Resonant Tunneling

Quantum Waveguides of Variable Cross-Section, Asymptotics, Numerics, and Applications

Lev Baskin, Pekka Neittaanmäki, Boris Plamenevskii, and Oleg Sarafanov

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Preface

Devices based on the phenomenon of electron resonant tunneling are widely used in electronics. Efforts are directed toward refining properties of resonance structures. There are prospects for building new nanosize electronics elements based on quantum dot systems. However, the role of resonance structure can also be given to a quantum wire of variable cross-section. Instead of an "electrode—quantum dot—electrode" system, one can use a quantum wire with two narrows. A waveguide narrow is an effective potential barrier for longitudinal electron motion along a waveguide. The part of the waveguide between two narrows becomes a "resonator", where electron resonant tunneling can occur. This phenomenon consists of the fact that, for an electron with energy E, the probability $T(E)$ to pass from one part of the waveguide to the other part through the resonator has a sharp peak at $E = E_{res}$, where E_{res} denotes a "resonant" energy. Such quantum resonators can find applications as elements of nanoelectronics devices and provide some advantages in regard to operation properties and production technology.

In the book, we study electron resonant tunneling in two- and three-dimensional quantum waveguides of variable cross-sections in the time-independent approach. We suggest mathematical models for resonant tunneling and develop asymptotic and numerical approaches for investigating the models. We also present schemes for several electronics devices based on the phenomenon of resonant tunneling. The book is addressed to mathematicians, physicists, and engineers interested in waveguide theory and its applications in electronics.

Contents

Chapter 1
Introduction

1.1 Resonance Structures

As a preliminary, we consider a brief example of resonant tunneling. A resonance structure where resonant tunneling can occur consists of a potential well bordered by two potential barriers, an electron source, and a drain. A "one-dimensional" model of such a structure is exemplified by the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Psi''(x) + U(x)\Psi(x) = E\Psi(x), \qquad -\infty < x < +\infty, \tag{1.1.1}$$

where $U(x) = U_1$ for $x \in [x_1, x_2]$, $U(x) = U_2$ for $[x_3, x_4]$, U_1 and U_2 are positive constants, and $x_1 < x_2 < x_3 < x_4$; moreover, $U(x) = 0$ for the rest x. The parts of U over $[x_1, x_2]$ and $[x_3, x_4]$ are called potential barriers and $[x_2, x_3]$ is a potential well; the barriers and the well comprise a resonator. An electron wave function Ψ satisfies Eq. (1.1.1), where E is the electron energy, m is the electron mass, and U is the electron energy. Besides, Ψ can be chosen to satisfy the equalities (see, e.g., [3, 13, 18])

$$\Psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{as } x < x_1, \\ te^{ikx} & \text{as } x > x_4, \end{cases}$$

where $k = (2mE/\hbar^2)^{1/2}$. For $x < x_1$, functions e^{ikx} and re^{-ikx} are considered as an incoming wave and a reflected wave, respectively, and, for $x > x_4$, te^{ikx} is a transmitted wave. The values $T(E) = |t(E)|^2$ and $R(E) = |r(E)|^2$ are called a transmission coefficient and a reflection coefficient. It turns out that $T(E) + R(E) = 1$; the $T(E)$ ($R(E)$) is interpreted as a probability for the electron to transmit through the resonator (to be reflected from the resonator). Under certain conditions, there exists a "resonance" E_{res}, $0 < E_{res} < \min\{U_1, U_2\}$, such that for $E = E_{res}$ the transmission coefficient T takes a maximal value (in particular, $T(E_{res}) = 1$ can be the case).

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Then with probability close to 1 the electron transmission can take place, under the barriers, through the resonator. This phenomenon is called resonant tunneling.

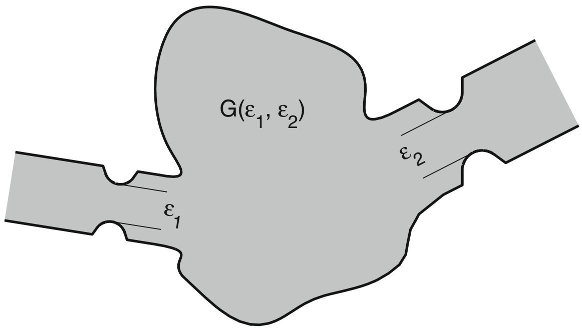
There is a variety of electronics devices (transistors, key devices, energy monochromators) based on resonant tunneling. Classic two-barrier resonant devices use the process of one-dimensional resonant tunneling. Efforts are directed towards refining production technology and operation properties of resonance structures. At present, electron tunneling is being studied intensively in the "metallic electrode-quantum dot-metallic electrode" systems (e.g., see [2, 45]). A quantum dot is a conductive domain of about 10 nm size and is separated from electrodes by "tunnel" intervals (vacuum gaps or dielectric layers). Owing to resonant tunneling, the conductivity of such a system can abruptly vary with voltage between the electrodes. There are prospects for building new nanosize electronics elements that are based on the aforementioned quantum dot systems and have a frequency-operating range of around 10^{12} Hz. However, the properties of such systems heavily depend on inevitable inhomogeneities of the electrode-vacuum and quantum dot-vacuum interfaces. Therefore, the production of the systems must satisfy not easily accessible accuracy conditions.

The role of resonant structures can be given to quantum wires. Resonant tunneling occurs as an electron propagates in a quantum waveguide (wire) of variable cross-section. Instead of an "electrode-quantum dot-electrode" system, one can use a quantum wire with two narrows. This can be explained heuristically by the following reasons. For simplicity, let us consider a waveguide whose cross-section is a disk. If the waveguide is a cylinder, the full energy of an electron is the sum $E = E_\perp + E_\|$, $E_\perp$ being the (quantized) transverse motion energy and $E_\|$ the longitudinal motion energy; $E_\perp$ is inversely proportional to the cross-section square. When a waveguide cross-section varies along the axis, the narrows of the waveguide play the role of effective barriers for the longitudinal motion. Indeed, the full energy E remains constant. One can consider $E = E_\perp + E_\|$ as an approximate relation. In a narrow, $E_\perp$ is increasing, so $E_\|$ is decreasing. For $E_\perp > E$, the electron wave function is exponentially decaying in the narrow just as it does in electron tunneling under a potential barrier. The part of the waveguide between two narrows becomes a "resonator," and conditions for electron resonant tunneling can occur. The tunneling consists of the fact that, for an electron with energy E, the probability $T(E)$ to pass from one part of the waveguide to the other through the resonator has a sharp peak at $E = E_{res}$, where E_{res} denotes a "resonant" energy. That resonant tunneling happens in deformed waveguides was confirmed by numerical experiments in [5, 32].

To analyze the operation of devices based on this phenomenon, it is important to know E_{res}, the behavior of $T(E)$ for E close to E_{res}, the height of the resonant peak, and its width at the half-height (which is inversely proportional to the so-called resonator quality factor). Approximate numerical calculations are effective only if the narrows of a waveguide are "not too narrow" so that the resonant peak is sufficiently wide. That is why, to obtain a detailed picture of the phenomenon, it is of value to use both numerical and asymptotics methods which complement each other.

We consider electron propagation in a waveguide with two cylindrical outlets to infinity and two narrows of small diameters ε_1 and ε_2 (Fig. 1.1). The boundary of

Fig. 1.1 The waveguide with narrows



the waveguide is assumed to be smooth. The electron motion is described by the Helmholtz equation (or the Pauli system for the electron motion in magnetic field). In particular, we obtain asymptotic formulas for the aforementioned characteristics of the resonant tunneling as ε_1 and ε_2 tend to zero.

Resonant devices based on quantum wires can provide advantages in regard to both operation properties and production technology. Such a device is homogeneous, i.e., it is made of one material only. When tunneling, an electron crosses no interfaces of dielectrics, electrodes, or vacuums. Therefore, the operation of the device is more stable under small perturbations of its geometry.

1.2 Scattering Matrix

The basic characteristics of electron resonant tunneling can be expressed in terms of a waveguide scattering matrix. Therefore, when studying tunneling, we mainly analyze the scattering matrix behavior. Chapters 2–4 define scattering matrices, describe their properties, and present a method for approximate computing of such matrices.

In these chapters, we consider waveguides of somewhat more complicated structure (with finitely many cylindrical outlets to infinity) than those in the studies of electron resonant tunneling in the subsequent chapters. In fact, this does not make the discussion more complicated, rather, it provides possibilities for introducing other applications (e.g., see the description of an electron flow switch for quantum nets in Chap. 10).

Chapter 2 presents a radiation principle for the Helmholtz equation in waveguides, that is the solvability of a boundary value problem with radiation conditions, the asymptotics of solutions at infinity, and the scattering matrix definition. In essence, there is given a version (for the Helmholtz equation) of the theory exposed in [37] for the general elliptic self-adjoint elliptic systems in domains with cylindrical ends. (Detailed references are given in the Bibliographical sketch; as a rule, in the body of the book, we restrict ourselves to technical references.)

We are now going to define a scattering matrix, and to this end we need to consider certain issues. Let G be a domain in $\mathbb{R}^{n+1}$, $n = 1, 2$, with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi^1_+ \cup \cdots \cup \Pi^{\mathcal{T}}_+$ of finitely many non-overlapping semi-cylinders

$$\Pi^r_+ = \{(y^r, t^r) : y^r \in \Omega^r,\ t^r > 0\},$$

where (y^r, t^r) are local coordinates in B Π^r_+ and Ω^r is a bounded domain in $\mathbb{R}^n$ (Fig. 1.2). We consider the boundary value problem

$$\begin{aligned} -\Delta\Psi(x) - \mu\Psi(x) = 0, \quad x \in G,\\ \Psi(x) = 0, \quad x \in \partial G, \end{aligned} \tag{1.2.1}$$

with $\Delta = \sum_{j=1}^{n+1} \partial^2/\partial x_j^2$. We suppose that, under certain conditions, electron wave functions satisfy (1.2.1); moreover, the functions are bounded and do not vanish at infinity. To describe the wave function behavior at infinity, we will use solutions to the problem in the cylinder

$$\begin{aligned} -(\Delta_{y,t} + \mu)u(y, t) = 0, \quad (y, t) \in \Omega \times \mathbb{R} = \Pi,\\ u(y, t) = 0, \quad (y, t) \in \partial\Pi, \end{aligned} \tag{1.2.2}$$

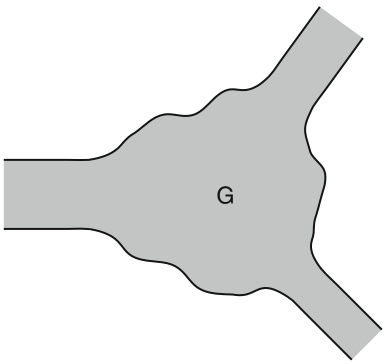
where Ω is a domain in $\mathbb{R}^n$ and

$$\Delta_{y,t} = \Delta_y + \partial_t^2, \quad \Delta_y = \partial_1^2 + \partial_2^2, \quad \partial_j = \partial/\partial y_j.$$

Straightforward calculation shows that the nonzero functions

$$\Omega \times \mathbb{R} \ni (y, t) \mapsto \exp(\pm i(\mu - \tau)^{1/2} t)\varphi(y)$$

Fig. 1.2 The waveguide



satisfy (1.2.2) if and only if

$$-(\Delta_y + \tau)\varphi(y) = 0, \quad y \in \Omega, \qquad (1.2.3)$$
$$\varphi(y) = 0, \quad y \in \partial\Omega,$$

that is, φ has to be an eigenfunction of problem (1.2.3), whereas τ is the corresponding eigenvalue. The eigenvalues of problem (1.2.3) form an increasing positive sequence $\tau_1 < \tau_2 < \cdots$ that tends to $+\infty$. Let us assume, for the time being, that $\tau_1 < \mu < \tau_2$ (recall that τ_1 is a simple eigenvalue). We denote by φ_1 an eigenfunction corresponding to τ_1, normalized by the condition

$$\int_\Omega |\varphi_1(y)|^2\, dy = 1,$$

and set

$$u_1^{\pm}(y,t) = (2|\lambda_1^{\mp}|)^{-1/2} \exp(i\lambda_1^{\mp} t)\varphi_1(y) \qquad (1.2.4)$$

with $\lambda_1^{\pm} = \pm(\mu - \tau_1)^{1/2}$. Functions (1.2.4) are bounded, satisfy (1.2.2), and do not decay at infinity. We will call the u_1^+ (u_1^-) a wave incoming from $+\infty$ (outgoing to $+\infty$).

For $\mu \in (\tau_2, \tau_3)$, besides $u_1^{\pm}$ in (1.2.4), we have waves of the form

$$u^{\pm}(y,t) = (2|\lambda_2^{\mp}|)^{-1/2} \exp(i\lambda_2^{\mp} t)\psi(y), \qquad (1.2.5)$$

where $\lambda_2^{\pm} = \pm(\mu - \tau_2)^{1/2}$ and ψ is an eigenfunction of problem (1.2.3) corresponding to τ_2. The number of pairs of the form (1.2.5) is equal to the multiplicity $\varkappa(\tau_2)$ of the eigenvalue τ_2; as an eigenfunction ψ, the elements $\psi_1, \dots, \psi_{\varkappa(\tau_2)}$ of a basis in the eigenspace of problem (1.2.3) have to be chosen, subject to the orthogonality and normalization conditions

$$\int_\Omega \psi_p(y)\overline{\psi_q(y)}\, dy = \delta_{p,q}, \quad p, q = 1, \dots, \varkappa(\tau_2).$$

In general, for $\mu \in (\tau_l, \tau_{l+1})$, the number of the wave pairs in the cylinder Π is equal to $\varkappa(\tau_1) + \cdots + \varkappa(\tau_l)$.

Let μ be different from the eigenvalues of problems (1.2.3) in $\Omega_1, \dots, \Omega_{\mathcal{T}}$. Given μ, we enumerate all wave pairs in the cylinders $\Pi_1, \dots, \Pi_{\mathcal{T}}$ by the same index $j = 1, 2, \dots, M$. Among electron functions in G, there exist $\Psi_1, \dots, \Psi_M$ that admit the representations

$$\Psi_l(x) = u_l^+(x) + \sum_{j=1}^{M} S_{lj} u_j^-(x) + O(\exp(-\varepsilon|x|)) \qquad (1.2.6)$$

for $|x| \to \infty$, sufficiently small positive ε, and $l = 1, \dots, M$. The matrix

$$S(\mu) = \|S_{lj}(\mu)\|_{j,l=1}^{M} \tag{1.2.7}$$

is called the scattering matrix.

Let us discuss the definition in more detail. The eigenvalues of problems (1.2.3) for $\Omega = \Omega_1, \dots, \Omega_{\mathcal{T}}$ are called the thresholds (of the waveguide G). Let τ_1 denote the minimal threshold; $\tau_1 > 0$. We have defined the scattering matrix $S = S(\mu)$ for $\mu > \tau_1$ except the thresholds; later, in Chap. 3, it will be defined at the thresholds as well. The set $[\tau_1, +\infty)$ is called the waveguide continuous spectrum. Thus, the S is a matrix-valued function on the continuous spectrum. The size $M = M(\mu)$ of $S(\mu)$ depends on μ, remains constant between two neighboring thresholds, and jumps at the thresholds increasing to $+\infty$ as μ tends to $+\infty$. It will be shown that, at any threshold τ, there exist both one-sided limits of $S(\mu)$ as $\mu \to \tau \pm 0$ and, moreover, the S is continuous from the right at the threshold τ.

The scattering matrix $S(\mu)$ is unitary for every $\mu \in [\tau_1, +\infty)$. Given μ, we consider a wave pair u_j^+, u_j^-, $j = 1, \dots, M(\mu)$, as a scattering channel. The $|S_{lj}(\mu)|^2$ is interpreted as the probability of an electron, incoming through the lth channel, to go out through the jth channel.

Remark 1.2.1 For the one-dimensional resonance structure (1.1.1) there are two related scattering channels: u_1^+, u_1^- and u_2^+, u_2^-, where $u_1^+(x) = e^{ikx}$ ($u_1^-(x) = e^{-ikx}$) is an incoming (outgoing) wave to the left of the resonator, and $u_2^+(x) = e^{-ikx}$ ($u_2^-(x) = e^{ikx}$) is an incoming (outgoing) wave to the right of the resonator. Thus, the scattering matrix is of size 2×2.

1.3 Method for Approximate Computation of Scattering Matrices

Next we are going to state the method employed for numerical simulation of resonant tunneling. In the introduction, we restrict ourselves to considering the scattering matrix on a finite interval of the continuous spectrum containing no thresholds. In Chap. 4, we modify the method to calculate the scattering matrix also in vicinity of thresholds and present a justification for the method in both of these situations.

Introduce the notation

$$\Pi_+^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r > R\}, \quad G^R = G \setminus \cup_{r=1}^{N} \Pi_+^{r,R}$$

for large R. Then $\partial G^R \setminus \partial G = \Gamma^R = \cup_r \Gamma^{r,R}$, where $\Gamma^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r = R\}$. We seek the row $(S_{l1}, \dots, S_{lM})$ of the scattering matrix $S = S(\mu)$. As approximation to the row, we take the minimizer of a quadratic functional. To construct such a functional, we consider the problem

$$\begin{aligned} -(\Delta+\mu)\mathcal{X}_l^R &= 0, \quad x \in G^R; \\ \mathcal{X}_l^R &= 0, \quad x \in \partial G^R \setminus \Gamma^R; \\ (\partial_\nu + i\zeta)\mathcal{X}_l^R &= (\partial_\nu + i\zeta)(u_l^+ + \sum\nolimits_{j=1}^M a_j u_j^-),\ x \in \Gamma^R, \end{aligned} \tag{1.3.1}$$

where $\zeta \in \mathbb{R}\backslash\{0\}$ is an arbitrary fixed number, ν is an outward normal, and $a_1, \dots, a_M$ are complex numbers.

Let us explain the origin of the problem. Being a solution to problem (1.2.1), the electron wave function Ψ_l satisfies the first two equations (1.3.1). The asymptotics (1.2.6) can be differentiated, so

$$(\partial_\nu + i\zeta)\Psi_l = (\partial_\nu + i\zeta)(u_l^+ + \sum_{j=1}^M a_j u_j^-) + O(e^{-\gamma R})$$

for $a_j = S_{lj}$. Thus, Ψ_l satisfies the last equation in (1.3.1) up to an exponentially small discrepancy. As an approximation for the row $(S_{l1}, \dots, S_{lM})$, we take the minimizer $a^0(R) = (a_1^0(R), \dots, a_M^0(R))$ of the functional

$$J_l^R(a_1, \dots, a_M) = \|\mathcal{X}_l^R - u_l^+ - \sum_{j=1}^M a_j u_j^-; L_2(\Gamma^R)\|^2, \tag{1.3.2}$$

where $\mathcal{X}_l^R$ is a solution to problem (1.3.1). One can expect that $a_j^0(R, \mu) \to S_{lj}(\mu)$ at exponential rate as $R \to \infty$ and $j = 1, \dots, M$.

Let $[\mu', \mu'']$ be an interval of the continuous spectrum without thresholds. In Chap. 4, we prove, in particular, that for all $R \geq R_0$ and $\mu \in [\mu', \mu'']$ there exists a unique minimizer $a(R, \mu) = (a_1(R, \mu), \dots, a_M(R, \mu))$ of functional (1.3.2) and the estimates

$$|a_j(R, \mu) - S_{lj}(\mu)| \leq c(\Lambda)e^{-\Lambda R}, \quad j = 1, \dots, M, \tag{1.3.3}$$

hold with some positive constants Λ and $c(\Lambda)$ independent of R and μ.

1.4 Asymptotic and Numerical Studies of Resonant Tunneling in 2D Waveguides for Electrons of Small Energy

Chapter 5 begins an asymptotic and numerical study of resonant tunneling. Electrons propagate in a $2D$ waveguide that coincides with an infinite strip in a plane having two identical narrows of the diameter ε and symmetric about the waveguide axis. Electron wave functions satisfy the Helmholtz equation in the strip and vanish at its boundary. The electron energy is supposed to be between the first and the second thresholds, so only one scattering channel relates to each of the waveguide outlets to infinity. The purpose is to obtain, as $\varepsilon \to 0$, the asymptotics for the resonant energy E_{res}, the transmission $T(E)$ and reflection $R(E)$ coefficients, and for the resonator quality factor.

It turns out that such asymptotic formulas depend on the limiting shape of the narrows. We assume that the limiting waveguide in a neighborhood of each narrow coincides with two cones intersecting only at their common vertex. We first construct an asymptotics of the corresponding electron wave function by the method of "compound" asymptotic expansions (the general theory of the method was exposed, e.g., in [30, 33]). The expansions contain terms of two kinds: the first kind terms depend on the "slow" variables x and approximate the wave function "far" from the narrows; the second kind terms depend on the "fast" variables x/ε and serve as an approximation in a neighborhood of the narrows. The terms are obtained by solving the so-called first and second kind limit problems, respectively. The analysis of the obtained expansions enables us to get asymptotic formulas for the mentioned characteristics of resonant tunneling.

Let us discuss the situation in more detail. To describe the domain $G(\varepsilon)$ in $\mathbb{R}^2$ occupied by the waveguide, we first introduce two auxiliary domains G and Ω in $\mathbb{R}^2$. The domain G is the strip

$$G = \mathbb{R} \times D = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} = (-\infty, +\infty); \, y \in D = (-l/2, l/2)\}.$$

We denote by K a double cone with vertex at the origin O that contains the x-axis and is symmetric about the coordinate axes. The set $K \cap S^1$, where S^1 is a unit circle, consists of two simple arcs. Assume that Ω contains the cone K and a neighborhood of its vertex; moreover, outside a large disk (centered at the origin) Ω coincides with K. The boundary $\partial\Omega$ of Ω is supposed to be smooth (see Fig. 1.3).

Denote by $\Omega(\varepsilon)$ the domain obtained from Ω by the contraction with center at O and coefficient ε. In other words, $(x, y) \in \Omega(\varepsilon)$ if and only if $(x/\varepsilon, y/\varepsilon) \in \Omega$. Let K_j and $\Omega_j(\varepsilon)$ stand for K and $\Omega(\varepsilon)$ shifted by the vector $\mathbf{r}_j = (x_j^0, 0)$, $j = 1, 2$. We assume that $|x_1^0 - x_2^0|$ is sufficiently large so the distance from $\partial K_1 \cap \partial K_2$ to G is positive. We put $G(\varepsilon) = G \cap \Omega_1(\varepsilon) \cap \Omega_2(\varepsilon)$ (Fig. 1.4).

The wave function of a free electron of energy k^2 satisfies the boundary value problem

$$\begin{aligned} -\Delta u(x, y) - k^2 u(x, y) = 0, \quad (x, y) \in G(\varepsilon), \\ u(x, y) = 0, \quad (x, y) \in \partial G(\varepsilon). \end{aligned}$$

Moreover, u is subject to certain radiation conditions at infinity (that correspond, for example, to an electron wave incoming from $-\infty$).

We set $G(0) = G \cap K_1 \cap K_2$ (Fig. 1.5); thus, $G(0)$ consists of three parts G_0, G_1, and G_2, where G_1 and G_2 are infinite domains, while G_0 is a bounded resonator.

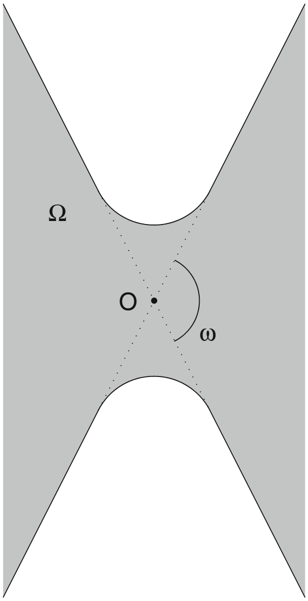


Fig. 1.3 Domain Ω

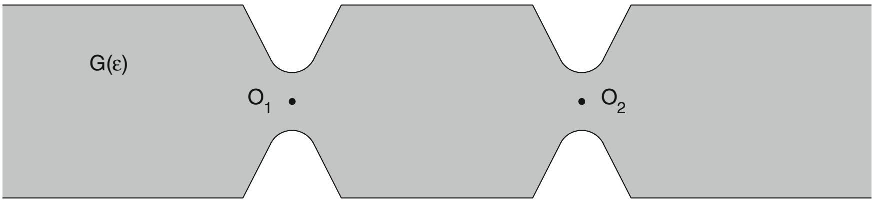


Fig. 1.4 Waveguide $G(\varepsilon)$

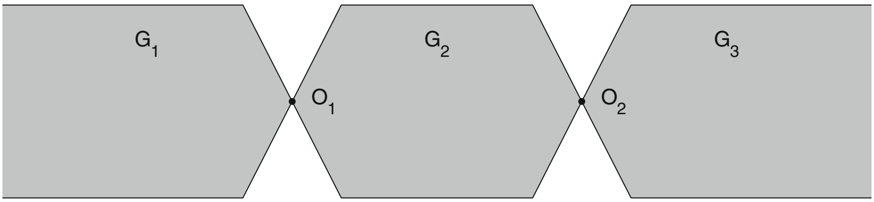


Fig. 1.5 The "limit waveguide" $G(0)$

The problems

$$\begin{aligned} -\Delta v(x, y) - k^2 v(x, y) &= f, \qquad (x, y) \in G_j, \\ v(x, y) &= 0, \qquad (x, y) \in \partial G_j, \end{aligned} \tag{1.4.1}$$

where $j = 0, 1, 2$, are called the first kind limit problems. In the domains Ω_j, $j = 1, 2$, we consider the boundary value problems

$$\begin{aligned} \Delta w(\xi_j, \eta_j) &= F(\xi_j, \eta_j), \quad && (\xi_j, \eta_j) \in \Omega_j, \\ w(\xi_j, \eta_j) &= 0, && (\xi_j, \eta_j) \in \partial\Omega_j, \end{aligned}$$

which are called the second kind limit problems; (ξ_j, η_j) are Cartesian coordinates with origin at O_j.

We denote by k_e^2 a simple eigenvalue of problem (1.4.1) in the resonator G_0 and by $k_r^2(\varepsilon)$ a resonance frequency such that $k_r^2(\varepsilon) \to k_e^2$ as $\varepsilon \to 0$. For $|k^2 - k_r^2| = O(\varepsilon^{2\pi/\omega})$ the asymptotic representations hold:

$$\begin{aligned} T(k, \varepsilon) &= \frac{1}{1 + P^2\left(\dfrac{k^2 - k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}\bigl(1 + O(\varepsilon^{2-\delta})\bigr), \\ k_r^2(\varepsilon) &= k_e^2 + Q\varepsilon^{2\pi/\omega} + O\bigl(\varepsilon^{2\pi/\omega+2-\delta}\bigr), \\ \Upsilon(\varepsilon) &= \frac{1}{P}\varepsilon^{4\pi/\omega}\bigl(1 + O(\varepsilon^{2-\delta})\bigr), \end{aligned}$$

where $T(k, \varepsilon)$ is the electron transmission coefficient and $\Upsilon(\varepsilon)$ is the width of the resonant peak at its half-height (which is inversely proportional to the resonator quality factor), δ being an arbitrarily small positive number; the P and Q are the products of several constants in the asymptotics of limit problem solutions near corners or at infinity.

Without numerical values of the constants, the asymptotic formulas provide only a qualitative picture. To find the constants, one has to solve numerically several boundary value problems. We state the problems and describe a way to solve them. When the constants are found, the asymptotics can be used as an approximate solution. However, it remains uncertain for what band of parameters the approximation is reliable. On the other hand, one should expect numerical approach to be efficient only if the waveguide narrows are not too small in diameter and if the resonant peak of the transition coefficient is sufficiently wide. Therefore a detailed picture of resonant tunneling can be achieved when the asymptotic and numerical approaches are combined. Independently of asymptotic approach, an approximation to the waveguide scattering matrix is calculated. For that purpose, we employ the method from Chap. 4. Then we can compare the asymptotics with calculated constants and the scattering matrix (the transition and reflection coefficients). It turns out, that there is an interval for ε, where the asymptotic and numerical results practically coincide. To the right of the interval, the asymptotics vanishes but the numerical method for calculation of the scattering matrix is effective; to the left of the interval, the numerical method is ill-conditioned while the asymptotics is reliable.

1.5 The Impact of a Finite Waveguide Work Function on Resonant Tunneling

This subject concludes Chap. 5. When considering electron transport in a waveguide, we assume that the electron wave functions vanish at the waveguide boundary. This means that an electron can not get out of the waveguide because of the infinite potential barrier at the boundary. In reality, the assumption has never been fulfilled: the surface potential barrier is always of a finite height and some electrons can penetrate through the waveguide boundary and go away some distance from the waveguide. In other words, in reality we deal with a waveguide of a finite work function. Due to this phenomenon, the effective widths of a waveguide and waveguide narrows are greater than their geometric widths. Therefore, to draw a conclusion about the adequacy of the boundary condition used in the mathematical model, we have to clarify the impact of a finite waveguide work function on the resonant tunneling.

To this end, we present some physics preliminaries concerning work functions, introduce a boundary value problem with regard to a finite work function, and analyze the problem numerically. The results show the need, when employing resonant tunneling in a waveguide with narrows, to restrict somewhat the range of narrow parameters and that of electron energy. In particular, by decreasing the narrow diameter at a resonator, one can not diminish the effective narrow diameter beyond a certain critical value. This restricts the possibility to improve the resonator quality factor by diminishing the narrow diameter. The angle of a wedge-like narrow should not be too small. However, increasing the angle causes an increase in the effective width of the potential barrier and a decrease in the width of the resonant peak. This increases the resonant tunneling time and affects the frequency properties of the system. Optimal angles for wedge-like narrows range between 20° and 35°.

1.6 Asymptotic Study of Resonant Tunneling in 3D Waveguides for Electrons of Small Energy

In Chap. 6, we consider 3D waveguide with two non-overlapping cylindrical outlets $\mathcal{C}_1$ and $\mathcal{C}_2$ to infinity; the axes of the outlets may be of any directions. There are two waveguide narrows, one narrow in $\mathcal{C}_1$ and the other one in $\mathcal{C}_2$. Generally, the narrow diameters ε_1 and ε_2 are different. The resonator (that is, the waveguide part between the narrows) can be of arbitrary form. The boundary of the waveguide is supposed to be smooth. We denote the waveguide by $G(\varepsilon_1, \varepsilon_2)$. The limit set $G(0, 0)$ consists of unbounded parts G_1, G_2, and a bounded resonator G_0. In a neighborhood of the point $O_j = \overline{G_0} \cap \overline{G_j}$, the set $G(0, 0)$ coincides with a double cone $\mathcal{K}_j$, $j = 1, 2$.

A wave function of a free electron of energy $E = \hbar^2 k^2/2m$ satisfies the boundary value problem

$$-\Delta u - k^2 u = 0 \text{ in } G(\varepsilon_1, \varepsilon_2), \quad u = 0 \text{ on } \partial G(\varepsilon_1, \varepsilon_2),$$

and certain radiation conditions at infinity. We consider the scattering of a wave coming from $\mathcal{C}_1$ and seek the resonant values $k_r = k_r(\varepsilon_1, \varepsilon_2)$ of the parameter k, where the transition coefficient $T = T(k, \varepsilon_1, \varepsilon_2)$ takes maximal values.

Let k_e^2 be a simple eigenvalue (between the first and second thresholds) of the boundary value problem in the resonator,

$$-\Delta v(x) - k^2 v(x) = f, \quad x \in G_0; \quad v(x) = 0, \quad x \in \partial G_0.$$

Near such an eigenvalue there is a resonant value $k_r(\varepsilon_1, \varepsilon_2)$ satisfying

$$k_r^2(\varepsilon_1, \varepsilon_2) = k_e^2 + \mathcal{D}_1 \varepsilon_1^{\nu_1} + \mathcal{D}_2 \varepsilon_2^{\nu_2} + O\big(\varepsilon_1^{\nu_1+\tau_1} + \varepsilon_2^{\nu_2+\tau_2}\big)$$

as $\varepsilon_1, \varepsilon_2 \to 0$. The coefficients $\mathcal{D}_1$ and $\mathcal{D}_2$ are constant, ν_j and τ_j are some positive numbers, $j = 1, 2$. Under the condition $|k^2 - k_r^2| = O\big(\varepsilon_1^{2\mu_{11}+1+\tau_1} + \varepsilon_2^{2\mu_{21}+1+\tau_2}\big)$, the transition coefficient $T_1(k, \varepsilon_1, \varepsilon_2)$ admits the asymptotics

$$\begin{aligned} T_1(k, \varepsilon_1, \varepsilon_2) = &\left(\frac{1}{4}\left(z + \frac{1}{z}\right)^2 + P^2 \left(\frac{k^2 - k_r^2}{\varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{2\mu_{21}+1}} \right)^2 \right)^{-1} \\ &\times \big(1 + O(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2})\big), \end{aligned}$$

where τ_j are the same as in (6.1.6), $z = Q\varepsilon_1^{2\mu_{11}+1}/\varepsilon_2^{2\mu_{21}+1}$, while P and Q are constant. The width of the resonant peak at its half-height (calculated for the principal part in the asymptotics of T) is

$$\Upsilon(\varepsilon_1, \varepsilon_2) = |\frac{1}{P}(z + \frac{1}{z})| \varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{2\mu_{21}+1} (1 + O(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2})).$$

1.7 Electron Resonant Tunneling in the Presence of Magnetic Fields

The presence of a magnetic field can essentially affect the basic characteristics of the resonant tunneling and bring new possibilities for applications in electronics. In particular, in the presence of a magnetic field, the tunneling phenomenon is feasible for producing spin-polarized electron flows consisting of electrons with spins of the same direction. In Chaps. 7 and 8 we consider the same 2D and 3D waveguides with narrows as in Chaps. 5 and 6, respectively. A part of the resonator is occupied by a homogeneous magnetic field. An electron wave function satisfies the Pauli equation in a waveguide and vanishes on its boundary. An electron energy is in between the first and the second thresholds. The asymptotics of basic resonant tunneling characteristics are presented as the narrow diameters tend to zero. Moreover, in Chap. 7, the asymptotic results for 2D waveguides are compared with numerical ones obtained by approximate computing the scattering matrix; there is an interval of ε (the narrow diameter) where the asymptotic and numerical results practically coincide. Using the approximate scattering matrix, we also observe the dependence of the tunneling characteristics on a magnetic field position in the resonator.

1.8 Numerical Simulation of High Energy Electron Resonant Tunneling, the Fano Resonances

Chapter 9 is devoted to the numerical simulation of high energy electron scattering. We consider multi-channel resonant tunneling. An electron wave of energy E incident on a resonator with transverse quantum number n passes through the resonator and arises with transverse number k; shortly, the wave passes from state n to state k. We denote by $T_{nk}(E)$ the transmission coefficient of the wave, calculate the dependence $E \to T_{nk}(E)$ by computing the scattering matrix $S(E)$, and obtain $T_{nk}(E) = |S_{nk}(E)|^2$, where $S_{nk}(E)$ is the entry of $S(E)$. The curve $E \to T_{nk}(E)$ can be sufficiently complicated and not always easily interpreted. To explain the curve, we consider $S_{nk}(E)$ as a probability amplitude and represent it in the form $S_{nk}(E) = \sum_s A_{nsk}(E)$, where $A_{nsk}(E)$ is the probability amplitude of the transmission from n to k through an intermediate state s; the summation is over all intermediate states (cf. [19]).

As before, we denote by $G(\varepsilon_1, \varepsilon_2)$ be a waveguide with two narrows and let G_0 be the closed resonator, that is, the bounded part of the limit waveguide $G(0, 0)$(see Fig. 1.5); generally, the resonator form may be arbitrary. We denote by $k_1^2 \le k_2^2 \le \cdots$ the eigenvalues of problem (1.4.1) with $j = 0$ numbered according to their multiplicities. Then the resonant energies of the waveguide $G(\varepsilon_1, \varepsilon_2)$ form the sequence $\mathrm{Re}E_1, \mathrm{Re}E_2, \dots,$ where $E_1, E_2, \dots$ can be viewed as the "perturbed" $k_1^2, k_2^2, \dots$ and $\mathrm{Im}E_j < 0$ for all $j = 1, 2, \dots$. The amplitude A_{nsk} admits the representation

$$A_{nsk}(E) = H_{nk}^{(s)}(E) + \frac{R_{nk}^{(s)}(E)}{E - E_s}$$

with continuous functions $E \to H_{nk}^{(s)}(E)$ and $E \to R_{nk}^{(s)}(E)$. In a small neighborhood of $\mathrm{Re}E_r$,

$$|S_{nk}(E)|^2 = |\sum_s A_{nsk}(E)|^2 \approx |H_{nk}(E_r) + \frac{R_{nk}(E_r)}{E - E_r}|^2 \equiv \mathcal{T}_{nk}(E),$$

where $H_{nk}(E_r)$ and $R_{nk}(E_r)$ are constant. We take the function $\mathcal{T}_{nk}(E)$ as an approximation to the calculated $|S_{nk}(E)|^2$ and find the constants $H_{nk}(E_r)$, $R_{nk}(E_r)$, and E_r by the method of least squares.

1.9 Asymptotic Analysis of Multichannel Resonant Tunneling

In Chap. 10, for electrons of high energy, we generalize the asymptotic theory exposed in Chap. 6. We present and justify the asymptotics of tunneling characteristics as the narrow diameters tend to zero.

1.10 Electronics Devices Based on Resonant Tunneling in Waveguides of Variable Cross-Sections

Chapter 11 presents electronics devices based on a quantum waveguide with narrows: transistors controlled by external electric field and magnetic field sensors controlled by external magnetic field. Besides, we describe an electron flow switch for quantum nets. The switch is not related to resonant tunneling, however, the description is based on analyzing the corresponding scattering matrix calculated by the method of Chap. 4. We present the switch to demonstrate the method.

Chapter 2
Waveguides. Radiation Principle. Scattering Matrices

First, we briefly outline the chapter content. Section 2.1 is devoted to the boundary value problem

$$\begin{aligned}(-\Delta-\mu)u(y,t)&=f(y,t),\quad (y,t)\in\Pi,\\ u(y,t)&=0,\quad (y,t)\in\partial\Pi,\end{aligned}\tag{2.0.1}$$

in the cylinder $\Pi=\{(y,t): y=(y_1,\dots,y_n)\in\Omega, t\in\mathbb{R}\}$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $\mu\in\mathbb{R}$. The Fourier transform

$$\widehat{v}(\lambda)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\exp(-i\lambda t)v(t)\,dt\tag{2.0.2}$$

reduces the problem to the family of problems depending on the parameter λ:

$$\begin{aligned}(-\Delta_y+\lambda^2-\mu)\widehat{u}(y,\lambda)&=\widehat{f}(y,\lambda),\qquad y\in\Omega,\\ \widehat{u}(y,\lambda)&=0,\qquad y\in\partial\Omega.\end{aligned}\tag{2.0.3}$$

If the inverse operator $\mathfrak{A}(\lambda,\mu)^{-1}$ of problem (2.0.3) exists for all $\lambda\in\mathbb{R}$, the μ being fixed, we obtain a solution u to problem (2.0.1) of the form

$$u(\cdot,t)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\exp(i\lambda t)\mathfrak{A}(\lambda,\mu)^{-1}\widehat{f}(\cdot,\lambda)\,d\lambda.\tag{2.0.4}$$

However, the spectrum of the pencil $\lambda\mapsto\mathfrak{A}(\lambda,\mu)$, that is, the set of numbers λ such that the operator $\mathfrak{A}(\lambda,\mu)$ is not invertible, consists of an imaginary number sequence accumulating at infinity and, for sufficiently large μ, additionally contains finitely many real numbers. Therefore, formula (2.0.4) can fail and we will use the complex Fourier transform

$$\widehat{v}(\lambda)=(2\pi)^{-1/2}\int_{\mathbb{R}}\exp(-i\lambda t)v(t)\,dt,\qquad \lambda\in\mathbb{R}+i\beta,$$

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where $\mathbb{R} + i\beta = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta\}$; there are the inversion formula

$$v(t) = (2\pi)^{-1/2} \int_{\mathbb{R}+i\beta} \exp(i\lambda t)\widehat{v}(\lambda)d\lambda$$

and the Parseval equality

$$\int_{\mathbb{R}} \exp(2\beta t)|v(t)|^2\, dt = \int_{\mathbb{R}+i\beta} |\widehat{v}(\lambda)|^2\, d\lambda.$$

Let us assume that the line $\mathbb{R} + i\beta$ is free from the spectrum of $\mathfrak{A}(\cdot, \mu)$ and the f in (2.0.1) satisfies the condition

$$\int_{\Pi} \exp(2\beta t)|f(y,t)|^2\, dydt = \int_{\mathbb{R}+i\beta} |\widehat{f}(y,\lambda)|^2\, dyd\lambda < \infty.$$

Then, according to Theorem 2.1.4, there exists a unique solution u to problem (2.0.1) such that

$$u(\cdot, t) = (2\pi)^{-1/2} \int_{\mathbb{R}+i\beta} \exp(i\lambda t)\mathfrak{A}(\lambda, \mu)^{-1}\widehat{f}(\cdot, \lambda)\, d\lambda \tag{2.0.5}$$

and the inequality

$$\sum_{|\alpha|+k\leq 2} \int_{\Pi} \exp(2\beta t)|\partial_t^k \partial_y^\alpha u(y,t)|^2\, dydt \leq C \int_{\Pi} \exp(2\beta t)|f(y,t)|^2\, dydt$$

holds with a constant C independent of f.

These considerations motivate the statement of the boundary value problem in the domain G with cylindrical ends

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= f(x), \quad x \in G, \\ u(x) &= 0, \quad x \in \partial G, \end{aligned} \tag{2.0.6}$$

in function spaces with weighted norms (see Fig. 1.2 and the definition of G just after the figure). For integer $l \geq 0$, we denote by $H^l(G)$ the Sobolev space with norm

$$\|v; H^l(G)\| = \Big(\sum_{j=0}^{l} \int_G \sum_{|\alpha|=j} |D_x^\alpha v(x)|^2\, dx\Big)^{1/2}.$$

For real β, we denote by ρ_β a smooth positive function on $\overline{G}$ given by the equality $\rho_\beta(x) = \exp(\beta|x|)$ for large $|x|$. We also introduce the space $H^l_\beta(G)$ with norm $\|u; H^l_\beta(G)\| = \|\rho_\beta u; H^l(G)\|$. Let $\dot{H}^2_\beta(G)$ denote the closure in $H^2_\beta(G)$ of the set of smooth functions in $\overline{G}$ that have compact supports in $\overline{G}$ and vanish on ∂G. The operator $u \mapsto (-\Delta - \mu)u$ of problem (2.0.6) implements a continuous mapping

$$\mathcal{A}_\beta(\mu) : \dot{H}^2_\beta(G) \to H^0_\beta(G).$$

We denote by $\ker\mathcal{A}_\beta(\mu)$ the kernel of $\mathcal{A}_\beta(\mu)$, i.e. the space $\{u \in \dot{H}^2_\beta(G) : \mathcal{A}_\beta(\mu)u = 0\}$, and we denote by $\operatorname{Im}\mathcal{A}_\beta(\mu)$ the range of $\mathcal{A}_\beta(\mu)$,

$$\operatorname{Im}\mathcal{A}_\beta(\mu) = \{f \in H^0_\beta(G) : f = \mathcal{A}_\beta(\mu)u,\, u \in \dot{H}^2_\beta(G)\}.$$

The operator $\mathcal{A}_\beta(\mu)$ is called Fredholm if $\operatorname{Im}\mathcal{A}_\beta(\mu)$ is closed, and $\ker\mathcal{A}_\beta(\mu)$ and $\operatorname{coker}\mathcal{A}_\beta(\mu) := H^0_\beta(G)/\operatorname{Im}\mathcal{A}_\beta(\mu)$ are finite-dimensional, where $H^0_\beta(G)/\operatorname{Im}\mathcal{A}_\beta(\mu)$ is the factor space $H^0_\beta(G)$ modulo $\operatorname{Im}\mathcal{A}_\beta(\mu)$. From Theorem 2.2.2 it follows that $\mathcal{A}_\beta(\mu)$ is Fredholm for all $\beta \in \mathbb{R}$ except a certain sequence accumulated at infinity. Moreover, $\dim(H^0_\beta(G)/\operatorname{Im}\mathcal{A}_\beta(\mu)) = \dim\ker\mathcal{A}_{-\beta}(\mu)$ and the index $\operatorname{Ind}\mathcal{A}_\beta(\mu)$ of $\mathcal{A}_\beta(\mu)$ can be defined by

$$\operatorname{Ind}\mathcal{A}_\beta(\mu) = \dim\ker\mathcal{A}_\beta(\mu) - \dim\ker\mathcal{A}_{-\beta}(\mu).$$

We describe the asymptotics at infinity of solutions to problem (2.0.6) and calculate the difference $\operatorname{Ind}\mathcal{A}_\beta(\mu) - \operatorname{Ind}\mathcal{A}_\gamma(\mu)$. Then we make use of these results when defining the scattering matrix and proving the existence of a unique solution to problem (2.0.6) subject to radiation conditions at infinity (the radiation principle).

2.1 Boundary Value Problem in a Cylinder

2.1.1 Statement of the Problem. Operator Pencil

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. In the cylinder $\Pi = \{(y, t) : y = (y_1, \dots, y_n) \in \Omega, t \in \mathbb{R}\}$, we consider the problem

$$\begin{gathered}(-\Delta - \mu)u(y,t) = f(y,t), \quad (y,t) \in \Pi, \\ u(y,t) = 0, \quad (y,t) \in \partial\Pi,\end{gathered} \tag{2.1.1}$$

where

$$\Delta = \Delta_y + \partial_t^2, \quad \Delta_y = \sum_{j=1}^n \partial_j^2, \quad \partial_j = \partial/\partial y_j.$$

We apply to problem (2.1.1) the Fourier transform

$$\widehat{v}(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp(-i\lambda t)v(t)\,dt \tag{2.1.2}$$

and obtain the family of boundary value problems depending on the parameter λ:

$$(-\Delta_y + \lambda^2 - \mu)\widehat{u}(y,\lambda) = \widehat{f}(y,\lambda), \qquad y \in \Omega, \tag{2.1.3}$$
$$\widehat{u}(y,\lambda) = 0, \qquad y \in \partial\Omega.$$

Now, we introduce an operator-valued function $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda,\mu)$ defined by the equality

$$\mathfrak{A}(\lambda,\mu)v(y) = (-\Delta_y + \lambda^2 - \mu)v(y), \qquad y \in \Omega, \tag{2.1.4}$$

for the functions v, smooth in $\overline{\Omega}$ and equal to zero on $\partial\Omega$; for the time being, the parameter μ is fixed. The function $\mathfrak{A}(\cdot,\mu)$ is called an operator pencil. A number $\lambda_0 \in \mathbb{C}$ is said to be an eigenvalue of $\mathfrak{A}(\cdot,\mu)$ if there exists a nontrivial solution φ^0 (an eigenvector) to the equation $\mathfrak{A}(\lambda_0,\mu)v = 0$, that is, the λ_0 and φ^0 satisfy the boundary value problem

$$(-\Delta_y + \lambda_0^2 - \mu)\varphi^0(y) = 0, \qquad y \in \Omega,$$
$$\varphi^0(y) = 0, \qquad y \in \partial\Omega.$$

We also consider the problem

$$(-\Delta_y - \mu)v(y) = 0, \quad y \in \Omega, \tag{2.1.5}$$
$$v(y) = 0, \quad y \in \partial\Omega,$$

with spectral parameter μ. The eigenvalues of problem (2.1.5) are called the thresholds of problem (2.1.1). The thresholds form a positive sequence $\tau_1 < \tau_2 < \dots$, which strictly increases to infinity. Any eigenvalue τ_l is of finite multiplicity, that is, there exist at most finitely many linearly independent eigenvectors corresponding to τ_l. Let us introduce the non-decreasing sequence $\{\mu_k\}_{k=1}^{\infty}$ of the eigenvalues of problem (2.1.5) counted according to their multiplicity. Generally speaking, the numbering of τ_l and that of μ_k are different; every μ_k coincides with one of the thresholds τ_l.

For any μ, the eigenvalues of the pencil $\lambda \mapsto \mathfrak{A}(\lambda,\mu)$ are defined by the equality $\lambda_k^{\pm}(\mu) = \pm(\mu - \mu_k)^{1/2}$; more precisely, we set $\lambda^{\pm}(\mu) = \pm i(\mu_k - \mu)^{1/2}$ for $\mu_k > \mu$ with $(\mu_k - \mu)^{1/2} > 0$ and $\lambda^{\pm}(\mu) = \pm i(\mu - \mu_k)^{1/2}$ for $\mu_k < \mu$ with $(\mu - \mu_k)^{1/2} > 0$. If $\mu = \mu_k$, we have $\lambda_k^{+}(\mu) = \lambda_k^{-}(\mu) = 0$; in such a case we will sometimes write $\lambda_k^{0}(\mu)$ instead of $\lambda_k^{\pm}(\mu)$. Moreover, we sometimes write simply $\lambda_k^{\pm}$ instead of $\lambda_k^{\pm}(\mu)$. For $\mu_{k-1} < \mu < \mu_k$, the $\lambda_k^{\pm}(\mu), \lambda_{k+1}^{\pm}(\mu), \dots$ are imaginary and the $\lambda_1^{\pm}(\mu), \dots, \lambda_{k-1}^{\pm}(\mu)$ are real. To the eigenvalues $\lambda_k^{\pm}$ there corresponds the same eigenvector φ_k, which is also an eigenvector of problem (2.1.5) corresponding to the eigenvalue μ_k. Any eigenvalue of the pencil $\mathfrak{A}(\cdot,\mu)$ coincides with one of the eigenvalues mentioned in this paragraph.

We denote by $H^l(\Omega)$ the Sobolev function space in Ω with norm

$$\|u\|_l = \Big(\int_\Omega \sum_{|\alpha|\le l} |\partial_y^\alpha u(y)|^2\, dy\Big)^{1/2}, \tag{2.1.6}$$

where $l = 0, 1, \dots$; in particular, $H^0(\Omega) = L_2(\Omega)$. Besides, we denote by $\dot{H}^2(\Omega)$ the closure in $H^2(\Omega)$ of the set of smooth functions in $\overline{\Omega}$ that vanish on $\partial\Omega$.

Let us consider the problem

$$\begin{aligned}(-\Delta_y + \lambda^2 - \mu)v(y) &= f(y), \qquad y \in \Omega,\\ v(y) &= 0, \qquad y \in \partial\Omega.\end{aligned} \tag{2.1.7}$$

Proposition 2.1.1 (e.g., see [1]) (i) *Assume that λ is not an eigenvalue of the pencil $\mathfrak{A}(\cdot, \mu)$, the μ being fixed. Then for any $f \in L_2(\Omega)$ there exists a unique solution $v \in \dot{H}^2(\Omega)$ to problem (2.1.7) and the inequality*

$$\sum_{j=0}^{2} |\lambda|^{2j} \|v\|_{2-j}^2 \le C\|f\|_0^2 \tag{2.1.8}$$

holds with a constant C independent of f.

(ii) *Let F be a closed subset in $\mathbb{C}$ that belongs to a strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < h < +\infty\}$ and contains no eigenvalues of the pencil $\mathfrak{A}(\cdot, \mu)$. Then, for any $\lambda \in F$, estimate (2.1.8) holds with a constant $C = C(F)$ that depends on F and remains independent of λ and f.*

Let λ_0 be an eigenvalue of $\mathfrak{A}(\cdot, \mu)$, and let φ^0 be an eigenvector corresponding to λ_0. Smooth functions $\varphi^1, \dots, \varphi^{m-1}$ on $\overline{\Omega}$ which vanish on $\partial\Omega$ and satisfy

$$\sum_{k=0}^{l} \frac{1}{k!} \partial_\lambda^k \mathfrak{A}(\lambda_0, \mu)\varphi^{l-k} = 0, \qquad l = 1, \dots, m-1, \tag{2.1.9}$$

are called generalized eigenvectors. The ordered collection $\varphi^0, \varphi^1, \dots, \varphi^{m-1}$ is said to be a Jordan chain corresponding to λ_0. Clearly, in view of (2.1.4), the relations (2.1.9) take the form

$$\begin{aligned}\mathfrak{A}(\lambda_0, \mu)\varphi^0 &= 0,\\ \mathfrak{A}(\lambda_0, \mu)\varphi^1 + 2\lambda_0\varphi^0 &= 0,\\ \mathfrak{A}(\lambda_0, \mu)\varphi^l + 2\lambda_0\varphi^{l-1} + 2\varphi^{l-2} &= 0, \qquad l = 2, \dots, m-1.\end{aligned} \tag{2.1.10}$$

There are no generalized eigenvectors for $\lambda_k^\pm \ne 0$. Indeed, assuming, for example, that a Jordan chain φ_k^0, φ_k^1 exists for $\lambda_k^+ \ne 0$, we obtain the equations

$$\begin{aligned}\mathfrak{A}(\lambda_k^+, \mu)\varphi_k^0 &= 0,\\ \mathfrak{A}(\lambda_k^+, \mu)\varphi_k^1 + 2\lambda_k^+\varphi_k^0 &= 0,\end{aligned}$$

which can be written in the form

$$(-\Delta_y - \mu_k)\varphi_k^0 = 0, \qquad (2.1.11)$$
$$(-\Delta_y - \mu_k)\varphi_k^1 + 2\lambda_k^+\varphi_k^0 = 0.$$

The boundary value problem

$$(-\Delta_y - \mu_k)v(y) = f(y), \quad y \in \Omega; \qquad v(y) = 0, \quad y \in \partial\Omega, \qquad (2.1.12)$$

has a solution, if and only if $(f, \psi)_\Omega = 0$ for each eigenvector ψ of this problem that corresponds to the eigenvalue μ_k; the $(f, \psi)_\Omega$ denotes the inner product in $L_2(\Omega)$. Therefore, there is no solution φ_k^1 to the equation (2.1.11) with $\lambda_k^+ \neq 0$. In the case of $\mu = \mu_k$, we have $\lambda_k^+ = \lambda_k^- = 0$ and a Jordan chain φ_k^0, φ_k^1. Both of these vectors satisfy the same homogeneous boundary value problem (2.1.12) with $f = 0$; the φ_k^0 must be nonzero, and the φ_k^1 may equal 0. It is easy to see from the equation (2.1.10) with $l = 0$ and $\lambda_0 = 0$ that there is no generalized eigenvector φ_k^3.

The operator function $\lambda \mapsto \mathfrak{A}(\lambda, \mu)^{-1} : L_2(\Omega) \to \dot{H}^2(\Omega)$ except the poles at the eigenvalues of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, \mu)$ is holomorphic everywhere. To describe the behavior of $\mathfrak{A}(\lambda, \mu)^{-1}$ in a neighborhood of the poles, we specify the general Keldysh's theorem for our problem. Let τ be an eigenvalue of problem (2.1.5) and let J be the geometric multiplicity of τ. We introduce a basis $\varphi^{(0,1)}, \dots, \varphi^{(0,J)}$ of the eigenspace corresponding to τ. For $\mu > \tau$, we denote by $\lambda^\pm = \lambda^\pm(\mu)$ the eigenvalues $\pm(\mu - \tau)^{1/2}$ of the pencil $\mathfrak{A}(\cdot, \mu)$. The multiplicity of each of the $\lambda^\pm$ is equal to J, and the eigenspace is spanned by $\varphi^{(0,1)}, \dots, \varphi^{(0,J)}$. According to the Keldysh theorem, in a neighborhood of λ^+ there holds the representation

$$\mathfrak{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^+)^{-1} \sum_{j=1}^{J} (\cdot,\ \psi^{(0,j)})_\Omega\, \varphi^{(0,j)} + \Gamma(\lambda), \qquad (2.1.13)$$

where $(u, v)_\Omega$ denotes the inner product in $L_2(\Omega)$, $\Gamma(\lambda) : L_2(\Omega) \to \dot{H}^2(\Omega)$ is a holomorphic function, and the $\psi^{(0,1)}, \dots, \psi^{(0,J)}$ are eigenvectors of the pencil $\mathfrak{A}(\cdot, \mu)$ that correspond to λ^+ (and, simultaneously, to λ^-) and satisfy the conditions

$$(\partial_\lambda \mathfrak{A}(\lambda^+, \mu)\varphi^{(0,j)}, \psi^{(0,k)})_\Omega = \delta_{jk}.$$

Since $\partial_\lambda \mathfrak{A}(\lambda^+, \mu) = 2\lambda^+$, we have $2\lambda^+(\varphi^{(0,j)}, \psi^{(0,k)})_\Omega = \delta_{jk}$ and, assuming $\|\varphi^{(0,j)}\|_0 = 1$, obtain $\psi^{(0,j)} = (2\lambda^+)^{-1}\varphi^{(0,j)}$. Therefore, representation (2.1.13) takes the form

$$\mathfrak{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^+)^{-1} \sum_{j=1}^{J} (2\lambda^+)^{-1} (\cdot,\ \varphi^{(0,j)})_\Omega \varphi^{(0,j)} + \Gamma(\lambda). \qquad (2.1.14)$$

To obtain a representation for $\mathfrak{A}(\lambda, \mu)^{-1}$ in a neighborhood of the pole λ^-, it suffices to change λ^+ for λ^- in (2.1.14).

For $\mu < \tau$, we set $\lambda^{\pm}(\mu) = i(\tau - \mu)^{1/2}$, where $(\tau - \mu)^{1/2} > 0$, and denote by $\varphi^{(0,1)}, \dots, \varphi^{(0,J)}$ an orthonormal basis in the eigenspace corresponding to $\lambda^{\pm}$ (recall that $\lambda^+(\mu)$ and $\lambda^-(\mu)$ have the same eigenspace). We arrive at the following assertion.

Proposition 2.1.2 *For any real $\mu \neq \tau$, the operator function $\lambda \mapsto \mathfrak{A}(\lambda, \mu)^{-1}$ admits the representation*

$$\mathfrak{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^{\pm})^{-1} \sum_{j=1}^{J} (2\lambda^{\pm})^{-1} (\cdot,\ \varphi^{(0,j)})_{\Omega} \varphi^{(0,j)} + \Gamma(\lambda) \tag{2.1.15}$$

in a neighborhood of $\lambda^{\pm} = \lambda^{\pm}(\mu)$ with holomorphic function $\lambda \mapsto \Gamma(\lambda) : L_2(\Omega) \to \dot{H}^2(\Omega)$.

Now, we suppose that $\mu = \tau$. Then $\lambda^0 = 0$ is an eigenvalue of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, \mu)$; the geometric multiplicity of λ^0 is equal to J. Let $\varphi^{(0,1)}, \dots, \varphi^{(0,J)}$ be a basis in the eigenspace corresponding to λ^0 and $\varphi^{(0,j)}, \varphi^{(1,j)}$ a Jordan chain, where $j = 1, \dots, J$. By the Keldysh theorem, in a neighborhood of the eigenvalue λ^0 there holds the representation

$$\mathfrak{A}(\lambda, \mu)^{-1} = \sum_{j=1}^{J} \sum_{k=1}^{2} (\lambda - \lambda^0)^{-k} \sum_{q=0}^{2-k} (\cdot,\ \psi^{(q,j)})_{\Omega} \varphi^{(2-k-q,j)} + \Gamma(\lambda), \tag{2.1.16}$$

where $\psi^{(0,j)}, \psi^{(1,j)}$, $j = 1, \dots, J$, is a collection of Jordan chains of the pencil $\mathfrak{A}(\cdot, \mu)$ that correspond to the λ^0 and satisfy the conditions

$$\sum_{p+q+r=2+\nu} \frac{1}{p!} (\partial_{\lambda}^{p} \mathfrak{A}(\lambda^0, \mu) \varphi^{(q,\sigma)}, \psi^{(r,\zeta)})_{\Omega} = \delta_{\sigma,\zeta} \delta_{0,\nu}, \tag{2.1.17}$$

with $\sigma, \zeta = 1, \dots, J$ and $\nu = 0, 1$; the operator function $\lambda \mapsto \Gamma(\lambda) : L_2(\Omega) \to \dot{H}^2(\Omega)$ is holomorphic in a neighborhood of λ^0. Let the basis $\varphi^{(0,1)}, \dots, \varphi^{(0,J)}$ be orthonormal and let every generalized eigenvector $\varphi^{(1,j)}$ be zero. Then (2.1.17) reduces to the relations

$$(\varphi^{(0,\sigma)}, \psi^{(0,\zeta)})_{\Omega} = \delta_{\sigma,\zeta}, \qquad (\varphi^{(0,\sigma)}, \psi^{(1,\zeta)})_{\Omega} + (\varphi^{(1,\sigma)}, \psi^{(0,\zeta)})_{\Omega} = 0.$$

The equalities $(\varphi^{(0,\sigma)}, \psi^{(0,\zeta)})_{\Omega} = \delta_{\sigma,\zeta}$ imply that $\psi^{(0,\sigma)} = \varphi^{(0,\sigma)}$ for $\sigma = 1, \dots, J$. Since $\varphi^{(1,\sigma)} = 0$, we have $(\varphi^{(0,\sigma)}, \psi^{(1,\zeta)})_{\Omega} = 0$, which leads to $\psi^{(1,\zeta)} = 0$ for $\zeta = 1, \dots, J$.

Proposition 2.1.3 *For $\mu = \tau$, in a neighborhood of $\lambda^0 = 0$ the operator function $\lambda \mapsto \mathfrak{A}(\lambda, \mu)^{-1}$ admits the representation*

$$\mathfrak{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^0)^{-2} \sum_{j=1}^{J} (\cdot, \varphi^{(0,j)})_{\Omega} \varphi^{(0,j)} + \Gamma(\lambda) \tag{2.1.18}$$

with holomorphic function $\lambda \mapsto \Gamma(\lambda) : L_2(\Omega) \to \dot{H}^2(\Omega)$.

2.1.2 The Solvability of the Problem in a Cylinder

Let $C_c^\infty(\overline{\Pi})$ denote the set of smooth functions with compact supports in $\overline{\Pi}$; as before, $\Pi = \{y, t) : y \in \Omega, t \in \mathbb{R}\}$. For $l = 0, 1, \dots$ and $\beta \in \mathbb{R}$, we introduce the space $H^l_\beta(\Pi)$ as the completion of $C_c^\infty(\overline{\Pi})$ in the norm

$$\|u; H^l_\beta(\Pi)\| = \Big(\sum_{|\alpha|+k \le l} \int_\Pi \exp(2\beta t) |\partial_t^k \partial_y^\alpha u(y, t)|^2 \, dy dt \Big)^{1/2}. \tag{2.1.19}$$

We denote by $\dot{H}^2_\beta(\Pi)$ the closure in $H^2_\beta(\Pi)$ of the set of smooth functions in $\overline{\Pi}$ that have compact supports in $\overline{\Pi}$ and vanish on $\partial\Pi$. The operator of problem (2.1.1) implements a continuous mapping

$$A_\beta(\mu) : \dot{H}^2_\beta(\Pi) \ni u \mapsto (-\Delta - \mu) u \in H^0_\beta(\Pi). \tag{2.1.20}$$

We will use the complex Fourier transform

$$\widehat{v}(\lambda) = (2\pi)^{-1/2} \int_{\mathbb{R}} \exp(-i\lambda t) v(t) \, dt, \qquad \lambda \in \mathbb{R} + i\beta, \tag{2.1.21}$$

where $\mathbb{R} + i\beta = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta\}$, the inversion formula

$$v(t) = (2\pi)^{-1/2} \int_{\mathbb{R}+i\beta} \exp(i\lambda t) \widehat{v}(\lambda) d\lambda, \tag{2.1.22}$$

and the Parseval equality

$$\int_{\mathbb{R}} \exp(2\beta t) |v(t)|^2 \, dt = \int_{\mathbb{R}+i\beta} |\widehat{v}(\lambda)|^2 \, d\lambda. \tag{2.1.23}$$

Theorem 2.1.4 *Let the line $\mathbb{R} + i\beta$ be free from the eigenvalues of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, \mu)$. Then, for any $f \in H^0_\beta(\overline{\Pi})$ there exists a unique solution $u \in \dot{H}^2_\beta(\Pi)$ to problem (2.1.1). The estimate*

$$\|u;\, H^2_\beta(\Pi)\| \leq C\|f;\, H^0_\beta(\Pi)\| \tag{2.1.24}$$

holds with a constant C independent of f.

Proof The Fourier transform (2.1.21) reduces problem (2.1.1) to the family of problems

$$\begin{aligned} (-\Delta_y + \lambda^2 - \mu)\widehat{u}(y, \lambda) = \widehat{f}(y, \lambda), \qquad & y \in \Omega, \\ \widehat{u}(\lambda, y) = 0, \qquad & y \in \partial\Omega, \end{aligned} \tag{2.1.25}$$

with $\lambda \in \mathbb{R} + i\beta$. This line contains no eigenvalues of the pencil $\mathfrak{A}(\cdot, \mu)$. Therefore, by Proposition 2.1.1, for any $\lambda \in \mathbb{R} + i\beta$ there exists a unique solution $\widehat{u}(\cdot, \lambda) := \mathfrak{A}(\lambda, \mu)^{-1}\widehat{f}(\cdot, \lambda)$ to problem (2.1.25), which subject to the inequality

$$\sum_{j=0}^{2} |\lambda|^{2j}\|\widehat{u}(\cdot, \lambda)\|^2_{2-j} \leq C\|\widehat{f}(\cdot, \lambda)\|^2_0 \tag{2.1.26}$$

and the constant C is independent of λ and $\widehat{f}(\lambda, \cdot)$. Consequently,

$$\int_{\mathbb{R}+i\beta} \sum_{j=0}^{2} |\lambda|^{2j}\|\widehat{u}(\cdot, \lambda)\|^2_{2-j}\, d\lambda \leq C \int_{\mathbb{R}+i\beta} \|\widehat{f}(\cdot, \lambda)\|^2_0\, d\lambda.$$

By virtue of (2.1.23), the left-hand side is equivalent to $\|u;\, H^2_\beta(\Pi)\|^2$ and the right-hand side is equal to $C\|f;\, H^0_\beta(\Pi)\|^2$. Thus, the function

$$u(\cdot, t) = (2\pi)^{-1/2} \int_{\mathbb{R}+i\beta} \exp{(i\lambda t)}\mathfrak{A}(\lambda, \mu)^{-1}\widehat{f}(\cdot, \lambda)\, d\lambda \tag{2.1.27}$$

satisfies problem (2.1.1) and admits estimate (2.1.24). □

2.1.3 Asymptotics of Solutions

Let us assume that f is a smooth function with compact support in $\overline{\Pi}$. Then the function $\lambda \mapsto \widehat{f}(\cdot, \lambda)$ is analytic on $\mathbb{C}$ and rapidly decaying in anystrip $\{\lambda \in \mathbb{C} :$

$|\mathrm{Im}\lambda| \leq h < \infty\}$ as $\lambda \to \infty$. The function $\widehat{u}(\cdot, \lambda) = \mathfrak{A}(\lambda, \mu)\widehat{f}(\cdot, \lambda)$ is analytic everywhere except at the poles of the function $\lambda \mapsto \mathfrak{A}(\lambda, \mu)$. Moreover, in view of inequality (2.1.26), $\widehat{u}(\cdot, \lambda)$ is also rapidly decaying in the aforementioned strip as $\lambda \to \infty$. Let β and γ be real numbers such that the lines $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta\}$ and $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \gamma\}$ contain no poles of $\mathfrak{A}(\cdot, \mu)$. Then, in a representation of the form (2.1.27), we can, using the residue theorem, change β for γ.

We now calculate the residues of the function

$$\lambda \mapsto F(\lambda) := (2\pi)^{-1/2} \exp(i\lambda t)\mathfrak{A}(\lambda, \mu)^{-1}\widehat{f}(\cdot, \lambda). \tag{2.1.28}$$

By Proposition 2.1.2,

$$\mathrm{res}\,F(\lambda)|_{\lambda=\lambda^{\pm}} = (2\pi)^{-1/2}\exp(i\lambda^{\pm}t)\sum_{j=1}^{J}(2\lambda^{\pm})^{-1}\int_{\Omega}\widehat{f}(y, \lambda^{\pm})\overline{\varphi^{(0,j)}(y)}\,dy\,\varphi^{(0,j)};$$

as before, $\lambda^{\pm} = \lambda^{\pm}(\mu)$, where $\lambda^{\pm}(\mu) = \pm(\mu - \tau)^{1/2}$ for $\mu > \tau$ and $\lambda^{\pm}(\mu) = \pm i(\tau - \mu)^{1/2}$ for $\mu < \tau$. For the real $\lambda^{\pm}$, we have

$$\begin{aligned}\int_{\Omega}\widehat{f}(y, \lambda^{\pm})\overline{\varphi^{(0,j)}(y)}\,dy &= (2\pi)^{-1/2}\int_{\mathbb{R}}\int_{\Omega}\exp(-i\lambda^{\pm}s)f(y, s)\overline{\varphi^{(0,j)}(y)}\,dyds\\ &= (2\pi)^{-1/2}(f, Z_j^{\pm})_{\Pi},\end{aligned}$$

where

$$Z_j^{\pm}(y, s) = \exp(i\lambda^{\pm}s)\varphi^{(0,j)}(y) \tag{2.1.29}$$

and $(\cdot, \cdot)_{\Pi}$ is the inner product in $L_2(\Pi)$. Thus, for the real $\lambda^{\pm}$,

$$\mathrm{res}\,F(\lambda)|_{\lambda=\lambda^{\pm}} = (2\pi)^{-1}\sum_{j=1}^{J}(2\lambda^{\pm})^{-1}(f, Z_j^{\pm})_{\Pi}Z_j^{\pm}. \tag{2.1.30}$$

For the imaginary $\lambda^{\pm}$, we obtain

$$\mathrm{res}\,F(\lambda)|_{\lambda=\lambda^{\pm}} = (2\pi)^{-1}\sum_{j=1}^{J}(2\lambda^{\pm})^{-1}(f, Z_j^{\mp})_{\Pi}Z_j^{\pm}, \tag{2.1.31}$$

where $Z_j^{\pm}(y, s)$ is defined by equality (2.1.29). By Proposition 2.1.3, for $\lambda^0 = 0$ we have

$$\operatorname{res} F(\lambda)|_{\lambda=\lambda^0} = (2\pi)^{-1/2} \sum_{j=1}^{J} \varphi^{(0,j)} \int_{\Omega} \big(it\widehat{f}(y,0) + \partial_\lambda \widehat{f}(y,0)\big) \overline{\varphi^{(0,j)}(y)}\, dy \tag{2.1.32}$$

$$= (2\pi)^{-1} \sum_{j=1}^{J} \varphi^{(0,j)} \int_{\mathbb{R}} \int_{\Omega} (it - is) f(y,s) \overline{\varphi^{(0,j)}(y)}\, dyds$$

$$= (2\pi)^{-1} \sum_{j=1}^{J} \Big((f, Z_j^0)_\Pi Z_j^1 + (f, Z_j^1)_\Pi Z_j^0 \Big)$$

with

$$Z_j^0(y,t) = \varphi^{(0,j)}(y), \qquad Z_j^1(y,t) = it\varphi^{(0,j)}(y). \tag{2.1.33}$$

Lemma 2.1.5 *Let $\lambda^\pm$ be an eigenvalue of $\mathfrak{A}(\cdot,\mu)$, $\operatorname{Im}\lambda^\pm \neq 0$, and $\beta < \operatorname{Im}\lambda^+ < \gamma$ ($\beta < \operatorname{Im}\lambda^- < \gamma$). Then, for any $Z = Z_j^-$ ($Z = Z_j^+$) in (2.1.29), the estimate*

$$|(f, Z)_\Pi| \le C(\|f; H_\beta^0(\Pi)\| + \|f; H_\gamma^0(\Pi)\|)$$

holds for $f \in H_\beta^0(\Pi) \cap H_\gamma^0(\Pi)$ with constant C independent of f. If $\beta < 0 < \gamma$ and $\lambda^\pm$ is a real eigenvalue, this estimate is also valid for any $Z_j^\pm$ in (2.1.29), $Z = Z_j^0$, and $Z = Z_j^1$ in (2.1.33).

Proof We choose $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$ such that $0 \le \eta_2(t) \le 1$, $\eta_2(t) = 1$ for $t \ge 1$, $\eta_2(t) = 0$ for $t \le -1$, and $\eta_1 + \eta_2 = 1$. For instance, we assume that $\operatorname{Im}\lambda^+ \neq 0$ and $\beta < \operatorname{Im}\lambda^+ < \gamma$. Then, for $Z = Z_j^-$, we have

$$|(f, Z)_\Pi| \le C \int_\Pi |f(y,s)| \exp(-s\operatorname{Im}\lambda^-)\, dyds$$
$$\le \int_\Pi \eta_1(s)|f(y,s)| \exp(\beta s) \exp(-s(\operatorname{Im}\lambda^- + \beta))\, dyds$$
$$+ \int_\Pi \eta_2(s)|f(y,s)| \exp(\gamma s) \exp(-s(\operatorname{Im}\lambda^- + \gamma))\, dyds.$$

Since $\operatorname{Im}\lambda^+ = -\operatorname{Im}\lambda^-$, we obtain $\operatorname{Im}\lambda^- + \gamma > 0$ and $\operatorname{Im}\lambda^- + \beta < 0$. Therefore,

$$|(f, Z)_\Pi| \le C\|\eta_1 f; H_\beta^0(\Pi)\| \left(\int_{-\infty}^0 \exp(-2s(\operatorname{Im}\lambda^- + \beta))\, ds \right)^{1/2}$$
$$+ C\|\eta_2 f; H_\gamma^0(\Pi)\| \left(\int_0^{+\infty} \exp(-2s(\operatorname{Im}\lambda^- + \gamma))\, ds \right)^{1/2}$$
$$\le C(\|f; H_\beta^0(\Pi)\| + \|f; H_\gamma^0(\Pi)\|). \qquad \square$$

The next theorem describes the asymptotics of a solution to problem (2.1.1) at infinity.

Theorem 2.1.6 *Let the lines $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta\}$ and $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \gamma\}$ be free from the eigenvalues of the pencil $\mathfrak{A}(\cdot, \mu)$ and $f \in H^0_\beta(\Pi) \cap H^0_\gamma(\Pi)$. Then*

$$u_\beta = u_\gamma + 2\pi i \mathfrak{S}(\beta, \gamma), \tag{2.1.34}$$

where u_β and u_γ are solutions to problem (2.1.1) in $\dot{H}^2_\beta(\Pi)$ and $\dot{H}^2_\gamma(\Pi)$ respectively, $\beta < \gamma$, and $\mathfrak{S}(\beta, \gamma)$ is the sum of the residues of function (2.1.28) in the strip $\{\lambda \in \mathbb{C} : \beta < \mathrm{Im}\lambda < \gamma\}$. All functions $Z^\pm_j$ in (2.1.29), Z^0_j, and Z^1_j in (2.1.33) satisfy homogeneous problem (2.1.1). Equality (2.1.34) can be taken as an asymptotics of $u_\beta(y, t)$ for $t \to +\infty$ and as an asymptotics of $u_\gamma(y, t)$ for $t \to -\infty$; the u_γ (u_β) plays the role of a remainder as t tends to $+\infty$ (to $-\infty$).

Proof For f in the set $C^\infty_c(\overline{\Pi})$ of smooth functions with compact support in $\overline{\Pi}$, equality (2.1.34) was discussed at the beginning of Sect. 2.1.3. By Lemma 2.1.5, the functionals $f \mapsto (f, Z)_{\overline{\Pi}}$ in $\mathfrak{S}(\beta, \gamma)$ are continuous on $H^0_\beta(\Pi) \cap H^0_\gamma(\Pi)$. Therefore, we can obtain (2.1.34) for $f \in H^0_\beta(\Pi) \cap H^0_\gamma(\Pi)$ by closing $C^\infty_c(\overline{\Pi})$ in the norm $\|f\| := \|f; H^0_\beta(\Pi)\| + \|f; H^0_\gamma(\Pi)\|$ of the space $H^0_\beta(\Pi) \cap H^0_\gamma(\Pi)$.

Straightforward calculation shows that the functions $Z^\pm_j$, Z^0_j, and Z^1_j satisfy homogeneous problem (2.1.1) (it also follows from (2.1.34) and the fact that the difference $u_1 - u_2$ satisfies this problem). □

We now rewrite (2.1.34) using a more detailed notation. Let $\lambda^\pm_k = \lambda^\pm_k(\mu)$ be the eigenvalue notation defined in the paragraph before formula (2.1.6). Besides, we assume that $Z^\pm_k$ corresponds to $\lambda^\pm_k(\mu)$, i.e., $Z^\pm_k(y, t) = \exp(i\lambda^\pm_k(\mu)t)\varphi_k(y)$, where φ_k is an eigenvector corresponding to $\lambda^\pm_k(\mu)$ etc. [see (2.1.29), (2.1.32), and (2.1.33)]. Then (2.1.34) takes the form

$$\begin{aligned} u_\beta - u_\gamma = &\sum_{\max\{0,\beta\}<\mathrm{Im}\lambda^+_k<\gamma} i(2\lambda^+_k)^{-1}(f, Z^-_k)_\Pi Z^+_k \\ &+ \sum_{0>\mathrm{Im}\lambda^-_k>\min\{0,\beta\}} i(2\lambda^-_k)^{-1}(f, Z^+_k)_\Pi Z^-_k \\ &+ \sum_{\lambda^\pm_k\in\mathbb{R}} i(2\lambda^\pm)^{-1}(f, Z^\pm_k)_\Pi Z^\pm_k \\ &+ \sum_{\lambda^0_k=0} i\left((f, Z^0_k)_\Pi Z^1_k + (f, Z^1_k)_\Pi Z^0_k\right) \end{aligned} \tag{2.1.35}$$

and the two last sums (corresponding to the real eigenvalues) are absent if $\beta\gamma \geq 0$.

The right-hand side of (2.1.35) is a linear combination of the solutions Z^+_k, Z^-_k, and so on, to homogeneous problem (2.1.1) (where $f = 0$). The coefficients $i(2\lambda^+_k)^{-1}(f, Z^-_k)_\Pi$, $i(2\lambda^-_k)^{-1}(f, Z^+_k)_\Pi$, and so on, of this linear combination are continuous functionals on the space $H^0_\beta(\Pi) \cap H^0_\gamma(\Pi)$.

The following simplifications of equality (2.1.35) for some special cases are evident. For $0 < \beta < \gamma$, (2.1.35) reduces to the form

$$u_\beta - u_\gamma = \sum_{\beta < \mathrm{Im}\lambda_k^+ < \gamma} i(2\lambda_k^+)^{-1}(f, Z_k^-)_\Pi Z_k^+.$$

In the case of $\beta < \gamma < 0$,

$$u_\beta - u_\gamma = \sum_{\beta < \mathrm{Im}\lambda_k^- < \gamma} i(2\lambda_k^-)^{-1}(f, Z_k^+)_\Pi Z_k^-.$$

If the strip $\beta < \mathrm{Im}\lambda < \gamma$ contains no eigenvalues of the pencil $\mathfrak{A}(\cdot, \mu)$, except the real ones, we have

$$u_\beta - u_\gamma = \sum_{\lambda_k^\pm \in \mathbb{R}} i(2\lambda^\pm)^{-1}(f, Z_k^\pm)_\Pi Z_k^\pm + \sum_{\lambda_k^0 = 0} i\left((f, Z_k^0)_\Pi Z_k^1 + (f, Z_k^1)_\Pi Z_k^0\right);$$

if, in addition, the μ is not a threshold, this equality takes the form

$$u_\beta - u_\gamma = \sum_{\lambda_k^\pm \in \mathbb{R}} i(2\lambda^\pm)^{-1}(f, Z_k^\pm)_\Pi Z_k^\pm,$$

which is the most important situation in the subsequent chapters.

2.2 Problem in a Domain G with Cylindrical Ends

2.2.1 Statement and Fredholm Property of the Problem

Let G be a domain in $\mathbb{R}^{n+1}$ with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi_+^1 \cup \cdots \cup \Pi_+^{\mathcal{T}}$ of finitely many non-overlapping semicylinders

$$\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r,\ t^r > 0\},$$

where (y^r, t^r) are local coordinates in Π_+^r and Ω^r is a bounded domain in $\mathbb{R}^n$. We consider the problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= f(x), \quad x \in G, \\ u(x) &= 0, \quad x \in \partial G. \end{aligned} \tag{2.2.1}$$

For integer $l \geq 0$, we denote by $H^l(G)$ the Sobolev space with norm

$$\|v; H^l(G)\| = \left(\sum_{j=0}^{l} \int_G \sum_{|\alpha|=j} |D_x^\alpha v(x)|^2 \, dx \right)^{1/2}.$$

We assume that $\beta = (\beta^1, \dots, \beta^{\mathcal{T}})$ with real β^r and denote by ρ_β a smooth positive on $\overline{G}$ function given on Π_+^r by the equality $\rho_\beta(y^r, t^r) = \exp(\beta^r t^r)$. We also introduce the space $H_\beta^l(G)$ with norm $\|u; H_\beta^l(G)\| = \|\rho_\beta u; H^l(G)\|$. Let $\dot{H}_\beta^2(G)$ denote the closure in $H_\beta^2(G)$ of the set of smooth functions in $\overline{G}$ that have compact supports in $\overline{G}$ and vanish on ∂G. The operator $u \mapsto (-\Delta - \mu)u$ of problem (2.2.1) implements a continuous mapping

$$\mathcal{A}_\beta(\mu) : \dot{H}_\beta^2(G) \to H_\beta^0(G). \tag{2.2.2}$$

We denote by $\ker \mathcal{A}_\beta(\mu)$ the kernel of $\mathcal{A}_\beta(\mu)$, i.e. the space $\{u \in \dot{H}_\beta^2(G) : \mathcal{A}_\beta(\mu)u = 0\}$, and denote by $\mathrm{Im}\mathcal{A}_\beta(\mu)$ the range of $\mathcal{A}_\beta(\mu)$,

$$\mathrm{Im}\mathcal{A}_\beta(\mu) = \{f \in H_\beta^0(G) : f = \mathcal{A}_\beta(\mu)u, u \in \dot{H}_\beta^2(G)\}.$$

Definition 2.2.1 The operator $\mathcal{A}_\beta(\mu)$ is called Fredholm if $\mathrm{Im}\mathcal{A}_\beta(\mu)$ is closed and $\ker\mathcal{A}_\beta(\mu)$ and $\mathrm{coker}\mathcal{A}_\beta(\mu) := H_\beta^0(G)/\mathrm{Im}\mathcal{A}_\beta(\mu)$ are finite-dimensional, where $H_\beta^0(G)/\mathrm{Im}\mathcal{A}_\beta(\mu)$ is the factor space $H_\beta^0(G)$ modulo $\mathrm{Im}\mathcal{A}_\beta(\mu)$.

