu \left(\frac{\mathfrak{p}v}{T}\right) \\
&= \rho D \left(\frac{\mathfrak{p}v}{T}\right) \\
&= \frac{\mathfrak{p}}{T} Dv + \rho v D \left(\frac{\mathfrak{p}}{T}\right).
\end{aligned} \tag{2.216}$$

The second line of the Pfaffian differential form (2.212) vanishes because if the normalization conditions for the nonequilibrium canonical forms are varied with respect to the parameters β, μ_a, $X_a^{(q)}$, there follows the vanishing differential form

$$\mathcal{E} DT^{-1} + vD\left(\mathfrak{p}T^{-1}\right) - \sum_{a=1}^{r} \mathfrak{c}_a D\left(\frac{\mu_a}{T}\right) + \sum_{a=1}^{r}\sum_{q\geq 1} \widehat{\Phi}_a^{(q)} \odot D\left(\frac{X_a^{(q)}}{T}\right) = 0. \tag{2.217}$$

This differential form will turn out to be the nonequilibrium Gibbs–Duhem (NGD) equation, when the parameters T^{-1}, $\mathfrak{p}$, μ_a, and $X_a^{(q)}$ are identified with their thermodynamic correspondents. Thus, by virtue of (2.217) the Pfaffian differential form (2.212) reduces to the differential form for $\widehat{\Psi}$

$$D\widehat{\Psi} = T^{-1}\left(D\mathcal{E} + \mathfrak{p}Dv - \sum_{a=1}^{r} \mu_a D\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1} X_a^{(q)} \odot D\widehat{\Phi}_a^{(q)}\right), \tag{2.218}$$

which is known as the extended Gibbs relation for $\widehat{\Psi}$. This differential form is an exact differential in the thermodynamic manifold, because, when the nonequilibrium Gibbs–Duhem equation is added side by side to it, (2.218) integrates exactly to a bilinear form in the nonequilibrium Gibbs manifold $\mathfrak{G} = \mathfrak{P} \cup \mathfrak{T} \cup \Psi$ for the calortropy density given in (2.207). For this reason the NGD equation (2.217) becomes the necessary and sufficient condition for the integrability of the extended Gibbs relation (2.218). The differential form (2.218) for $\widehat{\Psi}$ is in the same form as for relativistic gas mixture in the absence of radiation obtained in Chap. 1 of this Volume. This is in contrast to the Boltzmann entropy differential $D\mathcal{S}$, which, unless the system is in equilibrium, cannot be put into an exact Pfaffian differential in the Gibbs manifold, and it gives rise to the conclusion that the extended Gibbs relation does not hold for $\mathcal{S}$ and neither is there the nonequilibrium Gibbs–Duhem equation for $\mathcal{S}$.

When the parameters T, $\mathfrak{p}$, μ_a, and $X_a^{(q)}$ are elucidated of their physical (i.e., thermodynamic) meanings on correspondence with the phenomenological thermodynamics, the pair of differential forms, namely, the extended Gibbs relation and the nonequilibrium Gibbs–Duhem equation, provides the foundation of thermodynamics of irreversible processes for the system of matter and radiation. The following pair of vanishing circular integrals reinforces the aforementioned statement.

The Pfaffian differential form (2.218) for $\widehat{\Psi}$ gives rise to a vanishing circular (i.e., cyclic) integral when integrated over a path of an irreversible process in the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. In a local rest frame it may be written as [5]

$$\oint d\widehat{\Psi} = 0. \tag{2.219}$$

It is a representation of the second law of thermodynamics in an integral form in manifold $\mathfrak{P} \cup \mathfrak{T}$. And in this sense the thermodynamic theory of irreversible processes described in terms of generalized hydrodynamic equations in the Gibbs manifold $\mathfrak{G}$ is consistent with the laws of thermodynamics. In addition to this vanishing integral for the second law, there also holds a vanishing integral of internal energy differential in manifold $\mathfrak{P} \cup \mathfrak{T}$, representing the first law of thermodynamics

$$\oint d\mathcal{E} = 0. \tag{2.220}$$

We have shown that this is the case in the nonrelativistic formalism in the nonrelativistic theory chapters of Volume 1. The same conclusion robustly holds up even if the system is relativistic and away from equilibrium: *The thermodynamic laws thus may be represented by a pair of vanishing circular integrals over an irreversible path in the thermodynamic manifold.*

Further progress in the theory of transport processes in the system can be made if the unknowns $X_a^{(q)}$ are suitably approximated in a thermodynamically consistent manner, such that $D\widehat{\Psi}$ remains an exact Pfaffian differential in the Gibbs manifold $\mathfrak{G} = \mathfrak{P} \cup \mathfrak{T} \cup \Psi$ and the theory of macroscopic irreversible processes remains thereby thermodynamically consistent.

2.9.2 Thermodynamic Correspondence

The extended Gibbs relation (2.218)—a Pfaffian differential form—contains as-yet-undetermined parameters $T, \mathfrak{p}, \mu_a, X_a^{(q)}$, which must be elucidated of their operational (physical) meanings, so that it becomes a physically and practically useful equation. This objective is accomplished if we invoke the phenomenological irreversible thermodynamic extended Gibbs relation derived from the laws of thermodynamics and make correspondence of the statistical mechanical extended Gibbs relation (2.218) with the phenomenological one, so that the undetermined parameters therein are determined in correspondence to those in the phenomenological relations. That is, if we make the following correspondence between theoretical and thermodynamic observables

$$\begin{aligned}&\widehat{\Psi}|_{\mathrm{st}} \Leftrightarrow \widehat{\Psi}|_{\mathrm{th}}, \quad \mathcal{E}|_{\mathrm{st}} \Leftrightarrow \mathcal{E}|_{\mathrm{th}}, \quad v|_{\mathrm{st}} \Leftrightarrow v|_{\mathrm{th}}, \\ &\mathfrak{c}_a|_{\mathrm{st}} \Leftrightarrow \mathfrak{c}_a|_{\mathrm{th}}, \quad \widehat{\Phi}_a^{(q)}|_{\mathrm{st}} \Leftrightarrow \widehat{\Phi}_a^{(q)}|_{\mathrm{th}} \\ &\qquad (a = 1, \ldots, m, r)\,, \end{aligned} \tag{2.221}$$

then there would hold the correspondence between the theoretical parameters $\left(T, \mathfrak{p}, \mu_a, X_a^{(q)}\right)$ and the thermodynamic parameters appearing in the phenomenological extended Gibbs relation:

$$T|_{\mathrm{st}} \Leftrightarrow T|_{\mathrm{th}}, \quad \mathfrak{p}|_{\mathrm{st}} \Leftrightarrow \mathfrak{p}|_{\mathrm{th}}, \quad \mu_a|_{\mathrm{st}} \Leftrightarrow \mu_a|_{\mathrm{th}}, \quad X_a^{(q)}|_{\mathrm{st}} \Leftrightarrow X_a^{(q)}|_{\mathrm{th}}. \tag{2.222}$$

In (2.221) and (2.222) the subscripts st and th mean statistical mechanical and thermodynamic, respectively. These correspondences between theoretical (statistical mechanical) and thermodynamic (phenomenological) parameters fully endow the thermodynamic meanings on the corresponded parameters, and the nonequilibrium distribution functions are grounded on the thermodynamic principles and experiments. In the relativistic theory the correspondence presented relies on a conjecture that the phenomenological extended Gibbs relation is experimentally realizable, because the extended Gibbs relation is not as yet established experimentally. However, we may reason that the relativistic extended Gibbs relation has the nonrelativistic limit as $u/c \to 0$. And the validity of the nonrelativistic limit is sufficiently well demonstrated by numerous applications in hydrodynamics as discussed in Volume 1 of this work. Therefore the relativistic extension of extension of irreversible thermodynamics is plausible and reasonable.

2.9.3 Nonequilibrium Statistical Thermodynamics for Relativistic Quantum Gases

The formal theory based on the covariant kinetic equation and the nonequilibrium canonical form taken for the distribution function under the functional hypothesis have enabled us to formulate a formal theory of hydrodynamics and a formal theory of thermodynamics of irreversible processes in support of hydrodynamics for the system removed from equilibrium at arbitrary degree. These macroscopic theories must be provided with a molecular theory for various macroscopic observables on which the theories are built. We have seen in the case of nonrelativistic theories discussed in Volume 1 of this work that the stated goal is achieved if nonequilibrium statistical thermodynamics is formulated on the basis of nonequilibrium canonical form and the extended Gibbs relation descending therefrom, since the equilibrium theory is then seamlessly recovered from the nonequilibrium theory as the system tends to equilibrium. In this section we show that the stated goal can be achieved for relativistic quantum dilute gases with the help of the extended Gibbs relation (2.218) if nonequilibrium statistical thermodynamics is formulated for the gases. If we review the theory of a nonrelativistic noninteracting gas mixture presented in kinetic theory chapters of Volume 1 and the relativistic theory of relativistic gases in Chap. 1 of this Volume and compare them, it is clear how we may proceed in formulating the theory in the case of the system of matter and radiation, bearing in mind that we are dealing with quantum gas particles.

First, let us have a quick review of how equilibrium statistical thermodynamics is formulated for a system consisting of a mixture of quantum ideal gases. If the system consists of a mixture of noninteracting (uncorrelated) gases of N particles ($N = N_1 + N_2 + \cdots + N_r$), the total equilibrium density matrix—an equilibrium grand canonical form—is given by the formula

$$F_{\mathrm{e}}^{(N)} = \exp\left(-\beta\mathcal{H}^{(N)} - \ln\Gamma^{\mathrm{e}}\right), \tag{2.223}$$

where Γ^{e} is the local grand partition function

$$\Gamma^{\mathrm{e}} = \mathrm{Tr}\prod_{a=1}^{r}\prod_{j}\exp\left(-\beta\mathcal{H}_j^{(1)}\right) \tag{2.224}$$

with the Trace operation denoting integration over the entire N particle momentum space, and $\mathcal{H}_j^{(1)}$ are the single particle energy operators

$$\mathcal{H}_j^{(1)} = \varepsilon_{aj} - \mu_a \tag{2.225}$$

with ε_{aj} and μ_a denoting the jth energy eigenvalue and chemical potential of species a, respectively. If the tracing operation is performed according to the Pauli exclusion principle with regard to fermions and bosons, Γ^{e} can be written in the following products of Fermi–Dirac (FD) and Bose–Einstein contributions:

$$\Gamma^{\mathrm{e}} = \mathrm{Tr}\prod_{a\in FD}^{r}\prod_{j}\left(1 + e^{-\beta\left(\varepsilon_{ja}-\mu_a\right)}\right) \times \prod_{a\in BE}^{r}\prod_{j}\left(1 - e^{-\beta\left(\varepsilon_{ja}-\mu_a\right)}\right)^{-1}. \tag{2.226}$$

Since the distribution functions normalize to density, in relativistic theory it is necessary to insert the factor $c^{-2}U_\mu p_a^\mu$ and trace with the integrals over momentum spaces of particles as follows:

$$\mathrm{Tr}A \Rightarrow \left\langle c^{-2}U_\mu p_a^\mu A\right\rangle,$$

the angular brackets denoting integration over the momentum space of particle species a, weighted with equilibrium distribution function. For ideal (noninteracting) gases, Γ^{e} can then be written as

$$\begin{aligned}\ln\Gamma^{\mathrm{e}} &= \sum_{a\in FD}\left\langle c^{-2}U_\mu p_a^\mu \ln\left(1+e^{-\beta\mathcal{H}_a^{(1)}}\right)\right\rangle \\ &\quad + \sum_{a\in BE}\left\langle c^{-2}U_\mu p_a^\mu \ln\left(1-e^{-\beta\mathcal{H}_a^{(1)}}\right)^{-1}\right\rangle \end{aligned} \tag{2.227}$$

in the limit of treating the discrete sum as an integral. Here $\mathcal{H}_a^{(1)}$ now stands for $\mathcal{H}_a^{(1)} = p_a^\nu U_\nu - \mu_a$ in the case of relativistic gases.

If the nonequilibrium canonical form f_a^{c} is used, the exponential factors in (2.227) can be written as

$$1+e^{-\beta\mathcal{H}_a^{(1)}} = \frac{1}{1+\epsilon_a f_a^{\mathrm{e}}} \quad (\epsilon_a = -1 \text{ for fermions; } \epsilon_a = +1 \text{ for bosons})\,.$$

Hence, it is now possible to write $\ln\Gamma^{\mathrm{e}}$ in a simple unified form

$$\ln\Gamma^{\mathrm{e}} = \sum_{a=1}^{r}\left\langle c^{-2}U_\mu p_a^\mu \epsilon_a \ln\left(1+\epsilon_a f_a^{\mathrm{e}}\right)\right\rangle, \tag{2.228}$$

where the index a stands for fermion or boson species. Therefore if we define the global form of $\ln\Gamma^{\mathrm{e}}$ by the relation

$$\ln\Xi_{\mathrm{e}} = \int_V d\mathbf{r}\rho\sum_{a=1}^{r}\ln\Gamma_a^{\mathrm{e}}, \tag{2.229}$$

we find

$$\begin{aligned} k_B T\ln\Xi_{\mathrm{e}} &= \mathfrak{p}^{\mathrm{e}}V \\ &= \int_V d\mathbf{r}\sum_{a=1}^{r}\left\langle c^{-2}U_\mu p_a^\mu \epsilon_a \ln\left(1+\epsilon_a f_a^{\mathrm{e}}\right)\right\rangle, \end{aligned}$$

and hence

$$\mathfrak{p}^{\mathrm{e}}v = k_B T\sum_{a=1}^{r}\ln\Gamma_{\mathrm{e}a}^{(N)} = k_B T\sum_{a=1}^{r}\left\langle c^{-2}U_\mu p_a^\mu \epsilon_a \ln\left(1+\epsilon_a f_a^{\mathrm{e}}\right)\right\rangle \tag{2.230}$$

for the statistical mechanical expression for relativistic equilibrium pressure. This confirms (2.204) and (2.205), which have turned out to be statistical mechanical formulas for pressures of matter and radiation, respectively, from the standpoint of the ensemble theory of statistical mechanics. The distribution function (2.223) also gives rise to the equilibrium Gibbs relation and the accompanying theory of equilibrium statistical thermodynamics.

In the case of nonequilibrium relativistic ideal gases, in view of the fact that the nonequilibrium canonical form f_a^{c} taken under the functional hypothesis gives rise to the nonequilibrium Gibbs relation, which also tends to the equilibrium Gibbs relation, the nonequilibrium grand canonical form for the N noninteracting quantum particles may be sought for in the form

$$F_{\mathrm{c}}^{(N)} = \exp\left(-\beta\mathcal{H}^{(N)} - \ln\Gamma\right), \tag{2.231}$$

where $\mathcal{H}^{(N)}$ now contains nonequilibrium contributions

$$\begin{aligned} \mathcal{H}^{(N)} &= \sum_{a=FD \text{ or } BE} \sum_{j} \mathcal{H}_{aj}^{(1)} \\ &= \sum_{a=FD \text{ or } BE} \sum_{j} \left(p_{aj}^{\nu} U_{\nu} + \sum_{q\geq 1} X_{a\alpha\cdots\omega}^{(q)} h_{aj}^{(q)} - \mu_a \right) \end{aligned} \tag{2.232}$$

in the notation already defined earlier. Here FD and BE stand for Fermi–Dirac and Bose–Einstein particles. In (2.231) Γ is the local nonequilibrium grand partition function

$$\Gamma = \mathrm{Tr}\prod_{a=1}^{r}\prod_{j} \exp\left(\beta\mathcal{H}_{aj}^{(1)}\right) \tag{2.233}$$

with the trace operation denoting integration over all N particle momentum space. If tracing operation is performed according to the Pauli exclusion principle applied to Fermi–Dirac (FD) and Bose–Einstein particles, $\Gamma^{(N)}$ can be written in the following products of fermion and boson contributions:

$$\Gamma = \mathrm{Tr}\prod_{a\in FD}^{r}\prod_{j}\left(1 + e^{-\beta\mathcal{H}_{aj}^{(1)}}\right) \times \prod_{a\in BE}^{r}\prod_{j}\left(1 - e^{-\beta\mathcal{H}_{aj}^{(1)}}\right)^{-1}. \tag{2.234}$$

Since the distribution functions normalize to density, in the relativistic theory formulation it is necessary to insert the factor $c^{-2}U_{\mu}p_a^{\mu}$ and to replace trace with the integrals over momentum spaces of particles as follows:

$$\mathrm{Tr}A \Rightarrow \left\langle c^{-2}U_{\mu}p_a^{\mu}A\right\rangle$$

the angular brackets denoting integration over momentum space of particle species a. Thus we have

$$\begin{aligned} \ln\Gamma = &\sum_{a\in FD}\left\langle c^{-2}p_a^{\mu}U_{\mu}\ln\left(1 + e^{-\beta\mathcal{H}_{aj}^{(1)}}\right)\right\rangle \\ &+ \sum_{a\in BE}\left\langle c^{-2}p_a^{\mu}U_{\mu}\ln\left(1 - e^{-\beta\mathcal{H}_{aj}^{(1)}}\right)^{-1}\right\rangle \end{aligned} \tag{2.235}$$

in the limit of treating the discrete sum as an integral.

If the nonequilibrium canonical form f_a^{c} is represented by the form

$$f_a^{\mathrm{c}} = \frac{1}{e^{\beta \mathcal{H}_{aj}^{(1)}} - \epsilon_a}, \tag{2.236}$$

where

$$\epsilon_a = \begin{cases} -1 & \text{for fermions} \\ +1 & \text{for bosons} \end{cases},$$

then $\ln \Gamma$ in (2.235) can be written in a simple unified form

$$\ln \Gamma = \sum_{a=1}^{r} \epsilon_a \left\langle c^{-2} p_a^{\mu} U_{\mu} \ln \left(1 + \epsilon_a f_a^{\mathrm{c}}\right)\right\rangle, \tag{2.237}$$

where the index a stands for both fermion species or boson species. Therefore if the global form of $\ln \Gamma$ defined by relation

$$\ln \Xi = \int_V d\mathbf{r}\, \rho \sum_{a=1}^{r} \ln \Gamma_a \tag{2.238}$$

we find the equation of state for nonequilibrium relativistic pressure $\mathfrak{p}$ is given by the formula

$$\begin{aligned} k_B T \ln \Xi &= \mathfrak{p} V \\ &= \int_V d\mathbf{r} \sum_{a=1}^{r} \epsilon_a \left\langle c^{-2} p_a^{\mu} U_{\mu} \ln \left(1 + \epsilon_a f_a^{\mathrm{c}}\right)\right\rangle . \end{aligned} \tag{2.239}$$

It should be noted that the thermodynamic correspondence has been already made between the statistical thermodynamic parameters and thermodynamic parameters and, consequently, the thermodynamic meanings of various parameters have been established. On the basis of the thermodynamic correspondence we are identifying $\mathfrak{p}$ with the nonequilibrium thermodynamic pressure.

If the Legendre transformation

$$\mathfrak{p} v = \sum_{a=1}^{r} \mu_a \mathfrak{c}_a + T \widehat{\Psi} - \sum_{a=1}^{r} \sum_{q \geq 1} X_{a\,\alpha\cdots\sigma}^{(q)} D \widehat{\Phi}_a^{(q)\sigma\cdots\alpha} - \mathcal{E} \tag{2.240}$$

is made, then $\mathfrak{p} v$ may be regarded as a thermodynamic potential, so that the extended Gibbs relation (2.218)—the differential form—for $\widehat{\Psi}$ is transformed to the exact differential form for the thermodynamic potential $\mathfrak{p} v$:

$$D\left(\mathfrak{p}v\right)=\widehat{\Psi}DT+\mathfrak{p}Dv+\sum_{a=1}^{r}\mathfrak{c}_{a}D\mu_{a}-\sum_{a=1}^{r}\sum_{q\geq1}\widehat{\Phi}_{a}^{(q)}\odot DX_{a}^{(q)}. \tag{2.241}$$

Making use of this differential form for thermodynamic potential $\mathfrak{p}v$ or its statistical mechanical expression (grand partition function), which is an extended Gibbs relation (2.241), we obtain the thermodynamic variables in the thermodynamic manifold in the local rest frame in terms of derivatives of $\ln\Gamma$—namely, nonequilibrium partition function—as below:

$$\widehat{\Psi}=\left(\frac{\partial k_{B}T\ln\Gamma}{\partial T}\right)_{v,\mu,X}=k_{B}\ln\Gamma+k_{B}T\left(\frac{\partial\ln\Gamma}{\partial T}\right)_{v,\mu,X}, \tag{2.242}$$

$$\mathfrak{p}=k_{B}T\left(\frac{\partial\ln\Gamma}{\partial v}\right)_{T,\mu,X}, \tag{2.243}$$

$$\mathfrak{c}_{a}=k_{B}T\left(\frac{\partial\ln\Gamma}{\partial\mu_{a}}\right)_{T,v,\mu',X}, \tag{2.244}$$

$$\widehat{\Phi}_{a}^{(q)}=-k_{B}T\left(\frac{\partial\ln\Gamma}{\partial X_{a}^{(q)}}\right)_{T,v,\mu,X'}\quad\left(a=1,\ldots,m,r\right). \tag{2.245}$$

Together with the differential form (2.241), the relations (2.242)–(2.245) enable us to compute the variables $\widehat{\Psi}$, $\mathfrak{p}$, $\mathfrak{c}_a$, $\widehat{\Phi}_{a}^{(q)}$ and any other macroscopic variables of the Gibbs manifold $\mathfrak{G}$ from the knowledge of the nonequilibrium grand partition function Γ. The aforementioned derivative set, together with the extended Gibbs relation for $D\left(\mathfrak{p}v\right)$, therefore forms the foundation of the nonequilibrium statistical thermodynamics of a relativistic gas mixture of matter and radiation, since by utilizing relations (2.230)–(2.244) and Legendre transformations of variables in the thermodynamic manifold as well as accompanying Maxwell's relations between derivatives [23], it is possible to calculate any variables belonging to the Gibbs manifold in terms of the grand partition function. Thus the irreversible thermodynamic formalism of ideal relativistic gas mixtures of matter and radiation is now in place for nonequilibrium processes. And we can apply it to study irreversible thermodynamic phenomena in the system considered in the regime far removed from equilibrium at arbitrary degree.