Let us introduce an operator pencil $\lambda \to \mathfrak{A}^r(\lambda, \mu)$ defined by (2.1.4) for the domain Ω^r, $r = 1, \dots, \mathcal{T}$.

Theorem 2.2.2 (i) *Operator* (2.2.2) *is Fredholm if and only if the line* $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta^r\}$ *is free from the eigenvalues of the pencil* $\mathfrak{A}^r(\cdot, \mu)$ *for every* $r = 1, \dots, \mathcal{T}$.

(ii) $\dim(H_\beta^0(G)/\mathrm{Im}\mathcal{A}_\beta(\mu)) = \dim\ker\mathcal{A}_{-\beta}(\mu)$.

(iii) $f \in \mathrm{Im}\mathcal{A}_\beta(\mu)$ *if and only if* $(f, v)_G = 0$ *for all* $v \in \ker\mathcal{A}_{-\beta}(\mu)$*; here* $(\cdot, \cdot)_G$ *means the extension of the inner product in* $L_2(G)$ *by continuity to the pair* $H_\beta^0(G)$, $H_{-\beta}^0(G)$.

2.2.2 Asymptotics of Solutions

Theorem 2.2.3 *Let* u *be a solution to problem* (2.2.1) *such that* $u \in \dot{H}_\beta^2(G)$ *with* $\beta = (\beta^1, \dots, \beta^{\mathcal{T}})$. *Let* $\eta_r f \in H_{\gamma^r}^0(\Pi_+^r \cap G)$ *for a certain* r, *where* $\beta^r < \gamma^r$, η_r *denotes a smooth function with support in* $\overline{\Pi}_+^r \cap \overline{G}$, *and* $\eta_r(y^r, t^r) = 1$ *for* $t^r > T$ *with a large* T. *We assume the lines* $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \beta^r\}$ *and* $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \gamma^r\}$ *to be free from the eigenvalues of the pencil* $\mathfrak{A}^r(\cdot, \mu)$.

Then in Π_+^r for $t^r > T$ there holds the equality

$$u = \sum_{\max\{0,\beta^r\}<\mathrm{Im}\lambda_k^+<\gamma^r} c_k^+ Z_k^+ + \sum_{0>\mathrm{Im}\lambda_k^->\min\{0,\beta^r\}} c_k^- Z_k^- \\ + \sum_{\lambda_k^\pm\in\mathbb{R}} c_k^\pm Z_k^\pm + \sum_{\lambda_k^0=0} (c_k^1 Z_k^1 + c_k^0 Z_k^0) + v, \tag{2.2.3}$$

where the functions Z_k^+, Z_k^-, and so on, are defined in $\Pi := \Pi^r = \Omega^r \times \mathbb{R}$ like those in (2.1.35), the c_k^+, c_k^-, and so on, are some constant coefficients and $\eta_r v \in \dot{H}^2_{\gamma^r}(\Pi^r)$. (The two last sums (corresponding to the real eigenvalues) are absent if $\beta^r\gamma^r \geq 0$.)

Proof We have

$$(-\Delta - \mu)(\eta_r u)(x) = g(x), \qquad x \in \Pi^r, \\ (\eta_r u)(x) = 0, \qquad x \in \partial\Pi^r, \tag{2.2.4}$$

where $g = \eta_r f - 2\nabla\eta_r\nabla u - u\Delta\eta_r$. Because $\nabla\eta_r$ and $\Delta\eta_r$ have compact supports, the g belongs to $H^0_{\beta_1}(\Pi^r) \cap H^0_{\beta_2}(\Pi^r)$. Applying Theorems 2.1.4 and 2.1.6, we obtain (2.1.35) with $f = g$, $u_1 = \eta_r u$, and $v = u_1$. This leads to equality (2.2.3), where

$$c_k^+ = i(2\lambda_k^+)^{-1}(g, Z_k^-)_{\Pi^r}, \quad c_k^- = i(2\lambda_k^-)^{-1}(g, Z_k^+)_{\Pi^r}, \ldots, \quad c_k^0 = i(g, Z_k^1)_{\Pi^r}. \tag{2.2.5}$$

□

Note that, in the proof, the function g depends on f, u, and η_r. Therefore, formulas (2.2.5) do not present explicit expressions of the coefficients in (2.2.3) as functionals defined immediately for f in (2.2.1). Such expressions are given in Sect. 2.2.4.

2.2.3 *Properties of the Index Ind $\mathcal{A}_\beta(\mu)$ and of the Spaces* $\ker\mathcal{A}_\beta(\mu)$ *and* $\operatorname{coker}\mathcal{A}_\beta(\mu)$

Let $\mathcal{A}_\beta(\mu)$ be Fredholm (see Definition 2.2.1). The difference $\dim\ker\mathcal{A}_\beta(\mu) - \dim\operatorname{coker}\mathcal{A}_\beta(\mu)$ is called the index of $\mathcal{A}_\beta(\mu)$ and denoted by $\operatorname{Ind}\mathcal{A}_\beta(\mu)$. Assuming both of the operators $\mathcal{A}_\beta(\mu)$ and $\mathcal{A}_\gamma(\mu)$ to be Fredholm, we calculate, in particular, the difference $\operatorname{Ind}\mathcal{A}_\beta(\mu) - \operatorname{Ind}\mathcal{A}_\gamma(\mu)$ in terms of the spectrum of the pencils $\mathfrak{A}^r(\cdot, \mu)$.

Recall that, for any non-zero eigenvalue λ_0 of a pencil $\mathfrak{A}^r(\cdot, \mu)$, there are no generalized eigenvectors and, consequently, the full multiplicity of λ_0 is equal to its geometric multiplicity, i.e., the full multiplicity coincides with $\dim\ker\mathfrak{A}^r(\lambda_0, \mu)$. If $\lambda_0 = 0$ turns out to be an eigenvalue of a certain $\mathfrak{A}^r(\cdot, \mu)$, for any eigenvector $\varphi^0 \in \ker\mathfrak{A}^r(\lambda_0, \mu)$ there exists a generalized eigenvector and the full multiplicity of λ_0 equals its doubled geometric multiplicity.

Theorem 2.2.4 *Let* $\beta = (\beta^1, \dots, \beta^{\mathcal{T}_1}, \beta^{\mathcal{T}_2}, \dots, \beta^{\mathcal{T}})$ *and* $\gamma = (\gamma^1, \dots, \gamma^{\mathcal{T}_1}, \beta^{\mathcal{T}_2}, \dots, \beta^{\mathcal{T}})$*, where* $\beta^r < \gamma^r$ *for* $r = 1, \dots, \mathcal{T}_1$*, and let the lines* $\mathbb{R} + i\beta^r$ *and* $\mathbb{R} + i\gamma^r$ *be free from the eigenvalues of the pencil* $\mathfrak{A}^r(\cdot, \mu)$ *for* $r = 1, \dots, \mathcal{T}$*. We denote by* $\varkappa^r$ *the sum of the full multiplicities of the eigenvalues of the pencil* $\mathfrak{A}^r(\cdot, \mu)$ *in the strip* $\{\lambda \in \mathbb{C} : \beta^r < \operatorname{Im}\lambda < \gamma^r\}$ *with* $r = 1, \dots, \mathcal{T}_1$ *and set* $\varkappa = \varkappa^1 + \cdots + \varkappa^{\mathcal{T}_1}$*.*

Then

$$\dim\left(\ker\mathcal{A}_\beta(\mu)/\ker\mathcal{A}_\gamma(\mu)\right) + \dim\left(\ker\mathcal{A}_{-\gamma}(\mu)/\ker\mathcal{A}_{-\beta}(\mu)\right) = \varkappa, \quad (2.2.6)$$

$$\operatorname{Ind}\mathcal{A}_\beta(\mu) = \operatorname{Ind}\mathcal{A}_\gamma(\mu) + \varkappa. \quad (2.2.7)$$

Proof We number all functions of the form $\eta_r Z_k^\pm$, $\eta_r Z_k^0$, and $\eta_r Z_k^1$ that correspond to the eigenvalues of the pencil $\mathfrak{A}^r(\cdot, \mu)$ in the strip $\{\lambda \in \mathbb{C} : \beta^r < \operatorname{Im}\lambda < \gamma^r\}$, $r = 1, \dots, \mathcal{T}_1$, by the same index and obtain the collection $Z_1, \dots, Z_\varkappa$. According to Theorem 2.2.3, any function u in $\ker\mathcal{A}_\beta(\mu)$ admits the asymptotics

$$u = c_1 Z_1 + \cdots + c_\varkappa Z_\varkappa + v, \quad (2.2.8)$$

with constant coefficients c_j and v in $\dot{H}^2_\gamma(G)$. Therefore, there exist at most $\varkappa$ vectors in the space $\ker\mathcal{A}_\beta(\mu)$ linearly independent modulo $\ker\mathcal{A}_\gamma(\mu)$; we set $d := \dim\left(\ker\mathcal{A}_\beta(\mu)/\ker\mathcal{A}_\gamma(\mu)\right)$ and have $0 \le d \le \varkappa$. Without loss of generality, we assume that there exist vectors U_j in $\ker\mathcal{A}_\beta(\mu)$ such that

$$U_j = Z_j + \sum_{k=d+1}^{\varkappa} c_{jk} Z_k + v_j, \qquad j = 1, \dots, d, \quad (2.2.9)$$

where $c_{jk} = \text{const}$ and $v_j \in \dot{H}^2_\gamma(G)$. Clearly, the $U_1, \dots, U_d$ are linearly independent modulo $\ker\mathcal{A}_\gamma(\mu)$.

Let D denote $\dim\left(\ker\mathcal{A}_{-\gamma}(\mu)/\ker\mathcal{A}_{-\beta}(\mu)\right)$; we will now verify that $D = \varkappa - d$.

We first assume that $D < \varkappa - d$ and denote by $\varphi_1, \dots, \varphi_D$ a collection of vectors in $\ker\mathcal{A}_{-\gamma}(\mu)$ linearly independent modulo $\ker\mathcal{A}_{-\beta}(\mu)$. Then there exists a non-trivial linear combination $Z = c^0_{d+1} Z_{d+1} + \cdots + c^0_\varkappa Z_\varkappa$ such that $f := \mathcal{A}_\beta(\mu) Z \in H^0_\gamma(G)$ and, moreover, $(f, \varphi_j)_G = 0$ for $j = 1, \dots, D$. This and Theorem 2.2.2(iii) imply the existence of a function V satisfying $\mathcal{A}_\gamma(\mu) V = f$. Therefore, we have $U_0 := V - Z \in \ker\mathcal{A}_\beta(\mu)$ and the vectors $U_0, U_1, \dots, U_d$ are linearly independent modulo $\ker\mathcal{A}_\gamma(\mu)$, which contradicts $d = \dim\left(\ker\mathcal{A}_\beta(\mu)/\ker\mathcal{A}_\gamma(\mu)\right)$. Thus, we obtained the inequality $D \ge \varkappa - d$.

Now, we suppose that $D > \varkappa - d$. Let $\varphi_1, \dots, \varphi_D$ be a collection of elements in $\ker\mathcal{A}_{-\gamma}(\mu)$ linearly independent modulo $\ker\mathcal{A}_{-\beta}(\mu)$. We choose a collection $\Phi_1, \dots, \Phi_D$ in $H^0_\gamma(G)$ such that

$$(\Phi_j, \varphi_k)_G = \delta_{jk}, \quad j, k = 1, \dots, D,$$
$$(\Phi_j, \psi)_G = 0 \quad \textit{for all} \quad \psi \in \ker\mathcal{A}_{-\beta}(\mu).$$

Then there exists χ_j that satisfies $\mathcal{A}_\beta(\mu)\chi_j = \Phi_j$, where $j = 1, \ldots, D$. If needed, we can subtract from the χ_j a linear combination of $U_1, \ldots, U_d$ in (2.2.9) to provide the inclusions

$$\chi_j - \sum_{h=d+1}^{\varkappa} d_{jh} Z_h \in \dot{H}^2_\gamma(G), \quad j = 1, \ldots, D. \tag{2.2.10}$$

No nontrivial linear combination of $\chi_1, \ldots, \chi_D$ belongs to $\dot{H}^2_\gamma$; otherwise there is a linear combination of $\mathcal{A}_\gamma(\mu)\chi_j = \Phi_j$ orthogonal to all of the vectors $\varphi_1, \ldots, \varphi_D$, which is impossible in view of the choice of the $\Phi_1, \ldots, \Phi_D$. This and (2.2.10) imply that $D \leq \varkappa - D$. Therefore, we obtain the equality $D = \varkappa$ and, consequently, equality (2.2.6).

Let us verify formula (2.2.7). According to Theorem 2.2.2(ii), $\dim \operatorname{coker} \mathcal{A}_\beta(\mu) = \dim \ker \mathcal{A}_{-\beta}(\mu)$, hence $\operatorname{Ind} \mathcal{A}_\beta(\mu) = \dim \ker \mathcal{A}_\beta(\mu) - \dim \ker \mathcal{A}_{-\beta}(\mu)$, and the same with β replaced by γ. From (2.2.6) it follows that

$$\begin{aligned}\dim \ker \mathcal{A}_\beta(\mu) &= \dim \ker \mathcal{A}_\gamma(\mu) + d,\\ \dim \ker \mathcal{A}_{-\beta}(\mu) &= \dim \ker \mathcal{A}_{-\gamma}(\mu) + d - \varkappa,\end{aligned}$$

and therefore $\operatorname{Ind} \mathcal{A}_\beta(\mu) = \operatorname{Ind} \mathcal{A}_\gamma(\mu) + \varkappa$. $\square$

2.2.4 Calculation of the Coefficients in the Asymptotics

Now, we are in a position to obtain explicit expressions for the coefficients in (2.2.3). We will use the notation Z_j with $j = 1, \ldots, \varkappa$, defined at the beginning of the proof of Theorem 2.2.4, and introduce also

$$\begin{aligned} Z_j^* &:= (2\lambda_k^{\pm})^{-1} \eta_r Z_k^{\pm} \quad \textit{for} \quad Z_j = \eta_r Z_k^{\mp} \quad \textit{and} \quad \lambda_k^{\pm} \notin \mathbb{R};\\ Z_j^* &:= (2\lambda_k^{\pm})^{-1} \eta_r Z_k^{\pm} \quad \textit{for} \quad Z_j = \eta_r Z_k^{\pm} \quad \textit{and} \quad \lambda_k^{\pm} \in \mathbb{R} \setminus 0;\\ Z_j^* &:= \eta_r Z_k^0 \quad \textit{for} \quad Z_j = \eta_r Z_k^1 \quad \textit{and} \quad \lambda_k^0 = 0;\\ Z_j^* &:= \eta_r Z_k^1 \quad \textit{for} \quad Z_j = \eta_r Z_k^0 \quad \textit{and} \quad \lambda_k^0 = 0; \end{aligned} \tag{2.2.11}$$

the connection between Z_j and Z_j^* has been stated in (2.1.35).

We assume the hypotheses of Theorem 2.2.3 to be fulfilled and write the asymptotics of a solution $u \in \dot{H}^2_\beta$ in the form (2.2.8).

Proposition 2.2.5 *Let $V = Z_j^* + \eta_r v$, where Z_j^* is a function in (2.2.11) with a certain r and $v \in \dot{H}^2_{-\beta}(G)$. We suppose the V satisfies the equations*

$$\begin{aligned}(-\Delta - \mu)V(x) = 0, \quad & x = (y^r, t^r) \in \Pi^r_+, \ t^r > T,\\ V(x) = 0, \quad & x \in \partial\Pi^r_+ \cap \partial G, \ t^r > T.\end{aligned}$$

Then, for the coefficient c_j in (2.2.8) there holds the equality

$$c_j = i(\mathcal{A}_\beta(\mu)\eta_r u, V)_G. \tag{2.2.12}$$

Proof We set $\eta_{r,\varepsilon}(y^r, t^r) := \eta_r(y^r, \varepsilon t^r)$ with small positive ε and obtain

$$(\mathcal{A}_\beta(\mu)\eta_r u, V)_G = (\mathcal{A}_\beta(\mu)\eta_{r,\varepsilon} u, V)_G + (\mathcal{A}_\beta(\mu)(\eta_r - \eta_{r,\varepsilon})u, V)_G.$$

The function $(\eta_r - \eta_{r,\varepsilon})u$ vanishes on infinity, so we can integrate the second term on the right by parts :

$$\begin{aligned}(\mathcal{A}_\beta(\mu)(\eta_r - \eta_{r,\varepsilon})u, V)_G &= ((-\Delta - \mu)(\eta_r - \eta_{r,\varepsilon})u, V)_{\Pi^r_+}\\ &= ((\eta_r - \eta_{r,\varepsilon})u, (-\Delta - \mu)V)_{\Pi^r_+} = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}(\mathcal{A}_\beta(\mu)\eta_r u, V)_G = (\mathcal{A}_\beta(\mu)\eta_{r,\varepsilon} u, V)_G &= (\mathcal{A}_\beta(\mu)\eta_{r,\varepsilon} u, Z_j^*)_G\\ &+ (\mathcal{A}_\beta(\mu)\eta_{r,\varepsilon} u, V - Z_j^*)_G.\end{aligned} \tag{2.2.13}$$

According to Theorem 2.2.3,

$$c_j = i(\mathcal{A}_\beta(\mu)\eta_{r,\varepsilon} u, Z_j^*)_G. \tag{2.2.14}$$

Moreover,

$$|(\mathcal{A}_\beta(\mu)\eta_{r,\varepsilon} u, V - Z_j^*)_G| \le C\|\eta_{r,\varepsilon} u; H^2_\beta(G)\|\|v; H^0_{-\beta}\| \tag{2.2.15}$$

with a constant C independent of ε. Since $\|\eta_{r,\varepsilon} u; H^2_\beta(G)\| \to 0$ as $\varepsilon \to 0$, relations (2.2.13), (2.2.14), and (2.2.15) lead to (2.2.12). □

Proposition 2.2.6 *Let the hypotheses of Theorem 2.2.4 be fulfilled and let $U_1, \dots, U_d$ be vectors in $\ker\mathcal{A}_\beta(\mu)$ that satisfy (2.2.9), where $d = \dim\big(\ker\mathcal{A}_\beta(\mu)/\ker\mathcal{A}_\gamma(\mu)\big)$. Then there exist vectors $U^*_{d+1}, \dots, U^*_{\varkappa}$ in $\ker\mathcal{A}_{-\gamma}(\mu)$ such that*

$$U_k^* = Z_k^* - \sum_{j=1}^{d} \overline{c}_{jk} Z_j^* + v_k^*, \qquad k = d+1, \dots, \varkappa, \tag{2.2.16}$$

and $v_k^* \in \dot{H}^2_{-\beta}(G)$.

Proof By virtue of Theorem 2.2.4, there exist vectors $V_{d+1}, \dots, V_\varkappa$ in $\ker \mathcal{A}_{-\gamma}(\mu)$ linearly independent modulo $\ker \mathcal{A}_{-\beta}(\mu)$. According to Theorem 2.2.3, V_k admits the representation

$$V_k = \sum_{j=1}^{\varkappa} b_{kj} Z_j^* + v_k, \quad v_k \in H^2_{-\beta}(G), \quad k = d+1, \dots, \varkappa. \tag{2.2.17}$$

We have

$$0 = (\mathcal{A}_\beta(\mu) U_h, V_k)_G = (\mathcal{A}_\beta(\mu) \sum_{r=1}^{\mathcal{T}_1} \eta_r U_h, V_k)_G + (\mathcal{A}_\beta(\mu)(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r) U_h, V_k)_G. \tag{2.2.18}$$

The function $(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r) U_h$ belongs to $\dot{H}^2_\gamma(G)$ hence

$$\mathcal{A}_\beta(\mu)(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r) U_h = \mathcal{A}_\gamma(\mu)(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r) U_h.$$

Taking into account this equality, the relation $V_k \in \ker \mathcal{A}_{-\gamma}(\mu)$, and Theorem 2.2.2(iii), we obtain $\mathcal{A}_\gamma(\mu)(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r) U_h = 0$ and

$$0 = (\mathcal{A}_\beta(\mu) U_h, V_k)_G = (\mathcal{A}_\beta(\mu) \sum_{r=1}^{\mathcal{T}_1} \eta_r U_h, V_k)_G.$$

To calculate the right-hand side, we employ Proposition 2.2.5 and, in view of (2.2.9), arrive at

$$\overline{b}_{kh} + \sum_{j=d+1}^{\varkappa} c_{hj} \overline{b}_{kj} = 0, \quad h = 1, \dots, d, \quad k = d+1, \dots, \varkappa. \tag{2.2.19}$$

Therefore, the first d columns of the $(\varkappa - d) \times d$-matrix $b = \|b_{kp}\|$ are linear combinations of the rest $\varkappa - d$ columns. The rank of the matrix b is equal to $\varkappa - d$ because $V_{d+1}, \dots, V_\varkappa$ are linearly independent modulo $\ker \mathcal{A}_{-\beta}(\mu)$. It follows that the matrix $\|b_{kj}\|_{k,j=d+1}^{\varkappa}$ is nonsingular. This allows us to assume in (2.2.17) that $b_{kj} = \delta_{kj}$ for $k, j = d+1, \dots, \varkappa$. Then, by virtue (2.2.19), we obtain

$$\overline{b}_{kh} = -c_{hk}, \quad h = 1, \dots, d, \quad k = d+1, \dots, \varkappa,$$

which completes the proof. □

We now pass on to the basic theorem of this section. As before, we suppose that $\beta = (\beta^1, \dots, \beta^{\mathcal{T}_1}, \beta^{\mathcal{T}_2}, \dots, \beta^{\mathcal{T}})$ and $\gamma = (\gamma^1, \dots, \gamma^{\mathcal{T}_1}, \beta^{\mathcal{T}_2}, \dots, \beta^{\mathcal{T}})$, where $\beta^r < \gamma^r$ for $r = 1, \dots, \mathcal{T}_1$, and the lines $\mathbb{R} + i\beta^r$ and $\mathbb{R} + i\gamma^r$ contain no eigenvalues of the pencil $\mathfrak{A}^r(\cdot, \mu)$ for $r = 1, \dots, \mathcal{T}$. We denote by $\varkappa^r$ the sum of the full multiplicities of the eigenvalues of the pencil $\mathfrak{A}^r(\cdot, \mu)$ in the strip $\{\lambda \in \mathbb{C} : \beta^r < \operatorname{Im}\lambda < \gamma^r\}$ with $r = 1, \dots, \mathcal{T}_1$ and set $\varkappa = \varkappa^1 + \dots + \varkappa^{\mathcal{T}_1}$. We also keep the notation $d := \dim\left(\ker\mathcal{A}_\beta(\mu)/\ker\mathcal{A}_\gamma(\mu)\right)$.

Theorem 2.2.7 *Let $f \in H^0_\gamma(G)$ and let problem (2.2.1) have a solution in $\dot{H}^2_\beta(G)$. Then, for any constant $c_1, \dots, c_d$, there exists a solution $u \in \dot{H}^2_\beta(G)$ to problem (2.2.1) such that*

$$u = \sum_{j=1}^{d} c_j Z_j + \sum_{k=d+1}^{\varkappa} b_k Z_k + v,$$

where $v \in \dot{H}^2_\gamma(G)$ and $Z_1, \dots, Z_\varkappa$ are the same as in (2.2.8). The constant b_k with $k = d+1, \dots, \varkappa$ is defined by

$$b_k = i(f, U_k^*)_G + \sum_{h=1}^{d} c_h c_{hk},$$

where U_k^* belongs to $\ker\mathcal{A}_{-\gamma}(\mu)$ and satisfies (2.2.16), $k = d+1, \dots, \varkappa$.

Proof Let $w \in \dot{H}^2_\beta(G)$ be an arbitrary solution to problem (2.2.1) and let $U_1, \dots, U_d$ be the vectors in $\ker\mathcal{A}_\beta(\mu)$ defined by (2.2.9). We choose a linear combination $\mathcal{L}$ of $U_1, \dots, U_d$ such that $v := w + \mathcal{L}$ admits the representation

$$v = \sum_{j=d+1}^{\varkappa} a_j Z_j + \rho, \quad \rho \in \dot{H}^2_\gamma(G)$$

with constant coefficients a_j. Let us calculate the a_k. The function $v(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r)$ belongs to $\dot{H}^2_\gamma(G)$, which follows $(\mathcal{A}_\beta(\mu)v(1 - \sum_{r=1}^{\mathcal{T}_1} \eta_r), U_k^*)_G = 0$ (compare with the proof of Proposition 2.2.6). Therefore,

$$i(f, U_k^*)_G = i(\mathcal{A}_\beta(\mu)v, U_k^*)_G = \sum_{r=1}^{\mathcal{T}_1} i(\mathcal{A}_\beta(\mu)\eta_r v, U_k^*)_G.$$

From Proposition 2.2.5, it follows that the right-hand side is equal to a_k. The solution u in the statement of this theorem is defined by the equality $u = v + c_1 U_1 + \dots + c_d U_d$. □

2.3 Waves and Scattering Matrices

2.3.1 Waves

We start with the boundary value problem

$$\begin{aligned}(-\Delta-\mu)u(y,t)&=0, \quad (y,t)\in\Pi,\\ u(y,t)&=0, \quad (y,t)\in\partial\Pi,\end{aligned} \tag{2.3.1}$$

in the cylinder $\Pi=\{(y,t): y=(y_1,\dots,y_n)\in\Omega, t\in\mathbb{R}\}$ and with the operator pencil

$$\mathfrak{A}(\lambda,\mu)v(y)=(-\Delta_y+\lambda^2-\mu)v(y), \quad y\in\Omega; \quad v|_{\partial\Omega}=0. \tag{2.3.2}$$

Let $\{\mu_k\}_{k=1}^{\infty}$ be the non-decreasing sequence of the eigenvalues of the problem

$$\begin{aligned}(-\Delta_y-\mu)v(y)&=0, \quad y\in\Omega,\\ v(y)&=0, \quad y\in\partial\Omega.\end{aligned} \tag{2.3.3}$$

counted according to their multiplicity (see 2.1.1). We fix a real $\mu\neq\mu_k$, $k=1,2,\dots$, that is, the μ is not a threshold, and introduce the functions

$$u_k^{\pm}(y,t;\mu)=(2|\lambda_k^{\mp}|)^{-1/2}\exp(i\lambda_k^{\mp}t)\varphi_k(y) \tag{2.3.4}$$

with real $\lambda_k^{\pm}=\pm(\mu-\mu_k)^{1/2}$ in the cylinder Π; these functions satisfy problem (2.3.3). The u_k^+ (u_k^-) will be called a wave incoming from $+\infty$ (outgoing to $+\infty$). The number of the waves is equal to twice the number of μ_k (counted according to their multiplicities) such that $\mu_k<\mu$. Recall that $\lambda_k^{\pm}$ are eigenvalues of the pencil $\mathfrak{A}(\cdot,\mu)$ with the same eigenvector φ_k, which is also an eigenvector of problem (2.3.3) corresponding to the eigenvalue μ_k. The eigenvectors are orthogonal and normalized by the condition

$$(\varphi_j,\varphi_k)_\Omega=\delta_{jk}. \tag{2.3.5}$$

Let G be a domain in $\mathbb{R}^{n+1}$ introduced at the beginning of Sect. 2.2.1; we consider problem (2.2.1). With every Π_+^r, we associate a problem of the form (2.3.1) in the cylinder $\Pi^r=\{(y^r,t^r): y^r\in\Omega^r, t^r\in\mathbb{R}\}$. A number τ is called a threshold for problem (2.2.1) if the τ is a threshold for at least one of the problems in the cylinders Π^r. In this section, we consider problem (2.2.1) for a real μ different from the thresholds.

Let $\chi\in C^\infty(\mathbb{R})$ be a cut-off function, $\chi(t)=0$ for $t<0$ and $\chi(t)=1$ for $t>1$. We multiply each wave in Π^r by the function $t\mapsto\chi(t^r-t_0^r)$ with a certain (sufficiently large) $t_0^r>0$ and then extend the product by zero to the domain G. We denote the obtained functions by $v_1,\dots,v_{2M}$, where $2M$ is the number of all real eigenvalues of the pencils $\mathfrak{A}^1(\cdot, \mu), \dots, \mathfrak{A}^{\mathcal{T}}(\cdot, \mu)$ counted according to their (geometric) multiplicity.

For $l = 0, 1, \dots$ and $\delta \in \mathbb{R}$, we introduce the space $H^l_\delta(G)$ with norm $\|u; H^l_\delta(G)\| = \|\rho_\delta u; H^l(G)\|$, where ρ_δ denotes a smooth positive on $\overline{G}$ function given on $\Pi^r_+ \cap G$ by the equality $\rho_\delta(y^r, t^r) = \exp(\delta t^r)$; unlike a similar definition in 2.2.1, from now on we choose the weight index δ to be the same in all cylindrical ends. Let $\dot{H}^2_\delta(G)$ denote the closure in $H^2_\delta(G)$ of the set of smooth functions in $\overline{G}$ that have compact supports in $\overline{G}$ and vanish on ∂G. We now assume that the δ is positive and small so that the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \delta\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, except the real ones.

We denote by $\mathfrak{M}$ the linear space spanned by the functions $v_1, \dots, v_{2M}$ and introduce the quotient space $\mathcal{W}(\mu, G) := (\mathfrak{M} \dotplus \dot{H}^2_\delta(G))/\dot{H}^2_\delta(G)$. The elements in $\mathcal{W}(\mu, G)$ are called waves in G. We will often write $\mathcal{W}$ instead of $\mathcal{W}(\mu, G)$.

Proposition 2.3.1 *The bilinear form*

$$q(u, v) := ((-\Delta - \mu)u, v)_G - (u, (-\Delta - \mu)v)_G \tag{2.3.6}$$

makes sense for u and v in $\mathfrak{M} \dotplus \dot{H}^2_\delta(G)$. Moreover, if one of the elements u and v belongs to $\dot{H}^2_\delta(G)$, the $q(u, v)$ vanishes. Therefore, the form $q(\,,\,)$ is defined on $\mathcal{W} \times \mathcal{W}$. For any waves U and V in $\mathcal{W}$, there holds the equality $q(U, V) = -\overline{q(V, U)}$.

Proof Any function in $\mathfrak{M} \dotplus \dot{H}^2_\delta(G)$ is of the form $c_1 v_1 + \dots + c_{2M} v_{2M} + w$, where $c_1, \dots, c_{2M}$ are some constants and $w \in \dot{H}^2_\delta(G)$. The support of $(-\Delta - \mu)v_j$ is compact and $(-\Delta - \mu)w$ belongs to $H^0_\delta(G)$, so the right-hand side in (2.3.6) makes sense. Let us assume that u or v belongs to $\dot{H}^2_\delta(G)$; then, integrating by parts, we obtain the equality

$$((-\Delta - \mu)u, v)_G = (u, (-\Delta - \mu)v)_G.$$

Therefore, we can set

$$q(\tilde{u}, \tilde{v}) := q(u, v), \tag{2.3.7}$$

for any u and v in $u \in \mathfrak{M} \dotplus \dot{H}^2_\delta(G)$, where $\tilde{u}$ and $\tilde{v}$ denote the classes of u and v in $\mathcal{W}$. The equality $q(U, V) = -\overline{q(V, U)}$ follows immediately from (2.3.6) and (2.3.7). □

The number $q(U, U)$ is imaginary for any $U \in \mathcal{W}$. We call wave U outgoing (incoming) if $iq(U, U)$ is a positive (negative) number. Let $u^\pm$ be a wave of the form (2.3.4) in a cylinder Π^r_+. We extend the function $(y^r, t^r) \mapsto \chi(t^r - t^r_0)u^\pm(y^r, t^r, \mu)$ by zero to G and denote by $U^\pm$ the class in $\mathcal{W}$ of the obtained function in G. In this way, we define the waves $U^\pm_1, \dots, U^\pm_M$; as before, $2M = 2M(\mu)$ is equal to the number of all real eigenvalues of the pencils $\mathfrak{A}^1(\cdot, \mu), \dots, \mathfrak{A}^{\mathcal{T}}(\cdot, \mu)$ counted according to their multiplicity. Integrating by parts in expressions of the form $q(u, v)$, where u and v are representatives of the waves $U^\pm_j$, we arrive at the following

Proposition 2.3.2 *The U_j^+ (U_j^-), $j = 1, \dots, M$, are incoming (outgoing) waves. The collection $U_1^+, \dots, U_M^+, U_1^-, \dots, U_M^-$ forms a basis in the space $\mathcal{W}$ subject to the orthogonality and normalization conditions*

$$\begin{aligned} q(U_j, U_k) = 0 \quad \text{for} \quad j \neq k, \quad iq(U_j^+, U_j^+) = -1, \\ iq(U_j^-, U_j^-) = 1 \quad \text{for} \quad j = 1, \dots, M. \end{aligned} \tag{2.3.8}$$

2.3.2 Continuous Spectrum Eigenfunctions. The Scattering Matrix

In this section, we consider the parameter μ in an interval $[\mu', \mu'']$ that contains no thresholds of problem (2.2.1), where $\mu' > \tau_1$ and τ_1 is the first threshold. Therefore, there exists such a positive δ that, for all $\mu \in [\mu', \mu'']$, the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \delta\}$ is free from the eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, except the real ones.

Now, we introduce several definitions. If $u \in \ker\mathcal{A}_{-\delta}(\mu_0)$ and $u \notin L_2(G)$, the u is called a continuous spectrum eigenfunction of the problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) = 0, \quad x \in G, \\ u(x) = 0, \quad x \in \partial G, \end{aligned} \tag{2.3.9}$$

at the point μ_0. If $u \in \ker\mathcal{A}_{-\delta}(\mu_0)$, $u \neq 0$, and $u \in L_2(G)$, the u is said to be an eigenfunction and μ_0 is an eigenvalue of problem (2.3.9) embedded in the continuous spectrum; in fact, any such eigenfunction belongs to $\ker\mathcal{A}_\delta(\mu_0)$ (this can be derived from Theorem 2.2.3). For problem (2.3.9), it is known that the eigenvalues do not accumulate at finite distance. Therefore, the interval $[\mu', \mu'']$ contains finitely many eigenvalues at most. The number $\dim(\ker\mathcal{A}_{-\delta}(\mu)/\ker\mathcal{A}_\delta(\mu))$ is called the continuous spectrum multiplicity at μ. Equality (2.2.6) for $\beta = -\delta$ and $\gamma = \delta$ takes the form

$$\dim(\ker\mathcal{A}_{-\delta}(\mu)/\ker\mathcal{A}_\delta(\mu)) = M(\mu) \tag{2.3.10}$$

because, in this case, $\varkappa = 2M(\mu)$. The interval $[\mu', \mu'']$ contains no thresholds and therefore the continuous spectrum multiplicity is constant on this interval.

Any element $v \in \ker\mathcal{A}_{-\delta}(\mu)$ defines a certain class $\tilde{v}$ in the wave space $\mathcal{W}$; we let $\mathfrak{K}$ denote the image of $\ker\mathcal{A}_{-\delta}(\mu)$ in $\mathcal{W}$. The $\mathfrak{K}$ is a subspace in $\mathcal{W}$.

Theorem 2.3.3 *Let $U_1^+(\mu), \dots, U_M^+(\mu), U_1^-(\mu), \dots, U_M^-(\mu)$ be the same basis of $\mathcal{W}(\mu, G)$ as in Proposition 2.3.2. Then there exist bases $\tilde{\zeta}_1(\mu), \dots, \tilde{\zeta}_M(\mu)$ and $\tilde{\eta}_1(\mu), \dots, \tilde{\eta}_M(\mu)$ in $\mathfrak{K}(\mu)$ such that*

$$\tilde{\zeta}_j(\mu) = U_j^+(\mu) + \sum_{k=1}^{M} S_{jk}(\mu)U_k^-(\mu), \tag{2.3.11}$$

$$\tilde{\eta}_j(\mu) = U_j^-(\mu) + \sum_{k=1}^{M} T_{jk}(\mu)U_k^+(\mu). \tag{2.3.12}$$

The matrix $S(\mu) = \|S_{jk}(\mu)\|$ *is unitary and* $S(\mu)^{-1} = T(\mu) = \|T_{jk}(\mu)\|$.

Proof Let $v_1, \dots, v_M$ be linear independent elements in $\ker \mathcal{A}_{-\delta}(\mu)/\ker \mathcal{A}_\delta(\mu)$ and $\tilde{v}_1, \dots, \tilde{v}_M$ their classes in $\mathcal{W}$. We have

$$\tilde{v}_j = \sum_{k=1}^{M} m_{jk}^+ U_k^+ + \sum_{k=1}^{M} m_{jk}^- U_k^-, \qquad j = 1, \dots, M.$$

The matrices $\mathcal{M}^+ = \|m_{jk}^+\|$ and $\mathcal{M}^- = \|m_{jk}^-\|$ are nonsingular. Indeed, if, for instance, $\det \mathcal{M}^+ = 0$, there exists a nonzero $\tilde{v} \in \mathcal{W}$ with $v \in \ker \mathcal{A}_{-\delta}(\mu)$ such that

$$v = \sum_{k=1}^{M} a_k u_k + w,$$

where a_k is a constant, $\tilde{u}_k = U_k^-$, and $w \in \dot{H}_\delta^2(G)$. From (2.3.6) it follows that $q(v, v) = 0$. On the other hand,

$$q(v, v) = q\Big(\sum_{k=1}^{M} a_k u_k, \sum_{k=1}^{M} a_k u_k\Big) = \sum_{k=1}^{M} |a_k|^2 q(U_k, U_k) = i \sum_{k=1}^{M} |a_k|^2 \neq 0,$$

which is a contradiction. Therefore, there exist linear combinations $\tilde{\zeta}_j$ and $\tilde{\eta}_j$ of the $\tilde{v}_1, \dots, \tilde{v}_M$ that satisfy (2.3.11) and (2.3.12), respectively.

We now pass to verifying the second part of this theorem. For $\tilde{\zeta}_l$ and $\tilde{\eta}_m$ in (2.3.11) and (2.3.12), we choose representatives ζ_l and η_m in $\ker \mathcal{A}_{-\delta}(\mu)$. Then $q(\zeta_l, \eta_m) = 0$ and, moreover,

$$\begin{aligned} q(\zeta_l, \eta_m) &= q\Big(U_l^+ + \sum_{j=1}^{M} S_{lj} U_j^-, U_m^- + \sum_{k=1}^{M} T_{mk} U_k^+\Big) \\ &= q\Big(\sum_{j=1}^{M} S_{lj} U_j^-, U_m^-\Big) + q\Big(U_l^+, \sum_{k=1}^{M} T_{mk} U_k^+\Big) = i S_{lm} - i \overline{T}_{ml}, \end{aligned}$$

hence $S(\mu) = T^*(\mu)$. Let us consider

$$\tilde{v} := \sum_{j=1}^{M} S_{lj}\tilde{\eta}_j = \sum_{j=1}^{M} S_{lj}U_j^- + \sum_{j=1}^{M}\sum_{k=1}^{M} S_{lj}T_{jk}U_k^+.$$

The coefficients of the $\tilde{v}$ and $\tilde{\zeta}_j$ are the same at U_j^-, $j = 1, \dots, M$. Therefore, the coefficients also coincide at U_k^+, $k = 1, \dots, M$; otherwise, for the $\tilde{v} - \tilde{\zeta}_j$, we obtain a contradiction like that in the first part of the proof. Thus, we have $S(\mu)^{-1} = T(\mu)$. $\square$

Definition 2.3.4 The matrix $S(\mu) = \|S_{jk}(\mu)\|_{j,k=1}^{M}$ with entries in (2.3.11) is called the scattering matrix.

2.3.3 The Intrinsic Radiation Principle

Let $U_1^+(\mu), \dots, U_M^+(\mu), U_1^-(\mu), \dots, U_M^-(\mu)$ be the same basis of $\mathcal{W}(\mu, G)$ as in Proposition 2.3.2 and in Theorem 2.3.3. We choose any representatives u_j^- of U_j^-, $j = 1, \dots, M$ and denote by $\mathfrak{N}$ the linear hull of $u_1^-, \dots, u_M^-$. We define the norm of $u = \sum c_j u_j^- + v \in \mathfrak{N} \dotplus \dot{H}_\delta^2(G)$ with $c_j \in \mathbb{C}$ and $v \in \dot{H}_\delta^2(G)$ by

$$\|u\| = \sum |c_j| + \|v; H_\delta^2(G)\|.$$

Let $\mathbf{A}(\mu)$ be the restriction of the operator $\mathcal{A}_{-\delta}(\mu)$ to the space $\mathfrak{N} \dotplus \dot{H}_\delta^2(G)$. The map

$$\mathbf{A}(\mu) : \mathfrak{N} \dotplus \dot{H}_\delta^2(G) \to H_\delta^0(G)$$

is continuous. The following theorem provides the statement of problem (2.2.1) with intrinsic radiation conditions at infinity (the numbers μ and δ are supposed to satisfy the requirements given at the beginning of 2.3.2.

Theorem 2.3.5 *Let $z_1, \dots, z_d$ be a basis in the space* $\ker\mathcal{A}_\delta(\mu)$*, $f \in H_\delta^0(G)$ and $(f, z_j)_G = 0$, $j = 1, \dots, d$. Then:*

1. There exists a solution $u \in \mathfrak{N} \dotplus \dot{H}_\delta^2(G)$ of the equation $\mathbf{A}(\mu)u = f$ determined up to an arbitrary term in $\ker\mathcal{A}_\delta(\mu)$.

2. The inclusion

$$v \equiv u - c_1 u_1^- - \cdots - c_M u_M^- \in \dot{H}_\delta^2(G) \tag{2.3.13}$$

holds with $c_j = i(f, \tilde{\eta}_j)_G$.

3. The inequality

$$\|v; H^2_\delta(G)\| + |c_1| + \cdots + |c_M| \leq \mathrm{const}\left(\|f; H^0_\delta(G)\| + \|v; L_2(G^c)\|\right) \quad (2.3.14)$$

holds with v *and* $c_1, \ldots, c_M$ *in* (2.3.13)*, while* G^c *is a compact subset of* G*. A solution* u_0 *that is subject to the additional conditions* $(u_0, z_j)_G = 0$ *for* $j = 1, \ldots, d$ *is unique and satisfies* (2.3.14) *with right-hand side changed for* $\mathrm{const}\|f; H^0_\delta(G)\|$.

Proof Let us outline the proof. The operator $\mathcal{A}_{-\delta}(\mu)$ is Fredholm. Therefore, the orthogonality conditions $(f, z_j)_G = 0$, $j = 1, \ldots, d$, provide the existence of a solution $u \in \dot{H}^2_{-\delta}(G)$ to the equation $\mathcal{A}_{-\delta}(\mu)u = f$. Since $z_1, \ldots, z_d, \eta_1, \ldots, \eta_M$ form a basis in $\ker\mathcal{A}_{-\delta}(\mu)$, the general solution is of the form

$$u = u^0 + \sum_{j=1}^{M} a_j \eta_j + \sum_{k=1}^{d} b_k z_k \quad (2.3.15)$$

with a particular solution $u^0 \in \dot{H}^2_{-\delta}(G)$ and arbitrary constants a_k and b_k. According to Theorem 2.3.3, we have $\det\|T_{jk}\| \neq 0$. Therefore, in view of (2.3.12), we can obtain (2.3.13) by choosing the coefficients a_k. The equality $c_j = i(f, \tilde{\eta}_j)_G$ now follows from Theorem 2.2.7 and the relations $q(U^-_j, \eta_k) = -i\delta_{jk}$, $j, k = 1, \ldots, M$. In connections with estimate (2.3.14) see Theorems 5.3.5 and 5.1.4 in [37]. □

Chapter 3
Properties of Scattering Matrices in a Vicinity of Thresholds

We assume $\tau' < \tau''$ to be thresholds of problem (3.1.16) such that the interval (τ', τ'') contains the only threshold τ. We also suppose that the three thresholds relate to the same cylindrical end. On the interval (τ, τ''), one can choose a basis of incoming $w_1^+(\cdot, \mu), \dots, w_\varkappa^+(\cdot, \mu)$ and outgoing $w_1^-(\cdot, \mu), \dots, w_\varkappa^-(\cdot, \mu)$ waves with analytic functions $(\tau, \tau'') \ni \mu \mapsto w_j^\pm(\cdot, \mu)$ that admit the analytic continuation to (τ', τ''); here $\varkappa = \varkappa(\mu'')$ (recall that $\varkappa(\mu) = \text{const}$ for $\mu \in [\tau, \tau'')$). Such a basis is called stable at the threshold τ. For $\mu \in (\tau', \tau)$, some incoming waves and the same number of outgoing waves turn out to be exponentially growing as $x \to \infty$. On the interval (τ, τ''), in the space of continuous spectrum eigenfunctions, there exists a basis $\mathcal{Y}_1(\cdot, \mu), \dots, \mathcal{Y}_\varkappa(\cdot, \mu)$ satisfying the conditions

$$\mathcal{Y}_j(x, \mu) = w_j^+(x, \mu) - \sum_{k=1}^{M} \mathcal{S}_{jk}(\mu) w_k^-(x, \mu) + O(e^{-\varepsilon|x|}). \tag{3.0.1}$$

The functions $\mu \mapsto \mathcal{Y}_j(\cdot, \mu)$ and $\mu \mapsto \mathcal{S}_{jk}(\mu)$ are analytic and admit the analytic continuation to (τ', τ''). Unlike $S(\mu)$, the new matrix $\mathcal{S}(\mu) = \|\mathcal{S}_{jk}(\mu)\|$ keeps its size on this interval; the matrix is unitary for all $\mu \in (\tau', \tau'')$. The entries of $S(\mu)$ can be expressed in terms related only to the matrix $\mathcal{S}(\mu)$. In particular, this enables us to prove the existence of finite limits of $S(\mu)$ as $\mu \to \tau \pm 0$, to calculate the limits, and, in essence, to reduce (in Chap. 4) the approximate calculation of the matrix $S(\mu)$ with $\mu \in [\mu', \mu'']$ to that of the augmented matrix $\mathcal{S}(\mu)$.

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3.1 Augmented Space of Waves

3.1.1 Waves in a Cylinder

We start with the boundary value problem

$$\begin{aligned}(-\Delta - \mu)u(y,t) &= 0, \quad (y,t) \in \Pi,\\ u(y,t) &= 0, \quad (y,t) \in \partial\Pi,\end{aligned} \tag{3.1.1}$$

in the cylinder $\Pi = \{(y,t) : y = (y_1, \dots, y_n) \in \Omega, t \in \mathbb{R}\}$ and with the operator pencil

$$\mathfrak{A}(\lambda, \mu)v(y) = (-\Delta_y + \lambda^2 - \mu)v(y), \quad y \in \Omega; \quad v|_{\partial\Omega} = 0. \tag{3.1.2}$$

Let $\{\mu_k\}_{k=1}^{\infty}$ be the non-decreasing sequence of the eigenvalues of the problem

$$\begin{aligned}(-\Delta_y - \mu)v(y) &= 0, \quad y \in \Omega,\\ v(y) &= 0, \quad y \in \partial\Omega\end{aligned} \tag{3.1.3}$$

counted according to their multiplicity (see Chap. 2). We fix a real $\mu \neq \mu_k$, $k = 1, 2, \dots$, that is, the μ is not a threshold, and introduce a linear complex space $W(\mu)$ spanned by the functions

$$u_k^{\pm}(y,t;\mu) = (2|\lambda_k^{\mp}|)^{-1/2} \exp(i\lambda_k^{\mp} t)\varphi_k(y) \tag{3.1.4}$$

with real $\lambda_k^{\pm} = \pm(\mu - \mu_k)^{1/2}$. We will call $W(\mu)$ the space of waves. Its dimension is equal to twice the number of μ_k (counted according to their multiplicities) such that $\mu_k < \mu$. Recall that $\lambda_k^{\pm}$ are eigenvalues of the pencil $\mathfrak{A}(\cdot, \mu)$ with the same eigenvector φ_k, which is also an eigenvector of problem (3.1.3) corresponding to the eigenvalue μ_k. The eigenvectors are orthogonal and normalized by the condition

$$(\varphi_j, \varphi_k)_\Omega = \delta_{jk}. \tag{3.1.5}$$

Assume now that $\mu = \tau$ is a threshold and, consequently, μ is an eigenvalue of (3.1.3) with multiplicity $\varkappa \geq 1$. Then $\varkappa$ numbers μ_l satisfy $\mu_l = \tau$. For each l, the functions $\exp(i\lambda_l^{+} t)\varphi_l(y)$ and $\exp(i\lambda_l^{-} t)\varphi_l(y)$ coincide. Therefore, the number of linearly independent functions of the form $(y,t) \mapsto \exp(i\lambda_k^{\pm} t)\varphi_k(y)$ for $\mu = \tau$ is $\varkappa$ less than the number of such functions for μ satisfying $\tau < \mu < \tau + \beta$ with small $\beta > 0$. However, for a more general notion of the waves, the dimension of the space $W(\mu)$ is continuous from the right at the threshold. In such a case the definition of incoming and outgoing waves is based on reasons of energy, as in the Sommerfeld and Mandelstamm principles.

For the definition, we introduce the form

$$q_N(u, v) := ((-\Delta - \mu)u, v)_{\Pi(N)} + (u, -\partial_\nu v)_{\partial\Pi(N)\cap\partial\Pi} \\ - (u, (-\Delta - \mu)v)_{\Pi(N)} - (-\partial_\nu u, v)_{\partial\Pi(N)\cap\partial\Pi}, \tag{3.1.6}$$

where $\Pi(N) = \{(y, t) \in \Pi : t < N\}$, the number $\mu \in \mathbb{R}$ is for the time being not a threshold, $u = \chi f$ and $v = \chi g$, while f and g are any of the functions (3.1.4) corresponding to real $\lambda_k^\pm(\mu)$ (possibly, with distinct indices); χ denotes a smooth cut-off function, $\chi(t) = 0$ for $t < T - 1$ and $\chi(t) = 1$ for $t > T$ with $T < N$. Integrating by parts, we see that

$$iq_N(\chi u_k^\pm, \chi u_l^\mp) = 0 \quad \text{for all } k, l, \tag{3.1.7}$$

$$iq_N(\chi u_k^\pm, \chi u_l^\pm) = \mp\delta_{kl}, \tag{3.1.8}$$

so the result is independent of N and χ; in what follows we drop N but keep χ. We name the wave $u_k^+(u_k^-)$ incoming (outgoing) for $-(+)$ on the right in (3.1.8) and obtain the definition of incoming (outgoing) waves equivalent to the old definition.

We are going to construct a basis in the (augmented) space of waves "stable at a threshold". Let $\mu \in \mathbb{R}$ be a regular value of the spectral parameter of problem (3.1.3) and μ_m the eigenvalue with the greatest number satisfying $\mu_m < \mu$. We also assume that $\mu_l < \mu_{l+1} = \cdots = \mu_m$. Then the numbers $\tau' := \mu_l$, $\tau := \mu_{l+1} = \cdots = \mu_m$, and $\tau'' := \mu_{m+1}$ turn out to be three successive thresholds $\tau' < \tau < \tau''$ of problem (3.1.1) in cylinder Π. (We discuss the general situation; the cases $l + 1 = m$, $m = 1$, and so on, can be considered with evident simplifications.)

We set

$$w_k^\pm(y, t; \mu) = 2^{-1/2}\left(\frac{e^{it\sqrt{\mu-\mu_k}} + e^{-it\sqrt{\mu-\mu_k}}}{2} \mp \frac{e^{it\sqrt{\mu-\mu_k}} - e^{-it\sqrt{\mu-\mu_k}}}{2\sqrt{\mu - \mu_k}}\right)\varphi_k(y), \tag{3.1.9}$$

$$w_p^\pm(y, t; \mu) = u_p^\pm(y, t; \mu), \tag{3.1.10}$$

where $k = l + 1, \ldots, m$, $p = 1, \ldots, l$, and $u_p^\pm$ are defined in (3.1.4).

Proposition 3.1.1 *The functions $\mu \mapsto w_k^\pm(y, t; \mu)$, $k = l + 1, \ldots, m$, admit the analytic continuation to the whole complex plane. These analytic functions smoothly depend on the parameters $y \in \bar{\Omega}$ and $t \in \mathbb{R}$ (i.e., any derivatives in y and t are analytic functions as well).*

The functions $\mu \mapsto w_p^\pm(y, t; \mu)$ are analytic on the complex plane with cut along the ray $\{\mu \in \mathbb{R} : -\infty < \mu \le \mu_p\}$, $p = 1, \ldots, l$; they smoothly depend on y and t.

All the functions $w_k^\pm$, $k = 1, \ldots, m$, are solutions to problem (3.1.1). *For every μ in $(\tau' < \mu < +\infty)$ the functions* (3.1.9), (3.1.10) *satisfy the orthogonality and normalization conditions*

$$iq(\chi w_r^{\pm}(\cdot\,;\mu), \chi w_s^{\mp}(\cdot\,;\mu)) = 0 \quad \textit{for all} \quad r, s = 1, \dots, m, \tag{3.1.11}$$
$$iq(\chi w_r^{\pm}(\cdot\,;\mu), \chi w_s^{\pm}(\cdot\,;\mu)) = \mp\delta_{rs}. \tag{3.1.12}$$

Proof The first and second fractions in the parentheses in (3.1.9) can be decomposed in the series

$$\sum_{l\geq 0} \frac{(\mu_k-\mu)^l t^{2l}}{(2l)!} \quad \text{and} \quad it\sum_{l\geq 0} \frac{(\mu_k-\mu)^l t^{2l}}{(2l+1)!}, \tag{3.1.13}$$

which are absolutely and uniformly convergent on any compact $K \subset \{(\mu, t) : \mu \in \mathbb{C}, t \in \mathbb{R}\}$. This implies the analyticity properties of $w_k^{\pm}(y, t; \mu)$ for $k = l+1, \dots, m$. The corresponding assertions about $w_p^{\pm}(y, t; \mu)$ with $p = 1, \dots, l$ are evident.

It remains to verify the orthogonality and normalization conditions. We first assume that $\mu > \tau$ and consider, for instance, (3.1.12). If r and s are distinct, then the equalities (3.1.12) follow from the orthogonality of φ_r and φ_s (as well as (3.1.7) and (3.1.8)). In the case $r = s \leq l$, relation (3.1.8) contains the needed formula. Finally, assume that $r = s > l$ and substitute the expressions (3.1.9) into $q(\chi w_r^{\pm}, \chi w_s^{\pm})$. Setting $\lambda := \sqrt{\mu - \tau}$, we obtain

$$\begin{aligned} iq(\chi w_s^{\pm}, \chi w_s^{\pm}) = {}&\lambda^{-2}((\lambda \pm 1)\,(\lambda \mp 1) iq^{+-} + (\lambda \mp 1)\,(\lambda \pm 1) iq^{-+} \\ &+ (\lambda \mp 1)^2\, iq^{++} + (\lambda \pm 1)^2\, iq^{--}), \end{aligned} \tag{3.1.14}$$

where, for example, $q^{+-} = 2^{-3} q(\chi e^{it\lambda}\varphi_s, \chi e^{-it\lambda}\varphi_s)$, and so on. Taking account of (3.1.4), (3.1.7), and (3.1.8), we arrive at (3.1.12).

We now consider the function

$$\begin{aligned} \mathbb{C} \ni \mu \mapsto q_N(u, v; \mu) := {}&((-\Delta - \mu)u, v)_{\Pi(N)} + (u, -\partial_\nu v)_{\partial\Pi(N)\cap\partial\Pi} \\ &- (u, (-\Delta - \bar{\mu})v)_{\Pi(N)} - (-\partial_\nu u, v)_{\partial\Pi(N)\cap\partial\Pi}, \end{aligned} \tag{3.1.15}$$

where $\Pi(N)$, N, and χ are the same as in (3.1.6), $u = \chi w_r^{\pm}(\cdot\,;\mu)$, and $v = \chi w_s^{\mp}(\cdot\,;\bar{\mu}))$. Since u and $\bar{v}$ are analytic, the function $\mu \mapsto q_N(u, v; \mu)$ is analytic as well. Therefore, the equalities (3.1.12) (with $r = s > l$) are valid for all $\mu \in \mathbb{C}$. □

It follows from (3.1.9) that $w_k^{\pm}(y, t; \tau) = 2^{-1/2}(1 \mp it)\varphi_k(y)$, $k = l+1, \dots, m$, and, in the case $\mu < \tau$, the amplitudes of the waves exponentially grow as $t \to \infty$. The space spanned by waves (3.1.9) and (3.1.10) is called the augmented space of waves for $\tau' < \mu < \tau$ and denoted by $W_a(\mu)$. We let $W(\mu)$ denote the linear hull of functions (3.1.9) and (3.1.10) for $\tau \leq \mu < \tau''$ and the linear hull of functions (3.1.10) for $\tau' < \mu < \tau$. The lineal $W(\mu)$ is called the space of waves. An element $w \in W_a(\mu)$ (or $W(\mu)$) is called a wave incoming from $+\infty$ (outgoing to $+\infty$), if $iq(\chi w, \chi w) < 0$ $(iq(\chi w, \chi w) > 0)$.

The collection of waves $\{w^{\pm}\}_{k=1}^{m}$ defined by (3.1.9) and (3.1.10) is called a basis of waves stable in a neighborhood of the threshold τ. A basis of waves of the form (3.1.4) is by definition stable on (μ', μ'') if the interval $[\mu', \mu'']$ contains no thresholds.

3.1.2 Waves in Domain G

Let G be a domain in $\mathbb{R}^{n+1}$ with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi_+^1 \cup \cdots \cup \Pi_+^{\mathcal{T}}$ of finitely many non-overlapping semicylinders

$$\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r,\ t^r > 0\},$$

where (y^r, t^r) are local coordinates in $\mathrm{B}\Pi_+^r$ and Ω^r is a bounded domain in $\mathbb{R}^n$. We consider the problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) = 0, &\quad x \in G,\\ u(x) = 0, &\quad x \in \partial G. \end{aligned} \tag{3.1.16}$$

With every Π_+^r we associate a problem of the form (3.1.1) in the cylinder $\Pi^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r \in \mathbb{R}\}$. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function, $\chi(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t > 1$. We multiply each wave in Π^r by the function $t \mapsto \chi(t^r - t_0^r)$ with a certain $t_0^r > 0$ and then extend it by zero to the domain G. All functions (for all Π^r) obtained in such a way are called waves in G. A number τ is called a threshold for problem (3.1.16) if the τ is a threshold at least for one of problems of the form (3.1.1) in Π^r, $r = 1, \ldots, \mathcal{T}$. Let $\tau' < \tau < \tau''$ be three successive thresholds for problem (3.1.16); then the intervals (τ', τ) and (τ, τ'') are free from the thresholds.

For $\mu \in (\tau', \tau)$, we introduce the augmented space $\mathcal{W}_a(\mu, G)$ of waves in G as the union of the waves in G corresponding to those in $W_a(\mu)$ for Π^r, $r = 1, \ldots, \mathcal{T}$; if a space $W_a(\mu)$ is not introduced on the interval $\tau' < \mu < \tau$ for a certain Π^r (which means that the τ is not a threshold for problem (3.1.1) in such a cylinder), then, from this cylinder, we include into the space $\mathcal{W}_a(\mu, G)$ the waves generated by the elements of the corresponding $W(\mu)$. By definition, for $\mu \in (\tau', \tau'')$, the space $\mathcal{W}(\mu, G)$ of waves in G is the union of the waves in G that correspond to the waves in $W(\mu)$ for Π^r, $r = 1, \ldots, \mathcal{T}$.

The bases $\{u_j^{\pm}(\cdot, \mu)\}$ and $\{w_j^{\pm}(\cdot, \mu)\}$ of waves in $\mathcal{W}(\mu, G)$ and $\mathcal{W}_a(\mu, G)$ consist of the waves obtained in G from the basis waves in Π^r, $r = 1, \ldots, \mathcal{T}$. The basis waves in the spaces $\mathcal{W}(\mu, G)$ and $\mathcal{W}_a(\mu, G)$ are subject to orthogonality and normalization conditions like (3.1.7) and (3.1.8) or (3.1.11) and (3.1.12) with the form q in a cylinder replaced by the form q_G in G:

$$\begin{aligned} q_G(u, v) := {} & ((-\Delta - \mu)u, v)_G + (u, -\partial_\nu v)_{\partial G}\\ & - (u, (-\Delta - \mu)v)_G - (-\partial_\nu u, v)_{\partial G}. \end{aligned} \tag{3.1.17}$$

An element w in $\mathcal{W}_a(\mu, G)$ (or in $\mathcal{W}(\mu, G)$) is called a wave incoming from ∞ (outgoing to ∞), if $iq_G(\chi w, \chi w) < 0$ ($iq_G(\chi w, \chi w) > 0$).

A basis of waves in G is called stable near a value ν of the spectral parameter if the basis consists of bases in the cylinders $\Pi^1, \dots, \Pi^{\mathcal{T}}$ stable near ν.

3.2 Continuous Spectrum Eigenfunctions. Scattering Matrices

Let $\tau' < \tau < \tau''$ be three successive thresholds for problem (3.1.16). For the sake of simplicity, we assume that these three numbers are thresholds for a problem of the form (3.1.1) only in one of the cylinders $\Pi^1, \dots, \Pi^{\mathcal{T}}$, for instance in $\Pi^1 = \Omega^1 \times \mathbb{R}$. Moreover, we suppose that $\tau' = \mu_l$, $\tau = \mu_{l+1} = \dots = \mu_m$, and $\tau'' = \mu_{m+1}$, where μ_k are eigenvalues of problem (3.1.3) in Ω^1. Thus for $\Pi = \Pi^1$ we deal with the situation considered in 3.1.1.

3.2.1 Intrinsic and Expanded Radiation Principles

We consider the boundary value problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= f(x), \quad x \in G, \\ u(x) &= g(x), \quad x \in \partial G, \end{aligned} \tag{3.2.1}$$

and recall two correct statements of the problem with radiation conditions at infinity: the intrinsic and expanded radiation principles. In the first principle, the intrinsic radiation conditions contain only outgoing waves in the space $\mathcal{W}(\mu, G)$. The second (expanded) principle includes the outgoing waves in the augmented space $\mathcal{W}_a(\mu, G)$. We will apply the intrinsic principle with spectral parameter outside a neighborhood of the thresholds. In the vicinity of a threshold, we make use of the expanded principle employing the stable basis of waves in $\mathcal{W}_a(\mu, G)$ constructed in Sect. 3.1.

We first define the needed function spaces. For integer $l \geq 0$, we denote by $H^l(G)$ the Sobolev space with norm

$$\|v; H^l(G)\| = \left(\sum_{j=0}^{l} \int_G \sum_{|\alpha|=j} |D_x^\alpha v(x)|^2 \, dx \right)^{1/2},$$

and let $H^{l-1/2}(\partial G)$ with $l \geq 1$ stand for the space of traces on ∂G of the functions in $H^l(G)$. Assume that ρ_γ is a smooth positive on $\overline{G}$ function given on Π_+^r by the equality $\rho_\gamma(y^r, t^r) = \exp(\gamma t^r)$ with $\gamma \in \mathbb{R}$. We also introduce the spaces $H_\gamma^l(G)$ and

$H^{l-1/2}_\gamma(\partial G)$ with norms $\|u; H^l_\gamma(G)\| = \|\rho_\gamma u; H^l(G)\|$ and $\|v; H^{l-1/2}_\gamma(\partial G)\| = \|\rho_\gamma v; H^{l-1/2}(\partial G)\|$. The operator of problem (3.2.1) implements the continuous mapping

$$\mathcal{A}_\gamma(\mu) : H^2_\gamma(G) \to H^0_\gamma(G) \times H^{3/2}_\gamma(\partial G). \tag{3.2.2}$$

As is known, operator (3.2.2) is Fredholm if and only if the line $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \gamma\}$ is free of the eigenvalues of the pencils $\lambda \mapsto \mathfrak{A}^r(\lambda, \mu)$, $r = 1, \dots, \mathcal{T}$, where $\mathfrak{A}^r$ is a pencil of the form (3.1.2) for problem (3.1.1) in the cylinder Π^r. (An operator is called Fredholm if its range is closed and the kernel and cokernel are finite dimensional.)

We now proceed to the intrinsic radiation principle. Assume that μ does not coincide with a threshold, $\mu \in (\tau', \tau'')$, and $\mu \neq \tau$. Let δ denote a small positive number such that the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \delta\}$ contains only real eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$; we denote the number of such eigenvalues (counted with their multiplicities) by $2M = 2M(\mu)$. There exist collections of elements $\{Y^+_1(\cdot, \mu), \dots, Y^+_M(\cdot, \mu)\}$ and $\{Y^-_1(\cdot, \mu), \dots, Y^-_M(\cdot, \mu)\}$ in the kernel $\ker\mathcal{A}_{-\delta}(\mu)$ of $\mathcal{A}_{-\delta}(\mu)$ such that

$$\left(Y^+_j(\cdot, \mu) - u^+_j(\cdot, \mu) - \sum_{k=1}^{M} S_{jk}(\mu) u^-_k(\cdot, \mu)\right) \in H^2_\delta(G), \tag{3.2.3}$$

$$\left(Y^-_j(\cdot, \mu) - u^-_j(\cdot, \mu) - \sum_{k=1}^{M} T_{jk}(\mu) u^+_k(\cdot, \mu)\right) \in H^2_\delta(G), \tag{3.2.4}$$

where $S(\mu) = \|S_{jk}(\mu)\|$ is a unitary scattering matrix and $S(\mu)^{-1} = T(\mu) = \|T_{jk}(\mu)\|$. For future needs, we rewrite (3.2.3)–(3.2.4) in the form

$$\begin{aligned} Y^+_j(\cdot, \mu) &= u^+_j(\cdot, \mu) + \sum_{k=1}^{M} S_{jk}(\mu) u^-_k(\cdot, \mu) + O(e^{-\delta|x|}), \\ Y^-_j(\cdot, \mu) &= u^-_j(\cdot, \mu) + \sum_{k=1}^{M} T_{jk}(\mu) u^+_k(\cdot, \mu) + O(e^{-\delta|x|}). \end{aligned} \tag{3.2.5}$$

Every collection $\{Y^+_1(\cdot, \mu), \dots, Y^+_M(\cdot, \mu)\}$ and $\{Y^-_1(\cdot, \mu), \dots, Y^-_M(\cdot, \mu)\}$ is a basis modulo $\ker\mathcal{A}_\delta(\mu)$ in $\ker\mathcal{A}_{-\delta}(\mu)$. This means that any $v \in \ker\mathcal{A}_{-\delta}(\mu)$ is a linear combination of the functions $Y^+_1(\cdot, \mu), \dots, Y^+_M(\cdot, \mu)$ up to a term in $\ker\mathcal{A}_\delta(\mu)$; the same is true also for $Y^-_1(\cdot, \mu), \dots, Y^-_M(\cdot, \mu)$. If μ is not an eigenvalue of operator (3.2.2), that is, $\ker\mathcal{A}_\delta(\mu) = 0$, every collection $\{Y^+_j\}$ and $\{Y^-_j\}$ is a basis of $\ker\mathcal{A}_{-\delta}(\mu)$ in the usual sense.

The elements $Y(\cdot, \mu)$ in $\ker\mathcal{A}_{-\delta}(\mu)\backslash\ker\mathcal{A}_\delta(\mu)$ are called the continuous spectrum eigenfunctions of problem (3.1.16) corresponding to μ.

Denote by $\mathfrak{N}$ the linear hull $\mathfrak{L}(u_1^-, \dots, u_M^-)$. We define the norm of $u = \sum c_j u_j^- + v \in \mathfrak{N} \dot{+} H_\delta^2(G)$ with $c_j \in \mathbb{C}$ and $v \in H_\delta^2(G)$ by

$$\|u\| = \sum |c_j| + \|v; H_\delta^2(G)\|.$$

Let $\mathbf{A}(\mu)$ be the restriction of the operator $\mathcal{A}_{-\delta}(\mu)$ to the space $\mathfrak{N} \dot{+} H_\delta^2(G)$. The map

$$\mathbf{A}(\mu) : \mathfrak{N} \dot{+} H_\delta^2(G) \to H_\delta^0(G) \times H_\delta^{3/2}(\partial G) =: \mathcal{H}_\delta(G) \tag{3.2.6}$$

is continuous. The following theorem provides the statement of problem (3.2.1) with intrinsic radiation conditions at infinity (the numbers μ and δ are supposed to satisfy the requirements given above (3.2.3)).

Theorem 3.2.1 *Let $z_1, \dots, z_d$ be a basis in the space* $\ker\mathcal{A}_\delta(\mu)$*, $\{f, g\} \in \mathcal{H}_\delta(G)$ and $(f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G} = 0$, $j = 1, \dots, d$. Then:*

1. *There exists a solution $u \in \mathfrak{N} \dot{+} H_\delta^2(G)$ of the equation $\mathbf{A}(\mu)u = \{f, g\}$ determined up to an arbitrary term in* $\ker\mathcal{A}_\delta(\mu)$.

2. *The inclusion*

$$v \equiv u - c_1 u_1^- - \dots - c_M u_M^- \in H_\delta^2(G) \tag{3.2.7}$$

holds with $c_j = i(f, Y_j^-)_G + i(g, -\partial_\nu Y_j^-)_{\partial G}$.

3. *The inequality*

$$\|v; H_\delta^2(G)\| + |c_1| + \dots + |c_M| \le \text{const}\left(\|\{f, g\}; \mathcal{H}_\delta(G)\| + \|\rho_\delta v; L_2(G)\|\right). \tag{3.2.8}$$

holds with v and $c_1, \dots, c_M$ in (3.2.7). A solution u_0 that is subject to the additional conditions $(u_0, z_j)_G = 0$ for $j = 1, \dots, d$ is unique and satisfies (3.2.8) with right-hand side changed for $\text{const}\|\{f, g\}; \mathcal{H}_\delta(G)\|$.

4. *If $\{f, g\} \in \mathcal{H}_\delta(G) \cap \mathcal{H}_{\delta'}(G)$ and the strip $\{\lambda \in \mathbb{C} : \min\{\delta, \delta'\} \le \text{Im}\lambda \le \max\{\delta, \delta'\}\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, then the solutions $u \in \mathfrak{N} \dot{+} H_\delta^2(G)$ and $u' \in \mathfrak{N} \dot{+} H_{\delta'}^2(G)$ coincide, while the choice between δ and δ', in essence, affects only the constant in (3.2.8).*

Remark 3.2.2 In Theorem 3.2.1, one can take the numbers δ and "const" in (3.2.8) invariant for all μ in $[\mu', \mu''] \subset (\tau, \tau'')$ (in $[\mu', \mu''] \subset (\tau', \tau)$). If μ'' approaches τ'' (τ), the δ must tend to zero: an admissible interval for δ has to be narrowed because the imaginary eigenvalues of the pencils move closer to the real axis; the constant in (3.2.8) increases. From the proof of Theorem 3.2.1 in [37] one can see that the constant also increases when μ' approaches τ (or τ').

We now turn to the expanded radiation principle in a neighgborhood of τ. To this end, for problem (3.1.16), we construct a basis of waves stable at the threshold τ. We make up such a basis from the waves generated by functions (3.1.9) and (3.1.10), and from the waves corresponding to the real eigenvalues of the pencils

$\mathfrak{A}^r(\cdot, \mu)$, $r = 2, \dots, \mathcal{T}$. According to our assumption at the beginning of 3.2, the interval $[\tau', \tau'']$ contains no threshold for problems of the form (3.1.1) in the cylinders $\Pi^2, \dots, \Pi^{\mathcal{T}}$. Therefore, the number of real eigenvalues for each one of the pencils $\mathbb{R} \ni \lambda \to \mathfrak{A}^r(\lambda, \mu)$, $r = 2, \dots, \mathcal{T}$, remains invariant for $\mu \in [\tau', \tau'']$. Thus, when passing from the cylinder Π^1 to the domain G, the dimension of wave space increases by the same number for all $\mu \in (\tau', \tau'')$. We set $2L = \dim \mathcal{W}(\mu, G)$ for $\mu \in (\tau', \tau)$ and $2M = \dim \mathcal{W}(\mu, G)$ for $\mu \in (\tau, \tau'')$; then $M - L = m - l$, where m and l are the same as in (3.1.9) and (3.1.10), while $\dim \mathcal{W}_a(\mu, G) = 2M$ for $\mu \in (\tau', \tau)$.

We choose the number γ for the operators $\mathcal{A}_{\pm\gamma}(\mu)$ to be proper for all μ in a neighborhood of the threshold $\tau = \mu_m$. Let us explain such a choice. We have $\lambda_k^{\pm}(\mu) = \pm(\mu - \mu_k)^{1/2}$, $\mu_{l+1} = \dots = \mu_m$, so $\lambda_k^{\pm}(\tau) = 0$ with $k = l+1, \dots, m$. The interval of the imaginary axis with ends $-i(\mu_{m+1} - \mu_m)^{1/2}$, $i(\mu_{m+1} - \mu_m)^{1/2}$ punctured at the coordinate origin is free of the spectra of the pencils $\mathfrak{A}^q(\cdot, \mu_m)$, $q = 1, \dots, \mathcal{T}$. If μ moves a little along $\mathbb{R}$, the eigenvalues of the pencils $\mathfrak{A}^q(\cdot, \mu)$ slightly shift along the coordinate axes. Therefore, for a small $\alpha > 0$, there exists $\beta > 0$ such that, for $\mu \in (\mu_m - \beta, \mu_m + \beta)$, the intervals $iI_{\pm\alpha} := \pm i(\alpha, (\mu_{m+1} - \mu_m)^{1/2} - \alpha)$ are free of the spectra of the pencils $\mathfrak{A}^q(\cdot, \mu)$. So the lines $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda = \pm\gamma\}$ with $\gamma \in I_\alpha$ do not intersect the spectra of $\mathfrak{A}^q(\cdot, \mu)$, while the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \gamma\}$ contains only the real eigenvalues of the pencils and the numbers $\lambda_k^{\pm}(\mu) = \pm(\mu - \mu_k)^{1/2} = \pm(\mu - \mu_m)^{1/2}$ in (3.1.9), $k = l+1, \dots, m$.

Let $\mu \in (\tau - \beta, \tau + \beta)$, $\gamma \in I_\alpha$, and let $\{w_k^{\pm}(\cdot, \mu)\}$ be the stable basis of waves in G described in 3.1.1 and 3.1.2. In the kernel $\ker \mathcal{A}_{-\gamma}(\mu)$ of $\mathcal{A}_{-\gamma}(\mu)$, there exist collections of elements $\{\mathcal{Y}_1^+(\cdot, \mu), \dots, \mathcal{Y}_M^+(\cdot, \mu)\}$ and $\{\mathcal{Y}_1^-(\cdot, \mu), \dots, \mathcal{Y}_M^-(\cdot, \mu)\}$ such that

$$\left(\mathcal{Y}_j^+(\cdot, \mu) - w_j^+(\cdot, \mu) - \sum_{k=1}^{M} \mathcal{S}_{jk}(\mu) w_k^-(\cdot, \mu)\right) \in H_\gamma^2(G), \tag{3.2.9}$$

$$\left(\mathcal{Y}_j^-(\cdot, \mu) - w_j^-(\cdot, \mu) - \sum_{k=1}^{M} \mathcal{T}_{jk}(\mu) w_k^+(\cdot, \mu)\right) \in H_\gamma^2(G), \tag{3.2.10}$$

where $\mathcal{S}(\mu) = \|\mathcal{S}_{jk}(\mu)\|$ is the unitary and $\mathcal{S}(\mu)^{-1} = \mathcal{T}(\mu) = \|\mathcal{T}_{jk}(\mu)\|$. Every collection $\{\mathcal{Y}_1^+(\cdot, \mu), \dots, \mathcal{Y}_M^+(\cdot, \mu)\}$ and $\{\mathcal{Y}_1^-(\cdot, \mu), \dots, \mathcal{Y}_M^-(\cdot, \mu)\}$ is a basis (modulo $\ker \mathcal{A}_\gamma(\mu)$) in $\ker \mathcal{A}_{-\gamma}(\mu)$.

The elements $\mathcal{Y}(\cdot, \mu)$ in $\ker \mathcal{A}_{-\gamma}(\mu) \setminus \ker \mathcal{A}_\gamma(\mu)$ are called the continuous spectrum eigenfunctions of problem (3.1.16) corresponding to the number μ. The matrix $\mathcal{S}(\mu)$ (with $\mu \in (\tau - \beta, \tau)$) is referred to as the augmented scattering matrix.

Let $\mathfrak{K}$ denote the linear hull $\mathfrak{L}(w_1^-, \dots, w_M^-)$. We define a norm of $w = \sum c_j w_j^- + v \in \mathfrak{K} \dotplus H_\gamma^2(G)$, where $c_j \in \mathbb{C}$ and $v \in H_\gamma^2(G)$, by the equality

$$\|w\| = \sum |c_j| + \|v; H_\gamma^2(G)\|.$$

Let $\mathbf{A}(\mu)$ be the restriction of $\mathcal{A}_{-\gamma}(\mu)$ to the space $\mathfrak{R} \dotplus H^2_\gamma(G)$; then the mapping

$$\mathbf{A}(\mu) : \mathfrak{R} \dotplus H^2_\gamma(G) \to H^0_\gamma(G) \times H^{3/2}_\gamma(\partial G) =: \mathcal{H}_\gamma(G). \tag{3.2.11}$$

is continuous.

Theorem 3.2.3 *Let $\mu \in (\tau - \beta, \tau + \beta)$, $\gamma \in I_\alpha$, and let $\{w^\pm_k(\cdot, \mu)\}$ be the aforementioned basis of waves in G. Assume $z_1, \dots, z_d$ to be a basis in the space* $\ker \mathcal{A}_\gamma(\mu)$*, $\{f, g\} \in \mathcal{H}_\gamma(G)$ and $(f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G} = 0$, $j = 1, \dots, d$. Then:*

(1) *There exists a solution $w \in \mathfrak{R} \dotplus H^2_\gamma(G)$ to the equation $\mathbf{A}(\mu)w = \{f, g\}$ determined up to an arbitrary term in the lineal $\mathcal{L}(z_1, \dots, z_d)$.*

(2) *The inclusion*

$$v \equiv w - c_1 w^-_1 - \dots - c_M w^-_M \in H^2_\gamma(G) \tag{3.2.12}$$

holds with $c_j = i(f, \mathcal{Y}^-_j)_G + i(g, -\partial_\nu \mathcal{Y}^-_j)_{\partial G}$.

(3) *Such a solution w satisfies the inequality*

$$\|v; H^2_\gamma(G)\| + |c_1| + \dots + |c_M| \le \text{const}\left(\|\{f, g\}; \mathcal{H}_\gamma(G)\| + \|\rho_\gamma v; L_2(G)\|\right). \tag{3.2.13}$$

A solution w_0 that is subject to the conditions $(w_0, z_j)_G = 0$ for $j = 1, \dots, d$ is unique, and estimate (3.2.13) holds with the right-hand side changed for $\text{const}\|\{f, g\}; \mathcal{H}_\gamma(G)\|$.

(4) *If $\{f, g\} \in \mathcal{H}_\gamma(G) \cap \mathcal{H}_{\gamma'}(G)$ and the strip $\{\lambda \in \mathbb{C} : \min\{\gamma, \gamma'\} \le \operatorname{Im}\lambda \le \max\{\gamma, \gamma'\}\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, the solutions $w(\cdot, \mu) \in \mathfrak{R} \dotplus H^2_\gamma(G)$ and $w'(\cdot, \mu) \in \mathfrak{R} \dotplus H^2_{\gamma'}(G)$ of the equation $\mathbf{A}(\mu)w = \{f, g\}$ coincide, while the choice between γ and γ', in essence, affects only the constant in* (3.2.13).

We would like to extend relations of the form (3.2.9) and (3.2.10) to the interval (τ', τ'') with analytic functions $\mu \mapsto \mathcal{Y}^\pm_j(\mu)$. Unlike the situation in Remark 3.2.2, it is not possible, generally speaking, to extend (3.2.9) and (3.2.10) to any interval $[\mu', \mu''] \subset (\tau', \tau'')$ with the same index γ. However, to that purpose, one can use a finite collection of indices for various parts of $[\mu', \mu'']$. The following lemma explains how to compile such a collection.

Lemma 3.2.4 *For any interval $[\mu', \mu''] \subset (\tau', \tau'')$, there exists a finite covering $\{U_p\}^N_{p=0}$ consisting of open intervals and a collection of indices $\{\gamma^p\}^N_{p=0}$ subject to the following conditions (with a certain nonnegative number N):*

(1) *$\mu' \in U_0$, $\mu'' \in U_N$; $U_0 \cap U_p = \emptyset$, $p = 2, \dots, N$; $U_N \cap U_p = \emptyset$, $p = 0, \dots, N-2$; moreover, U_p overlaps only U_{p-1} and U_{p+1}, $1 \le p \le N-1$.*

(2) *The line $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \gamma^p\}$ is free of the spectra of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, for all $\mu \in U_p \cap [\mu', \mu'']$ and $p = 0, \dots, N$.*

(3) *The strip $\{\lambda \in \mathbb{C} : \gamma^p \le \operatorname{Im}\lambda \le \gamma^{p+1}\}$ is free of the spectra of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$, for all $\mu \in U_p \cap U_{p+1}$ and $p = 0, \dots, N-1$.*

(4) *The inequality* $|\mathrm{Im}(\mu-\tau)^{1/2}| < \gamma^p$ *holds for* $\mu \in U_p \cap [\mu', \mu'']$ *(recall that* $\pm(\mu-\tau)^{1/2}$ *are eigenvalues of* $\mathfrak{A}^1(\cdot,\mu)$ *and* $\tau = \mu_{l+1} = \cdots = \mu_m$*); there are no other eigenvalues of the pencils* $\mathfrak{A}^r(\cdot,\mu)$, $r = 1, \dots, \mathcal{T}$, *in the strip* $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| \le \gamma^p\}$, *except the real ones,* $p = 0, \dots, N$.

Proof Let us outline the proof. We consider an interval $[\mu', \mu'']$ and assume that $\tau \in (\mu', \mu'')$. Just before formulas (3.2.9) and (3.2.10), we have defined the interval $(\tau-\beta, \tau+\beta)$ that can be taken as an element of the desired covering. It was earlier shown that as an index γ for such an element one can choose any number in $I_\alpha = (\alpha, (\mu_{m+1}-\mu_m)^{1/2}-\alpha)$ with small positive α; the number β depends on α.

Let us take some $\nu \in (\tau, \tau+\beta)$. The eigenvalue $\lambda_m(\mu) = (\mu-\mu_m)^{1/2}$ of the pencil $\mathfrak{A}^1(\cdot,\mu)$ is real for $\mu > \nu$, the eigenvalue $\lambda_{m+1}(\mu) = i(\mu_{m+1}-\mu)^{1/2}$ of the pencil tends to zero when μ increases from ν to $\tau'' = \mu_{m+1}$, and the interval $\{z \in \mathbb{C} : z = it,\ 0 < t < (\mu_{m+1}-\mu'')^{1/2}\}$ of the imaginary axis remains free of the spectra of the pencils $\mathfrak{A}^r(\cdot,\mu)$, $\mu' \le \mu \le \mu''$, $r = 1, \dots, \mathcal{T}$. Therefore, the interval $(\nu, \tilde\nu)$ with $\mu'' < \tilde\nu < \tau''$ can serve as an element of the covering, and any number $\gamma \in (0, (\mu_{m+1}-\mu'')^{1/2})$ can be an index for the element. Finally, we choose the elements U_p to the left of the threshold τ so that the graphs of the functions $U_p \ni \mu \mapsto \gamma^p = \mathrm{const}$ are located between the graphs of the functions $(\tau', \tau) \ni \mu \mapsto \mathrm{Im}\lambda_k(\mu) = (\mu_k-\mu)^{1/2}$, $k = m, m+1$, and the indices form a decreasing sequence $\gamma^0 > \gamma^1 > \cdots$. $\square$

3.2.2 *Analyticity of Scattering Matrices with Respect to Spectral Parameter*

Let us consider the bases $\{\mathcal{Y}_j^+\}$ and $\{\mathcal{Y}_j^-\}$ in the spaces of continuous spectrum eigenfunctions (CSE) defined near the threshold τ (see (3.2.9) and (3.2.10)). We first show that the functions $\mu \mapsto \mathcal{Y}_j^\pm(\cdot,\mu)$ admit analytic extension to the interval (τ', τ''). In what follows, by the analyticity of a function on an interval we mean the possibility of analytic continuation of the function in a complex neighborhood of every point in the interval. Then we prove the analyticity of the scattering matrix $\mu \mapsto \mathcal{S}(\mu)$ on (τ', τ''). The analyticity does not exclude the existence of eigenvalues of problem (3.1.16) embedded into the continuous spectrum; however, the analyticity eliminates the arbitrariness in the choice of CSE. Moreover, we establish the analyticity of the elements $\mu \mapsto Y_j^\pm(\cdot,\mu)$ in (3.2.3) and (3.2.4) as well as the analyticity of the corresponding scattering matrix $\mu \mapsto S(\mu)$ on (τ', τ) and (τ, τ'').

In a neighborhood of any point of the interval (τ', τ''), one can define an operator $\mathcal{A}_\gamma(\mu)$, which is needed for relations like (3.2.9) and (3.2.10). The index γ has been provided by Lemma 3.2.4: the same number γ^p can serve for all $\mu \in U_p$. Therefore, for $\mu \in U_p$, there exist the families $\{\mathcal{Y}_j^\pm(\cdot,\mu)\} \subset \ker\mathcal{A}_{-\gamma^p}(\mu)$ satisfying relations like (3.2.9) and (3.2.10) with unitary matrix $\mathcal{S}(\mu)$, so Theorem 3.2.3 holds with $\mu \in U_p$. Thus, it suffices to prove the analyticity of the "local families" $\{\mathcal{Y}_j^\pm(\cdot,\mu)\}$ and that of the matrix $\mathcal{S}(\mu)$ on U_p and to verify the compatibility of such families on the intersections of neighborhoods.

We first obtain a representation of the operator $\mathbf{A}(\mu)^{-\mathbf{1}}$, where $\mathbf{A}(\mu)$ is operator (3.2.6) or (3.2.11), in a neighborhood of an eigenvalue of problem (3.1.16). To this end we recall some facts in the theory of holomorphic operator-valued functions (e.g., see [22, 23, 29]). Let $\mathcal{D}$ be a domain in a complex plane, B_1 and B_2 Banach spaces, and $\mathbb{A}$ a holomorphic operator-valued function $\mathcal{D} \ni \mu \mapsto \mathbb{A}(\mu) : B_1 \to B_2$. The spectrum of the function $\mathbb{A}(\cdot)$ is the set of points $\mu \in \mathcal{D}$ such that $\mathbb{A}(\mu)$ is a noninvertible operator. A number μ_0 is called an eigenvalue of $\mathbb{A}$ if there exists a nonzero vector $\varphi_0 \in B_1$ such that $\mathbb{A}(\mu_0)\varphi_0 = 0$; then φ_0 is called an eigenvector. Let μ_0 and φ_0 be an eigenvalue and an eigenvector. Elements $\varphi_1, \dots, \varphi_{m-1}$ are called generalized eigenvectors, if

$$\sum_{q=0}^{n} \frac{1}{q!}(\partial_\mu^q \mathbb{A})(\mu_0)\varphi_{n-q} = 0,$$

where $n = 1, \dots, m$. A holomorphic function $\mathbb{A}$ is said to be Fredholm if the operator $\mathbb{A}(\mu) : B_1 \to B_2$ is Fredholm for all $\mu \in \mathcal{D}$ and is invertible at least for one μ. The spectrum of a Fredholm function $\mathbb{A}$ consists of isolated eigenvalues of finite algebraic multiplicity. The holomorphic function $\mathbb{A}^*$ adjoint to $\mathbb{A}$ is defined on the set $\{\mu : \bar{\mu} \in \mathcal{D}\}$ by the equality $\mathbb{A}^*(\mu) = (\mathbb{A}(\bar{\mu}))^* : B_1^* \to B_2^*$. If one of the functions $\mathbb{A}$ and $\mathbb{A}^*$ is Fredholm, then the other one is also Fredholm. A number μ_0 is an eigenvalue of $\mathbb{A}$ if and only if $\bar{\mu}_0$ is an eigenvalue of $\mathbb{A}^*$; the algebraic and geometric multiplicities of $\bar{\mu}_0$ coincide with those of μ_0.

Let us consider the operator-valued function $\mu \mapsto \mathbf{A}(\mu)$ in (3.2.6) or (3.2.11) on an interval $[\mu', \mu'']$ that belongs to one of the intervals (τ', τ) or (τ, τ''). Taking account of Remark 3.2.2, we choose the same index δ in (3.2.6) and in Theorem 3.2.1 for all $\mu \in [\mu', \mu'']$. When considering the function $\mu \mapsto \mathbf{A}(\mu)$ in (3.2.11) on an interval $[\mu', \mu''] \subset (\tau', \tau'')$, we suppose the interval to be so small that Lemma 3.2.4 enables us to take the same γ in (3.2.11) and in Theorem 3.2.3 for all $\mu \in [\mu', \mu'']$. According to Proposition 3.1.1, the waves in the definitions of operators (3.2.6) and (3.2.11) are holomorphic in a complex neighborhood of the corresponding interval $[\mu', \mu'']$. Therefore, the functions $\mu \mapsto \mathbf{A}(\mu)$ in Theorems 3.2.1 and 3.2.3 are holomorphic in the same neighborhood.

Proposition 3.2.5 (i) *Let $\mu \mapsto \mathbf{A}(\mu)$ be the operator-valued function in Theorem 3.2.3, μ_0 an eigenvalue of operator (3.2.2), and $(z_1, \dots, z_d)$ a basis of* $\ker \mathcal{A}_\gamma(\mu_0)$. *Then, in a punctured neighborhood of μ_0, there holds the representation*

$$\mathbf{A}^{-1}(\mu)\{f, g\} = (\mu - \mu_0)^{-1}\mathbf{P}\{f, g\} + \mathbf{R}(\mu)\{f, g\}, \tag{3.2.14}$$

where $\{f, g\} \in \mathcal{H}_\gamma(G)$,

$$\mathbf{P}\{f, g\} = -\sum_{j=1}^{d} \big((f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G}\big)\, z_j, \tag{3.2.15}$$

and the function $\mathbf{R}(\mu) : \mathcal{H}_\gamma(G) \to \mathfrak{K} \dotplus H^2_\gamma(G)$ *is holomorphic in a neighborhood of* μ_0.

(ii) *Let* $\mu \mapsto \mathbf{A}(\mu)$ *be the operator-valued function in Theorem* 3.2.1, μ_0 *an eigenvalue of operator* (3.2.2) *in* (τ', τ) *or* (τ, τ''), *and* $(z_1, \dots, z_d)$ *a basis of* $\ker \mathcal{A}_\delta(\mu_0)$. *Then, in a punctured neighborhood of* μ_0, *there holds representation* (3.2.14), *where* $\mathbf{P}\{f, g\}$ *is defined by* (3.2.15) *and the function* $\mathbf{R}(\mu) : \mathcal{H}_\delta(G) \to \mathfrak{N} \dotplus H^2_\delta(G)$ *is holomorphic in a neighborhood of* μ_0.

Proof (i) By Theorem 3.2.3, (1), the operator $\mathbf{A}(\mu)$ is Fredholm at any $\mu \in [\mu', \mu'']$. We may consider $\mathbf{A}(\mu)$ as Fredholm in a neighborhood U (the Fredholm property is stable with respect to perturbations that are small in the operator norm). Moreover, the operator $\mathbf{A}(\mu)$ is invertible for all $\mu \in [\mu', \mu'']$ except the eigenvalues of the operator (3.2), which are real and isolated. Hence the function $\mu \mapsto \mathbf{A}(\mu)$ is Fredholm in a neighborhood of μ_0 in the complex plane. From Theorem 3.2.3, (4), it follows that the eigenspaces of operators (3.2.11) and (3.2.2) coincide, that is, $\ker \mathbf{A}(\mu_0) = \ker \mathcal{A}_\gamma(\mu_0) \subset H^2_\gamma(G)$. It is easy to verify that the operator-valued function $\mathbf{A}$ has no generalized eigenvectors at μ_0. Then the Keldysh theorem on the resolvent of holomorphic operator-valued function (see [23, 29]) provides the equality

$$\mathbf{A}^{-1}(\mu)\{f, g\} = (\mu - \mu_0)^{-1}\mathbf{T}\{f, g\} + \mathbf{R}(\mu)\{f, g\}; \tag{3.2.16}$$

here $\mathbf{T}\{f, g\} = \sum_{j=1}^{d} \langle \{f, g\}, \{\psi_j, \chi_j\}\rangle z_j$, the duality $\langle \cdot, \cdot \rangle$ on the pair $\mathcal{H}_\gamma(G)$, $\mathcal{H}_\gamma(G)^*$ is defined by $\langle \{f, g\}, \{\psi, \chi\}\rangle = (f, \psi)_G + (g, \chi)_{\partial G}$, and $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_{\partial G}$ are the extensions of the inner products on $L_2(G)$ and $L_2(\partial G)$ to the pairs $H^0_\gamma(G)$, $H^0_\gamma(G)^*$ and $H^{3/2}_\gamma(\partial G)$, $H^{3/2}_\gamma(\partial G)^*$, respectively. The elements $\{\psi_j, \chi_j\} \in \ker \mathbf{A}(\mu_0)^* \subset W(G; \gamma)^*$ are subject to the orthogonality and normalization conditions

$$\langle (\partial_\mu \mathbf{A})(\mu_0) z_j, \{\psi_k, \chi_k\}\rangle = \delta_{jk}, \quad j, k = 1, \dots, d. \tag{3.2.17}$$

Furthermore, $(\partial_\mu \mathbf{A})(\mu_0) z_j = \{-z_j, 0\} \in W(G; \gamma)$. The elements $\{\psi_k, \chi_k\}$ can be interpreted in terms of the Green formula and, in view of (3.2.17), rewritten in the form $\{\psi_k, \chi_k\} = \{-z_k, \partial_\nu z_k\}$ (e.g., see [37]). Now, $\mathbf{T}\{f, g\}$ coincides with $\mathbf{P}\{f, g\}$ in (3.2.15), and (3.2.16) takes the form (3.2.14).

(ii) One can repeat, with evident modifications, the argument in (i). □

We are now ready to discuss the analyticity of bases in the space of the continuous spectrum eigenfunctions. For instance, we proceed to the basis $\{\mathcal{Y}^+_j\}$ in (3.2.9). From the definition of the wave w^+_j in G (see 3.1.2), it follows that the function $G \ni x \mapsto w^+_j(x, \mu)$ is supported by one of the cylindrical ends of G,

$$-\Delta w^+_j(x, \mu) - \mu w^+_j(x, \mu) = f_j(x, \mu), \quad x \in G,$$
$$w^+_j(x, \mu) = 0, \quad x \in \partial G,$$

and the support of the function $x \mapsto f_j(x, \mu)$ is compact. Let us consider the equation

$$\mathbf{A}(\mu)w(\cdot, \mu) = \{f_j(\cdot, \mu), 0\} \tag{3.2.18}$$

on an interval $[\mu', \mu''] \subset (\tau', \tau'')$. We first assume that the interval $[\mu', \mu'']$ is free of the eigenvalues of the operator-valued function $\mu \mapsto \mathbf{A}(\mu)$. In view of Theorem 3.2.3, for all $\mu \in [\mu', \mu'']$, there exists a unique solution $w = v + c_1 w_1^- + \dots + c_M w_M^-$ to Eq. (3.2.18),

$$w(\cdot, \mu) = \{c_1(\mu), \dots, c_M(\mu), v(\cdot, \mu)\} \in \mathfrak{K} \dotplus H^2_\gamma(G). \tag{3.2.19}$$

Since the functions $\mu \mapsto \mathbf{A}(\mu)^{-1}$ and $\mu \mapsto f_j(\cdot, \mu)$ are holomorphic in a complex neighborhood of the interval $[\mu', \mu'']$, the components of the vector-valued function $\mu \mapsto w(\cdot, \mu)$ are holomorphic as well. Therefore, the analyticity of the function $\mu \mapsto \mathcal{Y}_j^+(\cdot, \mu)$ in the same neighborhood follows from the equality

$$\mathcal{Y}_j^+ = w_j^+ - w. \tag{3.2.20}$$

Assume now that the interval $[\mu', \mu'']$ contains an eigenvalue μ_0 of the operator-valued function $\mu \mapsto \mathbf{A}(\mu)$. We find the residue $\mathbf{P}\{f, g\}$ in (3.2.14) for $\{f, g\} = \{f_j, 0\}$ in the right-hand side of (3.2.18). For $z \in \ker \mathcal{A}_\gamma(\mu_0)$, we have

$$(f, z)_G + (g, -\partial_\nu z)_{\partial G} = (f_j, z)_G = (-\Delta w_j^+ - \mu w_j^+, z)_G = (w_j^+, -\Delta z - \mu z)_G = 0.$$

Hence $\mathbf{P}\{f_j, 0\} = 0$ and, by virtue of (3.2.14),

$$w(\cdot, \mu) = \mathbf{A}(\mu)^{-1}\{f_j, 0\} = \mathbf{R}(\mu)\{f_j, 0\},$$

which means that the function $\mu \mapsto w(\cdot, \mu)$ is analytic in a neighborhood of μ_0. This implies the analyticity of the function $\mu \mapsto \mathcal{Y}_j^+(\cdot, \mu)$.

The analyticity of the functions $\mu \mapsto \mathcal{Y}_j^-(\cdot, \mu)$ can be proved in the same way. When verifying the analyticity of functions of the form $\mu \mapsto Y_j^+(\cdot, \mu)$ and $\mu \mapsto Y_j^-(\cdot, \mu)$ in (3.2.3) and (3.2.4) in a complex neighborhood of the interval $[\mu', \mu''] \subset (\tau', \tau)$ or $[\mu', \mu''] \subset (\tau, \tau'')$, one has to make only an evident modification of the above argument.

Lemma 3.2.4 and Theorem 3.2.3, (4) enable us to extend formulas (3.2.9) and (3.2.10) to the whole interval (τ', τ'') for the analytic families $\mu \mapsto \mathcal{Y}_j^\pm(\cdot, \mu)$; however, one index γ has to be replaced by a collection of indices. Nonetheless, in a neighborhood of any given point $\mu \in (\tau', \tau'')$, one can do with one index γ. Remark 3.2.2 and Theorem 3.2.1, (4) allow to extend (3.2.3) and (3.2.4) to the intervals (τ', τ) and (τ, τ'') for the analytic families $\mu \mapsto Y_j^\pm(\cdot, \mu)$.

Theorem 3.2.6 *Let τ' and τ'' be thresholds of problem* (3.1.16) *such that $\tau' < \tau''$ and the interval (τ', τ'') contains the only threshold τ. We also suppose that the three thresholds relate to the same cylindrical end. Then:*

(i) *On the intervals (τ', τ) and (τ, τ''), there exist analytic bases $\{\mu \mapsto Y_j^{\pm}(\cdot, \mu)\}$ in the spaces of continuous spectrum eigenfunctions of problem* (3.1.16) *satisfying* (3.2.3) *and* (3.2.4) *with the scattering matrix $\mu \mapsto S(\mu)$ analytic on the mentioned intervals.*

(ii) *On the interval (τ', τ''), there exist analytic bases $\{\mu \mapsto \mathcal{Y}_j^{\pm}(\cdot, \mu)\}$ in the spaces of continuous spectrum eigenfunctions of problem* (3.1.16) *satisfying* (3.2.9) *and* (3.2.10) *with the scattering matrix $\mu \mapsto \mathcal{S}(\mu)$ analytic on (τ', τ'').*

Proof In view of the argument in 3.2.2, it suffices to verify the analyticity of the scattering matrices. For example, let us consider the matrix $\mu \mapsto \mathcal{S}(\mu)$. Equality (3.2.20), the representation $w = v + c_1 w_1^- + \cdots + c_M w_M^-$, and inclusion (3.2.19) lead to

$$\mathcal{Y}_j^+(\cdot, \mu) = w_j^+(\cdot, \mu) - \sum_{k=1}^{M} c_k(\mu) w_k^-(\cdot, \mu) \in H_\gamma^2(G).$$

Therefore, $\mathcal{S}_{jk}(\mu) = -c_k(\mu)$, $k = 1, \ldots, M$. It remains to take into account that the functions $\mu \mapsto c_k(\mu)$ are analytic on (τ', τ''). $\square$

For the basis $\{\mathcal{Y}_j^+(\cdot, \mu)\}_{j=1}^M$ (see Theorem 3.2.6, (ii)), we introduce the columns $\mathcal{Y}_{(\mathbf{1})}^+ = (\mathcal{Y}_1^+, \ldots, \mathcal{Y}_L^+)^t$ and $\mathcal{Y}_{(\mathbf{2})}^+ = (\mathcal{Y}_{L+1}^+, \ldots, \mathcal{Y}_M^+)^t$ and write down the scattering matrix in the form

$$\mathcal{S}(\mu) = \begin{pmatrix} \mathcal{S}_{(11)}(\mu) & \mathcal{S}_{(12)}(\mu) \\ \mathcal{S}_{(21)}(\mu) & \mathcal{S}_{(22)}(\mu) \end{pmatrix},$$

where $\mathcal{S}_{(11)}(\mu)$ is a block of size $L \times L$ and $\mathcal{S}_{(22)}(\mu)$ is a block of size $(M - L) \times (M - L)$, while $\mu \in (\tau', \tau'')$. We also set

$$D = ((\mu - \tau)^{1/2} + 1)/((\mu - \tau)^{1/2} - 1)$$

with $(\mu - \tau)^{1/2} = i(\tau - \mu)^{1/2}$ for $\mu \le \tau$ and $(\tau - \mu)^{1/2} \ge 0$. The next assertion will be of use in Sect. 3.3.

Lemma 3.2.7 *Assume that $\mu \in (\tau', \tau]$ and $\mathcal{S}(\mu)$ is the scattering matrix in Theorem* 3.2.6, (ii). *Then*

$$\ker(D + \mathcal{S}_{(22)}(\mu)) \subset \ker \mathcal{S}_{(12)}(\mu), \tag{3.2.21}$$

$$\operatorname{Im}(D + \mathcal{S}_{(22)}(\mu)) \supset \operatorname{Im} \mathcal{S}_{(21)}(\mu). \tag{3.2.22}$$

Therefore, the operator $\mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}$ is defined on $\operatorname{Im}(D + \mathcal{S}_{(22)}(\mu))$.

Proof Let us consider (3.2.21). We assume that $h \in \ker(D + \mathcal{S}_{(22)}(\mu))$ and $(0, h)^t \in \mathbb{C}^M$. Then

$$\begin{pmatrix} \mathcal{S}_{(11)}(\mu) & \mathcal{S}_{(12)}(\mu) \\ \mathcal{S}_{(21)}(\mu) & \mathcal{S}_{(22)}(\mu) \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} S_{(12)}(\mu)h \\ -Dh \end{pmatrix}.$$

Since the matrix $\mathcal{S}(\mu)$ is unitary and $|D| = 1$, we have $\|h\|^2 = \|\mathcal{S}_{(12)}(\mu)h\|^2 + \|h\|^2$, so $\mathcal{S}_{(12)}(\mu)h = 0$ and (3.2.21) is valid. Inclusion (3.2.22) is equivalent to

$$\ker(D + \mathcal{S}_{(22)}(\mu))^* \subset \ker \mathcal{S}_{(21)}(\mu)^*. \tag{3.2.23}$$

Moreover,

$$\mathcal{S}(\mu)^* = \begin{pmatrix} \mathcal{S}_{(11)}(\mu)^* & \mathcal{S}_{(21)}(\mu)^* \\ \mathcal{S}_{(12)}(\mu)^* & \mathcal{S}_{(22)}(\mu)^* \end{pmatrix}$$

and the matrix $\mathcal{S}(\mu)^*$ is unitary; therefore (3.2.23) may be proven by the same argument as (3.2.21). □

3.3 Other Properties of the Scattering Matrices

Here we clarify the connection between the matrices $\mathcal{S}(\mu)$ and $S(\mu)$ on the interval $\tau' < \mu < \tau$, prove the existence of the one-side finite limits of $S(\mu)$ as $\mu \to \tau \pm 0$, and describe the transformation of the scattering matrix under changes of basis in the space of waves $\mathcal{W}(\mu, G)$ for $\mu \in (\tau, \tau'')$.

3.3.1 The Connection Between $\mathcal{S}(\mu)$ and $S(\mu)$ for $\tau' < \mu < \tau$

Let us recall the description of the stable basis chosen for the definition of $\mathcal{S}(\mu)$. In the semicylinder Π^1_+, we introduce the functions

$$\Pi^1_+ \ni (y, t) \mapsto e^{\pm}_k(y, t; \mu) := \chi(t) \exp(\pm it\sqrt{\mu - \mu_k})\varphi_k(y), \tag{3.3.1}$$

where $k = l + 1, \dots, m$ (the notation is the same as in (3.1.9); as before, $\mu_{l+1} = \cdots = \mu_m = \tau$). We extend the functions by zero to the whole domain G and set

$$w^{\pm}_{L+j}(\cdot\,; \mu) = 2^{-1/2}\left(\frac{e^+_{l+j}(\cdot\,; \mu) + e^-_{l+j}(\cdot\,; \mu)}{2} \mp \frac{e^+_{l+j}(\cdot\,; \mu) - e^-_{l+j}(\cdot\,; \mu)}{2\sqrt{\mu - \mu_{l+j}}}\right) \tag{3.3.2}$$

for $j = 1, \dots, m-l = M-L$ (the equality $m-l = M-L$ was explained just after Remark 3.2.2). All the rest of the waves with supports in Π^1_+ obtained from functions (3.1.10) and the waves of the same type with supports in $\Pi^2_+, \dots, \Pi^{\mathcal{T}}_+$ we number by one index $j = 1, \dots, L$ and denote by $w^\pm_1(\cdot\,; \mu), \dots, w^\pm_L(\cdot\,; \mu)$. The obtained collection $\{w^\pm_1, \dots, w^\pm_M\}$ is a basis of waves in G stable in a neighborhood of the threshold τ. Finally, we introduce the columns $\mathbf{w}^\pm_{(\mathbf{1})} = (w^\pm_1, \dots, w^\pm_L)^t$, $\mathbf{w}^\pm_{(\mathbf{2})} = (w^\pm_{L+1}, \dots, w^\pm_M)^t$, and $(\mathbf{w}^\pm_{(\mathbf{1})}, \mathbf{w}^\pm_{(\mathbf{2})}) = (w^\pm_1, \dots, w^\pm_M)^t$, where t stands for matrix transposing.

The components of the vector $\mathbf{w}^\pm_{(\mathbf{1})}$ are bounded, while the components of $\mathbf{w}^\pm_{(\mathbf{2})}$ exponentially grow at infinity in Π^1_+. Setting $\mathbf{e}^\pm_{(\mathbf{1})} = (e^\pm_1, \dots, e^\pm_L)^t$ and $\mathbf{e}^\pm_{(\mathbf{2})} = (e^\pm_{L+1}, \dots, e^\pm_M)^t$, we arrive at

$$\mathbf{w}^\pm_{(\mathbf{2})} = D^\mp \mathbf{e}^+_{(\mathbf{2})} + D^\pm \mathbf{e}^-_{(\mathbf{2})} \tag{3.3.3}$$

with

$$D^\pm = ((\mu-\tau)^{1/2} \pm 1)/2\sqrt{2}(\mu-\tau)^{1/2}.$$

The following assertion is, in essence, contained in [38] and in [37], Ch. 12.

Proposition 3.3.1 *Let $\mu \in (\tau', \tau)$ and let $S(\mu)$ and $\mathcal{S}(\mu)$ be the scattering matrices in Theorem 3.2.6. Then*

$$S(\mu) = \mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{S}_{(21)}(\mu), \tag{3.3.4}$$

with

$$D = D^+/D^- = ((\mu-\tau)^{1/2}+1)/((\mu-\tau)^{1/2}-1).$$

Proof We verify (3.3.4). Let us write (3.2.9) in the form

$$\begin{aligned} \mathcal{Y}^+_{(\mathbf{1})} - \mathbf{w}^+_{(\mathbf{1})} - \mathcal{S}_{(11)}\mathbf{w}^-_{(\mathbf{1})} - \mathcal{S}_{(12)}\mathbf{w}^-_{(\mathbf{2})} \in H^2_\gamma(G), \\ \mathcal{Y}^+_{(\mathbf{2})} - \mathbf{w}^+_{(\mathbf{2})} - \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} - \mathcal{S}_{(22)}\mathbf{w}^-_{(\mathbf{2})} \in H^2_\gamma(G). \end{aligned} \tag{3.3.5}$$

Recall that the $\gamma > 0$ has been chosen according to Lemma 3.2.4, so the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \gamma\}$ contains the eigenvalues $\pm(\mu-\tau)^{1/2}$ of the pencil $\mathfrak{A}^1(\cdot, \mu)$. We take $\delta > 0$ such that the strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}\lambda| < \delta\}$ contains only the real eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \dots, \mathcal{T}$; then $\delta < \gamma$ and $H^2_\gamma(G) \subset H^2_\delta(G)$. Instead of $\mathbf{w}^\pm_{(\mathbf{2})}$, we substitute into (3.3.5) their expressions in (3.3.3); for the aforementioned δ, the vector-valued function $\mathbf{e}^+_{(\mathbf{2})}$ belongs to $H^2_\delta(G)$. As a result we obtain

$$\mathcal{Y}^+_{(\mathbf{1})} = \mathbf{w}^+_{(\mathbf{1})} + \mathcal{S}_{(11)}\mathbf{w}^-_{(\mathbf{1})} + \mathcal{S}_{(12)}D^-\mathbf{e}^-_{(\mathbf{2})} + \mathfrak{R}_{(\mathbf{1})}, \tag{3.3.6}$$

$$\mathcal{Y}^+_{(\mathbf{2})} = \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} + (D + \mathcal{S}_{(22)})D^-\mathbf{e}^-_{(\mathbf{2})} + \mathfrak{R}_{(\mathbf{2})}, \tag{3.3.7}$$

where $\mathfrak{R}_{(\mathbf{1})}, \mathfrak{R}_{(\mathbf{2})} \in H^2_\delta(G)$. We now introduce the orthogonal projector

$$\mathcal{P} : \mathbb{C}^{M-L} \to \operatorname{Im}(D + \mathcal{S}_{(22)}(\mu)).$$

Taking account of (3.2.22) and (3.3.7), we arrive at

$$\mathcal{P}\mathcal{Y}^+_{(\mathbf{2})} = \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} + (D + \mathcal{S}_{(22)})D^-\mathbf{e}^-_{(\mathbf{2})} + \mathcal{P}\mathfrak{R}_{(\mathbf{2})}. \tag{3.3.8}$$

We apply the operator $\mathcal{S}_{(12)}(\mu)(D+\mathcal{S}_{(22)}(\mu))^{-1}$ to both sides of (3.3.8) and subtract the resulting equality from (3.3.6). We then have

$$Z = \mathbf{w}^+_{(\mathbf{1})} + (\mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{S}_{(21)}(\mu))\mathbf{w}^-_{(\mathbf{1})} + R, \tag{3.3.9}$$

where

$$Z = \mathcal{Y}^+_{(\mathbf{1})} - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{P}\mathcal{Y}^+_{(\mathbf{2})}, \tag{3.3.10}$$

$$R = \mathfrak{R}_{(\mathbf{1})} - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{P}\mathfrak{R}_{(\mathbf{2})}. \tag{3.3.11}$$

The components of the vectors $\mathcal{Y}^+_{(\mathbf{1})}$ and $\mathcal{Y}^+_{(\mathbf{2})}$ satisfy problem (3.1.16); in view of (3.3.10), the same is true for the components of the vector Z. Moreover, from $\mathfrak{R}_{(\mathbf{1})}, \mathfrak{R}_{(\mathbf{2})} \in H^2_\delta(G)$, and (3.3.11) it follows that $R \in H^2_\delta(G)$. Hence the formula (3.3.9) describes the scattering of the vector $\mathbf{w}^+_{(\mathbf{1})}$ of incoming waves in the basis $\mathbf{w}^+_{(1)}, \mathbf{w}^-_{(1)}$ as well as (3.2.3), so we obtain (3.3.4). □

3.3.2 The Connection Between $\mathcal{S}(\mu)$ and $S(\mu)$ for $\tau < \mu < \tau''$

We consider two bases in the wave space $\mathcal{W}(\mu, G)$ for $\tau < \mu < \tau''$. One of the bases consists of the waves in G corresponding to functions of the form $u^\pm_q(\cdot, \mu)$ in (3.1.4), while the other one comprises the waves generated by the functions $w^\pm_q(\cdot, \mu)$ (see (3.1.9), (3.1.10)). As before, the scattering matrices, defined in these bases, are denoted by $S(\mu)$ and $\mathcal{S}(\mu)$ (see Theorem 3.2.6); this time, that is, for $\mu \in (\tau, \tau'')$, the matrices are of the same size $M \times M$.

The scattering matrices are independent of the choice of the cut-off function χ in the definition of the space $\mathcal{W}(\mu, G)$. Identifying "equivalent" waves, one can omit such a cut-off function from consideration. To this end, we introduce the quotien space

$$\dot{\mathcal{W}}(\mu, G) := (\mathcal{W}(\mu, G)\dot{+}H^2_\gamma(G))/H^2_\gamma(G).$$

Let $\dot{v}$ stand for the class in $\dot{\mathcal{W}}(\mu, G)$ with representative $v \in \mathcal{W}(\mu, G)$. In what follows, waves of the form $\chi u^\pm_q(\cdot, \mu)$ and $\chi w^\pm_q(\cdot, \mu)$ in G are denoted by $u^\pm_q(\cdot, \mu)$

and $w_q^{\pm}(\cdot,\mu)$. The collections $\{\dot{u}_q^{\pm}(\cdot,\mu)\}_{j=1}^{M}$ and $\{\dot{w}_k^{\pm}(\cdot,\mu)\}_{k=1}^{M}$ are bases in the space $\dot{\mathcal{W}}(\mu,G)$, so $\dim\dot{\mathcal{W}}(\mu,G)=2M$. The form $q_G(u,v)$ in (3.1.17) is independent of the choice of representatives in $\dot{u}$ and $\dot{v}$; hence it is defined on $\dot{\mathcal{W}}(\mu,G)\times\dot{\mathcal{W}}(\mu,G)$. From (3.1.7) and (3.1.8) it follows that

$$iq_G(\dot{u}_k^{\pm}(\cdot\,;\mu),\dot{u}_l^{\mp}(\cdot\,;\mu))=0 \quad \text{for all } k,l=1,\dots,M, \tag{3.3.12}$$

$$iq_G(\dot{u}_k^{\pm}(\cdot\,;\mu),\dot{u}_l^{\pm}(\cdot\,;\mu))=\mp\delta_{kl}, \tag{3.3.13}$$

and equalities (3.1.11) and (3.1.12) lead to

$$iq_G(\dot{w}_r^{\pm}(\cdot\,;\mu),\dot{w}_s^{\mp}(\cdot\,;\mu))=0 \quad \text{for all } r,s=1,\dots,M, \tag{3.3.14}$$

$$iq_G(\dot{w}_r^{\pm}(\cdot\,;\mu),\dot{w}_s^{\pm}(\cdot\,;\mu))=\mp\delta_{rs}. \tag{3.3.15}$$

Thus $\dot{\mathcal{W}}(\mu,G)$ turns out to be a $2M$-dimensional complex space with indefinite inner product $<\dot{u},\dot{v}>:=-iq_G(\dot{u},\dot{v})$. The projection

$$\pi:\mathcal{W}(\mu,G)\dot{+}H_\gamma^2(G)\to\dot{\mathcal{W}}(\mu,G) \tag{3.3.16}$$

maps the space of continuous spectrum eigenfunctions onto a subspace in $\dot{\mathcal{W}}(\mu,G)$ of dimension M; we denote the subspace by $\mathcal{E}(\mu)$.

Let $V_1,\dots,V_{2M}$ be a basis in $\dot{\mathcal{W}}(\mu,G)$ subject to the orthogonality and normalization conditions

$$<V_j,V_l>=\delta_{jl},\ <V_{j+M},V_{l+M}>=-\delta_{jl} \text{ for } j,l=1,\dots,M. \tag{3.3.17}$$

The elements $V_1,\dots,V_M$ are called incoming waves, and the elements $V_{M+1},\dots,V_{2M}$ are called outgoing waves. Assume that $X_1,\dots,X_M$ is a basis of $\mathcal{E}(\mu)$ that defines, in the basis of waves $V_1,\dots,V_{2M}$, the scattering matrix $\mathfrak{S}(\mu)$ of size $M\times M$ (compare with (3.2.3)). We represent the vectors X_j as coordinate rows and form the $M\times 2M$-matrix $X=(X_1,\dots,X_M)^t$ (which is a column of the letters $X_1,\dots,X_M$). Finally, let I denote the unit matrix of size $M\times M$. Then a relation of the form (3.2.3) leads to

$$X=(I,\mathfrak{S}(\mu))V, \tag{3.3.18}$$

where V is the $2M\times 2M$-matrix $(V_1,\dots,V_{2M})^t$ consisting of the coordinate rows of the vectors V_j and $(I,\mathfrak{S}(\mu))$ is a matrix of size $M\times 2M$.

Assume that $\tilde{V}_1,\dots,\tilde{V}_{2M}$ is another basis of waves subject to conditions of the form (3.3.17), $\tilde{X}_1,\dots,\tilde{X}_M$ is a basis of $\mathcal{E}(\mu)$, and $\tilde{\mathfrak{S}}(\mu)$ is the corresponding scattering matrix such that

$$\tilde{X}=(I,\tilde{\mathfrak{S}}(\mu))\tilde{V}. \tag{3.3.19}$$

We suppose that $\tilde{V}=\mathfrak{T}V$ and write down the $2M\times 2M$-matrix $\mathfrak{T}$ as $\mathfrak{T}=(\mathfrak{T}_{(k\,l)})_{k,l=1}^{2}$ with blocks $\mathfrak{T}_{(k\,l)}$ of size $M\times M$.

Lemma 3.3.2 *The matrices* $\mathfrak{T}_{(1\,1)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)}$ *and* $\mathfrak{T}_{(1\,2)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}$ *are invertible and*

$$\mathfrak{S}(\mu) = (\mathfrak{T}_{(1\,1)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)})^{-1}(\mathfrak{T}_{(1\,2)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}). \tag{3.3.20}$$

Proof For the bases $X_1, \dots, X_M$ and $\tilde{X}_1, \dots, \tilde{X}_M$ there exists a nonsingular $M \times M$-matrix B such that $\tilde{X} = BX$. Therefore, by virtue of (3.3.19), we have

$$BX = (I, \tilde{\mathfrak{S}}(\mu))\mathfrak{T}V.$$

Taking account of (3.3.18), we obtain $B(I, \mathfrak{S}(\mu))V = (I, \tilde{\mathfrak{S}}(\mu))\mathfrak{T}V$, so

$$B(I, \mathfrak{S}(\mu)) = (I, \tilde{\mathfrak{S}}(\mu))\mathfrak{T}.$$

Let us write this equality in the form

$$(B, B\mathfrak{S}(\mu)) = (\mathfrak{T}_{(1\,1)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)}, \mathfrak{T}_{(1\,2)} + \tilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}).$$

Now, the assertions of lemma are evident. □

We intend to make use of (3.3.20) taking as $\tilde{V}$ the image, under canonical projection (3.3.16), of the stable basis of $\mathcal{W}(\mu, G)$ in (3.2.9) and as V the image of the wave basis in (3.2.3). As $\tilde{\mathfrak{S}}(\mu)$ and $\mathfrak{S}(\mu)$, we choose $\mathcal{S}(\mu)$ and $S(\mu)$, respectively. Let us proceed to computing the matrix $\mathfrak{T}$ in the equality $\tilde{V} = \mathfrak{T}V$. In doing so, instead of $\tilde{V}$ and V we can consider their just-mentioned pre-images in $\mathcal{W}(\mu, G)$. We set

$$u_j := u_j^+, \quad u_{j+M} := u_j^-, \quad j = 1, \dots, M, \tag{3.3.21}$$

where $u_j^\pm$ are the waves in $\mathcal{W}(\mu, G)$ generated by functions of the form (3.1.4). We also introduce

$$\begin{gathered} w_j := w_j^+ = u_j^+, \quad w_{j+M} := w_j^- = u_j^-, \quad j = 1, \dots, L, \\ w_p := w_p^+, \quad w_{p+M} := w_p^-, \quad p = L+1, \dots, M, \end{gathered} \tag{3.3.22}$$

where $w_p^\pm$ are the waves in $\mathcal{W}(\mu, G)$ generated by functions (3.1.9). For the matrix $\mathfrak{T}$, the equality $w = \mathfrak{T}u$ holds with the columns $w = (w_1, \dots, w_{2M})^t$ and $u = (u_1, \dots, u_{2M})^t$. For convenience, we will here denote functions (3.1.9) in the same way as the waves $w_p^\pm$; let us write down these functions in the form

$$w_p^\pm(\mu) = 2^{-1/2}((e^{it\lambda} + e^{-it\lambda})/2) \mp (e^{it\lambda} - e^{-it\lambda})/2\lambda)\varphi_p,$$

where $\lambda = \sqrt{\mu - \tau}$ and τ is a threshold; we also write functions (3.1.4) in the form

$$u_p^\pm(\mu) = (2\lambda)^{-1/2} e^{\mp it\lambda}\varphi_p.$$

Then we have

$$w_p^{\pm} = (1/2)(u_p^{+}(\lambda^{1/2} \pm \lambda^{-1/2}) + u_p^{-}(\lambda^{1/2} \mp \lambda^{-1/2})), \quad p = L+1, \dots, M;$$

here by $w_p^{\pm}$ and $u_p^{\pm}$ one can mean the functions in the cylinder and the corresponding waves in the domain G alike. Together with (3.3.21) and (3.3.22), this leads to the following description of the blocks $\mathfrak{T}_{(ij)}$ of the matrix $\mathfrak{T}$.

Lemma 3.3.3 *Each of the matrices $\mathfrak{T}_{(ij)}$ consists of four blocks and is block-diagonal. The equalities*

$$\mathfrak{T}_{(11)}(\mu) = \mathfrak{T}_{(22)}(\mu) = \operatorname{diag}\{I_L,\ 2^{-1}(\lambda^{1/2} + \lambda^{-1/2})I_{M-L}\}, \tag{3.3.23}$$

$$\mathfrak{T}_{(21)}(\mu) = \mathfrak{T}_{(12)}(\mu) = \operatorname{diag}\{O_L,\ 2^{-1}(\lambda^{1/2} - \lambda^{-1/2})I_{M-L}\} \tag{3.3.24}$$

hold; I_K is the unit matrix of size $K \times K$, O_L is the zero matrix of size $L \times L$, and $\lambda = \sqrt{\mu - \tau}$ with $\mu \in (\tau, \tau'')$.

We return to (3.3.20) with $\mathcal{S}$ and S instead of $\tilde{\mathfrak{S}}$ and $\mathfrak{S}$. Let us divide the matrix $\mathcal{S}$ into four blocks with $\mathcal{S}_{(11)}$ of size $L \times L$ and $\mathcal{S}_{(22)}$ of size $(M-L) \times (M-L)$. We also set $d^{\pm} = 2^{-1}(\lambda^{1/2} \pm \lambda^{-1/2})$. Then

$$\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)} = \begin{pmatrix} I_L & \mathcal{S}_{(12)}d^- \\ O & \mathcal{S}_{(22)}d^- + I_{M-L}d^+ \end{pmatrix}. \tag{3.3.25}$$

According to Lemma 3.3.2, the matrix $\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)}$ is invertible, so the matrix $\mathcal{S}_{(22)}d^- + I_{M-L}d^+$ is invertible as well, therefore

$$(\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)})^{-1} = \begin{pmatrix} I_L & -\mathcal{S}_{(12)}d^-(\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1} \\ O & (\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1} \end{pmatrix}. \tag{3.3.26}$$

In view of (3.3.20), we now obtain

Proposition 3.3.4 *For $\mu \in (\tau, \tau'')$, the blocks $S_{(ij)}$ of the scattering matrix*

$$S(\mu) = (\mathfrak{T}_{(11)} + \mathcal{S}(\mu)\mathfrak{T}_{(21)})^{-1}(\mathfrak{T}_{(12)} + \mathcal{S}(\mu)\mathfrak{T}_{(22)})$$

admit the representations

$$S_{(11)} = \mathcal{S}_{(11)} - \mathcal{S}_{(12)}d^-(\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}\mathcal{S}_{(21)}, \tag{3.3.27}$$

$$S_{(12)} = \mathcal{S}_{(12)}d^+ - \mathcal{S}_{(12)}d^-(\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}(\mathcal{S}_{(22)}d^+ + I_{M-L}d^-), \tag{3.3.28}$$

$$S_{(21)} = (\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}\mathcal{S}_{(21)}, \tag{3.3.29}$$

$$S_{(22)} = (\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}(\mathcal{S}_{(22)}d^+ + I_{M-L}d^-). \tag{3.3.30}$$

3.3.3 The Limits of $S(\mu)$ as $\mu \to \tau \pm 0$

To calculate the one-sided limits of $S(\mu)$, we make use of (3.3.4) as $\mu \to \tau - 0$ and apply (3.3.27)–(3.3.30) as $\mu \to \tau + 0$. The computation procedure depends on whether the number 1 is an eigenvalue of the matrix $\mathcal{S}_{22}(\tau)$.

3.3.3.1 The Limits of $S(\mu)$ as $\mu \to \tau \pm 0$, Provided 1 Is Not an Eigenvalue of $\mathcal{S}_{(22)}(\tau)$

Recall that the functions $\mu \mapsto \mathcal{S}_{(kl)}(\mu)$ are analytic in a neighborhood of $\mu = \tau$. Therefore, from (3.3.4) it immediately follows that

$$\lim_{\mu\to\tau-0} S(\mu) = \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau). \tag{3.3.31}$$

Let us proceed to compute $\lim S(\mu)$ as $\mu \to \tau+0$. By virtue of (3.3.27) and (3.3.31),

$$\begin{aligned}\lim_{\mu\to\tau+0} S_{(11)}(\mu) &= \lim_{\mu\to\tau+0}(\mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(\mathcal{S}_{(22)}(\mu) + d^+(\mu)/d^-(\mu))^{-1}\mathcal{S}_{(21)}(\mu)) \qquad (3.3.32)\\ &= \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau) = \lim_{\mu\to\tau-0} S(\mu).\end{aligned}$$

According to (3.3.30),

$$\begin{aligned}\lim_{\mu\to\tau+0} S_{(22)}(\mu) &= \lim_{\mu\to\tau+0}(\mathcal{S}_{(22)} + d^+/d^-)^{-1}(\mathcal{S}_{(22)}d^+/d^- + 1) \qquad (3.3.33)\\ &= (\mathcal{S}_{(22)}(\tau) - 1)^{-1}(-\mathcal{S}_{(22)}(\tau) + 1) = -I_{M-L}.\end{aligned}$$

It follows from (3.3.29) that

$$S_{(21)}(\mu) = (\mathcal{S}_{(22)} + d^+/d^-)^{-1}\mathcal{S}_{(21)}/d^-.$$

Since $d^-(\mu) = 2^{-1}((\mu-\tau)^{1/2} - 1)/(\mu-\tau)^{1/4}$, we arrive at

$$S_{(21)}(\mu) = O((\mu-\tau)^{1/4}) \to 0 \quad \text{for} \quad \mu \to \tau + 0. \tag{3.3.34}$$

Finally, consider $S_{(12)}(\mu)$. We write (3.3.28) in the form

$$\begin{aligned}S_{(12)} &= \mathcal{S}_{(12)}d^+(1 - (\mathcal{S}_{(22)} + d^+/d^-)^{-1}(\mathcal{S}_{(22)} + d^-/d^+))\\ &= \mathcal{S}_{(12)}d^+(\mathcal{S}_{(22)} + d^+/d^-)^{-1}(d^+/d^- - d^-/d^+).\end{aligned}$$

In view of

$$d^+(\mu)(d^+/d^- - d^-/d^+) = 2(\mu - \tau)^{1/4}/((\mu - \tau)^{1/2} - 1),$$

we obtain

$$S_{(12)}(\mu) = O((\mu - \tau)^{1/4}) \to 0 \quad \text{for} \quad \mu \to \tau + 0. \tag{3.3.35}$$

3.3.3.2 The Limits of $S(\mu)$ as $\mu \to \tau \pm 0$, Provided 1 Is an Eigenvalue of $\mathcal{S}_{(22)}(\tau)$

We set $\lambda = \sqrt{\mu - \tau}$ with $\mu = \tau + \lambda^2$ and consider the function $\lambda \mapsto \Phi(\lambda) : \mathbb{C}^{M-L} \to \mathbb{C}^{M-L}$,

$$\Phi(\lambda) := \mathcal{S}_{(22)}(\mu) + d^+(\mu)/d^-(\mu) = \mathcal{S}_{(22)}(\tau + \lambda^2) + (\lambda + 1)/(\lambda - 1). \tag{3.3.36}$$

The number $\lambda = 0$ is an eigenvalue of the function $\lambda \mapsto \Phi(\lambda)$ if and only if 1 is an eigenvalue of the matrix $\mathcal{S}_{(22)}(\tau)$; in such a case $\ker(\mathcal{S}_{(22)}(\tau) - 1) = \ker \Phi(0)$. To calculate the limits of $S(\mu)$ as $\mu \to \tau \pm 0$, we need knowledge of the resolvent $\lambda \mapsto \Phi(\lambda)^{-1}$ in a neighborhood of $\lambda = 0$. Propositions 3.3.5 and 3.3.6 provide the required information.

Proposition 3.3.5 *There holds the equality*

$$\ker(\mathcal{S}_{(22)}(\tau) - 1) = \ker(\mathcal{S}_{(22)}(\tau)^* - 1). \tag{3.3.37}$$

Proof Assume that $h \in \ker(\mathcal{S}_{(22)}(\tau) - 1)$. Then, as was shown in the proof of Lemma 3.2.7, the vector $(0, h)^t \in \mathbb{C}^M$ belongs to $\ker(\mathcal{S}(\tau) - 1)$ and $\mathcal{S}_{(12)}(\tau)h = 0$. The same argument with $\mathcal{S}(\tau)^*$ instead of $\mathcal{S}(\tau)$ shows that the inclusion $g \in \ker(\mathcal{S}_{(22)}(\tau)^* - 1)$ implies $(0, g)^t \in \ker(\mathcal{S}(\tau)^* - 1)$ and $\mathcal{S}_{(21)}(\tau)^* g = 0$. Since $\mathcal{S}(\tau)^* = \mathcal{S}(\tau)^{-1}$, we have

$$\ker(\mathcal{S}(\tau) - 1) = \ker(\mathcal{S}(\tau)^* - 1). \tag{3.3.38}$$

Let $h_1, \dots, h_\varkappa$ be a basis of $\ker(\mathcal{S}_{(22)}(\tau) - 1)$ and $g_1, \dots, g_\varkappa$ a basis of $\ker(\mathcal{S}_{(22)}(\tau)^* - 1)$. We set $\tilde{h_j} = (0, h_j)^t$ and $\tilde{g_j} = (0, g_j)^t$. From (3.3.38) it follows that

$$\tilde{h_j}, \tilde{g_j} \in \ker(\mathcal{S}(\tau) - 1) = \ker(\mathcal{S}(\tau)^* - 1), \quad j = 1, \dots, \varkappa.$$

Therefore, any vector of the collection $h_1, \dots, h_\varkappa$ is a linear combination of the vectors $g_1, \dots, g_\varkappa$, and vice versa. □

Proposition 3.3.6 *Let* Φ *be the matrix function in* (3.3.36) *and* $\dim \ker \Phi(0) = \varkappa > 0$. *Then, in a punctured neighborhood of* $\lambda = 0$, *the resolvent* $\lambda \mapsto \Phi(\lambda)^{-1}$ *admits the representation*

$$\Phi(\lambda)^{-1} = -(2\lambda)^{-1} \sum_{j=1}^{\varkappa} \{\cdot, h_j\} h_j + \Gamma(\lambda); \tag{3.3.39}$$

here $h_1, \dots, h_\varkappa$ *is an orthonormal basis of* $\ker(\mathcal{S}_{(22)}(\tau) - 1)$, $\{u, v\}$ *is the inner product on the space* $\mathbb{C}^{M-L}$, *and* $\lambda \mapsto \Gamma(\lambda) : \mathbb{C}^{M-L} \to \mathbb{C}^{M-L}$ *is a matrix function holomorphic in a neighborhood of* $\lambda = 0$.

Proof It is known (e.g., see [22, 23]) that, under certain conditions, the resolvent $\mathfrak{A}(\lambda)^{-1}$ of a holomorphic operator function $\lambda \mapsto \mathfrak{A}(\lambda)$ in a punctured neighborhood of an isolated eigenvalue λ_0 admits the representation

$$\mathfrak{A}(\lambda)^{-1} = (\lambda - \lambda_0)^{-1} \sum_{j=1}^{\varkappa} (\cdot, \psi_j)\phi_j + \Gamma(\lambda), \tag{3.3.40}$$

where $\phi_1, \dots, \phi_\varkappa$ and $\psi_1, \dots, \psi_\varkappa$ are bases of the spaces $\ker\mathfrak{A}(\lambda_0)$ and $\ker\mathfrak{A}(\lambda_0)^*$ satisfying the orthogonality and normalization conditions

$$(\partial_\lambda \mathfrak{A}(\lambda_0)\phi_j, \psi_k) = \delta_{jk}, \quad j, k = 1, \dots, \varkappa, \tag{3.3.41}$$

and Γ is an operator function holomorphic in a neighborhood of λ_0. The formula (3.3.40) is related to the case where the operator function $\lambda \mapsto \mathfrak{A}(\lambda)$ has no generalized eigenvectors at the point λ_0. To justify (3.3.39), we have to show that there are no generalized eigenvectors of the function $\lambda \mapsto \Phi(\lambda)$ at the point $\lambda = 0$ and to verify agreement between (3.3.39) and (3.3.40).

We first take up the generalized eigenvectors. Assume that $0 \neq h^0 \in \ker \Phi(0)$. The equation $\Phi(0)h^1 + (\partial_\lambda \Phi)(0)h^0 = 0$ for a generalized eigenvector h^1 is of the form

$$(\mathcal{S}_{(22)}(\tau) - 1)h^1 = 2h^0.$$

The orthogonality of h^0 to the lineal $\ker(\mathcal{S}_{(22)}(\tau)^* - 1) = \ker(\mathcal{S}_{(22)}(\tau) - 1)$ is necessary for the solvability of this equation (see (3.3.37)). Since $0 \neq h^0 \in \ker \Phi(0) = \ker(\mathcal{S}_{(22)}(\tau) - 1)$, the solvability condition is not fulfilled, so the generalized eigenvectors do not exist.

Let us compare (3.3.39) and (3.3.40). We have $(\partial_\lambda \Phi)(0) = -2I_{M-L}$. Moreover, in view of (3.3.37), the bases $\phi_1, \dots, \phi_\varkappa$ and $\psi_1, \dots, \psi_\varkappa$ in (3.3.40) can be chosen to satisfy $\phi_j = -\psi_j = h_j/\sqrt{2}$ and, as $h_1, \dots, h_\varkappa$, there can be taken an orthonormal basis of $\ker(\mathcal{S}_{22}(\tau) - 1)$. Then

$$\{(\partial_\lambda \Phi)(0)\phi_j, \psi_k\} = \delta_{jk}, \quad j, k = 1, \dots, \varkappa,$$

and the representation (3.3.40) takes the form of (3.3.39). □

Let us calculate lim $S(\mu)$ as $\mu \to \tau - 0$. According to Lemma 3.2.7,

$$\operatorname{Im}(\mathcal{S}_{(22)}(\tau) - 1) \supset \operatorname{Im}\mathcal{S}_{(21)}(\tau).$$

Therefore, Proposition 3.3.5 leads to the equalities $\{\mathcal{S}_{(21)}(\tau) f, h_j\} = 0$ for any $f \in \mathbb{C}^L$ and $h_1, \dots, h_\varkappa$ in (3.3.39). Because the function $\mu \to \mathcal{S}_{(21)}(\mu)$ is analytic, we have $\mathcal{S}_{(21)}(\mu) = \mathcal{S}_{(21)}(\tau) + O(|\mu - \tau|)$; recall that $|\mu - \tau| = |\lambda|^2$. Applying (3.3.39), we obtain

$$(\mathcal{S}_{(22)}(\mu) + D(\mu))^{-1}\mathcal{S}_{(21)}(\mu) = \Gamma(\lambda)\mathcal{S}_{(21)}(\mu) + O(|\lambda|). \tag{3.3.42}$$

Now, from (3.3.4) it follows that

$$\lim_{\mu\to\tau-0} S(\mu) = \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)\Gamma(0)\mathcal{S}_{(21)}(\tau); \tag{3.3.43}$$

Lemma 3.2.7 allows to treat the right-hand side as the operator $\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)$ $(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau)$ (see (3.3.31)). For $\mu \to \tau - 0$, there holds the estimate

$$S(\mu) - (\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)\Gamma(0)\mathcal{S}_{(21)}(\tau)) = O(|\mu - \tau|^{(1/2)}). \tag{3.3.44}$$

Let us proceed to calculating the limits as $\mu \to \tau + 0$. We compute $\lim_{\mu\to\tau+0} S_{(11)}(\mu)$ in the same way as $\lim_{\mu\to\tau-0} S(\mu)$ and obtain

$$\lim_{\mu\to\tau+0} S_{(11)}(\mu) = \lim_{\mu\to\tau-0} S(\mu). \tag{3.3.45}$$

In view of (3.3.30),

$$\begin{aligned} S_{(22)}(\mu) &= \left(\mathcal{S}_{(22)}(\mu) + d^+/d^-\right)^{-1}\left(\mathcal{S}_{(22)}(\mu) + d^-/d^+\right) d^+/d^- \\ &= d^+/d^- + \left(\mathcal{S}_{(22)}(\mu) + d^+/d^-\right)^{-1}\left(d^-/d^+ - d^+/d^-\right) d^+/d^-. \end{aligned}$$

Applying resolvent representation (3.3.39), we write the last equality in the form

$$S_{(22)}(\mu) = \frac{\lambda+1}{\lambda-1}\left(I + \frac{2}{\lambda^2-1}\sum_{j=1}^{\varkappa}(\cdot, h_j)h_j - \frac{4\lambda}{\lambda^2-1}\Gamma(\lambda)\right). \tag{3.3.46}$$

Hence

$$\lim_{\mu\to\tau+0} S_{(22)}(\mu) = 2\sum_{j=1}^{\varkappa}(\cdot, h_j)h_j - I = P - Q, \tag{3.3.47}$$

where $P = \sum_{j=1}^{\varkappa} (\cdot, h_j)h_j$ is the orthogonal projector $\mathbb{C}^{M-L}$ onto ker $(\mathcal{S}_{(22)}(\tau) - 1)$ and $Q = I - P$. Moreover, for $\mu \to \tau + 0$, it follows from (3.3.46) that

$$S_{(22)}(\mu) - P + Q = O(|\mu - \tau|^{1/2}). \tag{3.3.48}$$

In accordance with (3.3.29),

$$S_{(21)}(\mu) = (\mathcal{S}_{22}(\mu) + I_{M-L}d^+/d^-)^{-1}\mathcal{S}_{(21)}/d^-.$$

Taking account of (3.3.42) and of $d^- = (\lambda - 1)/2\sqrt{\lambda}$, we obtain

$$S_{(21)}(\mu) = \big(\Gamma(\lambda)\mathcal{S}_{(21)}(\mu) + O(|\lambda|)\big)\, 2\sqrt{\lambda}/(\lambda - 1).$$

Consequently,

$$S_{(21)}(\mu) = O(|\mu - \tau|^{1/4}) \to 0 \quad \text{for} \quad \mu \to \tau + 0. \tag{3.3.49}$$

It remains to find the limit of $S_{(12)}(\mu)$. By virtue of (3.3.28),

$$S_{(12)}(\mu) = \mathcal{S}_{(12)}(\mu)d^+ \left(I - (\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1}(\mathcal{S}_{(22)}(\mu) + d^-/d^+)\right).$$

Since

$$(\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1}(\mathcal{S}_{(22)}(\mu) + d^-/d^+) = I - \frac{4\lambda}{\lambda^2 - 1}(\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1},$$

we arrive at

$$S_{(12)}(\mu) = \frac{2\sqrt{\lambda}}{\lambda - 1}\mathcal{S}_{(12)}(\mu)\left(-\frac{1}{2\lambda}\sum(\cdot, h_j)h_j + \Gamma(\lambda)\right).$$

Recall that $h_j \in \ker(\mathcal{S}_{22}(\tau) - 1) \subset \ker\mathcal{S}_{12}(\tau)$ (see (3.2.21)), $\mathcal{S}_{(12)}(\mu) = \mathcal{S}_{(12)}(\tau) + O(|\mu - \tau|)$, and $\mu - \tau = \lambda^2$. Therefore, as $\mu \to \tau + 0$ we have

$$S_{(12)}(\mu) = O(|\mu - \tau|^{1/4}) \to 0. \tag{3.3.50}$$

Chapter 4
Method for Computing Scattering Matrices

Section 4.1 is independent of Chap. 3. Section 4.2 is devoted to computing the scattering matrices in a neighborhood of a threshold and uses the results of Chap. 3. In fact, the scheme of the method in Sect. 4.2 is similar to that in Sect. 4.1; however, near a threshold we first calculate the augmented scattering matrix defined in a basis of waves stable at the threshold and then take into account its connection with the usual (not augmented) S-matrix.

4.1 A Method for Computing Scattering Matrices Outside Thresholds

4.1.1 *Statement of the Method*

We introduce the notation

$$\Pi_+^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r > R\}, \quad G^R = G \setminus \cup_{r=1}^N \Pi_+^{r,R}$$

for large R. Then $\partial G^R \setminus \partial G = \Gamma^R = \cup_r \Gamma^{r,R}$, where $\Gamma^{r,R} = \{(y^r, t^r) \in \Pi^r : t^r = R\}$. We seek the row $(S_{l1}, \dots, S_{lM})$ of the scattering matrix $S = S(\mu)$. As an approximation to the row we take a minimizer of a quadratic functional. To construct such a functional, we consider the problem

$$\begin{aligned}(-\Delta - \mu)\mathcal{X}_l^R &= 0, \quad x \in G^R;\\ \mathcal{X}_l^R &= 0, \quad x \in \partial G^R \setminus \Gamma^R;\\ (\partial_\nu + i\zeta)\mathcal{X}_l^R &= (\partial_\nu + i\zeta)(u_l^+ + \sum\nolimits_{j=1}^M a_j u_j^-),\ x \in \Gamma^R,\end{aligned} \tag{4.1.1}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$ is an arbitrary fixed number, ν is the outward normal, and $a_1, \dots, a_M$ are complex numbers.

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Let us explain the origin of the problem. The solution Y_l to the homogeneous problem (3.1.16) satisfies the first two Equation in (4.1.1). The asymptotics (3.2.5) can be differentiated, hence

$$(\partial_\nu + i\zeta)Y_l = (\partial_\nu + i\zeta)(u_l^+ + \sum_{j=1}^{M} a_j u_j^-) + O(e^{-\delta R})$$

for $a_j = S_{lj}$. Thus, Y_l gives an exponentially small discrepancy in the last equation (4.1.1). As an approximation to the row $(S_{l1}, \dots, S_{lM})$, we take a minimizer $a^0(R) = (a_1^0(R), \dots, a_M^0(R))$ of the functional

$$J_l^R(a_1, \dots, a_M) = \|\mathcal{X}_l^R - u_l^+ - \sum_{j=1}^{M} a_j u_j^-; L_2(\Gamma^R)\|^2, \tag{4.1.2}$$

where $\mathcal{X}_l^R$ is a solution to problem (5.6.4). One can expect that $a_j^0(R, \mu) \to S_{lj}(\mu)$ at exponential rate as $R \to \infty$ for $j = 1, \dots, M$. To find the dependence of $\mathcal{X}_l^R$ on $a_1, \dots, a_M$, we consider the problems

$$\begin{aligned} (-\Delta - \mu)v_j^\pm &= 0, \quad x \in G^R; \\ v_j^\pm &= 0, \quad x \in \partial G^R \setminus \Gamma^R; \\ (\partial_\nu + i\zeta)v_j^\pm &= (\partial_\nu + i\zeta)u_j^\pm, \; x \in \Gamma^R; \quad j = 1, \dots, M. \end{aligned} \tag{4.1.3}$$

It is evident that $\mathcal{X}_l^R = v_{l,R}^+ + \sum_j a_j v_{j,R}^-$, where $v_j^\pm = v_{j,R}^\pm$ are solutions to problems (4.1.3). Let us introduce the $M \times M$–matrices with entries

$$\begin{aligned} \mathcal{E}_{ij}^R &= \Big((v_i^- - u_i^-), (v_j^- - u_j^-)\Big)_{\Gamma^R}, \\ \mathcal{F}_{ij}^R &= \Big((v_i^+ - u_i^+), (v_j^- - u_j^-)\Big)_{\Gamma^R}, \end{aligned} \tag{4.1.4}$$

and set

$$\mathcal{G}_i^R = \big((v_i^+ - u_i^+), (v_i^+ - u_i^+)\big)_{\Gamma^R}.$$

The functional (4.1.2) can be written in the form

$$J_l^R(a, \mu) = \langle a\mathcal{E}^R(\mu), a\rangle + 2\mathrm{Re}\,\langle \mathcal{F}_l^R(\mu), a\rangle + \mathcal{G}_l^R(\mu),$$

where $\mathcal{F}_l^R$ is the lth row of the matrix $\mathcal{F}^R$ and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^M$. The minimum is attained at $a^0 = a^0(R, \mu)$ (a row) satisfying the system $a^0(R, \mu)\mathcal{E}^R + \mathcal{F}_l^R = 0$. Thus, as an approximation $S^R(\mu)$ to the scattering matrix $S(\mu)$, we take a solution of the equation $S^R\mathcal{E}^R + \mathcal{F}^R = 0$.

To justify the algorithm, we have to show that problems (4.1.3) are uniquely solvable for $\zeta \in \mathbb{R} \setminus \{0\}$ and large R, the matrix $\mathcal{E}^R$ is nonsingular, and the minimizer $a^0(R, \mu)$ of $J_l^R(\cdot, \mu)$ tends to the row $(S_{l1}(\mu), \dots, S_{lM}(\mu))$ of the scattering matrix as $R \to \infty$ and $\mu \in [\mu_1, \mu_2]$.

In the following theorem, the number $\zeta \in \mathbb{R} \setminus \{0\}$ participating in the definition of the functional $J_l^R(\cdot, \mu)$ is any (fixed) and the interval $[\mu_1, \mu_2]$ of the continuous spectrum of problem (3.1.16) is free of the thresholds and may contain eigenvalues whose eigenfunctions exponentially decay at infinity.

Theorem 4.1.1 *For all $\mu \in [\mu_1, \mu_2]$ and $R \geq R_0$, where R_0 is a sufficiently large number, there exists a unique minimizer $a(R, \mu) = (a_1(R, \mu), \dots, a_M(R, \mu))$ of the functional $J_l^R(a, \mu)$ in (4.1.2). The estimates*

$$|a_j(R, \mu) - S_{lj}(\mu)| \leqslant Ce^{-\delta R}, \quad j = 1, \dots, M,$$

hold with the same δ as in (3.2.5) and the constant C is independent of R and μ.

4.1.2 The Problem in G^R

Let us consider the problem

$$\begin{aligned} (-\Delta - \mu)u(x) &= 0, \quad x \in G^R; \\ u(x) &= 0, \quad x \in \partial G^R \setminus \Gamma^R; \\ (\partial_\nu + i\zeta)u(x) &= h(x), \quad x \in \Gamma^R, \end{aligned} \tag{4.1.5}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$, $\mu \in \mathbb{R}$, and $h \in L_2(\Gamma^R)$. We introduce a generalized solution of the problem. We set

$$\mathcal{H} = \{u \in H^1(G^R) : u \in C^2(\overline{G^R} \setminus \partial\Gamma^R); \qquad u(x) = 0, \quad x \in \partial G^R \setminus \Gamma^R\};$$

as usual, $H^1(G^R)$ denotes the Sobolev space in G^R with norm

$$\|u; H^1(G^R)\| = \Big(\int_{G^R} \sum_{|\alpha| \leq 1} |D^\alpha u(x)|^2 \, dx\Big)^{1/2}.$$

A function $u \in \mathcal{H}$ is a solution to problem (4.1.5) if and only if

$$((-\Delta - \mu)u, v)_{G^R} + (u, \partial_\nu v)_{\partial G^R \setminus \Gamma^R} + (\partial_\nu u + i\zeta u, v)_{\Gamma^R} = (h, v)_{\Gamma^R} \tag{4.1.6}$$

for all $v \in \mathcal{H}$. Integrating by parts and taking into account that u and v vanish on $\partial G^R \setminus \Gamma^R$, we can write (4.1.6) in the form

$$(\nabla u, \nabla v)_{G^R} - \mu(u, v)_{G^R} + i\zeta(u, v)_{\Gamma^R} = (h, v)_{\Gamma^R}. \tag{4.1.7}$$

Each term in (4.1.7) makes sense for u and v in $H^1(G^R)$. Let $\overline{\mathcal{H}}$ denote the closure of $\mathcal{H}$ in the norm of $H^1(G^R)$.

A function $u \in \overline{\mathcal{H}}$ is called a generalized solution to problem (4.1.5) if equality (4.1.7) holds for all $v \in \overline{\mathcal{H}}$.

Let $T : \overline{\mathcal{H}}(G^R) \to L_2(G^R)$ and $\mathcal{T} : H^{1/2}(\Gamma^R) \to L_2(\Gamma^R)$ be the embedding operators. It is known that these operators are compact. Besides, let $j : \overline{\mathcal{H}}(G^R) \to H^{1/2}(\Gamma^R)$ be the restriction operator. Then

$$(u, v)_{G^R} = (Tu, Tv)_{G^R} = [T^*Tu, v],$$

where $[\cdot, \cdot]$ is an inner product in $H^1(G^R)$ (or, what is the same, in $\overline{\mathcal{H}}(G^R)$) defined by the relation

$$[w, v] = (\nabla w, \nabla v)_{G^R} + (w, v)_{G^R}.$$

We also have

$$(u, v)_{\Gamma^R} = (\mathcal{T}ju, \mathcal{T}jv)_{\Gamma^R} = [j^*\mathcal{T}^*\mathcal{T}ju, v].$$

Now, equality (4.1.7) means that

$$[u, v] + [Vu, v] = [f, v] \tag{4.1.8}$$

for any $v \in \overline{\mathcal{H}}$; here

$$V = -(\mu + 1)T^*T + i\zeta j^*\mathcal{T}^*\mathcal{T}j$$

and $f = j^*\mathcal{T}^*h$. Therefore,

$$u + Vu = f. \tag{4.1.9}$$

Since the operator V is compact in $\overline{\mathcal{H}}$, the Fredholm alternative is valid for equation (4.1.9). Thus, to prove the unique solvability of this equation, it suffices to show that $\ker(I + V) = 0$.

Proposition 4.1.2 *For all $\mu \in \mathbb{R}$, $\zeta \in \mathbb{R} \setminus 0$, and $h \in L_2(\Gamma^R)$, problem (4.1.5) has a unique generalized solution $u \in \overline{\mathcal{H}}$.*

Proof Let us assume that $u \in \ker(I+V)$. Setting $v = u$ and $h = 0$ in relation (4.1.7), we obtain $\zeta(u, u)_{\Gamma^R} = 0$, that is, $u = 0$ on Γ^R. Problem (4.1.5) is elliptic; according to known properties of solutions to elliptic problems, the generalized solution is a smooth function on $\overline{G^R} \setminus \partial\Gamma^R$. From the third equation of problem (4.1.5) with $h = 0$, it follows that $\partial_\nu u = -i\zeta u = 0$ on Γ^R. Thus, u has zero Cauchy data on Γ^R. Therefore, $u \equiv 0$ in G^R by the unique continuation theorem (see [14], Part II, Sect. 1.4). $\square$

Proposition 4.1.3 *Let u be a generalized solution to problem* (4.1.5) *with right-hand side $h \in L_2(\Gamma^R)$. Then there holds the estimate*

$$\|u; L_2(\Gamma^R)\| \leq \frac{1}{|\zeta|}\|h : L_2(\Gamma^R)\|. \tag{4.1.10}$$

Proof Relation (4.1.7) for $v = u$ takes the form

$$(\nabla u, \nabla u)_{G^R} - \mu(u, u)_{G^R} + i\zeta(u, u)_{\Gamma^R} = (h, u)_{\Gamma^R}. \tag{4.1.11}$$

It follows that $\zeta(u, u)_{\Gamma^R} = \mathrm{Im}(h, u)_{\Gamma^R}$. Hence

$$\|u; L_2(\Gamma^R)\|^2 \leq \frac{1}{|\zeta|}\|h : L_2(\Gamma^R)\|\|u; L_2(\Gamma^R)\|.$$ $\square$

Proposition 4.1.4 *Let u be a generalized solution to problem* (4.1.5)*, where h is a smooth function on Γ^R and $h \in L_2(\Gamma^R)$. Then*

$$(\partial_\nu u, u)_{\Gamma^R} - (u, \partial_\nu u)_{\Gamma^R} = 0. \tag{4.1.12}$$

Proof From the assumptions of the proposition it follows that the u is a smooth function on $\overline{G^R} \setminus \partial\Gamma^R$. Therefore, the boundary condition on Γ^R can be understood in the classical sense. Setting $h = \partial_\nu u + i\zeta u$ in (4.1.11), we obtain

$$\zeta(u, u)_{\Gamma^R} = \mathrm{Im}(\partial_\nu u + i\zeta u, u)_{\Gamma^R} = \mathrm{Im}(\partial_\nu u, u)_{\Gamma^R} + \zeta(u, u)_{\Gamma^R}$$

and $\mathrm{Im}(\partial_\nu u, u)_{\Gamma^R} = 0$. $\square$

4.1.3 Justification of the Method for Computing the Scattering Matrix

To justify the method, we have to verify that the matrix $\mathcal{E}^R$ with entries (4.1.4) is nonsingular and the minimizer $a^0(R)$ of functional (4.1.2) tends to the lth row of the scattering matrix as $R \to \infty$.