2.10 Cumulant Expansion for the Dissipation Terms

2.10.1 Cumulant Expansion for the Calortropy Production

The description of generalized hydrodynamic equations is not as yet complete since the dissipation terms $\Lambda_{a}^{(\alpha)}$ of the nonconserved variable evolution equations still remain formally defined, but not explicitly evaluated in terms of variables in $\mathfrak{P}\cup\mathfrak{T}$. Therefore, the formal development in the theory of irreversible thermodynamics presented up to this point has not acquired desired explicit forms for them in terms of variables of $\mathfrak{P} \cup \mathfrak{T}$. However, if the transport processes are to be studied in the thermodynamic manifold their dependence on the variables in manifold $\mathfrak{P} \cup \mathfrak{T}$ must be explicitly given.

Since the dissipation terms are directly related to the calortropy production in the system, they must be evaluated such that the second law of thermodynamics is rigorously satisfied by suitable forms to be taken for them. Such a procedure is possible to attain if a cumulant expansion method is used [5, 18] for the calortropy production. Since the present covariant kinetic equations of radiation and matter are similar in their structure to the covariant kinetic equations for the case of matter alone discussed in Chap. 1 of this Volume, the cumulant expansion method is expected to be also similar. For this reason we will be brief and present only the final results with necessary definitions in Chap. 1 mentioned.

Let us define the symbol g by the formula

$$g = \frac{(mc^2/k_B T)^4}{c\lambda^2\rho^2}, \tag{2.246}$$

where λ is a parameter of dimension length—it may be taken for the mean free path or the interaction range or the mean size parameter of particles. This parameter g has a dimension of volume × time. If the calortropy production is scaled by k_B/g it can be rendered dimensionless:

$$\widehat{\sigma}_{\mathrm{c}} = \frac{g}{k_B}\sigma_{\mathrm{c}} \tag{2.247}$$

and similarly for the Boltzmann entropy production σ_{ent}:

$$\widehat{\sigma}_{\mathrm{ent}} = \frac{g}{k_B}\sigma_{\mathrm{ent}}. \tag{2.248}$$

Then analysis can be carried on for $\widehat{\sigma}_{\mathrm{c}}$ and $\widehat{\sigma}_{\mathrm{ent}}$ in the same manner as in the previous chapter. For example, a cumulant expansion method may be applied and a formally identical result is obtained.

To the first-order cumulant approximation, the reduced calortropy production is given by

$$\widehat{\sigma}_{\mathrm{c}} = \kappa \sinh \kappa \geq 0, \tag{2.249}$$

where κ^2 is the Rayleigh dissipation function given by a quadratic form of generalized potentials $X_a^{(q)}$. Since κ is a positive quadratic form of the generalized potentials $X_a^{(\alpha)}$, the first-order cumulant approximation for $\widehat{\sigma}_{\mathrm{c}}$ in (2.249) remains positive semidefinite for all approximations made for the generalized potentials, which may be determined from (2.245), for example. If the right hand side of (2.245) is calculated from the knowledge of Γ and then the relation is inverted, $X_a^{(\alpha)}$ is obtained as a function of $\Phi_a^{(q)}$. In general, their relation may be expressed as

$$X_a^{(\alpha)} = -g_a^{(q)} \Phi_a^{(q)}, \tag{2.250}$$

where $g_a^{(q)}$ is a scalar function of variables spanning the manifold $\mathfrak{P} \cup \mathfrak{T}$. In the lowest order of approximation $g_a^{(q)}$ is a scalar function of the variables of manifold $\mathfrak{T}$ but independent of the variables of manifold $\mathfrak{P}$. See (2.274) below for $g_a^{(q)}$. In the first-order cumulant approximation (2.249) for $\widehat{\sigma}_{\mathrm{c}}$, the cumulant κ is given by the formula

$$\kappa = \left[\sum_{a=1}^{r} \sum_{b=1}^{r} \sum_{\alpha \geq 1} \sum_{\gamma \geq 1} X_{a\,\mu\cdots\sigma}^{(\alpha)} R_{ab}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} X_{b\,\nu\cdots\omega}^{(\gamma)} \right]^{1/2}. \tag{2.251}$$

Here $R_{ab}^{(\alpha\gamma)}$ are given by the collision bracket integrals of the collision integrals $\mathfrak{R}_a\,[f_a]$ and $\mathfrak{R}_r\,[f_r]$ of the covariant kinetic equation for matter and radiation. They can be explicitly determined in terms of particulate dynamical information (e.g., interactions and collision cross sections of particles) if we make use of the linear steady-state evolution equations for nonconserved variables, namely, the linearized steady-state generalized hydrodynamic equations. In fact, so determined $R_{ab}^{(\alpha\gamma)}$ are found to be exactly the same as the collision bracket integrals appearing, for example, in the first-order Chapman–Enskog method [24] for the covariant kinetic equation. Their statistical formulas are as follows. With the notations for reduced momenta

$$\pi_a = c\beta p_a \tag{2.252}$$

and the abbreviation $[A \odot B]_{ab}$ for the integral

$$[A \odot B]_{ab} = G_a G_b \sum_s \int d^3\overline{\pi}_a \int d^3\overline{\pi}_b \int d^3\overline{\pi}_k^* \int d^3\overline{\pi}_l^* f_a^{\mathrm{e}} f_b^{\mathrm{e}} \overline{W}_{ab;kl}^{(s)} \xi_{abkl} A \odot B, \tag{2.253}$$

where $\overline{W}_{ab;kl}^{(s)}$ is a reduced transition rate

$$\overline{W}_{ab;kl}^{(s)} = \left(\frac{mk_B T}{\lambda^2} \right) W_{ab;kl}^{(s)} \tag{2.254}$$

and

$$\xi_{abkl} = \left[\left(1 + \epsilon_a f_a^{\mathrm{e}}\right) \left(1 + \epsilon_b f_b^{\mathrm{e}}\right) \left(1 + \epsilon_k f_k^{\mathrm{e}}\right) \left(1 + \epsilon_l f_l^{\mathrm{e}}\right) \right]^{-1}, \tag{2.255}$$

the collision bracket integrals are given by expressions

$$\begin{aligned} R_{aa}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} = \frac{1}{4}\beta^2 \Big[& \left(h_a^{(\alpha)} + h_{a'}^{(\alpha)} - h_a^{(\alpha)*} - h_{a'}^{(\alpha)*} \right)^{\sigma\cdots\mu} \\ & \times \left(h_a^{(\gamma)} + h_{a'}^{(\gamma)} - h_a^{(\gamma)*} - h_{a'}^{(\gamma)*} \right)^{\omega\cdots\nu} \Big]_{aa'} \end{aligned}$$

$$+\frac{1}{2}\beta^2 \sum_{b\neq a} \left[\left(h_a^{(\alpha)} - h_{a'}^{(\alpha)*} \right)^{\sigma\cdots\mu} \left(h_b^{(\gamma)} - h_{b'}^{(\gamma)*} \right)^{\omega\cdots\nu} \right]_{ab}, \tag{2.256}$$

$$R_{ab}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} = \beta^2 \sum_{a\neq b} \left[\left(h_a^{(\alpha)} - h_a^{(\alpha)*} \right)^{\sigma\cdots\mu} \left(h_b^{(\gamma)} - h_b^{(\gamma)*} \right)^{\omega\cdots\nu} \right]_{ab} \quad (a \neq b), \tag{2.257}$$

$$R_{rr}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} = \frac{1}{2}\beta^2 \sum_{a\neq r} \left[\left(h_r^{(\alpha)} - h_r^{(\alpha)*} \right)^{\sigma\cdots\mu} \left(h_r^{(\gamma)} - h_r^{(\gamma)*} \right)^{\omega\cdots\nu} \right]_{ai}, \tag{2.258}$$

$$R_{ra}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu} = \beta^2 \sum_{a\neq r} \left[\left(h_r^{(\alpha)} - h_i^{(\alpha)*} \right)^{\sigma\cdots\mu} \left(h_r^{(\gamma)} - h_i^{(\gamma)*} \right)^{\omega\cdots\nu} \right]_{ra}. \tag{2.259}$$

These tensor integrals can be simplified to products of scalar integrals and isotropic tensors as will be shown in the following.

2.10.2 Dissipation Terms in the First-Order Cumulant Approximation

Comparing (2.249) with (2.211) and with the help of the expression for the first-order cumulant κ, we now find approximate dissipation terms consistent with the inequality (2.249) for the calortropy production

$$\Lambda_a^{(q)\sigma\cdots\mu} = (\beta g)^{-1} \sum_{b=1}^{r} \sum_{\gamma\geq 1} R_{ab}^{(q\gamma)\sigma\cdots\mu\omega\cdots\nu} X_{b\,\nu\cdots\omega}^{(\gamma)} \left(\sinh\kappa/\kappa\right). \tag{2.260}$$

This is the first-cumulant approximation for $\Lambda_a^{(q)}$, which is also consistent with the H theorem in the limit of vanishing fluctuations of distribution function f_a from f_a^{c}: $\Delta f_a = f_a - f_a^{\mathrm{c}} = 0$, because, in that case, $\sigma_{\mathrm{ent}} = \sigma_{\mathrm{c}} \geq 0$ according to (2.249).

Higher order cumulant approximations for the dissipation terms may be obtained if the corresponding cumulant approximations are used for the reduced entropy production. It is straightforward to obtain them by following the procedure described in the literature [5, 18, 25, 26]. There now remains the task of determining the unknowns $X_a^{(\alpha)}$ in order to complete the formulation of thermodynamically consistent generalized hydrodynamic equations.

To proceed further it is useful to recast the collision bracket integrals $R_{aa}^{(\alpha\gamma)}$, etc., which are given above in unwieldy tensor product forms, to more computationally practical scalar forms. If we exploit the properties of isotropic tensors [18] this aim can be easily achieved for collision bracket integrals considered here, which involve tensors of rank 4 at most. We observe that $R_{ab}^{(q\gamma)}$ are expandable into isotropic tensors with scalar collision bracket integrals appearing as their coefficients; see, for example, Chap. 5, Ref. [18]. Thus we obtain

$$R_{ab}^{(11)\mu\nu\sigma\omega} = \mathcal{R}_{ab}^{(11)} \left[\frac{1}{2} \left(\Delta^{\mu\sigma} \Delta^{\nu\omega} + \Delta^{\mu\omega} \Delta^{\nu\sigma} \right) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\omega} \right]. \tag{2.261}$$

The coefficient $\mathcal{R}_{ab}^{(11)}$ is then a scalar obtained by contracting $R_{ab}^{(11)\mu\nu\sigma\omega}$ with the isotropic tensor $\left[\frac{1}{2}\left(\Delta_{\mu\sigma}\Delta_{\nu\omega} + \Delta_{\mu\omega}\Delta_{\nu\sigma}\right) - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\sigma\omega}\right]$:

$$\begin{aligned} \mathcal{R}_{ab}^{(11)} &= \frac{1}{5} R_{ab}^{(11)\mu\nu\sigma\omega} \left[\frac{1}{2} \left(\Delta^{\mu\sigma} \Delta^{\nu\omega} + \Delta^{\mu\omega} \Delta^{\nu\sigma} \right) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\omega} \right] \\ &= \frac{1}{10} \beta^2 \sum_{a \neq b} \left[\left(h_a^{(\alpha)} - h_a^{(\alpha)*} \right) : \left(h_b^{(\gamma)} - h_b^{(\gamma)*} \right) \right]_{ab} \quad (a \neq b) \end{aligned} \tag{2.262}$$

and similarly

$$\begin{aligned} \mathcal{R}_{aa}^{(11)} &= \frac{1}{5} R_{aa}^{(11)\mu\nu\sigma\omega} \left[\frac{1}{2} \left(\Delta^{\mu\sigma} \Delta^{\nu\omega} + \Delta^{\mu\omega} \Delta^{\nu\sigma} \right) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\omega} \right] \\ &= \frac{1}{10} \beta^2 \left\{ \frac{1}{2} \left[\Delta \left(h_a^{(1)} + h_{a'}^{(1)} \right) : \Delta \left(h_a^{(1)} + h_{a'}^{(1)} \right) \right]_{aa'} \right. \\ &\qquad \left. + \sum_{b \neq a} \left[\left(h_a^{(1)} - h_{a'}^{(1)*} \right) : \left(h_b^{(1)} - h_{b'}^{(1)*} \right) \right]_{ab} \right\}, \end{aligned} \tag{2.263}$$

where

$$\Delta \left(h_a^{(1)} + h_{a'}^{(1)} \right) = h_a^{(1)} + h_{a'}^{(1)} - h_a^{(1)*} - h_{a'}^{(1)*}.$$

Here the prime on the subscript a of $[A]_{aa'}$ means another particle of species a and the colon: between the tensors means contraction of rank 2 tensors. Since the collision bracket integral $R_{ab}^{(22)}$ is scalar for the case of $q = s = 2$, we simply obtain scalar collision bracket integrals:

$$\begin{aligned} \mathcal{R}_{ab}^{(22)} &= R_{ab}^{(22)} \\ &= \beta^2 \sum_{a \neq b} \left[\left(h_a^{(2)} - h_a^{(2)*} \right) \left(h_b^{(2)} - h_b^{(2)*} \right) \right]_{ab} \quad (a \neq b), \end{aligned} \tag{2.264}$$

$$\begin{aligned} \mathcal{R}_{aa}^{(22)} &= R_{aa}^{(22)} \\ &= \frac{1}{2} \beta^2 \left\{ \frac{1}{2} \left[\Delta \left(h_a^{(2)} + h_{a'}^{(2)} \right) \Delta \left(h_a^{(2)} + h_{a'}^{(2)} \right) \right]_{aa'} \right. \\ &\qquad \left. + \sum_{b \neq a} \left[\left(h_a^{(\alpha)} - h_{a'}^{(\alpha)*} \right) \left(h_b^{(s)} - h_{b'}^{(s)*} \right) \right]_{ab} \right\}, \end{aligned} \tag{2.265}$$

where

$$\Delta \left(h_a^{(2)} + h_{a'}^{(2)} \right) = h_a^{(2)} + h_{a'}^{(2)} - h_a^{(2)*} - h_{a'}^{(2)*}.$$

For $q, s = 3$ or 4, namely, for vector processes such as heat flow and diffusion the collision bracket integral $R_{ab}^{(q\gamma)\mu\nu}$ is expressible as

$$R_{ab}^{(q\gamma)\mu\nu} = \mathcal{R}_{ab}^{(q\gamma)} \Delta^{\mu\nu} \tag{2.266}$$

and hence upon contracting with $\Delta^{\mu\nu}$ we obtain

$$\begin{aligned} \mathcal{R}_{ab}^{(qs)} &= R_{ab}^{(qs)\mu\nu} \Delta^{\mu\nu} \\ &= \frac{1}{6}\beta^2 \sum_{a \neq b} \left[\left(h_a^{(q)} - h_a^{(q)*} \right) \cdot \left(h_b^{(s)} - h_b^{(s)*} \right) \right]_{ab} \quad (a \neq b)\,, \end{aligned} \tag{2.267}$$

$$\begin{aligned} \mathcal{R}_{aa}^{(qs)} = \frac{1}{6}\beta^2 \Bigg\{ & \frac{1}{2} \left[\Delta \left(h_a^{(q)} + h_{a'}^{(q)} \right) \cdot \Delta \left(h_a^{(s)} + h_{a'}^{(s)} \right) \right]_{aa'} \\ & + \sum_{b \neq a} \left[\left(h_a^{(q)} - h_{a'}^{(q)*} \right) \cdot \left(h_b^{(s)} - h_{b'}^{(s)*} \right) \right]_{ab} \Bigg\} \quad (q, s = 3, 4)\,, \end{aligned} \tag{2.268}$$

where the symbol $\cdot$ means the scalar product of vectors, namely, contraction of rank 1 tensors. These procedures of tensor contractions in the collision bracket integrals can be extended to the cases of higher rank tensors with progressively complicated results.[10] In the present chapter we are limiting the calculations up to rank 4 tensors, in other words, shear stresses $\Phi_a^{(q)\mu\nu}$, since higher rank tensors than 4 are not required in practice. It is, in principle, possible to cast the first-order cumulant κ into the quadratic form of generalized potentials $X_a^{(q)}$:

$$\kappa = \left[\sum_{a=1}^{r} \sum_{b=1}^{r} \sum_{q \geq 1} \sum_{s \geq 1} \mathcal{R}_{ab}^{(q\gamma)} \left(X_a^{(q)} \odot X_b^{(s)} \right) \right]^{1/2}, \tag{2.269}$$

where $X_a^{(\alpha)} \odot X_b^{(s)}$ is a scalar:

$$X_a^{(q)} \odot X_b^{(s)} = X_a^{(q)\mu\nu\cdots\sigma\omega} X_{b\,\omega\sigma\cdots\nu\mu}^{(s)}. \tag{2.270}$$

The generalized potentials $X_a^{(q)}$ in κ and f_a^{c} now must be determined, so that the whole structure of the theory we are pursuing becomes free from undetermined parameters. It is discussed in the next section.

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[10] For higher rank tensors isotropic tensors of higher rank, more complicated basis sets would be required. For the basis sets for higher order isotropic Cartesian tensors, see pages 97–98, Chap. 5, Ref. [18].

2.11 Generalized Potentials

We have managed the formulation of the theory, deferring the determination of the generalized potentials $X_a^{(q)}$ making up the nonequilibrium canonical forms (2.190) and (2.192). We have now reached the point to determine $X_a^{(q)}$ in terms of variables of manifold $\mathfrak{P}$.

The formal relations of nonequilibrium quantities are indicated by various relations of the macroscopic irreversible thermodynamics developed with the nonequilibrium canonical forms. One of them can be made use of to obtain the desired relation of $X_a^{(q)}$ to $\Phi_a^{(q)}$; see, especially, (2.245), which may be regarded as one of the most important among them in the present nonequilibrium statistical mechanical theory. Another mathematically satisfactory method of determining $X_a^{(q)}$ was discussed in Chap. 3 of Volume 1 and we may apply it to calculate it in terms of $\widehat{\Phi}_a^{(q)}$. Here we make use of (2.245) for the purpose in mind.