Proposition 4.1.5 *The matrix $\mathcal{E}^R$ with entries* (4.1.4) *is nonsingular for all $R \geqslant R_0$, where R_0 is a sufficiently large number.*

Proof Suppose that the required assertion is false. Then, for any R^0, there exists a number $R > R^0$ such that the matrix $\mathcal{E}^R$ is singular and the functions $\mathcal{U} = \sum_j c_j u_j^-$ and $\mathcal{V} = \sum_j c_j v_j^-$ are related by

$$\mathcal{U}(x) = \mathcal{V}(x), \quad x \in \Gamma^R, \tag{4.1.13}$$

where v_j^- is a solution to problem (4.1.3) and $\overrightarrow{c} = (c_1, \dots, c_M)$ is a vector with $|\overrightarrow{c}| = 1$. According to the equation on Γ^R in (4.1.3), we have

$$\partial_\nu \mathcal{U}(x) = \partial_\nu \mathcal{V}(x), \quad x \in \Gamma^R. \tag{4.1.14}$$

In view of Proposition 4.1.4,

$$(\partial_\nu \mathcal{V}, \mathcal{V})_{\Gamma^R} - (\mathcal{V}, \partial_\nu \mathcal{V})_{\Gamma^R} = 0.$$

Now, (4.1.13) and (4.1.14) lead to the equality

$$(\partial_\nu \mathcal{U}, \mathcal{U})_{\Gamma^R} - (\mathcal{U}, \partial_\nu \mathcal{U})_{\Gamma^R} = 0. \tag{4.1.15}$$

For the waves $u_j^\pm$ in G [see (2.3.4) and (2.3.5)], the relations

$$(\partial_\nu u_j^\pm, u_k^\pm)_{\Gamma^R} - (u_j^\pm, \partial_\nu u_k^\pm)_{\Gamma^R} = \mp i\delta_{jk}, \tag{4.1.16}$$
$$(\partial_\nu u_j^\pm, u_k^\mp)_{\Gamma^R} - (u_j^\pm, \partial_\nu u_k^\mp)_{\Gamma^R} = 0. \tag{4.1.17}$$

hold. From (4.1.15), (4.1.16), and (4.1.17), it follows that

$$0 = \sum_{j,k} c_j \overline{c}_k ((\partial_\nu u_j^-, u_k^-)_{\Gamma^R} - (u_j^-, \partial_\nu u_k^-)_{\Gamma^R}) = i \sum_j |c_j|^2 = i,$$

a contradiction. □

Proposition 4.1.6 *Suppose that a vector $a(R) = (a_1(R), \dots, a_M(R))$ minimizes the functional J_l^R in (4.1.2). Then*

$$J_l^R(a(R)) = O(e^{-2\delta R}) \quad \textit{as } R \to \infty, \tag{4.1.18}$$

where δ is the same number as in (3.2.5). For all $R \geqslant R_0$, the components of $a(R)$ are uniformly bounded, that is,

$$|a_j(R)| \leqslant \text{const} < \infty, \quad j = 1, \dots, M. \tag{4.1.19}$$

Proof Denote by Y_l^R the solution to problem (4.1.1), where for a_j, $j = 1, \dots, M$, the elements S_{lj} of the scattering matrix S of problem (3.1.16) are taken. The asymptotics (3.2.5) can be differentiated; hence

$$(\partial_\nu + i\zeta)(Y_l^R - Y_l)|_{\Gamma^R} = O(e^{-\delta R}).$$

Since $Y_l^R - Y_l$ satisfies the first two equations of problem (4.1.5) with $f = 0$ and $g = 0$, it follows that estimate (4.1.10) holds for $u = Y_l^R - Y_l$:

$$\|Y_l^R - Y_l; L_2(\Gamma^R)\| \leqslant |\zeta|^{-1} \|(\partial_\nu + i\zeta)(Y_l^R - Y_l); L_2(\Gamma^R)\| \leqslant ce^{-\delta R}.$$

This inequality, together with (3.2.5), provides the estimate

$$J_l^R(S_l) = \|Y_l^R - (u_l^+ + \sum_{j=1}^{M} S_{lj} u_j^-); L_2(\Gamma^R)\|^2 \leqslant ce^{-2\delta R}$$

with constant c independent of R. To obtain (4.1.18), it remains to note that $J_l^R(a(R)) \leqslant J_l^R(S_l)$

We proceed to estimating the minimizer $a(R)$. Let Z_l^R denote the solution to problem (4.1.1) corresponding to the vector $a(R) = (a_1(R), \dots, a_M(R))$. By Proposition 4.1.4,

$$(\partial_\nu Z_l^R, Z_l^R)_{\Gamma^R} - (Z_l^R, \partial_\nu Z_l^R)_{\Gamma^R} = 0. \tag{4.1.20}$$

In view of (4.1.18),

$$\|Z_l^R - (u_l^+ + \sum_{j=1}^{M} a_j(R) u_j^-); L_2(\Gamma^R)\| = O(e^{-\delta R}), \quad R \to \infty. \tag{4.1.21}$$

Since

$$(\partial_\nu + i\zeta) Z_l^R|_{\Gamma^R} = (\partial_\nu + i\zeta)(u_l^+ + \sum_{j=1}^{M} a_j(R) u_j^-)|_{\Gamma^R},$$

from (4.1.21) it follows that

$$\|\partial_\nu (Z_l^R - (u_l^+ + \sum_{j=1}^{M} a_j(R) u_j^-)); L_2(\Gamma^R)\| = O(e^{-\delta R}), \quad R \to \infty. \tag{4.1.22}$$

From (4.1.21) and (4.1.22) we derive that $Z_l^R = \varphi_l + T^R$, where $\varphi_l = u_l^+ + \sum a_j(R)u_j^-$ and $\|T^R; L_2(\Gamma^R)\| = O(e^{-\delta R})$ and $\|\partial_\nu T^R; L_2(\Gamma^R)\| = O(e^{-\delta R})$. We substitute the expression of Z_l^R into (4.1.20) and obtain

$$\begin{aligned}(\partial_\nu\varphi_l, \varphi_l)_{\Gamma^R} &- (\varphi_l, \partial_\nu\varphi_l)_{\Gamma^R}\\ &= (\varphi_l, \partial_\nu T^R)_{\Gamma^R} - (\partial_\nu T^R, \varphi_l)_{\Gamma^R} + (T^R, \partial_\nu\varphi_l)_{\Gamma^R} - (\partial_\nu\varphi_l, T^R)_{\Gamma^R}\\ &\quad + (T^R, \partial_\nu T^R)_{\Gamma^R} - (\partial_\nu T^R, T^R)_{\Gamma^R}.\end{aligned} \tag{4.1.23}$$

In view of (4.1.16) and (4.1.17), the left-hand side of (4.1.23) is calculated straightforwardly:

$$(\partial_\nu\varphi_l, \varphi_l)_{\Gamma^R} - (\varphi_l, \partial_\nu\varphi_l)_{\Gamma^R} = -i(1 - |a(R)|^2).$$

This and equality (4.1.23) lead to the estimate

$$|1 - |a(R)|^2| = (\|\varphi_l; L_2(\Gamma^R)\| + \|\partial_\nu\varphi_l; L_2(\Gamma^R)\|)O(e^{-\delta R}).$$

Moreover, taking into account the inequality $(\|\varphi_l; L_2(\Gamma^R)\| + \|\partial_\nu\varphi_l; L_2(\Gamma^R)\| \leq C(1 + |a(R)|)$, we obtain

$$|1 - |a(R)|^2| = (1 + |a(R)|O(e^{-\delta R})$$

and $|a(R)| = 1 + o(1)$ as $R \to \infty$. □

Proof of Theorem 4.1.1 Let Y_l, Z_l^R and $(a_1(R), \dots, a_M(R))$ be the same as in Proposition 4.1.6. We substitute $u = U_l := Y_l - Z_l^R$ into (4.1.12) and obtain

$$(\partial_\nu U_l, U_l)_{\Gamma^R} - (U_l, \partial_\nu U_l)_{\Gamma^R} = 0. \tag{4.1.24}$$

We set

$$\varphi_l = u_l^+ + \sum_{j=1}^{M} a_j(R)u_j^-, \quad \psi_l = u_l^+ + \sum_{j=1}^{M} S_{lj}u_j^- \tag{4.1.25}$$

and write U_l as

$$U_l = Y_l - Z_l^R = (Y_l - \psi_l) + (\psi_l - \varphi_l) + (\varphi_l - Z_l^R).$$

By virtue of (3.2.5), $(Y_l - \psi_l)|_{\Gamma^R} = O(e^{-\gamma R})$; the waves $u_j^{\pm}$ are bounded as well as the minimizer [see (4.1.19)], hence $\psi_l - \varphi_l = O(1)$. Taking into account also (4.1.21) and (4.1.22), we pass from (4.1.24) to the relation

$$(\partial_\nu(\psi_l - \varphi_l), \psi_l - \varphi_l)_{\Gamma^R} - (\psi_l - \varphi_l, \partial_\nu(\psi_l - \varphi_l))_{\Gamma^R} = O(e^{-\delta R}). \tag{4.1.26}$$

Here the left-hand side is calculated straightforwardly (by means of (4.1.25), (4.1.16), and (4.1.17)) and equals $i\sum_{j=1}^{M}|a_j(R)-S_{lj}|^2$. Finally, we obtain

$$\sum_{j=1}^{M}|a_j(R)-S_{lj}|^2 = O(e^{-\delta R}). \tag{4.1.27}$$

We now prove that the estimate

$$\sum_{j=1}^{M}|a_j(R)-S_{lj}|^2 = O(\exp -2\delta(1-2^{-N})R), \tag{4.1.28}$$

which coincides with (4.1.27) for $N=1$, is valid for any positive integer. It suffices to show that N in (4.1.28) can be replaced by $N+1$. Using (4.1.28), we obtain

$$\psi_l - \varphi_l = \sum_{j=1}^{M}(S_{lj}-a_j(R))u_j^- = O(\exp -\delta(1-2^{-N})R).$$

Let us employ this estimate instead of $\psi_l - \varphi_l = O(1)$ and pass from (4.1.24) to (4.1.26) with right-hand side changed for $O(\exp(-\delta(1-2^{-N})R-\delta R))$. Once again, calculating the left-hand side of (4.1.26), we have

$$\sum_{j=1}^{M}|a_j(R)-S_{lj}|^2 = O(\exp -2\delta(1-2^{-N-1})R). \tag{4.1.29}$$

Let us notice that, instead of δ, we could from the outset take a slightly greater number δ'. Choosing now a sufficiently large N, we obtain $2\delta'(1-2^{-N-1}) > 2\delta$. □

4.2 A Method for Computing Scattering Matrices in Vicinity of Thresholds

We now proceed to calculating the matrix $\mathcal{S}(\mu)$ in Theorem 3.2.6, (ii) with $\mu \in [\mu', \mu''] \subset (\tau', \tau'')$. The interval $[\mu', \mu'']$ contains the threshold τ and, possibly, some eigenvalues of operator (3.2.11). For the sake of simplicity, we suppose that the interval $[\mu', \mu'']$ is narrow enough to be contained in an open interval U_p provided by Lemma 3.2.4. Then the index γ in asymptotics (3.2.9) is independent of $\mu \in [\mu', \mu'']$ and satisfies $\sqrt{\tau-\mu'} < \gamma < \sqrt{\mu''-\tau''}$. We introduce the boundary value problem

$$
\begin{aligned}
-\Delta\mathcal{X}_j^R - \mu\mathcal{X}_j^R &= 0, \quad x \in G^R;\\
\mathcal{X}_j^R &= 0 \quad x \in \partial G^R \setminus \Gamma^R;\\
(\partial_n + i\zeta)\mathcal{X}_j^R &= (\partial_n + i\zeta)(w_j^+ + \sum\nolimits_{k=1}^M a_k w_k^-),\ x \in \Gamma^R,
\end{aligned}
\tag{4.2.1}
$$

where $w_j^\pm$ is stable basis (3.1.9), (3.1.10) in the space of waves, $\zeta \in \mathbb{R} \setminus \{0\}$, and $a_k \in \mathbb{C}$. As an approximation to the row $(\mathcal{S}_{j1}, \dots, \mathcal{S}_{jM})$, we suggest a minimizer $a^0(R) = (a_1^0(R), \dots, a_M^0(R))$ of the functional

$$
\mathcal{J}_j^R(a_1, \dots, a_M) = \|\mathcal{X}_j^R - w_j^+ - \sum_{k=1}^M a_k w_k^-; L_2(\Gamma^R)\|^2, \tag{4.2.2}
$$

where $\mathcal{X}_j^R$ is a solution of problem (4.2.1). Let us consider the problems

$$
\begin{aligned}
-\Delta z_j^\pm - \mu z_j^\pm &= 0, \quad x \in G^R;\\
z_j^\pm &= 0, \quad x \in \partial G^R \setminus \Gamma^R;\\
(\partial_n + i\zeta)z_j^\pm &= (\partial_n + i\zeta)w_j^\pm,\ x \in \Gamma^R; \quad j = 1, \dots, M,
\end{aligned}
$$

set

$$
\begin{aligned}
\mathcal{E}_{jk}^R &= \left(z_j^- - w_j^-, z_k^- - w_k^-\right)_{\Gamma^R},\\
\mathcal{F}_{jk}^R &= \left(z_j^+ - w_j^+, z_k^- - w_k^-\right)_{\Gamma^R},\\
\mathcal{G}_j^R &= \left(z_j^+ - w_j^+, z_j^+ - w_j^+\right)_{\Gamma^R},
\end{aligned}
\tag{4.2.3}
$$

and rewrite functional (4.2.2) in the form

$$
\mathcal{J}_j^R(a) = \langle a\mathcal{E}^R, a\rangle + 2\mathrm{Re}\,\langle \mathcal{F}_j^R, a\rangle + \mathcal{G}_j^R,
$$

where $\mathcal{F}_j^R$ is the jth row of the matrix $\mathcal{F}^R$. Thus the minimizer $a^0(R)$ is a solution to the system $a^0(R)\mathcal{E}^R + \mathcal{F}_j^R = 0$.

The justification of the method is similar to that in the previous section. The next proposition can be verified in the same way as Proposition 4.1.5.

Proposition 4.2.1 *The matrix $\mathcal{E}^R(\mu)$ with entries (4.2.3) is non-singular for all $\mu \in [\mu', \mu'']$ and $R \geqslant R_0$, where R_0 is sufficiently large number.*

Proposition 4.2.2 *Let $a^0(R, \mu) = (a_1^0(R, \mu), \dots, a_M^0(R, \mu))$ be a minimizer of the functional $\mathcal{J}_l^R$ in (4.2.2). Then*

$$
\mathcal{J}_l^R\left(a^0(R, \mu)\right) \leqslant Ce^{-2\gamma R} \quad \textit{for } R \to \infty, \tag{4.2.4}
$$

where the constant C is independent of $R \geqslant R_0$, $\mu \in [\mu', \mu'']$, and γ is the same number as in (3.2.9). For all $R \geqslant R_0$ and $\mu \in [\mu', \mu'']$, the components of the vector $a^0(R, \mu)$ are uniformly bounded,

$$|a_j^0(R, \mu)| \leqslant \text{const} < \infty, \quad j = 1, \dots, M.$$

Proof Relation (4.2.4) has been obtained in the same manner as 4.1.18. Let us verify the uniform boundedness of the minimizer $a^0(R, \mu)$. Denote by Z_l^R the solution of problem (4.2.1) corresponding to $a^0(R, \mu) = (a_1^0(R, \mu), \dots, a_M^0(R, \mu))$. Setting $u = v = Z_l^R$ in the Green formula, we obtain

$$(\partial_\nu Z_l^R, Z_l^R)_{\Gamma^R} - (Z_l^R, \partial_\nu Z_l^R)_{\Gamma^R} = 0. \tag{4.2.5}$$

By virtue of (4.2.4),

$$\|Z_l^R - (w_l^+ + \sum_{j=1}^{M} a_j(R, \mu) w_j^-); L_2(\Gamma^R)\| = O(e^{-\gamma R}), \quad R \to \infty, \tag{4.2.6}$$

uniformly with respect to μ. Since

$$(\partial_\nu + i\zeta) Z_l^R|_{\Gamma^R} = (\partial_\nu + i\zeta)(w_l^+ + \sum_{j=1}^{M} a_j^0(R) w_j^-)|_{\Gamma^R},$$

from (4.2.5) it follows that

$$\|\partial_\nu (Z_l^R - (w_l^+ + \sum_{j=1}^{M} a_j^0(R) w_j^-)); L_2(\Gamma^R)\| = O(e^{-\gamma R}), \quad R \to \infty. \tag{4.2.7}$$

Recall that, for $\mu > \tau$, the waves $w_l^\pm$ are bounded functions; for $\mu < \tau$, the waves $w_l^\pm$ with $L < l \leqslant M$ defined by (3.3.2) grow at infinity as $O(e^{\sqrt{\tau - \mu}\,|x|})$ and, for $\mu = \tau$, as $O(|x|)$. Moreover, $\sqrt{\tau - \mu'} < \gamma$.

We use (4.2.6) and (4.2.7) to reduce (4.2.5) to the form

$$\begin{aligned}(\partial_\nu \varphi_l, \varphi_l)_{\Gamma^R} - (\varphi_l, \partial_\nu \varphi_l)_{\Gamma^R} = (&\|\varphi_l; L_2(\Gamma^R)\| + \|\partial_\nu \varphi_l; L_2(\Gamma^R)\|) \\ &\times O(e^{-(\gamma - \sqrt{\tau - \mu} - \varepsilon) R}),\end{aligned}$$

where $\varphi_l = w_l^+ + \sum a_j^0(R) w_j^-$; as before, $\sqrt{\tau - \mu} = i\sqrt{\mu - \tau}$ for $\mu > \tau$, ε being an arbitrary small positive number. In view of (3.1.11) and (3.1.12), the left-hand side is equal to $-i(1 - \sum |a_j^0(R)|^2)$. Therefore,

$$|1-|a(R)|^2|=(1+|a(R)|)O(e^{-(\gamma-\sqrt{\tau-\mu}-\varepsilon)}),$$

which leads to $|a^0(R)|=1+o(1)$. $\square$

Theorem 4.2.3 *For all $R\geqslant R_0$, where R_0 is a sufficiently large number, and for all $\mu\in[\mu',\mu'']\subset(\tau',\tau'')$, there exists a unique minimizer $a^0(R,\mu)=(a_1^0(R,\mu),\dots,a_M^0(R,\mu))$ of the functional $\mathcal{J}_l^R$ in (4.1.2). The estimates*

$$\sum_{k=1}^{M}|\mathcal{S}_{jk}(\mu)-a_k^0(R,\mu)|\leqslant Ce^{-\Lambda R} \tag{4.2.8}$$

hold for all $R\geqslant R_0$, $\mu\in[\mu',\mu'']$, and $0<\Lambda\le\gamma-\sqrt{\tau-\mu'}$, where γ is the same as in (3.2.9) and the constant $C=C(\Lambda)$ is independent of R and μ.

Proof Let Y_l^R be a solution to problem (4.2.1), where a_j, $j=1,\dots,M$, are taken to be the entries $\mathcal{S}_{lj}$ of the scattering matrix $\mathcal{S}$, and let Z_l^R and $(a_1^0(R,\mu),\dots,a_M^0(R,\mu))$ be the same as in Proposition 4.2.2. We substitute $u=v=U_l:=\mathcal{Y}_l-Z_l^R$ into the Green formula. Since U_l satisfies the first two equations in (4.2.1), we have

$$(\partial_\nu U_l,U_l)_{\Gamma^R}-(U_l,\partial_\nu U_l)_{\Gamma^R}=0. \tag{4.2.9}$$

Setting

$$\varphi_l=w_l^+ +\sum_{j=1}^{M}a_j^0(R,\mu)w_j^-,\quad \psi_l=w_l^+ +\sum_{j=1}^{M}\mathcal{S}_{lj}(\mu)w_j^-, \tag{4.2.10}$$

we write down U_l in the form

$$U_l=\mathcal{Y}_l-Z_l^R=(\mathcal{Y}_l-\psi_l)+(\psi_l-\varphi_l)+(\varphi_l-Z_l^R).$$

Note that $(\mathcal{Y}_l-\psi_l)|_{\Gamma^R}=O(e^{-\gamma R})$ by virtue (3.2.9). Moreover, by Proposition 4.2.2, the components of the minimizer $a_j(R,\mu)$ are uniformly bounded. In view of (4.2.6) and (4.2.7), this leads from (4.2.9) to the relation

$$(\partial_\nu(\psi_l-\varphi_l),(\psi_l-\varphi_l))_{\Gamma^R}-((\psi_l-\varphi_l),\partial_\nu(\psi_l-\varphi_l))_{\Gamma^R}=O(e^{-(\gamma-\sqrt{\tau-\mu}-\varepsilon)R}), \tag{4.2.11}$$

where ε is an arbitrary small positive number. Straightforward calculation shows that the left-hand side is equal to $i\sum_{j=1}^{M}|a_j^0(R,\mu)-\mathcal{S}_{lj}(\mu)|^2$ (it suffices to use (4.2.9), (3.1.11), and (3.1.12)) and we arrive at

$$\sum_{j=1}^{M}|a_j^0(R,\mu)-\mathcal{S}_{lj}(\mu)|^2=O(e^{-(\gamma-\sqrt{\tau-\mu}-\varepsilon)R}). \tag{4.2.12}$$

We now prove the inequality

$$\sum_{j=1}^{M} |a_j(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O(e^{-2(\gamma-\sqrt{\tau-\mu}-\varepsilon)(1-2^{-N})R}), \tag{4.2.13}$$

which coincides with (4.2.12) as $N = 1$. It suffices to show that N in (4.2.13) can be replaced by $N + 1$. Using (4.2.13), we obtain

$$\psi_l - \varphi_l = \sum_{j=1}^{M} (\mathcal{S}_{lj} - a_j(R))u_j^- = O(\exp\{-(\gamma - \sqrt{\tau-\mu} - \varepsilon)(1 - 2^{-N})R\}).$$

Let us employ this estimate instead of $\psi_l - \varphi_l = O(e^{(\sqrt{\tau-\mu}+\varepsilon)\,|x|})$ and pass from (4.2.9) to (4.2.11) with right-hand side changed for $O(\exp(-\delta(1-2^{-N})R - \delta R))$ as $\delta = \gamma - \sqrt{\tau-\mu} - \varepsilon$. Once again calculating the left-hand side of (4.2.11), we have

$$\sum_{j=1}^{M} |a_j(R) - \mathcal{S}_{lj}|^2 = O(\exp\{-2(\gamma - \sqrt{\tau-\mu} - \varepsilon)(1 - 2^{-N-1})R\}). \tag{4.2.14}$$

Let us notice that, instead of γ, we could from the outset take a slightly greater number γ' such that $\gamma' - \varepsilon > \gamma$. Choosing now a sufficiently large N, we obtain $2(\gamma' - \sqrt{\tau-\mu} - \varepsilon)(1 - 2^{-N-1}) > 2(\gamma - \sqrt{\tau-\mu})$. It remains to replace $\sqrt{\tau-\mu}$ by $\max_{\mu\in[\mu',\mu'']} \sqrt{\tau-\mu} = \sqrt{\tau-\mu'}$. $\square$

In a neighborhood of the threshold τ, the matrix $\mathcal{S}(\mu)$ can be calculated by the presented method. Since the limits of $S(\mu)$ as $\mu \to \tau \pm 0$ are finite, the connection between $S(\mu)$ and $\mathcal{S}(\mu)$ allows us to calculate $S(\mu)$ for μ in vicinity of τ.

Chapter 5
Asymptotic and Numerical Studies of Resonant Tunneling in 2D-Waveguides for Electrons of Small Energy

In this chapter, we consider a $2D$-waveguide that coincides with a strip having two narrows of the same width ε symmetric about the waveguide axis. The resonant tunneling is discussed for electrons with energy between the first and the second thresholds, so only one incoming wave and one outgoing wave can propagate in every outlet of the waveguide; in other words, we deal with electrons of small energy. There are no external fields. We derive asymptotics for the resonant energy, for the transmission coefficient, and for the width of the resonant peak at its half-height as ε tends to zero. Then we compare the asymptotic results with those obtained by numerical calculation of the scattering matrix. Finally, we discuss the impact of a finite waveguide work function on the resonant tunneling and assess the mathematical model adequacy for the tunneling in quantum waveguides with narrows.

The scheme of the asymptotic analysis developed in the chapter will be generalized and implemented for $3D$-waveguides with resonator of arbitrary form and two narrows of width ε_1 and ε_2 (Chap. 6) and for the resonant tunneling in the presence of a magnetic field in the resonator (Chaps. 7 and 8).

5.1 Statement of the Problem

To describe the domain $G(\varepsilon)$ in $\mathbb{R}^2$ occupied by the waveguide, we first introduce two auxiliary domains G and Ω in $\mathbb{R}^2$. The domain G is the strip

$$G = \mathbb{R} \times D = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} = (-\infty, +\infty);\, y \in D = (-l/2, l/2)\}.$$

Let us define Ω. Denote by K a double cone with vertex at the origin O that contains the x-axis and is symmetric about the coordinate axes. The set $K \cap S^1$, where S^1 is a unit circle, consists of two simple arcs. Assume that Ω contains the cone K and a neighborhood of its vertex; moreover, outside a large disk (centered at the origin) Ω coincides with K. The boundary $\partial\Omega$ of Ω is supposed to be smooth (see Fig. 5.1).

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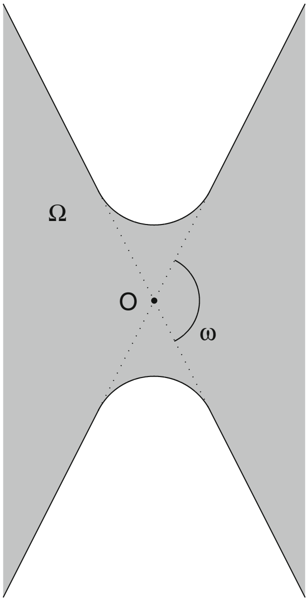

Fig. 5.1 Domain Ω

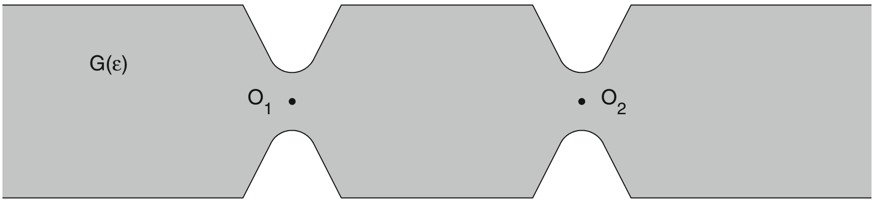

Fig. 5.2 Waveguide $G(\varepsilon)$

We now turn to the waveguide $G(\varepsilon)$. Denote by $\Omega(\varepsilon)$ the domain obtained from Ω by the contraction with center at O and coefficient ε. In other words, $(x, y) \in \Omega(\varepsilon)$ if and only if $(x/\varepsilon, y/\varepsilon) \in \Omega$. Let K_j and $\Omega_j(\varepsilon)$ stand for K and $\Omega(\varepsilon)$ shifted by the vector $\mathbf{r}_j = (x_j^0, 0)$, $j = 1, 2$. We assume that $|x_1^0 - x_2^0|$ is sufficiently large, so the distance from $\partial K_1 \cap \partial K_2$ to G is positive. We put (see Fig. 5.2)

$$G(\varepsilon) = G \cap \Omega_1(\varepsilon) \cap \Omega_2(\varepsilon).$$

The wave function of a free electron of energy k^2 satisfies the boundary value problem

$$\begin{aligned} -\Delta u(x, y) - k^2 u(x, y) &= 0, \quad (x, y) \in G(\varepsilon), \\ u(x, y) &= 0, \quad (x, y) \in \partial G(\varepsilon). \end{aligned} \tag{5.1.1}$$

Moreover, u is subject to radiation conditions at infinity. To formulate the conditions we need the problem

$$\begin{aligned} -\Delta v(y) - \lambda^2 v(y) &= 0, \quad y \in D, \\ v(-l/2) = v(l/2) &= 0. \end{aligned} \tag{5.1.2}$$

The eigenvalues λ_q^2 of this problem, where $q = 1, 2, \ldots$ are called the thresholds; they form the sequence $\lambda_q^2 = (\pi q/l)^2, q = 1, 2, \ldots$. We suppose that k^2 in (5.1.1) satisfies $(\pi/l)^2 < k^2 < (2\pi/l)^2$, i.e., k^2 is between the first and the second thresholds. Then, in the space of bounded wave functions, a basis is formed by the wave functions subject to the radiation conditions

$$u_1(x, y) = \begin{cases} U_1(x, y) + S_{11}(k)\, U_2(x, y) + O(e^{\delta x}), & x \to -\infty, \\ S_{12}(k)\, U_1(x, y) + O(e^{-\delta x}), & x \to +\infty; \end{cases} \tag{5.1.3}$$

$$u_2(x, y) = \begin{cases} S_{21}(k)\, U_2(x, y) + O(e^{\delta x}), & x \to -\infty, \\ U_2(x, y) + S_{22}(k)\, U_1(x, y) + O(e^{-\delta x}), & x \to +\infty. \end{cases} \tag{5.1.4}$$

In the strip G, the function $U_1(x, y) = e^{i\nu_1 x}\Psi_1(y)$ is a wave incoming from $-\infty$ and outgoing to $+\infty$, while $U_2(x, y) = e^{-i\nu_1 x}\Psi_1(y)$ is a wave going from $+\infty$ to $-\infty$. Here $\nu_1 = \sqrt{k^2 - \lambda_1^2}$; Ψ_1 is an eigenfunction of problem (5.1.2) that corresponds to the eigenvalue λ_1^2,

$$\Psi_1(y) = \sqrt{2/l\nu_1}\cos\lambda_1 y. \tag{5.1.5}$$

The matrix

$$S = \|S_{mj}\|_{m,j=1,2}$$

with elements from conditions (5.1.3) and (5.1.4) is called the scattering matrix; it is unitary. The values

$$R_1 = |S_{11}|^2, \quad T_1 = |S_{12}|^2$$

are called the reflection and transition coefficients, relatively, for the wave U_1 incoming to $G(\varepsilon)$ from $-\infty$. (Similar definitions can be given for the wave U_2 coming from $+\infty$.) The goal is to find a "resonant" value $k_r = k_r(\varepsilon)$ of the parameter k corresponding to the maximum of the transition coefficient and to describe the behavior of $T_m(k, \varepsilon)$, $m = 1, 2$, for k in a neighborhood of $k_r(\varepsilon)$ as $\varepsilon \to 0$.

5.2 Limit Problems

We derive the asymptotics of a wave function (i.e., the solution of problem (5.1.1) as $\varepsilon \to 0$) by use of the method of compound asymptotic expansions. To this end, we introduce "limit" boundary value problems independent of the parameter ε.

5.2.1 First Kind Limit Problems

Put $G(0) = G \cap K_1 \cap K_2$ (Fig. 5.3); thus, $G(0)$ consists of three parts, G_0, G_1, and G_2, where G_1 and G_2 are infinite domains, while G_0 is a bounded resonator.

The problems

$$\begin{aligned} -\Delta v(x, y) - k^2 v(x, y) = f, &\quad (x, y) \in G_j, \\ v(x, y) = 0, &\quad (x, y) \in \partial G_j, \end{aligned} \tag{5.2.1}$$

where $j = 0, 1, 2$, are called the first kind limit problems.

Now we introduce function spaces for problem (5.2.1) in G_0. Let ϕ_1 and ϕ_2 be smooth real functions in the closure $\overline{G}_0$ of G_0 such that $\phi_j = 1$ in a neighborhood of O_j, $j = 1, 2$, and $\phi_1^2 + \phi_2^2 = 1$. For $l = 0, 1, \ldots$ and $\gamma \in \mathbb{R}$, the space $V_\gamma^l(G_0)$ is the completion in the norm

$$\|v; V_\gamma^l(G_0)\| = \left(\int_{G_0} \sum_{|\alpha|=0}^{l} \sum_{j=1}^{2} \phi_j^2(x, y) r_j^{2(\gamma - l + |\alpha|)} |\partial^\alpha v(x, y)|^2 \, dx \, dy \right)^{1/2}$$

of the set of smooth functions in $\overline{G}_0$ which vanish near O_1 and O_2; here r_j is the distance between (x, y) and O_j, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, and $\partial^\alpha = \partial^{|\alpha|}/\partial x^{\alpha_1} \partial y^{\alpha_2}$. Proposition 5.2.1 follows from well-known general results, e.g., see [37, Chaps. 2 and 4, Sect. 1–3] or [33, Vol. 1, Chap. 1].

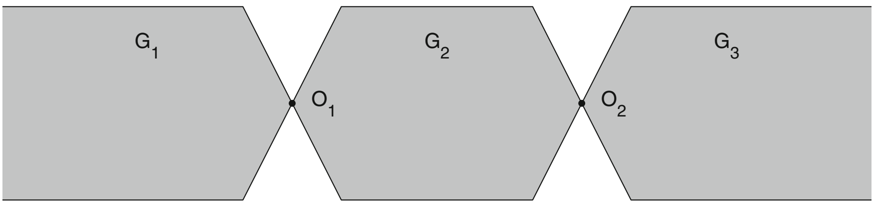

Fig. 5.3 The "limit waveguide" $G(0)$

Proposition 5.2.1 *Assume that $|\gamma - 1| < \pi/\omega$. Then, for every $f \in V^0_\gamma(G_0)$ and any k^2 except the positive increasing sequence $\{k^2_p\}_{p=1}^\infty$ of eigenvalues, $k^2_p \to \infty$, there exists a unique solution $v \in V^2_\gamma(G_0)$ to problem (5.2.1) in G_0. The estimate*

$$\|v; V^2_\gamma(G_0)\| \le c\, \|f; V^0_\gamma(G_0)\| \tag{5.2.2}$$

holds with a constant c independent of f. If f is a smooth function in $\overline{G}_0$ vanishing near O_1 and O_2 and v is any solution in $V^2_\gamma(G_0)$ of problem (5.2.1), then v is smooth in $\overline{G}_0$ except at O_1 and O_2 and admits the asymptotic representation

$$v(x, y) = \begin{cases} b_1 \widetilde{J}_{\pi/\omega}(kr_1)\Phi(\varphi_1) + O\big(r_1^{2\pi/\omega}\big), & r_1 \to 0; \\ b_2 \widetilde{J}_{\pi/\omega}(kr_2)\Phi(\pi - \varphi_2) + O\big(r_2^{2\pi/\omega}\big), & r_2 \to 0 \end{cases}$$

near the points O_1 and O_2, where (r_j, φ_j) are polar coordinates with center at O_j, b_j are some constant coefficients, $\widetilde{J}_\mu$ stands for the Bessel function multiplied by a constant so that $\widetilde{J}_\mu(kr) = r^\mu + o(r^\mu)$, and $\Phi(\varphi) = \pi^{-1/2}\cos(\pi\varphi/\omega)$.

If $k^2 = k^2_e$ is an eigenvalue of problem (5.2.1), then problem (5.2.1) will be solvable in G_0 if and only if $(f, v_e)_{G_0} = 0$ for any eigenfunction v_e corresponding to k^2_e. The condition being fulfilled, there exists a unique solution v to problem (5.2.1), which is orthogonal to the eigenfunctions and satisfies (5.2.2) (i.e., *the Fredholm alternative holds*).

We turn to problems (5.2.1) for $j = 1, 2$. Let $\chi_{0,j}$ and $\chi_{\infty,j}$ be smooth real functions in the closure $\overline{G}_j$ of G_j such that $\chi_{0,j} = 1$ in a neighborhood of O_j, $\chi_{0,j} = 0$ outside of a compact set, and $\chi^2_{0,j} + \chi^2_{\infty,j} = 1$. We also assume that the support $\operatorname{supp}\chi_{\infty,j}$ is located in the cylindrical part of G_j. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \ldots$, the space $V^l_{\gamma,\delta}(G_j)$ is the completion in the norm

$$\|v; V^l_{\gamma,\delta}(G_j)\| = \left(\int_{G_j} \sum_{|\alpha|=0}^{l} \big(\chi^2_{0,j} r_j^{2(\gamma - l + |\alpha|)} + \chi^2_{\infty,j}\exp(2\delta x)\big)|\partial^\alpha v|^2\, dx\, dy\right)^{1/2}$$

of the set of smooth functions in $\overline{G}_j$ having compact supports and vanishing near O_j.

Recall that, according to our assumption, k^2 lies between the first and the second thresholds, so that in every G_j there is only one outgoing wave. Let $U_1^- = U_2$ be the outgoing wave in G_1, and $U_2^- = U_1$ be the outgoing wave in G_2 (for the definition of U_j in G see Sect. 5.1). The next proposition follows, e.g., from [37, Theorem 5.3.5].

Proposition 5.2.2 *Let $|\gamma - 1| < \pi/\omega$ and suppose that there is no nontrivial solution to homogeneous problem* (5.2.1) (*where $f = 0$*) *in $V^2_{\gamma,\delta}(G_j)$ with arbitrary small positive δ. Then, for any $f \in V^0_{\gamma,\delta}(G_j)$, there exists a unique solution v to problem* (5.2.1) *that admits the representation*

$$v = u + A_j \chi_{\infty,j} U_j^-,$$

where $A_j = const$, $u \in V^2_{\gamma,\delta}(G_j)$, and δ is sufficiently small. Furthermore, the inequality

$$\|u; V^2_{\gamma,\delta}(G_j)\| + |A_j| \le c\,\|f; V^0_{\gamma,\delta}(G_j)\|,$$

holds with a constant c independent of f. If the function f is smooth and vanishes near O_j, then the solution v in G_1 admits the representation

$$v(x, y) = a_1 \widetilde{J}_{\pi/\omega}(kr_1)\Phi(\pi - \varphi_1) + O\big(r_1^{2\pi/\omega}\big), \quad r_1 \to 0,$$

and the solution in G_2 admits the representation

$$v(x, y) = a_2 \widetilde{J}_{\pi/\omega}(kr_2)\Phi(\varphi_2) + O\big(r_2^{2\pi/\omega}\big), \quad r_2 \to 0,$$

where a_j are some constants.

5.2.2 Second Kind Limit Problems

In the domains Ω_j, $j = 1, 2$ (introduced in Sect. 5.1) we consider the boundary value problems

$$\begin{aligned} \Delta w(\xi_j, \eta_j) &= F(\xi_j, \eta_j), && (\xi_j, \eta_j) \in \Omega_j, \\ w(\xi_j, \eta_j) &= 0, && (\xi_j, \eta_j) \in \partial\Omega_j, \end{aligned} \tag{5.2.3}$$

which are called the second kind limit problems; (ξ_j, η_j) are Cartesian coordinates with origin at O_j.

Let $\rho_j = \mathrm{dist}((\xi_j, \eta_j), O_j)$ and let $\psi_{0,j}$, $\psi_{\infty,j}$ be smooth real functions in $\overline{\Omega}_j$ such that $\psi_{0,j} = 1$ for $\rho_j < N/2$, $\psi_{0,j} = 0$ for $\rho_j > N$, and $\psi^2_{0,j} + \psi^2_{\infty,j} = 1$, N being a sufficiently large positive number. For $\gamma \in \mathbb{R}$ and $l = 0, 1, \ldots$, the space $V^l_\gamma(\Omega_j)$ is the completion in the norm

$$\|v; V^l_\gamma(\Omega_j)\| = \left(\int_{\Omega_j} \sum_{|\alpha|=0}^{l} \big(\psi_{0,j}(\xi_j, \eta_j)^2 + \psi_{\infty,j}(\xi_j, \eta_j)^2 \rho_j^{2(\gamma - l + |\alpha|)}\big)|\partial^\alpha v(\xi_j, \eta_j)|^2\, d\xi_j d\eta_j\right)^{1/2}$$

of the set $C^\infty_c(\overline{\Omega}_j)$ of smooth functions with compact supports in $\overline{\Omega}_j$. The next proposition is a corollary of [37, Theorem 4.3.6].

Proposition 5.2.3 *Let $|\gamma - 1| < \pi/\omega$. Then, for every $F \in V^0_\gamma(\Omega_j)$, there exists a unique solution $w \in V^2_\gamma(\Omega_j)$ to problem (5.2.3), and*

$$\|w; V^2_\gamma(\Omega_j)\| \le c\,\|F; V^0_\gamma(\Omega_j)\|$$

holds with a constant c independent of F. If $F \in C_c^\infty(\overline{\Omega}_j)$, the w is smooth in $\overline{\Omega}_j$ and admits the representation

$$w(\xi_j, \eta_j) = \begin{cases} d_j^l \rho_j^{-\pi/\omega} \Phi(\pi - \varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j < 0, \\ d_j^r \rho_j^{-\pi/\omega} \Phi(\varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j > 0, \end{cases} \tag{5.2.4}$$

as $\rho_j \to \infty$; here (ρ_j, φ_j) are polar coordinates in Ω_j with center at O_j, and function Φ is the same as in Proposition 5.2.1. The constant coefficients d_j^l and d_j^r are defined by

$$d_j^l = -(F, w_j^l)_\Omega, \quad d_j^r = -(F, w_j^r)_\Omega,$$

where w_j^l and w_j^r are unique solutions to homogeneous problem (5.2.3) such that, as $\rho_j \to \infty$,

$$w_j^l = \begin{cases} \left(\rho_j^{\pi/\omega} + \alpha \rho_j^{-\pi/\omega}\right) \Phi(\pi - \varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j < 0; \\ \beta \rho_j^{-\pi/\omega} \Phi(\varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j > 0; \end{cases} \tag{5.2.5}$$

$$w_j^r = \begin{cases} \beta \rho_j^{-\pi/\omega} \Phi(\pi - \varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j < 0; \\ \left(\rho_j^{\pi/\omega} + \alpha \rho_j^{-\pi/\omega}\right) \Phi(\varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j > 0; \end{cases} \tag{5.2.6}$$

the coefficients α and β depend only on the geometry of the set Ω and should be calculated.

5.3 Special Solutions to the First Kind Homogeneous Problems

Here we introduce special solutions to homogeneous problems (5.2.1) in G_j, $j = 0, 1, 2$. In the domain G_j, $j = 1, 2$, there exists a bounded solution V_j such that

$$V_j(x, y) = \begin{cases} U_j^+(x, y) + S_{jj}^0 U_j^-(x, y) + O(\exp(-\delta x)), & x \to \infty, \\ s_j \widetilde{J}_{\pi/\omega}(kr_j) \Phi_j(\varphi_j) + O\big(r^{2\pi/\omega}\big), & r \to 0, \end{cases} \tag{5.3.1}$$

with arbitrary small positive δ, $\Phi_1(\varphi_1) = \Phi(\pi - \varphi_1)$, and $\Phi_2(\varphi_2) = \Phi(\varphi_2)$. The scattering matrix in G_j consists of the only entry S_j^0, $|S_j^0| = 1$.

Let K^l be the part of the double cone K to the left of the coordinate origin, $K^l = \{(\xi, \eta) \in K : \xi < 0\}$. Let us consider the problem

$$\begin{aligned} -\Delta u - k^2 u = 0 \quad &\text{in } K^l, \\ u = 0 \quad &\text{on } \partial K^l. \end{aligned} \tag{5.3.2}$$

The function

$$v(r,\varphi)=\widetilde{N}_{\pi/\omega}(kr)\Phi(\pi-\varphi) \tag{5.3.3}$$

satisfies (5.3.2); $\widetilde{N}_{\pi/\omega}$ stands for the Neumann function multiplied by a constant such that

$$\widetilde{N}_{\pi/\omega}(kr)=r^{-\pi/\omega}+o(r^{-\pi/\omega})$$

and Φ is the same as in Proposition 5.2.1. Let $t\mapsto\Theta(t)$ be a cut-off function on $\mathbb{R}$ equal to 1 for $t<\delta/2$ and to 0 for $t>\delta$, δ being a small positive number. Introduce a solution

$$\mathbf{v}_1(x,y)=\Theta(r_1)v(r_1,\varphi_1)+\widetilde{v}_1(x,y) \tag{5.3.4}$$

of homogeneous problem (5.2.1) in G_1, where $\widetilde{v}_1$ solves (5.2.1) with $f=-[\Delta,\Theta]v$; the existence of $\widetilde{v}_1$ is provided by Proposition 5.2.2. Thus,

$$\mathbf{v}_1(x,y)=\begin{cases}\big(\widetilde{N}_{\pi/\omega}(kr_1)+a_1\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\pi-\varphi_1)+O(r_1^{3\pi/\omega}), & r_1\to 0,\\ A_1U_1^-(x,y)+O(e^{\delta x}), & x\to-\infty,\end{cases} \tag{5.3.5}$$

where $\widetilde{J}_{\pi/\omega}$ is the same as in Propositions 5.2.1 and 5.2.2. The constant $A\neq 0$ depends only on the geometry of the domain G_1 and should be calculated.

Define the solution $\mathbf{v}_2$ to the problem (5.2.1) in G_2 by $\mathbf{v}_2(x,y)=\mathbf{v}_1(d-x,y)$, where $d=\operatorname{dist}(O_1,O_2)$. Then

$$\mathbf{v}_2(x,y)=\begin{cases}\big(\widetilde{N}_{\pi/\omega}(kr_2)+a_2\widetilde{J}_{\pi/\omega}(kr_2)\big)\Phi(\varphi_2)+O(r_2^{3\pi/\omega}), & r_2\to 0,\\ A_2U_2^-(x,y)+O(e^{-\delta x}), & x\to+\infty;\end{cases} \tag{5.3.6}$$

where obviously $a_2=a_1$, $A_2=A_1e^{-i\nu_1 d}$.

Lemma 5.3.1 *The equalities $|A_j|^2=2\operatorname{Im}a_j$, $A_j=i\overline{s}_jS^0_{jj}$ hold.*

Proof Let $(u,v)_Q$ denote the integral $\int_Q u(x)\overline{v(x)}\,dx$ and let $G_{N,\delta}$ stand for the truncated domain $G_1\cap\{x>-N\}\cap\{r_1>\delta\}$. By the Green formula,

$$\begin{aligned}0&=(\Delta\mathbf{v}_1+k^2\mathbf{v}_1,\mathbf{v}_1)_{G_{N,\delta}}-(\mathbf{v}_1,\Delta\mathbf{v}_1+k^2\mathbf{v}_1)_{G_{N,\delta}}\\&=(\partial\mathbf{v}_1/\partial n,\mathbf{v}_1)_{\partial G_{N,\delta}}-(\mathbf{v}_1,\partial\mathbf{v}_1/\partial n)_{\partial G_{N,\delta}}=2i\operatorname{Im}(\partial\mathbf{v}_1/\partial n,\mathbf{v}_1)_E,\end{aligned}$$

where $E=(\partial G_{N,\delta}\cap\{x=-N\})\cup(\partial G_{N,\delta}\cap\{r_1=\delta\})$. Taking into account (5.3.5) as $x\to+\infty$ and (5.1.5), we have

$$\operatorname{Im}(\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta}\cap\{x=-N\}} = -\operatorname{Im}\int_{-l/2}^{l/2} A_1 \frac{\partial U_1^-}{\partial x}(x,y)\overline{A_1 U_1^-(x,y)}\Big|_{x=-N} dy + o(1)$$
$$= |A_1|^2 \nu_1 \int_{-l/2}^{l/2} |\Psi_1(y)|^2 dy + o(1) = |A_1|^2 + o(1).$$

Using (5.3.5) as $r_1 \to 0$ and the definition of Φ (see Proposition 5.2.1), we obtain

$$\operatorname{Im}(\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta}\cap\{r_1=\delta\}} = \operatorname{Im}\int_{\pi-\omega/2}^{\pi+\omega/2} \left[-\frac{\partial}{\partial r_1}\left(\widetilde{N}_{\pi/\omega}(kr_1) + a_1 \widetilde{J}_{\pi/\omega}(kr_1)\right)\right]$$
$$\times \left(\widetilde{N}_{\pi/\omega}(kr_1) + \overline{a_1}\widetilde{J}_{\pi/\omega}(kr_1)\right)|\Phi(\pi-\varphi_1)|^2 r_1\Big|_{r_1=\delta} d\varphi_1 + o(1)$$
$$= -(\operatorname{Im} a_1)\frac{2\pi}{\omega}\int_{\pi-\omega/2}^{\pi+\omega/2} |\Phi(\pi-\varphi_1)|^2 d\varphi_1 + o(1)$$
$$= -\operatorname{Im} a_1 + o(1).$$

Thus $|A_1|^2 - \operatorname{Im} a_1 + o(1) = 0$ as $N \to \infty$ and $\delta \to 0$. The relation for $j = 2$ can be verified in a similar way. To obtain $A_j = i\overline{s}_j S^0_{jj}$, it suffices in the same manner to apply the Green formula to the functions V_j and $\mathbf{v}_j$. □

Let k_e^2 be a simple eigenvalue for $-\Delta$ with Dirichlet boundary condition in G_0, and let v_e be an eigenfunction corresponding to k_e^2 and normalized by $\int_{G_0} |v_e|^2 dx = 1$. By Proposition 5.2.1,

$$v_e(x) \sim \begin{cases} b_1 \widetilde{J}_{\pi/\omega}(k_e r_1)\Phi(\varphi_1), & r_1 \to 0, \\ b_2 \widetilde{J}_{\pi/\omega}(k_e r_2)\Phi(\pi-\varphi_2), & r_2 \to 0. \end{cases} \quad (5.3.7)$$

We assume that $b_j \neq 0$; it is true, e.g., for the eigenfunction corresponding to the least eigenvalue of the resonator. Since the resonator is symmetric with respect to the mapping $(x, y) \mapsto (d-x, y)$, we have $q = b_1/b_2 = \pm 1$. For k^2 in a punctured neighborhood of k_e^2 separated from the other eigenvalues, we introduce solutions v_{0j} to homogeneous problem (5.2.1) in G_0 by

$$\mathbf{v}_{0j}(x, y) = \Theta(r_j)v(r_j, \varphi_j) + \widetilde{v}_{0j}(x, y), \quad j = 1, 2, \quad (5.3.8)$$

where v is defined by (5.3.3) and $\widetilde{v}_{0j}$ is the bounded solution to problem (5.2.1) in G_0 with $f_j(x, y) = [\Delta, \Theta(r_j)]v(r_j, \varphi_j)$.

Lemma 5.3.2 *In a neighborhood $V \subset \mathbb{C}$ of k_e^2 containing no eigenvalues of problem (5.2.1) in G_2 except k_e^2, the equalities $\mathbf{v}_{0j} = -b_j(k^2 - k_e^2)^{-1}v_e + \widehat{\mathbf{v}}_{0j}$ hold with b_j from (5.3.7) and functions $\widehat{\mathbf{v}}_{0j}$ analytic in $k^2 \in V$.*

Proof Let us first prove that $(\mathbf{v}_{0j}, v_e)_{G_0} = -b_j/(k^2 - k_e^2)$, with $\mathbf{v}_{0j}$ defined by (5.3.8). We have

$$(\triangle \mathbf{v}_{0j} + k^2 \mathbf{v}_{0j}, v_e)_{G_\delta} - (\mathbf{v}_{0j}, \triangle v_e + k^2 v_e)_{G_\delta} = -(k^2 - k_e^2)(\mathbf{v}_{0j}, v_e)_{G_\delta}$$

in the domain G_δ obtained from G_0 by cutting out the balls of radius δ centered at O_1 and O_2. Applying the Green formula as in the proof of Lemma 5.3.1, we arrive at $-(k^2 - k_e^2)(\mathbf{v}_{0j}, v_e)_{G_\delta} = b_j + o(1)$. It remains to let δ go to zero.

Since k_e^2 is a simple eigenvalue, we have

$$\widetilde{v}_{0j}(x) = \frac{B_j(k^2)}{k^2 - k_e^2} v_e(x) + \widehat{v}_j(x), \tag{5.3.9}$$

where $B_j(k^2)$ is independent of x and $\widehat{v}_j$ are some functions analytic in k^2 near the point $k^2 = k_e^2$. Multiplying (5.3.8) by v_e and taking into account (5.3.9), the proved formula for $(\mathbf{v}_{0j}, v_e)_{G_0}$, and the normalization condition $(v_e, v_e)_{G_0} = 1$, we find that $B_j(k^2) = -b_j + (k^2 - k_e^2)\widetilde{B}_j(k^2)$ with analytic function $\widetilde{B}_j$. Together with (5.3.9), this leads to the required statement. □

In view of Proposition 5.2.1,

$$\mathbf{v}_{01}(x, y) \sim \begin{cases} \big(\widetilde{N}_{\pi/\omega}(kr_1) + c_{11}(k)\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\varphi_1), & r_1 \to 0, \\ c_{12}(k)\widetilde{J}_{\pi/\omega}(kr_2)\Phi(\pi - \varphi_2), & r_2 \to 0, \end{cases} \tag{5.3.10}$$

$$\mathbf{v}_{02}(x, y) \sim \begin{cases} \big(c_{21}(k)\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\varphi_1), & r_1 \to 0, \\ \big(\widetilde{N}_{\pi/\omega}(kr_2) + c_{22}(k)\widetilde{J}_{\pi/\omega}(kr_2)\big)\Phi(\pi - \varphi_2), & r_2 \to 0. \end{cases} \tag{5.3.11}$$

According to Lemma 5.3.2 and relations (5.3.7),

$$c_{pq}(k) = -\frac{b_p b_q}{k^2 - k_e^2} + \widehat{c}_{pq}(k), \tag{5.3.12}$$

where $\widehat{c}_{pq}$ analytically depends on k^2 nearby k_e^2.

Lemma 5.3.3 *If* $\mathbf{v}_{01}$ *and* $\mathbf{v}_{02}$ *in* (5.3.10) *and* (5.3.11) *make sense for a number* k, *then* $c_{12}(k) = c_{21}(k)$.

Proof It suffices to apply the Green formula to $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$ in the same domain G_δ as in the proof of Lemma 5.3.2, to use (5.3.10) and (5.3.11), and to let δ tend to 0. □

5.4 Asymptotic Formulas

This section is devoted to derivation of the asymptotic formulas. In Sect. 5.4.1, we present the formula for the wave function (see (5.4.1)), explain its structure, and describe the solutions of the first kind limit problems involved in the formula. Construction of formula (5.4.1) is completed in Sect. 5.4.2, where the solutions to the second kind limit problems are given and the coefficients in the expressions for the solutions of the first kind limit problems are calculated. In Sect. 5.4.3, we analyze the expression for $\widetilde{s}_{12}$ obtained in 5.4.2 and derive formal asymptotics for the characteristics of resonant tunneling. Notice, that the remainders in (5.4.24)–(5.4.27) arose at the intermediate stage of considerations on simplifying the principal part of the asymptotics; they are not the remainders in the final asymptotic formulas. The "final" remainders are estimated in Sect. 5.5 (see Theorem 5.5.3). First, we derive the integral estimate (5.5.21) for the remainder in the formula (5.4.1), which proves to be sufficient to obtain more simplified estimates of the remainders in the formulas for the characteristics of resonant tunneling. The formula (5.4.1) and the estimate (5.5.21) are auxiliary and are investigated only to that extent which is necessary for deriving the asymptotic expressions for the characteristics of resonant tunneling.

5.4.1 Asymptotics of the Wave Function

In the waveguide $G(\varepsilon)$, we consider the scattering of the wave $U_1^+(x, y) = e^{i\nu_1 x}\Psi_1(y)$ incoming from $-\infty$. The wave function admits the representation

$$\begin{aligned} u(x, y; \varepsilon) = {} & \chi_{1,\varepsilon}(x, y)v_1(x, y; \varepsilon) \\ & + \Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon) + \chi_{0,\varepsilon}(x, y)v_0(x, y; \varepsilon) \\ & + \Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon) + \chi_{2,\varepsilon}(x, y)v_2(x, y; \varepsilon) + R(x, y; \varepsilon). \end{aligned} \tag{5.4.1}$$

Let us explain the notation and the structure of this formula. When composing the formula, we first describe the behavior of the wave function u outside of the narrows, where the solutions v_j to homogeneous problems (5.2.1) in G_j serve as approximations to u. The function v_j is a linear combination of the special solutions introduced in the previous section; v_1 and v_3 are subject to the same radiation conditions as u:

$$\begin{aligned} v_1(x, y; \varepsilon) = V_1(x, y) + C_{11}\mathbf{v}_1(x, y) \sim {} & U_1^+(x, y) \\ & + \widetilde{S}_{11}(\varepsilon)U_1^-(x, y), \quad x \to -\infty; \end{aligned} \tag{5.4.2}$$

$$v_0(x, y; \varepsilon) = C_{12}(\varepsilon)\mathbf{v}_{01}(x, y) + C_{13}(\varepsilon)\mathbf{v}_{02}(x, y); \tag{5.4.3}$$

$$v_2(x, y; \varepsilon) = C_{14}\mathbf{v}_2(x, y) \sim \widetilde{S}_{12}(\varepsilon)U_2^-(x, y), \quad x \to +\infty; \tag{5.4.4}$$

the approximations $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$ to the elements $S_{11}(\varepsilon)$, $S_{12}(\varepsilon)$ of the scattering matrix and the coefficients $C_{11}(\varepsilon), \dots, C_{14}(\varepsilon)$ are yet unknown. By $\chi_{j,\varepsilon}$ we denote cut-off functions defined by

$$\begin{aligned} & \chi_{1,\varepsilon}(x, y) = (1 - \Theta(r_1/\varepsilon))\,\mathbf{1}_{G_1}(x, y), \quad \chi_{2,\varepsilon}(x, y) = (1 - \Theta(r_2/\varepsilon))\,\mathbf{1}_{G_2}(x, y), \\ & \chi_{0,\varepsilon}(x, y) = (1 - \Theta(r_1/\varepsilon) - \Theta(r_2/\varepsilon))\,\mathbf{1}_{G_0}(x, y), \end{aligned}$$

where $r_j = \sqrt{x_j^2 + y_j^2}$, and (x_j, y_j) are the coordinates of a point (x, y) in the system obtained by shifting the origin to the point O_j; $\mathbf{1}_{G_j}$ is the indicator of G_j (equal to 1 in G_j and to 0 outside G_j); $\Theta(\rho)$ is the same cut-off function as in (5.3.4) (equal to 1 for $0 \leqslant \rho \leqslant \delta/2$ and to 0 for $\rho \geqslant \delta$, δ being a fixed positive number). Thus, $\chi_{j,\varepsilon}$ are defined on the whole waveguide $G(\varepsilon)$ as well as the functions $\chi_{j,\varepsilon} v_j$ in (5.4.1).

Being substituted to (5.1.1), the sum $\sum_{j=0}^{2} \chi_{j,\varepsilon} v_j$ gives a discrepancy in the right-hand side of the Helmholtz equation supported near the narrows. We compensate the principal part of the discrepancy by means of the second kind limit problems. Namely, the discrepancy supported in the neighborhood of the point O_j is rewritten into coordinates $(\xi_j, \eta_j) = (\varepsilon^{-1} x_j, \varepsilon^{-1} y_j)$ in the domain Ω_j and is taken as a right-hand side for the Laplace equation. The solutions w_j of the corresponding problem (5.2.3) are rewritten into coordinates (x_j, y_j) and multiplied by a cut-off function. As a result, the terms $\Theta(r_j) w_j(\varepsilon^{-1} x_j, \varepsilon^{-1} y_j; \varepsilon)$ arise in (5.4.1).

Proposition 5.2.3 provides the existence of solutions w_j decaying at infinity as $O(\rho_j^{-\pi/\omega})$ (see (5.2.4)). But those solutions will not lead us to the goal, because substitution of (5.4.1) into (5.1.1) gives a high-order discrepancy, which has to be compensated again. Therefore, we require the rate $w_j = O(\rho_j^{-3\pi/\omega})$ as $\rho_j \to \infty$. By Proposition 5.2.3, such a solution exists if the right-hand side of problem (5.2.3) satisfies the additional conditions

$$(F, w_j^l)_{\Omega_j} = 0, \quad (F, w_j^r)_{\Omega_j} = 0.$$

These conditions (two at each narrow) uniquely determine the coefficients $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$, and $C_{11}(\varepsilon), \dots, C_{14}(\varepsilon)$. The remainder $R(x, y; \varepsilon)$ is small in comparison with the principal part of (5.4.1) as $\varepsilon \to 0$.

5.4.2 Formulas for $\widetilde{S}_{11}$, $\widetilde{S}_{12}$, and C_{1j}

Now, let us specify the right-hand sides F_j of the problems (5.2.3) and find $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$, and $C_{1j}(\varepsilon)$. Substituting $\chi_{1,\varepsilon} v_1$ into (5.1.1), we get the discrepancy

$$-(\Delta + k^2)\chi_{1,\varepsilon} v_1 = -[\Delta, \chi_{\varepsilon,1}] v_1 - \chi_{\varepsilon,1}(\Delta + k^2) v_1 = -[\Delta, 1 - \Theta(\varepsilon^{-1} r_1)] v_1,$$

which is non-zero in the neighborhood of the point O_1, where v_1 can be replaced by asymptotics; the boundary condition in (5.1.1) is fulfilled. According to (5.4.2) and (5.3.1), (5.3.5)

$$\begin{aligned} v_1(x, y; \varepsilon) = &\big(a_1^-(\varepsilon)\widetilde{N}_{\pi/\omega}(kr_1) + a_1^+(\varepsilon)\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\pi - \varphi_1) \\ &+ O(r_1^{3\pi/\omega}), \quad r_1 \to 0, \end{aligned}$$

where

$$a_1^-(\varepsilon) = C_{11}, \quad a_1^+ = s_1 + C_{11}a_1. \tag{5.4.5}$$

We select the leading term in each summand, take $\rho_1 = r_1/\varepsilon$, and obtain

$$\begin{aligned}-(\Delta + k^2)\chi_{\varepsilon,1}v_1 &\sim -[\Delta, 1 - \Theta(\varepsilon^{-1}r_1)] \left(a_1^- r_1^{-\pi/\omega} + a_1^+ r_1^{\pi/\omega}\right) \Phi(\pi - \varphi_1)\\ &= -\varepsilon^{-2}[\Delta_{(\rho_1,\varphi_1)}, 1 - \Theta(\rho_1)] \left(a_1^- \varepsilon^{-\pi/\omega}\rho_1^{-\pi/\omega} + a_1^+ \varepsilon^{\pi/\omega}\rho_1^{\pi/\omega}\right) \Phi(\pi - \varphi_1).\end{aligned} \tag{5.4.6}$$

In the same way, taking account of (5.4.3), (5.3.10), and (5.3.11), we write the leading discrepancy of $\chi_{\varepsilon,0}v_0$ supported in a neighborhood of O_1:

$$-(\Delta+k^2)\chi_{\varepsilon,0}v_0 \sim -\varepsilon^{-2}[\Delta_{(\rho_1,\varphi_1)}, 1-\Theta(\rho_1)] \left(b_1^- \varepsilon^{-\pi/\omega}\rho_1^{-\pi/\omega} + b_1^+ \varepsilon^{\pi/\omega}\rho_1^{\pi/\omega}\right) \Phi(\varphi_1), \tag{5.4.7}$$

where

$$b_1^- = C_{12}(\varepsilon), \quad b_1^+ = C_{12}(\varepsilon)c_{11} + C_{13}(\varepsilon)c_{21}. \tag{5.4.8}$$

As a right-hand side F_1 of problem (5.2.3) in Ω_1, we take the function

$$\begin{aligned}F_1(\xi_1, \eta_1) = &- [\Delta, \zeta^-] \left(a_1^- \varepsilon^{-\pi/\omega}\rho_1^{-\pi/\omega} + a_1^+ \varepsilon^{\pi/\omega}\rho_1^{\pi/\omega}\right) \Phi(\pi - \varphi_1)\\ &- [\Delta, \zeta^+] \left(b_1^- \varepsilon^{-\pi/\omega}\rho_1^{-\pi/\omega} + b_1^+ \varepsilon^{\pi/\omega}\rho_1^{\pi/\omega}\right) \Phi(\varphi_1),\end{aligned} \tag{5.4.9}$$

where ζ^+ (respectively ζ^-) denotes the function $1 - \Theta$, first restricted to the domain $\xi_1 > 0$ (respectively $\xi_1 < 0$) and then extended by zero to the whole domain Ω_1. Let w_1 be the corresponding solution; then the term $\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon)$ in (5.4.1), being substituted in (5.1.1), compensates discrepancies (5.4.6) and (5.4.7).

Now, we use (5.4.3) and (5.4.4), (5.3.10) and (5.3.11), and (5.3.6) to find the right-hand side of problem (5.2.3) for $j = 2$:

$$\begin{aligned}F_2(\xi_2, \eta_2) = &- [\Delta, \zeta^-] \left(a_2^- \varepsilon^{-\pi/\omega}\rho_2^{-\pi/\omega} + a_2^+ \varepsilon^{\pi/\omega}\rho_2^{\pi/\omega}\right) \Phi(\pi - \varphi_2)\\ &- [\Delta, \zeta^+] \left(b_2^- \varepsilon^{-\pi/\omega}\rho_2^{-\pi/\omega} + b_2^+ \varepsilon^{\pi/\omega}\rho_2^{\pi/\omega}\right) \Phi(\varphi_2),\end{aligned}$$

where

$$\begin{aligned}&a_2^-(\varepsilon) = C_{13}(\varepsilon), \quad a_2^+(\varepsilon) = C_{12}(\varepsilon)c_{12} + C_{13}(\varepsilon)c_{22},\\ &b_2^-(\varepsilon) = C_{14}(\varepsilon), \quad b_2^+(\varepsilon) = C_{14}(\varepsilon)a_2.\end{aligned} \tag{5.4.10}$$

Lemma 5.4.1 *Let the solution w_j to problem (5.2.3) with right-hand side*

$$F_j(\xi_j,\eta_j) = -[\Delta,\zeta^-]\left(a_j^-\varepsilon^{-\pi/\omega}\rho_j^{-\pi/\omega} + a_j^+\varepsilon^{\pi/\omega}\rho_j^{\pi/\omega}\right)\Phi(\pi-\varphi_j)$$
$$-[\Delta,\zeta^+]\left(b_j^-\varepsilon^{-\pi/\omega}\rho_j^{-\pi/\omega} + b_j^+\varepsilon^{\pi/\omega}\rho_j^{\pi/\omega}\right)\Phi(\varphi_j),$$

$j=1,2$, be majorized by $O(\rho_j^{-3\pi/\omega})$ as $\rho_j\to\infty$. Then the relations

$$a_j^-\varepsilon^{-\pi/\omega} - \alpha a_j^+\varepsilon^{\pi/\omega} - \beta b_j^+\varepsilon^{\pi/\omega} = 0,\quad b_j^-\varepsilon^{-\pi/\omega} - \alpha b_j^+\varepsilon^{\pi/\omega} - \beta a_j^+\varepsilon^{\pi/\omega} = 0, \tag{5.4.11}$$

hold with α and β in (5.2.5) and (5.2.6).

Proof In view of Proposition 5.2.3, we have $w_j = O(\rho_j^{-3\pi/\omega})$ as $\rho_j\to\infty$ iff the right-hand side of the problem (5.2.3) satisfies the conditions

$$(F_j, w_j^l)_{\Omega_j} = 0,\quad (F_j, w_j^r)_{\Omega_j} = 0, \tag{5.4.12}$$

where w_j^l and w_j^r are solutions to the homogeneous problem (5.2.3), for which the expansions (5.2.5)–(5.2.6) hold. We introduce the functions $f_\pm$ on Ω_j by equalities $f_\pm(\rho_j,\varphi_j) = \rho_j^{\pm\pi/\omega}\Phi(\varphi_j)$. To derive (5.4.11) from (5.4.12), it suffices to check that

$$([\Delta,\zeta^-]f_-, w_j^l)_{\Omega_j} = ([\Delta,\zeta^+]f_-, w_j^r)_{\Omega_j} = -1,$$
$$([\Delta,\zeta^-]f_+, w_j^l)_{\Omega_j} = ([\Delta,\zeta^+]f_+, w_j^r)_{\Omega_j} = \alpha,$$
$$([\Delta,\zeta^+]f_-, w_j^l)_{\Omega_j} = ([\Delta,\zeta^-]f_-, w_j^r)_{\Omega_j} = 0,$$
$$([\Delta,\zeta^+]f_+, w_j^l)_{\Omega_j} = ([\Delta,\zeta^-]f_+, w_j^r)_{\Omega_j} = \beta.$$

Let us prove the first equality; the rest are treated analogously. Since $[\Delta,\zeta^+]f_-$ is compactly supported, in the calculation of $([\Delta,\zeta^-]f_-, w_j^l)_{\Omega_j}$ one may replace Ω_j by $\Omega_j^R = \Omega_j\cap\{\rho_j<R\}$ with sufficiently large R. Let E denote the set $\partial\Omega_j^R\cap\{\rho_j=R\}\cap\{\xi_j>0\}$. By the Green formula

$$([\Delta,\zeta^-]f_-, w_j^l)_{\Omega_j} = (\Delta\zeta^- f_-, w_j^l)_{\Omega_j^R} - (\zeta^- f_-, \Delta w_j^l)_{\Omega_j^R}$$
$$= (\partial f_-/\partial n, w_j^l)_E - (f_-, \partial w_j^l/\partial n)_E.$$

Considering (5.2.5) for $\xi_j<0$ and the definition of Φ from Proposition 5.2.1, we arrive at

$$([\Delta,\zeta^-]f_-,w_j^l)_{\Omega_j}=\left[\frac{\partial\rho_j^{-\pi/\omega}}{\partial\rho_j}(\rho_j^{\pi/\omega}+\alpha\rho_j^{-\pi/\omega})-\rho_j^{-\pi/\omega}\frac{\partial}{\partial\rho_j}(\rho_j^{\pi/\omega}+\alpha\rho_j^{-\pi/\omega})\right]\rho_j\Big|_{\rho_j=R}$$
$$\times\int_{\pi-\omega/2}^{\pi+\omega/2}\Phi(\pi-\varphi_j)^2d\varphi_j+o(1)$$
$$=-\frac{2\pi}{\omega}\int_{\pi-\omega/2}^{\pi+\omega/2}\Phi(\pi-\varphi_j)^2d\varphi_j+o(1)=-1+o(1).$$

It remains to pass to the limit as $R\to\infty$. □

Remark 5.4.2 The solutions w_j mentioned in Lemma 5.4.1 can be represented as linear combinations of functions independent of ε. Let w_j^l and w_j^r be the solutions of problem (5.2.3) specified by conditions (5.2.5) and (5.2.6) and let ζ^+ and ζ^- be the same cut-off functions as in (5.4.9). Put

$$\mathbf{w}_j^l=w_j^l-\zeta^-\left(\rho_j^{\pi/\omega}+\alpha\rho_j^{-\pi/\omega}\right)\Phi(\pi-\varphi_j)-\zeta^+\beta\rho_j^{-\pi/\omega}\Phi(\varphi_j),$$
$$\mathbf{w}_j^r=w_j^r-\zeta^-\beta\rho_j^{-\pi/\omega}\Phi(\pi-\varphi_j)-\zeta^+\left(\rho_j^{\pi/\omega}+\alpha\rho_j^{-\pi/\omega}\right)\Phi(\varphi_j).$$

A straightforward verification shows that

$$w_j=a_j^+\varepsilon^{\pi/\omega}\mathbf{w}_j^l+b_j^+\varepsilon^{\pi/\omega}\mathbf{w}_j^r. \tag{5.4.13}$$

It is convenient to write (5.4.11) in the form

$$(a_j^-,b_j^-)=(a_j^+,b_j^+)\,\Lambda\,\varepsilon^{2\pi/\omega},\quad \Lambda=\begin{pmatrix}\alpha&\beta\\\beta&\alpha\end{pmatrix}. \tag{5.4.14}$$

We use (5.4.5) and (5.4.8) to transform (5.4.14) with $j=1$ to the equality

$$(C_{11},C_{12})=(s_1+C_{11}a_1,C_{12}c_{11}+C_{13}c_{21})\,\Lambda\,\varepsilon^{2\pi/\omega}. \tag{5.4.15}$$

For $j=2$, taking (5.4.10) into account, we reduce (5.4.14) to

$$(C_{13},C_{14})=(C_{12}c_{12}+C_{13}c_{22},C_{14}\,a_2)\,\Lambda\,\varepsilon^{2\pi/\omega}. \tag{5.4.16}$$

Setting $\boldsymbol{\Lambda}=\operatorname{diag}\{\Lambda,\Lambda\}$,

$$a=\begin{pmatrix}a_1&0&0&0\\0&c_{11}&c_{12}&0\\0&c_{21}&c_{22}&0\\0&0&0&a_2\end{pmatrix}, \tag{5.4.17}$$

and combining the above relations for C_{1j}, we obtain

$$(C_{11}, C_{12}, C_{13}, C_{14}) = (s_1, 0, 0, 0)\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega} + (C_{11}, C_{12}, C_{13}, C_{14})\, a\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega},$$

hence

$$(C_{11}, C_{12}, C_{13}, C_{14})(I - a\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}) = (s_1, 0, 0, 0)\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}. \tag{5.4.18}$$

Let us calculate the inverse matrix for $I - a\mathbf{\Lambda}\varepsilon^{2\pi/\omega}$, assuming ε to be sufficiently small. From (5.3.12) it follows that

$$a(k) = -\frac{\mathbf{b}^*\mathbf{b}}{k^2 - k_e^2} + \widehat{a}(k),$$

where $\mathbf{b} = (0,\, b_1,\, b_2,\, 0)$ and the matrix $\widehat{a}$ is analytic near $k = k_e$ and defined by (5.4.17), whereas c_{pq} is replaced for $\widehat{c}_{pq}$. We have

$$\begin{aligned} I - a\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega} &= I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega} + \frac{\mathbf{b}^*\mathbf{b}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}}{k^2 - k_e^2} \\ &= \left(I + \frac{\mathbf{b}^*\mathbf{b}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}}{k^2 - k_e^2}\right)(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}); \end{aligned}$$

it is evident that $(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}$ exists for small ε. Straightforward calculation shows that

$$\left(I + \frac{\mathbf{b}^*\mathbf{c}}{k^2 - k_e^2}\right)^{-1} = I - \frac{\mathbf{b}^*\mathbf{c}}{k^2 - k_e^2 + \langle \mathbf{c}, \mathbf{b} \rangle}$$

for $\mathbf{c} = \mathbf{b}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^4$. Therefore,

$$\begin{aligned} (I - a\mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1} &= (I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1} \\ &\quad \times \left(I - \frac{\mathbf{b}^*\mathbf{b}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}}{k^2 - k_e^2 + \langle \mathbf{b}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}, \mathbf{b} \rangle}\right). \end{aligned}$$

This leads to

$$\begin{aligned} (C_{11}, C_{12}, C_{13}, C_{14}) &= (s_1, 0, 0, 0)\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}(I - a\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1} \\ &= (s_1, 0, 0, 0)\left(D - \frac{D\, \mathbf{b}^*\mathbf{b}\, D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b} \rangle}\right), \end{aligned} \tag{5.4.19}$$

where $\mathbf{b} = (0,\, b_1,\, b_2,\, 0)$, $D = \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega}(I - \widehat{a}\, \mathbf{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}$ and the matrix $\widehat{a}$ is analytic in k near k_e and defined by (5.4.17) with c_{pq} replaced by $\widehat{c}_{pq}$ (see (5.3.12)).

We now seek an approximation to the entries of the first row (S_{11}, S_{12}) of the scattering matrix. By virtue of (5.4.2) and (5.4.4),

$$(\widetilde{S}_{11}, \widetilde{S}_{12}) = (S_{11}^0 + C_{11}A_1, C_{14}A_2). \tag{5.4.20}$$

We set

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & A_2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 \end{pmatrix};$$

$S^0 = \operatorname{diag}(S_{11}^0, S_{22}^0)$; then, by Lemma 5.3.1, $A = is^*S^0$. In view of (5.4.20) and (5.4.19), we obtain

$$\begin{aligned}(\widetilde{S}_{11}, \widetilde{S}_{12}) &= (S_{11}^0, 0) + (C_{11}, C_{12}, C_{13}, C_{14})A \\ &= (S_{11}^0, 0) + i(s_1, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0.\end{aligned} \tag{5.4.21}$$

An approximation to the second row of the scattering matrix is of the form

$$(\widetilde{S}_{21}, \widetilde{S}_{22}) = (0, S_{22}^0) + i(0, 0, 0, s_2)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0. \tag{5.4.22}$$

Lemma 5.4.3 *The matrix $\widetilde{S}(\varepsilon)$ is unitary.*

Proof Let B temporarily denote the matrix $(I - a\Lambda\varepsilon^{2\pi/\omega})^{-1}\Lambda\varepsilon^{2\pi/\omega}$. Since $(S^0)^* S^0 = I$, the equalities

$$\begin{aligned}\widetilde{S}(\varepsilon)\widetilde{S}(\varepsilon)^* = \widetilde{S}(\varepsilon)(S^0)^*S^0\widetilde{S}(\varepsilon)^* &= (I + isBs^*)(I - isB^*s^*) \\ &= I + is(B - B^* - iBs^*sB^*)s^*\end{aligned}$$

hold. We have to show that $B - B^* - iBs^*sB^* = 0$. By Lemma 5.3.1,

$$a - a^* = iAA^* = i(is^*S^0)(is^*S^0)^* = is^*s$$

and, consequently,

$$\begin{aligned}B - B^* - iBs^*sB^* &= B - B^* - B(a - a^*)B^* \\ &= B(I + a^*B^*) - (I + Ba)B^*.\end{aligned}$$

Moreover,

$$I + Ba = I + (I - \Lambda a\varepsilon^{2\mu_1+1})^{-1}\Lambda a\varepsilon^{2\mu_1+1} = (I - \Lambda a\varepsilon^{2\mu_1+1})^{-1},$$
$$I + a^* B^* = (I + Ba)^* = (I - a^*\Lambda\varepsilon^{2\mu_1+1})^{-1},$$

whence

$$B(I + a^* B^*) - (I + Ba)B^* = 0.$$

□

5.4.3 Formulas for Resonant Tunneling Characteristics

The solutions of the first kind limit problems involved in (5.4.1) are defined for complex k as well. Expressions (5.4.21) and (5.4.22) for $\widetilde{S}$ have a pole k_p in the lower complex half-plane. To find k_p^2 we equate $k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle$ to zero and solve the equation for $k^2 - k_e^2$:

$$k^2 - k_e^2 = -\langle \mathbf{b}D, \mathbf{b}\rangle = -\varepsilon^{2\pi/\omega}\langle \mathbf{b}\Lambda\,(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}, \mathbf{b}\rangle. \tag{5.4.23}$$

Since the right-hand side of the last equation behaves like $O(\varepsilon^{2\pi/\omega})$ as $\varepsilon \to 0$, it may be solved by the successive approximation method. Considering the formulas $b_1 = \pm b_2$, $\operatorname{Im} a_1 = \operatorname{Im} a_2 = |s_1|^2/2$, which follow from the waveguide symmetry and Lemma 5.3.1, and discarding the lower order terms, we get $k_p^2 = k_r^2 - ik_i^2$, where

$$k_r^2 = k_e^2 - 2\alpha b_1^2\varepsilon^{2\pi/\omega} + O(\varepsilon^{4\pi/\omega}), \quad k_i^2 = \beta^2 b_1^2 |s_1(k_e^2)|^2\varepsilon^{4\pi/\omega} + O(\varepsilon^{6\pi/\omega}). \tag{5.4.24}$$

From (5.4.21) and (5.4.22), we obtain

$$\begin{aligned}
\widetilde{S}(k,\varepsilon) =& S^0(k) + is(k)\Lambda\, s^*(k)S^0(k)\varepsilon^{2\pi/\omega} - i\frac{s(k)\Lambda\,\mathbf{b}^*\mathbf{b}\,\Lambda\, s^*(k)S^0(k)}{k^2 - k_p^2}\varepsilon^{4\pi/\omega}\\
&+ O\left(\frac{\varepsilon^{6\pi/\omega}}{k^2 - k_p^2}\right)\\
=& \begin{pmatrix} S_{11}^0(k) & 0\\ 0 & S_{22}^0(k)\end{pmatrix} + i\begin{pmatrix} |s_1(k)|^2\alpha_1 S_{11}^0(k) & 0\\ 0 & |s_2(k)|^2\alpha_2 S_{22}^0(k)\end{pmatrix}\varepsilon^{2\pi/\omega}\\
&- \frac{i}{k^2 - k_p^2}\begin{pmatrix} |s_1(k)|^2 b_1^2\beta^2 S_{11}^0(k) & s_1(k)\overline{s_2(k)}b_1 b_2\beta^2 S_{22}^0(k)\\ s_2(k)\overline{s_1(k)}b_1 b_2\beta^2 S_{11}^0(k) & |s_2(k)|^2 b_2^2\beta^2 S_{22}^0(k)\end{pmatrix}\varepsilon^{4\pi/\omega}\\
&+ O\left(\frac{\varepsilon^{6\pi/\omega}}{k^2 - k_p^2}\right).
\end{aligned}$$

Let $k^2 - k_e^2 = O(\varepsilon^{2\pi/\omega})$, then $c\varepsilon^{4\pi/\omega} \leq |k^2 - k_p^2| \leq c\varepsilon^{2\pi/\omega}$, $s_j(k) = s_j(k_e) + O(\varepsilon^{2\pi/\omega})$, $S^0_{jj}(k) = S^0_{jj}(k_e) + O(\varepsilon^{2\pi/\omega})$, and

$$\begin{aligned}
\widetilde{S}_{12}(k,\varepsilon) &= -i\varepsilon^{4\pi/\omega}\frac{s_1(k)\overline{s_2(k)}b_1 b_2\beta^2 S^0_{22}(k)}{k^2-k_p^2}\left(1+O(\varepsilon^{2\pi/\omega})\right)\\
&= -\frac{q\dfrac{s_1(k_e)}{|s_1(k_e)|}\dfrac{\overline{s_2(k_e)}}{|s_2(k_e)|}S^0_{22}(k_e)}{1-iP\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}}\left(1+O(\varepsilon^{2\pi/\omega})\right), \qquad (5.4.25)\\
\widetilde{S}_{21}(k,\varepsilon) &= -i\varepsilon^{4\pi/\omega}\frac{\overline{s_1(k)}s_2(k)b_1 b_2\beta^2 S^0_{11}(k)}{k^2-k_p^2}\left(1+O(\varepsilon^{2\pi/\omega})\right)\\
&= -\frac{q\dfrac{\overline{s_1(k_e)}}{|s_1(k_e)|}\dfrac{s_2(k_e)}{|s_2(k_e)|}S^0_{11}(k_e)}{1-iP\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}}\left(1+O(\varepsilon^{2\pi/\omega})\right),
\end{aligned}$$

where $q = b_2/b_1$ and $P = (b_1^2\beta^2|s_1(k_e)|^2)^{-1}$. Thus,

$$\widetilde{T}_1(k,\varepsilon) = \widetilde{T}_2(k,\varepsilon) = |\widetilde{S}_{12}|^2 = \frac{1}{1+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}(1+O(\varepsilon^{2\pi/\omega})). \qquad (5.4.26)$$

The obtained approximation $\widetilde{T}_j$ to the transition coefficient T_j has a peak at $k^2 = k_r^2$ whose width at its half-height is

$$\widetilde{\Upsilon}(\varepsilon) = \frac{2}{P}\varepsilon^{4\pi/\omega}. \qquad (5.4.27)$$

5.5 Justification of the Asymptotics

Introduce functional spaces for the problem

$$-\Delta u - k^2 u = f \quad \text{in } G(\varepsilon), \quad u = 0 \quad \text{on } \partial G(\varepsilon). \qquad (5.5.1)$$

Let Θ be the same function as in (5.3.4) and let the cut-off functions η_j, $j = 0, 1, 2$, be nonzero in G_j and satisfy the relation $\eta_1(x,y) + \Theta(r_1) + \eta_0(x,y) + \Theta(r_2) + \eta_2(x,y) = 1$ in $G(\varepsilon)$. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G(\varepsilon))$ is the completion in the norm

$$\|u; V^l_{\gamma,\delta}(G(\varepsilon))\|$$
$$= \left(\int_{G(\varepsilon)} \sum_{|\alpha|=0}^{l} \left(\sum_{j=1}^{2} \Theta^2(r_j)\,(r_j^2+\varepsilon_j^2)^{\gamma-l+|\alpha|} + \eta_1^2 e^{2\delta|x|} + \eta_0 + \eta_2^2 e^{2\delta|x|}\right) |\partial^\alpha v|^2\, dx\, dy\right)^{1/2}$$

of the set of smooth functions compactly supported on $\overline{G(\varepsilon)}$. Denote by $V^{0,\perp}_{\gamma,\delta}$ the space of function f, analytic in k^2, with values in $V^0_{\gamma,\delta}(G(\varepsilon))$ that satisfy, at $k^2 = k_e^2$, the condition $(\chi_{0,\varepsilon^\sigma} f, v_e)_{G_0} = 0$ with a small $\sigma > 0$; here k_e^2 is a simple eigenvalue of problem (5.2.1) in G_0, and v_e is an eigenfunction corresponding to k_e^2.

Proposition 5.5.1 *Let k_r^2 be a resonance, $k_r^2 \to k_e^2$ as $\varepsilon \to 0$, and let $|k^2 - k_r^2| = O(\varepsilon^{2\pi/\omega})$. Let γ satisfy the condition $\pi/\omega - 2 < \gamma - 1 < \pi/\omega$, $f \in V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$, and let u be a solution to problem (5.5.1) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-;$$

here $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma,\delta}(G(\varepsilon))$ for small $\delta > 0$. Then

$$\|\widetilde{u}; V^2_{\gamma,\delta}(G(\varepsilon))\| + |A_1^-| + |A_2^-| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \tag{5.5.2}$$

where c is a constant independent of f and ε.