Expanding $\ln \Gamma_a$ in power series in $X_a^{(q)}$, we obtain

$$k_B T \left(\frac{\partial \ln \Gamma_a}{\partial X_a^{(q)}} \right)_{T,v,\mu,X'} = \left\langle c^{-2} U_\mu p_a^\mu f_a^{\mathrm{e}} \left(h_a^{(q)} + \beta h_a^{(q)} h_a^{(q)} \odot X_a^{(q)} + \cdots \right) \right\rangle, \tag{2.271}$$

which, to the lowest nonvanishing order in $X_a^{(q)}$, yields the formula

$$k_B T \left(\frac{\partial \ln \Gamma_a}{\partial X_a^{(q)}} \right)_{T,v,\mu,X'} = \beta \left\langle c^{-2} U_\mu p_a^\mu f_a^{\mathrm{e}} h_a^{(q)} h_a^{(q)} \odot X_a^{(q)} \right\rangle + O\left(X_a^{(q)2} \right). \tag{2.272}$$

Thus finally, to the lowest nonvanishing order the generalized potentials are given by the linear relation

$$X_a^{(q)} = -\left(1/g_a^{(q)} \right) \Phi_a^{(q)}, \tag{2.273}$$

where

$$g_a^{(q)} = \frac{1}{2} I_s \beta \left\langle f_a^{\mathrm{e}} c^{-2} U_\mu p_a^\mu h_a^{(q)} \odot h_a^{(q)} \right\rangle \ (a = 1, \ldots, m, r) \tag{2.274}$$

with I_s denoting symmetry-related factor for integrals of tensors

$$I_s = \begin{cases} \frac{1}{5} & \text{for second-rank tensors} \\ \frac{1}{3} & \text{for first-rank tensors} \\ 1 & \text{for scalar} \end{cases} . \tag{2.275}$$

It should be noted that the rank of tensor $X_a^{(q)}$ is the same as that of $\Phi_a^{(q)}$. The approximation for $X_a^{(q)}$ (2.273) leaves the first-order cumulant approximation for the calortropy entropy production positive semidefinite. In this manner, the moment method employed here for relativistic Boltzmann–Nordholm–Uehling–Uhlenbeck equation is made complete to an approximation consistent with $\sigma_{\mathrm{c}} \geq 0$. When the approximate relation for $X_a^{(q)}$ is used in the dissipation terms, the evolution equations formally obtained earlier are closed with respect to $\left\{\Phi_a^{(q)}\right\}$ and ready for solution because all the undetermined parameters are given formal relations to $\left\{\Phi_a^{(q)}\right\}$, by means of which we can determined them in terms of variables spanning the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. The transport properties then can be calculated from the solutions of the evolution equations—i.e., the generalized hydrodynamic equations for fluxes and conserved variables. The generalized hydrodynamic equations thus obtained, therefore, provide a mathematical framework—in the form of closed field equations—to describe transport processes attendant on irreversible processes in a system of radiation and matter in a manner consistent with the thermodynamic laws.

2.12 Generalized Hydrodynamics

The combined set of conservation laws, (2.163)–(2.168), and nonconserved variable evolution equations [(2.169) and (2.173)] constitutes the generalized hydrodynamics of radiation and matter. Having been appropriately subjected to the laws of thermodynamics, they are thermodynamically consistent. They are the fruits of labor expended so far to make generalized potentials and dissipation terms explicit with regard to variables spanning manifold $\mathfrak{P} \cup \mathfrak{T}$, so that the generalized hydrodynamic equations properly describe irreversible processes in the system consistently with the thermodynamic laws, given the initial and boundary conditions for variables belonging to $\mathfrak{P} \cup \mathfrak{T}$. They are now ready for hydrodynamic applications. It would be useful to summarize them in the following.

2.12.1 Conservation Laws

The equation of continuity for the total density is

$$D\rho = -\rho \nabla_\mu U^\mu. \tag{2.276}$$

The conserved variables balance equations are:

$$\rho D\mathfrak{c}_{\mathrm{m}} = -\partial_\mu J_{\mathrm{m}}^\mu + c^{-2} J_{\mathrm{m}}^\mu D U_\mu + \Lambda_{\mathrm{m}}^{(n)}, \tag{2.277}$$

$$\rho D\mathcal{E}_{\mathrm{m}} = -\partial_\mu Q_{\mathrm{m}}^\mu + P_{\mathrm{m}}^{\mu\nu} \nabla_\nu U_\mu + c^{-2} Q_{\mathrm{m}}^\mu D U_\mu + U_\nu \Lambda_{\mathrm{m}}^\nu, \tag{2.278}$$

$$\begin{aligned} c^{-2}\mathcal{E}_{\mathrm{m}}\rho D U^\mu &= \nabla^\mu p_{\mathrm{m}} + \Delta_\sigma^\mu \partial_\nu \Pi_{\mathrm{m}}^{\sigma\nu} \\ &\quad + c^{-2}\left(P_{\mathrm{m}}^{\mu\nu} D U_\nu - \Delta_\nu^\mu D Q_{\mathrm{m}}^\nu - Q_{\mathrm{m}}^\mu \nabla_\nu U^\nu - Q_{\mathrm{m}}^\nu \nabla_\nu U^\mu\right) \end{aligned} \tag{2.279}$$

for matter and

$$\rho D\mathfrak{c}_r = -\partial_\nu J_r^\nu + c^{-2} J_r^\mu DU_\mu + \Lambda_r^{(n)}, \tag{2.280}$$

$$\rho D\mathcal{E}_r = -\partial_\nu Q_r^\nu + P_r^{\mu\nu}\nabla_\nu U_\mu + c^{-2} Q_r^\nu DU_\nu + U_\mu \Lambda_r^\mu \tag{2.281}$$

for the radiation. Here $\mathcal{E}_r$ is the radiation energy per material particle. We emphasize that the radiation momentum equation is not listed here, since it is related to the radiation heat flux evolution equation which appears as one of the evolution equations for $\widehat{\Phi}_a^{(q)\mu\nu\cdots l}$.

2.12.2 Evolution Equations for Nonconserved Variables

Coupled to the balance equations of the conserved variables, there are the evolution equations for nonconserved variables

$$\rho D\widehat{\Phi}_a^{(q)\mu\cdots\nu} = -\nabla_\sigma \Theta_a^{(q)\sigma\mu\cdots\nu} + \mathfrak{Z}_a^{(q)\mu\nu\cdots l} + \Lambda_a^{(q)\mu\nu\cdots l}(x), \tag{2.282}$$

$$\Theta_a^{(q)\sigma\mu\cdots\nu} = \Delta_\gamma^\mu \psi_a^{(q)\gamma\sigma\mu\cdots\nu} \quad (a = 1, \cdots, m, r; q \geq 1) \tag{2.283}$$

which holds for either matter or radiation. This is the generic form of nonconserved variable evolution equations. The kinematic terms and dissipation terms are presented separately for matter and radiation below.

Kinematic Terms for Matter

With the definitions for symbols

$$\begin{aligned}
[\nabla U]^{(2)\mu\nu} &= \frac{1}{2}\left(\nabla^\mu U^\nu + \nabla^\nu U^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}\nabla^\tau U_\tau, \\
[A \cdot B]^{(2)\mu\nu} &= \frac{1}{2}\left(A^{\mu\sigma}B_\sigma^\nu + A^{\nu\sigma}B_\sigma^\mu\right) - \frac{1}{3}\Delta^{\mu\nu}A^{\sigma\tau}B_{\sigma\tau},
\end{aligned} \tag{2.284}$$

the kinematic terms for matter species are collected below:

Modified Kinematic Term for Shear Stress

$$\begin{aligned}
\mathfrak{Z}_a^{(1)\mu\nu} = &-2\left[P_a \cdot \nabla U\right]^{(2)\mu\nu} - 2c^{-2}\left[Q_r' \cdot DU\right]^{(2)} - 2c^{-2}\left[\mathfrak{J}_r DU\right]^{(2)\mu\nu} \\
&- 2\left[P_a U \cdot DU\right]^{(2)} + c^{-2}\left[UU\right]^{(2)\mu\nu} P_a^{\tau\varepsilon}\nabla_\varepsilon U_\tau \\
&+ c^{-2}\left(U^\mu P_a^{\nu\omega} DU_\omega + U^\nu P_a^{\mu\omega} DU_\omega\right) + c^{-2}\left[UU\right]^{(2)\mu\nu} Q_r'^\omega DU_\omega \\
&- \frac{1}{3}c^{-2}\left(U^\mu Q_a'^\varepsilon \nabla_\varepsilon U^\nu + U^\nu Q_a'^\varepsilon \nabla_\varepsilon U^\mu\right) + c^{-2}\left[UU\right]^{(2)\mu\nu} Q_a'^\omega DU_\omega \\
&- \frac{1}{3}c^{-2}\left(U^\mu \mathfrak{J}_a^\varepsilon \nabla_\varepsilon U^\nu + U^\nu \mathfrak{J}_a^\varepsilon \nabla_\varepsilon U^\mu\right) + 2c^{-2}\left[UU\right]^{(2)\mu\nu} \mathfrak{J}_a^\omega DU_\omega
\end{aligned}$$

$$
\begin{aligned}
&- c^{-4} U^\mu U^\nu Q'^{\omega}_a DU_\omega + \frac{1}{3}\left\langle f_a \frac{p^\varepsilon_a \left(p^\sigma_a p_{a\sigma}\right)}{\left(U_\lambda p^\lambda_a\right)} \right\rangle \left(U^\mu \nabla_\varepsilon U^\nu + U^\nu \nabla_\varepsilon U^\mu\right) \\
&- c^2 \left\langle f_a \frac{p^\mu_a p^\nu_a \left[p_a p_a\right]^{(2)\omega\varepsilon}}{\left(U_\lambda p^\lambda_a\right)^2} \right\rangle \nabla_\varepsilon U_\omega,
\end{aligned}
\tag{2.285}
$$

where

$$
\left[p_a p_a\right]^{(2)\omega\varepsilon} = p^\omega_a p^\varepsilon_a - \frac{1}{3}\Delta^{\mu\nu}\left(p^\sigma_a p_{a\sigma}\right) = p^\omega_a p^\varepsilon_a - \frac{1}{3} m_a^2 c^2 \Delta^{\mu\nu}.
$$

Modified Kinematic Term for Excess Normal Stress

$$
\begin{aligned}
\mathfrak{Z}^{(2)}_a = &-\rho_a D\left(p_a/\rho_a\right) - \frac{2}{3} P^{\mu\varepsilon}_a \nabla_\varepsilon U_\mu - \frac{2}{3} c^{-2} Q^\omega_a DU_\omega - J^\varepsilon_a \nabla_\varepsilon \left(p_a/\rho_a\right) \\
&- c^{-2}\left(p_a/\rho_a\right) J^\omega_a DU_\omega + \frac{1}{3}\left[P^{\mu\varepsilon}_a - c^2 \left\langle f_a \frac{p^\omega_a p^\varepsilon_a \left(p^\mu_a p_{a\mu}\right)}{\left(U_\lambda p^\lambda_a\right)^2}\right\rangle\right] \nabla_\varepsilon U_\omega
\end{aligned}
\tag{2.286}
$$

where

$$
\left(p^\mu_a p_{a\mu}\right) = m_a^2 c^2.
$$

Modified Kinematic Term for Heat Flux

$$
\begin{aligned}
\mathfrak{Z}^{(3)\mu}_a = &\; P^{\mu\omega}_a DU_\omega - Q'^{\sigma}_a \nabla_\sigma U^\mu - P^{\mu\sigma}_a \nabla_\sigma \widehat{h}'_a - c^{-2} \mathfrak{J}^\sigma_a \nabla_\sigma U^\mu \\
&+ \widehat{h}'_a J^\sigma_a \nabla_\sigma U^\mu + \rho_a \left(\widehat{h}'_a - \mathcal{E}_a\right) DU^\mu - J^\mu_a D\widehat{h}'_a \\
&- c^{-2} U^\mu \left(Q^\sigma_a DU_\sigma + c^{-2}\mathfrak{J}^\sigma_a DU_\sigma - \widehat{h}'_a J^\omega_a DU_\omega\right) \\
&- U^\mu \left(P^{\sigma\varepsilon}_a \nabla_\sigma U_\varepsilon - J^\sigma_a \nabla_\sigma \widehat{h}'_a\right) \\
&+ c^2 \widehat{h}'_a \left\langle f_a \frac{p^\mu_a p^\omega_a p^\sigma_a}{\left(U_\lambda p^\lambda_a\right)^2}\right\rangle \nabla_\sigma U_\omega + \left[P^{\mu\sigma}_a - c^2 \left\langle f_a \frac{p^\mu_a p^\sigma_a}{\left(U_\lambda p^\lambda_a\right)}\right\rangle\right] \nabla_\sigma \widehat{h}'_a.
\end{aligned}
\tag{2.287}
$$

Modified Kinematic Term for Diffusion Flux

$$
\mathfrak{Z}^{(4)\mu}_a = -\rho_a DU^\mu - J^\sigma_a \nabla_\sigma U^\mu - c^{-2} U^\mu J^\omega_a DU_\omega - c^2 \left\langle f_a \frac{p^\mu_a p^\omega_a p^\sigma_a}{\left(U_\lambda p^\lambda_a\right)^2}\right\rangle \nabla_\sigma U_\omega .
\tag{2.288}
$$

In this case the divergence term is added to the kinematic term because $\Theta^{(4)\mu}_a$ gives rise to a variable in the manifold $\mathfrak{P} \cup \mathfrak{T}$.

Modified Kinematic Terms for Radiation

The modified kinematic terms for radiation are collected below.

Modified Kinematic Term for Shear Stress

$$\begin{aligned}\mathfrak{Z}_r^{(1)\mu\nu} = &-2\left[P_r\cdot\nabla U\right]^{(2)\mu\nu} - 2c^{-2}\left[Q'_r\cdot DU\right]^{(2)} - 2c^{-2}\left[\mathfrak{J}_r DU\right]^{(2)\mu\nu}\\ &-2\left[P_r U\cdot DU\right]^{(2)} + c^{-2}\left[UU\right]^{(2)\mu\nu} P_r^{\tau\varepsilon}\nabla_\varepsilon U_\tau\\ &+c^{-2}\left(U^\mu P_r^{\nu\omega} DU_\omega + U^\nu P_r^{\mu\omega} DU_\omega\right) + c^{-2}\left[UU\right]^{(2)\mu\nu} Q_r'^{\omega} DU_\omega\\ &-c^{-4}U^\mu U^\nu Q_r'^{\omega} DU_\omega + c^{-2}\left(\left[UU\right]^{(2)\mu\nu} - c^{-2}U^\mu U^\nu\right)\mathfrak{J}_r^\omega DU_\omega\\ &-\frac{1}{3}c^{-2}\left(U^\mu Q_r'^{\varepsilon}\nabla_\varepsilon U^\nu + U^\nu Q_r'^{\varepsilon}\nabla_\varepsilon U^\mu\right)\\ &-\frac{1}{3}c^{-2}\left(U^\mu J_r^\varepsilon\nabla_\varepsilon U^\nu + U^\nu J_a^\varepsilon\nabla_\varepsilon U^\mu\right) - c^2\left\langle f_r\frac{p_r^\mu p_r^\nu p_r^\omega p_r^\varepsilon}{\left(U_\lambda p_r^\lambda\right)^2}\right\rangle\nabla_\varepsilon U_\omega.\end{aligned}\tag{2.289}$$

Modified Kinematic Term for Excess Normal Stress

$$\begin{aligned}\mathfrak{Z}_r^{(2)} = &-\rho_r D\left(p_r/\rho_r\right) - \frac{1}{3}P_r^{\mu\varepsilon}\nabla_\varepsilon U_\mu - J_r^\varepsilon\nabla_\varepsilon\left(p_r/\rho_r\right) - c^{-2}\left(p_r/\rho_r\right)J_r^\omega DU_\omega\\ &-\frac{2}{3}c^{-2}\mathfrak{J}_r^\omega DU_\omega - \frac{2}{3}c^{-2}Q_r'^{\omega}DU_\omega.\end{aligned}\tag{2.290}$$

Modified Kinematic Term for Heat Flux

$$\begin{aligned}\mathfrak{Z}_r^{(3)\mu} = &-P_r^{\mu\sigma}\nabla_\sigma h_r - Q_r'^{\sigma}\nabla_\sigma U^\mu - c^{-2}\mathfrak{J}_r^\sigma\nabla_\sigma U^\mu\\ &+P_r^{\omega\mu}DU_\omega + \rho_r p_r DU^\mu - J_r^\mu Dh_r + h_r J_r^\sigma\nabla_\sigma U^\mu\\ &-U^\mu P_r^{\sigma\omega}\nabla_\sigma U_\omega - U^\mu Q_r'^{\sigma}DU_\sigma - c^{-2}U^\mu\mathfrak{J}_r^\sigma DU_\sigma\\ &+U^\mu J_r^\sigma\nabla_\sigma h_r + c^{-2}U^\mu h_r J_r^\omega DU_\omega + c^2 h_r\left\langle f_r\frac{p_r^\mu p_r^\omega p_r^\sigma}{\left(U_\lambda p_r^\lambda\right)^2}\right\rangle\nabla_\sigma U_\omega\\ &+2h_r\left\langle f_r\frac{p_r^\omega p_r^\sigma}{\left(U_\lambda p_r^\lambda\right)}\right\rangle U^\mu\nabla_\sigma U_\omega + \left[P_r^{\mu\sigma} - c^2\left\langle f_r\frac{p_r^\mu p_r^\sigma}{\left(U_\lambda p_r^\lambda\right)}\right\rangle\right]\nabla_\sigma h_r.\end{aligned}\tag{2.291}$$

The third and fourth lines proportional to U^μ on the right vanish in the local rest frame. They are relativistic effects as are the integrals.

Modified Kinematic Term for Diffusion Flux

$$\mathfrak{Z}_a^{(4)\mu} = -\rho_a DU^\mu - J_a^\sigma\nabla_\sigma U^\mu - c^{-2}U^\mu J_a^\omega DU_\omega - c^2\left\langle f_a\frac{p_a^\mu p_a^\omega p_a^\sigma}{\left(U_\lambda p_a^\lambda\right)^2}\right\rangle\nabla_\sigma U_\omega.\tag{2.292}$$

The integrals represent relativistic effects. In this case of $\mathfrak{Z}_r^{(4)\mu}$, the divergence term

$$-\,\partial_\sigma \Omega_r^{(4)\sigma\mu} = J_r^\mu \nabla_\sigma U^\sigma - c^{-2} U^\mu J_r^\omega DU_\omega + \left\langle f_r \frac{p_r^\omega p_r^\mu}{\left(p_r^\lambda U_\lambda\right)} \right\rangle DU_\omega \tag{2.293}$$

has been added to $\mathcal{Z}_r^{(4)\mu}$ to obtain $\mathfrak{Z}_a^{(4)\mu}$ since the term consists of variables in the manifold $\mathfrak{P}$.

The kinematic terms for matter and radiation are different despite the formally rather similar moments $h_i^{(q)\mu\cdots\nu}$ $(a = 1, 2, \ldots m, r)$ because $\left(p_i^\sigma p_{i\,\sigma}\right) = m_a^2 c^2$ for matter whereas $\left(p_i^\sigma p_{i\,\sigma}\right) = 0$ for radiation.

Dissipation Terms $\Lambda_a^{(q)\mu\cdots\nu}$

The dissipation terms accompanying the kinematic terms in the generic evolution equation are calculated in the first-order cumulant approximation. The first-order cumulant approximation combined with the first-order approximation for the generalized potentials given in (2.273) is found sufficient for many applications in the case of nonrelativistic generalized hydrodynamic equations applied so far. For this reason it is logical to consider a similar level of approximations in the case of the relativistic theory. With the generalized potentials approximated to the linear order in fluxes as given in (2.273) the dissipation terms are given by the formula

$$\Lambda_a^{(q)\mu\cdots\nu} = -\sum_{b=1}^{\{r,m\}} \sum_{s=1}^{4} \left(\beta g g_b^{(s)}\right)^{-1} \mathfrak{R}_{ab}^{(qs)} \Phi_b^{(s)\mu\cdots\nu} \left(\sinh\kappa/\kappa\right)$$
$$(a = 1, \ldots, m, r)\,, \tag{2.294}$$

where the coefficients $\mathfrak{R}_{ab}^{(qs)}$ are defined by

$$\mathfrak{R}_{ab}^{(qs)} = g_a^{(q)} \mathcal{R}_{ab}^{(qs)} g_b^{(s)} \tag{2.295}$$

in terms of the collision bracket integrals $\mathcal{R}_{ab}^{(qs)}$ given in (2.262)–(2.265), (2.267), and (2.268). Recalling the definition of the first-order cumulant κ, we now find that κ is given, in the first order approximation (2.273) for generalized potentials, by the quadratic form of $\Phi_b^{(s)}$:

$$\kappa^2 = \sum_{a,b=1}^{\{r,m\}} \sum_{q,s\geq 1} \mathfrak{R}_{ab}^{(qs)} \Phi_a^{(s)\mu\cdots\nu} \Phi_{b\nu\cdots\mu}^{(s)}. \tag{2.296}$$

Note that this is a generalized Rayleigh dissipation function. With the dissipation terms given in (2.294) we are now ready to explore some useful approximate forms for the nonconserved variable evolution equations presented above.

2.12.3 Linear Constitutive Equations and Transport Coefficients

In order to derive the statistical mechanical formulas for transport coefficients of relativistic gases in terms of dynamical quantities such as collision cross sections we must first obtain appropriate constitutive equations for linear transport processes. This aim is achieved if we linearize the nonconserved variable evolution equations presented earlier with respect to the thermodynamic forces and if the open hierarchy of evolution equations is closed by setting $\Theta_i^{(1)\sigma\mu\nu}$, $\Theta_i^{(2)\sigma}$, and $\Theta_i^{(3)\sigma\mu}$ $(i = 1, 2, \ldots, m, r)$ equal to zero or quantities belonging to the thermodynamic manifold $\mathfrak{P} \cup \mathfrak{T}$. By means of such closures, the hierarchy of evolution equations is closed at the tensorial moment of rank 2. The linear transport coefficients can then be identified from the steady-state nonconserved variable evolution equations.