Proof Step A. First we construct an auxiliary function u_p. As mentioned above, $\widetilde{S}$ has a pole $k_p^2 = k_r^2 - ik_i^2$ (see (5.4.24)). Let us multiply the solutions to the limit problems, involved in (5.4.1), by $g := -(k^2 - k_e^2 + \langle \mathbf{b}D(k), \mathbf{b}\rangle)/\langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle$, put $k = k_p$, and denote the resulting functions by adding the subscript p. In view of (5.4.19) and the equality $(s_1, 0, 0, 0)D\mathbf{b}^* = \langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle$, we get

$$g\,(C_{11}, C_{12}, C_{13}, C_{14})|_{k=k_p} = \mathbf{b}D(k_p) = (b_1\beta, b_1\alpha, b_2\alpha, b_2\beta)\varepsilon^{2\pi/\omega} + O(\varepsilon^{4\pi/\omega}). \tag{5.5.3}$$

This and (5.4.2), (5.4.4) lead to

$$v_{1p}(x, y; \varepsilon) = g\,C_{11}|_{k=k_p}\mathbf{v}_1(x, y; k_p) = \varepsilon^{2\pi/\omega}\left(b_1\beta + O\left(\varepsilon^{2\pi/\omega}\right)\right)\mathbf{v}_1(x, y; k_p), \tag{5.5.4}$$
$$v_{2p}(x, y; \varepsilon) = g\,C_{14}|_{k=k_p}\mathbf{v}_2(x, y; k_p) = \varepsilon^{2\pi/\omega}\left(b_2\beta + O\left(\varepsilon^{2\pi/\omega}\right)\right)\mathbf{v}_2(x, y; k_p);$$

the dependence of k_p on ε is not shown. According to (5.4.3) and Lemma 5.3.2,

$$v_{0p}(x, y; \varepsilon) = -\frac{(g\,C_{12}b_1 + g\,C_{13}b_2)|_{k=k_p}}{k_p^2 - k_e^2} v_e(x, y) + g\,C_{12}|_{k=k_p}\widehat{\mathbf{v}}_{01}(x, y)$$
$$+ g\,C_{13}|_{k=k_p}\widehat{\mathbf{v}}_{02}(x, y).$$

Taking into account (5.4.19), we obtain

$$
\begin{aligned}
C_{12}b_1 + C_{13}b_2 &= (C_{11}, C_{12}, C_{13}, C_{14})\mathbf{b}^* = (s_1, 0, 0, 0)D\mathbf{b}^* \\
&\quad \times \left(1 - \frac{\langle \mathbf{b}D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right) \\
&= (k^2 - k_e^2)\frac{\langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}.
\end{aligned}
\tag{5.5.5}
$$

Hence,

$$
\begin{aligned}
v_{0p}(x, y; \varepsilon) = v_e(x, y) &+ \varepsilon^{2\pi/\omega}(b_1\alpha + O(\varepsilon^{2\pi/\omega}))\widehat{\mathbf{v}}_{01}(x, y) \\
&+ \varepsilon^{2\pi/\omega}(b_2\alpha + O(\varepsilon^{2\pi/\omega}))\widehat{\mathbf{v}}_{02}(x, y).
\end{aligned}
$$

Finally, using (5.4.13) and formulas (5.4.5), (5.4.8), (5.4.10) for a_j^+ and b_j^+, we find

$$
\begin{aligned}
w_{1p}(\xi_1, \eta_1; \varepsilon) &= (gC_{11})|_{k=k_p} a_1 \varepsilon^{\pi/\omega}\mathbf{w}_1^l(\xi_1, \eta_1) \\
&\quad + (gC_{12}c_{11} + gC_{13}c_{21})|_{k=k_p}\varepsilon^{\pi/\omega}\mathbf{w}_1^r(\xi_1, \eta_1), \\
w_{2p}(\xi_2, \eta_2; \varepsilon) &= (gC_{22}c_{11} + gC_{23}c_{21})|_{k=k_p}\varepsilon^{\pi/\omega}\mathbf{w}_2^l(\xi_2, \eta_2) \\
&\quad + (gC_{14})|_{k=k_p} a_2 \varepsilon^{\pi/\omega}\mathbf{w}_2^r(\xi_2, \eta_2).
\end{aligned}
$$

Compare the equalities (5.3.12), (5.5.5), and (5.5.3); then

$$
\begin{aligned}
(gC_{12}c_{1j} + gC_{j3}c_{2j})|_{k=k_p} &= -b_j \frac{(g\,C_{12}b_1 + g\,C_{13}b_2)|_{k=k_p}}{k_p^2 - k_e^2} \\
&\quad + (gC_{12}\widehat{c}_{1j} + gC_{j3}\widehat{c}_{2j})|_{k=k_p} \\
&= b_j + O(\varepsilon^{2\pi/\omega}),
\end{aligned}
$$

where $j = 1, 2$. Thus

$$
\begin{aligned}
w_{1p}(\xi_1, \eta_1; \varepsilon) &= \varepsilon^{3\pi/\omega}(a_1 b_1 \beta + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_1^l(\xi_1, \eta_1) \\
&\quad + \varepsilon^{\pi/\omega}(b_1 + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_1^r(\xi_1, \eta_1),
\end{aligned}
\tag{5.5.6}
$$

$$
\begin{aligned}
w_{2p}(\xi_2, \eta_2; \varepsilon) &= \varepsilon^{\pi/\omega}(b_2 + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_2^l(\xi_2, \eta_2) \\
&\quad + \varepsilon^{3\pi/\omega}(a_2 b_2 \beta + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_2^r(\xi_2, \eta_2).
\end{aligned}
\tag{5.5.7}
$$

We set

$$
\begin{aligned}
u_p(x,y;\varepsilon) = \Xi(x,y)\Big[&\chi_{1,\varepsilon}(x,y)v_{1p}(x,y;\varepsilon) + \Theta(\varepsilon^{-2\sigma}r_1)w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon)\\
&+ \chi_{0,\varepsilon}(x,y)v_{0p}(x,y;\varepsilon) + \Theta(\varepsilon^{-2\sigma}r_2)w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;k,\varepsilon)\\
&+ \chi_{2,\varepsilon}(x,y)v_{2p}(x,y;k,\varepsilon)\Big],
\end{aligned}
\tag{5.5.8}
$$

where Ξ is a cut-off function in $G(\varepsilon)$ that is equal to 1 on the set $G(\varepsilon)\cap\{|x|<R\}$ and to 0 on $G(\varepsilon)\cap\{|x|>R+1\}$ for a large $R>0$; σ is such that $2\sigma<1$. The principal part of the norm of u_p is given by $\chi_{0,\varepsilon}v_{0p}$. Considering the definitions of v_{0p} and $\widehat{\mathbf{v}}_{0j}$ (see Sect. 5.2) and Lemma 5.3.2, we obtain $\|\chi_{0,\varepsilon}v_{0p}\| = \|v_e\| + o(1)$.

Step B. Let us show that

$$
\|(\Delta + k_p^2)u_p;\, V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{\pi/\omega+\kappa},
\tag{5.5.9}
$$

where $\kappa = \min\{\pi/\omega, 3\pi/\omega-\sigma_1, \gamma+1\}$, $\sigma_1 = 2\sigma(3\pi/\omega-\gamma+1)$. If $\pi/\omega<\gamma+1$ and σ is small so that $2\pi/\omega>\sigma_1$, we have $\kappa=\pi/\omega$. One can take $\sigma<1/4$, since, due to $\pi/\omega-\gamma<1$ and $1<\pi/\omega$,

$$
\sigma_1 = 2\sigma(3\pi/\omega-\gamma+1) < 2\sigma(2\pi/\omega+2) < 8\sigma\pi/\omega < 2\pi/\omega.
$$

In view of (5.5.8),

$$
\begin{aligned}
(\Delta+k_p^2)u_p(x,y;\varepsilon) = {}&[\Delta,\chi_{1,\varepsilon}]\Big(v_{1p}(x,y;\varepsilon) - b_1\beta\varepsilon^{2\pi/\omega}(r_1^{-\pi/\omega} + a(k_p)r_1^{\pi/\omega})\Phi(\pi-\varphi_1)\Big)\\
&+ [\Delta,\Theta]w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon) + k_p^2\Theta(\varepsilon^{-2\sigma}r_1)w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon)\\
&+ [\Delta,\chi_{0,\varepsilon}]\Big(v_{0p}(x,y;\varepsilon) - \Theta(r_1)\big(b_{1p}^-(\varepsilon)r_1^{-\pi/\omega} + b_{1p}^+(\varepsilon)r_1^{\pi/\omega}\big)\Phi(\pi-\varphi_1)\\
&\qquad - \Theta(r_2)\big(a_{2p}^-(\varepsilon)r_2^{-\pi/\omega} + a_{2p}^+(\varepsilon)r_2^{\pi/\omega}\big)\Phi(\varphi_2)\Big)\\
&+ [\Delta,\Theta]w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;\varepsilon) + k_p^2\Theta(\varepsilon^{-2\sigma}r_2)w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;\varepsilon)\\
&+ [\Delta,\chi_{2,\varepsilon}]\Big(v_{2p}(x,y;\varepsilon) - b_2\beta\varepsilon^{2\pi/\omega}(r_2^{-\pi/\omega} + a(k_p)r_2^{\pi/\omega})\Phi(\varphi_2)\Big)\\
&+ [\Delta,\Xi]v_{1p}(x,y;\varepsilon) + [\Delta,\Xi]v_{2p}(x,y;\varepsilon),
\end{aligned}
$$

where $b_{1p}^- = O(\varepsilon^{2\pi/\omega})$, $b_{1p}^+ = b_1 + O(\varepsilon^{2\pi/\omega})$, $a_{2p}^- = O(\varepsilon^{2\pi/\omega})$, and $a_{2p}^+ = b_2 + O(\varepsilon^{2\pi/\omega})$. Taking into account the asymptotics of $\mathbf{v}_1$ as $r_1\to 0$ and passing to the variables $(\xi_1,\eta_1) = (\varepsilon^{-1}x_1, \varepsilon^{-1}y_1)$, we obtain

$$
\begin{aligned}
&\Big\|(x,y)\mapsto[\Delta,\chi_{1,\varepsilon}]\Big(\mathbf{v}_1(x,y) - (r_1^{-\pi/\omega} + a(k_p)r_1^{\pi/\omega})\Phi(\pi-\varphi_1)\Big);\, V^0_{\gamma,\delta}(G(\varepsilon))\Big\|^2\\
&\qquad\le c\int_{G(\varepsilon)}(r_1^2+\varepsilon^2)^\gamma\left|[\Delta,\chi_{1,\varepsilon}]r_1^{-\pi/\omega+2}\Phi(\pi-\varphi_1)\right|^2dxdy \le c\varepsilon^{2(\gamma-\pi/\omega+1)}.
\end{aligned}
$$

This and (5.5.4) imply the estimate

$$\Big\|(x,y)\mapsto[\Delta,\chi_{1,\varepsilon}]\Big(v_{1p}(x,y)-(r_1^{-\pi/\omega}+a(k_p)r_1^{\pi/\omega})\Phi(\pi-\varphi_1)\Big);\\ V^0_{\gamma,\delta}(G(\varepsilon))\Big\|\le c\varepsilon^{\gamma+\pi/\omega+1}.$$

Likewise,

$$\Big\|(x,y)\mapsto[\Delta,\chi_{0,\varepsilon}]\Big(v_{0p}(x,y)-\Theta(r_1)\big(b^-_{1p}(\varepsilon)r_1^{-\pi/\omega}+b^+_{1p}(\varepsilon)r_1^{\pi/\omega}\big)\Phi(\pi-\varphi_1)\\ -\ \Theta(r_2)\big(a^-_{2p}(\varepsilon)r_2^{-\pi/\omega}+a^+_{2p}(\varepsilon)r_2^{\pi/\omega}\big)\Phi(\varphi_2)\Big)\Big\|\le c\varepsilon^{\gamma+\pi/\omega+1},$$
$$\Big\|(x,y)\mapsto[\Delta,\chi_{2,\varepsilon}]\Big(v_{2p}(x,y)-(r_2^{-\pi/\omega}+a(k_p)r_2^{\pi/\omega})\Phi(\varphi_2)\Big);V^0_{\gamma,\delta}(G(\varepsilon))\Big\|\\ \le c\varepsilon^{\gamma+\pi/\omega+1}.$$

It is evident, that

$$\Big\|[\Delta,\Xi]v_{jp};V^0_{\gamma,\delta}(G(\varepsilon))\Big\|\le c\varepsilon^{2\pi/\omega},\quad j=1,2.$$

Further, since $\mathbf{w}^l_j$ behaves like $O(\rho_j^{-3\pi/\omega})$ at infinity,

$$\int_{G(\varepsilon)}(r_j^2+\varepsilon^2)^\gamma\left|[\Delta,\Theta]\mathbf{w}^l_j(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j)\right|^2dx_jdy_j\\ \le c\int_{K_j}(r_j^2+\varepsilon^2)^\gamma\left|[\Delta,\Theta](\varepsilon^{-1}r_j)^{-3\pi/\omega}\Phi_2(\varphi_j)\right|^2dx_jdy_j\\ \le c\varepsilon^{2(3\pi/\omega-\sigma_1)},$$

where $\sigma_1=2\sigma(3\pi/\omega-\gamma+1)$. A similar inequality holds with $\mathbf{w}^l_j$ replaced by $\mathbf{w}^r_j$. Considering (5.5.6)–(5.5.7), we obtain

$$\Big\|[\Delta,\Theta]w_{jp};V^0_{\gamma,\delta}(G(\varepsilon))\Big\|\le c\varepsilon^{4\pi/\omega-\sigma_1}.$$

Finally, using (5.5.6)–(5.5.7) once again, taking into account the estimate

$$\int_{G(\varepsilon)}(r_j^2+\varepsilon^2)^\gamma\left|\Theta(\varepsilon^{-2\sigma}r_j)\mathbf{w}^l_j(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j)\right|^2dx_jdy_j\\ =\varepsilon^{2\gamma+2}\int_\Omega(\rho_j^2+1)^\gamma\left|\Theta(\varepsilon^{1-2\sigma}\rho_j)\mathbf{w}^l_j(\xi_j,\eta_j)\right|^2d\xi_jd\eta_j\le c\varepsilon^{2\gamma+2},$$

and a similar estimate for $\mathbf{w}_j^r$, we derive

$$\left\|(x,y)\mapsto \Theta(\varepsilon^{-2\sigma}r_j)w_{jp}(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j);\, V^0_{\gamma,\delta}(G(\varepsilon))\right\| \le c\varepsilon^{\pi/\omega+\gamma+1}.$$

Combining the obtained estimates, we arrive at (5.5.9).

Step C. This part contains somewhat modified arguments from the proof of Theorem 5.1.1 in [33]. Let us write the right-hand side of problem (5.5.1) in the form

$$\begin{aligned} f(x,y) &= f_1(x,y;\varepsilon)+f_0(x,y;\varepsilon)+f_2(x,y;\varepsilon)\\ &\quad -\varepsilon^{-\gamma-1}F_1(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon_1)-\varepsilon^{-\gamma-1}F_2(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;\varepsilon),\end{aligned}$$

where

$$\begin{aligned} &f_l(x,y;\varepsilon)=\chi_{l,\varepsilon^\sigma}(x,y)f(x,y),\\ &F_j(\xi_j,\eta_j;\varepsilon)=-\varepsilon^{\gamma+1}\Theta(\varepsilon^{1-\sigma}\rho_j)f(x_{O_j}+\varepsilon\xi_j,y_{O_j}+\varepsilon\eta_j),\end{aligned}$$

(x,y) are arbitrary Cartesian coordinates, (x_{O_j},y_{O_j}) stand for the coordinates of O_j in the system (x,y), and x_j,y_j were introduced in Sect. 5.4. From the definition of the norms, it follows that

$$\begin{aligned} &\|f_1;V^0_{\gamma,\delta}(G_1)\|+\|f_0;V^0_\gamma(G_0)\|+\|f_2;V^0_{\gamma,\delta}(G_2)\|\\ &+\|F_j;V^0_\gamma(\Omega_j)\|\le c\|f;V^0_{\gamma,\delta}(G(\varepsilon))\|. \end{aligned} \tag{5.5.10}$$

We consider solutions v_l and w_j to the limit problems

$$\begin{aligned} -\Delta v_l-k^2v_l=f_l \text{ in } G_l, &\qquad v_l=0 \text{ on } \partial G_l,\\ \Delta w_j=F_j \text{ in } \Omega_j, &\qquad w_j=0 \text{ on } \partial\Omega_j, \end{aligned}$$

respectively; moreover, the v_l with $l=1,2$ satisfy the intrinsic radiation conditions at infinity, and the v_0 is subject to the condition $(v_0,v_e)_{G_0}=0$. According to Propositions 5.2.1, 5.2.2, and 5.2.3, the problems in G_l and Ω_j are uniquely solvable and

$$\begin{aligned} \|v_0;V^2_\gamma(G_0)\|&\le c_0\|f_0;V^0_\gamma(G_0)\|,\\ \|v_l;V^2_{\gamma,\delta,-}(G_l)\|&\le c_l\|f_l;V^0_{\gamma,\delta}(G_l)\|,\ l=1,2,\\ \|w_j;V^2_\gamma(\Omega_j)\|&\le C_j\|F_j;V^0_\gamma(\Omega_j)\|,\ j=1,2, \end{aligned} \tag{5.5.11}$$

where c_l and C_j are independent of ε. We set

$$\begin{aligned}U(x, y; \varepsilon) = {}&\chi_{1,\varepsilon}(x, y)v_1(x, y; \varepsilon) + \varepsilon^{-\gamma+1}\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon)\\ &+ \chi_{0,\varepsilon}(x, y)v_0(x, y; \varepsilon) + \varepsilon^{-\gamma+1}\Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon)\\ &+ \chi_{2,\varepsilon}(x, y)v_2(x, y; \varepsilon).\end{aligned}$$

Estimates (5.5.10) and (5.5.11) lead to

$$\|U; V^2_{\gamma,\delta,-}(G(\varepsilon))\| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\| \tag{5.5.12}$$

with c independent of ε. Let R_ε denote the mapping $f \mapsto U$.

Let us show that $-(\Delta + k^2)R_\varepsilon = I + S_\varepsilon$, where S_ε is an operator in $V^0_{\gamma,\delta}(G(\varepsilon))$ of small norm. We have

$$\begin{aligned}(\Delta + k^2)R_\varepsilon f(x, y) = (\Delta + k^2)U(x, y; \varepsilon) = {}&-f(x, y) + [\Delta, \chi_{1,\varepsilon}]v_1(x, y; \varepsilon)\\ &+ \varepsilon^{-\gamma+1}[\Delta, \Theta]w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon) + k^2\varepsilon^{-\gamma+1}\Theta(r_1))\\ &\times w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon) + [\Delta, \chi_{0,\varepsilon}]v_0(x, y; \varepsilon)\\ &+ \varepsilon^{-\gamma+1}[\Delta, \Theta]w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon) + k^2\varepsilon^{-\gamma+1}\Theta(r_2)\\ &\times w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon) + [\Delta, \chi_{2,\varepsilon}]v_2(x, y; \varepsilon).\end{aligned} \tag{5.5.13}$$

Let d be a positive number such that $\gamma - d + \pi/\omega - 1 > 0$. On the support of the function $[\Delta, \chi_{1,\varepsilon}]v_1$ the estimate $(x_1^2 + y_1^2)^{1/2} = O(\varepsilon)$ holds, therefore,

$$\begin{aligned}&\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^d\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma-d,\delta}(G_1)\|\\ &\le c\varepsilon^d\|v_1; V^2_{\gamma-d,\delta}(G_1)\|.\end{aligned}$$

This and (5.5.11) lead to

$$\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^d\|f_1; V^0_{\gamma-d,\delta}(G_1)\|.$$

Moreover, $f_1 = 0$ outside the zone $c\varepsilon^\sigma \le (x_1^2 + y_1^2)^{1/2} \le C\varepsilon^\sigma$, therefore,

$$\|f_1; V^0_{\gamma-d,\delta}(G_1)\| \le c\varepsilon^{-d\sigma}\|f_1; V^0_{\gamma,\delta}(G_1)\|.$$

The two last estimates together with (5.5.10) show that

$$\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{5.5.14}$$

In a similar way, we obtain

$$\|[\Delta, \chi_{l,\varepsilon}]v_l; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \quad l = 0, 2. \tag{5.5.15}$$

We now assume in addition that the d satisfies $\gamma + d - \pi/\omega - 1 < 0$. Because the support of the function $[\Delta_{(\xi_j,\eta_j)}, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j; \varepsilon)$, $j = 1, 2$, belongs to the domain $c\varepsilon^{-1} \le (\xi_j^2 + \eta_j^2)^{1/2} \le C\varepsilon^{-1}$,

$$\begin{aligned}
&\|(\xi_j, \eta_j) \mapsto [\Delta_{\xi_j,\eta_j}, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j; \varepsilon);\ V^0_\gamma(\Omega_j)\| \\
&\qquad \le c\varepsilon^d \|(\xi_j, \eta_j) \mapsto [\Delta_{\xi_j,\eta_j}, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j; \varepsilon);\ V^0_{\gamma+d}(\Omega_j)\| \\
&\qquad \le c\varepsilon^d \|w_j;\ V^2_{\gamma+d}(\Omega_j)\|.
\end{aligned}$$

Now, taking into account (5.5.11), we obtain

$$\begin{aligned}
\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto &[\Delta, \Theta(r_j)]w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon);\ V^0_{\gamma,\delta}(G(\varepsilon))\| \\
&\le c\varepsilon^d \|F_j;\ V^0_{\gamma+d}(\Omega_j)\|.
\end{aligned}$$

Since $F_j = 0$ for $(\xi_j^2 + \eta_j^2)^{1/2} > c\varepsilon^{-\sigma}$,

$$\|F_j;\ V^0_{\gamma+d}(\Omega_j)\| \le c\varepsilon^{-d\sigma} \|F_j;\ V^0_\gamma(\Omega_j)\|. \tag{5.5.16}$$

Consequently,

$$\begin{aligned}
\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto &[\Delta, \Theta(r_j)]w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon);\ V^0_{\gamma,\delta}(G(\varepsilon))\| \\
&\le c\varepsilon^{d(1-\sigma)} \|f;\ V^0_{\gamma,\,\delta}(G(\varepsilon))\|.
\end{aligned} \tag{5.5.17}$$

It remains to estimate the middle terms of the two last lines in (5.5.13). We have

$$\begin{aligned}
\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto \Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon);\ &V^0_{\gamma,\delta}(G(\varepsilon))\| \\
&= \varepsilon^2 \|(\xi_j, \eta_j) \mapsto \Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j; \varepsilon);\ V^0_\gamma(\Omega_j)\| \\
&\le \varepsilon^2 \|(\xi_j, \eta_j) \mapsto \Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j; \varepsilon);\ V^2_{\gamma+2}(\Omega_j)\| \\
&\le c\varepsilon^d \|w_j;\ V^2_{\gamma+d}(\Omega_j)\|;
\end{aligned}$$

in the last inequality we took into account that $\Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j; \varepsilon) = 0$ for $\rho_j \ge c\varepsilon^{-1}$; besides, we assume that $2 - d > 0$. In view of (5.5.11), (5.5.16), and (5.5.10), we obtain

$$\begin{aligned}
\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto &\Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon);\ V^0_{\gamma,\delta}(G(\varepsilon))\| \\
&\le c\varepsilon^{d(1-\sigma)} \|f;\ V^0_{\gamma,\,\delta}(G(\varepsilon))\|.
\end{aligned} \tag{5.5.18}$$

Thus, (5.5.13)–(5.5.15) and (5.5.17)–(5.5.18) lead to the inequality

$$\| - (\Delta + k^2)R_\varepsilon f - f;\ V^0_{\gamma,\,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)} \|f;\ V^0_{\gamma,\,\delta}(G(\varepsilon))\|,$$

which means that $-(\Delta + k^2)R_\varepsilon = I + S_\varepsilon$ and the norm of the operator S_ε in the space $V^0_{\gamma,\delta}(G(\varepsilon))$ admits the estimate $\|S_\varepsilon\| \le c\varepsilon^{d(1-\sigma)}$.

Step D. Let us recall that the operator S_ε is defined on the subspace $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. We also need the range of the operator S_ε be included in $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. To this end, we replace the mapping R_ε by $\widetilde{R}_\varepsilon : f \mapsto U(f) + a(f)u_p$, the u_p was constructed in Step A, and $a(f)$ is a constant. Then $-(\Delta + k^2)\widetilde{R}_\varepsilon = I + \widetilde{S}_\varepsilon$ with $\widetilde{S}_\varepsilon = S_\varepsilon - a(\cdot)(\Delta + k^2)u_p$. As $k = k_e$, the condition $(\chi_{0,\varepsilon^\sigma}\widetilde{S}_\varepsilon f, v_e)_{G_0} = 0$ implies $a(f) = (\chi_{0,\varepsilon^\sigma} S_\varepsilon f, v_e)_{G_0}/(\chi_{0,\varepsilon^\sigma}(\Delta+k_e^2)u_p, v_e)_{G_0}$. Now, we prove that $\|\widetilde{S}_\varepsilon\| \le c\|S_\varepsilon\|$, where c is independent of ε and k. We have

$$\|\widetilde{S}_\varepsilon f\| \le \|S_\varepsilon f\| + |a(f)|\,\|(\Delta + k^2)u_p\|.$$

Estimate (5.5.9) (with $\gamma > \pi/\omega - 1$ and $2\pi/\omega > \sigma_1$), the formula for k_p, and the condition $k^2 - k_e^2 = O\big(\varepsilon^{2\pi/\omega}\big)$ imply the inequalities

$$\|(\Delta + k^2)u_p; V^0_{\gamma,\delta}\| \le |k^2 - k_p^2|\,\|u_p; V^0_{\gamma,\delta}\| + \|(\Delta + k_p^2)u_p; V^0_{\gamma,\delta}\| \le c\varepsilon^{2\pi/\omega}.$$

Since the supports of the functions $(\Delta + k_p^2)u_p$ and $\chi_{0,\varepsilon^\sigma}$ are disjoint, we obtain

$$|(\chi_{0,\varepsilon^\sigma}(\Delta + k_e^2)u_p, v_e)_{G_0}| = |(k_e^2 - k_p^2)(u_p, v_e)_{G_0}| \ge c\varepsilon^{2\pi/\omega}.$$

Moreover, $\gamma - 1 < \pi/\omega$ and, consequently,

$$|(\chi_{0,\varepsilon^\sigma} S_\varepsilon f, v_e)_{G_0}| \le \|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\|\,\|v_e; V^0_{-\gamma}(G_0)\| \le c\|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\|.$$

Hence,

$$|a(f)| \le c\varepsilon^{-2\pi/\omega}\|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\|$$

and $\|\widetilde{S}_\varepsilon f\| \le c\|S_\varepsilon f\|$. Thus, the operator $I + \widetilde{S}_\varepsilon$ in $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$ is invertible, which is also true for the operator of problem (5.5.1):

$$A_\varepsilon : u \mapsto -\Delta u - k^2 u : \mathring{V}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon)) \mapsto V^{0,\perp}_{\gamma,\delta}(G(\varepsilon)),$$

the $\mathring{V}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon))$ consists of the elements in $V^2_{\gamma,\delta,-}(G(\varepsilon))$ that vanish on $\partial G(\varepsilon)$, and the operator $-\Delta - k^2$ takes $\mathring{V}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon))$ to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$ to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. The inverse operator $A_\varepsilon^{-1} = \widetilde{R}_\varepsilon(I + \widetilde{S}_\varepsilon)^{-1}$ is bounded uniformly with respect to ε and k. Therefore, the inequality (5.5.2) holds with c independent of ε and k. $\square$

Consider a solution u_1 to the homogeneous problem (5.1.1) defined by

$$u_1(x,y) = \begin{cases} U_1^+(x,y) + S_{11}\, U_1^-(x,y) + O(\exp(\delta x)), & x \to -\infty, \\ S_{12}\, U_2^-(x,y) + O(\exp(-\delta x)), & x \to +\infty. \end{cases}$$

Let S_{11} and S_{12} be the elements of the scattering matrix determined by this solution. Denote by $\widetilde{u}_{1,\sigma}$ the function defined by (5.4.1) with $\Theta(r_j)$ replaced by $\Theta(\varepsilon_j^{-2\sigma} r_j)$ and remainder R removed; $\widetilde{S}_{11}$, $\widetilde{S}_{12}$ are the same as in (5.4.21).

Theorem 5.5.2 *Let the hypotheses of Proposition 5.5.1 be fulfilled. Then the inequalities*

$$|S_{11} - \widetilde{S}_{11}| + |S_{12} - \widetilde{S}_{12}| \le c|\widetilde{S}_{12}|\varepsilon^{2-\delta}, \tag{5.5.19}$$

$$|S_{21} - \widetilde{S}_{21}| + |S_{22} - \widetilde{S}_{22}| \le c|\widetilde{S}_{22}|\varepsilon^{2-\delta} \tag{5.5.20}$$

hold with a constant c, independent of ε *and k,* δ *being an arbitrarily small positive number.*

Proof For example, we verify (5.5.19). The difference $R = u_1 - \widetilde{u}_{1,\sigma}$ is in the space $V^2_{\gamma,\delta,-}(G(\varepsilon))$ and $f_1 := -(\Delta + k^2)(u_1 - \widetilde{u}_{1,\sigma})$ belongs to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. By Proposition 5.5.1,

$$\|R; V^2_{\gamma,\delta,-}(G(\varepsilon))\| \le c\,\|f_1; V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{5.5.21}$$

Let us show that

$$\|f_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c|\widetilde{S}_{12}|(\varepsilon^{\gamma-\pi/\omega+1} + \varepsilon^{2\pi/\omega-\sigma_1}), \tag{5.5.22}$$

where $\sigma_1 = 2\sigma(3\pi/\omega - \gamma + 1)$. (Then estimate (5.5.19) follows from (5.5.21) and (5.5.22) with $\gamma = \pi/\omega + 1 - \delta$ and $\sigma_1 = \delta$.) Arguing, as in the proof of Proposition 5.5.1, Step B, we obtain the estimate

$$\begin{aligned} \|f_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c(\varepsilon^{\gamma+1} + \varepsilon^{3\pi/\omega-\sigma_1}) \\ \times \max_{j=1,2}(|a_j^-(\varepsilon)|\varepsilon^{-\pi/\omega} + |a_j^+(\varepsilon)|\varepsilon^{\pi/\omega} + |b_j^-(\varepsilon)|\varepsilon^{-\pi/\omega} \\ + |b_j^+(\varepsilon)|\varepsilon^{\pi/\omega}). \end{aligned}$$

From (5.4.11) it follows that

$$|a_j^-(\varepsilon)| + |b_j^-(\varepsilon)| \le c\varepsilon^{2\pi/\omega}(|a_j^+(\varepsilon)| + |b_j^+(\varepsilon)|).$$

Using (5.4.5) and (5.4.10) for a_1^+, b_2^+ and taking account of relations (5.4.19) and (5.4.25) and of the fact that $k^2 - k_p^2 = O(\varepsilon^{2\pi/\omega})$, we obtain

$$|a_1^+(\varepsilon)| + |b_2^+(\varepsilon)| \le c\frac{\varepsilon^{2\pi/\omega}}{|k^2 - k_p^2|} \le c\varepsilon^{-2\pi/\omega}|\widetilde{S}_{12}(\varepsilon)|.$$

Analogously, using (5.4.5) and (5.4.8) for a_2^+, b_1^+ and the relations (5.3.12), (5.5.5), we get

$$\begin{aligned}|a_1^+(\varepsilon)| + |b_2^+(\varepsilon)| \le\ &\max_{j=1,2}\left|-b_j\frac{\langle(s_1, 0, 0, 0)D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle\mathbf{b}D, \mathbf{b}\rangle} + C_{12}\widehat{c}_{1j} + C_{13}\widehat{c}_{2j}\right|\\ &\le c\frac{\varepsilon^{2\pi/\omega}}{|k^2 - k_p^2|} \le c\varepsilon^{-2\pi/\omega}|\widetilde{S}_{12}(\varepsilon)|.\end{aligned}$$

Combining the above inequalities, we arrive at (5.5.22) and, consequently, at (5.5.19). □

Let us recall some notations. We denote by k_e^2 a simple eigenvalue of problem (5.2.1) in the resonator G_0 and by $k_r^2(\varepsilon)$ a resonance frequency such that $k_r^2(\varepsilon) \to k_e^2$ as $\varepsilon \to 0$. Moreover, let b_j be the constants in asymptotics (5.3.7) of an eigenfunction corresponding to the eigenvalue k_e^2 and $s_j(k)$ the constant in asymptotics (5.3.1) of the special solution V_j for $r_j \to 0$, $j = 1, 2$. Finally, the constants α and β are defined by (5.2.5) and (5.2.6). We set $P = (b_1^2\beta^2|s_1(k_e)|^2)^{-1}$; this is the same constant as in (6.4.27) and (6.4.29).

Theorem 5.5.3 *For $|k^2 - k_r^2| = O(\varepsilon^{2\pi/\omega})$, the asymptotic expansions*

$$\begin{aligned}T_1(k, \varepsilon) = T_2(k, \varepsilon) &= \frac{1}{1 + P^2\left(\dfrac{k^2 - k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}\bigl(1 + O(\varepsilon^{2-\delta})\bigr),\\ k_r^2(\varepsilon) &= k_e^2 + 2b_1^2\alpha\varepsilon^{2\pi/\omega} + O\bigl(\varepsilon^{2\pi/\omega+2-\delta}\bigr),\\ \Upsilon(\varepsilon) &= \left|\frac{1}{P}\right|\varepsilon^{4\pi/\omega}\bigl(1 + O(\varepsilon^{2-\delta})\bigr)\end{aligned}$$

hold: $\Upsilon(\varepsilon)$ is the width of the resonant peak at its half-height (the so-called resonant quality factor), δ being an arbitrarily small positive number.

Proof Theorem 5.5.2 leads to $|S_{p2} - \widetilde{S}_{p2}| \le c|\widetilde{S}_{p2}|\varepsilon^{2-\delta}$ with a positive δ. Therefore

$$|T_p - \widetilde{T}_p| \le c|\widetilde{S}_{p2}||S_{p2} - \widetilde{S}_{p2}| \le c\widetilde{T}_p\varepsilon^{2-\delta}$$

and, in view of (5.4.26),

$$T_p(k,\varepsilon)=\widetilde{T}_p(1+O(\varepsilon^{2-\delta}))=\frac{1}{1+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}\big(1+O(\varepsilon^{2-\delta})\big).$$

This leads to formulas for $k_r^2(\varepsilon)$ and $\Upsilon(\varepsilon)$. □

5.6 Comparison of Asymptotic and Numerical Results

The principal parts of the asymptotic formulas in Theorem 5.5.3 contain the constants b_1, $|s_1(k_e)|$, α, and β. To find them, one has to solve numerically several boundary value problems. We state the problems and describe a way to solve them. Then the asymptotics with calculated constants and the numerically found scattering matrix are compared.

5.6.1 Problems and Methods for Numerical Analysis

5.6.1.1 Calculation of b_1

To find b_1 in (5.3.7), we solve the spectral problem

$$-\Delta v-k^2v=0\quad\text{in } G_0,\qquad v=0\quad\text{on } \partial G_0,$$

by FEM, as usual. Let v_e be an eigenfunction corresponding to k_e^2 and normalized by

$$\int_{G_0}|v_e(x,y)|^2\,dxdy=1.$$

Then b_1 can be determined (approximately) by

$$b_1=\varepsilon^{-\pi/\omega}\frac{v_e(\varepsilon,0)}{\Phi(0)}=\sqrt{\pi}\varepsilon^{-\pi/\omega}v_e(\varepsilon,0).$$

5.6.1.2 Calculation of $|s_1|$

The constant $s_1\neq 0$ has arisen in the asymptotics (5.3.1) of the solution V_1 to homogeneous problem (5.2.1) in G_1. Denote the truncated domain $G_1\cap\{(x_1,y_1): x_1>-R\}$ by G_1^R and put $\Gamma^R:=\partial G_1^R\cap\{(x_1,y_1): x_1=-R\}$. Now we introduce the problem

$$\begin{aligned} -\Delta V(x_1, y_1) - k^2 V(x_1, y_1) &= 0, && (x_1, y_1) \in G_1^R; \\ V(x_1, y_1) &= 0, && (x_1, y_1) \in \partial G_1^R \setminus \Gamma^R; \\ \partial_n V(x_1, y_1) + i\nu_1 V(x_1, y_1) &= 2i\nu_1 e^{i\nu_1 R}\Psi_1(y_1), && (x_1, y_1) \in \Gamma^R; \end{aligned} \tag{5.6.1}$$

the function Ψ_1 is defined in (5.1.5). The solution V is found by FEM. One may put

$$s_1 = \sqrt{\pi}\varepsilon^{-\pi/\omega} V(-\varepsilon, 0).$$

5.6.1.3 Calculation of α and β

Let us introduce the boundary value problem for calculation of α and β in (5.2.6). Denote the truncated domain $\Omega \cap \{(r, \varphi) : r < R\}$ by Ω^R and put $\Gamma^R := \partial\Omega \cap \{(r, \varphi) : r = R\}$. Consider the problem

$$\begin{aligned} \Delta w(\xi, \eta) &= 0, && (\xi, \eta) \in \Omega^R; \\ w(\xi, \eta) &= 0, && (\xi, \eta) \in \partial\Omega^R \setminus \Gamma^R; \\ \partial_n w(\xi, \eta) + \zeta w(\xi, \eta) &= g(\xi, \eta), && (\xi, \eta) \in \Gamma^R. \end{aligned} \tag{5.6.2}$$

If w is a solution and $\zeta > 0$, then

$$\|w; L_2(\Gamma^R)\| \leqslant \zeta^{-1} \|g; L_2(\Gamma^R)\|. \tag{5.6.3}$$

Indeed, substitute $u = v = w$ to the Green formula

$$\begin{aligned} (\Delta u, v)_{\Omega^R} = \ & (\partial_n u, v)_{\partial\Omega^R} - (\nabla u, \nabla v)_{\Omega^R} \\ & = (\partial_n u, v)_{\partial\Omega^R \setminus \Gamma^R} + (\partial_n u + \zeta u, v)_{\Gamma^R} - \zeta(u, v)_{\Gamma^R} - (\nabla u, \nabla v)_{\Omega^R}, \end{aligned}$$

and get

$$0 = (g, w)_{\Gamma^R} - \zeta \|w; L_2(\Gamma^R)\|^2 - \|\nabla w; L_2(\Omega^R)\|^2.$$

From this and the obvious chain of inequalities

$$\begin{aligned} \zeta \|w; L_2(\Gamma^R)\|^2 &\leqslant \zeta \|w; L_2(\Gamma^R)\|^2 + \|\nabla w; L_2(\Omega^R)\|^2 = (g, w)_{\Gamma^R} \\ &\leqslant \|w; L_2(\Gamma^R)\| \, \|g; L_2(\Gamma^R)\| \end{aligned}$$

we obtain (5.6.3). Denote the left (right) part of Γ^R by Γ_-^R (Γ_+^R). Let W be the solution of (5.6.2) as $\zeta = \pi/\omega R$, $g|_{\Gamma_-^R} = 0$, and $g|_{\Gamma_+^R} = (2\pi/\omega)R^{(\pi/\omega)-1}\Phi(\varphi)$. Let, in addition, w^r be a solution to homogeneous problem (5.2.3) in the domain Ω with asymptotics of the form (5.2.6). Since the asymptotics can be differentiated, $w_r - W$ satisfies (5.6.2) with $g = O(R^{-(3\pi/\omega)-1})$. According to (5.6.3),

$$\|w_r - W; L_2(\Gamma^R)\| \leqslant c\frac{\omega R}{\pi} R^{-(3\pi/\omega)-1} = c' R^{-3\pi/\omega}$$

as $R \to +\infty$. We find W with FEM and determine β by the equality

$$\beta = \frac{W(-R,0)}{\Phi(0)} R^{\pi/\omega} = \sqrt{\pi}\, W(-R,0) R^{\pi/\omega}.$$

Obviously, $\|(w_r - R^{\pi/\omega}\Phi(\varphi)) - (W - R^{\pi/\omega}\Phi(\varphi)); L_2(\Gamma^R)\| \leqslant c' R^{-3\pi/\omega}$, therefore we put

$$\alpha = \frac{W(R,0) - R^{\pi/\omega}\Phi(0)}{\Phi(0)} R^{\pi/\omega} = \sqrt{\pi}\, W(R,0) R^{\pi/\omega} - R^{2\pi/\omega}.$$

5.6.1.4 Calculation of the Scattering Matrix

Let us describe the method for calculation of the scattering matrix, considering electrons of energy between the first and the second thresholds only. Then in (5.1.5) we have $M = 1$. We put

$$G(\varepsilon, R) = G(\varepsilon) \cap \{(x,y) : -R < x < d + R\},$$
$$\Gamma_1^R = \partial G(\varepsilon, R) \cap \{(x,y) : x = -R\}, \quad \Gamma_2^R = \partial G(\varepsilon, R) \cap \{(x,y) : x = d + R\}$$

for large R. As an approximation to the row (S_{11}, S_{12}) of the scattering matrix $S = S(k)$, we take the minimizer of a quadratic functional. To construct such a functional, we consider the problem

$$\begin{aligned}
-\Delta\mathcal{X}^R - k^2\mathcal{X}^R &= 0 \quad \text{in}\, G(\varepsilon, R),\\
\mathcal{X}^R &= 0 \quad \text{on}\, \partial G(\varepsilon, R) \setminus (\Gamma_1^R \cup \Gamma_2^R),\\
(\partial_n + i\zeta)\mathcal{X}^R &= i(-\nu_1 + \zeta)e^{-i\nu_1 R}\Psi_1(y) + a_1\, i(\nu_1 + \zeta)e^{i\nu_1 R}\Psi_1(y)\, \Gamma_1^R,\\
(\partial_n + i\zeta)\mathcal{X}^R &= a_2\, i(\nu_1 + \zeta)e^{i\nu_1(d+R)}\Psi_1(y)\, \Gamma_2^R,
\end{aligned} \tag{5.6.4}$$

where $\zeta \in \mathbb{R}\setminus\{0\}$ is an arbitrary fixed number, and a_1, a_2 are complex numbers. As an approximation to the row (S_{11}, S_{12}), we take the minimizer $a^0(R) = (a_1^0(R), a_2^0(R))$ of the functional

$$\begin{aligned}
J^R(a_1, a_2) = \|\mathcal{X}^R - e^{-i\nu_1 R}\Psi_1 - a_1 e^{i\nu_1 R}\Psi_1; L_2(\Gamma_1^R)\|^2 \\
+ \|\mathcal{X}^R - a_2\, e^{i\nu_1(d+R)}\Psi_1; L_2(\Gamma_2^R)\|^2,
\end{aligned} \tag{5.6.5}$$

where $\mathcal{X}^R$ is a solution to problem (5.6.4). From Theorem 4.1.1 it follows that $a_j^0(R,k) \to s_{1j}(k)$ with exponential rate as $R \to \infty$. More precisely, there exist constants Λ and C, such that $|a_j^0(R,k) - S_{1j}(k)| \leqslant C\exp(-\Lambda R)$, $j = 1, 2$, for all $k^2 \in [\mu_1, \mu_2]$ and sufficiently large R; the interval $[\mu_1, \mu_2]$ of the continuous spectrum of problem (5.1.1) lies between the first and the second thresholds and does not contain the thresholds. To express $\mathcal{X}^R$ by means of a_1, a_2, we consider the problems

$$\begin{aligned} &-\Delta v_1^\pm - k^2 v_1^\pm = 0 \quad \text{in} \quad G(\varepsilon, R), \quad v_1^\pm = 0 \quad \text{on} \quad \partial G(\varepsilon, R)\backslash(\Gamma_1^R \cup \Gamma_2^R),\\ &(\partial_n + i\zeta)v_1^\pm = i(\mp\nu_1 + \zeta)e^{\mp i\nu_1 R}\Psi_1 \quad \text{on}\Gamma_1^R, \quad (\partial_n + i\zeta)v_1^\pm = 0 \quad \text{on}\Gamma_2^R, \end{aligned} \tag{5.6.6}$$

and

$$\begin{aligned} &-\Delta v_2^\pm - k^2 v_2^\pm = 0 \quad \text{in} \quad G(\varepsilon, R), \quad v_2^\pm = 0 \quad \text{on} \quad \partial G(\varepsilon, R)\backslash(\Gamma_1^R \cup \Gamma_2^R),\\ &(\partial_n + i\zeta)v_2^\pm = 0 \quad \text{on } \Gamma_1^R, \quad (\partial_n + i\zeta)v_2^\pm = i(\mp\nu_2 + \zeta)e^{\mp i\nu_2 (d+R)}\Psi_2 \quad \text{on } \Gamma_2^R. \end{aligned} \tag{5.6.7}$$

Let $v_j^\pm = v_{j,R}^\pm$ be solutions to problems (5.6.6), (5.6.1); then $\mathcal{X}^R = v_{1,R}^+ + \sum_j a_j v_{j,R}^-$. Now, functional (5.6.5) can be written in the form

$$J^R(a;k) = \langle a\mathcal{E}^R(k), a\rangle + 2\text{Re}\,\langle \mathcal{F}_1^R(k), a\rangle + \mathcal{G}_1^R(k),$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{C}^2$ and $\mathcal{E}^R$ denotes the 2×2-matrix with entries

$$\begin{aligned} \mathcal{E}_{11}^R &= \Big((v_1^- - e^{i\nu_1 R}\Psi_1), (v_1^- - e^{i\nu_1 R}\Psi_1)\Big)_{\Gamma_1^R} + \big(v_1^-, v_1^-\big)_{\Gamma_2^R},\\ \mathcal{E}_{12}^R &= \Big((v_1^- - e^{i\nu_1 R}\Psi_1), v_2^-\Big)_{\Gamma_1^R} + \Big(v_1^-, (v_2^- - e^{i\nu_1(d+R)}\Psi_1)\Big)_{\Gamma_2^R},\\ \mathcal{E}_{21}^R &= \Big(v_2^-, (v_1^- - e^{i\nu_1 R}\Psi_1)\Big)_{\Gamma_1^R} + \Big((v_2^- - e^{i\nu_1(d+R)}\Psi_1), v_1^-\Big)_{\Gamma_2^R},\\ \mathcal{E}_{22}^R &= \big(v_2^-, v_2^-\big)_{\Gamma_1^R} + \Big((v_2^- - e^{i\nu_1(d+R)}\Psi_1), (v_2^- - e^{i\nu_1(d+R)}\Psi_1)\Big)_{\Gamma_2^R}, \end{aligned}$$

$\mathcal{F}^R(k)$ is the row $(\mathcal{F}_{11}^R(k), \mathcal{F}_{12}^R(k))$, and $\mathcal{G}_1^R(k)$ is the number defined by the equalities

$$\begin{aligned} \mathcal{F}_{11}^R &= \Big((v_1^+ - e^{-i\nu_1 R}\Psi_1), (v_1^- - e^{i\nu_1 R}\Psi_1)\Big)_{\Gamma_1^R} + \big(v_1^+, v_1^-\big)_{\Gamma_2^R},\\ \mathcal{F}_{12}^R &= \Big((v_1^+ - e^{-i\nu_1 R}\Psi_1), v_2^-)\Big)_{\Gamma_1^R} + \Big(v_1^+, (v_2^- - e^{i\nu_1(d+R)}\Psi_j)\Big)_{\Gamma_2^R},\\ \mathcal{G}_1^R &= \Big((v_1^+ - e^{-i\nu_1 R}\Psi_1), (v_1^+ - e^{-i\nu_1 R}\Psi_1)\Big)_{\Gamma_1^R} + \big(v_1^+, v_1^+\big)_{\Gamma_2^R}. \end{aligned}$$

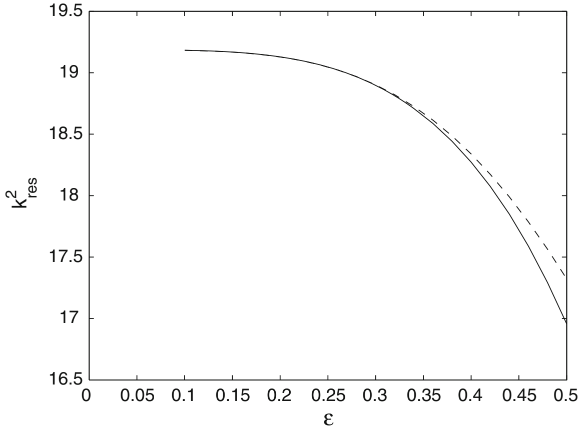

Fig. 5.4 Asymptotic description $k^2_{res,a}(\varepsilon)$ (*solid curve*) and numerical description $k^2_{res,n}(\varepsilon)$ (*dashed curve*) for resonant energy $k^2_{res}(\varepsilon)$

The minimizer $a^0 = (a^0_1(R, k), a^0_2(R, k))$ satisfies $a^0\mathcal{E}^R + \mathcal{F}^R_1 = 0$. A solution to this equation serves as an approximation to the first row of the scattering matrix. As an approximation to the scattering matrix $S(k)$, one can take a solution $S^R = S^R(k)$ to a matrix equation of the form $S^R\mathcal{E}^R + \mathcal{F}^R = 0$. Choosing $\zeta = -\nu_1$, we obtain $v^-_1 = v^-_2 = 0$, $\mathcal{E}^R = (1/\nu_1)\mathrm{Id}$, and $S^R = -\nu_1\mathcal{F}^R$.

5.6.2 Comparison of Asymptotic and Numerical Results

Let us compare the asymptotics $k^2_{res,a}(\varepsilon)$ of resonant energy $k^2_{res}(\varepsilon)$ and the approximate value $k^2_{res,n}(\varepsilon)$ obtained by a numerical method. Figure 5.4 shows good agreement of the values for $0.1 \leqslant \varepsilon \leqslant 0.5$. We have

$$|k^2_{res,a}(\varepsilon) - k^2_{res,n}(\varepsilon)|/k^2_{res,a}(\varepsilon) \leqslant 10^{-3}$$

for $0.1 \leqslant \varepsilon \leqslant 0.3$, and only for $\varepsilon = 0.5$ the ratio approaches 2×10^{-2}. For $\varepsilon < 0.1$ the numerical method is ill-conditioned. This is caused by the fact that the waveguide tends to the‘limit’ (see Fig. 5.3), on which the problems for calculation of the scattering matrix are incorrect (ill-posed). This means that the round-off errors cause the larger deviations in the solution, and at some ε we get a random vector instead of the sought-for vector of coefficients of the piecewise polynomial function. The asymptotics moves this ‘incorrectness’ out of the numerical part (i.e., the problems for the constants that have to be solved numerically) and thus remains efficient at $\varepsilon \to 0$.

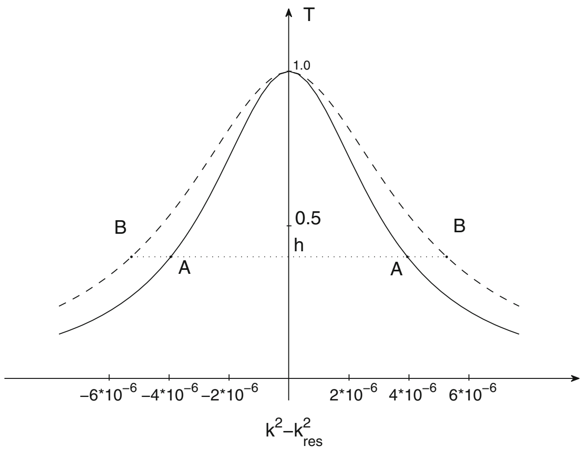

Fig. 5.5 The shape of the resonant peak for $\varepsilon = 0.2$: asymptotic description $T_a(k^2 - k^2_{res,a})$ (*solid curve*) and numerical description $T_n(k^2 - k^2_{res,n})$ (*dashed curve*) for transition coefficient $T(k^2 - k^2_{res})$. The width of the resonant peak at height h: asymptotic $\Delta_a(h, \varepsilon) = AA$; numerical $\Delta_n(h, \varepsilon) = BB$

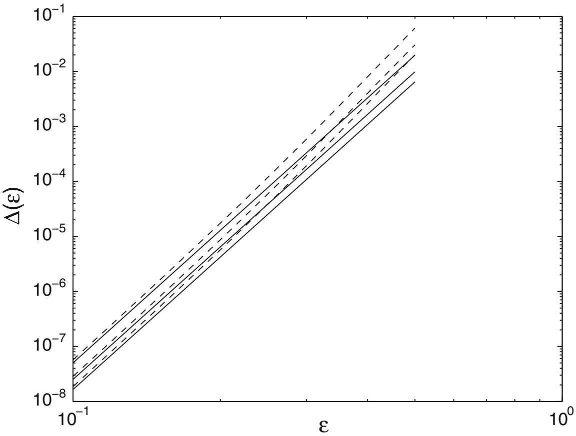

Fig. 5.6 The dependence of the width $\Delta(h, \varepsilon)$ of resonant peak on ε for various heights h (*dashed line* for numerical description, *solid line* for asymptotic description): the *upper pair* of lines for $h = 0.2$; the *middle lines* for $h = 0.5$; the *bottom lines* for $h = 0.7$

The difference between the asymptotic and numerical values becomes more significant as ε increases going out of the interval; the asymptotics becomes unreliable. The numerical method shows that for $\varepsilon \geq 0.5$ the resonant peak turns out to be so wide that the resonant tunneling phenomenon dies out by itself. The forms of "asymptotic" and "numerical" resonant peaks are almost the same (see Fig. 5.5). The difference between the peaks is quantitatively depicted in Fig. 5.6. Moreover, it turns out that the ratio of the width $\Delta_n(h, \varepsilon)$ of the numerical peak at height h to $\Delta_a(h, \varepsilon)$ of the asymptotic peak is independent of h. The ratio as a function in ε is displayed in Fig. 5.7.

Note that for $\varepsilon = 0.1$ (i.e., at the left end of the band where the numerical and asymptotic results can be compared) the disparity of the results is more significant for the width of the resonant peak than that for the resonant energy.

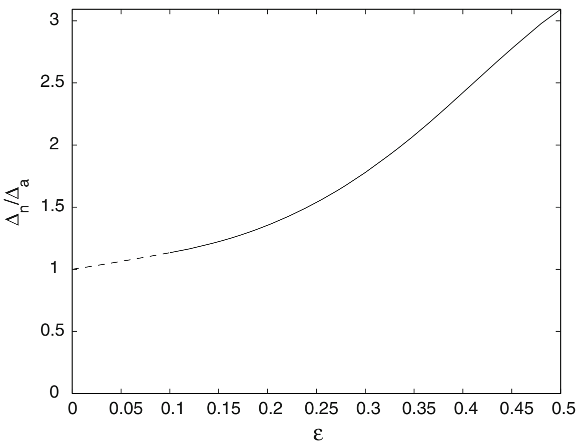

Fig. 5.7 Ratio $\Delta_n(h, \varepsilon)/\Delta_a(h, \varepsilon)$ as a function of ε. The ratio is independent of h within the accuracy of the analysis

5.7 The Impact of a Finite Waveguide Work Function on Resonant Tunneling

To describe electron transport in a waveguide, we assumed in Sects. 5.1–5.6 that the electron wave functions vanish at the waveguide boundary. This means that, being in the waveguide, an electron can not cross the waveguide boundary because of the infinite potential barrier. In reality, the assumption has never been fulfilled: generally, electrons can penetrate through the waveguide boundary and go some distance away from the waveguide. Therefore, we have to clarify how this phenomenon affects the resonant tunneling.

5.7.1 Preliminaries

In a crystal, the electric field of positive ions of the lattice impedes electrons from escaping through the crystal surface. This field acts in a narrow layer near the surface; the layer is called the surface potential barrier. Thus, being in the crystal, an electron is in a potential well. Some energy is required to remove such an electron from the well.

Considering a moving electron of the minimal kinetic energy in a large crystal, the electron is at the bottom of the potential well. To withdraw the electron from the crystal, the energy required is equal to the height of the surface potential barrier. This energy is called the full work function W_a of the crystal.

Let us describe the structure of the surface potential barrier in more detail. Within a small distance x_0 from the crystal surface, an electron is subject to the almost constant force of interaction with a surface layer of positive ions of the lattice. At a distance x from the surface, $x_0 < x < l$, the mirror interaction force acts on the electron,

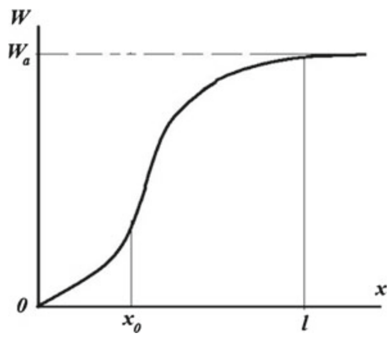

Fig. 5.8 The dependence of the electron potential energy outside a crystal on the distance to the cristal surface. The electron potential energy inside the crystal is assumed to be zero

Table 5.1 Total work functions of some metals

Metal	Cs	Ba	Mo	W	Pt	Ag	Zn	Ni
Wa, eV	3.4	4.9	10.0	10.4	11.3	∼ 14	15.5	∼ 16

that is, the force of interaction with a surface positive charge induced by inter-electron repulsion. The mirror interaction force is proportional to x^{-2}. Qualitatively, the x-dependence of the electron potential energy is shown in Fig. (5.8). The x_0 is considerably less than the lattice period and usually ranges between 0.03 and 0.1 nm. The width l of the transition zone is in the range 0.3–0.5 nm.

Experimental values of the full work function for some metals are presented in Table 5.1 ([21]).

Now, we consider an electron of the maximal kinetic energy in a metal at temperature $T = 0$. The minimal energy required to withdraw the electron from a solid and to place it just beyond the surface potential barrier (that is, at the distance l from the surface) is called the effective work function of the solid and denoted by $e\varphi$, where $-e$ is the electron charge.

The effective work function $e\varphi$ plays an important role in the description of an electron withdrawing from a solid. In what follows, the effective work function is frequently called the work function.

At $T = 0$, the maximal electron kinetic energy is called the Fermi energy (the Fermi level) and is denoted by W_F. Figure (5.9) shows the structure of the potential barrier near a metallic surface, the conductivity band and the effective work function $e\varphi = W_a - W_F$. (The potential energy of an electron is defined up to a constant term; in vacuum, the energy is assumed to be zero.)

In the semiconductors, the electrons are located in the conductivity band (above the Fermi level) and in the valence band (below the Fermi level) Fig. (5.10). In this figure, the energy difference χ of the vacuum level and the conductivity band bottom level is called the electron affinity. For an electron at the conductivity band bottom, the χ is equal to the minimal energy required to withdraw the electron from the solid.

The work function of various materials ranges between 1 and 5 eV. However, for the most part of the materials used in nanotechnology, this range is 3–5 eV. Table 5.2

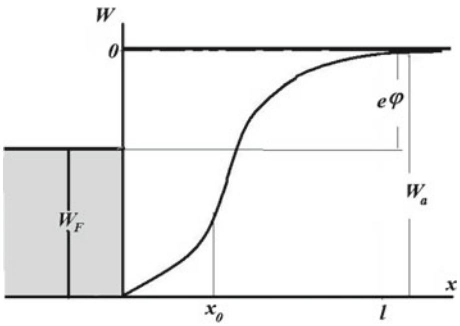

Fig. 5.9 The surface potential barrier and the conductivity band of a metal

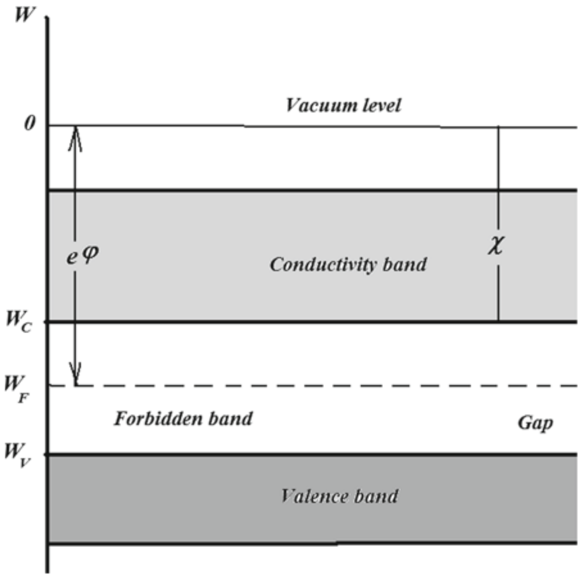

Fig. 5.10 The work function and the electron affinity of a semiconductor

Table 5.2 The work function of some metals

Metal	Cs	Ba	Nb	Au	Ag	Cu	W	Pt
eφ, eV	1.81	2.49	4.0	4.3	4.3	4.4	4.54	5.35

shows the work function for some metals, and Table 5.3 depicts the effective work function and the electron affinity for the most-used semiconductors.

In spite of the complicated structure of the surface potential barrier, the “rectangular” barriers are in common use; for all practical purposes, this is a feasible approximation.

Table 5.3 The work function and the electron affinity of some semiconductors

Semiconductors	Si	Ge	GaAs
$e\varphi$, eV	4.5	4.3	4.7
χ, eV	4.05	4.0	4.07

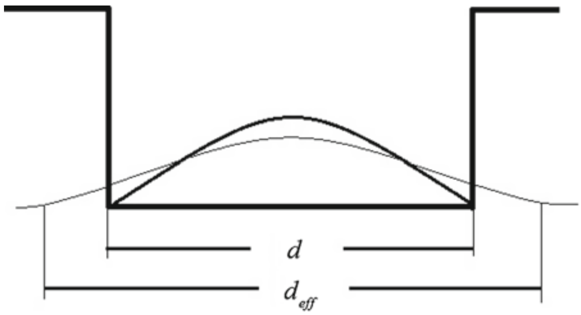

Fig. 5.11 The increase in the waveguide effective width caused by electron penetration under the surface potential barrier

5.7.2 A Qualitative Analysis of a Finite Work Function Impact on Electron Transport

We first consider a waveguide that coincides with an infinite strip $\Pi = \{(x, y) \in \mathbb{R}^2 : -\infty < x < +\infty, -d/2 < y < d/2\}$. In the case of a finite work function, electrons can penetrate through the surface potential barrier; this leads to an increase in the effective diameter of the waveguide cross-section (Fig. 5.11) and to a decrease in the threshold energies. We will take account of the change in thresholds to estimate the work function impact. We assume that the electron potential energy is zero inside the waveguide Π and equal to $U = \text{constant} > 0$ outside Π. For the semiconductors, the U is the electron affinity χ and, for the metals, the U equals the effective work function.

Let us introduce the problem

$$
\begin{aligned}
-\frac{\hbar^2}{2m^*}\psi''(y) &= E_\perp \psi(y), \quad |y| < d/2,\\
-\frac{\hbar^2}{2m}\psi''(y) + U\psi(y) &= E_\perp \psi(y), \quad |y| > d/2,
\end{aligned}
\tag{5.7.1}
$$

where $E_\perp$ is a spectral parameter, m is the electron mass, and m^* is the effective electron mass. The function ψ and its derivative ψ' are supposed to be continuous at $y = \pm d/2$ and, moreover, $\psi(y) \to 0$ as $|y| \to +\infty$. The interval $(0, U)$ may contain only isolated eigenvalues of problem (5.7.1). These eigenvalues represent the waveguide threshold energies not exceeding the U and calculated with regard to the finite work function (see [20, 35]).

Setting $\lambda^2 = (2m^*/\hbar^2)E_\perp$ and $\mu^2 = (2m/\hbar^2)(U - E_\perp)$, we write equations (5.7.1) in the form

$$\psi''(y) + \lambda^2\psi(y) = 0, \quad |y| < d/2,$$
$$\psi''(y) - \mu^2\psi(y) = 0, \quad |y| > d/2.$$

The λ and μ satisfy the transcendental equation

$$\tan(\lambda d) = 2\lambda\mu(\lambda^2 - \mu^2)^{-1}. \tag{5.7.2}$$

Therefore, to obtain approximate values of the mentioned threshold energies, it suffices to solve approximately this equation.

Typically, the U comprises several eV and for lower thresholds $E_\perp(n) \approx \pi^2\hbar^2 n^2/2m^*d^2$. For $m = m^*$ and $d = 10$ nm, we even have $E_\perp(6) \approx 0.2$ eV, which is, roughly, 20 times less than the work function. Thus, $U \gg E_\perp(n)$ for lower thresholds, $\mu \gg \lambda$, and we can restrict ourselves to considering the first approximation only. For several lower thresholds, $\mu^2 \approx (2m/\hbar^2)U$. From Eq. (5.7.2) it follows that

$$\tan((2m^*\hbar^{-2}E_\perp)^{1/2}d) \approx -2(E_\perp/U)^{1/2}. \tag{5.7.3}$$

The right-hand side is small, therefore,

$$(2m^*\hbar^{-2}E_\perp)^{1/2}d = n\pi + \delta, \quad \delta \ll 1.$$

In view of (5.7.3), $\delta \approx -2(E_\perp/U)^{1/2}$ hence

$$E_\perp(n) \approx \frac{\pi^2\hbar^2 n^2}{2m^*d^2}\left(1 + \left(\frac{2\hbar^2}{md^2U}\right)^{1/2}\right)^{-2}.$$

The factor

$$\eta = \left(1 + \left(\frac{2\hbar^2}{md^2U}\right)^{1/2}\right)^{-2}$$

shows the work function impact on $E_\perp(n)$. Assuming the impact to be small, we can take it into account by changing the real waveguide width d for the effective width

$$d_{eff} = d\left(1 + \left(\frac{2\hbar^2}{md^2U}\right)^{1/2}\right).$$

For $d = 10$ nm and $U = 4$ eV, we obtain $\eta = 0.96$. This justifies our assumption that the work function impact on $E_\perp(n)$ is small. However, for very thin waveguides, especially with small work functions, this assumption is less reliable. For $d = 3$ nm

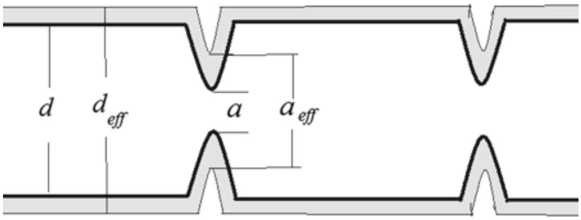

Fig. 5.12 The increase in the waveguide effective width caused by an electron wave function penetration under the surface potential barrier. The waveguide widening is much more significant at the waveguide narrows

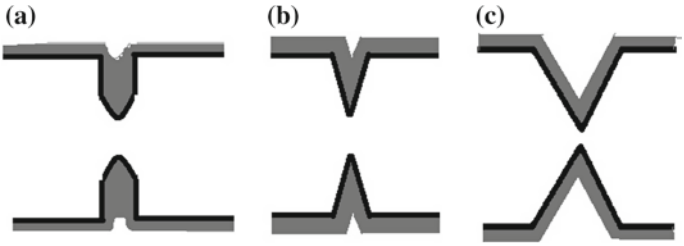

Fig. 5.13 Qualitative picture of a wave function outside a waveguide in the vicinity of narrows

and $U = 2$ eV, we obtain $\eta = 0.84$. For large n, the relation $U \gg E_\perp(n)$ is false and the work function impact is much more significant.

Let us now turn to a waveguide of variable cross-section. It is clear that the most notable impact of the waveguide work function should be expected at the waveguide narrows (Fig. 5.12). The effective diameter of a narrow remains greater than a certain positive value, even though the real diameter tends to zero. A finite work function essentially restricts a choice of narrow forms. Indeed, for a wedge-like narrow, the parts of an electron wave function, corresponding to different sides of the wedge, overlap (Fig. 5.13); therefore, for a wedge with a small angle, the narrow practically vanishes. These qualitative considerations are confirmed by the results of numerical simulations in Sect. 5.7.3.

5.7.3 Numerical Simulation of Resonant Tunneling with Regard to the Waveguide Work Function

Now, we pass on to numerical simulation of electron resonant tunneling in waveguides with finite work function. The waveguide geometry is the same as in Sect. 5.6. To take into account the electron penetration under the potential barrier, we embed the waveguide G_0 in the strip G of a sufficiently large width; in our calculations, the strip width is equal to 5 times the waveguide width d. Outside the waveguide,

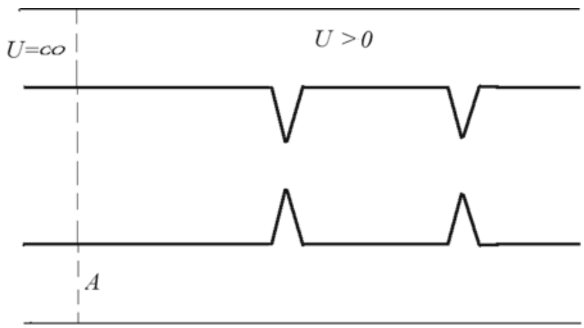

Fig. 5.14 The geometry of a system used for numerical simulation of a work function impact on resonant tunneling

the wave function of an electron decays exponentially with characteristic penetration depth $\delta = \hbar/\sqrt{mU}$, the U is the electron potential energy, and $U > 0$ outside the waveguide; for the semiconductors, U is equal to the electron affinity χ. Therefore, the wave function at the strip boundary is by factor $\exp(-d\sqrt{mU}/\hbar)$ less than that at the waveguide boundary. Even for small work functions and thin waveguides, the inequality $d\sqrt{mU}/\hbar > 10$ holds, so we assume the electron wave function to be zero at the strip boundary. Inside the waveguide, the electron potential energy equals zero. Between the boundaries of the waveguide and the strip, to the left of plane A (Fig. 5.14), the electron potential energy is chosen to be infinite, while to the right of plane A the energy is equal to the material work function. Inside the waveguide, an incident wave is of the same form as in Sect. 5.6 . We consider the scattering of the incident wave in the waveguide shown in Fig. 5.14. The width of the narrow is equal to 0.2 times the width of the waveguide; the angle at the narrow is 0.1π. In such a waveguide, the impact of the finite work function is clearly recognizable.

The wave function satisfies the boundary value problem

$$\begin{aligned} -\Delta U - k^2 U &= 0 \text{ in } G_0, \\ -\Delta U - (k^2 - D^2)U &= 0 \text{ in } G \setminus G_0, \\ U &= 0 \text{ on } \partial G, \end{aligned}$$

where $D = d/2\delta$, and the radiation conditions

$$U(x, y) = e^{i\lambda_m x}\Phi_m(y) + \sum_{j=1}^{n_{\max}} S_{mj} e^{-i\lambda_j x}\Phi_j(y) + O(e^{-\varepsilon|x|}) \qquad \text{as } x \to -\infty,$$

$$U(x, y) = \sum_{j=1}^{n_{\max}} S_{m,j+n_{\max}} e^{i\lambda_j x}\Phi_j(y) + O(e^{-\varepsilon|x|}) \qquad \text{as } x \to +\infty,$$

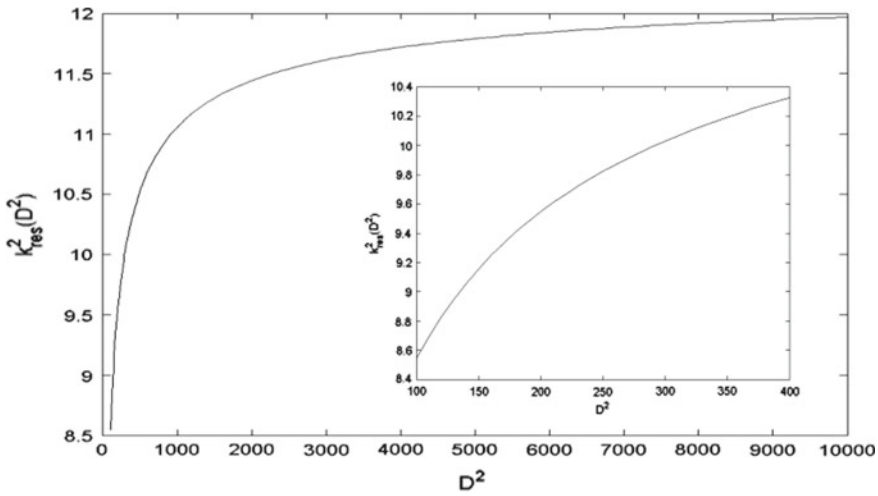

Fig. 5.15 D-dependence of the resonant center k_{res}

where $\varepsilon > 0$, $\lambda_m > 0$, and Φ_m are solutions to the problem

$$
\begin{aligned}
-\Phi''(y) - (k^2 - \lambda^2)\Phi(y) &= 0, \ |y| < d/2, \\
-\Phi''(y) - (k^2 - D^2 - \lambda^2)\Phi(y) &= 0, \ y \in (-5d/2, -d/2) \cup (d/2, 5d/2), \\
\Phi(y) &= 0, \ y = \pm 5d/2,
\end{aligned}
$$

and the functions Φ and Φ' are supposed to be continuous at $y = \pm d/2$. In addition, Φ_m are normalized by

$$\int_{-5d/2}^{5d/2} |\Phi_m(y)|^2 dy = \frac{1}{2\lambda_m}.$$

We study the D-dependence of the basic characteristics of resonant tunneling. The D ranges between 10 and 100. The minimal value of D corresponds to the waveguide width $d = 3$ nm and $U = 2$ eV, while $d = 20$ nm and $U = 5$ eV for the maximal D. The range $20 < D < 50$ of the greatest practical utility corresponds to $d \approx 10$ nm and U in the range 2–5 eV. Figure 5.15 shows k_{res} and Fig. 5.16 depicts the width Δ of the resonant peak. For sufficiently small D, the impact of the finite work function manifests itself in a certain shift of the resonant level and, mainly, in a sharp widening of the resonant peak. A choice of too small diameters of the waveguide narrows at the resonator causes a significant increase in the effective narrow diameters and in the resonator volume, which notably affects the resonator quality factor.

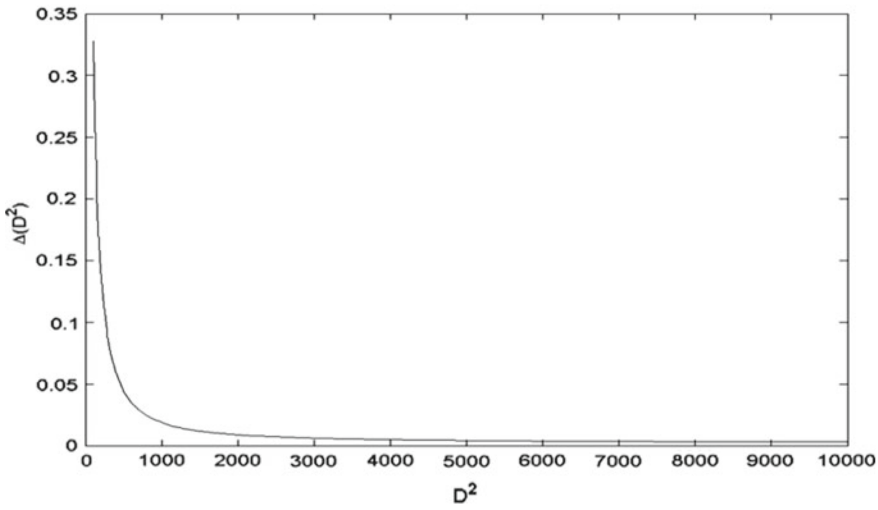

Fig. 5.16 D-dependence of the resonant peak width

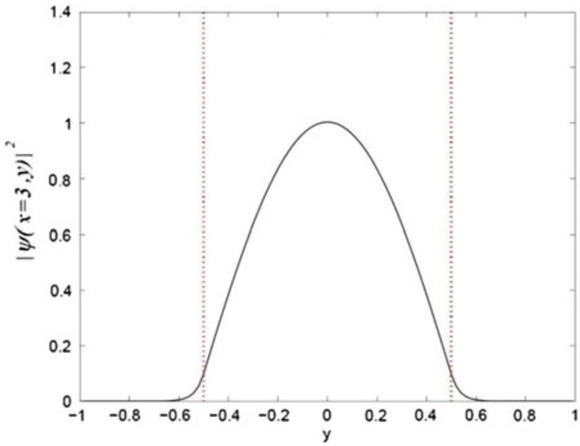

Fig. 5.17 Distribution of $|\psi(\cdot, k_{res})|^2$ (where $\psi(x, y, k)$ is an electron wave function) at the waveguide cross-section outside the resonator

The above results lead to the following conclusions:

1. Far from narrows, a wave function penetrates through waveguide walls in a distance significantly smaller than the diameter of the waveguide (Fig. 5.17). In a neighborhood of narrows, the situation changes (Fig. 5.18), that's why, decreasing the narrow diameter at the resonator, one can not diminish the effective narrow diameter beyond a certain critical value. This restricts the possibility to improve the resonator quality factor by diminishing the narrow diameter.

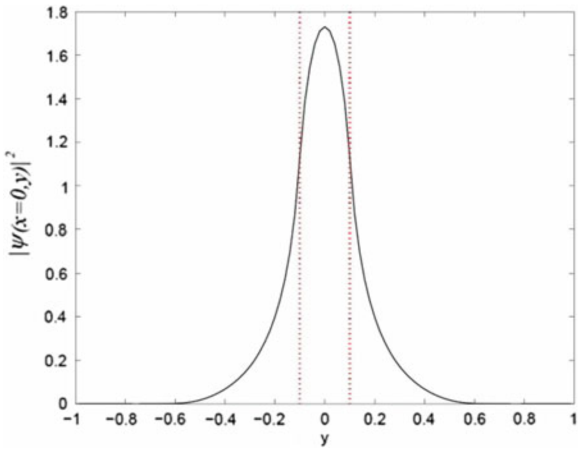

Fig. 5.18 Distribution of $|\psi(\cdot, k_{res})|^2$ (where $\psi(x, y, k)$ is an electron wave function) at the narrow cross-section

2. The distance between the vertical sides of a rectangular waveguide narrow should be significant (say, more than 1 nm), see Fig. 5.13a.
3. The angle of a wedge-like narrow should be sufficiently large. However, increasing the angle causes an increase in the effective width of the potential barrier (Fig. 5.13b) and a decrease in the width of the resonant peak. This increases the resonant tunneling time and affects the frequency properties of the system. Optimal angles for wedge-like narrows range between 20° and 35° (Fig. 5.13c).
4. When choosing a waveguide material, one should prefer that of maximal work function. For instance, for a waveguide of the cross-section diameter ≈ 10 nm, a wedge-like narrow of diameter >3 nm and angle $\approx 30°$, made of a material with electron affinity ≈ 4 eV (e.g., Si), the finite work function impact manifests itself as a negligible shift in the resonant levels and a small decrease in the resonator quality factor.

Chapter 6
Asymptotics of Resonant Tunneling in 3D Waveguides for Electrons of Small Energy

In this chapter, we consider electron propagation in a waveguide with two cylindric outlets to infinity and two narrows of small diameters ε_1 and ε_2. The boundary of the waveguide is assumed to be smooth. The electron motion is described by the Helmholtz equation. The electron energy is supposed to be between the first and the second thresholds. We generalize and implement the asymptotic approach developed in Chap. 5. The basic results are presented by Theorem 6.4.5.

6.1 Statement of the Problem and Outline of the Results

To describe the waveguide, we first introduce three domains G, Ω_1, and Ω_2 in $\mathbb{R}^3$ independent of the parameters ε_1 and ε_2. Let G be a domain in $\mathbb{R}^3$ that, outside a large ball, coincides with the union of two nonoverlapping half-cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ with bounded cross-sections D_1 and D_2, respectively. The boundary ∂G of G is smooth, and ∂D_1 and ∂D_2 are simple contours. Let us consider the domain Ω_1 (Fig. 5.1). We denote by K_1 and L_1 open cones in $\mathbb{R}^3$ that are symmetric to each other about their common vertex, that is, $K_1 \cup L_1$ is a double cone. The cone K_1 (resp., L_1) cuts out on the unit sphere centered at the vertex a domain $S(K_1)$ (resp., $S(L_1)$) bounded by a smooth contour. We suppose that Ω_1 contains both cones K_1 and L_1 as well as a neighborhood of their vertex; moreover, outside a large ball (with center at the vertex), Ω_1 coincides with $K_1 \cup L_1$; the boundary of Ω_1 is smooth. The domain Ω_2 is described like Ω_1 with cones K_2 and L_2.

We now consider the waveguide $G(\varepsilon_1, \varepsilon_2)$ (Fig. 1.1). For the time being, we let O_1 and O_2 be arbitrary (interior) points of the domain G placed (for the sake of simplicity) in the half-cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We now introduce orthogonal coordinates $x^j = (x_1^j, x_2^j, x_3^j)$ with origin O_j and axis x_1^j parallel to the generatrices of the half-cylinder $\mathcal{C}_j$, $j = 1, 2$; the positive half-axis x_1^j lies inside $\mathcal{C}_j$. The domain Ω_j is located so that the vertex of K_j and L_j coincides with O_j and the positive half-axis x_1^j lies inside K_j. From now on, we assume that the points O_1 and O_2

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are disposed far enough from the "noncylindrical" part of G so that the connected component of the set $\partial G \cap \partial L_j$ nearest to O_j coincides with $\partial \mathcal{C}_j \cap \partial L_j$. We denote by $\Omega_j(\varepsilon_j)$ the domain obtained from Ω_j by the contraction with center at O_j and coefficient $\varepsilon_j > 0$. In other words, $x^j \in \Omega_j(\varepsilon_j)$ if and only if $(x^j/\varepsilon_j) \in \Omega_j$. Let $G(\varepsilon_1, \varepsilon_2)$ be the domain obtained from G by changing $\mathcal{C}_1$ and $\mathcal{C}_2$ for $\mathcal{C}_1 \cap \Omega_1(\varepsilon_1)$ and $\mathcal{C}_2 \cap \Omega_2(\varepsilon_2)$, respectively.

A wave function of a free electron of energy $E = \hbar^2 k^2/2m$ satisfies the boundary value problem

$$-\Delta u - k^2 u = 0 \text{ in } G(\varepsilon_1, \varepsilon_2), \quad u = 0 \text{ on } \partial G(\varepsilon_1, \varepsilon_2). \tag{6.1.1}$$

Before formulating radiation conditions at infinity, let us make some comments on the boundary condition. The waveguide boundary is a potential barrier for an electron. The electron wave function exponentially decays outside the waveguide. The characteristic depth of electron penetration under the barrier is about 0.1 nm for the typical electron work function of 4–5 eV. The width of realistic waveguides (even at the narrows) is a few nanometers. Therefore, we neglect the electron penetration under the barrier and assume that $u = 0$ on $\partial G(\varepsilon_1, \varepsilon_2)$.

To formulate the radiation conditions, we need the boundary value problem on the cross-section D_j of the semicylinder $\mathcal{C}_j$, $j = 1, 2$:

$$-\Delta v - \lambda^2 v = 0 \text{ in } D_j, \quad v = 0 \text{ on } \partial D_j. \tag{6.1.2}$$

The eigenvalues $\lambda^2_{j\,m}$ of this problem, where $m = 1, 2, \ldots$, are called the thresholds; they form an increasing sequence of positive numbers tending to $+\infty$. We denote by Ψ_{jm} an eigenfunction of the problem (6.1.2) that corresponds to the eigenvalue $\lambda^2_{j\,m}$ and is normalized by

$$2\nu_{j\,m} \int_{D_j} |\Psi_{j\,m}(x_2, x_3)|^2 dx_2\, dx_3 = 1 \tag{6.1.3}$$

with $\nu_{j\,m} = \sqrt{k^2 - \lambda^2_{j\,m}}$. In this chapter, we discuss only the situation where the parameter k^2 is "between the first and second thresholds" or, more precisely, in the interval $(\lambda^2_{11}, \lambda^2_{12}) \cap (\lambda^2_{21}, \lambda^2_{22})$ (supposed to be nonempty). The function U_1^+ defined in the semicylinder $\mathcal{C}_1$ by $U_1^+(x^1) = \exp(-i\nu_{11}x_1^1)\Psi_{11}(x_2^1, x_3^1)$ is a wave coming in $\mathcal{C}_1$ from infinity (recall that the positive half-axis x_1^1 lies in $\mathcal{C}_1$). The function $U_2^+(x^2) = \exp(-i\nu_{22}x_1^2)\Psi_{21}(x_2^2, x_3^2)$ is a wave coming from infinity in $\mathcal{C}_2$. The outgoing waves U_m^-, $m = 1, 2$, are obtained from the incoming ones by complex conjugation: $U_m^- = \overline{U_m^+}$.