Linear Constitutive Equations

Upon imposing the closure and linearizing the kinematic terms with respect to thermodynamic driving forces—spatial gradients of velocity, temperature, pressure, or concentrations—and fluxes and setting $q_n(\kappa) = 1$ in the dissipation terms of evolution equations, we obtain linearized constitutive equations for fluxes. They take the following forms

$$\rho D\widehat{\Phi}_a^{(1)\mu\nu} = -\chi_a^{(1)\mu\nu} - \left(\beta g g_a^{(1)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab}^{(11)} \Phi_b^{(1)\mu\nu}, \tag{2.297}$$

$$\rho D\widehat{\Phi}_a^{(2)} = -\chi_a^{(2)} - \left(\beta g g_a^{(2)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab} \Phi_b^{(2)}, \tag{2.298}$$

$$\rho D\widehat{\Phi}_a^{(3)\mu} = -\chi_a^{(3)} - \left(\beta g g_a^{(3)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(3s)} \Phi_a^{(s)\mu}, \tag{2.299}$$

$$\rho D\widehat{\Phi}_a^{(4)\mu} = -\chi_a^{(4)} - \left(\beta g g_a^{(4)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(4s)} \Phi_b^{(s)\mu}, \tag{2.300}$$

where the subscript a runs over all species, i.e., $a = 1, \ldots, m, r$ and the thermodynamic forces $\chi_a^{(1)\mu\nu}$, etc. are defined by linear thermodynamic gradients

$$\chi_a^{(1)\mu\nu} = 2\mathfrak{p}_a \left[\nabla U\right]^{(2)\mu\nu}, \tag{2.301}$$

$$\chi_a^{(2)} = \frac{5}{3}\mathfrak{p}_a \nabla_\mu U^\mu, \tag{2.302}$$

$$\chi_a^{(3)} = \mathfrak{p}_a \widehat{C}_p \nabla^\mu \ln T, \tag{2.303}$$

$$\chi_a^{(4)} = \rho_a \nabla^\mu \ln \mathfrak{c}_a. \tag{2.304}$$

These thermodynamic forces are the linear approximations of the kinematic terms $\mathfrak{z}_a^{(q)\mu\cdots\nu}$ with respect to the thermodynamic gradients. It should be noted that in the case of diffusion flux evolution equations the divergence term $\nabla_\sigma \varphi_a^{(4)\mu\sigma}$ is combined with the kinematic term $\mathcal{Z}_a^{(4)\mu}$ to obtain the modified kinematic term.

At the steady state of this set of differential equations we obtain the linear steady-state constitutive equations

$$-\chi_a^{(1)\mu\nu} - \left(\beta g g_a^{(1)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab}^{(11)} \Phi_b^{(1)\mu\nu} = 0, \tag{2.305}$$

$$-\chi_a^{(2)} - \left(\beta g g_a^{(2)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \mathfrak{R}_{ab} \Phi_b^{(2)} = 0, \tag{2.306}$$

$$-\chi_a^{(3)} - \left(\beta g g_a^{(3)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(3s)} \Phi_a^{(s)\mu} = 0, \tag{2.307}$$

$$-\chi_a^{(4)} - \left(\beta g g_a^{(4)}\right)^{-1} \sum_{b=1}^{\{m,r\}} \sum_{s=3,4} \mathfrak{R}_{ab}^{(4s)} \Phi_b^{(s)\mu} = 0. \tag{2.308}$$

This set is algebraic with respect to fluxes $\Phi_a^{(q)}$. Since the first two of the set are not coupled to the rest of the equations, they are easily solved for the fluxes. We thus obtain

$$\Phi_a^{(1)\mu\nu} = -\sum_{b=1}^{\{m,r\}} [\mathfrak{B}]_{ab}^{-1} \, [\nabla U]^{(2)\mu\nu}, \tag{2.309}$$

$$\Phi_a^{(2)} = -\sum_{b=1}^{\{m,r\}} [\mathfrak{V}]_{ab}^{-1} \, \nabla_\mu U^\mu, \tag{2.310}$$

where $\mathfrak{B}$ and $\mathfrak{V}$ are $r \times r$ square matrices consisting of elements $\mathfrak{R}_{ab}^{(11)}$ and $\mathfrak{R}_{ab}^{(22)}$, respectively:

$$\mathfrak{B}_{ab} = \frac{\mathfrak{R}^{(11)}}{2\mathfrak{p}_a \beta g}, \tag{2.311}$$

$$\mathfrak{V}_{ab} = \frac{3\mathfrak{R}^{(22)}}{5\mathfrak{p}_a \beta g}. \tag{2.312}$$

Since the third and fourth equations of the set are coupled, we construct a $2r \times 2r$ square matrix $\mathfrak{S}$ consisting of $r \times r$ square submatrices $\left[\mathfrak{R}_{ab}^{(33)} / (\beta g)\right]$, $\left[\mathfrak{R}_{ab}^{(44)} / \beta g\right]$, $\left[\mathfrak{R}_{ab}^{(34)} / \beta g\right]$, and $\left[\mathfrak{R}_{ab}^{(43)} / \beta g_a^{(4)}\right]$:

$$\mathfrak{S} = \begin{pmatrix} \mathfrak{T} & \mathfrak{H} \\ \mathfrak{K} & \mathfrak{D} \end{pmatrix}, \tag{2.313}$$

where the submatrices are defined by $r \times r$ square matrices[11] made up of collision bracket integrals:

$$\mathfrak{T} = \left[\frac{\mathfrak{R}_{ab}^{(33)}}{\beta g} \right], \tag{2.314}$$

$$\mathfrak{H} = \left[\frac{\mathfrak{R}_{ab}^{(34)}}{\beta g} \right], \tag{2.315}$$

$$\mathfrak{K} = \left[\frac{\mathfrak{R}_{ab}^{(43)}}{\beta g} \right], \tag{2.316}$$

$$\mathfrak{D} = \left[\frac{\mathfrak{R}_{ab}^{(44)}}{\beta g} \right]. \tag{2.317}$$

We also construct a $2r$ dimensional column vector spanned by 2 r -dimensional thermodynamic force vectors of $T\nabla^\mu \ln T$ and $\nabla^\mu \ln \mathfrak{c}_j$:

$$\mathfrak{F}^t = (T\nabla^\mu \ln T, \ldots, T\nabla^\mu \ln T, \nabla^\mu \ln \mathfrak{c}_1, \ldots, \nabla^\mu \ln \mathfrak{c}_r)^t \,. \tag{2.318}$$

Then in the notation defined, the solution of the third and fourth equations of the set may be written in matrix form

$$\Phi = \mathfrak{S}^{-1}\mathfrak{F}, \tag{2.319}$$

where Φ denotes the $(2r-1)$ dimensional column vector

$$\Phi = \left(\Phi^{(3)}, \ldots, \Phi^{(4)}, \ldots\right). \tag{2.320}$$

Explicitly written out in components, the solutions of (2.307) and (2.308) are given by

$$\widehat{\Phi}_a^{(3)\mu} = -\sum_{j=1}^{r} \frac{|\mathfrak{S}|_{aj}}{\det|\mathfrak{S}|} T\nabla^\mu \ln T - \sum_{j=1}^{r} \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} \nabla^\mu \ln \mathfrak{c}_j, \tag{2.321}$$

$$\widehat{\Phi}_a^{(4)\mu} = -\sum_{j=1}^{r} \frac{|\mathfrak{S}|_{a+r,j}}{\det|\mathfrak{S}|} T\nabla^\mu \ln T - \sum_{j=1}^{r} \frac{|\mathfrak{S}|_{a+r,r+j}}{\det|\mathfrak{S}|} \nabla^\mu \ln \mathfrak{c}_j, \tag{2.322}$$

where the subscript a runs over all species including photons and $|\mathfrak{S}|_{aj}$ is the (aj) cofactor of matrix $\mathfrak{S}$. The solutions for the linear constitutive equations

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[11] It should be noted here that we have not eliminated the dependent diffusion flux in view of the fact that the number and range of photon spectrum is indeterminate in the present model.

(2.309)–(2.322) enable us to identify the linear transport coefficients of matter and radiation species under consideration.

Linear Transport Coefficients of Matter And Radiation

The linear transport coefficients are now easily identified with matrix elements of the solutions of the constitutive equations obtained above. The shear viscosity η^0 is found to have the statistical mechanical formula

$$\eta^0 = \sum_{a=1}^{r} \eta_a^0 := \sum_{a,b=1}^{r} \frac{1}{2} (\mathfrak{B})_{ab}^{-1} ; \tag{2.323}$$

the bulk viscosity ζ the formula

$$\eta_{\mathrm{B}}^0 = \sum_{a} \eta_{\mathrm{B}a}^0 := \sum_{a,b=1}^{r} (\mathfrak{V})_{ab}^{-1} ; \tag{2.324}$$

the thermal conductivity λ the formula

$$\lambda_a^0 = \sum_{j=1}^{r} \lambda_{aj}^0 := \sum_{j=1}^{r} \frac{|\mathfrak{S}|_{aj}}{\det |\mathfrak{S}|} ; \tag{2.325}$$

the diffusion coefficients D_{ab} the formula

$$D_a^0 = \sum_{j=1}^{r} D_{aj}^0 := \sum_{j=1}^{r} \frac{|\mathfrak{S}|_{a,r+j}}{\det |\mathfrak{S}|} ; \tag{2.326}$$

and the thermal diffusivity the formulas

$$D_{ab}^{\mathrm{d}0} = \frac{|\mathfrak{S}|_{a,r+b}}{\det |\mathfrak{S}|} ; \tag{2.327}$$

$$D_{ab}^{\mathrm{t}0} = \frac{|\mathfrak{S}|_{a+r,b}}{\det |\mathfrak{S}|} . \tag{2.328}$$

Here the species index a runs over the material species and photons.

In the nonrelativistic theory, the linear transport processes calculated by the method of generalized hydrodynamic equations are found to agree completely with their equivalents calculated in the first-order Chapman–Enskog theory. Therefore the present relativistic theory results for the linear transport coefficients are also expected to give the first-order Chapman–Enskog theory results for the relativistic gases of matter and radiation (photons). It can be shown that the first-order Chapman–Enskog method of solution for the covariant kinetic equation for the system of interest indeed gives the same linear transport coefficients.

2.12.4 Quasilinear Constitutive Equations and Transport Coefficients

If the kinematic terms are linearized with respect to the thermodynamic gradients on the one hand and, on the other hand, the dissipation terms $\Lambda_a^{(q)}$ are approximated in the first-order cumulant form together with the first-order generalized potentials as in (2.294), we obtain a model for nonconserved variable evolution equations. When the model is combined with the conserved variable evolution equations, we obtain a model for generalized hydrodynamic equations, which we call quasilinear generalized hydrodynamic equations. The model for the nonconserved variable evolution equations may be written as below:

$$\rho D\widehat{\Phi}_a^{(1)\mu\nu} = -\chi_a^{(1)\mu\nu} - \frac{1}{g_a^{(1)}} \sum_{b=1}^{\{m,r\}} \mathfrak{B}_{ab} \Phi_b^{(1)\mu\nu} q_n(\kappa), \tag{2.329}$$

$$\rho D\widehat{\Phi}_a^{(2)} = -\chi_a^{(2)} - \frac{1}{g_a^{(2)}} \sum_{b=1}^{\{m,r\}} \mathfrak{V}_{ab} \Phi_b^{(2)} q_n(\kappa), \tag{2.330}$$

$$\rho D\widehat{\Phi}_a^{(3)\mu} = -\chi_a^{(3)} - \frac{1}{g_a^{(3)}} \sum_{b=1}^{\{m,r\}} \left(\mathfrak{T}_{ab} \Phi_a^{(3)\mu} + \mathfrak{H}_{ab} \Phi_b^{(4)\mu}\right) q_n(\kappa), \tag{2.331}$$

$$\rho D\widehat{\Phi}_a^{(4)\mu} = -\chi_a^{(4)} - \frac{1}{g_a^{(4)}} \sum_{b=1}^{\{m,r\}} \left(\mathfrak{K}_{ab} \Phi_b^{(3)\mu} + \mathfrak{D}_{ab} \Phi_b^{(4)\mu}\right) q_n(\kappa), \tag{2.332}$$

where the index a runs over all species—i.e., matter and radiation. This set of equations differs from the linear evolution equations (2.297)–(2.300) by the presence of the nonlinear factor $q_n(\kappa)$. Therefore, they are still highly nonlinear and expected to describe flow processes far removed from equilibrium in fluids subjected to high thermodynamic gradients as in the case of nonrelativistic flow processes studied [5, 18, 27, 28] and discussed in Chap. 9 of Volume 1.

The quasilinear generalized hydrodynamic equations therefore include the linearized generalized hydrodynamic equations in the limit of $q_n(\kappa) \to 1$ as $\kappa \to 0$, namely, in the vicinity of equilibrium. Furthermore, they also include the classical hydrodynamic equations—the Navier–Stokes, Fourier, and Fick equations, if the linear transport processes are at steady state.

Quasilinear Transport Coefficients

The steady-state solutions of the quasilinear evolution equations (2.329)–(2.332) provide the constitutive relations for quasilinear transport processes and the associated nonlinear transport coefficients. The steady state solutions in question are as follows:

$$\Phi_a^{(1)\mu\nu} = -\eta_a \left[\nabla U\right]^{(2)\mu\nu}, \tag{2.333}$$

$$\Phi_a^{(2)} = -\eta_{\mathrm{B}a} \nabla_\mu U^\mu, \tag{2.334}$$

$$\Phi_a^{(3)\mu} = -\lambda_a T \nabla^\mu \ln T - \sum_{j=1}^{r} D_{aj}^{\mathrm{d}} \nabla^\mu \ln \mathfrak{c}_j \ \ (a = 1, \ldots, m, r)\,, \tag{2.335}$$

$$\begin{aligned} \Phi_a^{(4)\mu} &= -D_a^{\mathrm{t}} T \nabla^\mu \ln T - \sum_{j=1}^{r} D_{aj} \nabla^\mu \ln \mathfrak{c}_j \\ &\quad (a = r+1, \ldots m-1, r)\,. \end{aligned} \tag{2.336}$$

The nonlinear transport coefficients η_a, $\eta_{\mathrm{B}a}$, λ_a, D_{aj}^{d}, D_a^{t}, D_{aj}—the coefficients to the thermodynamic gradients—are defined by the formulas given below:

$$\eta_a = \sum_{b=1}^{r} (\mathfrak{B})_{ab}^{-1}\, q_L\left(\kappa_L\right) := \eta_a^0 q_L\left(\kappa_L\right), \tag{2.337}$$

$$\eta_{\mathrm{B}a} = \sum_{b=1}^{r} (\mathfrak{V})_{ab}^{-1}\, q_L\left(\kappa_L\right) := \eta_{\mathrm{B}a}^0 q_L\left(\kappa_L\right), \tag{2.338}$$

$$\lambda_a = \sum_{j=1}^{r} \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} q_L\left(\kappa_L\right) := \lambda_a^0 q_L\left(\kappa_L\right), \tag{2.339}$$

$$D_{aj}^{\mathrm{d}} = \frac{|\mathfrak{S}|_{a,r+j}}{\det|\mathfrak{S}|} q_L\left(\kappa_L\right) := D_{aj}^{\mathrm{d}0} q_L\left(\kappa_L\right), \tag{2.340}$$

$$D_a^{\mathrm{t}} = \sum_{j=1}^{r} \frac{|\mathfrak{S}|_{a+r,r+j}}{\det|\mathfrak{S}|} q_L\left(\kappa_{L\kappa}\right) := D_a^{\mathrm{t}0} q_L\left(\kappa_{L\kappa}\right), \tag{2.341}$$

$$D_{aj} = \frac{|\mathfrak{S}|_{a+r,r+j}}{\det|\mathfrak{S}|} q_L\left(\kappa_L\right) := D_{aj}^0 q_L\left(\kappa_L\right), \tag{2.342}$$

where $q_L\left(\kappa_L\right) = \sinh^{-1}\kappa_L/\kappa_L$ with κ_L denoting[12] the quadratic form of thermodynamic gradients of the matter-radiation system under consideration.

The nonlinear transport coefficients $\eta_a, \ldots, D_{aj}$ depends on thermodynamic gradients present in the fluid undergoing nonlinear transport processes described by the evolution equations (2.329)–(2.332). They in fact diminish to zero as ${\kappa_L}^{-1+\delta}$ $(\delta > 0)$ as $\kappa_L \to \infty$. For example, in the case of viscosity there appears a shear thinning effect as the shear rate increases—namely, the fluid is non-Newtonian. Notice that, as their nonrelativistic counterparts do, they also break the Curie principle because the flow process in question can be affected by a thermodynamic gradient of a different spatial symmetry of the flow. In the case of nonrelativistic flow phenomena, they can adequately account for numerous flow behavior in rheology, nonlinear charge carrier transport phenomena, rarefied gas flow, etc. Therefore in the light of the nonlinear transport coefficients for nonrelativistic flows just mentioned the relativistic

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[12] For the detail of getting $q_L(\kappa_L)$ from the nonlinear factor $q_n(\kappa)$ and κ_L, see Chap. 1 of this Volume.

quasilinear transport coefficients listed above and the quasilinear generalized hydrodynamic equations could be also of interest to relativistic flow phenomena in relativistic gases far removed from equilibrium.

2.13 Nonrelativistic Limits of Relativistic Generalized Hydrodynamic Equations

The nonrelativistic limits of the relativistic generalized hydrodynamic equations of matter and radiation and the related quantities presented earlier in the present chapter can be obtained by making use of the methods employed for a similar subject in the absence of radiation in the previous chapter of this Volume. Thus obtained would be the classical radiation hydrodynamic equations for flow processes in matter interacting with radiation. Of particular interest to us here is the radiation part of hydrodynamic equations, because it will provide us classical radiation hydrodynamic equations which have not been discussed in the previous chapters in this work. Since the material part of the classical hydrodynamic equations would be expected to be similar to the classical (nonrelativistic) limits studied in the previous chapter, we will focus our attention to the radiation part of the generalized hydrodynamic equations.

Since the nonrelativistic limits of various operators and quantities considered in the evolution equations in the previous chapter should also apply to the present generalized hydrodynamic equations of matter and radiation, they will be used to obtain the desired results for our goal here. It is convenient to list a few of relevant nonrelativistic limits of quantities below:

$$\begin{aligned} p_a^0 &= m_a c\left(1 + p_a^2/m_a^2 c^2\right)^{1/2} \quad (a = 1, 2, \ldots, m) \\ &\to m_a c, \text{ as } u/c \to 0 \end{aligned} \tag{2.343}$$

in the case of matter particles. Since the hydrodynamic velocity U^μ may be written for the whole system as

$$U^\mu = \left(1 - \frac{u^2}{c^2}\right)^{1/2} (c, \mathbf{u}) = \gamma\,(c, \mathbf{u}) \to (c, \mathbf{u}) \tag{2.344}$$

in the limit of $u/c \to 0$, it is clear that the space-components of U^μ is reduced to the nonrelativistic hydrodynamic velocity in the limit. With this preparation it is straightforward to obtain the nonrelativistic limits of the relativistic generalized hydrodynamic equations. Using the aforementioned limit we find in the case of matter

$$\begin{aligned} p_a^\mu U_\mu &= \gamma\left(c p_a^0 - \mathbf{p}_a \cdot \mathbf{u}\right) \\ &= m_a c^2 + \frac{1}{2} m C_a^2 + O\left(c^{-2}\right), \end{aligned} \tag{2.345}$$

where

$$\mathbf{C}_a = \mathbf{v}_a - \mathbf{u}$$

is the nonrelativistic peculiar velocity of species a, and the $m_a c^2$ term can be neglected in the nonrelativistic theory. This means that the Jüttner function for matter has the following limit

$$f_a^{\mathrm{e}} = \exp\left[-\beta\left(p_a^\mu U_\mu - \mu_a\right)\right] = \lim_{u/c\to 0} \exp\left[-\beta\left(\frac{1}{2}m\mathbf{C}_a^2 - \mu_a\right)\right]. \tag{2.346}$$

Therefore the Maxwell–Boltzmann distribution function is the nonrelativistic limit of the Jüttner distribution function.

The number density tends to the nonrelativistic number density for both matter species and photons:

$$\begin{aligned}\rho_a &= c^{-2}U_\mu N_a^\mu = c^{-1}\int \frac{d^3 p_a}{p_a^0} U_\mu p_a^\mu f_a \\ &= \int d^3 p_a f_a + O(c^{-1}) \\ &= n_a^{(\mathrm{nr})} + O(c^{-1}) \quad (a = 1, 2, \ldots, m, r)\,.\end{aligned} \tag{2.347}$$

Here $n_a^{(\mathrm{nr})}$ denotes the number density in the nonrelativistic kinetic theory.