There exist (smooth) solutions u_m, $m = 1, 2$, to problem (6.1.1) satisfying the radiation conditions

$$u_1(x) = \begin{cases} U_1^+(x^1) + S_{11}\, U_1^-(x^1) + O(\exp(-\delta x_1^1)), & x_1^1 \to +\infty, \\ S_{12}\, U_2^-(x^2) + O(\exp(-\delta x_1^2), & x_1^2 \to +\infty, \end{cases} \tag{6.1.4}$$

$$u_2(x) = \begin{cases} S_{21}\, U_1^-(x^1) + O(\exp(-\delta x_1^1)), & x_1^1 \to +\infty, \\ U_2^+(x^2) + S_{22}\, U_2^-(x^2) + O(\exp(-\delta x_1^2), & x_1^2 \to +\infty, \end{cases} \tag{6.1.5}$$

with sufficiently small positive δ. The scattering matrix $S = \|S_{pq}\|_{p,q=1,2}$ is unitary.

We consider the scattering of the wave coming from $\mathcal{C}_1$ and seek the resonant values $k_r = k_r(\varepsilon_1, \varepsilon_2)$ of the parameter k, where the transition coefficient $T_1 = T_1(k, \varepsilon_1, \varepsilon_2) = |S_{12}|^2$ takes the maximal values. Moreover, we are interested in the behavior of $k_r(\varepsilon_1, \varepsilon_2)$, $T_1(k, \varepsilon_1, \varepsilon_2)$ and that of the reflection coefficient $R_1 = R_1(k, \varepsilon_1, \varepsilon_2) = |S_{11}|^2$, as $\varepsilon_1, \varepsilon_2 \to 0$.

To outline the results, we present some formulas obtained in the chapter. The limit domain $G(0,0)$ consists of the unbounded parts G_1, G_2 and the bounded resonator G_0. Let $S(L_j)$ be the domain that the cone L_j cuts out on the unit sphere centered at O_j and let $0 < \mu_{j1} < \mu_{j2} < \cdots$ stand for the numbers such that $\mu_{jm}(\mu_{jm}+1)$ are the eigenvalues of the Dirichlet problem for the Beltrami operator in $S(L_j)$, $(m = 1, 2, \dots)$. Assume that k_e^2 is any eigenvalue (lying between the first and second thresholds) of the boundary value problem in the resonator,

$$-\Delta v(x) - k^2 v(x) = f, \quad x \in G_0; \quad v(x) = 0, \quad x \in \partial G_0.$$

Near such an eigenvalue, there is a resonant value $k_r(\varepsilon_1, \varepsilon_2)$ satisfying

$$k_r^2(\varepsilon_1, \varepsilon_2) = k_e^2 + \mathcal{D}_1 \varepsilon_1^{2\mu_{11}+1} + \mathcal{D}_2 \varepsilon_2^{2\mu_{21}+1} + O\big(\varepsilon_1^{2\mu_{11}+1+\tau_1} + \varepsilon_2^{2\mu_{21}+1+\tau_2}\big) \tag{6.1.6}$$

as $\varepsilon_1, \varepsilon_2 \to 0$. The coefficients $\mathcal{D}_1$ and $\mathcal{D}_2$ are constant, $\tau_j = \min\{\mu_{j2} - \mu_{j1}, 2 - \sigma_j\}$, and σ_j are small positive numbers; for more detail, see Theorem 6.4.5.

Under the condition $|k^2 - k_r^2| = O\big(\varepsilon_1^{2\mu_{11}+1+\tau_1} + \varepsilon_2^{2\mu_{21}+1+\tau_2}\big)$, the transition coefficient $T_1(k, \varepsilon_1, \varepsilon_2)$ satisfies

$$T_1(k, \varepsilon_1, \varepsilon_2) = \left(\frac{1}{4}\left(z + \frac{1}{z}\right)^2 + P^2 \left(\frac{k^2 - k_r^2}{\varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{2\mu_{21}+1}} \right)^2 \right)^{-1} \big(1 + O(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2})\big),$$

where τ_j are the same as in (6.1.6), $z = Q\varepsilon_1^{2\mu_{11}+1}/\varepsilon_2^{2\mu_{21}+1}$, and P and Q are constant. (For further detail, we again refer to Theorem 6.4.5.)

Finally, the width of the resonant peak at its half-height (calculated for the principal part in the asymptotics of T_1) is $\Upsilon(\varepsilon_1, \varepsilon_2) = |(z + z^{-1})/P| \varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{2\mu_{21}+1} (1 + O(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2}))$.

6.2 Limit Problems

To derive an asymptotics of a wave function (i.e., a solution to problem (6.1.1)) as $\varepsilon_1, \varepsilon_2 \to 0$, we use the method of compound asymptotic expansions. To this end, we introduce "limit" boundary value problems independent of the parameters ε_1 and ε_2. Actually, the reader could skip every mention of function spaces in the section; this will not prevent the reader from understanding the rest of the text apart from Sects. 4.4 and 5.3.

6.2.1 *First Kind Limit Problems*

Recall that the limit domain $G(0, 0)$ consists of the unbounded parts G_1, G_2 and the bounded resonator G_0. The problems

$$-\Delta v(x) - k^2 v(x) = f, \quad x \in G_j; \quad v(x) = 0, \quad x \in \partial G_j, \tag{6.2.1}$$

are called the first kind limit problems, where $j = 0, 1, 2$.

We introduce function spaces for the problem (6.2.1) in G_0. Let ϕ_1 and ϕ_2 be smooth real functions in the closure $\overline{G}_0$ of G_0 such that $\phi_j = 1$ in a neighborhood of O_j, $j = 1, 2$, and $\phi_1^2 + \phi_2^2 = 1$. For $l = 0, 1, \dots$ and $\gamma_j \in \mathbb{R}$, the space $V^l_{\gamma_1,\gamma_2}(G_0)$ is the completion in the norm

$$\|v; V^l_{\gamma_1,\gamma_2}(G_0)\| = \left(\int_{G_0} \sum_{|\alpha|=0}^{l} \sum_{j=1}^{2} \phi_j^2(x) r_j(x)^{2(\gamma_j - l + |\alpha|)} |\partial^\alpha v(x)|^2 \, dx \right)^{1/2} \tag{6.2.2}$$

of the set of smooth functions in $\overline{G}_0$ vanishing near O_1 and O_2; here $r_j(x) = \operatorname{dist}(x, O_j)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$. Proposition 6.2.1 follows from the well-known general results; e.g., see [37, Chapters 2 and 4, Sections 1–3] or [33, v.1, Chapter 1].

Proposition 6.2.1 (i) *Assume that $|\gamma_j - 1| < \mu_{j\,1} + 1/2$, where $\mu_{j\,1}$ ($j = 1, 2$) is the same as in (6.1.6). Then, for every $f \in V^0_{\gamma_1,\gamma_2}(G_0)$ and any k^2, except the positive increasing sequence $\{k_p^2\}_{p=1}^\infty$ of eigenvalues, $k_p^2 \to \infty$, there exists a unique solution $v \in V^2_{\gamma_1,\gamma_2}(G_0)$ to problem (6.2.1) in G_0. The estimate*

$$\|v; V^2_{\gamma_1,\gamma_2}(G_0)\| \le c\|f; V^0_{\gamma_1,\gamma_2}(G_0)\| \tag{6.2.3}$$

holds with a constant c independent of f.

(ii) *Let f be a smooth function in $\overline{G}_0$ vanishing near O_1 and O_2 and let v be any solution in $V^2_{\gamma_1,\gamma_2}(G_0)$ of problem (6.2.1). Then v is smooth in $\overline{G}_0$ except at O_1 and O_2 and admits the asymptotic representations*

$$v(x) = b_j \frac{1}{\sqrt{r_j}} \widetilde{J}_{\mu_{j1}+1/2}(kr_j)\Phi^L_{j1}(\varphi_j) + O\big(r_j^{\mu_{j2}}\big), \quad r_j \to 0, \; j = 1, 2$$

near the points O_1 and O_2, where (ρ_j, φ_j) are polar coordinates with center at O_j, b_j are some constant coefficients, and $\widetilde{J}_\mu$ denotes the Bessel function multiplied by a constant such that $r^{-1/2}\widetilde{J}_{\mu_{j1}+1/2}(kr) = r^{\mu_{j1}} + o(r^{\mu_{j1}})$; the Φ^L_{j1} is an eigenfunction of the Beltrami operator corresponding to the eigenvalue $\mu_{j1}(\mu_{j1}+1)$ and normalized by the condition

$$(2\mu_{j1}+1)\int_{S(L_j)} |\Phi^L_{j1}(\varphi)|^2 d\varphi = 1.$$

(iii) *Assume that $k^2 = k_e^2$ is an eigenvalue of problem* (6.2.1). *Then problem* (6.2.1) *in G_0 is solvable if and only if $(f, v_e)_{G_0} = 0$ for any eigenfunction v_e corresponding to k_e^2. Under such conditions, there exists for problem* (6.2.1) *a unique solution v that is orthogonal to the eigenfunctions and satisfies* (6.2.3) *(i.e., the Fredholm alternative holds).*

We turn to problems (6.2.1) for $j = 1, 2$. Let $\chi_{0,j}$ and $\chi_{\infty,j}$ be smooth real functions in the closure $\overline{G}_j$ of G_j such that $\chi_{0,j} = 1$ in a neighborhood of O_j, $\chi_{0,j}$ vanishes outside a compact set and $\chi^2_{0,j} + \chi^2_{\infty,j} = 1$. We also assume that the support $\operatorname{supp}\chi_{\infty,j}$ is located in the cylindrical part $\mathcal{C}_j$ of G_j. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G_j)$ is the completion in the norm

$$\|v; V^l_{\gamma,\delta}(G_j)\| = \left(\int_{G_j} \sum_{|\alpha|=0}^{l} \big(\chi^2_{0,j} r_j^{2(\gamma-l+|\alpha|)} + \chi^2_{\infty,j}\exp(2\delta x_1^j)\big)|\partial^\alpha v|^2\, dx^j\right)^{1/2} \tag{6.2.4}$$

of the set of smooth functions in $\overline{G}_j$ vanishing near O_j and having compact supports.

Let $S(K_j)$ be the domain that the cone K_j cuts out on the unit sphere centered at O_j. Since the domains $S(K_j)$ and $S(L_j)$ are symmetric, the eigenvalues of the Dirichlet problem for the Beltrami operator in $S(K_j)$ coincide with $\mu_{jm}(\mu_{jm}+1)$, $m = 1, 2, \dots$. Recall that, according to our assumption, k^2 lies between the first and the second thresholds, so in every G_j there is the only outgoing wave U_j^-. The next proposition follows, e.g., from [37, Theorem 5.3.5].

Proposition 6.2.2 *Assume that $|\gamma - 1| < \mu_{j1} + 1/2$ and, moreover, there is no nontrivial solution to the homogeneous problem* (6.2.1) *(where $f = 0$) in $V^2_{\gamma,\delta}(G_j)$ with arbitrary small positive δ. Then, for any $f \in V^0_{\gamma,\delta}(G_j)$, there exists a unique solution v to problem* (6.2.1) *that admits the representation*

$$v = u + A_j\chi_{\infty,j}U_j^-,$$

where $A_j = const$, $u \in V^2_{\gamma,\delta}(G_j)$ with a sufficiently small δ, and the estimate

$$\|u; V^2_{\gamma,\delta}(G_j)\| + |A_j| \le c\|f; V^0_{\gamma,\delta}(G_j)\| \tag{6.2.5}$$

holds with a constant c independent of f. If, in addition, f is smooth and vanishes near O_j, the solution v satisfies

$$v(x^j) = a_j \frac{1}{\sqrt{r_j}} \widetilde{J}_{\mu_{j1}+1/2}(kr_j)\Phi^K_{j1}(\varphi_j) + O\big(r_j^{\mu_{j2}}\big), \quad r_j \to 0,$$

where a_j is a constant and Φ^K_{j1} denotes an eigenfunction to the Beltrami operator corresponding to $\mu_{j1}(\mu_{j1}+1)$ and normalized by

$$(2\mu_{j1}+1)\int_{S(K_j)} |\Phi^K_{j1}(\varphi)|^2 d\varphi = 1.$$

6.2.2 Second Kind Limit Problems

In the domains Ω_j, $j = 1, 2$, introduced in Sect. 6.1, we consider the boundary value problems

$$\Delta w(\xi^j) = F(\xi^j) \text{ in } \Omega_j, \quad w(\xi^j) = 0 \text{ on } \partial\Omega_j, \tag{6.2.6}$$

which are called the second kind limit problems; by $\xi^j = (\xi^j_1, \xi^j_2, \xi^j_3)$ we mean Cartesian coordinates with origin at O_j.

Let $\rho_j(\xi^j) = \operatorname{dist}(\xi^j, O_j)$ and let $\psi_{0,j}$, $\psi_{\infty,j}$ be smooth real functions in $\overline{\Omega}_j$ such that $\psi_{0,j}$ equals 1 for $\rho_j < N/2$, vanishes for $\rho_j > N$, and $\psi^2_{0,j} + \psi^2_{\infty,j} = 1$, the N being a sufficiently large positive number. For $\gamma \in \mathbb{R}$ and $l = 0, 1, \ldots$, the space $V^l_\gamma(\Omega_j)$ is the completion in the norm

$$\|v; V^l_\gamma(\Omega_j)\| = \left(\int_{\Omega_j} \sum_{|\alpha|=0}^{l} \big(\psi_{0,j}(\xi^j)^2 + \psi_{\infty,j}(\xi^j)^2 \rho_j(\xi^j)^{2(\gamma-l+|\alpha|)}\big)|\partial^\alpha v(\xi^j)|^2\, d\xi^j\right)^{1/2} \tag{6.2.7}$$

of the set $C^\infty_c(\overline{\Omega}_j)$ of compactly supported smooth functions in $\overline{\Omega}_j$. The next proposition follows from [37, Theorem 4.3.6].

Proposition 6.2.3 *Let $|\gamma - 1| < \mu_{j1} + 1/2$. Then, for every $F \in V^0_\gamma(\Omega_j)$, there exists a unique solution $w \in V^2_\gamma(\Omega_j)$ to problem (6.2.6), and*

$$\|w; V^2_\gamma(\Omega_j)\| \le c\|F; V^0_\gamma(\Omega_j)\| \tag{6.2.8}$$

holds with a constant c independent of F. For $F \in C_c^\infty(\overline{\Omega}_j)$, the w is infinitely differentiable in $\overline{\Omega}_j$ and admits the representation

$$w(\xi^j) = d_j^K \rho_j^{-\mu_{j1}-1} \Phi_{j1}^K(\varphi_j) + O\big(\rho_j^{-\mu_{j2}-1}\big), \quad \rho_j \to \infty \tag{6.2.9}$$

in the cone K_j; here (ρ_j, φ_j) are polar coordinates in Ω_j with center at O_j, the μ_{jp} and Φ_{j1}^K are the same as in Proposition 6.2.2, and d_j^K is a constant coefficient. In the cone L_j, a similar expansion holds with d_j^L and Φ_{j1}^L instead of d_j^K and Φ_{j1}^K. The d_j^K and d_j^L are defined by

$$d_j^K = -(F, w_j^K)_{\Omega_j}, \quad d_j^L = -(F, w_j^L)_{\Omega_j},$$

where w_j^K and w_j^L are unique solutions to homogeneous problem (6.2.6) such that, as $\rho_j \to \infty$,

$$w_j^K = \begin{cases} \big(\rho_j^{\mu_{j1}} + \alpha_j \rho_j^{-\mu_{j1}-1}\big)\Phi_{j1}^K(\varphi_j) + O\big(\rho_j^{-\mu_{j2}-1}\big) & \text{in } K_j, \\ \beta_j \rho_j^{-\mu_{j1}-1} \Phi_{j1}^L(\varphi_j) + O\big(\rho_j^{-\mu_{j2}-1}\big) & \text{in } L_j, \end{cases} \tag{6.2.10}$$

$$w_j^L = \begin{cases} \beta_j \rho_j^{-\mu_{j1}-1} \Phi_{j1}^K(\varphi_j) + O\big(\rho_j^{-\mu_{j2}-1}\big) & \text{in } K_j, \\ \big(\rho_j^{\mu_{j1}} + \alpha_j \rho_j^{-\mu_{j1}-1}\big)\Phi_{j1}^L(\varphi_j) + O\big(\rho_j^{-\mu_{j2}-1}\big) & \text{in } L_j, \end{cases} \tag{6.2.11}$$

the coefficients α_j, β_j being constant.

6.3 Tunneling in a Waveguide with One Narrow

The purpose of this section is to carry out preliminary constructions which will be of use in further steps but not related to the phenomenon of resonance. We thereby lighten the exposition of the next section and, in so doing, demonstrate the compound asymptotics method in a more simpler situation. We consider the electron motion in a waveguide $G(\varepsilon)$ with one narrow. To describe $G(\varepsilon)$, we assume that $G = D \times \mathbb{R}$, where D is a bounded domain in $\mathbb{R}^2$ and ∂D is a smooth simple contour. A double cone $K \cup L$ with vertex $O \in G$, domains Ω and $\Omega(\varepsilon)$ are defined like $K_1 \cup L_1$, Ω_1, and $\Omega_1(\varepsilon)$ in Sect. 6.1. We set $G(\varepsilon) = G \cap \Omega(\varepsilon)$. The limit waveguide $G(0)$ consists of two components; either of them has one conical point and one cylindrical end at infinity. We denote the components by G_1 and G_2.

6.3.1 Special Solutions to the First Kind Homogeneous Problems

In the domain G_1, there exists a bounded solution V_1 satisfying the radiation condition

$$V_1(x) = U_1^+(x) + S_{11}^0 U_1^-(x) + O(\exp(-\delta x_1)), \quad x_1 \to +\infty, \tag{6.3.1}$$

with arbitrary small positive δ. The scattering matrix in G_1 consists of the only entry S_{11}^0, $|S_{11}^0| = 1$. The solution V_1 serves as a first approximation to the wave function u_1 determined by radiation conditions (6.1.4). In a neighborhood of O, there is the asymptotics

$$V_1(x) = s_1 \frac{1}{\sqrt{r}} \widetilde{J}_{\mu_1+1/2}(kr)\Phi_1^K(\varphi) + O\big(r^{\mu_2}\big), \quad r \to 0. \tag{6.3.2}$$

In G_2, we consider analogous solution admitting the expansions

$$V_2(x) = \begin{cases} U_2^-(x) + S_{22}^0 U_2^-(x) + O(e^{\delta x_1}), & x_1 \to -\infty, \\ s_2 \dfrac{1}{\sqrt{r}} \widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^L(\varphi) + O(r^{\mu_2}), & r \to 0. \end{cases} \tag{6.3.3}$$

In either of the domains G_1 and G_2, we assume that the homogeneous problem (6.2.1) (with $f = 0$) has no nontrivial bounded solutions exponentially decaying at infinity. In what follows, to construct an asymptotics of a wave function, we will use special solutions to the problem unbounded near the point O.

Let us consider the problem

$$-\Delta u - k^2 u = 0 \text{ in } K, \quad u = 0 \text{ on } \partial K. \tag{6.3.4}$$

The function

$$v_1^K(r,\varphi) = \frac{1}{\sqrt{r}} \widetilde{N}_{\mu_1+1/2}(kr)\Phi_1^K(\varphi) \tag{6.3.5}$$

satisfies (6.3.4); $\widetilde{N}_\mu$ stands for the Neumann function multiplied by a constant such that

$$\frac{1}{\sqrt{r}} \widetilde{N}_{\mu_1+1/2}(kr) = r^{-\mu_1-1} + o(r^{-\mu_1-1}),$$

μ_1 and Φ_1^K are the same as in Proposition 6.2.2. Let $t \mapsto \Theta(t)$ be a cut-off function on $\mathbb{R}$ equal to 1 for $t < \delta/2$ and to 0 for $t > \delta$ with a small positive δ. We introduce a solution

$$\mathbf{v}_1(x) = \Theta(r) v_1^K(x) + \widetilde{v}_1(x) \tag{6.3.6}$$

of homogeneous problem (6.2.1) in G_1, where $\widetilde{v}_1$ is the solution provided by Proposition 6.2.2 for problem (6.2.1) with $f = [\triangle, \Theta] v_1^K$. Thus,

$$\mathbf{v}_1(x) = \begin{cases} \dfrac{1}{\sqrt{r}} \big(\widetilde{N}_{\mu_1+1/2}(kr) + a_1 \widetilde{J}_{\mu_1+1/2}(kr)\big) \Phi_1^K(\varphi) + O(r^{\mu_2}), & r \to 0, \\ A_1 U_1^-(x) + O(e^{-\delta x_1}), & x_1 \to +\infty, \end{cases} \tag{6.3.7}$$

where $\widetilde{J}_\mu$ is the same as in Propositions 6.2.1 and 6.2.2. In G_2, analogous solution $\mathbf{v}_2$ admits

$$\mathbf{v}_2(x) = \begin{cases} \dfrac{1}{\sqrt{r}} \big(\widetilde{N}_{\mu_1+1/2}(kr) + a_2 \widetilde{J}_{\mu_1+1/2}(kr)\big) \Phi_1^L(\varphi) + O(r^{\mu_2}), & r \to 0, \\ A_2 U_2^-(x) + O(e^{\delta x_1}), & x_1 \to -\infty. \end{cases} \tag{6.3.8}$$

Lemma 6.3.1 *The equalities* $|A_j|^2 = 2\,\mathrm{Im}\, a_j$, $A_j = i\overline{s_j} S^0_{jj}$ *hold.*

Proof We prove the Lemma as $j = 1$; the case $j = 2$ can be treated in a similar way. Let $(u, v)_Q$ denote the integral $\int_Q u(x)\overline{v(x)}\, dx$ and let $G_{N,\delta}$ stand for the truncated domain $G_1 \cap \{x_1 < N\} \cap \{r > \delta\}$. By the Green formula,

$$\begin{aligned} 0 &= (\Delta \mathbf{v}_1 + k^2 \mathbf{v}_1, \mathbf{v}_1)_{G_{N,\delta}} - (\mathbf{v}_1, \Delta \mathbf{v}_1 + k^2 \mathbf{v}_1)_{G_{N,\delta}} \\ &= (\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta}} - (\mathbf{v}_1, \partial \mathbf{v}_1/\partial n)_{\partial G_{N,\delta}} = 2i\, \mathrm{Im}\, (\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_E \end{aligned}$$

with $E = (\partial G_{N,\delta} \cap \{x_1 = N\}) \cup (\partial G_{N,\delta} \cap \{r = \delta\})$. Taking into account (6.3.7) as $x_1 \to +\infty$ and (6.1.3), we have

$$\begin{aligned} \mathrm{Im}\, (\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta} \cap \{x_1 = N\}} &= \mathrm{Im} \int_{D_1} A_1 \frac{\partial U_1^-}{\partial x_1}(x) \overline{A_1 U_1^-(x)} \Big|_{x_1 = N} dx_2\, dx_3 + o(1) \\ &= |A_1|^2 \nu_1 \int_{D_1} |\Psi_1(x_2, x_3)|^2 dx_2\, dx_3 + o(1) \\ &= |A_1|^2/2 + o(1). \end{aligned}$$

Using (6.3.7) as $r \to 0$ and the normalization of Φ_1^K (see Proposition 6.2.2), we obtain

$$\begin{aligned} \mathrm{Im}\, (\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta} \cap \{r = \delta\}} = &\,\mathrm{Im} \int_{S(K)} \left[-\frac{\partial}{\partial r} \frac{1}{\sqrt{r}} \big(\widetilde{N}_{\mu_1+1/2}(kr) + a_1 \widetilde{J}_{\mu_1+1/2}(kr)\big) \right] \\ &\times \frac{1}{\sqrt{r}} \big(\widetilde{N}_{\mu_1+1/2}(kr) + \overline{a}_1 \widetilde{J}_{\mu_1+1/2}(kr)\big) |\Phi_1^K(\varphi)|^2 r^2 \Big|_{r=\delta} d\varphi + o(1) \\ = &- (\mathrm{Im}\, a_1)(2\mu_1 + 1) \int_{G_{N,\delta}} |\Phi_1^K(\varphi)|^2 d\varphi + o(1) = -\mathrm{Im}\, a_1 + o(1). \end{aligned}$$

Thus, $|A_1|^2/2 - \operatorname{Im} a_1 + o(1) = 0$ as $N \to \infty$ and $\delta \to 0$, which implies the first equality of this lemma. To obtain the second one, we apply the Green formula in the domain $G_{N,\delta}$ for the functions $\mathbf{v_1}$ and V_1 and arrive at $iA_1\overline{S^0_{11}} + \overline{s_1} + o(1) = 0$ with $N \to \infty$ and $\delta \to 0$. It remains to take into account that $\overline{S^0_{11}} = 1/S^0_{11}$. □

6.3.2 Passing Through the Narrow

Let v_1 and v_2 satisfy the homogeneous first kind limit problems in G_1 and G_2, respectively, and let

$$v_1 = \frac{1}{\sqrt{r}}\big(a_1^- \widetilde{N}_{\mu_1+1/2}(kr) + a_1^+ \widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^K(\varphi) + O(r^{\mu_2}), \quad r \to 0, \tag{6.3.9}$$

$$v_2 = \frac{1}{\sqrt{r}}\big(a_2^- \widetilde{N}_{\mu_1+1/2}(kr) + a_2^+ \widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^L(\varphi) + O(r^{\mu_2}), \quad r \to 0. \tag{6.3.10}$$

We assume that a wave function in $G(\varepsilon)$ is approximated, outside of a neighborhood of the narrow, by v_1 in G_1 and by v_2 in G_2. To find a relation between $a_1^{\pm}$ and $a_2^{\pm}$, we construct the principal term of the wave function asymptotics as $\varepsilon \to 0$.

Let us employ to this end the compound asymptotics method. We introduce a cut-off function $\chi_{\varepsilon,1}(x) = \big(1 - \Theta(\varepsilon^{-1}r)\big)\,\mathbf{1}_{G_1}(x)$, where Θ is the same as in (6.3.6) and $\mathbf{1}_{G_1}$ is the indicator of G_1 (equal to one in G_1 and to zero outside G_1). Extend $\chi_{\varepsilon,1}v_1$ by zero to the whole $G(\varepsilon)$ and substitute to problem (6.1.1) with $G(\varepsilon_1, \varepsilon_2)$ changed for $G(\varepsilon)$. We obtain the discrepancy

$$-(\triangle + k^2)\chi_{\varepsilon,1}v_1 = -[\triangle, \chi_{\varepsilon,1}]v_1 - \chi_{\varepsilon,1}(\triangle + k^2)v_1 = -[\triangle, 1 - \Theta(\varepsilon^{-1}r)]v_1,$$

while the boundary condition is fulfilled. The discrepancy differs from 0 only near the narrow, where v_1 can be replaced by its asymptotics. Then, with $\rho = \varepsilon^{-1}r$,

$$\begin{aligned} -(\triangle + k^2)\chi_{\varepsilon,1}v_1 &\sim -[\triangle, 1 - \Theta(\varepsilon^{-1}r)]\left(a_1^- r^{-\mu_1-1} + a_1^+ r^{\mu_1}\right)\Phi_1^K(\varphi) \\ &= -\varepsilon^{-2}[\triangle_{(\rho,\varphi)}, \zeta^K(\rho)]\left(a_1^- \varepsilon^{-\mu_1-1}\rho^{-\mu_1-1} + a_1^+ \varepsilon^{\mu_1}\rho^{\mu_1}\right)\Phi_1^K(\varphi); \end{aligned}$$

ζ^K denotes the function $1 - \Theta$ first restricted to the cone K and then extended by zero to the whole Ω. Similarly, we introduce the cut-off function $\chi_{\varepsilon,2}(x) = \big(1 - \Theta(\varepsilon^{-1}r)\big)\,\mathbf{1}_{G_2}(x)$ and extend $\chi_{\varepsilon,2}v_2$ by zero to $G(\varepsilon)$. Then, by virtue of (6.3.10),

$$-(\triangle + k^2)\chi_{\varepsilon,2}v_2 \sim -\varepsilon^{-2}[\triangle_{(\rho,\varphi)}, \zeta^L(\rho)]\left(a_2^- \varepsilon^{-\mu_1-1}\rho^{-\mu_1-1} + a_2^+ \varepsilon^{\mu_1}\rho^{\mu_1}\right)\Phi_2^L(\varphi),$$

where $\zeta^L = 1 - \Theta - \zeta^K$. We also introduce the solution w of problem (6.2.6) in Ω with right-hand side

$$F(\rho,\varphi) = -[\Delta,\zeta^K]\left(a_1^-\varepsilon^{-\mu_1-1}\rho_1^{-\mu_1-1} + a_1^+\varepsilon^{\mu_1}\rho_1^{\mu_1}\right)\Phi_1^K(\varphi)$$
$$-[\Delta,\zeta^L]\left(a_2^-\varepsilon^{-\mu_1-1}\rho_1^{-\mu_1-1} + a_2^+\varepsilon^{\mu_1}\rho_1^{\mu_1}\right)\Phi_1^L(\varphi) \qquad (6.3.11)$$

and substitute $\chi_{\varepsilon,1}(x)v_1(x) + \Theta(r)w(\varepsilon^{-1}x) + \chi_{\varepsilon,2}(x)v_2(x)$ into (6.1.1)(for the waveguide $G(\varepsilon)$):

$$\begin{aligned}
&(\triangle + k^2)\left(\chi_{\varepsilon,2}(x)v_2(x) + \Theta(r)w(\varepsilon^{-1}x) + \chi_{\varepsilon,2}(x)v_2(x)\right)\\
&= [\triangle,\chi_{\varepsilon,1}(x)]\left(v_1(x) - \left(a_1^- r^{-\mu_1-1} + a_1^+ r^{\mu_1}\right)\Phi_1^K(\varphi)\right)\\
&\quad + [\triangle,\Theta(r)]w(\varepsilon^{-1}x) + k^2\Theta(r)w(\varepsilon^{-1}x)\\
&\quad + [\triangle,\chi_{\varepsilon,2}(x)]\left(v_2(x) - \left(a_2^- r^{-\mu_1-1} + a_2^+ r^{\mu_1}\right)\Phi_1^L(\varphi)\right).
\end{aligned}$$

Thus, the principal terms of the discrepancies, originated from the terms $\chi_{\varepsilon,j}v_j$, are compensated. It will be shown in the proof of Theorem 6.3.6 that the term $k^2\Theta w$ is small. For $[\triangle,\Theta(r)]w$ to be small, the w must rapidly decay at infinity. Proposition 6.2.3 provides a solution w satisfying the estimate $w = O(\rho^{-\mu_1-1})$ as $\rho \to +\infty$. However, in this case, $[\Delta,\Theta(r)]w$ is of the same order as the terms already compensated. Therefore, we require the estimate $w = O(\rho^{-\mu_2-1})$ as $\rho \to +\infty$.

Lemma 6.3.2 *Let the solution w of problem* (6.2.6) *with right-hand side*

$$F(\xi) = -[\Delta,\zeta^K]\left(a_1^-\varepsilon^{-\mu_1-1}\rho_j^{-\mu_1-1} + a_1^+\varepsilon^{\mu_1}\rho^{\mu_1}\right)\Phi_1^K(\varphi)$$
$$-[\Delta,\zeta^L]\left(a_2^-\varepsilon^{-\mu_1-1}\rho^{-\mu_1-1} + a_2^+\varepsilon^{\mu_1}\rho^{\mu_1}\right)\Phi_1^L(\varphi)$$

admit the estimate $O(\rho^{-\mu_2-1})$ as $\rho\to\infty$. Then

$$a_1^-\varepsilon^{-\mu_1-1} - \alpha a_1^+\varepsilon^{\mu_1} - \beta a_2^+\varepsilon^{\mu_1} = 0, \quad a_2^-\varepsilon^{-\mu_1-1} - \alpha a_2^+\varepsilon^{\mu_1} - \beta a_1^+\varepsilon^{\mu_1} = 0, \qquad (6.3.12)$$

where α and β are the coefficients in (6.2.10) *and* (6.2.11).

Proof By Proposition 6.2.3, $w = O(\rho^{-\mu_2-1})$ as $\rho\to\infty$, if and only if the right-hand side of problem (6.2.6) satisfies the conditions

$$(F, w^K)_\Omega = 0, \quad (F, w^L)_\Omega = 0, \qquad (6.3.13)$$

where w^K and w^L are the solutions to homogeneous problem (6.2.6) with expansions (6.2.10) and (6.2.11). In Ω, we introduce the functions

$$f_\pm^K(\rho,\varphi) = \rho^{\pm(\mu_1+1/2)-1/2}\Phi_1^K(\varphi), \quad f_\pm^L(\rho,\varphi) = \rho^{\pm(\mu_1+1/2)-1/2}\Phi_1^L(\varphi).$$

To derive (6.3.12) from (6.3.13), it suffices to verify that

$$
\begin{aligned}
([\Delta, \zeta^K] f_-^K, w^K)_\Omega &= ([\Delta, \zeta^L] f_-^L, w^L)_\Omega = -1, \\
([\Delta, \zeta^K] f_+^K, w^K)_\Omega &= ([\Delta, \zeta^L] f_+^L, w^L)_\Omega = \alpha, \\
([\Delta, \zeta^L] f_-^L, w^K)_\Omega &= ([\Delta, \zeta^K] f_-^K, w^L)_\Omega = 0, \\
([\Delta, \zeta^L] f_+^L, w^K)_\Omega &= ([\Delta, \zeta^K] f_+^K, w^L)_\Omega = \beta.
\end{aligned}
$$

Let us check the first equality; the other ones can be considered in a similar way. The support of $[\Delta, \zeta^K] f_-^K$ is compact, so when calculating $([\Delta, \zeta^K] f_-^K, w^K)_\Omega$, one can replace Ω by $\Omega^R = \Omega \cap \{\rho < R\}$ with sufficiently large R. Let E denote the set $\partial\Omega^R \cap \{\rho = R\} \cap K$. By the Green formula,

$$
\begin{aligned}
([\Delta, \zeta^K] f_-^K, w^K)_\Omega &= (\Delta \zeta^K f_-^K, w^K)_{\Omega^R} - (\zeta^K f_-^K, \Delta w^K)_{\Omega^R} \\
&= (\partial f_-^K/\partial n, w^K)_E - (f_-^K, \partial w^K/\partial n)_E.
\end{aligned}
$$

Taking into account (6.2.10) in K and the definition Φ_1^K in Proposition 6.2.2, we obtain

$$
\begin{aligned}
&([\Delta, \zeta^K] f_-^K, w^K)_\Omega \\
&\quad = \left[\frac{\partial \rho^{-\mu_1 - 1}}{\partial \rho} (\rho^{\mu_1} + \alpha \rho^{-\mu_1 - 1}) - \rho^{-\mu_1 - 1} \frac{\partial}{\partial \rho} (\rho^{\mu_1} + \alpha \rho^{-\mu_1 - 1}) \right] \rho^2 \bigg|_{\rho = R} \\
&\quad \times \int_{S(K)} \Phi_1^K(\varphi)^2 d\varphi + o(1) = -(2\mu_1 + 1) \int_{S(K)} \Phi_1^K(\varphi)^2 d\varphi + o(1) = -1 + o(1).
\end{aligned}
$$

It remains to let $R \to \infty$. □

Remark 6.3.3 The solution w mentioned in Lemma 6.3.2 can be written as a linear combination of certain model functions independent of ε. We present the corresponding expression, which will be needed in the next section for estimating the remainders of asymptotic formulas. Let w^K and w^L be the solutions to problem (6.2.6) defined by (6.2.10) and (6.2.11) and let ζ^K, ζ^L be the same cut-off functions as in (6.3.11). We set

$$
\begin{aligned}
\mathbf{w}^K &= w^K - \zeta^K \left(\rho^{\mu_1} + \alpha \rho^{-\mu_1 - 1} \right) \Phi_1^K(\varphi) - \zeta^L \beta \rho^{-\mu_1 - 1} \Phi_1^L(\varphi), \\
\mathbf{w}^L &= w^L - \zeta^K \beta \rho^{-\mu_1 - 1} \Phi_1^K(\varphi) - \zeta^L \left(\rho^{\mu_1} + \alpha \rho^{-\mu_1 - 1} \right) \Phi_1^L(\varphi).
\end{aligned}
$$

A straightforward verification shows that

$$
w = a_1^+ \varepsilon^{\mu_1} \mathbf{w}^K + a_2^+ \varepsilon^{\mu_1} \mathbf{w}^L. \tag{6.3.14}
$$

□

It is convenient to write (6.3.12) in the form

$$(a_1^-, a_2^-) = (a_1^+, a_2^+)\,\Lambda\,\varepsilon^{2\mu_1+1}, \quad \Lambda = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \tag{6.3.15}$$

6.3.3 Formal Asymptotics

Here we obtain the asymptotics of the amplitudes of the reflected and transited waves as $\varepsilon \to 0$. Let the wave function u_1, defined by asymptotics (6.1.4), be approximated in G_1 by the solution $v_1 = V_1 + C_{11}\mathbf{v}_1$ and in G_2 by the solution $v_2 = C_{12}\mathbf{v}_2$ of the homogeneous limit problem. The special solutions V_1, $\mathbf{v}_1$, and $\mathbf{v}_2$ were defined in 6.3.1. For the time being, the constants C_{11} and C_{12} are unknown; we will find them when compensating the principle terms of discrepancy. According to (6.3.2) and (6.3.7), we have, as $r \to 0$,

$$v_1 = \frac{1}{\sqrt{r}}\big(C_{11}\widetilde{N}_{\mu_1+1/2}(kr) + (s_1 + C_{11}a_1)\widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^K(\varphi) + O(r^{\mu_2}), \quad r \to 0,$$
$$v_2 = \frac{1}{\sqrt{r}}\big(C_{12}\widetilde{N}_{\mu_1+1/2}(kr) + C_{12}\,a_2\widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^L(\varphi) + O(r^{\mu_2}), \quad r \to 0,$$

that is, v_1 and v_2 admit expansions (6.3.9) and (6.3.10) with the constants

$$(a_1^-, a_2^-) = (C_{11}, C_{12}), \quad (a_1^+, a_2^+) = (s_1 + C_{11}a_1, C_{12}\,a_2). \tag{6.3.16}$$

As was shown in Sect. 6.3.2, the constants must satisfy the relation

$$(C_{11}, C_{12}) = (s_1 + C_{11}a_1, C_{12}\,a_2)\,\Lambda\,\varepsilon^{2\mu_1+1}.$$

We introduce the matrix $a = \operatorname{diag}(a_1, a_2)$ and, taking into account that $\Lambda\,(I - A\,\Lambda)^{-1} = (I - \Lambda\,A\,)^{-1}\Lambda$ for $A = a\,\varepsilon^{2\mu_1+1}$, obtain

$$(C_{11}(\varepsilon), C_{12}(\varepsilon)) = (s_1, 0)(I - \Lambda\,a\,\varepsilon^{2\mu_1+1})^{-1}\Lambda\varepsilon^{2\mu_1+1}. \tag{6.3.17}$$

By virtue of (6.3.1) and (6.3.7) for $x_1^1 \to +\infty$,

$$v_1(x^1) = U_1^+(x^1) + (S_{11}^0 + C_{11}(\varepsilon)A_1)U_1^-(x^1) + O(\exp(-\delta x_1^1)), \quad x_1^1 \to +\infty,$$
$$v_2(x^2) = C_{12}(\varepsilon)A_2U_2^-(x^2) + O(\exp(-\delta x_1^2)), \quad x_1^2 \to +\infty.$$

This provides an approximation $(\widetilde{S}_{11}, \widetilde{S}_{12})$ to the first line of the scattering matrix:

$$(\widetilde{S}_{11}(\varepsilon), \widetilde{S}_{12}(\varepsilon)) = (S_{11}^0 + C_{11}(\varepsilon)A_1, C_{12}(\varepsilon)A_2) = (S_{11}^0, 0) + (C_{11}(\varepsilon), C_{12}(\varepsilon))A,$$

where $A = \operatorname{diag}(A_1, A_2)$. We set $s = \operatorname{diag}(s_1, s_2)$ and $S^0 = \operatorname{diag}(S^0_{11}, S^0_{22})$, then by Lemma 6.3.1

$$A = is^* S^0. \tag{6.3.18}$$

In view of (6.3.17) and (6.3.18), we obtain

$$(\widetilde{S}_{11}(\varepsilon), \widetilde{S}_{12}(\varepsilon)) = (S^0_{11}, 0) + i(s_1, 0)(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1} \Lambda\, s^* S^0 \varepsilon^{2\mu_1+1}. \tag{6.3.19}$$

An approximation to the wave function is of the form

$$\widetilde{u}_1(x; \varepsilon) = \chi_{\varepsilon,1}(x) v_1(x; \varepsilon) + \Theta(r) w(\varepsilon^{-1} x; \varepsilon) + \chi_{\varepsilon,2}(x) v_2(x; \varepsilon), \tag{6.3.20}$$

where, owing to (6.3.14),

$$v_1(x; \varepsilon) = V_1(x) + C_{11}(\varepsilon)\mathbf{v}_1(x), \tag{6.3.21}$$

$$w(\xi; \varepsilon) = a_1^+(\varepsilon)\varepsilon^{\mu_1} \mathbf{w}^K(\xi) + a_2^+(\varepsilon)\varepsilon^{\mu_1} \mathbf{w}^L(\xi), \tag{6.3.22}$$

$$v_2(x; \varepsilon) = C_{12}(\varepsilon)\mathbf{v}_2(x). \tag{6.3.23}$$

From (6.3.16) and (6.3.17) it follows that

$$(a_1^+(\varepsilon), a_2^+(\varepsilon)) = (s_1, 0) + (s_1, 0)(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1} \Lambda\, a\, \varepsilon^{2\mu_1+1}. \tag{6.3.24}$$

Taking account of the equality $I + (I - B)^{-1} B = (I - B)^{-1}$ for $B = a\, \Lambda\, \varepsilon^{2\mu_1+1}$, we have

$$(a_1^+(\varepsilon), a_2^+(\varepsilon)) = (s_1, 0)\, (I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}.$$

An approximation $\widetilde{u}_2$ to wave function (6.1.5) is derived in the same way. It takes the form of (6.3.20), where

$$v_1(x; \varepsilon) = C_{21}(\varepsilon)\mathbf{v}_1(x),$$

$$w(\xi; \varepsilon) = a_1^+(\varepsilon)\varepsilon^{\mu_1} \mathbf{w}^K(\xi) + a_2^+(\varepsilon)\varepsilon^{\mu_1} \mathbf{w}^L(\xi),$$

$$v_2(x; \varepsilon) = V_2(x) + C_{22}(\varepsilon)\mathbf{v}_2(x).$$

The functions v_1 and v_2 admit, near the point O, expansions of the form (6.3.9) and (6.3.10) with constants

$$(a_1^-, a_2^-) = (C_{21}, C_{22}), \quad (a_1^+, a_2^+) = (C_{21} a_1, s_2 + C_{22}\, a_2),$$

related by the equality

$$(C_{21}, C_{22}) = (C_{21} a_1, s_2 + C_{22}\, a_2)\, \Lambda\, \varepsilon^{2\mu_1+1}$$

(see (6.3.15)). It follows that

$$(C_{21}(\varepsilon), C_{22}(\varepsilon)) = (0, s_2)(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}\, \Lambda\, \varepsilon^{2\mu_1+1},$$
$$(a_1^+(\varepsilon), a_2^+(\varepsilon)) = (0, s_2)\,(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}.$$

Using expansions (6.3.1) and (6.3.7) for $x_1^1 \to +\infty$ and the formulas for C_{21} and C_{22}, we obtain an approximation to the second line of the scattering matrix:

$$(\widetilde{S}_{21}(\varepsilon), \widetilde{S}_{22}(\varepsilon)) = (0, S_{22}^0) + i(0, s_2)(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1} \Lambda s^* S^0 \varepsilon^{2\mu_1+1}. \quad (6.3.25)$$

We set $\widetilde{S} = \|\widetilde{S}_{pq}\|_{p,q=1,2}$ and unite (6.3.19) and (6.3.25) into the matrix equality

$$\widetilde{S}(\varepsilon) = S^0 + is(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1} \Lambda s^* S^0 \varepsilon^{2\mu_1+1}. \quad (6.3.26)$$

Lemma 6.3.4 *The matrix $\widetilde{S}(\varepsilon)$ is unitary.*

Proof Let us temporarily denote the matrix $(I - a\,\Lambda\,\varepsilon^{2\mu_1+1})^{-1}\Lambda$ by B. Since $(S^0)^* S^0 = I$, we obtain

$$\widetilde{S}(\varepsilon)\widetilde{S}(\varepsilon)^* = \widetilde{S}(\varepsilon)(S^0)^* S^0 \widetilde{S}(\varepsilon)^* = (I + isBs^*\varepsilon^{2\mu_1+1})(I - isB^*s^*\varepsilon^{2\mu_1+1})$$
$$= I + is(B - B^* - iBs^*sB^*\varepsilon^{2\mu_1+1})s^*\varepsilon^{2\mu_1+1}.$$

We have to verify that $B - B^* - iBs^*sB^*\varepsilon^{2\mu_1+1} = 0$. By Lemma 6.3.1,

$$a - a^* = iAA^* = i(is^*S^0)(is^*S^0)^* = is^*s$$

and, consequently,

$$B - B^* - iBs^*sB^*\varepsilon^{2\mu_1+1} = B - B^* - B(a - a^*)B^*\varepsilon^{2\mu_1+1}$$
$$= B(I + a^*B^*\varepsilon^{2\mu_1+1}) - (I + Ba\varepsilon^{2\mu_1+1})B^*.$$

We have

$$I + Ba\varepsilon^{2\mu_1+1} = I + (I - \Lambda a\varepsilon^{2\mu_1+1})^{-1}\Lambda a\varepsilon^{2\mu_1+1} = (I - \Lambda a\varepsilon^{2\mu_1+1})^{-1},$$
$$I + a^*B^*\varepsilon^{2\mu_1+1} = (I + Ba\varepsilon^{2\mu_1+1})^* = (I - a^*\Lambda\varepsilon^{2\mu_1+1})^{-1},$$

whence $B(I + a^*B^*\varepsilon^{2\mu_1+1}) - (I + Ba\varepsilon^{2\mu_1+1})B^* = 0$. □

We set $\widetilde{T}_1(\varepsilon) = |\widetilde{S}_{12}(\varepsilon)|^2$ and $\widetilde{T}_2(\varepsilon) = |\widetilde{S}_{21}(\varepsilon)|^2$. According to (6.3.26),

$$\widetilde{S}(\varepsilon) = S^0 + is\Lambda s^* S^0 \varepsilon^{2\mu_1+1} + O(\varepsilon^{4\mu_1+2})$$
$$= \begin{pmatrix} S_{11}^0 & 0 \\ 0 & S_{22}^0 \end{pmatrix} + \begin{pmatrix} |s_1|^2\alpha S_{11}^0 & s_1\overline{s_2}\beta S_{22}^0 \\ s_2\overline{s_1}\beta S_{11}^0 & |s_2|^2\alpha S_{22}^0 \end{pmatrix} \varepsilon^{2\mu_1+1} + O(\varepsilon^{4\mu_1+2}).$$

Therefore,

$$\widetilde{T}_1(\varepsilon) = \widetilde{T}_2(\varepsilon) = |s_1|^2|s_2|^2\beta^2\varepsilon^{4\mu_1+2} + O(\varepsilon^{6\mu_1+3}). \tag{6.3.27}$$

By Lemma 6.3.4, $\widetilde{R}_p(\varepsilon) + \widetilde{T}_p(\varepsilon) = 1$ with $\widetilde{R}_p(\varepsilon) = |\widetilde{S}^0_{pp}(\varepsilon)|^2$, $p = 1, 2$, and

$$\widetilde{R}_1(\varepsilon) = \widetilde{R}_2(\varepsilon) = 1 - |s_1|^2|s_2|^2\beta^2\varepsilon^{4\mu_1+2} + O\left(\varepsilon^{6\mu_1+3}\right).$$

We emphasize that the remainders in the above formulas denote the summands which were omitted in the explicit expressions for approximations and do not show the distinction between the kept terms and the real values of the coefficients we are interested in. We estimate this distinction in the next section (cf. Corollary 6.3.7).

6.3.4 The Estimate of the Remainder

We now introduce function spaces for the problem

$$-\Delta u - k^2 u = f \text{ in } G(\varepsilon), \quad u = 0 \text{ on } \partial G(\varepsilon). \tag{6.3.28}$$

Let Θ be the same as was introduced before (6.3.6), and let η_j, $j = 1, 2$, be supported by G_j and satisfy $\eta_1(x) + \Theta(r) + \eta_2(x) = 1$ in $G(\varepsilon)$. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G(\varepsilon))$ is the completion in the norm

$$\|v; V^l_{\gamma,\delta}(G(\varepsilon))\| = \left(\int_{G(\varepsilon)} \sum_{|\alpha|=0}^{l} \left(\Theta^2\,(r^2+\varepsilon^2)^{\gamma-l+|\alpha|} + \sum_{j=1}^{2} \eta_j^2 e^{2\delta x_1^j}\right) |\partial^\alpha v|^2\, dx\right)^{1/2} \tag{6.3.29}$$

of the set of smooth functions in $\overline{G(\varepsilon)}$ having compact supports. The next proposition can be proved by a somewhat simplified argument from the proof of Proposition 6.4.3 below (which, in turn, is a modification of the proof of [33, Theorem 5.1.1]).

Proposition 6.3.5 *Let $|\gamma - 1| < \mu_1 + 1/2$, $f \in V^0_{\gamma,\delta}(G(\varepsilon))$, and let u be a solution to (6.3.28) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-,$$

where $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma,\delta}(G(\varepsilon))$, δ being a small positive number. Then

$$\|\widetilde{u}; V^2_{\gamma,\delta}(G(\varepsilon))\| + |A_1^-| + |A_2^-| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\| \tag{6.3.30}$$

holds with a constant c independent of f and ε.

Theorem 6.3.6 *Under the hypotheses of Proposition 6.2.2, the inequality*

$$|S_{p1}(\varepsilon) - \widetilde{S}_{p1}(\varepsilon)| + |S_{p2}(\varepsilon) - \widetilde{S}_{p2}(\varepsilon)| \le c(\varepsilon^{\mu_2+1} + \varepsilon^{\gamma+3/2})\varepsilon^{\mu_1} \tag{6.3.31}$$

holds, where $p = 1, 2$, $\gamma > 0$, $|\gamma - 1| < \mu_1 + 1/2$, and the constant c is independent of ε.

Proof Let us verify, for instance, the inequality for $p = 1$. The difference $u_1 - \widetilde{u}_1$ satisfies (6.3.28), where, in view of (6.3.20),

$$\begin{aligned} f(x;\varepsilon) = {} & [\triangle, \chi_{\varepsilon,1}]\Big(v_1(x;\varepsilon) - (a_1^-(\varepsilon)r^{-\mu_1-1} + a_1^+(\varepsilon)r^{\mu_1})\Phi_1^K(\varphi)\Big) \\ & + [\triangle, \chi_{\varepsilon,2}]\Big(v_2(x;\varepsilon) - (a_2^-(\varepsilon)r^{-\mu_1-1} + a_2^+(\varepsilon)r^{\mu_1})\Phi_1^L(\varphi)\Big) \\ & + [\triangle, \Theta]w(\varepsilon^{-1}x;\varepsilon) + k^2\Theta(r)w(\varepsilon^{-1}x;\varepsilon). \end{aligned} \tag{6.3.32}$$

Moreover, the asymptotics of $u - \widetilde{u}$ contains only outgoing waves. To apply Proposition 6.3.5, we estimate $\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|$. In view of the asymptotics of v_1 as $r \to 0$, we have

$$\begin{aligned} & \|x \mapsto [\triangle, \chi_{\varepsilon,1}]\Big(v_1(x;\varepsilon) - (a_1^-(\varepsilon)r^{-\mu_1-1} + a_1^+(\varepsilon)r^{\mu_1})\Phi_1^K(\varphi)\Big); V^0_{\gamma,\delta}(G(\varepsilon))\|^2 \\ & \quad \le c\int_{G(\varepsilon)} (r^2+\varepsilon^2)^\gamma \Big|[\triangle, \chi_{\varepsilon,1}](a_1^-(\varepsilon)r^{-\mu_1+1} + a_1^+(\varepsilon)r^{\mu_1+2})\Phi_1^K(\varphi)\Big|^2 dx. \end{aligned}$$

Passing to the variables $\xi = \varepsilon^{-1}x$ and taking into account (6.3.15), we obtain

$$\begin{aligned} & \|x \mapsto [\triangle, \chi_{\varepsilon,1}]\Big(v_1(x;\varepsilon) - (a_1^-(\varepsilon)r^{-\mu_1-1} + a_1^+(\varepsilon)r^{\mu_1})\Phi_1^K(\varphi)\Big); V^0_{\gamma,\delta}(G(\varepsilon))\| \\ & \quad \le c\varepsilon^{\gamma+3/2}\Big(|a_1^-(\varepsilon)|\varepsilon^{-\mu_1-1} + |a_1^+(\varepsilon)|\varepsilon^{\mu_1}\Big) \le c|a_1^+(\varepsilon)|\varepsilon^{\gamma+\mu_1+3/2}. \end{aligned}$$

Analogously,

$$\begin{aligned} & \|x \mapsto [\triangle, \chi_{\varepsilon,2}]\Big(v_2(x;\varepsilon) - (a_2^-(\varepsilon)r^{-\mu_1-1} + a_2^+(\varepsilon)r^{\mu_1})\Phi_1^L(\varphi)\Big); V^0_{\gamma,\delta}(G(\varepsilon))\| \\ & \quad \le c|a_2^+(\varepsilon)|\varepsilon^{\gamma+\mu_1+3/2}. \end{aligned}$$

In view of (6.3.22) and of the fact that $\mathbf{w}^\pm$ behaves as $O(\rho^{-\mu_2-1})$ at infinity, we get the estimate

$$\begin{aligned} & \int_{G(\varepsilon)} (r^2+\varepsilon^2)^\gamma \Big|[\triangle, \Theta]w(\varepsilon^{-1}x;\varepsilon)\Big|^2 dx \le c\varepsilon^{2\mu_1}\left(|a_1^+(\varepsilon)| + |a_2^+(\varepsilon)|\right)^2 \\ & \quad \times \left(\int_K (r^2+\varepsilon^2)^\gamma \Big|[\triangle, \Theta](\varepsilon^{-1}r)^{-\mu_2-1}\Phi_2^K(\varphi)\Big|^2 dx\right. \end{aligned}$$

$$+\int_L (r^2+\varepsilon^2)^\gamma \left|[\triangle,\Theta](\varepsilon^{-1}r)^{-\mu_2-1}\Phi_2^L(\varphi)\right|^2 dx\Bigg)$$
$$\leq c\left(|a_1^+(\varepsilon)|+|a_2^+(\varepsilon)|\right)^2 \varepsilon^{2\mu_1+2\mu_2+2}.$$

Finally, again due to (6.3.22), we see that

$$\int_{G(\varepsilon)} (r^2+\varepsilon^2)^\gamma \left|\Theta(r)w(\varepsilon^{-1}x;\varepsilon)\right|^2 dx = \varepsilon^{2\gamma+3}\int_\Omega (\rho^2+1)^\gamma\,|\Theta(\varepsilon\rho)w(\xi;\varepsilon)|^2\,d\xi$$
$$\leq c\left(|a_1^+(\varepsilon)|+|a_2^+(\varepsilon)|\right)^2 \varepsilon^{2\mu_1+2\gamma+3}.$$

Combining the obtained estimates, we arrive at

$$\|f;V^0_{\gamma,\delta}(G(\varepsilon))\| \leq c\big(|a_1^+(\varepsilon)|+|a_2^+(\varepsilon)|\big)\big(\varepsilon^{\mu_2+1}+\varepsilon^{\gamma+3/2}\big)\varepsilon^{\mu_1}. \tag{6.3.33}$$

Now, let us apply Proposition 6.3.5 to the function $u_1-\widetilde{u}_1$. In (6.3.30), the u and A_j^- must be replaced by $u_1-\widetilde{u}_1$ and $S_{1j}-\widetilde{S}_{1j}$. From (6.3.33) and the estimates $a_1^+=O(\varepsilon^{2\mu_1+1})$ and $a_2^+=O(1)$ (cf. (6.3.24)), we obtain

$$|S_{11}(\varepsilon)-\widetilde{S}_{11}(\varepsilon)|+|S_{12}(\varepsilon)-\widetilde{S}_{12}(\varepsilon)| \leq \|u-\widetilde{u};V^2_{\gamma,\delta,-}(G(\varepsilon))\| \leq c(\varepsilon^{\mu_2+1}+\varepsilon^{\gamma+3/2})\varepsilon^{\mu_1}.$$

□

Corollay 6.3.7 *The asymptotic formulas*

$$T_p(\varepsilon)=|s_1|^2|s_2|^2\beta^2\varepsilon^{4\mu_1+2}+O\left(\varepsilon^{4\mu_1+2+\tau}\right),$$
$$R_p(\varepsilon)=1-|s_1|^2|s_2|^2\beta^2\varepsilon^{4\mu_1+2}+O\left(\varepsilon^{4\mu_1+2+\tau}\right)$$

hold with $p=1,2$ *and* $\tau=\min\{\mu_2-\mu_1,\,2-\sigma\}$, *where* σ *is a small positive number.*

Proof Theorem 6.3.6 leads to $|S_{p2}-\widetilde{S}_{p2}|\leq c\varepsilon^{2\mu_1+1+\tau}$ with $\tau=\min\{\mu_2-\mu_1,$ $2-\sigma\}$ and $\sigma=\mu_1+3/2-\gamma$. Therefore,

$$|T_p-\widetilde{T}_p|\leq c|\widetilde{S}_{p2}||S_{p2}-\widetilde{S}_{p2}|\leq c\varepsilon^{4\mu_1+2+\tau}.$$

To obtain the formula for T_p, it remains to take account of (6.3.27). The expansion for R_p follows now from the equality $R_p+T_p=1$. □

6.4 Tunneling in a Waveguide with Two Narrows

We turn to the waveguide $G(\varepsilon_1, \varepsilon_2)$ with two narrows. The limit domain $G(0, 0)$ consists of the infinite domains G_1, G_2, and the bounded "resonator" G_0. We assume that k^2 varies in a neighborhood of an eigenvalue k_e^2 of limit problem (6.2.1) in G_0. For the sake of simplicity, the eigenvalue is supposed to be simple.

6.4.1 Special Solutions to the Problem in the Resonator

Let k_e^2 be a simple eigenvalue for the operator $-\Delta$ with Dirichlet boundary condition in G_0 and let v_e be an eigenfunction corresponding to k_e^2 and normalized by $\int_{G_0} |v_e|^2 dx = 1$. By Proposition 6.2.1,

$$v_e(x) \sim \begin{cases} b_1 r_1^{-1/2} \widetilde{J}_{\mu_{11}+1/2}(k_e r_1)\Phi_{11}^L(\varphi_1) & \text{near } O_1, \\ b_2 r_2^{-1/2} \widetilde{J}_{\mu_{21}+1/2}(k_e r_2)\Phi_{21}^L(\varphi_2) & \text{near } O_2, \end{cases} \tag{6.4.1}$$

where (r_j, φ_j) are polar coordinates centered at O_j, $\mu_{j1}(\mu_{j1} + 1)$, and Φ_{j1}^L are the first eigenvalue and eigenfunction of the Laplace–Beltrami operator on the base of L_j normalized by $(2\mu_{j1} + 1) \int |\Phi_{j1}^L|^2 d\varphi = 1$. For k^2, in a punctured neighborhood of k_e^2 separated from the other eigenvalues of the problem in the resonator, we introduce solutions $\mathbf{v}_{0j}$ to the homogeneous problem (6.2.1) in G_0 by

$$\mathbf{v}_{0j}(x) = \Theta(r_j) v_{j1}^L(r_j, \varphi_j) + \widetilde{v}_{0j}(x), \quad j = 1, 2, \tag{6.4.2}$$

where $t \mapsto \Theta(t)$ is a cut-off function on $\mathbb{R}$ equal to 1 for $t < \delta/2$ and to 0 for $t > \delta$ with a small positive δ, v_{j1}^L are defined by (6.3.5) with L_j instead of K_j, and $\widetilde{v}_{0j}$ is the bounded solution to the problem (6.2.1) in G_0 for $f = [\triangle, \Theta] v_{j1}^L$.

Lemma 6.4.1 *In a neighborhood $V \subset \mathbb{C}$ of k_e^2 containing no eigenvalues of problem (6.2.1) in G_0 except k_e^2, the relations $\mathbf{v}_{0j}(x) = -b_j (k^2 - k_e^2)^{-1} v_e(x) + \widehat{\mathbf{v}}_{0j}(x)$ hold with b_j in (6.4.1) and functions $\widehat{\mathbf{v}}_{0j}$ analytic in $k^2 \in V$.*

Proof Let us first prove that $(\mathbf{v}_{0j}, v_e)_{G_0} = -b_j/(k^2 - k_e^2)$ with $\mathbf{v}_{0j}$ defined by (6.4.2). We have

$$(\triangle \mathbf{v}_{0j} + k^2 \mathbf{v}_{0j}, v_e)_{G_\delta} - (\mathbf{v}_{0j}, \triangle v_e + k^2 v_e)_{G_\delta} = -(k^2 - k_e^2)(\mathbf{v}_{0j}, v_e)_{G_\delta}$$

in the domain G_δ obtained from G_0 by cutting out the balls of radius δ centered at O_1 and O_2. Applying the Green formula as in the proof of Lemma 6.3.1, we arrive at $-(k^2 - k_e^2)(\mathbf{v}_{0j}, v_e)_{G_\delta} = b_j + o(1)$. It remains to let δ go to zero.

Since k_e^2 is a simple eigenvalue, we have

$$\widetilde{v}_{0j}(x) = \frac{B_j(k^2)}{k^2 - k_e^2} v_e(x) + \widehat{v}_j(x), \tag{6.4.3}$$

where $B_j(k^2)$ is independent of x and $\widehat{v}_j$ are some functions analytic in k^2 near the point $k^2 = k_e^2$. Multiplying (6.4.2) by v_e and taking into account (6.4.3), the proved formula for $(\mathbf{v}_{0j}, v_e)_{G_0}$, and the normalization condition $(v_e, v_e)_{G_0} = 1$, we find that $B_j(k^2) = -b_j + (k^2 - k_e^2)\widetilde{B}_j(k^2)$ with analytic function $\widetilde{B}_j$. Together with (6.4.3), this leads to the required statement. $\square$

Owing to Proposition 6.2.1,

$$\mathbf{v}_{01}(x) \sim \begin{cases} r_1^{-1/2}\big(\widetilde{N}_{\mu_{11}+1/2}(kr_1) + c_{11}(k)\widetilde{J}_{\mu_{11}+1/2}(kr_1)\big)\Phi_{11}^L(\varphi_1), & r_1 \to 0, \\ c_{12}(k) r_2^{-1/2}\widetilde{J}_{\mu_{21}+1/2}(kr_2)\Phi_{21}^L(\varphi_2), & r_2 \to 0, \end{cases} \tag{6.4.4}$$

$$\mathbf{v}_{02}(x) \sim \begin{cases} r_1^{-1/2} c_{21}(k)\widetilde{J}_{\mu_{11}+1/2}(kr_1)\Phi_{11}^L(\varphi_1), & r_1 \to 0, \\ r_2^{-1/2}\big(\widetilde{N}_{\mu_{21}+1/2}(kr_2) + c_{22}(k)\widetilde{J}_{\mu_{21}+1/2}(kr_2)\big)\Phi_{21}^L(\varphi_2), & r_2 \to 0. \end{cases} \tag{6.4.5}$$

In view of Lemma 6.4.1 and asymptotics (6.4.1), we obtain

$$c_{pq}(k) = -\frac{b_p b_q}{k^2 - k_e^2} + \widehat{c}_{pq}(k), \tag{6.4.6}$$

where $\widehat{c}_{pq}$ analytically depends on k^2 in a neighborhood of k_e^2.

Lemma 6.4.2 *If* $\mathbf{v}_{01}$ *and* $\mathbf{v}_{02}$ *in* (6.4.4) *and* (6.4.5) *make sense for a number k, then* $c_{12}(k) = c_{21}(k)$.

Proof It suffices to apply the Green formula to $\mathbf{v}_{01}$ and $\mathbf{v}_{02}$ in the same domain G_δ as in the proof of Lemma 6.4.1, to use (6.4.4) and (6.4.5), and to let δ tend to 0. $\square$

6.4.2 Formal Asymptotics

Let us consider the wave function u_1 in $G(\varepsilon_1, \varepsilon_2)$ satisfying

$$u_1(x; k, \varepsilon_1, \varepsilon_2) \sim \begin{cases} U_1^+(x^1; k) + S_{11}(k, \varepsilon_1, \varepsilon_2)\, U_1^-(x^1; k), & x_1^1 \to +\infty, \\ S_{12}(k, \varepsilon_1, \varepsilon_2)\, U_2^-(x^2; k), & x_1^2 \to +\infty. \end{cases}$$

In G_j, $j = 0, 1, 2$, u_1 is approximated by the solutions v_j to (6.2.1) such that

$$v_1 = V_1 + C_{11}\mathbf{v}_1, \quad v_0 = C_{12}\mathbf{v}_{01} + C_{13}\mathbf{v}_{02}, \quad v_2 = C_{14}\mathbf{v}_2, \tag{6.4.7}$$

the V_1 and $\mathbf{v}_1$ are defined in 6.3.1, the $\mathbf{v}_{0j}$ is defined in 6.4.1, and the $\mathbf{v}_2$ is an analog of the $\mathbf{v}_1$ in the domain G_2. The constants C_{1j} depend on ε_1, ε_2 and k. According to (6.3.2), (6.4.4), (6.4.5), and (6.3.7) for $r \to 0$,

$$v_1 = \frac{1}{\sqrt{r_1}}\big(C_{11}\widetilde{N}_{\mu_{11}+1/2}(kr_1) + (s_1 + C_{11}a_1)\widetilde{J}_{\mu_{11}+1/2}(kr_1)\big)\Phi_{11}^K(\varphi) + O(r_1^{\mu_{12}}), \quad r_1 \to 0,$$

$$v_0 = \begin{cases} \dfrac{1}{\sqrt{r_1}}\big(C_{12}\widetilde{N}_{\mu_{11}+1/2}(kr_1) + (C_{12}c_{11} + C_{13}c_{21})\widetilde{J}_{\mu_{11}+1/2}(kr_1)\big)\Phi_{11}^L(\varphi_1) + O(r_1^{\mu_{12}}), & r_1 \to 0, \\ \dfrac{1}{\sqrt{r_2}}\big(C_{13}\widetilde{N}_{\mu_{21}+1/2}(kr_2) + (C_{12}c_{12} + C_{13}c_{22})\widetilde{J}_{\mu_{21}+1/2}(kr_2)\big)\Phi_{21}^L(\varphi_2) + O(r_2^{\mu_{22}}), & r_2 \to 0. \end{cases}$$

$$v_2 = \frac{1}{\sqrt{r_2}}\big(C_{14}\widetilde{N}_{\mu_1+1/2}(kr_2) + C_{14}\,a_2\widetilde{J}_{\mu_1+1/2}(kr_2)\big)\Phi_{21}^L(\varphi) + O(r_2^{\mu_{22}}), \quad r_2 \to 0.$$

For every narrow, we introduce a matrix Λ_j (like the matrix Λ in (6.3.15)). Applying Lemma 6.3.2, we obtain

$$(C_{11}, C_{12}) = (s_1 + C_{11}a_1, C_{12}c_{11} + C_{13}c_{21})\,\Lambda_1\,\varepsilon_1^{2\mu_{11}+1}$$

for the first narrow and

$$(C_{13}, C_{14}) = (C_{12}c_{12} + C_{13}c_{22}, C_{14}\,a_2)\,\Lambda_2\,\varepsilon_2^{2\mu_{21}+1}$$

for the second narrow. The corresponding solutions of the second kind limit problems are of the form (see (6.3.14))

$$w_1(\xi^1) = (s_1 + C_{11}a_1)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^K(\xi^1) + (C_{12}c_{11} + C_{13}c_{21})\varepsilon_1^{\mu_{11}}\mathbf{w}_1^L(\xi^1), \tag{6.4.8}$$

$$w_2(\xi^2) = C_{14}a_2\varepsilon_2^{\mu_{21}}\mathbf{w}_2^K(\xi^2) + (C_{12}c_{12} + C_{13}c_{22})\varepsilon_2^{\mu_{21}}\mathbf{w}_2^L(\xi^2), \tag{6.4.9}$$

where $\mathbf{w}_j^K$ and $\mathbf{w}_j^L$ are analogs for the domains Ω_j, $j = 1, 2$, of the functions defined in Remark 6.3.3. We set $\Lambda = \operatorname{diag}\{\Lambda_1, \Lambda_2\}$, $\mathcal{E} = \operatorname{diag}\{\varepsilon_1^{2\mu_{11}+1},\ \varepsilon_1^{2\mu_{11}+1},\ \varepsilon_2^{2\mu_{21}+1},\ \varepsilon_2^{2\mu_{21}+1}\}$, and

$$a = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \tag{6.4.10}$$

and, combining the relations obtained for C_{1j}, we arrive at the equality

$$(C_{11}, C_{12}, C_{13}, C_{14}) = (s_1, 0, 0, 0)\,\Lambda\,\mathcal{E} + (C_{11}, C_{12}, C_{13}, C_{14})\,a\,\Lambda\,\mathcal{E}.$$

Thus,

$$(C_{11}, C_{12}, C_{13}, C_{14})(I - a \Lambda \mathcal{E}) = (s_1, 0, 0, 0) \Lambda \mathcal{E}. \tag{6.4.11}$$

Let us calculate the inverse matrix of $I - a\Lambda\mathcal{E}$, assuming ε_1 and ε_2 to be sufficiently small. It follows from (6.4.6) that

$$a(k) = -\frac{\mathbf{b}^*\mathbf{b}}{k^2 - k_e^2} + \widehat{a}(k),$$

where $\mathbf{b} = (0,\ b_1,\ b_2,\ 0)$ and the matrix $\widehat{a}$ is analytic near $k = k_e$ and is defined by equality (6.4.10) with c_{pq} changed for $\widehat{c}_{pq}$. We have

$$I - a \Lambda \mathcal{E} = I - \widehat{a} \Lambda \mathcal{E} + \frac{\mathbf{b}^*\mathbf{b} \Lambda \mathcal{E}}{k^2 - k_e^2} = \left(I + \frac{\mathbf{b}^*\mathbf{b} \Lambda \mathcal{E}(I - \widehat{a} \Lambda \mathcal{E})^{-1}}{k^2 - k_e^2}\right)(I - \widehat{a} \Lambda \mathcal{E});$$

it is evident that the matrix $(I - \widehat{a} \Lambda \mathcal{E})^{-1}$ exists for small ε_1 and ε_2. Straightforward calculation shows that

$$\left(I + \frac{\mathbf{b}^*\mathbf{c}}{k^2 - k_e^2}\right)^{-1} = I - \frac{\mathbf{b}^*\mathbf{c}}{k^2 - k_e^2 - \langle \mathbf{c}, \mathbf{b}\rangle}$$

for $\mathbf{c} = \mathbf{b} \Lambda \mathcal{E}(I - \widehat{a} \Lambda \mathcal{E})^{-1}$, where $\langle \cdot, \cdot \rangle$ is the inner product in the space $\mathbb{C}^4$. Thus,

$$(I - a\Lambda\mathcal{E})^{-1} = (I - \widehat{a} \Lambda \mathcal{E})^{-1} \left(I - \frac{\mathbf{b}^*\mathbf{b} \Lambda \mathcal{E}(I - \widehat{a} \Lambda \mathcal{E})^{-1}}{k^2 - k_e^2 + \langle \mathbf{b} \Lambda \mathcal{E}(I - \widehat{a} \Lambda \mathcal{E})^{-1}, \mathbf{b}\rangle}\right).$$

Using this equality and (6.4.11), we find the constants C_{1j}:

$$\begin{aligned}(C_{11}, C_{12}, C_{13}, C_{14}) &= (s_1, 0, 0, 0) \Lambda \mathcal{E}(I - a \Lambda \mathcal{E})^{-1} \\ &= (s_1, 0, 0, 0) \left(D - \frac{D\, \mathbf{b}^*\mathbf{b}\, D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right),\end{aligned} \tag{6.4.12}$$

where $D = \Lambda \mathcal{E}(I - \widehat{a} \Lambda \mathcal{E})^{-1}$; thereby, we have constructed an approximation to the wave function u_1. However, before presenting this approximation, we modify the solution v_0 of the limit problem in the resonator and the solutions w_j of the second kind limit problems; we show that these solutions do not have a pole at $k^2 = k_e^2$. In view of Lemma 6.4.1,

$$v_0(x) = C_{12}\mathbf{v}_{01}(x) + C_{13}\mathbf{v}_{02}(x) = C_{12}\widehat{\mathbf{v}}_{01}(x) + C_{13}\widehat{\mathbf{v}}_{02}(x) - \frac{C_{12}b_1 + C_{13}b_2}{k^2 - k_e^2} v_e(x).$$

By virtue of (6.4.12),

$$C_{12}b_1+C_{13}b_2=(C_{11},C_{12},C_{13},C_{14})\mathbf{b}^*=(s_1,0,0,0)\left(D\mathbf{b}^*-\frac{D\,\mathbf{b}^*\mathbf{b}\,D\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right).$$

Since $D\,\mathbf{b}^*\mathbf{b}\,D\mathbf{b}^*=D\,\mathbf{b}^*\langle\mathbf{b}D,\mathbf{b}\rangle$, we have

$$C_{12}b_1+C_{13}b_2=(s_1,0,0,0)D\,\mathbf{b}^*\frac{k^2-k_e^2}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle} \tag{6.4.13}$$

and

$$v_0(x)=C_{12}\widehat{\mathbf{v}}_{01}(x)+C_{13}\widehat{\mathbf{v}}_{02}(x)-\frac{(s_1,0,0,0)D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}v_e(x). \tag{6.4.14}$$

Let us modify (6.4.8) and (6.4.9) for w_j. Taking into account (6.4.6) and (6.4.13), we obtain

$$\begin{aligned}C_{12}c_{1j}+C_{13}c_{2j}&=C_{12}\widehat{c}_{1j}+C_{13}\widehat{c}_{2j}-\frac{b_j(C_{12}b_1+C_{13}b_2)}{k^2-k_e^2}\\&=C_{12}\widehat{c}_{1j}+C_{13}\widehat{c}_{2j}-\frac{b_j(s_1,0,0,0)D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle},\end{aligned} \tag{6.4.15}$$

whence

$$w_1(\xi^1)=(s_1+C_{11}a_1)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^K(\xi^1)+\left(C_{12}\widehat{c}_{11}+C_{13}\widehat{c}_{21}-\frac{b_1(s_1,0,0,0)D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^L(\xi^1), \tag{6.4.16}$$

$$w_2(\xi^2)=C_{14}a_2\varepsilon_2^{\mu_{21}}\mathbf{w}_2^K(\xi^2)+\left(C_{12}\widehat{c}_{12}+C_{13}\widehat{c}_{22}-\frac{b_2(s_1,0,0,0)D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right)\varepsilon_2^{\mu_{21}}\mathbf{w}_2^L(\xi^2). \tag{6.4.17}$$

Finally, we present the asymptotics of the wave function. Let the cut-off functions $t\mapsto\Theta(t)$ and $x^j\mapsto\chi_{\varepsilon_j,j}(x^j)$, $j=1,2$, be the same as in Sects. 3.2 and 3.3. We introduce $x\mapsto\chi_{\varepsilon_1,\varepsilon_2}(x)=\mathbf{1}_{G_0}(x)\,(1-\Theta(r_1/\varepsilon_1))\,(1-\Theta(r_2/\varepsilon_2))$, where $\mathbf{1}_{G_0}$ is the characteristic function of G_0. The principal term $\widetilde{u}_1$ of the asymptotics of the wave function u_1 is of the form

$$\begin{aligned}\widetilde{u}_1(x;k,\varepsilon_1,\varepsilon_2)=&\,\chi_{1,\varepsilon_1}(x^1)v_1(x^1;k,\varepsilon_1,\varepsilon_2)+\Theta(r_1)w_1(\varepsilon_1^{-1}x^1;k,\varepsilon_1,\varepsilon_2)\\&+\chi_{\varepsilon_1,\varepsilon_2}(x)v_0(x;k,\varepsilon_1,\varepsilon_2)+\Theta(r_2)w_2(\varepsilon_2^{-1}x^2;k,\varepsilon_1,\varepsilon_2)\\&+\chi_{2,\varepsilon_2}(x^2)v_2(x^2;k,\varepsilon_1,\varepsilon_2),\end{aligned} \tag{6.4.18}$$

where the solutions v_1 and v_2 of the first kind limit problems are defined by relations (6.4.7), v_0 is given by (6.4.14), and the solutions w_1 and w_2 of the second kind limit problems are defined by (6.4.16) and (6.4.17).

We now find an approximation $\widetilde{S}_{ij}$ to the entries of the scattering matrix $S = (S_{ij})_{i,j=1}^2$. By virtue of (6.3.1) and (6.3.7) for $x_1^1 \to +\infty$,

$$
\begin{aligned}
&v_1(x^1) = U_1^+(x^1) + (S_{11}^0 + C_{11}(\varepsilon)A_1)U_1^-(x^1) + O(\exp(-\delta x_1^1)), \quad x_1^1 \to +\infty,\\
&v_2(x^2) = C_{14}(\varepsilon)A_2U_2^-(x^2) + O(\exp(-\delta x_1^2)), \quad x_1^2 \to +\infty,
\end{aligned}
$$

whence

$$
(\widetilde{S}_{11}, \widetilde{S}_{12}) = (S_{11}^0 + C_{11}A_1, C_{14}A_2). \tag{6.4.19}
$$

We set

$$
A = \begin{pmatrix} A_1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & A_2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 & 0 & 0 & 0\\ 0 & 0 & 0 & s_2 \end{pmatrix};
$$

as before, let $S^0 = \operatorname{diag}(S_{11}^0, S_{22}^0)$; then, by Lemma 6.3.1, equality (6.3.18) remains valid. Taking into account (6.4.19), (6.4.12), and (6.3.18), we obtain

$$
\begin{aligned}
(\widetilde{S}_{11}, \widetilde{S}_{12}) &= (S_{11}^0, 0) + (C_{11}, C_{12}, C_{13}, C_{14})A\\
&= (S_{11}^0, 0) + i(s_1, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0,
\end{aligned} \tag{6.4.20}
$$

where $D = \Lambda\,\mathcal{E}(I - \widehat{a}\,\Lambda\,\mathcal{E})^{-1}$. The approximation

$$
(\widetilde{S}_{21}, \widetilde{S}_{22}) = (0, S_{22}^0) + i(0, 0, 0, s_2)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0. \tag{6.4.21}
$$

to the second row of the scattering matrix S is derived from the asymptotics of the wave function u_2,

$$
u_2(x; k, \varepsilon_1, \varepsilon_2) \sim \begin{cases} S_{21}(k, \varepsilon_1, \varepsilon_2)\,U_1^-(x^1; k), & x_1^1 \to +\infty,\\ U_2^+(x^1; k) + S_{22}(k, \varepsilon_1, \varepsilon_2)\,U_2^-(x^2; k), & x_1^2 \to +\infty. \end{cases}
$$

The principal term $\widetilde{u}_2$ of the asymptotics takes the form of (6.4.18), where

$$
\begin{aligned}
v_1(x^1) &= C_{21}\mathbf{v}_1(x^1),\\
w_1(\xi^1) &= C_{21}a_1\varepsilon_1^{\mu_{11}}\mathbf{w}_1^K(\xi^1) + \left(C_{12}\widehat{c}_{11} + C_{13}\widehat{c}_{21} - \frac{b_1(s_1, 0, 0, 0)D\,\mathbf{b}^*}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^L(\xi^1),
\end{aligned}
$$

$$v_0(x) = C_{22}\widehat{\mathbf{v}}_{01}(x) + C_{23}\widehat{\mathbf{v}}_{02}(x) - \frac{(0,0,0,s_2)D\,\mathbf{b}^*}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle} v_e(x),$$

$$w_2(\xi^2) = (s_2 + C_{24}a_2)\varepsilon_2^{\mu_{21}}\mathbf{w}_2^K(\xi^2) + \left(C_{12}\widehat{c}_{12} + C_{13}\widehat{c}_{22} - \frac{b_2(s_1,0,0,0)D\,\mathbf{b}^*}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)\varepsilon_2^{\mu_{21}}\mathbf{w}_2^L(\xi^2),$$

$$v_2(x^2) = V_2(x^2) + C_{24}\mathbf{v}_2(x^2),$$

the constants C_{2j} are given by

$$(C_{21}, C_{22}, C_{23}, C_{24}) = (0,0,0,s_2)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right).$$

Combining relations (6.4.20) and (6.4.21), we obtain the approximation $\widetilde{S} = \|\widetilde{S}_{pq}\|_{p,q=1}^2$ to the scattering matrix S:

$$\widetilde{S}(k, \varepsilon_1, \varepsilon_2) = S^0(k) + is(k)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*(k)S^0(k), \quad (6.4.22)$$

where $D = D(k, \varepsilon_1, \varepsilon_2) = \Lambda\,\mathcal{E}(\varepsilon_1, \varepsilon_2)(I - \widehat{a}(k)\,\Lambda\,\mathcal{E}(\varepsilon_1, \varepsilon_2))^{-1}$; the k_e^2 and $\mathbf{b}$ are independent of k, ε_1, ε_2. Arguing as in the proof of Lemma 6.3.4 and taking into account $(\Lambda\,\mathcal{E})^* = \Lambda\,\mathcal{E}$ and $a^* - a = is^*s$ (Lemma 6.3.1), one can verify that the matrix $\widetilde{S}$ is unitary.