The covariant operators D and ∇^μ have the nonrelativistic limit as follows :

$$\begin{aligned}D &= U^\mu \partial_\mu = U^0\partial_0 + U^k\partial_k \\ &= \partial_t + \mathbf{u}\cdot\nabla + O(c^{-1}) \\ &:= \frac{d}{dt} + O(c^{-1}).\end{aligned} \tag{2.348}$$

Namely, the relativistic substantial time derivative $\mathfrak{D} = \rho D$ tends to the nonrelativistic substantial time derivative $\rho\frac{d}{dt}$ used in classical fluid mechanics. The space-component of the operator ∇^μ is given by spatial gradient $\nabla_j = \partial/\partial x_j$ in the limit of $u/c \to 0$:

$$\begin{aligned}\nabla^j &= -\Delta^{jk}\partial_k - \Delta^{j0}\partial_0 \\ &= \frac{\partial}{\partial x_j} + O(c^{-1}) \quad (k, j = 1, 2, 3)\end{aligned} \tag{2.349}$$

Therefore there hold the limits

$$\nabla_\mu U^\nu \to \partial u_\nu/\partial x_\mu; \qquad \nabla^\mu U_\nu \to -\partial u_\nu/\partial x_\mu. \tag{2.350}$$

Let us then analyze the nonrelativistic limit of the Boltzmann entropy four-flow. We have defined the Boltzmann entropy in the relativistic kinetic theory[13]

$$S^{\mu}(x) = -k_B c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^\mu f_\alpha \ln f_\alpha \tag{2.351}$$

and

$$\begin{aligned} S := \rho \mathcal{S} &= c^{-2} U_\mu S^\mu(x) \\ &= -k_B \sum_{\alpha=1}^{\{m,r\}} \int d^3 p_\alpha f_\alpha \ln f_\alpha^{(\mathrm{nr})} + O(c^{-1}) \\ &= S^{(\mathrm{nr})} + O(c^{-1}), \end{aligned} \tag{2.352}$$

where $S^{(\mathrm{nr})}$ is the Boltzmann entropy of a nonrelativistic gas mixture and $f_\alpha^{(\mathrm{nr})}$ is the nonrelativistic distribution function. The space-component of Boltzmann entropy flux $J_{\mathrm{s}}^{\mu} = S^\mu - SU^\mu$ is given by

$$\begin{aligned} J_{\mathrm{s}}^{k} &= S^k - SU^k \\ &= -k_B \sum_{i=1}^{\{m,r\}} \int d^3 p_i \mathbf{C}_i^k f_i^{(\mathrm{nr})} \ln f_i^{(\mathrm{nr})} + O(c^{-1}) \\ &:= J_{\mathrm{s}k}^{(\mathrm{nr})} + O(c^{-1}), \end{aligned} \tag{2.353}$$

where $J_{\mathrm{s}k}^{(\mathrm{nr})}$ is the nonrelativistic Boltzmann entropy flux. The Boltzmann entropy production has the limit

$$\begin{aligned} \sigma_{\mathrm{ent}} &= -k_B c \sum_{i=1}^{\{m,r\}} \sum_{i=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} \ln f_\alpha \mathfrak{R}_{\alpha\beta}\left[f_\alpha, f_\beta\right] \\ &= -k_B \sum_{i=1}^{\{m,r\}} \sum_{i=1}^{\{m,r\}} \int d^3 p_\alpha \ln f_\alpha \mathfrak{R}_{\alpha\beta}\left[f_\alpha^{(\mathrm{nr})}, f_\beta^{(\mathrm{nr})}\right] + O(c^{-1}) \end{aligned}$$

---

[13] For example, de Groot et al. [29] define the Boltzmann entropy four-flow by the formula

$$S^{\mu}(x) = -k_B c \sum_{a=1}^{\{m,r\}} \int \frac{d^3 p_a}{p_a^0} p_a^\mu f_a \left(\ln f_a - 1\right).$$

We define $S^\mu(x)$ by

$$S^{\mu}(x) = -k_B c \sum_{a=1}^{\{m,r\}} \int \frac{d^3 p_a}{p_a^0} p_a^\mu f_a \ln f_a$$

without the (-1) factor, which we find superfluous.

$$= \sigma_{\mathrm{ent}}^{(\mathrm{nr})} + O(c^{-1}), \tag{2.354}$$

where $\sigma_{\mathrm{ent}}^{(\mathrm{nr})}$ is the entropy production of nonrelativistic gas expressed in terms of nonrelativistic distribution functions.

The dissipation terms are given by the nonrelativistic limit

$$\begin{aligned}
\Lambda_{a}^{(q)\mu\cdots\nu} &= c \sum_{\beta=1}^{\{m,r\}} \int \frac{d^3 p_\beta}{p_\beta^0} h_\beta^{(q)\mu\cdots\nu} \Re_\beta \left[f_\beta) \right] \\
&= \sum_{\beta=1}^{\{m,r\}} \int d^3 p_\beta h_\beta^{(q)\mu\cdots\nu} \Re_\beta \left[f_\beta^{(\mathrm{nr})} \right] + O(c^{-1}) \\
&:= \Lambda_{(\mathrm{nr})\alpha}^{(q)\mu\cdots\nu} + O\left(c^{-1}\right).
\end{aligned} \tag{2.355}$$

In order to show the nonrelativistic limit of the relativistic generalized hydrodynamic equations, it is necessary to analyze the energy-momentum tensor and other macroscopic variables. Since

$$T^{\mu\nu} = c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^\mu p_\alpha^\nu f_\alpha, \tag{2.356}$$

then its (00) component is given by the limit

$$\begin{aligned}
T^{00} &= c \sum_{a=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^0 p_\alpha^0 f_\alpha^{(\mathrm{nr})} \\
&= nmc^2 + \mathcal{E}_k + \mathcal{E}_r + O(c^{-1}).
\end{aligned} \tag{2.357}$$

On ignoring the rest energy part, it becomes the energy density of a sum of energies of matter and radiation.

Let us consider the components T^{0k} $(k = 1, 2, 3)$:

$$\begin{aligned}
T^{0k} = T^{k0} &= c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^0 p_\alpha^k f_\alpha \\
&= c \sum_{\alpha=1}^{\{m,r\}} \int d^3 p_\alpha p_\alpha^k f_\alpha,
\end{aligned} \tag{2.358}$$

that is, T^{0k}/c becomes the momentum density of a nonrelativistic gas when $u/c \to 0$. Moreover,

$$
\begin{aligned}
T^{kj} &= c \sum_{\alpha=1}^{\{m,r\}} \int \frac{d^3 p_\alpha}{p_\alpha^0} p_\alpha^k p_\alpha^j f_\alpha \\
&= T_{\mathrm{m}}^{(\mathrm{nr})kj} + T_r^{(\mathrm{nr})kj} + O(c^{-1}) \quad (k, j = 1, 2, 3)\,,
\end{aligned}
\tag{2.359}
$$

where $T_{\mathrm{m}}^{(\mathrm{nr})kj}$ and $T_r^{(\mathrm{nr})kj}$ denote the nonrelativistic energy-momentum tensors of matter and radiation.

With the help of the aforementioned properties it is easy to show that the divergence term $\partial_\tau \left(\Delta_\sigma^\tau {}_a^{(q)\sigma\mu\cdots\nu}\right)$ and the kinematic term $Z_a^{(q)\mu\cdots\nu}$ have the nonrelativistic limits as follows:

$$
\begin{aligned}
\partial_\tau \left\langle \Delta_\sigma^\tau f_\alpha {}_\alpha^{(q)\sigma\mu\cdots\nu} \right\rangle &= -\nabla \cdot \int d^3 p_\alpha \mathbf{C}_\alpha f_\alpha^{(\mathrm{nr})} h_\alpha^{(q)\mu\cdots\nu} + O(c^{-1}) \\
&= -\nabla \cdot \boldsymbol{\psi}_\alpha^{(q)\mu\cdots\nu} + O(c^{-1})
\end{aligned}
$$

where $\mathbf{C}_\alpha$ is the peculiar velocity and

$$
\boldsymbol{\psi}_\alpha^{(q)\mu\cdots\nu} = \int d^3 p_\alpha \mathbf{C}_\alpha f_\alpha^{(\mathrm{nr})} h_\alpha^{(q)\mu\cdots\nu}.
$$

The nonrelativistic limit of $Z_a^{(\alpha)\mu\cdots\nu}$ for matter species is given by

$$
\begin{aligned}
Z_a^{(q)\mu\cdots\nu} &= \left\langle p_a^\sigma \partial_\sigma h_a^{(q)\mu\cdots\nu} f_a \right\rangle \\
&= \left\langle f_a^{(\mathrm{nr})} (d_t + \mathbf{C}_a \cdot \nabla) h_a^{(q)\mu\cdots\nu} \right\rangle + O(c^{-1}),
\end{aligned}
\tag{2.360}
$$

for which we have used the limits given below:

$$
\Delta_\mu^0 p_a^\mu = \left(g_\mu^0 - c^{-2} U^0 U_\mu\right) p_a^\mu = c^{-1} \mathbf{u} \cdot \mathbf{p}_a + O(c^{-1}), \tag{2.361}
$$

$$
\Delta_\mu^k p_a^\mu = \left(g_\mu^k - c^{-2} U^k U_\mu\right) p_a^\mu = -m_\alpha C_a^k + O(c^{-1}), \tag{2.362}
$$

$$
\Delta_{\mu\nu} p_a^\mu p_a^\nu = m_a^2 C_a^2 + O(c^{-1}). \tag{2.363}
$$

Therefore, the nonconserved variable evolution equation (2.169), namely, generic evolution equation, tends to the nonrelativistic limit

$$
\begin{aligned}
n^{(\mathrm{nr})} D \widehat{\Phi}_{(\mathrm{nr})a}^{(q)\mu\cdots\nu} &= -\nabla \cdot \boldsymbol{\psi}_a^{(q)\mu\cdots\nu} + \left\langle f_a^{(\mathrm{nr})} (d_t + \mathbf{C}_a \cdot \nabla) h_a^{(q)\mu\cdots\nu} \right\rangle + \Lambda_{(\mathrm{nr})a}^{(q)\mu\cdots\nu} \\
&\quad (a = 1, 2, \ldots, m)\,.
\end{aligned}
\tag{2.364}
$$

Here $d_t = \partial_t + \mathbf{u} \cdot \nabla$ is the nonrelativistic substantial time derivative. Upon explicitly working out the second term on the right of (2.364) we obtain the nonrelativistic evolution equations for nonconserved variables of matter and radiation. For notational brevity we will omit the superscript or subscript nr as well as $\mu \cdots \nu$ from the so-obtained equations with understanding all quantities involved are nonrelativistic.

They are summarized below :

$$
\begin{aligned}
n\frac{d}{dt}\widehat{\Phi}_a^{(1)} &= -\nabla\cdot\boldsymbol{\psi}_a^{(1)} - 2\left[\mathbf{J}_a d_t \mathbf{u}\right]^{(2)} - 2\left[\Pi_a\cdot\nabla\mathbf{u}\right]^{(2)} \\
&\qquad - \frac{2}{3}\Pi_a\left(\nabla\cdot\mathbf{u}\right) - 2\widetilde{\Delta}_a\left[\nabla\mathbf{u}\right]^{(2)} - 2p_a\left[\nabla\mathbf{u}\right]^{(2)} + \Lambda_a^{(1)},
\end{aligned}
\tag{2.365}
$$

$$
\begin{aligned}
n\frac{d}{dt}\widehat{\Phi}_a^{(2)} &= -\nabla\cdot\boldsymbol{\psi}_a^{(2)} - \frac{2}{3}\mathbf{J}_a\cdot d_t\mathbf{u} - \frac{2}{3}\Pi_a : \left[\nabla\mathbf{u}\right]^{(2)} - \frac{2}{3}\widetilde{\Delta}_a\left(\nabla\cdot\mathbf{u}\right) \\
&\qquad - p_a d_t \ln\left(p_a v^{5/3}\right) - \nabla\cdot\left(\mathbf{J}_a p_a/\rho_a\right) + \Lambda_a^{(2)},
\end{aligned}
\tag{2.366}
$$

$$
\begin{aligned}
n\frac{d}{dt}\widehat{\Phi}_a^{(3)} &= -\nabla\cdot\boldsymbol{\psi}_a^{(3)} - d_t\mathbf{u}\cdot\left(\mathbf{P}_a - p_a\boldsymbol{\delta}\right) + \mathbf{Q}_a'\cdot\nabla\mathbf{u} - \mathbf{P}_a\cdot\nabla\widehat{h}_a \\
&\qquad + \boldsymbol{\phi}_a^{(3)} : \nabla\mathbf{u} - \mathbf{J}_a d_t\widehat{h}_a + \Lambda_a^{(3)},
\end{aligned}
\tag{2.367}
$$

$$
n\frac{d}{dt}\widehat{\Phi}_a^{(4)} = -\nabla\cdot\mathbf{P}_a - p_a d_t\mathbf{u} - \mathbf{J}_a\cdot\nabla\mathbf{u} + \Lambda_a^{(4)},
\tag{2.368}
$$

where the subscript a stands for matter species and the superscript (nr) on $n^{(\mathrm{nr})}$ is dropped for brevity of notation. The number density n multiplying the substantial derivative on the left-hand side of (2.365)–(2.368) would be replaced by mass density ρ, if the density $\widehat{\Phi}_a^{(q)}$ of $\Phi_a^{(q)}$ were redefined with respect to mass density instead of the number density n, which was compelled in the relativistic theory. In the equations presented above, the following definitions of symbols are used for brevity of notation:

$$
\boldsymbol{\psi}_a^{(3)} = \left\langle f_a m_a \mathbf{C}_a\mathbf{C}_a\mathbf{C}_a\right\rangle .
\tag{2.369}
$$

In the case of radiation, we obtain the nonrelativistic limits of the evolution equations for nonconserved variables as follows:

$$
\begin{aligned}
n\frac{d}{dt}\widehat{\Phi}_r^{(1)} &= -\nabla\cdot\boldsymbol{\psi}_r^{(1)} - 2\left[\mathbf{J}_r d_t \mathbf{u}\right]^{(2)} - 2\left[\Pi_r\cdot\nabla\mathbf{u}\right]^{(2)} \\
&\qquad - \frac{2}{3}\Pi_r\left(\nabla\cdot\mathbf{u}\right) - 2\widetilde{\Delta}_r\left[\nabla\mathbf{u}\right]^{(2)} - 2p_r\left[\nabla\mathbf{u}\right]^{(2)} + \Lambda_r^{(1)},
\end{aligned}
\tag{2.370}
$$

$$
\begin{aligned}
n\frac{d}{dt}\widehat{\Phi}_r^{(2)} &= -\nabla\cdot\boldsymbol{\psi}_r^{(2)} - \frac{2}{3}\mathbf{J}_r\cdot d_t\mathbf{u} - \frac{2}{3}\Pi_r : \left[\nabla\mathbf{u}\right]^{(2)} - \frac{2}{3}\widetilde{\Delta}_r\left(\nabla\cdot\mathbf{u}\right) \\
&\qquad - p_r d_t \ln\left(p_r v^{5/3}\right) - \nabla\cdot\left(\mathbf{J}_r p_r/\rho_r\right) + \Lambda_r^{(2)},
\end{aligned}
\tag{2.371}
$$

$$
\begin{aligned}
n\frac{d}{dt}\widehat{\Phi}_r^{(3)} &= -\nabla\cdot\boldsymbol{\psi}_r^{(3)} - d_t\mathbf{u}\cdot\left(\mathbf{P}_r - p_r\boldsymbol{\delta}\right) + \mathbf{Q}_r'\cdot\nabla\mathbf{u} \\
&\qquad - \mathbf{P}_r\cdot\nabla h_r - \mathbf{J}_r d_t h_r + \Lambda_r^{(3)},
\end{aligned}
\tag{2.372}
$$

$$
n\frac{d}{dt}\widehat{\Phi}_r^{(4)} = -\nabla\cdot\mathbf{P}_r - p_r d_t\mathbf{u} - \mathbf{J}_r\cdot\nabla\mathbf{u} + \Lambda_r^{(4)},
\tag{2.373}
$$

In this sense, the relativistic generalized hydrodynamic equations we have derived are reduced to their nonrelativistic counterparts—namely, the nonrelativistic radiation hydrodynamic equations—under the condition $u/c \to 0$, which we compactly summarize as below:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{u}, \tag{2.374}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot \mathbf{P}, \tag{2.375}$$

$$\rho \frac{\partial \mathcal{E}}{\partial t} = -\nabla \cdot \mathbf{Q} - \mathbf{P} : \nabla \mathbf{u}, \tag{2.376}$$

$$\rho \frac{\partial}{\partial t} \Phi_\alpha^{(q)} = -\nabla \cdot \psi_\alpha^{(q)} + \mathcal{Z}_\alpha^{(q)} + \Lambda_\alpha^{(q)} \qquad (q = 1, 2, \ldots, 4,\ etc.;\ \alpha = 1, 2, \ldots, m, r)\,. \tag{2.377}$$

The matter part of the nonrelativistic generalized hydrodynamic equations are well known results in the scheme of the modified moment method for nonrelativistic gases [5, 18]. The radiation part of the generalized hydrodynamic equations is new and one of important objectives of the present chapter on radiation and matter.

2.14 Relative Boltzmann Entropy Density and Fluctuations

The relative Boltzmann entropy density and its balance equation can be calculated similarly to the calortropy density and its balance equation obtained in the previous section. However, the relative Boltzmann entropy density balance equation is expected to be not integrable in the thermodynamic manifold.

For the task we face here for the relative Boltzmann entropy it is necessary to have the distribution function f_a explicitly as a function of momentum four flow, which of course would be forthcoming if the covariant kinetic equation were solved explicitly, but it is not available at this point in the development of theory. Nevertheless, we may propose a way to determine f_a and therewith deduce possible nature and properties of the relative Boltzmann entropy. Therefore we look for f_a in a form isomorphic mathematically to the nonequilibrium canonical form f_a^{c}, but the parameters therein are not the same as β, μ_a, and $X_a^{(q)}$ appearing in f_a^{c}. Therefore we postulate f_a in the form

$$f_a = \left\{ \exp\left[\beta \left(p_a^\mu U_\mu + \sum_{q \geq 1} X_a^{(q)\mu\cdots\sigma} h_{a\,\nu\cdots\omega}^{(q)} - \mu_a \right) - \epsilon_a \right] \right\}^{-1} \qquad (a = 1, \ldots, m, r), \tag{2.378}$$

which should be contrasted to the nonequilibrium canonical form

$$f_a^{\mathrm{c}} = \left\{ \exp\left[\beta^t \left(p_a^{\mu} U_{\mu} + \sum_{q\geq 1} X_a^{t(q)\mu\cdots\sigma} h^{(q)}_{a\,\nu\cdots\omega} - \mu_a^t \right) - \epsilon_a \right] \right\}^{-1}$$
$$(a = 1, \ldots, m, r), \qquad (2.379)$$

in which the superscript t is affixed on the parameters β^t, μ_a^t, and $X_a^{t(q)\mu\cdots\sigma}$ to distinguish them from β, μ_a, and $X_a^{(q)}$. The superscript t stands for thermodynamics and the meanings of β^t, μ_a^t, and $X_a^{t(q)\mu\cdots\sigma}$ are fixed according to the thermodynamic correspondence discussed in connection with the calortropy. Therefore the burden of difference between the exact solution f_a of the covariant kinetic equation and the nonequilibrium canonical form f_a^{c} falls on the parameters $\left(\beta, \mu_a, X_a^{(q)}\right)$, and $\Delta f_a := f_a - f_a^{\mathrm{c}}$ may be regarded as a functional of fluctuations of the parameters from the thermodynamic parameters $\left(\beta^t, \mu_a^t, X_a^{t(q)}\right)$. This viewpoint therefore motivates us to define the fluctuations of parameters:

$$\delta\beta = \beta - \beta^t, \quad \delta\left(\beta\mu_a\right) = \beta\mu_a - \beta^t\mu_a^t, \quad \delta\left(\beta X_a^{(q)}\right) = \beta X_a^{(q)} - \beta^t X_a^{t(q)} \qquad (2.380)$$

to which we add the fluctuation in $\beta\mathfrak{p}$:

$$\delta\left(\beta\mathfrak{p}\right) = \beta\mathfrak{p} - \beta^t\mathfrak{p}^t, \qquad (2.381)$$

which will be found necessary.