We denote by k_p the pole of the matrix $\widetilde{S}$, that is, k_p satisfies the equation $k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle = 0$. We substitute the expression for D and obtain

$$k^2 - k_e^2 = \langle \mathbf{b}\Lambda\,\mathcal{E}(I - \widehat{a}(k)\,\Lambda\,\mathcal{E})^{-1}, \mathbf{b}\rangle; \quad (6.4.23)$$

here $\mathcal{E} = \operatorname{diag}(\varepsilon_1^{2\mu_{11}+1}, \varepsilon_1^{2\mu_{11}+1}, \varepsilon_2^{2\mu_{21}+1}, \varepsilon_2^{2\mu_{21}+1})$ and $\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2)$ with

$$\Lambda_j = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j & \alpha_j \end{pmatrix}, \quad j = 1,\ 2.$$

Since ε_1 and ε_2 are small, a solution to Eq. (6.4.23) can be found by the successive approximaton method. We have $k_p^2 = k_r^2 - ik_i^2$, where

$$\begin{aligned} k_r^2 &= k_e^2 - \langle \mathbf{b}\,\Lambda\,\mathcal{E}, \mathbf{b}\rangle + O\left(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}\right) \\ &= k_e^2 - \alpha_1 b_1^2\varepsilon_1^{2\mu_{11}+1} - \alpha_2 b_2^2\varepsilon_2^{2\mu_{21}+1} + O\left(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}\right), \end{aligned} \quad (6.4.24)$$

$$\begin{aligned} k_i^2 &= \operatorname{Im}\langle \mathbf{b}\,\Lambda\,\mathcal{E}\,\widehat{a}(k_e)\,\Lambda\,\mathcal{E}, \mathbf{b}\rangle + O\left(\varepsilon_1^{6\mu_{11}+3} + \varepsilon_2^{6\mu_{21}+3}\right) \\ &= \frac{1}{2}|s_1(k_e)|^2 b_1^2\beta_1^2\varepsilon_1^{4\mu_{11}+2} + \frac{1}{2}|s_2(k_e)|^2 b_2^2\beta_2^2\varepsilon_2^{4\mu_{21}+2} + O\left(\varepsilon_1^{6\mu_{11}+3} + \varepsilon_2^{6\mu_{21}+3}\right); \end{aligned} \quad (6.4.25)$$

in the last equality, we used the relation $\operatorname{Im} a_j = |s_j|^2/2$, $j = 1, 2$, which follows from Lemma 6.3.1. We suppose the constants b_j and s_j to be distinct from zero. Then, by virtue of (6.4.25), $|k^2 - k_p^2| \geqslant c(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})$ for all real k.

Let us find the principal terms of the power series in ε_1 and ε_2 for the matrix $\widetilde{S}$. To this end we verify that

$$\frac{1}{k^2 - k_e^2 + \langle \mathbf{b} D(k), \mathbf{b}\rangle} = \frac{1 + O(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})}{k^2 - k_p^2} \tag{6.4.26}$$

uniformly with respect to k in any interval that is placed between the first and second thresholds and contains no eigenvalues of the resonator except k_e. Indeed, since $k_p^2 - k_e^2 = \langle \mathbf{b} D(k_p), \mathbf{b}\rangle$, we have

$$\begin{aligned}\frac{1}{k^2 - k_e^2 + \langle \mathbf{b} D(k), \mathbf{b}\rangle} - \frac{1}{k^2 - k_p^2} &= \frac{k_e^2 - \langle \mathbf{b} D(k), \mathbf{b}\rangle - k_p^2}{(k^2 - k_e^2 + \langle \mathbf{b} D(k), \mathbf{b}\rangle)(k^2 - k_p^2)} \\ &= -\frac{\langle \mathbf{b}(D(k) - D(k_p)), \mathbf{b}\rangle}{(k^2 - k_e^2 + \langle \mathbf{b} D(k), \mathbf{b}\rangle)(k^2 - k_p^2)}.\end{aligned}$$

Applying the Hilbert identity

$$(I - A)^{-1} - (I - B)^{-1} = (I - A)^{-1}(A - B)(I - B)^{-1}$$

for $A = \widehat{a}(k)\,\Lambda\,\mathcal{E}$ and $B = \widehat{a}(k_p)\,\Lambda\,\mathcal{E}$, we obtain

$$\begin{aligned}\frac{D(k) - D(k_p)}{k^2 - k_p^2} &= \Lambda\,\mathcal{E}\frac{(I - \widehat{a}(k)\,\Lambda\,\mathcal{E})^{-1} - (I - \widehat{a}(k_p)\,\Lambda\,\mathcal{E})^{-1}}{k^2 - k_p^2} \\ &= \frac{\Lambda\,\mathcal{E}(I - \widehat{a}(k)\,\Lambda\,\mathcal{E})^{-1}(\widehat{a}(k) - \widehat{a}(k_p))\Lambda\,\mathcal{E}(I - \widehat{a}(k_p)\,\Lambda\,\mathcal{E})^{-1}}{k^2 - k_p^2} \\ &= D(k)\frac{\widehat{a}(k) - \widehat{a}(k_p)}{k^2 - k_p^2}D(k_p).\end{aligned}$$

From the last two equalities and the estimate $D = O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$, it follows that

$$\frac{1}{k^2 - k_e^2 + \langle \mathbf{b} D(k), \mathbf{b}\rangle} - \frac{1}{k^2 - k_p^2} = \frac{O(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})}{k^2 - k_e^2 + \langle \mathbf{b} D(k), \mathbf{b}\rangle},$$

which leads to (6.4.26). We substitute (6.4.26) into (6.4.22) and take into account that $D = \Lambda\,\mathcal{E} + O(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})$. Then

$$\widetilde{S}(k,\varepsilon_1,\varepsilon_2) \sim S^0(k) + is(k)\Lambda\,\mathcal{E}s^*(k)S^0(k) - i\frac{s(k)\Lambda\,\mathcal{E}\,\mathbf{b}^*\mathbf{b}\,\Lambda\,\mathcal{E}s^*(k)S^0(k)}{k^2-k_p^2}$$
$$= \begin{pmatrix} S^0_{11}(k) & 0 \\ 0 & S^0_{22}(k)\end{pmatrix} + i\begin{pmatrix} |s_1(k)|^2\alpha_1 S^0_{11}(k)\varepsilon_1^{2\mu_{11}+1} & 0 \\ 0 & |s_2(k)|^2\alpha_2 S^0_{22}(k)\varepsilon_2^{2\mu_{21}+1}\end{pmatrix} - \frac{i}{k^2-k_p^2}$$
$$\times\begin{pmatrix} |s_1(k)|^2 b_1^2\beta_1^2 S^0_{11}(k)\varepsilon_1^{4\mu_{11}+2} & s_1(k)\overline{s_2(k)}b_1b_2\beta_1\beta_2 S^0_{22}(k)\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1} \\ s_2(k)\overline{s_1(k)}b_1b_2\beta_1\beta_2 S^0_{11}(k)\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1} & |s_2(k)|^2 b_2^2\beta_2^2 S^0_{22}(k)\varepsilon_2^{4\mu_{21}+2}\end{pmatrix},$$

where we dropped the terms that admit the estimate $O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})$ uniformly with respect to k. For $(k^2-k_p^2)^{-1}=O(1)$, the third term can be neglected as well; however, it must be taken into account for small $k^2-k_p^2$.

Let us choose a more narrow interval for k^2, assuming $k^2-k_r^2 = O\big(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1}\big)$. Using relations (6.4.24), (6.4.25), $S^0_{jj}(k)=S^0_{jj}(k_e)+O(k^2-k_e^2)$, and $s_j(k)=s_j(k_e)+O(k^2-k_e^2)$, we obtain

$$\widetilde{S}_{12}(k,\varepsilon_1,\varepsilon_2) = \frac{i\dfrac{s_1(k_e)}{|s_1(k_e)|}\dfrac{\overline{s_2(k_e)}}{|s_2(k_e)|}}{\dfrac{i}{2}\left(z+\dfrac{1}{z}\right)+P\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}}\big(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\big), \tag{6.4.27}$$

$$\widetilde{S}_{21}(k,\varepsilon_1,\varepsilon_2) = \frac{i\dfrac{s_2(k_e)}{|s_2(k_e)|}\dfrac{\overline{s_1(k_e)}}{|s_1(k_e)|}}{\dfrac{i}{2}\left(z+\dfrac{1}{z}\right)+P\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}}\big(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\big), \tag{6.4.28}$$

where

$$z=\frac{b_1\beta_1|s_1(k_e)|\varepsilon_1^{2\mu_{11}+1}}{b_2\beta_2|s_2(k_e)|\varepsilon_2^{2\mu_{21}+1}},\qquad P=\frac{1}{b_1b_2\beta_1\beta_2|s_1(k_e)||s_2(k_e)|}.$$

Now, we find approximations to the transmission and reflection coefficients:

$$\widetilde{T}_j(k,\varepsilon_1,\varepsilon_2) = \frac{1}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\big(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\big),$$
$$\widetilde{R}_j(k,\varepsilon_1,\varepsilon_2) = \frac{\dfrac{1}{4}\left(z-\dfrac{1}{z}\right)^2+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\big(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\big).$$

It is easy to see that $\widetilde{T}_j$ has a peak at $k^2 = k_r^2$ whose width at its half-height is

$$\widetilde{\Upsilon}(\varepsilon_1, \varepsilon_2) = |(z + z^{-1})/P|\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}. \tag{6.4.29}$$

6.4.3 The Estimate of the Remainder

We introduce function spaces for the problem

$$-\Delta u - k^2 u = f \quad \text{in } G(\varepsilon_1, \varepsilon_2), \quad u = 0 \quad \text{on } \partial G(\varepsilon_1, \varepsilon_2). \tag{6.4.30}$$

Let Θ be the same as in (6.3.6), and let η_j, $j = 0, 1, 2$, be supported by G_j and satisfy $\eta_1(x) + \Theta(r_1) + \eta_0(x) + \Theta(r_2) + \eta_2(x) = 1$ in $G(\varepsilon_1, \varepsilon_2)$. For $\gamma_1, \gamma_2 \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \ldots$, the space $V^l_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ is the completion in the norm

$$\|u; V^l_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|$$
$$= \left(\int_{G(\varepsilon_1,\varepsilon_2)} \sum_{|\alpha|=0}^{l} \left(\sum_{j=1}^{2} \Theta^2(r_j)\,(r_j^2 + \varepsilon_j^2)^{\gamma_j - l + |\alpha|} + \sum_{j=1}^{2} \eta_j^2 e^{2\delta x_1^j} + \eta_0^2\right)|\partial^\alpha v|^2\, dx\right)^{1/2} \tag{6.4.31}$$

of the set of smooth functions in $\overline{G(\varepsilon_1, \varepsilon_2)}$ with compact supports. Denote by $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}$ the space of functions f that are analytic in k^2, take values in $V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$, and, for $k^2 = k_e^2$, satisfy $(\chi_{\varepsilon_1^\sigma,\varepsilon_2^\sigma} f, v_e)_{G_0} = 0$ with small $\sigma > 0$.

Proposition 6.4.3 *Let k_r^2 be a resonance, $k_r^2 \to k_e^2$ as $\varepsilon_1, \varepsilon_2 \to 0$, and let $|k^2 - k_r^2| = O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$. Assume that γ_1, γ_2 satisfy $\mu_{j1} - 3/2 < \gamma_j - 1 < \mu_{j1} + 1/2$, $f \in V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ and u is a solution to (6.4.30) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-;$$

here $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ with small $\delta > 0$. Then

$$\|\widetilde{u}; V^2_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| + |A_1^-| + |A_2^-| \le c\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|, \tag{6.4.32}$$

where c is a constant independent of f and ε_1, ε_2.

Proof Step A. First, we construct an auxiliary function u_p. The solutions of the first kind limit problems involved in (6.4.18) are defined for complex k^2 as well. Multiply the limit problem solutions involved in $\widetilde{u}_1$ by $g := -(k^2 - k_e^2 + \langle \mathbf{b}D(k), \mathbf{b}\rangle)/\langle(s_1, 0, 0, 0)D, \mathbf{b}\rangle$, put $k = k_p$, and denote the resulting functions by adding the subscript p. Then

$$\begin{aligned} g(C_{11}, C_{12}, C_{13}, C_{14})|_{k=k_p} &= \mathbf{b}D(k_p) \\ &= (b_1\beta_1\varepsilon_1^{2\mu_{11}+1}, b_1\alpha_1\varepsilon_1^{2\mu_{11}+1}, b_2\alpha_2\varepsilon_2^{2\mu_{21}+1}, b_2\beta_2\varepsilon_2^{2\mu_{21}+1}) \\ &\quad + O(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}) \end{aligned}$$

and, in view of (6.4.7) and (6.4.14)–(6.4.17),

$$\begin{aligned} v_{jp}(x;\varepsilon_1,\varepsilon_2) &= \varepsilon_j^{2\mu_{j1}+1}\left(b_j\beta_j + O\left(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\right)\right)\mathbf{v}_j(x;k_p), \\ v_{0p}(x;\varepsilon_1,\varepsilon_2) &= v_e(x) + \varepsilon_1^{2\mu_{11}+1}\left(b_1\alpha_1 + O\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)\right)\widehat{\mathbf{v}}_{01}(x;k_p) \\ &\quad + \varepsilon_2^{2\mu_{21}+1}\left(b_2\alpha_2 + O\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)\right)\widehat{\mathbf{v}}_{02}(x;k_p), \\ w_{jp}(\xi^j;\varepsilon_1,\varepsilon_2) &= b_j\varepsilon_j^{\mu_{j1}}\Big[\varepsilon_j^{2\mu_{j1}+1}\left(a_j(k_p)\beta_j + O\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)\right)\mathbf{w}_j^K(\xi^j) \\ &\quad + \left(1 + O\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)\right)\mathbf{w}_j^L(\xi^j)\Big], \end{aligned}$$

where $j = 1, 2$; the dependence of k_p on $\varepsilon_1, \varepsilon_2$ is not shown. We set

$$\begin{aligned} u_p(x;\varepsilon_1,\varepsilon_2) = \Xi(x)\Big[&\chi_{1,\varepsilon_1}(x^1)v_{1p}(x^1;\varepsilon_1,\varepsilon_2) + \Theta(\varepsilon_1^{-2\sigma}r_1)w_{1p}(\varepsilon_1^{-1}x^1;\varepsilon_1,\varepsilon_2) \\ &+ \chi_{\varepsilon_1,\varepsilon_2}(x)v_{0p}(x;\varepsilon_1,\varepsilon_2) + \Theta(\varepsilon_2^{-2\sigma}r_2)w_2(\varepsilon_2^{-1}x^2;k,\varepsilon_1,\varepsilon_2) \\ &+ \chi_{2,\varepsilon_2}(x^2)v_2(x^2;k,\varepsilon_1,\varepsilon_2)\Big], \end{aligned}$$

where Ξ is a cut-off function in $G(\varepsilon_1,\varepsilon_2)$ that equals 1 on the set $G(\varepsilon_1,\varepsilon_2) \cap \{|x| < R\}$ and 0 on $G(\varepsilon_1,\varepsilon_2) \cap \{|x| > R+1\}$ for a large $R > 0$. The principal part of the norm of u_p is given by $\chi_{\varepsilon_1,\varepsilon_2}v_{0p}$. Taking into account the definitions of v_{0p} and of $\widehat{\mathbf{v}}_{0j}$ (see Sect. 6.4.1), we obtain $\|\chi_{\varepsilon_1,\varepsilon_2}v_{0p}\| = \|v_e\| + o(1)$. Note that $(\triangle + k_p^2)u_p$ is nonzero only in the region $\{r_1 < c_1\varepsilon_1^{2\sigma}\} \cup \{r_2 < c_2\varepsilon_2^{2\sigma}\}$. Arguing as in the proof of Theorem 6.3.6, we obtain

$$\|(\triangle + k_p^2)u_p; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\ \varepsilon_2))\| \le c\left(\varepsilon_1^{\mu_{11}+\kappa_1} + \varepsilon_2^{\mu_{21}+\kappa_2}\right), \tag{6.4.33}$$

where $\kappa_j = \min\{\mu_{j1}+1, \mu_{j2}+1-\sigma_j, \gamma_j+3/2\}$, $\sigma_j = 2\sigma(\mu_{j2}-\gamma_j+3/2)$. When $\mu_{j1}-3/2 < \gamma_j - 1$ and σ is small such that $\mu_{j2}-\mu_{j1} > \sigma_j$, we have $\kappa_j = \mu_{j1}+1$.

Step B. This part contains somewhat modified arguments from the proof of Theorem 5.1.1 in [33]. Let $\|v; V^2_{\gamma_j,\delta,-}(G_j)\|^2$ and $\|v; V^2_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1,\varepsilon_2))\|^2$ denote the left-hand side of (6.2.5) (with $\gamma = \gamma_j$) and (6.4.32). Rewrite the right-hand side of (6.4.30) in the form

$$\begin{aligned} f(x) = f_1(x;\varepsilon_1) + f_0(x;\varepsilon_1,\varepsilon_2) + f_2(x;\varepsilon_2) \\ + \varepsilon_1^{-\gamma_1-3/2}F_1(\varepsilon_1^{-1}x^1;\varepsilon_1) + \varepsilon_2^{-\gamma_2-3/2}F_2(\varepsilon_2^{-1}x^2;\varepsilon_2), \end{aligned}$$

where

$$f_0(x;\varepsilon_1,\varepsilon_2)=\chi_{\varepsilon_1^\sigma,\varepsilon_2^\sigma}(x)f(x),\quad f_j(x;\varepsilon_j)=\chi_{\varepsilon_j^\sigma,j}(x)f(x),$$
$$F_j(\xi^j;\varepsilon_j)=\varepsilon_j^{\gamma_j+3/2}\Theta(\varepsilon_j^{1-\sigma}\rho_j)f(x_{O_j}+\varepsilon_j\xi^j);$$

x are arbitrary Cartesian coordinates; x_{O_j} denotes the coordinates of O_j in the system x; x^j are introduced in Sect. 6.1. From the definition of the norms it follows that

$$\|f_0;V^0_{\gamma_1,\gamma_2}(G_0)\|+\|f_j;V^0_{\gamma_j,\delta}(G_j)\|+\|F_j;V^0_\gamma(\Omega_j)\|\le c\|f;V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\|. \tag{6.4.34}$$

Consider solutions v_0, v_j, and w_j of the problems

$$\begin{aligned}-\triangle v_0-k^2v_0&=f_0 \text{ in } G_0, & v_0&=0 \text{ on } \partial G_0,\\ -\triangle v_j-k^2v_j&=f_j \text{ in } G_j, & v_j&=0 \text{ on } \partial G_j,\\ \triangle w_j&=F_j \text{ in } \Omega_j, & w_j&=0 \text{ on } \partial\Omega_j,\end{aligned}$$

respectively; moreover, v_j satisfies the natural radiation conditions at infinity. Owing to Propositions 6.2.1, 6.2.2, and 6.2.3, the problems in G_0, G_j, and Ω_j, $j=1,2$ are uniquely solvable and

$$\begin{aligned}\|v_0;V^2_{\gamma_1,\gamma_2}(G_0)\|&\le\widetilde{c}_0\|f_0;V^0_{\gamma_1,\gamma_2}(G_0)\|,\\ \|v_j;V^2_{\gamma_j,\delta,-}(G_j)\|&\le\widetilde{c}_j\|f_j;V^0_{\gamma_j,\delta}(G_j)\|,\\ \|w_j;V^2_{\gamma_j}(\Omega_j)\|&\le\widetilde{C}_j\|F_j;V^0_{\gamma_j}(\Omega_j)\|,\end{aligned} \tag{6.4.35}$$

where $\widetilde{c}_0$, $\widetilde{c}_j$, and $\widetilde{C}_j$ are independent of $\varepsilon_1,\varepsilon_2$. We set

$$\begin{aligned}U(x;\varepsilon_1,\varepsilon_2)=\;&\chi_{\varepsilon_1,1}(x)v_1(x;\varepsilon_1,\varepsilon_2)+\varepsilon_1^{-\gamma_1+1/2}\Theta(r_1)w_1(\varepsilon_1^{-1}x^1;\varepsilon_1,\varepsilon_2)\\ &+\chi_{\varepsilon_1,\varepsilon_2}(x)v_0(x;\varepsilon_1,\varepsilon_2)+\varepsilon_2^{-\gamma_2+1/2}\Theta(r_2)w_2(\varepsilon_2^{-1}x^2;\varepsilon_1,\varepsilon_2)\\ &+\chi_{\varepsilon_2,2}(x)v_2(x;\varepsilon_1,\varepsilon_2).\end{aligned}$$

The estimates (6.4.34) and (6.4.35) lead to

$$\|U;V^2_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1,\varepsilon_2))\|\le c\|f;V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))\| \tag{6.4.36}$$

with c independent of $\varepsilon_1,\varepsilon_2$.

Let us show that $-(\Delta+k^2)R_{\varepsilon_1,\varepsilon_2}=I+S_{\varepsilon_1,\varepsilon_2}$, where $S_{\varepsilon_1,\varepsilon_2}$ is an operator in $V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\varepsilon_2))$ of small norm. We have

$$
\begin{aligned}
(\Delta + k^2)R_{\varepsilon_1,\varepsilon_2} f(x) = (\Delta + k^2)U(x; \varepsilon_1, \varepsilon_2) = &-f(x) + [\Delta, \chi_{1,\varepsilon_1}]v_1(x; \varepsilon_1, \varepsilon_2)\\
&+ \varepsilon_1^{-\gamma_1+1/2}[\Delta, \Theta]w_1(\varepsilon_1^{-1}x^1; \varepsilon_1, \varepsilon_2)\\
&+ k^2\varepsilon_1^{-\gamma_1+1/2}\Theta(r_1)w_1(\varepsilon_1^{-1}x^1; \varepsilon_1, \varepsilon_2)\\
&+ [\Delta, \chi_{0,\varepsilon_1,\varepsilon_2}]v_0(x; \varepsilon_1, \varepsilon_2)\\
&+ \varepsilon_2^{-\gamma_2+1/2}[\Delta, \Theta]w_2(\varepsilon_2^{-1}x^2; \varepsilon_1, \varepsilon_2)\\
&+ k^2\varepsilon_2^{-\gamma_2+1/2}\Theta(r_2)w_2(\varepsilon_2^{-1}x^2; \varepsilon_1, \varepsilon_2)\\
&+ [\Delta, \chi_{2,\varepsilon_2}]v_2(x; \varepsilon_1, \varepsilon_2).
\end{aligned}
\tag{6.4.37}
$$

Let d be a positive number such that $\gamma_j - 1 > \mu_{j1} + 1/2 - d$. On the support of the function $[\Delta, \chi_{1,\varepsilon}]v_1$ the estimate $r_1 = O(\varepsilon_1)$ holds, therefore,

$$\|[\Delta, \chi_{1,\varepsilon_1}]v_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \le c\varepsilon_1^d \|[\Delta, \chi_{1,\varepsilon_1}]v_1; V^0_{\gamma_1-d,\delta}(G_1)\| \le c\varepsilon_1^d \|v_1; V^2_{\gamma_1-d,\delta}(G_1)\|.$$

This and (6.4.35) lead to

$$\|[\Delta, \chi_{1,\varepsilon_1}]v_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \le c\varepsilon_1^d \|f_1; V^0_{\gamma_1-d,\delta}(G_1)\|.$$

Moreover, $f_1 = 0$ outside the zone $c\varepsilon_1^\sigma \le r_1 \le C\varepsilon_1^\sigma$, therefore,

$$\|f_1; V^0_{\gamma_1-d,\delta}(G_1)\| \le c\varepsilon_1^{-d\sigma} \|f_1; V^0_{\gamma_1,\delta}(G_1)\|.$$

The two last estimates together with (6.4.34) show that

$$\|[\Delta, \chi_{1,\varepsilon_1}]v_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \le c\varepsilon_1^{d(1-\sigma)} \|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|. \tag{6.4.38}$$

In a similar way, we obtain

$$\|[\Delta, \chi_{0,\varepsilon_1,\varepsilon_2}]v_0; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \le c(\varepsilon_1^{d(1-\sigma)} + \varepsilon_2^{d(1-\sigma)})\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|, \tag{6.4.39}$$

$$\|[\Delta, \chi_{2,\varepsilon_2}]v_2; V^0_{\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \le c\varepsilon_2^{d(1-\sigma)} \|f; V^0_{\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|. \tag{6.4.40}$$

We now assume in addition that the d satisfies $-\mu_{j1} - 1/2 + d < \gamma_j - 1$. Because the support of the function $[\Delta_{\xi^j}, \Theta(\varepsilon_j\rho_j)]w_j(\xi^j; \varepsilon_1, \varepsilon_2)$, $j = 1, 2$, belongs to the domain $c\varepsilon_j^{-1} \le \rho_j \le C\varepsilon_j^{-1}$,

$$
\begin{aligned}
&\|\xi^j \mapsto [\Delta_{\xi^j}, \Theta(\varepsilon_j\rho_j)]w_j(\xi^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_j}(\Omega_j)\|\\
&\quad \le c\varepsilon_j^d \|\xi^j \mapsto [\Delta_{\xi^j}, \Theta(\varepsilon_j\rho_j)]w_j(\xi^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_j+d}(\Omega_j)\| \le c\varepsilon_j^d \|w_j; V^2_{\gamma_j+d}(\Omega_j)\|.
\end{aligned}
$$

Now, taking into account (6.4.35), we obtain

$$\varepsilon_j^{-\gamma_j+1/2}\|x^j \mapsto [\Delta, \Theta(r_j)]w_j(\varepsilon_j^{-1}x^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \\ \leq c\varepsilon_j^d \|F_j; V^0_{\gamma_j+d}(\Omega_j)\|.$$

Since $F_j = 0$ for $\rho_j > c\varepsilon_j^{-\sigma}$,

$$\|F_j; V^0_{\gamma_j+d}(\Omega_j)\| \leq c\varepsilon_j^{-d\sigma}\|F_j; V^0_{\gamma_j}(\Omega_j)\|. \tag{6.4.41}$$

Consequently,

$$\varepsilon_j^{-\gamma_j+1/2}\|x^j \mapsto [\Delta, \Theta(r_j)]w_j(\varepsilon_j^{-1}x^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \\ \leq c\varepsilon_j^{d(1-\sigma)}\|f; V^0_{\gamma_1,\gamma_j,\,\delta}(G(\varepsilon_1, \varepsilon_2))\|. \tag{6.4.42}$$

It remains to estimate the middle terms of the two last lines in (6.4.37). We have

$$\varepsilon_j^{-\gamma_j+1/2}\|x^j \mapsto \Theta(r_j)w_j(\varepsilon_j^{-1}x^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \\ = \varepsilon_j^2\|\xi^j \mapsto \Theta(\varepsilon_j\rho_j)w_j(\xi^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_j}(\Omega_j)\| \\ \leq \varepsilon_j^2\|\xi^j \mapsto \Theta(\varepsilon_j\rho_j)w_j(\xi^j; \varepsilon_1, \varepsilon_2); V^2_{\gamma_j+2}(\Omega_j)\| \leq c\varepsilon_j^d\|w_j; V^2_{\gamma_j+d}(\Omega_j)\|;$$

in the last inequality we took into account that $\Theta(\varepsilon_j\rho_j)w_j(\xi^j; \varepsilon_1, \varepsilon_2) = 0$ for $\rho_j \geq c\varepsilon_j^{-1}$; besides, we assume that $2 - d > 0$. In view of (6.4.35), (6.4.41), and (6.4.34), we obtain

$$\varepsilon_j^{-\gamma_j+1/2}\|x^j \mapsto \Theta(r_j)w_j(\varepsilon^{-1}x^j; \varepsilon_1, \varepsilon_2); V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \\ \leq c\varepsilon_j^{d(1-\sigma)}\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|. \tag{6.4.43}$$

Thus, (6.4.37)–(6.4.40) and (6.4.42) and (6.4.43) lead to the inequality

$$\|-(\Delta+k^2)R_{\varepsilon_1,\varepsilon_2}f-f; V^0_{\gamma_1,\gamma_2,\,\delta}(G(\varepsilon_1, \varepsilon_2))\| \leq c(\varepsilon_1^{d(1-\sigma)} + \varepsilon_2^{d(1-\sigma)})\|f; V^0_{\gamma_1,\gamma_2,\,\delta}(G(\varepsilon_1, \varepsilon_2))\|,$$

which means that $(\Delta + k^2)R_{\varepsilon_1,\varepsilon_2} = I + S_{\varepsilon_1,\varepsilon_2}$ and the norm of the operator $S_{\varepsilon_1,\varepsilon_2}$ in the space $V^0_{\gamma_1,\gamma_2,\,\delta}(G(\varepsilon_1, \varepsilon_2))$ admits the estimate $\|S_{\varepsilon_1,\varepsilon_2}\| \leq c(\varepsilon_1^{d(1-\sigma)} + \varepsilon_2^{d(1-\sigma)})$.

Step C. Recall that the operator $S_{\varepsilon_1,\varepsilon_2}$ is defined on the subspace $V^{0,\perp}_{\gamma_1,\gamma_2,\,\delta}(G(\varepsilon_1,$ $\varepsilon_2))$. We need the image of the operator $S_{\varepsilon_1,\varepsilon_2}$ to be included in $V^{0,\perp}_{\gamma_1,\gamma_2,\,\delta}(G(\varepsilon_1, \varepsilon_2))$, too. To this end, replace the mapping $R_{\varepsilon_1,\,\varepsilon_2}$ by $\widetilde{R}_{\varepsilon_1,\,\varepsilon_2} : f \mapsto U(f) + a(f)u_p$, where u_p is constructed in **A**, $a(f)$ being a constant. Then $-(\Delta + k^2)\widetilde{R}_{\varepsilon_1,\varepsilon_2} = I + \widetilde{S}_{\varepsilon_1,\varepsilon_2}$, with $\widetilde{S}_{\varepsilon_1,\,\varepsilon_2} = S_{\varepsilon_1,\varepsilon_2} - a(\cdot)(\Delta + k^2)u_p$. The condition $(\chi_{\varepsilon_1^\sigma,\,\varepsilon_2^\sigma}\widetilde{S}_{\varepsilon_1,\,\varepsilon_2}f, v_e)_{G_0} = 0$ as

$k = k_e$ gives $a(f) = (\chi_{\varepsilon_1^\sigma,\,\varepsilon_2^\sigma} S_{\varepsilon_1,\,\varepsilon_2} f, v_e)_{G_0} / (\chi_{\varepsilon_1^\sigma,\,\varepsilon_2^\sigma} (\triangle + k_e^2) u_p, v_e)_{G_0}$. Prove that $\|\widetilde{S}_{\varepsilon_1,\,\varepsilon_2}\| \le c\|S_{\varepsilon_1,\,\varepsilon_2}\|$, c being independent of $\varepsilon_1,\ \varepsilon_2, k$. We have

$$\|\widetilde{S}_{\varepsilon_1,\,\varepsilon_2} f\| \le \|S_{\varepsilon_1,\,\varepsilon_2} f\| + |a(f)|\, \|(\triangle + k^2) u_p\|.$$

The estimate (6.4.33) (with $\gamma_j > \mu_{j1} - 1/2$ and $\mu_{j2} - \mu_{j1} > \sigma_j$), the formula for k_p, and the condition $k^2 - k_e^2 = O\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)$ imply

$$\begin{aligned}\|(\triangle + k^2)u_p;\, V^0_{\gamma_1,\gamma_2,\delta}\| &\le |k^2 - k_p^2|\, \|u_p;\, V^0_{\gamma_1,\gamma_2,\delta}\| + \|(\triangle + k_p^2)u_p;\, V^0_{\gamma_1,\gamma_2,\delta}\| \\ &\le c\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big).\end{aligned}$$

Since the supports of the functions $(\triangle + k_p^2)u_p$ and $\chi_{\varepsilon_1^\sigma,\,\varepsilon_2^\sigma}$ do not intersect, we have

$$|(\chi_{\varepsilon_1^\sigma,\varepsilon_2^\sigma}(\triangle + k_e^2)u_p, v_e)_{G_0}| = |(k_e^2 - k_p^2)(u_p, v_e)_{G_0}| \ge c\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big).$$

Further, $\gamma_j - 1 < \mu_{j1} + 1/2$, so

$$\begin{aligned}|(\chi_{\varepsilon_1^\sigma,\,\varepsilon_2^\sigma} S_{\varepsilon_1,\,\varepsilon_2} f, v_e)_{G_0}| &\le \|S_{\varepsilon_1,\,\varepsilon_2} f;\, V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\ \varepsilon_2))\|\, \|v_e;\, V^0_{-\gamma_1,-\gamma_2}(G_0)\| \\ &\le c\|S_{\varepsilon_1,\,\varepsilon_2} f;\, V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1,\ \varepsilon_2))\|.\end{aligned}$$

Hence,

$$|a(f)| \le c\|S_{\varepsilon_1,\varepsilon_2} f;\, V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| / \big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)$$

and $\|\widetilde{S}_{\varepsilon_1,\varepsilon_2} f\| \le c\|S_{\varepsilon_1,\varepsilon_2} f\|$. Thus the operator $I + \widetilde{S}_{\varepsilon_1,\varepsilon_2}$ in $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ is invertible, which is also true for the operator of problem (6.4.30):

$$A_{\varepsilon_1,\varepsilon_2} : u \mapsto -\triangle u - k^2 u :\ \mathring{V}^{2,\perp}_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1, \varepsilon_2)) \mapsto V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2));$$

here $\mathring{V}^{2,\perp}_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1, \varepsilon_2))$ denotes the space of elements of $V^2_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1, \varepsilon_2))$ that vanish on $\partial G(\varepsilon_1, \varepsilon_2)$ and are sent by the operator $-\triangle - k^2$ to $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}$. The inverse operator $A^{-1}_{\varepsilon_1,\varepsilon_2} = \widetilde{R}_{\varepsilon_1,\varepsilon_2}(I + \widetilde{S}_{\varepsilon_1,\varepsilon_2})^{-1}$ is bounded uniformly with respect to $\varepsilon_1, \varepsilon_2, k$. Hence, the inequality (6.4.32) holds with c independent of $\varepsilon_1, \varepsilon_2, k$. $\square$

Consider the solution u_1 of the homogeneous problem (6.1.1) defined by (6.1.4). Let S_{11} and S_{12} be the entries of the scattering matrix determined by this solution. Denote by $\widetilde{u}_{1,\sigma}$ the function defined by (6.4.18) with $\Theta(r_j)$ replaced by $\Theta(\varepsilon_j^{-2\sigma} r_j)$. The $\widetilde{S}_{11}$, $\widetilde{S}_{12}$ are the same as in Sect. 6.4.2.

Theorem 6.4.4 *Let the hypotheses of Propositions* 6.2.2 *and* 6.4.3 *be fulfilled and assume that the coefficients* s_j *in* (6.3.1), (6.3.3) *and the coefficients* b_j *in* (6.4.1) *are nonzero. Then*

$$|S_{p1} - \widetilde{S}_{p1}| + |S_{p2} - \widetilde{S}_{p2}|$$
$$\leq c \frac{(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})\varepsilon_1^{\mu_{11}}\big(\varepsilon_1^{\mu_{12}+1} + \varepsilon_1^{\gamma_1+3/2}\big) + \varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{\mu_{21}}\big(\varepsilon_2^{\mu_{22}+1} + \varepsilon_2^{\gamma_2+3/2}\big)}{|k^2 - k_r^2| + \varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}}$$

with c independent of $\varepsilon_1, \varepsilon_2, k$*;* $p = 1, 2$.

Proof Let for instance $p = 1$. The difference $u_1 - \widetilde{u}_{1,\sigma}$ is in the space $V^2_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1, \varepsilon_2))$, and $f_1 := -(\triangle + k^2)(u_1 - \widetilde{u}_{1,\sigma})$ is in $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$. By Proposition 6.4.3,

$$|S_{11}(\varepsilon) - \widetilde{S}_{11}(\varepsilon)| + |S_{12}(\varepsilon) - \widetilde{S}_{12}(\varepsilon)| \leq \|u_1 - \widetilde{u}_1; V^2_{\gamma_1,\gamma_2,\delta,-}(G(\varepsilon_1, \varepsilon_2))\|$$
$$\leq c\,\| f_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|.$$

Arguing as in the proof of Theorem 6.3.6 (cf. (6.3.33)), we obtain that

$$\| f_1; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| \leq c\Big(\big(|a_1^+| + |b_1^+|\big)\varepsilon_1^{\mu_{11}}\big(\varepsilon_1^{\mu_{12}+1-\sigma_1} + \varepsilon_1^{\gamma_1+3/2}\big)$$
$$+ \big(|a_2^+| + |b_2^+|\big)\varepsilon_2^{\mu_{21}}\big(\varepsilon_2^{\mu_{22}+1-\sigma_2} + \varepsilon_2^{\gamma_2+3/2}\big)\Big).$$

where $(a_1^+, b_1^+) = (s_1 + C_{11}a_1, C_{12}c_{11} + C_{13}c_{21})$ and $(b_2^+, a_2^+) = (C_{12}c_{12} + C_{13}c_{22}, C_{14}\,a_2)$. From (6.4.12), (6.4.15), (6.4.26) in view of $(s_1, 0, 0, 0)D = O(\varepsilon_1^{2\mu_{11}+1})$, it follows that

$$|C_{1j}| \leq c\varepsilon_1^{2\mu_{11}+1}\left(1 + \frac{\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}}{|k^2 - k_p^2|}\right), \quad |C_{12}c_{1j} + C_{13}c_{2j}| \leq c\frac{\varepsilon_1^{2\mu_{11}+1}}{|k^2 - k_p^2|},$$

and

$$|a_1^+| + |b_1^+| \leq c\frac{\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}}{|k^2 - k_p^2|}, \quad |a_2^+| + |b_2^+| \leq c\frac{\varepsilon_1^{2\mu_{11}+1}}{|k^2 - k_p^2|}.$$

Combining the above inequalities we obtain the required estimate. □

We denote by k_e^2 an eigenvalue of problem (6.2.1) in the resonator G_0 and by $k_r^2(\varepsilon)$ a resonance frequency such that $k_r^2(\varepsilon) \to k_e^2$ as $\varepsilon \to 0$. Moreover, let b_j be the constants in asymptotics (6.4.1) of an eigenfunction corresponding to the eigenvalue k_e^2 and $s_j(k)$ the constant in asymptotics (6.3.1) and (6.3.3) of the special solution V_j for $r_j \to 0$, $j = 1, 2$. Finally, the constants α and β are defined by (6.2.10) and (6.2.11). We set $P = (b_1 b_2 \beta_1 \beta_2 |s_1(k_e)|\,|s_2(k_e)|)^{-1}$ and $z = b_1\beta_1|s_1(k_e)|\varepsilon_1^{2\mu_{11}+1}/b_2\beta_2|s_2(k_e)|\varepsilon_2^{2\mu_{21}+1}$; these are the same values as in (6.4.27) and (6.4.29).

Theorem 6.4.5 *For* $|k^2 - k_r^2| = O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$*, the asymptotic expansions*

$$T_p(k, \varepsilon_1, \varepsilon_2) = \frac{1}{\frac{1}{4}\left(z + \frac{1}{z}\right)^2 + P^2\left(\frac{k^2 - k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\bigl(1 + O(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2})\bigr),$$

$$k_r^2(\varepsilon_1, \varepsilon_2) = k_e^2 - \alpha_1 b_1^2 \varepsilon_1^{2\mu_{11}+1} - \alpha_2 b_2^2 \varepsilon_2^{2\mu_{21}+1} + O\bigl(\varepsilon_1^{2\mu_{11}+1+\tau_1} + \varepsilon_2^{2\mu_{21}+1+\tau_2}\bigr),$$

$$\Upsilon(\varepsilon_1, \varepsilon_2) = \left|\frac{z + z^{-1}}{P}\right| \varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}\bigl(1 + O(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2})\bigr)$$

hold: $\tau_j = \min\{\mu_{j2} - \mu_{j1}, 2 - \sigma_j\}$ *and* σ_j *are small positive numbers.*

Proof From Theorems 6.4.4 and (6.4.27) we obtain

$$|T_1 - \widetilde{T}_1| \le c\left|\frac{S_{12} - \widetilde{S}_{12}}{\widetilde{S}_{12}}\right| \widetilde{T}_1 \le c\widetilde{T}_1\bigl(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2} + \varepsilon_2^{\tau_2+2\mu_{21}+1}/\varepsilon_1^{2\mu_{11}+1}\bigr)$$

with $\tau_j = \min\{\mu_{j2} - \mu_{j1}, 2 - \sigma_j\}, \sigma_j = \mu_{j1} + 3/2 - \gamma_j$, $j = 1, 2$. When $\varepsilon_1^{2\mu_{11}+1} \ge \varepsilon_2^{2\mu_{12}+1}$, we get the desired expansion for T_1. Assume that $\varepsilon_1^{2\mu_{11}+1} \le \varepsilon_2^{2\mu_{21}+1}$. In the analogous way we can obtain

$$|T_2 - \widetilde{T}_2| \le c\widetilde{T}_2\bigl(\varepsilon_1^{\tau_1} + \varepsilon_2^{\tau_2} + \varepsilon_1^{\tau_1+2\mu_{11}+1}/\varepsilon_2^{2\mu_{21}+1}\bigr)$$

with the same τ_j. As is known, $T_1 = T_2$, and it is easy to see that $\widetilde{T}_1 = \widetilde{T}_2$ (indeed, all characteristics of both narrows are interchangeably included in the formulas for $\widetilde{T}_p$). This leads to the required expansion for T_1 as $\varepsilon_1^{2\mu_{11}+1} \le \varepsilon_2^{2\mu_{21}+1}$. The formulas for k_r^2 and Υ follow from that for T_1. □

Chapter 7
Resonant Tunneling in 2D Waveguides in Magnetic Field

The presence of a magnetic field can essentially affect the basic characteristics of the resonant tunneling and bring new possibilities for applications in electronics. In particular, in the presence of a magnetic field, the tunneling phenomenon is feasible for producing spin-polarized electron flows consisting of electrons with spins of the same direction. In Chap. 11, we describe magnetic field sensors based on resonant tunneling in magnetic field.

We consider the same 2D waveguide with two narrows as in Chap. 5 and suppose that a part of the resonator is occupied by a homogeneous magnetic field. An electron wave function satisfies the Pauli equation in the waveguide and vanishes on its boundary. An electron energy is between the first and the second thresholds. The asymptotics of the basic resonant tunneling characteristics are presented as the narrow diameter ε tends to zero. The asymptotic results are compared with numerical ones obtained by approximate computing the scattering matrix; there is an interval of ε where the asymptotic and numerical results practically coincide. Using the approximate scattering matrix, we also observe the dependence of the tunneling characteristics on a magnetic field position in the resonator.

7.1 Statement of the Problem

To describe the domain $G(\varepsilon)$ in $\mathbb{R}^2$ occupied by the waveguide, we first introduce two auxiliary domains G and Ω in $\mathbb{R}^2$. The domain G is the strip

$$G = \mathbb{R} \times D = \{(x, y) \in \mathbb{R}^2 : \ x \in \mathbb{R}; \ y \in D = (-l/2, l/2)\}.$$

Let us define Ω. Denote by K a pair of opposite angles with vertex at the origin O. Assume that K is symmetric about the origin and contains the axis x. The set $K \cap S^1$, where S^1 is a unit circle, consists of two simple arcs. Assume that Ω contains K and a neighborhood of its vertex. Moreover, outside a sufficiently large disc the set Ω coincides with K. The boundary $\partial\Omega$ of Ω is supposed to be smooth (see Fig. 5.1).

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We now turn to the waveguide $G(\varepsilon)$. Denote by $\Omega(\varepsilon)$ the domain obtained from Ω by contraction with center at O and coefficient ε. In other words, $(x, y) \in \Omega(\varepsilon)$ if and only if $(x/\varepsilon, y/\varepsilon) \in \Omega$. Let K_j and $\Omega_j(\varepsilon)$ stand for K and $\Omega(\varepsilon)$ shifted by the vector $\mathbf{r}_j = (x_j^0, 0)$, $j = 1, 2$. We assume that $|x_1^0 - x_2^0|$ is sufficiently large so that the distance from $\partial K_1 \cap \partial K_2$ to G is positive. We put (see Fig. 1.4) $G(\varepsilon) = G \cap \Omega_1(\varepsilon) \cap \Omega_2(\varepsilon)$. Consider the equations

$$(-i\nabla + \mathbf{A})^2 u \pm Hu - k^2 u = 0, \tag{7.1.1}$$

which are 2D counterparts of the equations describing the motion of electrons of spin $\pm 1/2$ in a magnetic field parallel to z-axis. Here $\nabla = (\partial_x, \partial_y)^T$; $H = \partial_x A_y - \partial_y A_x$. Let H depend only on $\rho = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$, and let $H(\rho) = 0$ as $\rho > R$, where R is a positive constant. Then we can put $\mathbf{A} = A(\rho)\mathbf{e}_\psi$, where $\mathbf{e}_\psi = \rho^{-1}(-y + y_0, x - x_0)$ and

$$A(\rho) = \frac{1}{\rho} \int_0^{\min\{\rho, R\}} t H(t)\, dt.$$

It is evident, that the equality $\partial_x A_y - \partial_y A_x = H$ defines $\mathbf{A}$ up to a summand of the form ∇f.

Let (ρ, ψ) be polar coordinates in the plane xy centered at (x_0, y_0), the angle ψ being measured from a ray parallel to x-axis. Introduce $f(x, y) = c\psi$, where $c = \int_0^R t H(t)\, dt$. We assume that $-\pi/2 < \psi < 3\pi/2$. The function f is uniquely determined in the waveguide for $|x - x_0| > 0$, moreover, $\nabla f = \mathbf{A}$ for $|x - x_0| > R$. Let $\tau(t)$ be a cut-off function on $\mathbb{R}_+$, equal to 1 as $t > R + 2\delta$ and to 0 as $t < R + \delta$, δ being a positive constant. Put $\mathbf{A}'(x, y) = \mathbf{A}(x, y) - \nabla(\tau(|x - x_0|) f(x, y))$. Then $\partial_x A'_y - \partial_y A'_x = \partial_x A_y - \partial_y A_x = H$ and $\mathbf{A}' = 0$ as $|x - x_0| > R + 2\delta$. The wave function $u' = u \exp\{i\tau f\}$ satisfies (7.1.1) with $\mathbf{A}$ replaced by $\mathbf{A}'$. As $|x - x_0| > R + 2\delta$ the Eq. (7.1.1) with new potential $\mathbf{A}'$ reduces to the Helmholtz equation

$$-\Delta u' - k^2 u' = 0.$$

In what follows we omit the primes in the notations. We look for solutions to (7.1.1) satisfying the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial G(\varepsilon). \tag{7.1.2}$$

The obtained boundary value problems are self-adjoint with respect to the Green formulas

$$\begin{aligned} ((-i\nabla + \mathbf{A})^2 u \pm Hu - k^2 u, v)_{G(\varepsilon)} - (u, (-i\nabla + \mathbf{A})^2 v \pm Hv - k^2 v)_{G(\varepsilon)} \\ + (u, (-\partial_n - A_n)v)_{\partial G(\varepsilon)} - ((-\partial_n - A_n)u, v)_{\partial G(\varepsilon)} = 0, \end{aligned}$$

where A_n is a projection of $\mathbf{A}$ onto the outward normal to $\partial G(\varepsilon)$; $u, v \in C_0^\infty(G(\varepsilon))$. Additionally, we require u to satisfy some radiation conditions at infinity. To formulate the conditions, we consider the problem

$$\begin{aligned} &-\Delta v(y) - \lambda^2 v(y) = 0, \qquad y \in (-l/2, l/2); \\ &v(-l/2) = v(l/2) = 0. \end{aligned} \tag{7.1.3}$$

The eigenvalues λ_q^2 of this problem are called thresholds; they form the sequence $\lambda_q^2 = (\pi q/l)^2$, $q = 1, 2, \ldots$. Let us consider the Eq. (7.1.1) with "+". We suppose that k^2 in (7.1.1) satisfies $(\pi/l)^2 < k^2 < (2\pi/l)^2$, i.e., k^2 is between the first and the second thresholds. Then, in the space of bounded wave functions, a basis is formed by the wave functions subject to the radiation conditions

$$u_1^+(x, y) = \begin{cases} U_1(x, y) + S_{11}^+(k)\, U_2(x, y) + O(e^{\delta x}), & x \to -\infty, \\ S_{12}^+(k)\, U_1(x, y) + O(e^{-\delta x}), & x \to +\infty; \end{cases} \tag{7.1.4}$$

$$u_2^+(x, y) = \begin{cases} S_{21}^+(k)\, U_2(x, y) + O(e^{\delta x}), & x \to -\infty, \\ U_2(x, y) + S_{22}^+(k)\, U_1(x, y) + O(e^{-\delta x}), & x \to +\infty. \end{cases} \tag{7.1.5}$$

In the strip G, the function $U_1(x, y) = e^{i\nu_1 x}\Psi_1(y)$ is a wave incoming from $-\infty$ and outgoing to $+\infty$, while $U_2(x, y) = e^{-i\nu_1 x}\Psi_1(y)$ is a wave going from $+\infty$ to $-\infty$. Here $\nu_1 = \sqrt{k^2 - \lambda_1^2}$; Ψ_1 is an eigenfunction of problem (7.1.3) that corresponds to the eigenvalue λ_1^2,

$$\Psi_1(y) = \sqrt{2/l\nu_1}\cos \lambda_1 y. \tag{7.1.6}$$

The matrix

$$S^+ = \| S_{mj}^+ \|_{m,j=1,2}$$

with elements from conditions (7.1.4) and (7.1.5) is called the scattering matrix; it is unitary. The values

$$R_1^+ = |S_{11}^+|^2, \qquad T_1^+ = |S_{12}^+|^2$$

are called the reflection and transition coefficients, relatively, for the wave U_1 incoming to $G(\varepsilon)$ from $-\infty$. Similar definitions can be given for the wave U_2 coming from $+\infty$. The scattering matrix S^- and the reflection and transition coefficients R_m^-, T_m^- for the Eq. (7.1.1) with "−" are introduced in the same way.

The goal is to find a "resonant" value $k_r^\pm = k_r^\pm(\varepsilon)$ of the parameter k, where the transition coefficient takes at its maximum, and to describe the behavior of $T_m^\pm(k, \varepsilon)$ in a neighborhood of $k_r^\pm(\varepsilon)$ as $\varepsilon \to 0$.

7.2 The Limit Problems

We construct the asymptotics of the wave function (i.e., the solution of (7.1.1)) as $\varepsilon \to 0$ by the compound asymptotics method. To this end, we introduce "limit" boundary value problems independent of the parameter ε. We suppose the domain occupied by the magnetic field to be localized in the resonator, the part of the waveguide between the narrows. Furthermore, we assume that $|x_j - x_0| > R + 2$, $j = 1, 2$, so the vector potential $\mathbf{A}$ differs from zero only on a domain inside the resonator. Then, outside the resonator and, in particular, near the narrows, the sought wave function satisfies the Helmholtz equation. That is why this section coincides with Sect. 7.2 except several details. The only distinction is the discussion of the limit problem in G_0, where magnetic field presents. Nevertheless, we repeat here the description of necessary properties of limit problems for the convenience of the reader.

7.2.1 First Kind Limit Problems

Let $G(0) = G \cap K_1 \cap K_2$ (Fig. 5.3); therefore, $G(0)$ consists of three parts, G_1, G_2 and G_0, where G_1 and G_2 are infinite domains, and G_0 is a bounded resonator.

The boundary value problems

$$\begin{aligned} -\Delta v(x, y) - k^2 v(x, y) &= f(x, y), \; (x, y) \in G_j, \\ v(x, y) &= 0, \qquad\quad (x, y) \in \partial G_j, \end{aligned} \tag{7.2.1}$$

where $j = 1, 2$, and

$$\begin{aligned} (-i\nabla + \mathbf{A}(x, y))^2 v(x, y) \pm H(\rho) v(x, y) - k^2 v(x, y) &= f(x, y), \; (x, y) \in G_0, \\ v(x, y) &= 0, \qquad\quad (x, y) \in \partial G_0, \end{aligned} \tag{7.2.2}$$

are called first kind limit problems.

We introduce function spaces for the problem (7.2.2) in G_0. Let ϕ_1 and ϕ_2 be smooth real functions in the closure $\overline{G}_0$ of G_0 such that $\phi_j = 1$ in a neighborhood of O_j, $j = 1, 2$, and $\phi_1^2 + \phi_2^2 = 1$. For $l = 0, 1, \dots$ and $\gamma \in \mathbb{R}$ the space $V_\gamma^l(G_0)$ is the completion in the norm

$$\|v; V_\gamma^l(G_0)\| = \left(\int_{G_0} \sum_{|\alpha|=0}^{l} \sum_{j=1}^{2} \phi_j^2(x, y) r_j^{2(\gamma - l + |\alpha|)} |\partial^\alpha v(x, y)|^2 \, dx \, dy \right)^{1/2} \tag{7.2.3}$$

of the set of smooth functions in $\overline{G}_0$ vanishing near O_1 and O_2; here r_j is the distance from (x, y) to the origin O_j, $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, and $\partial^\alpha = \partial^{|\alpha|}/\partial x^{\alpha_1} \partial y^{\alpha_2}$.

Proposition 7.2.1 follows from the well-known general results; e.g., see [37, Chaps. 2 and 4, Sect. 1–3] or [33, vol. 1, Chap. 1].

Proposition 7.2.1 *Assume that* $|\gamma - 1| < \pi/\omega$. *Then, for* $f \in V^0_\gamma(G_0)$ *and arbitrary* k^2, *except the positive increasing sequence* $\{k_p^2\}_{p=1}^\infty$ *of eigenvalues,* $k_p^2 \to \infty$, *there exists a unique solution* $v \in V^2_\gamma(G_0)$ *to the problem* (7.2.1) *in* G_0. *The estimate*

$$\|v;\ V^2_\gamma(G_0)\| \le c\|f;\ V^0_\gamma(G_0)\| \tag{7.2.4}$$

holds with a constant c *independent of* f. *If* f *is a smooth function in* $\overline{G}_0$ *vanishing near* O_1 *and* O_2, *and* v *is any solution in* $V^2_\gamma(G_0)$ *to the problem* (7.2.2), *then* v *is smooth in* $\overline{G}_0$ *except at* O_1 *and* O_2 *and admits the asymptotic representation*

$$v(x,y) = \begin{cases} b_1 \widetilde{J}_{\pi/\omega}(kr_1)\Phi(\varphi_1) + O\big(r_1^{2\pi/\omega}\big), & r_1 \to 0, \\ b_2 \widetilde{J}_{\pi/\omega}(kr_2)\Phi(\pi - \varphi_2) + O\big(r_2^{2\pi/\omega}\big), & r_2 \to 0, \end{cases}$$

near the points O_1 *and* O_2, *where* (r_j, φ_j) *are polar coordinates centered at* O_j, b_j *are some constants coefficients,* $\widetilde{J}_\mu$ *stands for the Bessel function multiplied by a constant such that* $\widetilde{J}_\mu(kr) = r^\mu + o(r^\mu)$, *and* $\Phi(\varphi) = \pi^{-1/2}\cos(\pi\varphi/\omega)$.

Let $k^2 = k_e^2$ *be an eigenvalue of the problem* (7.2.2). *Then the problem* (7.2.2) *in* G_0 *is solvable if and only if* $(f, v_e)_{G_0} = 0$ *for any eigenfunction* v_e *corresponding to* k_e^2. *These conditions being fulfilled, there exists a unique solution* v *to the problem* (7.2.2) *that is orthogonal to the eigenfunctions and satisfies* (7.2.4) *(i.e., the Fredholm alternative holds).*

We turn to problems (7.2.1) for $j = 1, 2$. Let $\chi_{0,j}$ and $\chi_{\infty,j}$ be smooth real functions in the closure $\overline{G}_j$ of G_j such that $\chi_{0,j} = 1$ in a neighborhood of O_j, $\chi_{0,j} = 0$ outside a compact set, and $\chi_{0,j}^2 + \chi_{\infty,j}^2 = 1$. We also assume that the support supp $\chi_{\infty,j}$ is located in the strip G. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$ the space $V^l_{\gamma,\delta}(G_j)$ is the completion in the norm

$$\|v;\ V^l_{\gamma,\delta}(G_j)\| = \left(\int_{G_j} \sum_{|\alpha|=0}^{l} \big(\chi_{0,j}(x,y)^2 r_j^{2(\gamma - l + |\alpha|)} + \chi_{\infty,j}(x,y)^2 \exp(2\delta x)\big)|\partial^\alpha v(x,y)|^2\, dx\, dy\right)^{1/2} \tag{7.2.5}$$

of the set of smooth functions with compact supports on $\overline{G}_j$ vanishing near O_j.

Recall that, by assumption, k^2 is between the first and the second thresholds, therefore in each domain G_j there exists only one outgoing wave. Let $U_1^- = U_2$ be the outgoing wave in G_1 and let $U_2^- = U_1$ be the outgoing wave in G_2 (the definitions of U_j and G are given in Sect. 7.1). The next proposition follows, e.g., from Theorem 5.3.5 in [37].

Proposition 7.2.2 *Let $|\gamma - 1| < \pi/\omega$ and suppose that there is no nontrivial solution to the homogeneous problem (7.2.1) (where $f = 0$) in $V^2_{\gamma,\delta}(G_j)$ with arbitrarily small positive δ. Then, for any $f \in V^0_{\gamma,\delta}(G_j)$, there exists a unique solution v to (7.2.1) that admits the representation*

$$v = u + A_j \chi_{\infty,j} U_j^-,$$

where $A_j = \mathrm{const}$, $u \in V^2_{\gamma,\delta}(G_j)$, and δ is sufficiently small; herewith the estimate

$$\|u; V^2_{\gamma,\delta}(G_j)\| + |A_j| \le c\|f; V^0_{\gamma,\delta}(G_j)\|, \tag{7.2.6}$$

holds with a constant c independent of f. If f is smooth and vanishes near O_j, then the solution v to the problem in G_1 satisfies

$$v(x, y) = a_1 \widetilde{J}_{\pi/\omega}(kr_1)\Phi(\pi - \varphi_1) + O\big(r_1^{2\pi/\omega}\big), \quad r_1 \to 0,$$

and the solution to the problem in G_2 satisfies

$$v(x, y) = a_2 \widetilde{J}_{\pi/\omega}(kr_2)\Phi(\varphi_2) + O\big(r_2^{2\pi/\omega}\big), \quad r_2 \to 0,$$

where a_j are some constants.

7.2.2 Second Kind Limit Problems

In the domains Ω_j, $j = 1, 2$, introduced in Sect. 7.1, we consider the boundary value problems

$$\begin{aligned} \Delta w(\xi_j, \eta_j) &= F(\xi_j, \eta_j), \ (\xi_j, \eta_j) \in \Omega_j, \\ w(\xi_j, \eta_j) &= 0, \qquad\qquad\ (\xi_j, \eta_j) \in \partial\Omega_j, \end{aligned} \tag{7.2.7}$$

which are called second kind limit problems; (ξ_j, η_j) stands for Cartesian coordinates with origin at O_j.

Let $\rho_j = \mathrm{dist}((\xi_j, \eta_j), O_j)$ and let $\psi_{0,j}$, $\psi_{\infty,j}$ be smooth real functions in $\overline{\Omega}_j$ such that $\psi_{0,j} = 1$ for $\rho_j < N/2$, $\psi_{0,j} = 0$ for $\rho_j > N$, and $\psi^2_{0,j} + \psi^2_{\infty,j} = 1$, where N is a sufficiently large positive number. For $\gamma \in \mathbb{R}$ and $l = 0, 1, \dots$, the space $V^l_\gamma(\Omega_j)$ is the completion in the norm

$$\|v;\ V^l_\gamma(\Omega_j)\| = \left(\int_{\Omega_j} \sum_{|\alpha|=0}^{l} \big(\psi_{0,j}(\xi_j, \eta_j)^2 + \psi_{\infty,j}(\xi_j, \eta_j)^2 \rho_j^{2(\gamma - l + |\alpha|)}\big)|\partial^\alpha v(\xi_j, \eta_j)|^2 \, d\xi_j d\eta_j\right)^{1/2} \tag{7.2.8}$$

of the set $C_c^\infty(\overline{\Omega}_j)$ of smooth functions compactly supported in $\overline{\Omega}_j$. The next proposition is a corollary of Theorem 4.3.6 in [37].

Proposition 7.2.3 *Let $|\gamma - 1| < \pi/\omega$. Then for $F \in V_\gamma^0(\Omega_j)$ there exists a unique solution $w \in V_\gamma^2(\Omega_j)$ to (7.2.7) such that the estimate*

$$\|w;\ V_\gamma^2(\Omega_j)\| \le c\|F;\ V_\gamma^0(\Omega_j)\|, \tag{7.2.9}$$

holds with a constant c independent of F. If $F \in C_c^\infty(\overline{\Omega}_j)$, then w is smooth on $\overline{\Omega}_j$ and admits the representation

$$w(\xi_j, \eta_j) = \begin{cases} d_j^l \rho_j^{-\pi/\omega} \Phi(\pi - \varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j < 0, \\ d_j^r \rho_j^{-\pi/\omega} \Phi(\varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j > 0, \end{cases} \tag{7.2.10}$$

as $\rho_j \to \infty$; here (ρ_j, φ_j) are polar coordinates in Ω_j with center at O_j, and Φ is the same as in Proposition 7.2.1. The constants d_j^l and d_j^r are found with the formulas

$$d_j^l = -(F, w_j^l)_\Omega, \qquad d_j^r = -(F, w_j^r)_\Omega,$$

where w_j^l and w_j^r are the unique solutions to (7.2.7) with $F = 0$ satisfying

$$w_j^l = \begin{cases} \left(\rho_j^{\pi/\omega} + \alpha\rho_j^{-\pi/\omega}\right) \Phi(\pi - \varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j < 0; \\ \beta\rho_j^{-\pi/\omega} \Phi(\varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j > 0; \end{cases} \tag{7.2.11}$$

$$w_j^r = \begin{cases} \beta\rho_j^{-\pi/\omega} \Phi(\pi - \varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j < 0; \\ \left(\rho_j^{\pi/\omega} + \alpha\rho_j^{-\pi/\omega}\right) \Phi(\varphi_j) + O\big(\rho_j^{-3\pi/\omega}\big), & \xi_j > 0; \end{cases} \tag{7.2.12}$$

as $\rho_j \to \infty$; the coefficients α and β depend only on the geometry of Ω and have to be calculated.

7.3 Special Solutions to Homogeneous First Kind Limit Problems

In each of the domains G_j, $j = 0, 1, 2$, we introduce special solutions to the homogeneous problems (7.2.1) and (7.2.2). These solutions are necessary for the construction of the wave function asymptotics in the next section. It follows from Propositions 7.2.1 and 7.2.2 that the bounded solutions to the homogeneous problems (7.2.1) and (7.2.2) are trivial (except the eigenfunctions of the problem in the resonator). Therefore, we consider only solutions unbounded in the neighborhood of O_j.

We now introduce special solutions to homogeneous problems (7.2.1) in G_j, $j = 0, 1, 2$. In the domain G_j, $j = 1, 2$, there exists a bounded solution V_j such that

$$V_j(x, y) = \begin{cases} U_j^+(x, y) + S_{jj}^0 U_j^-(x, y) + O(\exp(-\delta x)), & x \to \infty, \\ s_j \widetilde{J}_{\pi/\omega}(kr_j)\Phi_j(\varphi_j) + O\big(r^{2\pi/\omega}\big), & r \to 0, \end{cases} \tag{7.3.1}$$

with arbitrary small positive δ, $\Phi_1(\varphi_1) = \Phi(\pi - \varphi_1)$, and $\Phi_2(\varphi_2) = \Phi(\varphi_2)$. The scattering matrix in G_j consists of the only entry S_j^0, $|S_j^0| = 1$.

Let K^l be the part of the double cone K to the left of the coordinate origin, $K^l = \{(\xi, \eta) \in K : \xi < 0\}$. Let us consider the problem

$$\begin{aligned} -\Delta u - k^2 u = 0 \quad &\text{in } K^l, \\ u = 0 \quad &\text{on } \partial K^l. \end{aligned} \tag{7.3.2}$$

The function

$$v(r, \varphi) = \widetilde{N}_{\pi/\omega}(kr)\Phi(\pi - \varphi) \tag{7.3.3}$$

satisfies (7.3.2); $\widetilde{N}_{\pi/\omega}$ stands for the Neumann function multiplied by a constant such that

$$\widetilde{N}_{\pi/\omega}(kr) = r^{-\pi/\omega} + o(r^{-\pi/\omega})$$

and Φ is the same as in Proposition 7.2.1. Let $t \mapsto \Theta(t)$ be a cut-off function on $\mathbb{R}$ equal to 1 for $t < \delta/2$ and to 0 for $t > \delta$, δ being a small positive number. Introduce a solution

$$\mathbf{v}_1(x, y) = \Theta(r_1)v(r_1, \varphi_1) + \widetilde{v}_1(x, y) \tag{7.3.4}$$

of homogeneous problem (5.2.1) in G_1, where $\widetilde{v}_1$ solves (7.2.1) with $f = -[\Delta, \Theta]v$; the existence of $\widetilde{v}_1$ is provided by Proposition 7.2.2. Thus,

$$\mathbf{v}_1(x, y) = \begin{cases} \big(\widetilde{N}_{\pi/\omega}(kr_1) + a_1\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\pi - \varphi_1) + O(r_1^{3\pi/\omega}), & r_1 \to 0, \\ A_1 U_1^-(x, y) + O(e^{\delta x}), & x \to -\infty, \end{cases} \tag{7.3.5}$$

where $\widetilde{J}_{\pi/\omega}$ is the same as in Propositions 7.2.1 and 7.2.2, and the constant $A \neq 0$ depends only on the geometry of the domain G_1 and should be calculated.

Define the solution $\mathbf{v}_2$ to the problem (7.2.1) in G_2 by $\mathbf{v}_2(x, y) = \mathbf{v}_1(d - x, y)$, where $d = \operatorname{dist}(O_1, O_2)$. Then

$$\mathbf{v}_2(x, y) = \begin{cases} \big(\widetilde{N}_{\pi/\omega}(kr_2) + a_2\widetilde{J}_{\pi/\omega}(kr_2)\big)\Phi(\varphi_2) + O(r_2^{3\pi/\omega}), & r_2 \to 0, \\ A_2 U_2^-(x, y) + O(e^{-\delta x}), & x \to +\infty; \end{cases} \tag{7.3.6}$$

where obviously $a_2 = a_1$, $A_2 = A_1 e^{-i\nu_1 d}$.

Lemma 7.3.1 *The equalities* $|A_j|^2 = 2\mathrm{Im}\, a_j$, $A_j = i\bar{s}_j S^0_{jj}$ *hold.*

Let $k^2_{e,\pm}$ be a simple eigenvalue of (7.2.2) in G_0 and let $v^\pm_e$ be a corresponding eigenfunction normalized by $\int_{G_0} |v^\pm_e|^2 dx\, dy = 1$. By Proposition 7.2.1

$$v^\pm_e(x) \sim \begin{cases} b^\pm_1 \widetilde{J}_{\pi/\omega}(k_{e,\pm} r_1)\Phi(\varphi_1), & r_1 \to 0, \\ b^\pm_2 \widetilde{J}_{\pi/\omega}(k_{e,\pm} r_2)\Phi(\pi - \varphi_2), & r_2 \to 0. \end{cases} \tag{7.3.7}$$

We assume that $b^\pm_j \neq 0$. For $H = 0$, it is true, e.g., for the eigenfunction corresponding to the least eigenvalue of the resonator. For nonzero H, this condition may not hold. For k^2 in a punctured neighborhood of $k^2_{e,\pm}$ separated from the other eigenvalues, we introduce solutions $\mathbf{v}^\pm_{0j}$ to the homogeneous problem (7.2.2) by

$$\mathbf{v}^\pm_{0j}(x, y) = \Theta(r_j) v(r_j, \varphi_j) + \widetilde{v}^\pm_{0j}(x, y), \qquad j = 1, 2, \tag{7.3.8}$$

where v is defined by (7.3.3), and $\widetilde{v}^\pm_{0j}$ is the bounded solution to the problem (7.2.2) with

$$f_j(x, y) = [\Delta, \Theta(r_j)] v(r_j, \varphi_j).$$

Lemma 7.3.2 *In a neighborhood* $V \subset \mathbb{C}$ *of* $k^2_{e,\pm}$ *containing no eigenvalues of the problem* (7.2.2) *in* G_0 *except* $k^2_{e,\pm}$, *the equalities* $\mathbf{v}^\pm_{0j} = -\overline{b^\pm_j}(k^2 - k^2_{e,\pm})^{-1} v^\pm_e + \widehat{\mathbf{v}}^\pm_{0j}$ *hold with* $b^\pm_j$ *in* (7.3.7) *and functions* $\widehat{\mathbf{v}}^\pm_{0j}$ *analytic in* $k^2 \in V$.

Proof First check the equality $(\mathbf{v}^\pm_{0j}, v^\pm_e)_{G_0} = -\overline{b^\pm_j}/(k^2 - k^2_{e,\pm})$, where $\mathbf{v}^\pm_{0j}$ are defined by (7.3.8). We have

$$(\Delta \mathbf{v}^\pm_{0j} + k^2 \mathbf{v}^\pm_{0j}, v^\pm_e)_{G_\delta} - (\mathbf{v}^\pm_{0j}, \Delta v^\pm_e + k^2 v^\pm_e)_{G_\delta} = -(k^2 - k^2_{e,\pm})(\mathbf{v}^\pm_{0j}, v^\pm_e)_{G_\delta};$$

the domain G_δ is obtained from G_0 by excluding discs with radius δ and centers O_1 and O_2. Using the Green formula, as in Lemma 5.3.1, we get the equality

$$-(k^2 - k^2_{e,\pm})(\mathbf{v}^\pm_{0j}, v^\pm_e)_{G_\delta} = \overline{b^\pm_j} + o(1).$$

It remains to let δ tend to zero.

Since $k^2_{e,\pm}$ is a simple eigenvalue, we have

$$\widetilde{v}^\pm_{0j} = \frac{B^\pm_j(k^2)}{k^2 - k^2_{e,\pm}} v^\pm_0 + \widehat{v}^\pm_{0j}, \tag{7.3.9}$$

where $B^\pm_j(k^2)$ does not depend on (x, y), and $\widehat{v}^\pm_{0j}$ are some functions analytic with respect to k^2 near the point $k^2 = k^2_{e,\pm}$. Multiplying (7.3.8) by $v^\pm_e$ and taking into account (7.3.9), the obtained formula for $(\mathbf{v}^\pm_{0j}, v^\pm_e)_{G_0}$, and the condition

$(v_e^\pm, v_e^\pm)_{G_0} = 1$, we get the equality $B_j^\pm(k^2) = -\overline{b_j^\pm} + (k^2 - k_{e,\pm}^2)\widetilde{B}_j^\pm(k^2)$, where $\widetilde{B}_j^\pm$ are some analytic functions. Together with (7.3.9) that leads to the required statement. $\square$

In view of Proposition 7.2.1,

$$\mathbf{v}_{01}^\pm(x,y) \sim \begin{cases} \big(\widetilde{N}_{\pi/\omega}(kr_1) + c_{11}^\pm(k)\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\varphi_1), & r_1 \to 0, \\ c_{12}^\pm(k)\widetilde{J}_{\pi/\omega}(kr_2)\Phi(\pi - \varphi_2), & r_2 \to 0, \end{cases} \tag{7.3.10}$$

$$\mathbf{v}_{02}^\pm(x,y) \sim \begin{cases} \big(c_{21}^\pm(k)\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\varphi_1), & r_1 \to 0, \\ \big(\widetilde{N}_{\pi/\omega}(kr_2) + c_{22}^\pm(k)\widetilde{J}_{\pi/\omega}(kr_2)\big)\Phi(\pi - \varphi_2), & r_2 \to 0. \end{cases} \tag{7.3.11}$$

According to Lemma 7.3.2 and relations (7.3.7),

$$c_{pq}^\pm(k) = -\frac{\overline{b_p^\pm} b_q^\pm}{k^2 - k_{e,\pm}^2} + \widehat{c}_{pq}^\pm(k), \tag{7.3.12}$$

where $\widehat{c}_{pq}^\pm$ analytically depends on k^2 near $k_{e,\pm}^2$.

Lemma 7.3.3 *If* $\mathbf{v}_{01}^\pm$ *and* $\mathbf{v}_{02}^\pm$ *in* (7.3.10) *and* (7.3.11) *make sense for a number* k*, then* $c_{12}^\pm(k) = \overline{c_{21}^\pm(k)}$.

Proof It suffices to apply the Green formula to $\mathbf{v}_{01}^\pm$ and $\mathbf{v}_{02}^\pm$ in the same domain G_δ as in the proof of Lemma 5.3.2, to use (7.3.10) and (7.3.11), and to let δ tend to 0. $\square$

7.4 Asymptotic Formulas

This section is devoted to the derivation of the asymptotic formulas. In Sect. 7.4.1, we present a formula for the wave function (see (7.4.1)), explain its structure, and describe the solutions of the first kind limit problems involved in the formula. The construction of formula (7.4.1) is completed in Sect. 7.4.2, where the solutions to the second kind limit problems are given and the coefficients in the expressions for the solutions of the first kind limit problems are calculated. In Sect. 7.4.3, we analyze the expression for $\widetilde{S}_{12}$ obtained in 7.4.2 and derive formal asymptotics for the characteristics of resonant tunneling. Notice that the remainders in the formulas (7.4.23) and (7.4.26) arose in the intermediate stage of considerations while simplifying the principal part of the asymptotics; they are not the remainders in the final asymptotic formulas. The "final" remainders are estimated in Sect. 7.5 (see Theorem 7.5.3). For brevity, in this section we omit "±" in the notations bearing in mind any one of the Eqs. (7.1.1) and not specifying, which is considered.

7.4.1 Asymptotics of the Wave Function

In the waveguide $G(\varepsilon)$, we consider the scattering of the wave $U(x, y) = e^{i\nu_1 x}\Psi_1(y)$, incoming from $-\infty$ (see (7.1.6)). The corresponding wave function admits the representation

$$\begin{aligned} u(x, y; \varepsilon) = \; & \chi_{1,\varepsilon}(x, y)v_1(x, y; \varepsilon) + \Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon) \\ & + \chi_{0,\varepsilon}(x, y)v_0(x, y; \varepsilon) + \Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon) \\ & + \chi_{2,\varepsilon}(x, y)v_2(x, y; \varepsilon) + R(x, y; \varepsilon). \end{aligned} \tag{7.4.1}$$

Let us explain the notation and the structure of this formula. When composing the formula, we first describe the behavior of the wave function u outside of the narrows, where the solutions v_j to the homogeneous problems (7.2.1) in G_j serve as approximations to u. The function v_j is a linear combination of the special solutions introduced in the previous section; v_1 and v_2 are subject to the same radiation conditions as u:

$$\begin{aligned} v_1(x, y; \varepsilon) = V_1(x, y) + C_{11}\mathbf{v}_1(x, y) \sim U_1^+(x, y) & \\ + \widetilde{S}_{11}(\varepsilon)U_1^-(x, y), \quad x \to -\infty; & \end{aligned} \tag{7.4.2}$$

$$v_0(x, y; \varepsilon) = C_{12}(\varepsilon)\mathbf{v}_{01}(x, y) + C_{13}(\varepsilon)\mathbf{v}_{02}(x, y); \tag{7.4.3}$$

$$v_2(x, y; \varepsilon) = C_{14}\mathbf{v}_2(x, y) \sim \widetilde{S}_{12}(\varepsilon)U_2^-(x, y), \quad x \to +\infty; \tag{7.4.4}$$

the approximations $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$ to the scattering matrix entries $S_{11}(\varepsilon)$, $S_{12}(\varepsilon)$ and the coefficients $C_{11}(\varepsilon), \dots, C_{14}(\varepsilon)$ are yet unknown. By $\chi_{j,\varepsilon}$ we denote the cut-off functions defined by

$$\begin{aligned} \chi_{1,\varepsilon}(x, y) &= (1 - \Theta(r_1/\varepsilon))\,\mathbf{1}_{G_1}(x, y), \\ \chi_{2,\varepsilon}(x, y) &= (1 - \Theta(r_2/\varepsilon))\,\mathbf{1}_{G_2}(x, y), \\ \chi_{0,\varepsilon}(x, y) &= (1 - \Theta(r_1/\varepsilon) - \Theta(r_2/\varepsilon))\,\mathbf{1}_{G_0}(x, y), \end{aligned}$$

where $r_j = \sqrt{x_j^2 + y_j^2}$, and (x_j, y_j) are the coordinates of a point (x, y) in the system obtained by shifting the origin to the point O_j; $\mathbf{1}_{G_j}$ is the indicator of G_j (equal to 1 in G_j and to 0 outside G_j); $\Theta(\rho)$ is the same cut-off function as in (7.3.4) (equal to 1 for $0 \leqslant \rho \leqslant \delta/2$ and to 0 for $\rho \geqslant \delta$, δ being a fixed positive number). Thus, $\chi_{j,\varepsilon}$ are defined on the whole waveguide $G(\varepsilon)$ as well as the functions $\chi_{j,\varepsilon}v_j$ in (7.4.1).

Being substituted to (7.1.1), the sum $\sum_{j=0}^{2} \chi_{j,\varepsilon}v_j$ gives a discrepancy in the right-hand side of the Helmholtz equation supported near the narrows. We compensate the principal part of the discrepancy by means of the second kind limit problems. Namely, the discrepancy supported near O_j is rewritten into coordinates $(\xi_j, \eta_j) = (\varepsilon^{-1}x_j, \varepsilon^{-1}y_j)$ in the domain Ω_j and is taken as a right-hand side for the Laplace equation. The solution w_j of the corresponding problem (6.2.6) is rewritten into coordinates (x_j, y_j) and multiplied by a cut-off function. As a result, the terms $\Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon)$ arise in (7.4.1).

Proposition 7.2.3 provides the existence of solutions w_j decaying at infinity as $O(\rho_j^{-\pi/\omega})$ (see (7.2.10)). But those solutions will not lead us to the goal, because substitution of (7.4.1) into (7.1.1) gives a discrepancy of high order, which has to be compensated again. Therefore, we require the rate $w_j = O(\rho_j^{-3\pi/\omega})$ as $\rho_j \to \infty$. By Proposition 7.2.3, such a solution exists if the right-hand side of the problem (7.2.7) satisfies the additional conditions

$$(F, w_j^l)_{\Omega_j} = 0, \qquad (F, w_j^r)_{\Omega_j} = 0.$$

These conditions (two in each narrow) uniquely determine the coefficients $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$, $C_{11}(\varepsilon), \dots, C_{14}(\varepsilon)$. The remainder $R(x, y; \varepsilon)$ is small in comparison with the principal part of (7.4.1) as $\varepsilon \to 0$.