Then inserting (2.378) and (2.379) into the relative Boltzmann entropy density balance equation (2.188) we obtain the equation for $\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}]$:

$$D\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}] = -k_B \left[\delta\beta D\mathcal{E} + \delta\left(\beta\mathfrak{p}\right) Dv - \sum_{a=1}^{r} \delta\left(\beta\mu_a\right) D\mathfrak{c}_a + \sum_{a=1}^{r}\sum_{q\geq 1} \left(\beta X_a^{(q)}\right) \delta \odot D\widehat{\Phi}_a^{(q)} \right] + \rho^{-1}\Delta\Xi_{\mathrm{r}}, \qquad (2.382)$$

where $\Delta\Xi_{\mathrm{r}}$ is given by

$$\begin{aligned} \Delta\Xi_{\mathrm{r}} = & -k_B\beta\left(c^{-2} Q^{\mu} DU_{\mu} + P^{\mu\nu}\nabla_{\nu}U_{\mu} - \mathfrak{p}\nabla_{\nu}U^{\nu} + \sum_{a=1}^{r}\mu_a\nabla_{\nu}J_a^{\nu} \right) \\ & - k_B \sum_{a=1}^{r}\sum_{q\geq 1} \beta X_a^{(q)} \odot \left\langle f_a \partial_{\mu} p_a^{\mu} h_a^{(q)} \right\rangle \\ & - k_B \sum_{a=1}^{r} \partial_{\mu}\left(\Delta_{\nu}^{\mu} \left\langle \epsilon_a p_a^{\nu} \ln\left(1 + \epsilon_a f_a\right) \right\rangle \right). \end{aligned} \qquad (2.383)$$

This balance equation for $\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}]$ is similar to the Pfaffian differential form for calortropy but for $\rho^{-1}\Delta\Xi_{\mathrm{r}}$, which does not vanish. Because of the term $\rho^{-1}\Delta\Xi_{\mathrm{r}}$ the balance equation for $\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}]$ is not integrable in the thermodynamic manifold, but the fluctuations in parameters $\left\{\delta\beta, \delta\left(\beta\mu_a\right), \delta\left(\beta X_a^{(q)}\right), \delta\left(\beta\mathfrak{p}\right)\right\}$ are determining factors of the relative Boltzmann entropy density and ultimately the distribution function f_a. In the limits of

$$\delta\beta, \delta\left(\beta\mu_a\right), \delta\left(\beta X_a^{(q)}\right), \delta\left(\beta\mathfrak{p}\right) \to 0$$

Equation (2.382) reduces to the form

$$D\widehat{\mathcal{S}}_{\mathrm{r}}[f|f^{\mathrm{c}}] = \rho^{-1}\Delta\Xi_{\mathrm{r}}, \tag{2.384}$$

which may be integrated along the path of the irreversible process of under consideration, but its value would be path-dependent. A stochastic theory may be pursued along the line suggested in the previous chapter of this volume. However, we will not pursue such a study here, leaving to future study on the subject.

2.15 Concluding Remarks

We have formulated a covariant kinetic theory for a mixture consisting of matter and radiation by putting the material particles and photons on an equal footing. This kinetic theory has an attendant theory of irreversible processes in a system of radiation and material gases, which is consistent with the laws of thermodynamics. The present formulation is achieved by treating the system as a dilute gas mixture of photons and material particles which interact with each other according to the dynamical laws of mechanics and, especially, at the collisional level, by quantum mechanics. The covariant kinetic equations used are the Boltzmann equations suitably generalized to accommodate the quantum nature of radiation, at least, with regard to their collisions. By applying the modified moment method, thermodynamically consistent solutions for the kinetic equations are obtained and a theory of irreversible thermodynamics is formulated therewith for the system. As it was the case for nonradiative systems, the Boltzmann entropy differential is found to be nonexact—namely, not integrable—in the thermodynamic manifold $\mathfrak{P}\cup\mathfrak{T}$ if the system is away from equilibrium, but the Pfaffian differential form for calortropy is exact in manifold $\mathfrak{P}\cup\mathfrak{T}$ and therefore serves for formulation of irreversible thermodynamics and hydrodynamics consistent with a local form of the thermodynamic laws. Therefore we now have fundamental equations for thermodynamics of irreversible processes in a system of radiation and matter undergoing nonequilibrium processes under the aegis of thermodynamic principles.

A theory of radiative transport processes can be developed by the means of the flux evolution equations presented and study of hydrodynamics can be made. The present covariant formulation removes the weakness inherent to the nonrelativistic kinetic theory reported in Ref. [1] in which photons are treated nonrelativistically. The covariant generalized hydrodynamic equations derived from the relativistic Boltzmann equation have better balanced structures, although more difficult to solve in practice. The generalized hydrodynamic equations, namely, the conservation equations and the flux evolution equations, for the system of radiation and matter can be used to describe irreversible thermodynamic and hydrodynamic processes occurring far removed from equilibrium. The quasilinear generalized hydrodynamic equations presented appear to be applicable to and practicable for nonlinear systems. The theory of transport processes obtained is applied to study radiation transport phenomena in the next chapter.

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Chapter 3
Radiative Transport Coefficients and Their Mutual Relations

In this chapter[1] we apply the covariant kinetic theory of quantum dilute relativistic gases developed in Chap. 2 of this Volume and, in particular, the theory of transport processes to explicitly calculate transport coefficients of photons interacting with matter. A model will be taken for collision processes so as to facilitate explicit computations to compare with empirical results observed in the laboratory. The results of the present application therefore will serve as a validation of the covariant kinetic theory employed, at least, in one aspect of the theory of transport processes in matter and radiation we have formulated in the previous chapter.

In the phenomenological theory of radiative energy transfer [1] the radiative transport coefficients are expressed in terms of the Rossland coefficient, which phenomenologically accounts for radiative absorption by matter interacting with radiation. The radiative transport coefficients so expressed stand in constant ratios independent of material parameters. For example, the ratio of the radiative shear viscosity η_r^0 to the radiative bulk viscosity $\eta_{\mathrm{B}r}^0$ is

$$\frac{\eta_r^0}{\eta_{\mathrm{B}r}^0} = \frac{3}{5},$$

whereas the ratio of the radiative shear viscosity to the radiative thermal conductivity λ_r^0 is

$$\frac{\eta_{\mathrm{B}r}^0}{\lambda_r^0} = \frac{1}{5c^2},$$

where c is the speed of light. If the thermal conductivity is defined with respect to the temperature gradient ∇T instead of $\nabla \ln T$, the latter ratio must be multiplied by T. In the case of material gas, there exist similar relations and they are called Eucken ratios

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[1] This chapter is based on an unpublished paper by K. Mao and B.C. Eu, which was also a part of the Ph.D. thesis of Kefei Mao, 1993, McGill University, Montreal, Quebec, Canada, under the supervision of B.C. Eu.

B.C. Eu, *Kinetic Theory of Nonequilibrium Ensembles,*
Irreversible Thermodynamics, and Generalized Hydrodynamics,
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[2]. Such ratios of radiative transport coefficients not only enable us to compute one radiative transport coefficient from another, but also serve as an internal consistency check for the kinetic theory formulated to study radiative transport processes of interest.

According to the theory formulated in Chap. 2 of this Volume, a covariant relativistic kinetic theory for a nonequilibrium system of radiation and matter provides a molecular theory foundation for the relativistic irreversible processes arising from the interaction of radiation and matter. It also furnishes a method of calculating various radiative and material transport coefficients in terms of the transition probabilities of elementary dynamical processes involving the material particles and photons. The formulas obtained allow explicit calculations of the transport coefficients once dynamical quantities, such as differential cross sections, are known in sufficient detail.

In this chapter, we first present various radiative transport coefficients in terms of the collision bracket integrals reminiscent of the collision bracket integrals appearing in the Chapman–Enskog method of solution for the nonrelativistic Boltzmann equation [3] for a material gas. Using the statistical mechanical formulas for the radiative transport coefficients and applying them to a photon-electron system, we calculate the radiative transport coefficients and their respective ratios. By assuming that the elementary collision dynamical process is the Compton scattering, we explicitly calculate various radiative transport coefficients. The ratios thus calculated are found to be in agreement with the phenomenological values mentioned earlier.

3.1 Linear Constitutive Equations and Transport Coefficients

To begin the discussion on transport coefficients, we summarize the evolution equations for material and radiation fluxes $\Phi_m^{(\alpha)}$ and $\Phi_r^{(\delta)}$ in one place:

$$\rho D\widehat{\Phi}_{\mathrm{m}}^{(q)\mu\nu\cdots l} = -\nabla_\sigma\, \varphi_{\mathrm{m}}^{(q)\mu\nu\cdots l\sigma} + \mathfrak{Z}_{\mathrm{m}}^{(q)\mu\nu\cdots l} + \Lambda_{\mathrm{m}}^{(q)\mu\nu\cdots l} \;\; (q = 1, \cdots, 4)\,, \tag{3.1}$$

$$\rho D\widehat{\Phi}_{r}^{(\delta)\mu\nu\cdots l} = -\nabla_\sigma\, \varphi_{r}^{(q)\mu\nu\cdots l\sigma} + \mathfrak{Z}_{r}^{(\delta)\mu\nu\cdots l} + \Lambda_{r}^{(\delta)\mu\nu\cdots l} \;\; (\delta = 1, \cdots, 4)\,. \tag{3.2}$$

These equations are coupled to material conservation laws (2.163)–(2.166) and radiation concentration fraction and energy balance equations (2.167) and (2.168) of Chap. 2 of the present Volume. The combined set of conservation equations and flux evolution equations constitute generalized hydrodynamic equations for the system of matter and radiation. The solutions of these equations subject to suitable initial and boundary conditions describe flow processes of the relativistic fluid (gas) in interaction with radiation. On the basis of the evolution equations presented we have derived kinetic theory formulas for various transport coefficients for the system of matter and radiation in the previous chapter. We list the relevant transport coefficients in the following.

3.1.1 Relevant Linear Transport Coefficients

In this work we are interested in three kinds of transport coefficients, for which we list the kinetic theory results below:

Shear viscosity:

$$\eta_{ab}^{0} = \left(\beta g \mathfrak{p}_b / \rho g_a^{(1)}\right) \left[\mathfrak{R}_{ab}^{(11)}\right]_{ab}^{-1}, \tag{3.3}$$

Bulk viscosity:

$$\eta_{\mathrm{B}ab}^{0} = \left(5\mathfrak{p}_b \beta g / 3\rho g_a^{(2)}\right) \left[\mathfrak{R}_{ab}^{(22)}\right]_{ab}^{-1}, \tag{3.4}$$

Thermal conductivity:

$$\lambda_{ab}^{0} = \left(c^2 \mathfrak{p}_b \beta g / \rho g_a^{(3)}\right) \left[\mathfrak{R}_{ab}^{(33)}\right]_{ab}^{-1}. \tag{3.5}$$

In practice, since the component transport coefficients are generally not measured, perhaps, except for photons, the transport coefficients defined in (3.3)–(3.5) must be summed over species to obtain the corresponding coefficients measured for matter as a whole in the laboratory. The transport coefficients for matter are given by the formulas

$$\eta_{\mathrm{m}}^{0} = \sum_{a,b \neq r} \eta_{ab}^{0}, \tag{3.6}$$

$$\zeta_{\mathrm{m}}^{0} = \sum_{a,b \neq r} \zeta_{ab}^{0}, \tag{3.7}$$

$$\lambda_{\mathrm{m}}^{0} = \sum_{a,b \neq r} \lambda_{ab}^{0}. \tag{3.8}$$

Here the summation is limited to the matter species only. In the case of radiation, if the coupling terms $\left[\mathfrak{R}_{ab}^{(11)}\right]_{rb}^{-1}$, etc. are neglected, then the radiative transport coefficients are given by the formulas

Shear viscosity:

$$\eta_{r}^{0} = \left(\beta g \mathfrak{p}_r / \rho g_r^{(1)}\right) \left[\mathfrak{R}_{ab}^{(11)}\right]_{rr}^{-1}, \tag{3.9}$$

Bulk viscosity:

$$\zeta_{r}^{0} = \beta g \left(5\mathfrak{p}_r / 3\rho g_r^{(2)}\right) \left[\mathfrak{R}_{ab}^{(22)}\right]_{rr}^{-1}, \tag{3.10}$$

Thermal conductivity:

$$\lambda_{r}^{0} = \beta g \left(c^2 \mathfrak{p}_r / \rho g_r^{(3)}\right) \left[\mathfrak{R}^{(33)}\right]_{rr}^{-1}. \tag{3.11}$$

Especially, in connection with (3.9)–(3.11) we note that

$$g_r^{(1)} = \frac{4}{15}\beta c^{-2}\left\langle\left(U_\mu p_r^\mu\right)^3 f_r^{\mathrm{e}}\right\rangle = 2.96\mathfrak{p}_r, \tag{3.12}$$

$$g_r^{(2)} = \frac{1}{9}\beta c^{-2}\left\langle\left(U_\mu p_r^\mu\right)^3 f_r^{\mathrm{e}}\right\rangle = 1.23\mathfrak{p}_r, \tag{3.13}$$

$$\begin{aligned} g_r^{(3)} &= \frac{1}{3}\beta c^{-2}\left[\left\langle\left(U_\mu p_r^\mu\right)^3 f_r^{\mathrm{e}}\right\rangle - 2a_r\overline{h}_r\left\langle\left(U_\mu p_r^\mu\right)^2 f_r^{\mathrm{e}}\right\rangle + \left(a_r\overline{h}_r\right)^2\left\langle U_\mu p_r^\mu f_r^{\mathrm{e}}\right\rangle\right] \\ &= 18.81c^2\mathfrak{p}_r, \end{aligned} \tag{3.14}$$

where

$$a_r = \Delta_\sigma^\mu\Delta_\tau^\nu\left\langle p_r^\sigma p_r^\tau f_r^{\mathrm{e}}\right\rangle g_{\mu\nu}\left[\beta\Delta_\sigma^\mu\Delta_\tau^\nu\left\langle p_r^\sigma p_r^\tau\left(p_r^\lambda U^\lambda\right) f_r^{\mathrm{e}}\right\rangle g_{\mu\nu}\right]^{-1}, \tag{3.15}$$

$$\overline{h}_r = 4\pi^4 k_B T/90\zeta\left(3\right), \tag{3.16}$$

$$\left\langle\left(U_\mu p_r^\mu\right)^n f_r^{\mathrm{e}}\right\rangle = \frac{2c}{\hbar^3}\int\frac{d^3p_r}{p_r^0}\left(U_\mu p_r^\mu\right)^n f_r^{\mathrm{e}}\ \ (n = 1, 2, 3)\,. \tag{3.17}$$

Here $\zeta\left(3\right)$ is a zeta function [4]. The rest of notations is the same as that in Chap. 2 of this Volume.

In the phenomenological theory the radiative transport coefficients are given in terms of the phenomenological Rossland coefficient $\overline{\kappa}_{\mathrm{RC}}$ as follows [1]:

$$\overline{\eta}_r^0 = \frac{4aT^4}{15c\overline{\kappa}_{\mathrm{RC}}}, \tag{3.18}$$

$$\overline{\zeta}_r^0 = \frac{4aT^4}{9c\overline{\kappa}_{\mathrm{RC}}}, \tag{3.19}$$

$$\overline{\lambda}_r^0 = \frac{4aT^4}{3\overline{\kappa}_{\mathrm{RC}}}, \tag{3.20}$$

where the parameter a is given by

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3}, \tag{3.21}$$

the so-called radiation constant. Therefore these radiative transport coefficients are related to each other by a proportionality constant. For example, we have

$$\frac{\overline{\eta}_r^0}{\overline{\zeta}_r^0} = \frac{3}{5}, \tag{3.22}$$

$$\frac{\overline{\eta}_r^0}{\overline{\lambda}_r^0} = \frac{1}{5c^2}. \tag{3.23}$$

Comparison of $\overline{\eta}_r^0$ and η_r^0 yields the Rossland coefficient $\overline{\kappa}_{\mathrm{RC}}$ in terms of the collision bracket integral $\mathfrak{R}_{rr}^{(11)}$ in the present theory:

$$\overline{\kappa}_{\mathrm{RC}} = \frac{4k_B T g_r^{(1)}}{5cg}\mathfrak{R}_{rr}^{(11)}. \tag{3.24}$$

Similar identifications can be made by means of $\overline{\zeta}_r^0$ and $\overline{\lambda}_r^0$, but they give the equivalent forms since different collision bracket integrals appearing in the transport coefficients are related to each other.

3.1.2 Relevant Collision Bracket Integrals

Since the collision bracket integrals appearing in the transport coefficients in general form are written out explicitly once again here:

$$\begin{aligned}\mathfrak{R}_{rr}^{(q\gamma)} = {} & \frac{cg}{h^6 g_r^{(q)} g_r^{(\gamma)}} \sum_{a,b\neq r}\sum_{s=1}^{r} G_r \int d^3\overline{p}_r \int d^3\overline{p}_a \int d^3\overline{p}_r^* \int d^3\overline{p}_b^* f_r^{\mathrm{e}} f_a^{\mathrm{e}} \\ & \times \xi_{rar^*b^*} W^{(s)}_{ra;r^*b^*}\left(p_r p_a | p_r^* p_b^*\right) \frac{1}{2}\beta^2 \left(h_r^{(q)} - h_r^{(q)*}\right) \odot \left(h_r^{(\gamma)} - h_r^{(\gamma)*}\right),\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathfrak{R}_{ra}^{(q\gamma)} = {} & \frac{cg}{h^6 g_r^{(q)} g_a^{(\gamma)}} \sum_{b\neq r}\sum_{s=1}^{r} G_r \int d^3\overline{p}_r \int d^3\overline{p}_a \int d^3\overline{p}_r^* \int d^3\overline{p}_a^* f_r^{\mathrm{e}} f_b^{\mathrm{e}} \\ & \times \xi_{rar^*b^*} W^{(s)}_{ra;r^*b^*}\left(p_r p_a | p_r^* p_b^*\right) \beta^2 \left(h_r^{(q)} - h_r^{(q)*}\right) \odot \left(h_r^{(\gamma)} - h_r^{(\gamma)*}\right),\end{aligned} \tag{3.26}$$

Here f_r^{e} is taken for the Planck distribution function

$$f_r^{\mathrm{e}} = \left[\exp(\beta p_r^{\mu} U_{\mu}) - 1\right]^{-1}, \tag{3.27}$$

but because the electron is treated classically so that the result can be compared with the literature value using the Compton scattering, we have approximated f_a^{e} for electrons with the classical distribution function

$$f_a^{\mathrm{e}} = \exp\left[-\beta\left(p_i^{\mu} U_{\mu} - \mu_a^{\mathrm{e}}\right)\right], \tag{3.28}$$

where the chemical potential μ_a^{e} is given by

$$e^{-\beta\mu_a^{\mathrm{e}}} = \sum_{a\neq r} \frac{4\pi m_a^2}{\rho} K_2(m_a c^2 \beta) k_B T \tag{3.29}$$

The collision bracket integral $\mathfrak{R}_{ri}^{(\alpha\gamma)}$ given in (3.25) requires a comment. In (3.25), the first collision bracket integral involving r and r' (namely, $i = r'$) in (3.25) is absent because photons do not directly interact with each other and therefore there is no collision event corresponding to the collision bracket integral $[\cdots]_{rr'}$, which appears in the case of matter particle collision bracket integrals.

The collision bracket integrals can be reduced if a more specific form is assumed for the transition probability $W_{ra;r^*j^*}^{(s)}$. In this chapter we will take into account only the Compton scattering. Therefore, the sum over the collision processes denoted by the index s will be reduced to a term and the corresponding transition probability will be denoted by the elastic scattering component $W_{ra;r^*j^*}^{(e)}$ related to the Compton scattering cross section of the electron. Evaluation of the collision bracket integrals for such a scattering process is described in the next section.

3.2 Evaluation of the Collision Bracket Integrals

In order to be specific, we shall consider a photon-electron system where the electron is treated as a relativistic classical particle whereas the photon is treated quantum mechanically. Only the electron-photon elastic scattering is taken into consideration. Therefore the question of divergence associated with the long-range Coulomb scattering does not arise and hence the Coulomb logarithm is not present in the collision integrals. Although the present treatment is specific to the photon-electron system, the method used is basically the same as for plasmas in general and relativistic particles. Therefore the system considered is a physically realistic example covered by the present kinetic theory. The procedure used for computing the collision bracket integrals is similar to the work of de Groot et al. [5].

There are only two sets of four-momenta for the present system, namely, those of the photon and electron. They will be distinguished by the subscripts r and e, respectively. As a preparation to discuss the collision process of an electron and photon

$$\hbar\omega\left(r\right) + e \rightarrow \hbar\omega^*\left(r^*\right) + e^*, \tag{3.30}$$

we first introduce the transformation of four-momenta

$$P^\mu = p_r^\mu + p_e^\mu = p_r^{*\mu} + p_e^{*\mu}, \tag{3.31}$$

$$Q^\mu = \overline{\Delta}^{\mu\nu}\left(p_{r\nu} - p_{e\nu}\right), \quad Q^{*\mu} = \overline{\Delta}^{\mu\nu}\left(p_{r\nu}^* - p_{e\nu}^*\right), \tag{3.32}$$

where $\overline{\Delta}^{\mu\nu}$ is a projector defined by

$$\overline{\Delta}^{\mu\nu} = g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \tag{3.33}$$

with $P^2 = P^\mu P_\mu$. We will also denote by Q^2 the length of the four-relative momentum $Q^2 = Q^\mu Q_\mu$. From (3.31)–(3.33) we can show the following relations:

$$P^\mu Q_\mu = 0, \quad \overline{\Delta}^{\mu\nu} P_\nu = 0, \quad \overline{\Delta}^{\mu\nu} Q_\nu = Q^\mu. \tag{3.34}$$

Therefore the total four-momentum P^μ is perpendicular to the relative four-momentum Q^μ. The four-momenta p_r^μ, etc. can be decomposed into the orthogonal components P^μ and Q^μ as follows:

$$P_r^\mu = \frac{1}{2}\left(1 + d_{re}\right) P^\mu + \frac{1}{2} Q^\mu, \tag{3.35}$$

$$P_e^\mu = \frac{1}{2}\left(1 - d_{re}\right) P^\mu - \frac{1}{2} Q^\mu, \tag{3.36}$$

$$P_e^{*\mu} = \frac{1}{2}\left(1 + d_{re}\right) P^\mu + \frac{1}{2} Q^{*\mu}, \tag{3.37}$$

$$P_e^{*\mu} = \frac{1}{2}\left(1 - d_{re}\right) P^\mu - \frac{1}{2} Q^{*\mu}, \tag{3.38}$$

where

$$d_{re} = \frac{\left(m_r^2 - m_e^2\right) c^2}{P^2} = -\frac{m_e^2 c^2}{P^2} \tag{3.39}$$

because the photon mass is equal to zero. Since the collision process under consideration is elastic, the electron masses are the same before and after the collision. Hence

$$Q^2 = -\left(1 + \frac{m_e^4 c^4}{P^4}\right) P^2 + 2 m_e^2 c^2 = Q^{*2}. \tag{3.40}$$

In order to facilitate the integration of the collision bracket integrals, we will change the variables from p_r^μ, p_e^μ to P^μ, Q^μ, etc. The Jacobian of this transformation is

$$\frac{\partial\left(P, Q\right)}{\partial\left(p_r, p_e\right)} = \frac{\partial\left(P^*, Q^*\right)}{\partial\left(p_r^*, p_e^*\right)} = 16.$$

The transition probability $W^{(e)}$ can be written in terms of the cross section $\sigma\left(P, \Theta\right)$ as follows:

$$W^{(e)} = P^2 \sigma\left(P, \Theta\right) \delta^{(4)}\left(P^\mu - P^{*\mu}\right), \tag{3.41}$$

where $\sigma\left(P, \Theta\right)$ is the differential cross section. For the Compton scattering it is given by the formula [5]:

$$\sigma\,(P,\Theta) = \frac{1}{2} r_0^2\,(1-\xi)\times \\ \left\{1 + \frac{\xi^2\,(1-x)^2}{4\left[1-\frac{1}{2}\xi\,(1-x)\right]} + \frac{1+\left(1-\frac{1}{2}\xi\right)(1-x)}{1-\frac{1}{2}\xi\,(1-x)}\right\}, \tag{3.42}$$

where

$$\xi = \frac{\left(P^2 - m_e^2 c^2\right)}{P^2}, \quad r_0 = \frac{e^2}{m_e c^2}, \quad x = \cos\Theta, \tag{3.43}$$

Θ being the scattering angle defined by

$$\cos\Theta = \frac{\mathbf{Q}\cdot\mathbf{Q}^*}{Q^2}. \tag{3.44}$$

The cross section is expanded in ξ and only up to the quadratic term in ξ will be retained in the subsequent calculations:

$$\sigma\,(P,\Theta) = \frac{1}{2} r_0^2 \left[1 + x^2 + C_1(x)\xi + C_2(x)\xi^2 + O(\xi^3)\right], \tag{3.45}$$

where

$$C_1(x) = x(1-x^2) - (1+x^2),$$
$$C_2(x) = \frac{1}{4}(1-x)\left\{(1-x)\left[1+(1+x)(1+3x)\right] - 4x(1+x)\right\}, \text{ etc.}$$

The volume element of the space may be written as

$$d^3\overline{p}_r d^3\overline{p}_e d^3\overline{p}_r^* d^3\overline{p}_e^* = dM(P)dM(P^*)dM(Q)dM(Q^*), \tag{3.46}$$

where

$$dM(P) = d^4P\theta\left(P^0\right)\theta\left(P^2 - m_e^2c^2\right), \tag{3.47}$$

$$dM(Q) = d^4Q\delta\left(P\cdot Q\right)\delta\left[Q^2 + \left(1 + \frac{m_e^4c^4}{P^4}\right)P^2 - 2m_e^2c^2\right] \tag{3.48}$$

with $\theta(y)$ denoting the Heaviside step function of y. The volume elements $dM(P^*)$ and $dM(Q^*)$ have similar forms only with P^* and Q^* replacing P and Q, respectively.

With the help of the preceding preparation, we are able to write the collision bracket integrals in more useful forms. We begin with $\mathfrak{R}_{rr}^{(11)}$. It may be written in the form

$$\mathcal{R}_{rr}^{(11)} = \frac{gc\beta^2 G_r}{30h^6 g_r^{(1)2}} e^{\overline{\mu}^0} \int dM(P)dM(P^*)dM(Q)dM(Q^*) \times e^{-\beta P^\mu U_\mu} F(P, Q) W^{(e)} \left[U_\mu \left(Q^\mu - Q^{*\mu} \right) \right]^2, \tag{3.49}$$

where

$$\overline{\mu}^{\mathrm{e}} = \beta \left(\mu_r^{\mathrm{e}} + \mu_e^{\mathrm{e}} \right) = \beta \mu_e^{\mathrm{e}}, \tag{3.50}$$

$$F(P, Q) = \frac{1}{\left(1 - e^{-\beta p_r^\mu U_\mu} \right)^2}. \tag{3.51}$$

Recall that the radiation chemical potential is equal to zero at equilibrium. Substitution of the expression for $W^{(e)}$ into (2.123), the collision bracket integral $\mathfrak{R}_{rr}^{(11)}$ can be written in the form

$$\mathfrak{R}_{rr}^{(11)} = \frac{gc\beta^2 G_r}{30h^6 g_r^{(1)2}} e^{\overline{\mu}^{\mathrm{e}}} \left(\mathfrak{R}_r^{(1)} + \mathfrak{R}_r^{(2)} + 2\mathfrak{R}_r^{(3)} \right), \tag{3.52}$$

where

$$\mathfrak{R}_r^{(1)} = \int dM(P)dM(P^*)dM(Q)dM(Q^*) e^{-\beta P^\mu U_\mu} \times F(P, Q) P^2 \sigma(P, \Theta) \left(U_\mu Q^\mu \right)^2 \delta^{(4)} \left(P^\mu - P^{*\mu} \right), \tag{3.53}$$

$$\mathfrak{R}_r^{(2)} = \int dM(P)dM(P^*)dM(Q)dM(Q^*) e^{-\beta P^\mu U_\mu} \times F(P, Q) P^2 \sigma(P, \Theta) \left(U_\mu Q^{*\mu} \right)^2 \delta^{(4)} \left(P^\mu - P^{*\mu} \right), \tag{3.54}$$

$$\mathfrak{R}_r^{(3)} = -\int dM(P)dM(P^*)dM(Q)dM(Q^*) e^{-\beta P^\mu U_\mu} \times F(P, Q) P^2 \sigma(P, \Theta) U_\mu U_\nu Q^\mu Q^{*\mu} \delta^{(4)} \left(P^\mu - P^{*\mu} \right). \tag{3.55}$$

In the subsequent calculation we will put $F(P, Q)$ equal to unity to an approximation. This approximation is tantamount to the condition that β is such that

$$\beta P^\mu U_\mu \gg 1.$$

The correction terms can be calculated in the same manner as for the case of $F(P, Q) = 1$.

It is convenient to define the integral $I(a, b, c|P)$ by

$$I(a, b, c|P) = (\beta P c)^2 \int dM(Q)dM(Q^*) \sigma(P, \Theta) \left(\beta U_\mu Q^\mu \right)^a \times \left(\beta U_\mu Q^{*\mu} \right)^b \left(-\beta^2 c^2 Q^\mu Q_\mu^* \right)^c. \tag{3.56}$$

In the center-of-momentum frame the four-momentum P^μ is time-like whereas the relative momentum Q^μ is space-like:

$$P^\mu = \left(P^0, 0\right) \tag{3.57}$$

and

$$Q^\mu = (0, \mathbf{Q}), \qquad Q^{*\mu} = \left(0, \mathbf{Q}^*\right). \tag{3.58}$$

We assume that the center-of-momentum is oriented in such a way that the space component of $U^\mu = \left(U^0, \mathbf{U}\right)$ is parallel to the z axis. Then, since $c^2 = \left(U^0\right)^2 - (U^z)^2$ and U^0 and U^z may be written as

$$U^0 = U \cdot \mathbf{P}/P \tag{3.59}$$

and

$$U^z = \left[(U \cdot P)^2 / P^2 - c^2\right]^{1/2}. \tag{3.60}$$

Furthermore, if the spherical coordinate angles are denoted by (θ, ϕ) for $\widehat{\mathbf{Q}} = \mathbf{Q}/\left|\mathbf{Q}\right|$ and (θ^*, ϕ^*) for $\widehat{\mathbf{Q}}^* = \mathbf{Q}^*/\left|\mathbf{Q}^*\right|$, respectively, then

$$\widehat{\mathbf{Q}} \cdot \mathbf{U} = \mathbf{U}^z \cos\theta, \qquad \widehat{\mathbf{Q}}^* \cdot \mathbf{U}^* = \mathbf{U}^z \cos\theta^* \tag{3.61}$$

and

$$x = \widehat{\mathbf{Q}} \cdot \widehat{\mathbf{Q}}^* = \cos\Theta = \cos\theta\cos\theta^* + \sin\theta\sin\theta^*\cos\left(\phi - \phi^*\right). \tag{3.62}$$

In such a frame, the collision cross section may be expanded in Legendre polynomials $P_l(\cos\Theta)$:

$$\left(\widehat{\mathbf{Q}} \cdot \widehat{\mathbf{Q}}^*\right)^c \sigma\left(P, \widehat{\mathbf{Q}} \cdot \widehat{\mathbf{Q}}^*\right) = \sum_{l=0}^{\infty} (2l+1)\, \sigma\left(c, l|\beta P c\right) P_l(\cos\Theta), \tag{3.63}$$

where

$$\sigma\left(c, l|\beta P c\right) = \frac{1}{2}\int_{-1}^{1} dx x^c P_l(x) \sigma\left(P, x\right). \tag{3.64}$$

With solid angles $d\Omega$ and $d\Omega^*$ defined by

$$d\Omega = \sin\theta d\theta d\phi, \qquad d\Omega^* = \sin\theta^* d\theta^* d\phi^*, \tag{3.65}$$

and the abbreviation

$$K(a, b, l) = (4\pi)^{-2} \int d\Omega d\Omega^* \left(-\cos\theta\right)^a \left(-\cos\theta^*\right)^b P_l(x) \tag{3.66}$$

the integral $I(a,b,c|P)$ may be written as

$$I(a,b,c|P) = 4\pi^2 \left(i\beta c Q\right)^{(a+b+1)} \left(i\beta c Q^*\right)^{(b+c+1)} \left[(P\cdot U)^2/c^2P^2 - 1\right]^{(a+b)/2} \times \sum_{l=0}^{\infty} (2l+1)\,\sigma\left(c,l|\beta Pc\right) K\left(a,b,l\right). \tag{3.67}$$

It is now straightforward to obtain $\mathfrak{R}_r^{(1)}$ in terms of $I(a,b,c|P)$:

$$\begin{aligned}\mathfrak{R}_r^{(1)} &= \beta^{-4}c^{-2}\int dM(P)\, I(2,0,0|P)\exp\left(-\beta p_r^{\mu}U_{\mu}\right) \\ &= \frac{8}{9}\pi^2c^2r_0^2\int dM(P)\exp\left(-\beta p_r^{\mu}U_{\mu}\right) Q^2Q^{*2}(1-\xi) \\ &\quad\times\left[(P\cdot Q)^2/c^2P^2-1\right].\end{aligned} \tag{3.68}$$

To evaluate this integral, we define the reduced variables

$$\tau = \beta p_r^{\mu}U_{\mu}, \qquad \upsilon = \beta P c. \tag{3.69}$$

Then, when the hydrodynamic four-velocity U^{μ} is taken purely time-like, the dimensionless four-vector may be written as

$$\beta p_r^{\mu} c = \left(\tau, \mathbf{1}\left(\tau^2-\upsilon^2\right)^{1/2}\right) \tag{3.70}$$

and the volume integral element $dM(P)$ is expressible in the spherical coordinates as

$$(\beta c)^4\, d^4P = (\beta c)^4\, \upsilon\left(\tau^2-\upsilon^2\right)^{1/2} d\tau d\upsilon d\boldsymbol{\Omega}. \tag{3.71}$$

Furthermore,

$$Q^2Q^{*2} = (\beta c)^{-4}\,\upsilon^{-4}\left(\tau^2-z^2\right)^4, \tag{3.72}$$

$$1-\xi = 1-\left(P^2-m_e^2c^2\right)/P^2 = z^2\upsilon^{-2}, \tag{3.73}$$

here z being defined by

$$z = \frac{m_e c^2}{k_B T}. \tag{3.74}$$

In terms of these variables the integral may be written in the form

$$\mathfrak{R}_r^{(1)} = \frac{8\pi^2c^2r_0^2z^2}{9(\beta c)^8}\int_z^{\infty}\int_{\upsilon}^{\infty} d\tau\int d\Omega\upsilon^{-7}\left(\upsilon^2-z^2\right)^4\left(\tau^2-\upsilon^2\right)^{3/2}e^{-\tau}. \tag{3.75}$$

By using the integral representation of the modified Bessel function $K_n(z)$ [4, 6], we may write (3.75) in the form

$$\mathfrak{R}_r^{(1)} = \frac{32\pi^3 c^2 r_0^2 z^2}{3\,(\beta c)^8} \int_z^\infty dv v^{-5} \left(v^2 - z^2\right)^4 K_2(v). \tag{3.76}$$

With the help of the asymptotic expansion of $K_2(v)$ in the limit of large z, we obtain

$$\mathfrak{R}_r^{(1)} \approx \frac{2048}{3\sqrt{2}} (4\pi)^2 \sqrt{\pi z} c^2 \sigma_T c^2 (\beta c)^{-8} e^z, \tag{3.77}$$

where σ_T is the Thomson cross section of the electron [7] :

$$\sigma_T = \frac{8}{3\pi} r_0^2 \tag{3.78}$$

with $r_0 = \left(\pi e^2/c^2 m_e\right) = 0.6552\cdots \times 10^{-29}$ m^2. Owing to the symmetry of the integral, it is easy to show

$$\mathfrak{R}_r^{(2)} = \mathfrak{R}_r^{(1)}. \tag{3.79}$$

Taking the same procedure as for $\mathfrak{R}_r^{(1)}$, we obtain $\mathfrak{R}_r^{(3)}$ as below:

$$\mathfrak{R}_r^{(3)} = \frac{6344}{9\sqrt{2}} (4\pi)^2 \sqrt{\frac{\pi}{z}} c^2 (\beta c)^{-8} e^z. \tag{3.80}$$

From (3.77) and (3.80) we obtain the following ratio

$$\frac{\mathfrak{R}_r^{(3)}}{\mathfrak{R}_r^{(1)}} = \frac{1.03}{z}. \tag{3.81}$$

With these relations we finally obtain the collision bracket integral in the form

$$\mathfrak{R}_{rr}^{(11)} = \frac{g c \beta^2 G_r}{15 h^6 g_r^{(1)2}} e^{\beta\mu_e} \mathfrak{R}_r^{(1)} \left(1 + \frac{1.03}{z}\right). \tag{3.82}$$

The shear viscosity of the photon gas is then given by

$$\eta_r^0 = 0.0305 \frac{k_B T}{c \sigma_T}, \tag{3.83}$$

for which the term of order z^{-1} [see (3.74) for z.] is neglected as will be for other transport coefficients.

The collision bracket integral $\mathfrak{R}_{rr}^{(22)}$ can be evaluated in a manner similar to the procedure used for $\mathfrak{R}_{rr}^{(11)}$. Here we omit the details. The final expression for $\mathfrak{R}_{rr}^{(22)}$ is given by

$$\begin{aligned}\mathfrak{R}_{rr}^{(22)} &= \frac{15}{36}\frac{gc\beta^2}{15h^6 g_r^{(2)2}} e^{\overline{\mu}^0} \int dM(P)dM(P^*)dM(Q)dM(Q^*) \\ &\quad \times e^{-\beta P^\mu U_\mu} F(P,Q) W^{(e)} \left[U_\mu \left(Q^\mu - Q^{*\mu}\right)\right]^2 \\ &= \frac{15}{36}\left(\frac{g_r^{(1)}}{g_r^{(2)}}\right)^2 \mathfrak{R}_{rr}^{(11)}. \end{aligned} \tag{3.84}$$

Therefore, if the approximate value for $\mathfrak{R}_{rr}^{(11)}$ given by (3.82) is used, the bulk viscosity of nonequilibrium photon gas is given by the formula

$$\zeta_r^0 = 0.0508 \frac{k_B T}{c\sigma_T}. \tag{3.85}$$

The collision bracket integral $\mathfrak{R}_{rr}^{(33)}$ can be also calculated similarly. In fact, it is related to $\mathfrak{R}_{rr}^{(11)}$ as follows:

$$\begin{aligned}\mathfrak{R}_{rr}^{(33)} &= \frac{5}{4}\frac{gc\beta^2}{15h^6 g_r^{(3)2}} e^{\overline{\mu}^0} \int dM(P)dM(P^*)dM(Q)dM(Q^*) \\ &\quad \times e^{-\beta P^\mu U_\mu} \Gamma(P,Q) W^{(e)} \left[U_\mu \left(Q^\mu - Q^{*\mu}\right)\right]^2 \\ &= \frac{5}{4}c^2\left(\frac{g_r^{(1)}}{g_r^{(3)}}\right)^2 \mathfrak{R}_{rr}^{(11)}. \end{aligned} \tag{3.86}$$

Thus the thermal conductivity of the photon gas to the same approximation as for the viscosity is given by the formula

$$\lambda_r^0 = 0.154 \frac{k_B T}{c\sigma_T}. \tag{3.87}$$

It is useful to remark that the radiative transport coefficients given in (3.83), (3.85) and (3.87) are independent of the photon and electron densities as are the gas transport coefficients in the Chapman–Enskog approximation, namely, the linear transport coefficients of dilute gases.

We obtain the ratios of the transport coefficients as follows:

$$\frac{\eta_r^0}{\zeta_r^0} = \frac{3}{5}\frac{g_r^{(2)}\mathfrak{R}_{rr}^{(22)}}{g_r^{(1)}\mathfrak{R}_{rr}^{(11)}}, \tag{3.88}$$

$$\frac{\eta_r^0}{\lambda_r^0} = \frac{1}{c^2}\frac{g_r^{(3)}\mathfrak{R}_{rr}^{(33)}}{g_r^{(1)}\mathfrak{R}_{rr}^{(11)}}. \tag{3.89}$$

In view of the relations (3.84) and (3.86), we easily find the following universal ratios of transport coefficients:

$$\frac{\eta_r^0}{\zeta_r^0} = \frac{3}{5}, \tag{3.90}$$

$$\frac{\eta_r^0}{\lambda_r^0} = \frac{1}{5c^2}\,(1 - 0.0165)\,. \tag{3.91}$$

The ratio in (3.90) is the same as the phenomenological theory value whereas the ratio (3.91) is less than 2 % off the phenomenological theory value. This difference is attributed to the approximate values obtained for the various integrals in the expression for $g_r^{(3)}$ presented earlier. For all practical purposes the ratio may be said to be in agreement with the phenomenological value.

In conclusion, we have computed the radiative transport coefficients for the photon-electron system by treating the electron as a relativistic classical particle in the limit of large z. We have also computed the ratios of radiative transport coefficients which are in agreement with the phenomenological theory values. The kinetic theory values of the ratios support the kinetic theory model presented for the system of photons and material gases which are displaced from equilibrium and interact with each other. The present theory provides a well defined molecular theory methods of computing the parameters in the phenomenological theory and, especially, macroscopic observables for nonequilibrium radiation and the phenomenological coefficients such as the Rossland coefficient. Furthermore, although only for an aspect of transport properties of matter and radiation, the agreement between theoretical and experimental results offers implications of broader significance to the radiation hydrodynamics of matter in interaction with radiation, because radiation hydrodynamic equations are anchored on the transport coefficients relevant to flow processes and correctly behaved transport coefficients are essential for theoretical prediction of correctly behaved flow by the hydrodynamic equations employed.

3.3 Concluding Remarks

The principal objective of Chap. 2 of this volume was to develop a kinetic theory for irreversible processes, thermodynamics of irreversible processes, and hydrodynamics for a system of radiation and matter removed from equilibrium. Instead of using the usual radiation energy transfer equation we have postulated covariant kinetic equations for (Wigner) distribution functions for photons and matter interacting with each other. More specifically, the kinetic equations are the covariant adaptation of the Boltzmann–Nordholm–Uehling–Uhlenbeck kinetic equation for quantum particles, which takes into account the exchange symmetries of quantum particles in their dynamical description of distribution functions. This way, the system of radiation and matter is regarded as a mixture of photons and material particles which interact with each other according to the law of mechanics for the system including the quantum nature of particles involved.

In the present chapter we have looked for a way to validate the theory in a rather specific aspect by applying the theory of transport processes in radiation and matter deduced from the kinetic equation employed. Accordingly, we have calculated the ratios of radiation transport coefficients and compared the theoretical results with experimentally observed values. We have shown the comparisons are successful within the limits of approximations made use of for calculations. With the assurance of the present validation made, we may proceed to apply the theories developed in the previous chapter to study theory of transport processes, hydrodynamics, and irreversible thermodynamics for systems of radiation and matter removed far from equilibrium, all of which descend from the covariant kinetic equation postulated.

On the whole, the Chaps. 2 and 3 of this Volume demonstrate the importance of the kinetic theory of radiation and matter as a basis for explaining irreversible processes for systems consisting of photons and material particles. In essence, the present formalism puts the statistical mechanics of nonequilibrium thermodynamics on the par with the Gibbs ensemble theory of equilibrium statistical thermodynamics in the sense that all thermodynamic functions and evolution equations are expressed in terms of $X_i^{(\alpha)}$ and $X_r^{(\alpha)}$, which must be ultimately obtained by solving the generalized hydrodynamic equations of radiation and matter, just as all equilibrium thermodynamic functions and relations are expressed in the Gibbs ensemble theory in terms of a partition function which must be computed for each and every system in the end. The generalized hydrodynamic equations, namely, the conservation equations and the flux evolution equations, presented for the system of radiation and matter can be used to describe irreversible and hydrodynamic processes occurring far from equilibrium.

Finally, we provide some discussions on the validity of kinetic theory for describing irreversible processes of radiation and matter. In particular, we may ask, what approximations, in the underlying physics, are contained in the kinetic equations for photons?

The most important approximation as far as the photon kinetic equation is concerned is that we consider photons basically as point particles which satisfy Bose-Einstein statistics. However, according to quantum mechanics photon has dual character (particle-like and wave-like), therefore, photons also exhibit wave behavior. In other words, a photon in reality is a wave packet. For the point particle picture of a photon to be valid, it is necessary that the spread of the wave packet in phase space (momentum and coordinate) be small [8]. This means that the spread must be small compared to the resolution of interest in the coordinate space ($\mathbf{x}$) and momentum space ($\mathbf{p}_r$) or ($\nu, \mathbf{n}$). Since the photon distribution function is written as a function of variables $\mathbf{x}$, ν and $\mathbf{n}$, it is sufficient to specify the phase space coordinates of the center of wave packet and any information concerning the distribution about this center is irrelevant. Owing to the Heisenberg uncertainty principle the wave packet spreads in spatial and momentum space cannot both be made arbitrarily small at the same time. These considerations impose a maximum possible resolution on the spatial and momentum coordinates. In fact, a kinetic equation for photons cannot describe the strong wave behavior manifested in a diffraction and reflection since it does not take into account the wave behavior of photons. These phenomena depend on interference among the waves arising from different scattering centers which scatter the same photon.

The kinetic equation of photons also neglects the effects of refraction and dispersion. It is known that a photon will move at less than the vacuum speed of light in matter with a refraction index other than unity. In particular, if the refractive index is a function of position, the photon will not stream in straight lines between collisions but will undergo (continuous) refraction. In addition, if the refractive index is time dependent, a photon will continuously change its frequency as it streams between collisions. The origin of these effect is due to an interference phenomenon of the scattering of photons which is discussed by Feynman et al. [9] A discussion on the validity of Boltzmann equation for material particles can be found in the book of Smith and Jensen [10].

Nevertheless, the present kinetic theory of radiation and matter is essential to understand some macroscopic phenomena of nonequilibrium systems consisting of photons and material particles. Note that the Maxwell equations are dynamical theory for radiation like Newton's law for classical particles whereas kinetic theory for photons provides a statistical description for photons in which irreversible processes are involved. The role of kinetic theory for photons is similar to that of Boltzmann equations for classical particles. Maxwell's theory and kinetic theory present different levels of description of physical systems.

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List of Symbols

The mathematical symbols appearing infrequently or only in one section or a few pages are not included in this list. Such symbols are generally defined locally. The location of a symbol is given when it occurs for the first time either by the equation number or the page number or redefined in a slightly different context or equation.

Italic and Roman Symbols

a_{SB}	Stefan–Boltzmann constant	p. 110, 112
$\mathbf{C}_a$	Nonrelativ. peculiar velocity	p. 17, 18, 19
$\widehat{C}_{pa}$	Constant p heat capacity/mass	p. 62, 63, 71, 72, 77
$C(f_a f_b)$	Boltzmann–Nordholm–Uehling–Uhlenbeck (BNUU) collision term	p. 102
$C_r(f_a f_b \vert f_k^* f_l^*)$	BNUU radiation collision term	p. 102
$C_{ab}\left(f_a, f_b\right)$	Collision intergral	p. 4
D_{aj}^0, D_a^0	Diffusion coefficient	p. 66, 161
D_{aj}	Nonlinear diffusion coeff.	p. 163, 164
$D_{aj}^{(dh)0}$	Diffusive heat conductivity	p. 66
$D_{aj}^{(dh)}$	Nonlinear diff heat conductivity	p. 73
$D_{aj}^{(td)0}$	Thermal diffusivity	p. 66
$D_{aj}^{(td)}$	Nonlinear thermal diffusivity	p. 73
E_a	Energy eigenvalue of an atom	p. 100
E, E_a,	Scalar energy,	p. 9, 12, 119
$\mathcal{E}$, $\mathcal{E}_a$	Energy density	p. 9, 12, 37, 68, 112, 116, 127, 137, 153
$\mathcal{E}^{\mathrm{e}}$	Equil. internal energy density	p. 106, 107, 108
$\mathcal{E}_r^{\mathrm{e}}$	Radiation energy density	p. 110, 112

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$F_{\rm L}^{\mu}$	Lorentz force	p. 31
f_a	Distribution function	p. 3, 101, 170
$f_a^{\rm e}(p_a)$	Equilibrium distribution function	p. 30, 32, 33, 35, 105, 179
$f_r^{\rm e}$	Equil. radiation distribution func.	p. 109, 110, 179
$f_a^{\rm c}$	Nonequil. canonical form of f_a	p. 40, 41, 53, 133
$f_{\rm c}^{(N)}$	Nonequil. canonical form of an ensemble	p. 41
$f_r^{\rm c}$	Nonequil. radiation distr. function	p. 134
$f_{\rm c}^{(N)}$	N-particle nonequil. canon. form	p. 41
$F_{\rm c}^{(N)}$	Nonequilibrium canonical form of N particle ensemble	p. 144
$F_{\rm e}^{(N)}$	Equil. grand partition function	p. 142
$\boldsymbol{g}^{\mu\nu}$	Metric tensor	p. 3, 13
$\boldsymbol{g}_a^{(q)}$	Eq. (1.314)	p. 61, 148, 157
$\boldsymbol{g}$	Eq. (1.275)	p. 54, 147
$h_a^{(q)\mu\cdots\nu}$	Nonconserved variable moment	p. 18, 19, 79
$h_r^{(q)\mu\cdots\nu}$	Photon variable moment	p. 124
$\hbar$	Planck constant	p. 99
$\mathcal{H}_a$	Exponent of $f_a^{\rm c}$	p. 133
J_a^{μ}, $\widehat{J}_a^{\mu}$	Number diffusion flux	p. 9, 10, 12, 14, 115, 125
$\mathbf{j}_a$	Particle flux of a	p. 5
J_s^{μ}	Boltzmann entropy flux four-flow	p. 29, 30, 104, 166
$J_{\rm c}^{\mu}$	Caloertropy flux four-flow	p. 43, 44, 130
$\mathbf{J}_r\,[f\,\vert\,f^{\rm c}]$	Relative Boltzmann entropy flux	p. 45, 131
$\mathbf{k}_r$	Photon wave vector	p. 100
$M_a^{(q)\mu\cdots\nu}$	Relativis. moment of a	p. 18
n_a	Number density	p. 5
N_a^{μ}	Particle four-flow of a	p. 5, 6, 8, 113, 122, 138
N^{μ}	Total number four-flow	p. 7, 8, 13
p_a,	Hydrostatic pressure	p. 11, 33
p	Pressure	p. 47, 52
$p^{\rm e}$	Equil. pressure	p. 33, 36, 106
p_a^{μ}	Four-momentum	p. 2, 99
p_a^0	Time component of p_a^{μ}	p. 2
$\overline{\mathbf{p}}_b$	$:= \mathbf{p}_b/p_b^0$	p. 4
$\mathbf{p}_r$, p_r^{μ}	Photon momentum	p. 100, 101, 123, 125
$P_a^{\mu\nu}$	Pressure tensor	p. 11, 12, 78, 117
$\overline{P}^{\mu\nu}$	Decomposition of pressure tensor	p. 15
$\mathbf{q}_r^{\mu}$	Dimensionless photon momentum	p. 123

Q_a^{μ}	Heat flux four-flow	p. 9, 10, 77
Q_{e}^{μ}	Projection of $T_{\mathrm{e}}^{\mu\nu}$	p. 107
$Q_a^{\prime\mu}$	Modified heat flux 4-flow	p. 9, 10, 16, 65, 66
$\mathbf{q}_r$, q_r^{μ}	Dimensionless photon momentum	p. 123
$q_L\ (\kappa)$	Nonlinear factor of thermdy. grad. Γ_κ	p. 72, 163
$q_n\ (\kappa)$	Nonlinear factor	p. 59
$\mathcal{R}^{(\pm)}$	Eqs. (1.281) and (1.282)	p. 55
$\mathbb{R}_{ab}^{(qs)\mu\cdots\sigma\nu\cdots\omega}$	Collision bracket integral	p. 56, 57
$\widehat{\mathbb{R}}_{ab}^{(qs)}$	Contracted coll. bracket integral	p. 57
$R_{ab}^{(\alpha\gamma)\sigma\cdots\mu\omega\cdots\nu}$	Collision bracket integral tensor	p. 148
$\mathcal{R}_{ab}^{(\alpha\gamma)}$	Scalar collision bracket integral	p. 150, 151
S^{μ}	Boltzmann entropy 4-flow	p. 27, 103, 104, 162
$S^{\mathrm{e}\mu}$	Equil. Boltzmann entropy 4-flow	p. 36
S	Scalar Boltzmann entropy	p. 27, 104, 166
S^{e}	Equil. Boltzmann entropy	p. 33, 34, 106, 107, 108
$S_{\mathrm{r}}^{\mu}[f\,\vert\,f^{\mathrm{c}}]$, $\widehat{S}_{\mathrm{r}}[f\,\vert\,f^{\mathrm{c}}]$	Relative Boltzmann entropy, Rel. B. entropy density	p. 45, 82, 131
T^{e}	Equilibrium temperature	p. 34, 36, 37, 38, 108, 110
T	Temperature	p. 51, 135
$T_{\mathrm{ea}}^{\mu\nu}$	Equil. energy-momentum tensor	p. 11, 36, 111
$T_a^{\mu\nu}$	Energy-momentum tensor	p. 5, 6, 11, 12, 112, 116, 163
U^{μ}	Hydrodynamic velocity	p. 3, 7, 13, 14, 67, 73, 114, 124, 159
U_{LR}^{μ}	Rest frame U^{μ}	p. 14
$\mathbf{u}_a$	Particle velocity of a	p. 5
$\mathbf{u}$	Total particle velocity	p. 5, 7
$\mathbf{v}_a$	Velocity of particle a	p. 5
v, v_a	Specific volume, of a	p. 9, 108, 116
$W_{ab}(p_a p_b\vert p_a^* p_b^*)$	Transition rate	p. 4, 103
x^{μ}	Covariant four vector	p. 2
x_{ab}	$:= \omega_a + \omega_b$	p. 54
$X_a^{(q)\beta\gamma\cdots\omega}$	General. potent. of a and order q	p. 61, 148, 151
$Z_a^{(q)\mu\cdots\nu}$	Kinematic term	p. 21, 79, 121, 125, 168

German Symbols

$\mathfrak{B}_{ab}$	Eq. (1.310)	p. 60, 62, 70, 159
$\mathfrak{c}_a$	Mass fraction of a	p. 9, 12, 15, 52, 66, 106, 115, 116, 122, 145, 146
$\mathfrak{C}_a^{\mu}$	Relativ. peculiar moment	p. 18, 19
$\mathfrak{D}_{ab}$	Eq. (1.309), diffusion coeff.	p. 60, 62, 70, 72, 162
$\mathfrak{D}$	Relativistic substantial time derivative	p. 20
$\mathfrak{D}_{ab}$	Diffusion coefficient	p. 60
$\mathfrak{D}$	Matrix of $\mathfrak{D}_{ab}$	p. 65, 160
$\mathfrak{J}_a^{\mu}$	Mech. energy diffusion flow	p. 10, 115
$\mathfrak{h}_a$	Nonequil. enthalpy density	p. 9
$\mathfrak{H}_{ab}$	Eq. (1.300)	p. 60, 62, 63, 70, 72, 162
$\mathfrak{H}$	Matrix of $\mathfrak{H}_{ab}$	p. 65
$\mathfrak{K}$	Matrix of $\mathfrak{K}_{ab}$	p. 65
$\mathfrak{K}_{ab}$	Eq. (1.309)	p. 60, 62, 63, 70, 72, 162
$\mathfrak{p}$, $\mathfrak{p}_a$	Nonequil. pressure, (1.42)	p. 9, 11, 62, 63, 136, 138, 145, 146
$\mathfrak{p}^{\mathrm{e}}$, $\mathfrak{p}_i^{\mathrm{e}}$	Hydrostatic pressure	p. 111, 143
$\mathfrak{R}_{ab}\,[f_a f_b]$	Wang–Uhlenbeck coll. term	p. 101, 102, 111
$\mathfrak{R}_a\,[f_a]$	Collision term	p. 101
$\mathfrak{R}_{ab}^{(qs)}$	Eq. (1.300)	p. 58, 59, 60, 62, 63, 64, 69, 70
$\mathfrak{T}_{ab}$	Eq. (1.309)	p. 60, 62, 63, 70, 71, 72, 74
$\mathfrak{T}$	Matrix of $\mathfrak{T}_{ab}$	p. 65
$\mathfrak{V}_{ab}$	Eq. (1.310)	p. 60, 62, 63, 70, 71
$\mathfrak{z}_a^{(q)\mu\cdots\nu}$	Modified kinematic term	p. 21, 22, 80, 122, 126, 127, 128

Greek Symbols

β_{e}	Inverse temperature, $1/k_B T^{\mathrm{e}}$	p. 35, 36, 37
β	Inverse temperature, $1/k_B T$	p. 133
γ	$:= 1/\sqrt{1-(u/c)^2}$	p. 8
Γ_a^{e}	Normalization factor of $f_a^{\mathrm{e}}(p_a)$	p. 33, 34, 35
Γ_r^{e}	Loc. equil. grand parti. f. of photon	p. 109
Γ_a	Normalization factor of $f_a^{\mathrm{c}}(p_a)$	p. 40, 42
Γ	Normalization factor of $f_{\mathrm{c}}^{(N)}$	p. 42, 46, 47, 49, 52
Γ^{e}	Normalization factor of $F_{\mathrm{e}}^{(N)}$	p. 142, 143
$\Delta^{\mu\nu}$	Projector tensor	p. 3, 11, 13, 14, 17
$\widetilde{\Delta}_a$	Excess normal stress	p. 11, 12, 16, 64, 68
$\widehat{\Delta}_a$	Excess normal stress	p. 68
$\Theta_a^{(q)\sigma\mu\cdots\nu}$	Divegence term	p. 22, 24, 25, 122, 127, 154
κ, κ_2, κ_3	First, second, third cumulant, Rayleigh dissipation function	p. 56, 59, 60, 61, 68, 69, 70
$\overline{\kappa}_R$	Rossland coefficient	p. 179
λ_{ab}^0, λ_a^0	Thermal conductivity	p. 66, 161, 175, 177, 187
λ_{ab}, λ_a	Nonlinear thermal conductivity	p. 73, 163
$\Lambda_a^{(q)\sigma\mu\cdots\nu}$	Dissipation term of a and order q	p. 20, 21, 22, 49, 59, 121, 122, 125, 126, 127, 128, 157, 167, 168
μ_a^{e}	Equil. chemical potential of a	p. 32, 34, 35, 36
μ_r^{e}	Photon chemical potential	p. 108
η_a^0, η^0	Viscosity	p. 64, 66, 161, 175, 177, 179, 187
η_a	Non-Newtonian viscosity	p. 162, 163
η_{Ba}^0, η_B^0	Bulk viscosity	p. 64, 66, 175, 177
η_B	Nonlinear bulk viscosity	p. 73, 163
Ξ_{e}	Equil. grand canon. partition func.	p. 107, 143
Ξ	Nonequil. grand partition func.	p. 42, 47, 145
$\Pi^{\mu\nu}$	Shear stress tensor	p. 12, 16, 64, 68, 70, 117, 122
ρ	Hydrodynamic density	p. 7, 8, 9, 12, 14, 15, 16, 20, 21
ρ_r	Photon number density	p. 109, 110
σ_{ent}	Boltzmann entropy production	p. 28, 29, 30, 104, 105

σ_{c}	Calortropy production	p. 43, 44, 55, 56, 59, 130, 131
$\overline{\sigma}_{\mathrm{c}}$	Reduced calortropy production	p. 55
$\sigma_{\mathrm{r}}[f\,\vert\,f^{\mathrm{c}}]$	Relative Boltzmann entropy prod.	p. 45
Σ_{c}	Kinematic term of calortropy bal. eq.	p. 130, 131
Υ_a	Normalization factor of f_a	p. 81, 82
$\psi_a^{(q)\mu\cdots\nu}$	Supermoment of moment $h_a^{(q)\mu\cdots\nu}$	p. 16, 20, 22, 80, 120, 121
Ψ, $\widehat{\Psi}$	Scalar calortropy, density of Ψ	p. 43, 44, 45, 46, 47, 48, 49, 50, 129, 136
$\widehat{\Psi}_m$, $\widehat{\Psi}_r$	Calortropy density	p. 136
$\Phi_a^{(q)\mu\cdots\nu}$	Flux tensor, projection of supermoment onto U_σ	p. 16, 20, 21, 22, 25, 41, 49, 61, 71, 120, 121, 122, 124, 125, 127, 128
χ_a	Thermodynamic gradient	p. 158, 159, 162
$\Omega_a^{(q)\sigma\mu\cdots\nu}$	Projection of supermoment	p. 20, 21
ω	Photon frequency	p. 100

Mathematical Operations

$\langle\langle A\rangle\rangle$	Collision average	p. 55
d_t	$:= \partial_t + \mathbf{u}\cdot\nabla$	p. 75
D	$:= U^\mu\partial_\mu$	p. 13, 75, 165
D_{LR}	Local rest frame $\partial/\partial t$	p. 14
$\mathfrak{D}$	Relativ. subst. time derivat.	p. 20, 115
∂^μ, ∂_μ	Covariant, contravariant gradient operator	p. 3, 13, 75
∇^μ, ∇_μ	Covariant, contravariant spatial gradient operator	p. 13
∇_{LR}^k	Local rest frame $-\partial/\partial x^k$	p. 14
M_a	Mean value of observable M_a	p. 6
$[\mathbf{P}]^{(2)}$	Traceless symmetric part of $\mathbf{P}$	p. 22
$[\nabla U]^{(2)\mu\nu}$	Eq. (2.275)	p. 158
$[A^{\mu\cdots\sigma}B^{\nu\cdots\omega}]_{ab}$	Collision bracket integral	p. 57, 148
$\langle A_a\,(x,\,p_a)\rangle$	Mean value, relativistic	p. 103
$[p_a p_a]^{(2)\omega\varepsilon}$	$:= p_a^\omega p_a^\varepsilon - \frac{1}{3}\Delta^{\omega\varepsilon}\left(p_a^\sigma p_{a\sigma}\right)$	p. 155

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