7.4.2 Formulas for $\widetilde{S}_{11}$, $\widetilde{S}_{12}$, and C_{1j}

Now let us specify the right-hand sides F_j of the problems (7.2.7) and find $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$, and $C_{1j}(\varepsilon)$. Substituting $\chi_{1,\varepsilon} v_1$ into (7.1.1), we get the discrepancy

$$-(\Delta + k^2)\chi_{1,\varepsilon} v_1 = -[\Delta, \chi_{\varepsilon,1}]v_1 - \chi_{\varepsilon,1}(\Delta + k^2)v_1 = -[\Delta, 1 - \Theta(\varepsilon^{-1} r_1)]v_1,$$

which is non-zero in the neighborhood of the point O_1, where v_1 can be replaced by asymptotics; the boundary condition (7.1.2) is fulfilled. According to (7.4.2) and (7.3.1), (7.3.5)

$$v_1(x, y; \varepsilon) = \big(a_1^-(\varepsilon)\widetilde{N}_{\pi/\omega}(kr_1) + a_1^+(\varepsilon)\widetilde{J}_{\pi/\omega}(kr_1)\big)\Phi(\pi - \varphi_1) + O(r_1^{3\pi/\omega}), \quad r_1 \to 0,$$

where

$$a_1^-(\varepsilon) = C_{11}, \qquad a_1^+ = s_1 + C_{11}a_1. \tag{7.4.5}$$

We select the leading term in each summand, take $\rho_1 = r_1/\varepsilon$, and obtain

$$\begin{aligned}-(\Delta + k^2)\chi_{\varepsilon,1} v_1 &\sim -[\Delta, 1 - \Theta(\varepsilon^{-1} r_1)]\left(a_1^- r_1^{-\pi/\omega} + a_1^+ r_1^{\pi/\omega}\right)\Phi(\pi - \varphi_1)\\ &= -\varepsilon^{-2}[\Delta_{(\rho_1,\varphi_1)}, 1 - \Theta(\rho_1)]\left(a_1^- \varepsilon^{-\pi/\omega}\rho_1^{-\pi/\omega} + a_1^+ \varepsilon^{\pi/\omega}\rho_1^{\pi/\omega}\right)\Phi(\pi - \varphi_1).\end{aligned} \tag{7.4.6}$$

In the same way, taking account of (7.4.3), (7.3.10) and (7.3.11), we write the leading discrepancy of $\chi_{\varepsilon,2} v_2$ supported in a neighborhood of O_1:

$$(\Delta + k^2)\chi_{\varepsilon,1} v_1 \sim \varepsilon^{-2}[\Delta_{(\rho_1,\varphi_1)}, 1 - \Theta(\rho_1)]\left(b_1^- \varepsilon^{-\pi/\omega}\rho_1^{-\pi/\omega} + b_1^+ \varepsilon^{\pi/\omega}\rho_1^{\pi/\omega}\right)\Phi(\varphi_1), \tag{7.4.7}$$

where

$$b_1^- = C_{12}(\varepsilon), \qquad b_1^+ = C_{12}(\varepsilon)c_{11} + C_{13}(\varepsilon)c_{21}. \tag{7.4.8}$$

As a right-hand side F_1 of problem (7.2.7) in Ω_1, we take the function

$$\begin{aligned} F_1(\xi_1, \eta_1) = &- [\Delta, \zeta^-] \left(a_1^- \varepsilon^{-\pi/\omega} \rho_1^{-\pi/\omega} + a_1^+ \varepsilon^{\pi/\omega} \rho_1^{\pi/\omega} \right) \Phi(\pi - \varphi_1) \\ &- [\Delta, \zeta^+] \left(b_1^- \varepsilon^{-\pi/\omega} \rho_1^{-\pi/\omega} + b_1^+ \varepsilon^{\pi/\omega} \rho_1^{\pi/\omega} \right) \Phi(\varphi_1), \end{aligned} \tag{7.4.9}$$

where ζ^+ (respectively ζ^-) denotes the function $1 - \Theta$, first restricted to the domain $\xi_1 > 0$ (respectively $\xi_1 < 0$) and then extended by zero to the whole domain Ω_1. Let w_1 be the corresponding solution; then the term $\Theta(r_1) w_1(\varepsilon^{-1} x_1, \varepsilon^{-1} y_1; \varepsilon)$ in (7.4.1), being substituted in (7.1.1), compensates discrepancies (7.4.6) and (7.4.7).

Now, we use (7.4.3) and (7.4.4), (7.3.10) and (7.3.11), and (7.3.6) to find the right-hand side of problem (7.2.7) for $j = 2$:

$$\begin{aligned} F_2(\xi_2, \eta_2) = &- [\Delta, \zeta^-] \left(a_2^- \varepsilon^{-\pi/\omega} \rho_2^{-\pi/\omega} + a_2^+ \varepsilon^{\pi/\omega} \rho_2^{\pi/\omega} \right) \Phi(\pi - \varphi_2) \\ &- [\Delta, \zeta^+] \left(b_2^- \varepsilon^{-\pi/\omega} \rho_2^{-\pi/\omega} + b_2^+ \varepsilon^{\pi/\omega} \rho_2^{\pi/\omega} \right) \Phi(\varphi_2), \end{aligned}$$

where

$$\begin{aligned} a_2^-(\varepsilon) &= C_{13}(\varepsilon), \qquad a_2^+(\varepsilon) = C_{12}(\varepsilon)c_{12} + C_{13}(\varepsilon)c_{22}, \\ b_2^-(\varepsilon) &= C_{14}(\varepsilon), \qquad b_2^+(\varepsilon) = C_{14}(\varepsilon)a_2. \end{aligned} \tag{7.4.10}$$

Lemma 7.4.1 *Let the solution w_j to problem* (7.2.7) *with right-hand side*

$$\begin{aligned} F_j(\xi_j, \eta_j) = &- [\Delta, \zeta^-] \left(a_j^- \varepsilon^{-\pi/\omega} \rho_j^{-\pi/\omega} + a_j^+ \varepsilon^{\pi/\omega} \rho_j^{\pi/\omega} \right) \Phi(\pi - \varphi_j) \\ &- [\Delta, \zeta^+] \left(b_j^- \varepsilon^{-\pi/\omega} \rho_j^{-\pi/\omega} + b_j^+ \varepsilon^{\pi/\omega} \rho_j^{\pi/\omega} \right) \Phi(\varphi_j), \end{aligned}$$

$j = 1, 2$, *be majorized by* $O(\rho_j^{-3\pi/\omega})$ *as* $\rho_j \to \infty$. *Then the relations*

$$a_j^- \varepsilon^{-\pi/\omega} - \alpha a_j^+ \varepsilon^{\pi/\omega} - \beta b_j^+ \varepsilon^{\pi/\omega} = 0, \qquad b_j^- \varepsilon^{-\pi/\omega} - \alpha b_j^+ \varepsilon^{\pi/\omega} - \beta a_j^+ \varepsilon^{\pi/\omega} = 0, \tag{7.4.11}$$

hold with α and β in (7.2.11) *and* (7.2.12).

Remark 7.4.2 The solutions w_j mentioned in Lemma 7.4.1 can be represented as linear combinations of functions independent of ε. Let w_j^l and w_j^r be the solutions of problem (7.2.7) specified by conditions (7.2.11) and (7.2.12), and let ζ^+ and ζ^- be the same cut-off functions as in (7.4.9). Put

$$\begin{aligned} \mathbf{w}_j^l &= w_j^l - \zeta^- \left(\rho_j^{\pi/\omega} + \alpha \rho_j^{-\pi/\omega} \right) \Phi(\pi - \varphi_j) - \zeta^+ \beta \rho_j^{-\pi/\omega} \Phi(\varphi_j), \\ \mathbf{w}_j^r &= w_j^r - \zeta^- \beta \rho_j^{-\pi/\omega} \Phi(\pi - \varphi_j) - \zeta^+ \left(\rho_j^{\pi/\omega} + \alpha \rho_j^{-\pi/\omega} \right) \Phi(\varphi_j). \end{aligned}$$

A straightforward verification shows that

$$w_j = a_j^+ \varepsilon^{\pi/\omega} \mathbf{w}_j^l + b_j^+ \varepsilon^{\pi/\omega} \mathbf{w}_j^r. \tag{7.4.12}$$

It is convenient to write (7.4.11) in the form

$$(a_j^-, b_j^-) = (a_j^+, b_j^+)\, \Lambda\, \varepsilon^{2\pi/\omega}, \qquad \Lambda = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \tag{7.4.13}$$

We use (7.4.5) and (7.4.8) to transform (7.4.13) with $j = 1$ to the equality

$$(C_{11}, C_{12}) = (s_1 + C_{11} a_1, C_{12} c_{11} + C_{13} c_{21})\, \Lambda\, \varepsilon^{2\pi/\omega}. \tag{7.4.14}$$

For $j = 2$, taking (7.4.10) into account, we reduce (7.4.13) to

$$(C_{13}, C_{14}) = (C_{12} c_{12} + C_{13} c_{22}, C_{14}\, a_2)\, \Lambda\, \varepsilon^{2\pi/\omega}. \tag{7.4.15}$$

Setting $\boldsymbol{\Lambda} = \operatorname{diag}\{\Lambda, \Lambda\}$,

$$a = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \tag{7.4.16}$$

and combining the above relations for C_{1j}, we obtain

$$(C_{11}, C_{12}, C_{13}, C_{14}) = (s_1, 0, 0, 0)\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega} + (C_{11}, C_{12}, C_{13}, C_{14})\, a\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega},$$

hence

$$(C_{11}, C_{12}, C_{13}, C_{14})(I - a\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega}) = (s_1, 0, 0, 0)\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega}. \tag{7.4.17}$$

Let us calculate the inverse matrix for $I - a\boldsymbol{\Lambda}\varepsilon^{2\pi/\omega}$, assuming ε to be sufficiently small. From (7.3.12) it follows that

$$a(k) = -\frac{\mathbf{b}^* \mathbf{b}}{k^2 - k_e^2} + \widehat{a}(k),$$

where $\mathbf{b} = (0,\, b_1,\, b_2,\, 0)$ and the matrix $\widehat{a}$ is analytic near $k = k_e$ and defined by (7.4.16), whereas c_{pq} is replaced for $\widehat{c}_{pq}$. We have

$$\begin{aligned} I - a\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega} &= I - \widehat{a}\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega} + \frac{\mathbf{b}^* \mathbf{b}\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega}}{k^2 - k_e^2} \\ &= \left(I + \frac{\mathbf{b}^* \mathbf{b}\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega} (I - \widehat{a}\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega})^{-1}}{k^2 - k_e^2} \right) (I - \widehat{a}\, \boldsymbol{\Lambda}\, \varepsilon^{2\pi/\omega}); \end{aligned}$$

it is evident that $(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}$ exists for small ε. Straightforward calculation shows that

$$\left(I + \frac{\mathbf{b}^*\mathbf{c}}{k^2 - k_e^2}\right)^{-1} = I - \frac{\mathbf{b}^*\mathbf{c}}{k^2 - k_e^2 - \langle \mathbf{c}, \mathbf{b}\rangle}$$

for $\mathbf{c} = \mathbf{b}\,\Lambda\,\varepsilon^{2\pi/\omega}(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^4$. Therefore,

$$(I - a\Lambda\,\varepsilon^{2\pi/\omega})^{-1} = (I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}\left(I - \frac{\mathbf{b}^*\mathbf{b}\,\Lambda\,\varepsilon^{2\pi/\omega}(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}}{k^2 - k_e^2 + \langle \mathbf{b}\,\Lambda\,\varepsilon^{2\pi/\omega}(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}, \mathbf{b}\rangle}\right).$$

This leads to

$$\begin{aligned}(C_{11}, C_{12}, C_{13}, C_{14}) &= (s_1, 0, 0, 0)\,\Lambda\,\varepsilon^{2\pi/\omega}(I - a\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}\\ &= (s_1, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right),\end{aligned} \tag{7.4.18}$$

where $\mathbf{b} = (0, b_1, b_2, 0)$, $D = \Lambda\,\varepsilon^{2\pi/\omega}(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}$, and the matrix $\widehat{a}$ is analytic in k near k_e and defined by (7.4.16) with c_{pq} replaced by $\widehat{c}_{pq}$ (see (7.3.12)).

We now seek an approximation to the entries of the first row (S_{11}, S_{12}) of the scattering matrix. By virtue of (7.4.2) and (7.4.4),

$$(\widetilde{S}_{11}, \widetilde{S}_{12}) = (S_{11}^0 + C_{11}A_1, C_{14}A_2). \tag{7.4.19}$$

We set

$$A = \begin{pmatrix} A_1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & A_2\end{pmatrix}, \qquad s = \begin{pmatrix} s_1 & 0 & 0 & 0\\ 0 & 0 & 0 & s_2\end{pmatrix};$$

$S^0 = \operatorname{diag}(S_{11}^0, S_{22}^0)$; then, by Lemma 7.3.1, $A = is^*S^0$. In view of (7.4.19) and (7.4.18), we obtain

$$\begin{aligned}(\widetilde{S}_{11}, \widetilde{S}_{12}) &= (S_{11}^0, 0) + (C_{11}, C_{12}, C_{13}, C_{14})A\\ &= (S_{11}^0, 0) + i(s_1, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0.\end{aligned} \tag{7.4.20}$$

An approximation to the second row of the scattering matrix is of the form

$$(\widetilde{S}_{21}, \widetilde{S}_{22}) = (0, S_{22}^0) + i(0, 0, 0, s_2)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0. \tag{7.4.21}$$

Lemma 7.4.3 *The matrix $\widetilde{S}(\varepsilon)$ is unitary.*

7.4.3 Formulas for Resonant Tunneling Characteristics

The solutions of the first kind limit problems involved in (7.4.1) are defined for complex k as well. Expressions (7.4.20) and (7.4.21) for $\widetilde{S}$ have a pole k_p in the lower complex half-plane. To find k_p^2 we equate $k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle$ to zero and solve the equation for $k^2 - k_e^2$:

$$k^2 - k_e^2 = -\langle \mathbf{b}D, \mathbf{b}\rangle = -\varepsilon^{2\pi/\omega}\langle \mathbf{b}\Lambda\,(I - \widehat{a}\,\Lambda\,\varepsilon^{2\pi/\omega})^{-1}, \mathbf{b}\rangle. \tag{7.4.22}$$

Since the right-hand side of the last equation behaves like $O(\varepsilon^{2\pi/\omega})$ as $\varepsilon \to 0$, it may be solved by the successive approximation method. Considering the formulas $\operatorname{Im} a_j = |s_j|^2/2$, which follow from the waveguide symmetry and Lemma 7.3.1, and discarding the lower order terms, we get $k_p^2 = k_r^2 - ik_i^2$, where

$$\begin{aligned} k_r^2 &= k_e^2 - \alpha(|b_1|^2 + |b_2|^2)\varepsilon^{2\pi/\omega} + O(\varepsilon^{4\pi/\omega}), \\ k_i^2 &= \frac{1}{2}\beta^2(|b_1|^2 + |b_2|^2)|s_1(k_e^2)|^2\varepsilon^{4\pi/\omega} + O(\varepsilon^{6\pi/\omega}). \end{aligned} \tag{7.4.23}$$

From (7.4.20) and (7.4.21), we obtain

$$\begin{aligned} \widetilde{S}(k,\varepsilon) &= S^0(k) + is(k)\Lambda\, s^*(k)S^0(k)\varepsilon^{2\pi/\omega} - i\frac{s(k)\Lambda\,\mathbf{b}^*\mathbf{b}\,\Lambda\, s^*(k)S^0(k)}{k^2 - k_p^2}\varepsilon^{4\pi/\omega} + O\left(\frac{\varepsilon^{6\pi/\omega}}{k^2 - k_p^2}\right) \\ &= \begin{pmatrix} S_{11}^0(k) & 0 \\ 0 & S_{22}^0(k) \end{pmatrix} + i\begin{pmatrix} |s_1(k)|^2\alpha_1 S_{11}^0(k) & 0 \\ 0 & |s_2(k)|^2\alpha_2 S_{22}^0(k) \end{pmatrix}\varepsilon^{2\pi/\omega} \\ &\quad - \frac{i}{k^2 - k_p^2}\begin{pmatrix} |s_1(k)|^2|b_1|^2\beta^2 S_{11}^0(k) & s_1(k)\overline{s_2(k)}\,\overline{b_1}b_2\beta^2 S_{22}^0(k) \\ s_2(k)\overline{s_1(k)}b_1\overline{b_2}\beta^2 S_{11}^0(k) & |s_2(k)|^2|b_2|^2\beta^2 S_{22}^0(k) \end{pmatrix}\varepsilon^{4\pi/\omega} \\ &\quad + O\left(\frac{\varepsilon^{6\pi/\omega}}{k^2 - k_p^2}\right). \end{aligned}$$

Let $k^2 - k_e^2 = O(\varepsilon^{2\pi/\omega})$, then $c\varepsilon^{4\pi/\omega} \le |k^2 - k_p^2| \le c\varepsilon^{2\pi/\omega}$, $s_j(k) = s_j(k_e) + O(\varepsilon^{2\pi/\omega})$, $S_{jj}^0(k) = S_{jj}^0(k_e) + O(\varepsilon^{2\pi/\omega})$, and

$$\begin{aligned} \widetilde{S}_{12}(k,\varepsilon) &= -i\varepsilon^{4\pi/\omega}\frac{s_1(k)\overline{s_2(k)}\,\overline{b_1}b_2\beta^2 S_{22}^0(k)}{k^2 - k_p^2}\left(1 + O(\varepsilon^{2\pi/\omega})\right) \\ &= -\frac{\dfrac{s_1(k_e)}{|s_1(k_e)|}\dfrac{\overline{s_2(k_e)}}{|s_2(k_e)|}\dfrac{\overline{b_1}}{|b_1|}\dfrac{b_2}{|b_2|}S_{22}^0(k_e)}{\dfrac{1}{2}\left(\dfrac{|b_1|}{|b_2|} + \dfrac{|b_2|}{|b_1|}\right) - iP\dfrac{k^2 - k_r^2}{\varepsilon^{4\pi/\omega}}}\left(1 + O(\varepsilon^{2\pi/\omega})\right), \end{aligned} \tag{7.4.24}$$

$$\widetilde{S}_{21}(k,\varepsilon) = -i\varepsilon^{4\pi/\omega}\frac{\overline{s_1(k)}s_2(k)b_1\overline{b_2}\beta^2 S^0_{11}(k)}{k^2-k_p^2}\Big(1+O(\varepsilon^{2\pi/\omega})\Big)$$
$$= -\frac{\dfrac{\overline{s_1(k_e)}}{|s_1(k_e)|}\dfrac{s_2(k_e)}{|s_2(k_e)|}\dfrac{b_2}{|b_2|}\dfrac{\overline{b_2}}{|b_2|}S^0_{11}(k_e)}{\dfrac{1}{2}\left(\dfrac{|b_1|}{|b_2|}+\dfrac{|b_2|}{|b_1|}\right)-iP\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}}\Big(1+O(\varepsilon^{2\pi/\omega})\Big),$$

where $P = \big(|b_1||b_2|\beta^2|s_1(k_e)|^2\big)^{-1}$. Thus,

$$\widetilde{T}_1(k,\varepsilon) = \widetilde{T}_2(k,\varepsilon) = |\widetilde{S}_{12}|^2 = \frac{1}{\dfrac{1}{4}\left(\dfrac{|b_1|}{|b_2|}+\dfrac{|b_2|}{|b_1|}\right)^2+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\pi/\omega}}\right)^2}(1+O(\varepsilon^{2\pi/\omega})). \tag{7.4.25}$$

The obtained approximation $\widetilde{T}_j$ to the transition coefficient T_j has a peak at $k^2 = k_r^2$ whose width at its half-height is

$$\widetilde{\Upsilon}(\varepsilon) = \left(\frac{|b_1|}{|b_2|}+\frac{|b_2|}{|b_1|}\right)\frac{1}{P}\varepsilon^{4\pi/\omega}. \tag{7.4.26}$$

7.5 Justification of the Asymptotics

As in the previous section, here we omit "±" the notations and do not specify which equation of (7.1.1) is considered. We return to the full notations in Theorem 7.5.3.

We now introduce functional spaces for the problem

$$(-i\nabla+\mathbf{A})^2u \pm Hu - k^2u = f \quad \text{in } G(\varepsilon), \qquad u=0 \quad \text{on } \partial G(\varepsilon). \tag{7.5.1}$$

Recall that the functions $\mathbf{A}$ and H are compactly supported, and, besides, they are nonzero only in the resonator at some distance from the narrows. Let Θ be the same function as in (7.3.4) and let the cut-off functions η_j, $j = 0, 1, 2$, be nonzero in G_j and satisfy the relation $\eta_1(x,y)+\Theta(r_1)+\eta_0(x,y)+\Theta(r_2)+\eta_2(x,y) = 1$ in $G(\varepsilon)$. For $\gamma\in\mathbb{R}$, $\delta>0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G(\varepsilon))$ is the completion in the norm

$$\|u; V^l_{\gamma,\delta}(G(\varepsilon))\|$$
$$= \left(\int_{G(\varepsilon)}\sum_{|\alpha|=0}^{l}\left(\sum_{j=1}^{2}\Theta^2(r_j)\,(r_j^2+\varepsilon_j^2)^{\gamma-l+|\alpha|}+\eta_1^2e^{2\delta|x|}+\eta_0+\eta_2^2e^{2\delta|x|}\right)|\partial^\alpha v|^2\,dx\,dy\right)^{1/2} \tag{7.5.2}$$

of the set of smooth functions compactly supported on $\overline{G(\varepsilon)}$. Denote by $V^{0,\perp}_{\gamma,\delta}$ the space of function f, analytic in k^2, with values in $V^0_{\gamma,\delta}(G(\varepsilon))$ that satisfy, at $k^2 = k_e^2$, the condition $(\chi_{0,\varepsilon^\sigma} f, v_e)_{G_0} = 0$ with a small $\sigma > 0$.

Proposition 7.5.1 *Let k_r^2 be a resonance, $k_r^2 \to k_e^2$ as $\varepsilon \to 0$, and let $|k^2 - k_r^2| = O(\varepsilon^{2\pi/\omega})$. Let γ satisfy the condition $\pi/\omega - 2 < \gamma - 1 < \pi/\omega$, $f \in V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$, and let u be a solution to problem (7.5.1) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-;$$

here $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma,\delta}(G(\varepsilon))$ for small $\delta > 0$. Then

$$\|\widetilde{u}; V^2_{\gamma,\delta}(G(\varepsilon))\| + |A_1^-| + |A_2^-| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \tag{7.5.3}$$

where c is a constant independent of f and ε.

Proof Step A. First we construct an auxiliary function u_p. As mentioned above, $\widetilde{S}$ has a pole $k_p^2 = k_r^2 - ik_i^2$ (see (7.4.23)). Let us multiply the solutions to the limit problems, involved in (7.4.1), by $g := -(k^2 - k_e^2 + \langle \mathbf{b}D(k), \mathbf{b}\rangle)/\langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle$, put $k = k_p$, and denote the resulting functions by adding the subscript p. In view of (7.4.18) and the equality $(s_1, 0, 0, 0)D\mathbf{b}^* = \langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle$, we get

$$g\,(C_{11}, C_{12}, C_{13}, C_{14})|_{k=k_p} = \mathbf{b}D(k_p) = (b_1\beta, b_1\alpha, b_2\alpha, b_2\beta)\varepsilon^{2\pi/\omega} + O(\varepsilon^{4\pi/\omega}). \tag{7.5.4}$$

This and (7.4.2), (7.4.4) lead to

$$v_{1p}(x, y; \varepsilon) = g\,C_{11}|_{k=k_p}\mathbf{v}_1(x, y; k_p) = \varepsilon^{2\pi/\omega}\left(b_1\beta + O\left(\varepsilon^{2\pi/\omega}\right)\right)\mathbf{v}_1(x, y; k_p), \tag{7.5.5}$$

$$v_{2p}(x, y; \varepsilon) = g\,C_{14}|_{k=k_p}\mathbf{v}_2(x, y; k_p) = \varepsilon^{2\pi/\omega}\left(b_2\beta + O\left(\varepsilon^{2\pi/\omega}\right)\right)\mathbf{v}_2(x, y; k_p);$$

the dependence of k_p on ε is not shown. According to (7.4.3) and Lemma 7.3.2,

$$\begin{aligned} v_{0p}(x, y; \varepsilon) = &-\frac{(g\,C_{12}\overline{b_1} + g\,C_{13}\overline{b_2})|_{k=k_p}}{k_p^2 - k_e^2} v_e(x, y) \\ &+ g\,C_{12}|_{k=k_p}\widehat{\mathbf{v}}_{01}(x, y) + g\,C_{13}|_{k=k_p}\widehat{\mathbf{v}}_{02}(x, y). \end{aligned}$$

Taking into account (7.4.18), we obtain

$$\begin{aligned} &C_{12}b_1 + C_{13}b_2 = (C_{11}, C_{12}, C_{13}, C_{14})\mathbf{b}^* = \\ &(s_1, 0, 0, 0)D\mathbf{b}^*\left(1 - \frac{\langle \mathbf{b}D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right) = (k^2 - k_e^2)\frac{\langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}. \end{aligned} \tag{7.5.6}$$

Hence,

$$v_{0p}(x,y;\varepsilon) = v_e(x,y) + \varepsilon^{2\pi/\omega}(b_1\alpha + O(\varepsilon^{2\pi/\omega}))\widehat{\mathbf{v}}_{01}(x,y) \\ + \varepsilon^{2\pi/\omega}(b_2\alpha + O(\varepsilon^{2\pi/\omega}))\widehat{\mathbf{v}}_{02}(x,y).$$

Finally, using (7.4.12) and formulas (7.4.5), (7.4.8), (7.4.10) for a_j^+ and b_j^+, we find

$$\begin{aligned} w_{1p}(\xi_1,\eta_1;\varepsilon) &= (gC_{11})|_{k=k_p} a_1 \varepsilon^{\pi/\omega}\mathbf{w}_1^l(\xi_1,\eta_1) \\ &\quad + (gC_{12}c_{11} + gC_{13}c_{21})|_{k=k_p}\varepsilon^{\pi/\omega}\mathbf{w}_1^r(\xi_1,\eta_1), \\ w_{2p}(\xi_2,\eta_2;\varepsilon) &= (gC_{22}c_{11} + gC_{23}c_{21})|_{k=k_p}\varepsilon^{\pi/\omega}\mathbf{w}_2^l(\xi_2,\eta_2) \\ &\quad + (gC_{14})|_{k=k_p} a_2\varepsilon^{\pi/\omega}\mathbf{w}_2^r(\xi_2,\eta_2). \end{aligned}$$

Compare the equalities (7.3.12), (7.5.4) and (7.5.6), then

$$(gC_{12}c_{1j} + gC_{j3}c_{2j})|_{k=k_p} = -b_j \frac{(g\,C_{12}\overline{b_1} + g\,C_{13}\overline{b_2})|_{k=k_p}}{k_p^2 - k_e^2} \\ + (gC_{12}\widehat{c}_{1j} + gC_{j3}\widehat{c}_{2j})|_{k=k_p} = b_j + O(\varepsilon^{2\pi/\omega}),$$

where $j = 1, 2$. Thus

$$\begin{aligned} w_{1p}(\xi_1,\eta_1;\varepsilon) &= \varepsilon^{3\pi/\omega}(a_1b_1\beta + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_1^l(\xi_1,\eta_1) && (7.5.7) \\ &\quad + \varepsilon^{\pi/\omega}(b_1 + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_1^r(\xi_1,\eta_1), \\ w_{2p}(\xi_2,\eta_2;\varepsilon) &= \varepsilon^{\pi/\omega}(b_2 + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_2^l(\xi_2,\eta_2) && (7.5.8) \\ &\quad + \varepsilon^{3\pi/\omega}(a_2b_2\beta + O(\varepsilon^{2\pi/\omega}))\mathbf{w}_2^r(\xi_2,\eta_2). \end{aligned}$$

We set

$$\begin{aligned} u_p(x,y;\varepsilon) = \Xi(x,y)\big[&\chi_{1,\varepsilon}(x,y)v_{1p}(x,y;\varepsilon) + \Theta(\varepsilon^{-2\sigma}r_1)w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon) \\ &+ \chi_{0,\varepsilon}(x,y)v_{0p}(x,y;\varepsilon) + \Theta(\varepsilon^{-2\sigma}r_2)w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;k,\varepsilon) \\ &+ \chi_{2,\varepsilon}(x,y)v_{2p}(x,y;k,\varepsilon)\big], \end{aligned} \tag{7.5.9}$$

where Ξ is a cut-off function in $G(\varepsilon)$ that is equal to 1 on the set $G(\varepsilon) \cap \{|x| < R\}$ and to 0 on $G(\varepsilon) \cap \{|x| > R+1\}$ for a large $R > 0$; σ is such that $2\sigma < 1$. The principal part of the norm of u_p is given by $\chi_{0,\varepsilon}v_{0p}$. Considering the definitions of v_{0p} and $\widehat{\mathbf{v}}_{0j}$ (see Sect. 7.2) and Lemma 7.3.2, we obtain $\|\chi_{0,\varepsilon}v_{0p}\| = \|v_e\| + o(1)$.

Step B. Let us show that

$$\|((-i\nabla + \mathbf{A})^2 \pm H - k_p^2)u_p; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{\pi/\omega+\kappa}, \tag{7.5.10}$$

where $\kappa = \min\{\pi/\omega, 3\pi/\omega - \sigma_1, \gamma+1\}$, $\sigma_1 = 2\sigma(3\pi/\omega - \gamma + 1)$. If $\pi/\omega < \gamma+1$ and σ is small so that $2\pi/\omega > \sigma_1$, we have $\kappa = \pi/\omega$.

In view of (7.5.9),

$$
\begin{aligned}
&((-i\nabla+\mathbf{A})^2 \pm H - k_p^2)u_p(x,y;\varepsilon)\\
&= -[\Delta,\chi_{1,\varepsilon}]\Big(v_{1p}(x,y;\varepsilon) - b_1\beta\varepsilon^{2\pi/\omega}(r_1^{-\pi/\omega}+a(k_p)r_1^{\pi/\omega})\Phi(\pi-\varphi_1)\Big)\\
&\quad - [\Delta,\Theta]w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon) - k_p^2\Theta(\varepsilon^{-2\sigma}r_1)w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon)\\
&\quad - [\Delta,\chi_{0,\varepsilon}]\Big(v_{0p}(x,y;\varepsilon) - \Theta(r_1)\big(b_{1p}^-(\varepsilon)r_1^{-\pi/\omega}+b_{1p}^+(\varepsilon)r_1^{\pi/\omega}\big)\Phi(\pi-\varphi_1)\\
&\qquad\qquad - \Theta(r_2)\big(a_{2p}^-(\varepsilon)r_2^{-\pi/\omega}+a_{2p}^+(\varepsilon)r_2^{\pi/\omega}\big)\Phi(\varphi_2)\Big)\\
&\quad - [\Delta,\Theta]w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;\varepsilon) - k_p^2\Theta(\varepsilon^{-2\sigma}r_2)w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;\varepsilon)\\
&\quad - [\Delta,\chi_{2,\varepsilon}]\Big(v_{2p}(x,y;\varepsilon) - b_2\beta\varepsilon^{2\pi/\omega}(r_2^{-\pi/\omega}+a(k_p)r_2^{\pi/\omega})\Phi(\varphi_2)\Big)\\
&\qquad\qquad - [\Delta,\Xi]v_{1p}(x,y;\varepsilon) - [\Delta,\Xi]v_{2p}(x,y;\varepsilon),
\end{aligned}
$$

where $b_{1p}^- = O(\varepsilon^{2\pi/\omega})$, $b_{1p}^+ = b_1 + O(\varepsilon^{2\pi/\omega})$, $a_{2p}^- = O(\varepsilon^{2\pi/\omega})$, and $a_{2p}^+ = b_2 + O(\varepsilon^{2\pi/\omega})$. Taking into account the asymptotics of $\mathbf{v}_1$ as $r_1 \to 0$ and passing to the variables $(\xi_1,\eta_1) = (\varepsilon^{-1}x_1, \varepsilon^{-1}y_1)$, we obtain

$$
\begin{aligned}
&\Big\|(x,y)\mapsto[\Delta,\chi_{1,\varepsilon}]\Big(\mathbf{v}_1(x,y) - (r_1^{-\pi/\omega}+a(k_p)r_1^{\pi/\omega})\Phi(\pi-\varphi_1)\Big);V^0_{\gamma,\delta}(G(\varepsilon))\Big\|^2\\
&\qquad\le c\int_{G(\varepsilon)}(r_1^2+\varepsilon^2)^\gamma\left|[\Delta,\chi_{1,\varepsilon}]r_1^{-\pi/\omega+2}\Phi(\pi-\varphi_1)\right|^2dxdy \le c\varepsilon^{2(\gamma-\pi/\omega+1)}.
\end{aligned}
$$

This and (7.5.5) imply the estimate

$$
\begin{aligned}
&\Big\|(x,y)\mapsto[\Delta,\chi_{1,\varepsilon}]\Big(v_{1p}(x,y) - (r_1^{-\pi/\omega}+a(k_p)r_1^{\pi/\omega})\Phi(\pi-\varphi_1)\Big);V^0_{\gamma,\delta}(G(\varepsilon))\Big\|\\
&\le c\varepsilon^{\gamma+\pi/\omega+1}.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\Big\|(x,y)\mapsto[\Delta,\chi_{0,\varepsilon}]\Big(v_{0p}(x,y) - \Theta(r_1)\big(b_{1p}^-(\varepsilon)r_1^{-\pi/\omega}+b_{1p}^+(\varepsilon)r_1^{\pi/\omega}\big)\Phi(\pi-\varphi_1)\\
&\qquad\qquad - \Theta(r_2)\big(a_{2p}^-(\varepsilon)r_2^{-\pi/\omega}+a_{2p}^+(\varepsilon)r_2^{\pi/\omega}\big)\Phi(\varphi_2)\Big)\Big\| \le c\varepsilon^{\gamma+\pi/\omega+1},\\
&\Big\|(x,y)\mapsto[\Delta,\chi_{2,\varepsilon}]\Big(v_{2p}(x,y) - (r_2^{-\pi/\omega}+a(k_p)r_2^{\pi/\omega})\Phi(\varphi_2)\Big);V^0_{\gamma,\delta}(G(\varepsilon))\Big\|\\
&\le c\varepsilon^{\gamma+\pi/\omega+1}.
\end{aligned}
$$

It is evident, that

$$
\left\|[\Delta,\Xi]v_{jp};V^0_{\gamma,\delta}(G(\varepsilon))\right\| \le c\varepsilon^{2\pi/\omega},\quad j=1,2.
$$

Further, since $\mathbf{w}_j^l$ behaves like $O(\rho_j^{-3\pi/\omega})$ at infinity,

$$\int_{G(\varepsilon)} (r_j^2+\varepsilon^2)^\gamma \left|[\Delta,\Theta]\mathbf{w}_j^l(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j)\right|^2 dx_j dy_j$$
$$\leq c\int_{K_j} (r_j^2+\varepsilon^2)^\gamma \left|[\Delta,\Theta](\varepsilon^{-1}r_j)^{-3\pi/\omega}\Phi_2(\varphi_j)\right|^2 dx_j dy_j \leq c\varepsilon^{2(3\pi/\omega-\sigma_1)},$$

where $\sigma_1 = 2\sigma(3\pi/\omega-\gamma+1)$. A similar inequality holds with $\mathbf{w}_j^l$ replaced by $\mathbf{w}_j^r$. Considering (7.5.7) and (7.5.8), we obtain

$$\left\|[\Delta,\Theta]w_{jp};\, V^0_{\gamma,\delta}(G(\varepsilon))\right\| \leq c\varepsilon^{4\pi/\omega-\sigma_1}.$$

Finally, using (7.5.7) and (7.5.8) once again, taking into account the estimate

$$\int_{G(\varepsilon)} (r_j^2+\varepsilon^2)^\gamma \left|\Theta(\varepsilon^{-2\sigma}r_j)\mathbf{w}_j^l(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j)\right|^2 dx_j dy_j$$
$$=\varepsilon^{2\gamma+2}\int_{\Omega} (\rho_j^2+1)^\gamma \left|\Theta(\varepsilon^{1-2\sigma}\rho_j)\mathbf{w}_j^l(\xi_j,\eta_j)\right|^2 d\xi_j d\eta_j \leq c\varepsilon^{2\gamma+2},$$

and a similar estimate for $\mathbf{w}_j^r$, we derive

$$\left\|(x,y)\mapsto \Theta(\varepsilon^{-2\sigma}r_j)w_{jp}(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j);\, V^0_{\gamma,\delta}(G(\varepsilon))\right\| \leq c\varepsilon^{\pi/\omega+\gamma+1}.$$

Combining the obtained estimates, we arrive at (7.5.10).

Step C. This part contains somewhat modified arguments from the proof of Theorem 5.1.1 in [33]. Let us write the right-hand side of problem (7.5.1) in the form

$$f(x,y) = f_1(x,y;\varepsilon)+f_0(x,y;\varepsilon)+f_2(x,y;\varepsilon)$$
$$-\varepsilon^{-\gamma-1}F_1(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1;\varepsilon_1)-\varepsilon^{-\gamma-1}F_2(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2;\varepsilon),$$

where

$$f_l(x,y;\varepsilon)=\chi_{l,\varepsilon^\sigma}(x,y)f(x,y),$$
$$F_j(\xi_j,\eta_j;\varepsilon)=-\varepsilon^{\gamma+1}\Theta(\varepsilon^{1-\sigma}\rho_j)f(x_{O_j}+\varepsilon\xi_j,y_{O_j}+\varepsilon\eta_j),$$

(x,y) are arbitrary Cartesian coordinates, (x_{O_j},y_{O_j}) stand for the coordinates of O_j in the system (x,y), and x_j, y_j were introduced in Sect. 7.4. From the definition of the norms, it follows that

$$\|f_1;\, V^0_{\gamma,\delta}(G_1)\|+\|f_0;\, V^0_\gamma(G_0)\|+\|f_2;\, V^0_{\gamma,\delta}(G_2)\|+\|F_j;\, V^0_\gamma(\Omega_j)\| \leq c\|f;\, V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{7.5.11}$$

We consider solutions v_l and w_j to the limit problems

$$\begin{aligned}(-i\nabla+\mathbf{A})^2 v_0 \pm H v_0 - k^2 v_0 &= f_0 \text{ in } G_0, \qquad v_0 = 0 \text{ on } \partial G_0,\\ -\Delta v_l - k^2 v_l &= f_l \text{ in } G_l, \qquad v_l = 0 \text{ on } \partial G_l,\\ \Delta w_j &= F_j \text{ in } \Omega_j, \qquad w_j = 0 \text{ on } \partial\Omega_j,\end{aligned}$$

respectively; moreover, the v_l with $l = 1, 2$ satisfy the intrinsic radiation conditions at infinity, and the v_0 is subject to the condition $(v_0, v_e)_{G_0} = 0$. According to Propositions 7.2.1, 7.2.2 and 7.2.3, the problems in G_l and Ω_j are uniquely solvable and

$$\begin{aligned}\|v_0; V^2_\gamma(G_0)\| &\le c_0\|f_0; V^0_\gamma(G_0)\|,\\ \|v_l; V^2_{\gamma,\delta,-}(G_l)\| &\le c_l\|f_l; V^0_{\gamma,\delta}(G_l)\|, \ l = 1, 2,\\ \|w_j; V^2_\gamma(\Omega_j)\| &\le C_j\|F_j; V^0_\gamma(\Omega_j)\|, \ j = 1, 2,\end{aligned} \tag{7.5.12}$$

where c_l and C_j are independent of ε. We set

$$\begin{aligned}U(x, y; \varepsilon) = {}& \chi_{1,\varepsilon}(x, y)v_1(x, y; \varepsilon) + \varepsilon^{-\gamma+1}\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon)\\ &+ \chi_{0,\varepsilon}(x, y)v_0(x, y; \varepsilon) + \varepsilon^{-\gamma+1}\Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon)\\ &+ \chi_{2,\varepsilon}(x, y)v_2(x, y; \varepsilon).\end{aligned}$$

Estimates (7.5.11) and (7.5.12) lead to

$$\|U; V^2_{\gamma,\,\delta,-}(G(\varepsilon))\| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\| \tag{7.5.13}$$

with c independent of ε. Let R_ε denote the mapping $f \mapsto U$.

Let us show that $((-i\nabla + \mathbf{A})^2 \pm H - k^2)R_\varepsilon = I + S_\varepsilon$, where S_ε is an operator in $V^0_{\gamma,\delta}(G(\varepsilon))$ of small norm. We have

$$\begin{aligned}&((-i\nabla+\mathbf{A})^2 \pm H - k^2)R_\varepsilon f(x, y)\\ &= f(x, y) - [\Delta, \chi_{1,\varepsilon}]v_1(x, y; \varepsilon) - \varepsilon^{-\gamma+1}[\Delta, \Theta]w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon)\\ &\quad - k^2\varepsilon^{-\gamma+1}\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1; \varepsilon) - [\Delta, \chi_{0,\varepsilon}]v_0(x, y; \varepsilon)\\ &\quad - \varepsilon^{-\gamma+1}[\Delta, \Theta]w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon)\\ &\quad - k^2\varepsilon^{-\gamma+1}\Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2; \varepsilon) - [\Delta, \chi_{2,\varepsilon}]v_2(x, y; \varepsilon).\end{aligned} \tag{7.5.14}$$

Let d be a positive number such that $\gamma - d + \pi/\omega - 1 > 0$. On the support of the function $[\Delta, \chi_{1,\varepsilon}]v_1$ the estimate $(x_1^2 + y_1^2)^{1/2} = O(\varepsilon)$ holds, therefore,

$$\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^d\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma-d,\delta}(G_1)\| \le c\varepsilon^d\|v_1; V^2_{\gamma-d,\delta}(G_1)\|.$$

This and (7.5.12) lead to

$$\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^d \|f_1; V^0_{\gamma-d,\delta}(G_1)\|.$$

Moreover, $f_1 = 0$ outside the zone $c\varepsilon^\sigma \le (x_1^2 + y_1^2)^{1/2} \le C\varepsilon^\sigma$, therefore,

$$\|f_1; V^0_{\gamma-d,\delta}(G_1)\| \le c\varepsilon^{-d\sigma} \|f_1; V^0_{\gamma,\delta}(G_1)\|.$$

The two last estimates together with (7.5.11) show that

$$\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)} \|f; V^0_{\gamma,\,\delta}(G(\varepsilon))\|. \tag{7.5.15}$$

In a similar way, we obtain

$$\|[\Delta, \chi_{l,\varepsilon}]v_l; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)} \|f; V^0_{\gamma,\,\delta}(G(\varepsilon))\|, \qquad l = 0, 2. \tag{7.5.16}$$

We now assume in addition that the d satisfies $\gamma + d - \pi/\omega - 1 < 0$. Because the support of the function $[\Delta_{(\xi_j,\eta_j)}, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j; \varepsilon)$, $j = 1, 2$, belongs to the domain $c\varepsilon^{-1} \le (\xi_j^2 + \eta_j^2)^{1/2} \le C\varepsilon^{-1}$,

$$\begin{aligned}
&\|(\xi_j, \eta_j) \mapsto [\Delta_{\xi_j,\eta_j}, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j; \varepsilon); V^0_\gamma(\Omega_j)\| \\
&\le c\varepsilon^d \|(\xi_j, \eta_j) \mapsto [\Delta_{\xi_j,\eta_j}, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j; \varepsilon); V^0_{\gamma+d}(\Omega_j)\| \le c\varepsilon^d \|w_j; V^2_{\gamma+d}(\Omega_j)\|.
\end{aligned}$$

Now, taking into account (7.5.12), we obtain

$$\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto [\Delta, \Theta(r_j)]w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^d \|F_j; V^0_{\gamma+d}(\Omega_j)\|.$$

Since $F_j = 0$ for $(\xi_j^2 + \eta_j^2)^{1/2} > c\varepsilon^{-\sigma}$,

$$\|F_j; V^0_{\gamma+d}(\Omega_j)\| \le c\varepsilon^{-d\sigma} \|F_j; V^0_\gamma(\Omega_j)\|. \tag{7.5.17}$$

Consequently,

$$\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto [\Delta, \Theta(r_j)]w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)} \|f; V^0_{\gamma,\,\delta}(G(\varepsilon))\|. \tag{7.5.18}$$

It remains to estimate the middle terms of the two last lines in (7.5.14). We have

$$\begin{aligned}
&\varepsilon^{-\gamma+1}\|(x_j, y_j) \mapsto \Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\| \\
&= \varepsilon^2\|(\xi_j, \eta_j) \mapsto \Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j; \varepsilon); V^0_\gamma(\Omega_j)\| \\
&\le \varepsilon^2\|(\xi_j, \eta_j) \mapsto \Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j; \varepsilon); V^2_{\gamma+2}(\Omega_j)\| \le c\varepsilon^d \|w_j; V^2_{\gamma+d}(\Omega_j)\|;
\end{aligned}$$

in the last inequality, we took into account that $\Theta(\varepsilon\rho_j)w_j(\xi_j,\eta_j;\varepsilon)=0$ for $\rho_j\geq c\varepsilon^{-1}$; besides, we assume that $2-d>0$. In view of (7.5.12), (7.5.17), and (7.5.11), we obtain

$$\varepsilon^{-\gamma+1}\|(x_j,y_j)\mapsto\Theta(r_j)w_j(\varepsilon^{-1}x_j,\varepsilon^{-1}y_j;\varepsilon);V^0_{\gamma,\delta}(G(\varepsilon))\|\leq c\varepsilon^{d(1-\sigma)}\|f;V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{7.5.19}$$

Thus, (7.5.14)–(7.5.16), (7.5.18) and (7.5.19) lead to the inequality

$$\|((-i\nabla+\mathbf{A})^2\pm H-k^2)R_\varepsilon f-f;V^0_{\gamma,\delta}(G(\varepsilon))\|\leq c\varepsilon^{d(1-\sigma)}\|f;V^0_{\gamma,\delta}(G(\varepsilon))\|,$$

which means that $((-i\nabla+\mathbf{A})^2\pm H-k^2)R_\varepsilon=I+S_\varepsilon$ and the norm of the operator S_ε in the space $V^0_{\gamma,\delta}(G(\varepsilon))$ admits the estimate $\|S_\varepsilon\|\leq c\varepsilon^{d(1-\sigma)}$.

Step D. Let us recall that the operator S_ε is defined on the subspace $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. We also need the range of the operator S_ε be included in $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. To this end, we replace the mapping R_ε by $\widetilde{R}_\varepsilon: f\mapsto U(f)+a(f)u_p$, the u_p was constructed in Step A, and $a(f)$ is a constant. Then $((-i\nabla+\mathbf{A})^2\pm H-k^2)\widetilde{R}_\varepsilon=I+\widetilde{S}_\varepsilon$ with $\widetilde{S}_\varepsilon=S_\varepsilon+a(\cdot)((-i\nabla+\mathbf{A})^2\pm H-k^2)u_p$. As $k=k_e$, the condition $(\chi_{0,\varepsilon^\sigma}\widetilde{S}_\varepsilon f,v_e)_{G_0}=0$ implies

$$a(f)=-(\chi_{0,\varepsilon^\sigma}S_\varepsilon f,v_e)_{G_0}/(\chi_{0,\varepsilon^\sigma}((-i\nabla+\mathbf{A})^2\pm H-k_e^2)u_p,v_e)_{G_0}.$$

Now we prove that $\|\widetilde{S}_\varepsilon\|\leq c\|S_\varepsilon\|$, where c is independent of ε and k. We have

$$\|\widetilde{S}_\varepsilon f\|\leq\|S_\varepsilon f\|+|a(f)|\,\|((-i\nabla+\mathbf{A})^2\pm H+k^2)u_p\|.$$

Estimate (7.5.10) (with $\gamma>\pi/\omega-1$ and $2\pi/\omega>\sigma_1$), the formula for k_p, and the condition $k^2-k_e^2=O\big(\varepsilon^{2\pi/\omega}\big)$ imply the inequalities

$$\begin{aligned}\|((-i\nabla+\mathbf{A})^2\pm H-k^2)u_p;V^0_{\gamma,\delta}\|&\leq|k^2-k_p^2|\,\|u_p;V^0_{\gamma,\delta}\|\\&\quad+\|((-i\nabla+\mathbf{A})^2\pm H-k_p^2)u_p;V^0_{\gamma,\delta}\|\leq c\varepsilon^{2\pi/\omega}.\end{aligned}$$

Since the supports of the functions $((-i\nabla+\mathbf{A})^2\pm H-k_p^2)u_p$ and $\chi_{0,\varepsilon^\sigma}$ are disjoint, we obtain

$$|(\chi_{0,\varepsilon^\sigma}((-i\nabla+\mathbf{A})^2\pm H-k_e^2)u_p,v_e)_{G_0}|=|(k_e^2-k_p^2)(u_p,v_e)_{G_0}|\geq c\varepsilon^{2\pi/\omega}.$$

Moreover, $\gamma-1<\pi/\omega$ and, consequently,

$$|(\chi_{0,\varepsilon^\sigma}S_\varepsilon f,v_e)_{G_0}|\leq\|S_\varepsilon f;V^0_{\gamma,\delta}(G(\varepsilon))\|\,\|v_e;V^0_{-\gamma}(G_0)\|\leq c\|S_\varepsilon f;V^0_{\gamma,\delta}(G(\varepsilon))\|.$$

Hence,

$$|a(f)|\leq c\varepsilon^{-2\pi/\omega}\|S_\varepsilon f;V^0_{\gamma,\delta}(G(\varepsilon))\|$$

and $\|\widetilde{S}_\varepsilon f\| \le c\|S_\varepsilon f\|$. Thus, the operator $I+\widetilde{S}_\varepsilon$ in $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$ is invertible, which is also true for the operator of problem (7.5.1):

$$A_\varepsilon : u \mapsto (-i\nabla + \mathbf{A})^2 u \pm Hu - k^2 u : \overset{\circ}{V}{}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon)) \mapsto V^{0,\perp}_{\gamma,\delta}(G(\varepsilon)),$$

the $\overset{\circ}{V}{}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon))$ consists of the elements in $V^2_{\gamma,\delta,-}(G(\varepsilon))$ that vanish on $\partial G(\varepsilon)$, and the operator $(-i\nabla+\mathbf{A})^2 \pm H - k^2$ takes $\overset{\circ}{V}{}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon))$ to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$ to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. The inverse operator $A_\varepsilon^{-1} = \widetilde{R}_\varepsilon(I+\widetilde{S}_\varepsilon)^{-1}$ is bounded uniformly with respect to ε and k. Therefore, the inequality (7.5.3) holds with c independent of ε and k. $\square$

We consider solution u_1 and u_2 to the homogeneous problem (7.1.1) and (7.1.2) defined by

$$u_1(x,y) = \begin{cases} U_1^+(x,y) + S_{11}\,U_1^-(x,y) + O(\exp(\delta x)), & x \to -\infty,\\ S_{12}\,U_2^-(x,y) + O(\exp(-\delta x)), & x \to +\infty;\end{cases}$$

$$u_2(x,y) = \begin{cases} S_{21}\,U_1^-(x,y) + O(\exp(\delta x)), & x \to -\infty,\\ U_2^+(x,y) + S_{22}\,U_2^-(x,y) + O(\exp(-\delta x)), & x \to +\infty.\end{cases}$$

Let S_{lm} be the elements of the scattering matrix determined by these solutions; $\widetilde{S}_{11}$, $\widetilde{S}_{12}$ are the same as in (7.4.20) and (7.4.21).

Theorem 7.5.2 *Let the hypotheses of Proposition 7.5.1 be fulfilled. Then the inequalities*

$$|S_{11} - \widetilde{S}_{11}| + |S_{12} - \widetilde{S}_{12}| \le c|\widetilde{S}_{12}|\varepsilon^{2-\delta},$$
$$|S_{21} - \widetilde{S}_{21}| + |S_{22} - \widetilde{S}_{22}| \le c|\widetilde{S}_{22}|\varepsilon^{2-\delta}$$

hold with a constant c, independent of ε and k, δ being an arbitrarily small positive number.

Now we return to the detailed notations introduced in the first three sections. We denote by $k^2_{e,\pm}$ an eigenvalue of problem (7.2.1) in the resonator G_0 and by $k^2_{r,\pm}(\varepsilon)$ a resonance frequency such that $k^2_{r,\pm}(\varepsilon) \to k^2_{e,\pm}$ as $\varepsilon \to 0$. Moreover, let $b_j^\pm$ be the constants in asymptotics (7.3.7) of an eigenfunction corresponding to the eigenvalue $k^2_{e,\pm}$ and $s_j(k)$ the constant in asymptotics (7.3.1) of the special solution V_j for $r_j \to 0$, $j = 1, 2$. Finally, the constants α and β are defined by (7.2.11) and (7.2.12). We set $P_\pm = (|b_1||b_2|\beta^2|s_1(k_e)|^2)^{-1}$; this is the same constant as in (7.4.24)–(7.4.26). Theorem 7.5.2 and formulas (7.4.25) and (7.4.26) lead to the next statement.

Theorem 7.5.3 *For $|k^2 - k^2_{r,\pm}| = O(\varepsilon^{2\pi/\omega})$, the asymptotic expansions*

$$
\begin{aligned}
T^{\pm}(k,\varepsilon) &= \frac{1}{\dfrac{1}{4}\left(\dfrac{|b_1^{\pm}|}{|b_2^{\pm}|}+\dfrac{|b_2^{\pm}|}{|b_1^{\pm}|}\right)^2 + P_{\pm}^2\left(\dfrac{k^2-k_{r,\pm}^2}{\varepsilon^{4\pi/\omega}}\right)^2}(1+O(\varepsilon^{2-\delta})),\\
k_{r,\pm}^2 &= k_{0,\pm}^2 - \alpha(|b_1^{\pm}|^2+|b_2^{\pm}|^2)\varepsilon^{2\pi/\omega} + O\big(\varepsilon^{2\pi/\omega+2-\delta}\big),\\
\Upsilon^{\pm}(\varepsilon) &= \left(\frac{|b_1^{\pm}|}{|b_2^{\pm}|}+\frac{|b_2^{\pm}|}{|b_1^{\pm}|}\right)P_{\pm}^{-1}\varepsilon^{4\pi/\omega}\big(1+O(\varepsilon^{2-\delta})\big),
\end{aligned}
$$

hold, $\Upsilon^{\pm}(\varepsilon)$ is the width of the resonant peak at its half-height, and δ is an arbitrarily small positive number.

7.6 Comparison of Asymptotic and Numerical Approaches

The principal parts of the asymptotic formulas in Theorem 7.5.3 contain the constants $b_j^{\pm}$, $|s_1|$, α, β. To find them one has to solve numerically several boundary value problems. In this section, we state the problems and describe a way to solve them. We also outline a method for computing the waveguide scattering matrix S taken from Chap. 4. Then we compare the asymptotics, having calculated constants and the numerically found scattering matrix.

7.6.1 Problems and Methods for Numerical Analysis

7.6.1.1 Calculation of $b_j^{\pm}$

To find $b_j^{\pm}$, we solve the spectral problem

$$
\begin{aligned}
(-i\nabla + \mathbf{A}(x,y))^2 v(x,y) \pm H(\rho)v(x,y) - k^2 v(x,y) = 0 \ \text{in} \quad G_0,&\\
v(x,y) = 0 \ \text{on} \ \partial G_0,&
\end{aligned}
\tag{7.6.1}
$$

by FEM, as usual. Let v_e be an eigenfunction corresponding to k_e^2 and normalized by

$$\int_{G_0} |v_e(x,y)|^2\,dxdy = 1.$$

We have

$$
v_e(x,y) \sim \begin{cases} b_1^{\pm} r_1^{\pi/\omega}\Phi(\varphi_1) & \text{as } r_1 \to 0,\\ b_2^{\pm} r_2^{\pi/\omega}\Phi(\pi-\varphi_2) & \text{as } r_2 \to 0, \end{cases}
\tag{7.6.2}
$$

where (ρ_j, φ_j) are polar coordinates centered in O_j, and $\Phi(\theta) = \pi^{-1/2}\cos(\pi\theta/\omega)$. Then $b_1^\pm$ and $b_2^\pm$ in (7.6.2) can be defined by

$$b_1^\pm = \epsilon^{-\pi/\omega}\frac{v_e(\epsilon,0)}{\Phi(0)} = \sqrt{\pi}\epsilon^{-\pi/\omega}v_e(\epsilon,0), \qquad b_2^\pm = \sqrt{\pi}\epsilon^{-\pi/\omega}v_e(d-\epsilon,0),$$

where ϵ is a small positive number.

7.6.1.2 Calculation of $|s_1|$

To calculate $|s_1|$, we must solve numerically the problem

$$\begin{aligned} -\Delta v(x,y) - k^2 v(x,y) = 0 \ \text{in} \quad G_1, \\ v(x,y) = 0 \ \text{on} \ \partial G_1, \end{aligned} \tag{7.6.3}$$

with conditions

$$\begin{aligned} &v(x,y) \sim s_1\rho^{\pi/\omega}\Phi(\pi-\varphi) && \text{as} \quad \rho \to 0, \\ &v(x,y) = \left(e^{i\nu_1 x} + S_{11}^0 e^{-i\nu_1 x}\right)\Psi_1(y) + O(e^{-\gamma|x|}) && \text{as } x \to -\infty, \end{aligned} \tag{7.6.4}$$

where (ρ, φ) are polar coordinates centered in O_1. We denote the truncated domain

$$G_1 \cap \{(x,y) : x > -D\}$$

by G_1^D and the artificial part of the boundary $\partial G_1^D \cap \{(x,y) : x = -D\}$ by Γ^D.

Consider the following problem

$$\begin{aligned} -\Delta v^D(x,y) - k^2 v^D(x,y) = 0 \qquad & \text{in} \qquad G_1^D, \\ v^D(x,y) = 0 \qquad & \text{on } \partial G_1^D \backslash \Gamma^D, \\ (\partial_n + i\nu_1)\, v^D(x,y) = f(x,y) \ & \text{on} \qquad \Gamma^D. \end{aligned} \tag{7.6.5}$$

If $\nu_1 \in \mathbb{R}\backslash 0$ and $f \in L_2(\Gamma^D)$, the problem has a unique solution v^D, and v^D satisfies the inequality

$$\|v^D\|_{G_1^D} \leqslant C_1\|f\|_{\Gamma^D},$$

where $\|v^D\|_{G_1^D} := \|v^D\|_{L_2(G_1^D)}$; similar notation is used for the norms and the inner products below.

Let v be a solution to the problem (7.6.3), (7.6.4) and let V be a solution to the problem (7.6.5) with $f = 2i\nu_1 e^{i\nu_1 D}\Psi_1(y)$. Then $u = v - V$ satisfies (7.6.5) with $f = O(e^{-\gamma D})$. Hence, $\|v - V\|_{G_1^D} \leqslant C_0 e^{-\gamma D}$.

We find V with FEM and put

$$s_1 = \sqrt{\pi}\epsilon^{-\pi/\omega}\overline{V}(-\epsilon, 0).$$

7.6.1.3 Calculation of α and β

To calculate α and β, we consider the boundary value problem

$$\begin{aligned} \Delta w(\xi,\eta) &= 0 \text{ in } \quad \Omega, \\ w(\xi,\eta) &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{7.6.6}$$

with the following conditions at infinity

$$w(\xi,\eta) = \begin{cases} (\rho^{\pi/\omega} + \alpha\rho^{-\pi/\omega})\Phi(\varphi) + O(\rho^{-3\pi/\omega}) \text{ as } \rho \to \infty,\ \xi > 0, \\ \beta\rho^{-\pi/\omega}\Phi(\pi-\varphi) + O(\rho^{-3\pi/\omega}) \text{ as } \rho \to \infty,\ \xi < 0, \end{cases} \tag{7.6.7}$$

where (ρ,φ) are polar coordinates centered in O_1. Introduce the notations

$$\begin{aligned} \Omega^D &= \Omega \cap \{(\rho,\varphi) : \rho < D\}, \\ \Gamma^D &= \partial\Omega^D \cap \{(\rho,\varphi) : \rho = D\}. \end{aligned}$$

Consider the problem

$$\begin{aligned} \Delta w^D(\xi,\eta) &= 0 && \text{in} \quad \Omega^D, \\ w^D(\xi,\eta) &= 0 && \text{on } \partial\Omega^D\backslash\Gamma^D, \\ (\partial_n + \zeta)\, w^D(\xi,\eta) &= g(\xi,\eta) && \text{on} \quad \Gamma^D. \end{aligned} \tag{7.6.8}$$

If w^D is a solution and $\zeta > 0$, then

$$\|w^D\|_{\Gamma^D} \leqslant \zeta^{-1}\|g\|_{\Gamma^D}. \tag{7.6.9}$$

Denote the left-hand part of Γ^D by Γ^D_- and the right-hand part of Γ^D by Γ^D_+. Let W satisfy (7.6.8) with $\zeta = \pi/\omega D$, $g|_{\Gamma^D_-} = 0$, $g|_{\Gamma^D_+} = (2\pi/\omega)D^{(\pi/\omega)-1}\Phi(\varphi)$. Since the conditions (7.6.7) can be differentiated, $w - W$ satisfies (7.6.8) with $g = O(D^{-(3\pi/\omega)-1})$. According to (7.6.9),

$$\|w - W\|_{\Gamma^D} \leqslant c\frac{\omega D}{\pi}D^{-(3\pi/\omega)-1} = c'D^{-3\pi/\omega}$$

as $D \to +\infty$. We find W with FEM and take

$$\beta = \frac{W(-D,0)}{\Phi(0)}D^{\pi/\omega} = \sqrt{\pi}\,W(-D,0)D^{\pi/\omega}.$$

Obviously, $\|(w - D^{\pi/\omega}\Phi(\varphi)) - (W - D^{\pi/\omega}\Phi(\varphi))\|_{\Gamma^D} \leqslant c'D^{-3\pi/\omega}$; therefore, we put

$$\alpha = \frac{W(D,0) - D^{\pi/\omega}\Phi(0)}{\Phi(0)}D^{\pi/\omega} = \sqrt{\pi}\,W(D,0)D^{\pi/\omega} - D^{2\pi/\omega}.$$

Now that the coefficients in the asymptotic formulas have been calculated, we can find the asymptotics for a quantitative description of the polarization process. However, the formulas are designed for sufficiently small narrows' diameters. Thus, it remains to estimate the range of ε where asymptotics works. To this end, we calculate the scattering matrix by employing the method suggested in Chap. 4. Here we present the needed description of the method. First, we introduce

$$\begin{aligned} G(\varepsilon, D) &= G(\varepsilon) \cap \{(x, y) : -D < x < d + D\}, \\ \Gamma_1^D &= \partial G(\varepsilon, D) \cap \{(x, y) : x = -D\}, \\ \Gamma_2^D &= \partial G(\varepsilon, D) \cap \{(x, y) : x = d + D\} \end{aligned}$$

for large D. As an approximation to the row (S_{11}, S_{12}) of the scattering matrix $S(k)$, we take the minimizer of a quadratic functional. To construct such a functional, we consider the problem

$$\begin{aligned} (-i\nabla + \mathbf{A})^2 \mathcal{X}_\pm^D \pm H\mathcal{X}_\pm^D - k^2 \mathcal{X}_\pm^D = 0 \text{ in } &\quad G(\varepsilon, D), \\ \mathcal{X}_\pm^D = 0 \text{ on } \partial G(\varepsilon, D) \setminus (\Gamma_1^D \cup \Gamma_2^D)&, \end{aligned} \tag{7.6.10}$$

$$\begin{aligned} (\partial_n + i\zeta)\mathcal{X}_\pm^D &= i(-\nu_1 + \zeta)e^{-i\nu_1 D}\Psi_1(y) + a_1\, i(\nu_1 + \zeta)e^{i\nu_1 D}\Psi_1(y) \text{ on } \Gamma_1^D, \\ (\partial_n + i\zeta)\mathcal{X}_\pm^D &= a_2\, i(\nu_1 + \zeta)e^{i\nu_1(d+D)}\Psi_1(y) \quad \text{on } \Gamma_2^D, \end{aligned} \tag{7.6.11}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$ is an arbitrary fixed number, and a_1, a_2 are complex numbers. As approximation to the row (S_{11}, S_{12}), we take the minimizer $a^0(D) = (a_1^0(D), a_2^0(D))$ of the functional

$$J^D(a_1, a_2) = \left\| \mathcal{X}_\pm^D - e^{-i\nu_1 D}\Psi_1 - a_1 e^{i\nu_1 D}\Psi_1 \right\|_{\Gamma_1^D}^2 + \left\| \mathcal{X}_\pm^D - a_2\, e^{i\nu_1(d+D)}\Psi_1 \right\|_{\Gamma_2^D}^2, \tag{7.6.12}$$

where $\mathcal{X}_\pm^D$ is a solution to problem (7.6.10). From the results of Chap. 4, it follows that $a_j^0(D, k) \to S_{1j}(k)$ with exponential rate as $D \to \infty$. More precisely, there exist positive constants Λ and C such that

$$|a_j^0(D, k) - S_{1j}(k)| \leqslant Ce^{-\Lambda D}, \qquad j = 1, 2,$$

for all $k^2 \in [\mu_1, \mu_2]$ and sufficiently large D; the interval $[\mu_1, \mu_2]$ of continuous spectrum of the problem (7.1.1) lies between the first and the second thresholds and does not contain the thresholds. (Note that application of the method is not hindered by possible presence on the interval $[\mu_1, \mu_2]$ of eigenvalues of the problem (7.1.1) corresponding to eigenfunctions exponentially decaying at infinity.) To express $\mathcal{X}_\pm^D$ by means of a_1, a_2, we consider the problems

$$\begin{aligned}
(-i\nabla+\mathbf{A})v_{\pm,1}^{\pm} \pm Hv_{\pm,1}^{\pm} - k^2 v_{\pm,1}^{\pm} &= 0 && \text{in} && G(\varepsilon, D),\\
v_{\pm,1}^{\pm} &= 0 && \text{on} && \partial G(\varepsilon, D) \setminus (\Gamma_1^D \cup \Gamma_2^D),\\
(\partial_n + i\zeta)v_{\pm,1}^{\pm} &= i(\mp\nu_1+\zeta)e^{\mp i\nu_1 D}\Psi_1 && \text{on} && \Gamma_1^D,\\
(\partial_n + i\zeta)v_{\pm,1}^{\pm} &= 0 && \text{on} && \Gamma_2^D
\end{aligned} \tag{7.6.13}$$

and

$$\begin{aligned}
(-i\nabla+\mathbf{A})v_{\pm,2}^{\pm} \pm Hv_{\pm,2}^{\pm} - k^2 v_{\pm,2}^{\pm} &= 0 && \text{in} && G(\varepsilon, D),\\
v_{\pm,2}^{\pm} &= 0 && \text{on} && \partial G(\varepsilon, D) \setminus (\Gamma_1^D \cup \Gamma_2^D),\\
(\partial_n + i\zeta)v_{\pm,2}^{\pm} &= 0 && \text{on} && \Gamma_1^D,\\
(\partial_n + i\zeta)v_{\pm,2}^{\pm} &= i(\mp\nu_1+\zeta)e^{\mp i\nu_1 (d+D)}\Psi_1 && \text{on} && \Gamma_2^D.
\end{aligned} \tag{7.6.14}$$

In $v_{\pm,j}^{\pm}$, the upper and lower $\pm$ correspond to $\mp$ in the condition on $\Gamma_1^D \cup \Gamma_2^D$ and to the sign in the Pauli equation, respectively. Let us express $\mathcal{X}_{\pm,m}^D$ by means of the solutions $v_{\pm,j}^{\pm}$ to problems (7.6.13) and (7.6.14). We have $\mathcal{X}_{\pm}^D = v_{\pm}^{+} + a_1 v_{\pm,1}^{-} + a_2 v_{\pm,2}^{-}$. The functional (7.6.12) can be rewritten in the form

$$J^D(a,k) = \langle a\mathcal{E}^D(k), a\rangle + 2\,\mathrm{Re}\left(\langle \mathcal{F}_1^D(k), a\rangle\right) + \mathcal{G}_1^D(k),$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{C}^2$, and $\mathcal{E}^D$ stands for the 2×2–matrix with entries

$$\begin{aligned}
\mathcal{E}_{11}^D &= \left((v_{\pm,1}^{-} - e^{i\nu_1 D}\Psi_1), (v_{\pm,1}^{-} - e^{i\nu_1 D}\Psi_1)\right)_{\Gamma_1^D} + \left(v_{\pm,1}^{-}, v_{\pm,1}^{-}\right)_{\Gamma_2^D},\\
\mathcal{E}_{1,2}^D &= \left((v_{\pm,1}^{-} - e^{i\nu_1 D}\Psi_1), v_{\pm,2}^{-}\right)_{\Gamma_1^D} + \left(v_{\pm,1}^{-}, (v_{\pm,2}^{-} - e^{i\nu_1 (d+D)}\Psi_1)\right)_{\Gamma_2^D},\\
\mathcal{E}_{2,1}^D &= \left(v_{\pm,2}^{-}, (v_{\pm,1}^{-} - e^{i\nu_1 D}\Psi_1)\right)_{\Gamma_1^D} + \left((v_{\pm,2}^{-} - e^{i\nu_1 (d+D)}\Psi_1), v_{\pm,1}^{-}\right)_{\Gamma_2^D},\\
\mathcal{E}_{2,2}^D &= \left(v_{\pm,2}^{-}, v_{\pm,2}^{-}\right)_{\Gamma_1^D} + \left((v_{\pm,2}^{-} - e^{i\nu_1 (d+D)}\Psi_1), (v_{\pm,2}^{-} - e^{i\nu_1 (d+D)}\Psi_1)\right)_{\Gamma_2^D};
\end{aligned}$$

$\mathcal{F}_1^D(k)$ is the row $(\mathcal{F}_{11}^D(k), \mathcal{F}_{12}^D(k))$ and $\mathcal{G}_1^D(k)$ is the number defined by

$$\begin{aligned}
\mathcal{F}_{11}^D &= \left((v_{\pm,1}^{+} - e^{-i\nu_1 D}\Psi_1), (v_{\pm,1}^{-} - e^{i\nu_1 D}\Psi_1)\right)_{\Gamma_1^D} + \left(v_{\pm,1}^{+}, v_{\pm,1}^{-}\right)_{\Gamma_2^D},\\
\mathcal{F}_{12}^D &= \left((v_{\pm,1}^{+} - e^{-i\nu_1 D}\Psi_1), v_{\pm,2}^{-}\right)_{\Gamma_1^D} + \left(v_{\pm,1}^{+}, (v_{\pm,2}^{-} - e^{i\nu_1 (d+D)}\Psi_1)\right)_{\Gamma_2^D},\\
\mathcal{G}_1^D &= \left((v_{\pm,1}^{+} - e^{-i\nu_1 D}\Psi_1), (v_{\pm,1}^{+} - e^{-i\nu_1 D}\Psi_1)\right)_{\Gamma_1^D} + \left(v_{\pm,1}^{+}, v_{\pm,1}^{+}\right)_{\Gamma_2^D},
\end{aligned}$$

The minimizer $a^0 = (a_1^0(D,k), a_2^0(D,k))$ satisfies $a^0\mathcal{E}^D + \mathcal{F}_1^D = 0$. The solution to this equation serves as an approximation to the first row of the scattering matrix. In the same way, one can show that the approximation to the scattering matrix $S(k)$ is the solution $S^D = S^D(k)$ to the matrix equation of the form $S^D\mathcal{E}^D + \mathcal{F}^D = 0$. If one chooses $\zeta = -\nu_1$, then $v_{\pm,1}^- = v_{\pm,2}^- = 0$, $\mathcal{E}^D = (1/\nu_1)\mathrm{Id}$, and $S^D = -\nu_1\mathcal{F}^D$.

7.6.2 Comparison of Asymptotic and Numerical Results

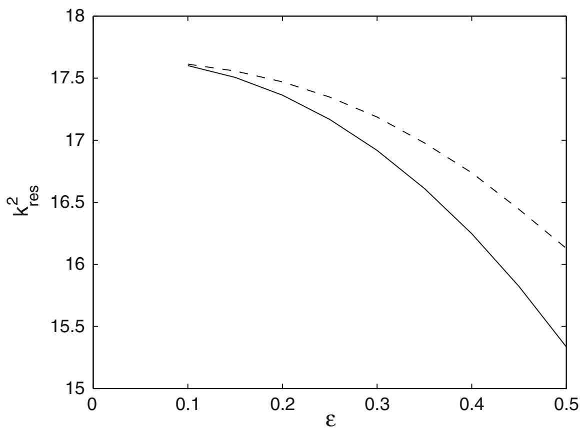
Let us compare the asymptotics $k^2_{res,a}(\varepsilon)$ of resonant energy $k^2_{res}(\varepsilon)$ and the approximate value $k^2_{res,n}(\varepsilon)$ obtained by the numerical method.

The 'numerical' and 'asymptotic' resonant energies are shown in Fig. 7.1. The discrepancy between the curves depends on the magnetic field H_0 and the narrows' opening ω. Numerical resonance is calculated by the iteration process, the asymptotic resonant energy is taken for the initial value.

The shapes of "asymptotic" and "numerical" resonant peaks are almost the same (see Fig. 7.2). The difference between the peaks is quantitatively depicted in Fig. 7.3 (note the logarithmic scale on the axes). Moreover, it turns out that the ratio of the width $\Delta_n(h,\varepsilon)$ of the numerical peak at height h to the width $\Delta_a(h,\varepsilon)$ of the asymptotic peak is independent of h. The ratio as function of ε is displayed in Fig. 7.4.

The obtained data show that asymptotic and numerical methods give equivalent results at the band of the narrows' diameters $0.1 < \varepsilon < 0.5$ (see Figs. 7.1 and 7.3). The numerical method becomes ill-conditioned as $\varepsilon < 0.1$. However, the asymptotics remains reliable at such a condition. On the other hand, the asymptotics gives way to the numerical method as the diameter increases.

Fig. 7.1 Asymptotic description $k^2_{res,a}(\varepsilon)$ (*solid curve*) and numerical description $k^2_{res,n}(\varepsilon)$ (*dashed curve*) for resonant energy $k^2_{res}(\varepsilon)$



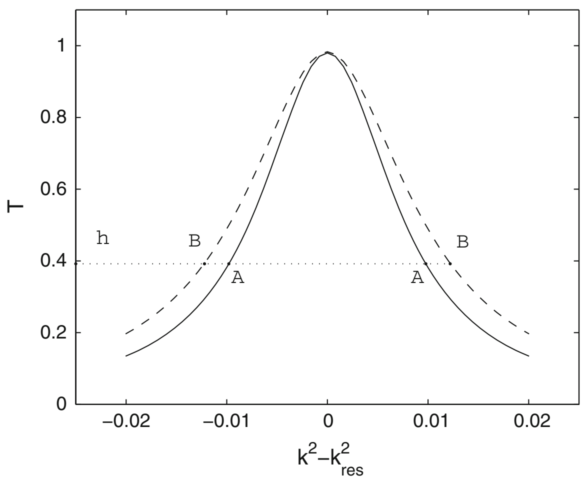


Fig. 7.2 Transition coefficient for $\varepsilon = 0.2$, asymptotic description $T_a(k^2 - k^2_{res,a})$ (*solid curve*) and numerical description $T_n(k^2 - k^2_{res,n})$ (*dashed curve*). The width of the resonant peak at height h: asymptotic $\Delta_a(h, \varepsilon) = AA$; numerical $\Delta_n(h, \varepsilon) = BB$

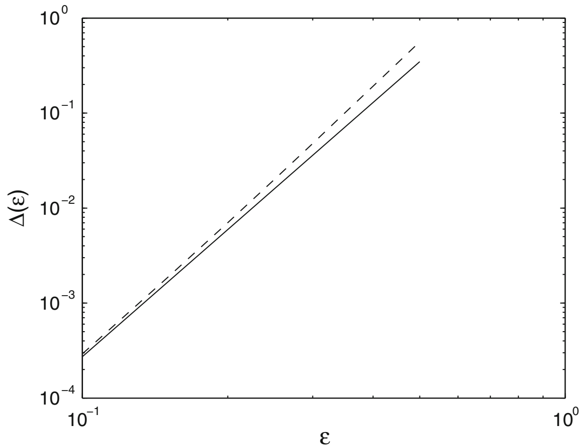


Fig. 7.3 The width $\Delta(\varepsilon)$ of the resonant peak at half-height of the peak (*dashed line* for numerical description, *solid line* for asymptotic description)

7.6.3 Dependence of Resonant Tunneling on the Magnetic Field Location in the Resonator

In the above numerical simulation results, the center of the magnetic field domain coincides with the resonator center. We will illustrate, by the Aharonov-Bohm effect, the dependence of resonant tunneling on the position of a magnetic patch in the resonator. Let $T(k^2_{res})$ denote the maximal value of the transmission coefficients T (at $k^2 = k^2_{res}$). Figure 7.5 depicts $T(k^2_{res})$ versus the magnetic flux of the patch for four values of the magnetic patch shift in the direction perpendicular to the waveguide axis. If the patch center belongs to the waveguide axis, the $T(k^2_{res})$ vanishes for certain values of the patch magnetic flux. The reason is that the electron waves streaming around the magnetic patch have the same amplitudes and the phases differing by

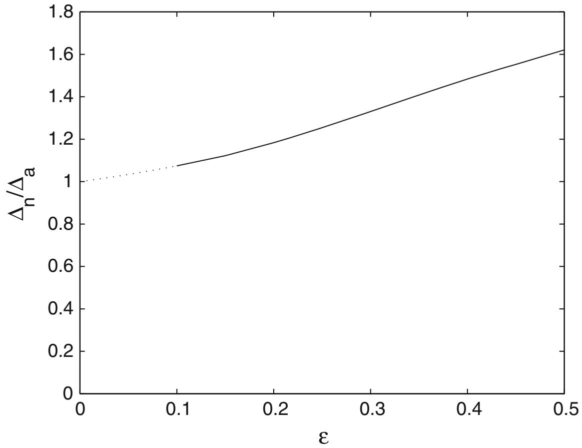


Fig. 7.4 Ratio $\Delta_n(h, \varepsilon)/\Delta_a(h, \varepsilon)$ as function in ε. The ratio is independent of h within the accuracy of the analysis

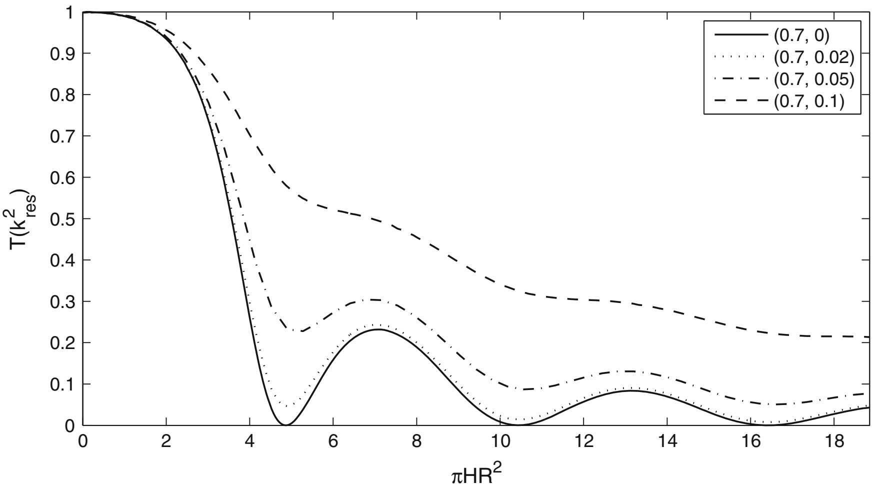


Fig. 7.5 $T(k_{res}^2)$ versus the magnetic flux of the patch for four values of the magnetic patch shift in the direction perpendicular to the waveguide axis. The legend shows the coordinates of the magnetic patch center

$(2q+1)\pi$. When the patch is shifted in the direction perpendicular to the waveguide axis, such a cancellation does not occur and the transmission probability does not vanish. We assume that the waveguide width is equal to 1 and the patch radius is equal to 0.2. Then Fig. 7.5 shows, in particular, that for the patch center shifted by 0.1, the Aharonov-Bohm effect is practically absent.

Chapter 8
Effect of Magnetic Field on Resonant Tunneling in 3D Waveguides of Variable Cross-Section

8.1 Introduction

In this chapter, we consider a three-dimensional waveguide that, far from the coordinate origin, coincides with a cylinder G containing the axis x. The cross-section of G is a two-dimensional domain (of an arbitrary form) with smooth boundary. The waveguide has two narrows of small diameter ε. The waveguide part between the narrows plays the role of a resonator and there can arise conditions for electron resonant tunneling. This phenomenon consists of the fact that, for an electron with energy E, the probability $T(E)$ to pass from one part of the waveguide to the other through the resonator has a sharp peak at $E = E_{res}$, where E_{res} denotes the "resonant" energy. To analyse the operation of devices based on resonant tunneling, it is important to know E_{res}, the behavior of $T(E)$ for E close to E_{res}, the height of the resonant peak, etc.

The presence of a magnetic field can essentially affect the basic characteristics of the resonant tunneling and bring new possibilities for applications in electronics. In particular, in the presence of a magnetic field, the tunneling phenomenon is feasible for producing spin-polarized electron flows consisting of electrons with spins of the same direction. We suppose that a part of the resonator has been occupied by the magnetic field generated by an infinite solenoid with axis orthogonal to the axis x. Electron wave function satisfies the Pauli equation in the waveguide and vanishes at its boundary (the work function of the waveguide is supposed to be sufficiently large, so such a boundary condition has been justified). Moreover, we assume that only one incoming wave and one outgoing wave can propagate in each cylindrical outlet of the waveguide. In other words, we do not discuss the multichannel electron scattering and consider only electrons with energy between the first and the second thresholds. We take ε as small parameter and obtain asymptotic formulas for the aforementioned characteristics of the resonant tunneling as $\varepsilon \to 0$. It turns out that such formulas depend on the limiting form of the narrows. We suppose that, in a neighborhood of each narrow, the limiting waveguide coincides with a double cone symmetric about the vertex.

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Section 8.2 contains the statement of the problem. In Sect. 8.3, we introduce so-called "limit" boundary value problems, which are independent of the parameter ε. Some model solutions to the problems are studied in Sect. 8.4. The solutions will be used in Sect. 8.5 to construct asymptotic formulas for appropriate wave functions. In the same section, we investigate the asymptotics of the wave functions and derive asymptotic formulas for the main characteristics of the resonant tunneling. The remainders in the asymptotic formulas are estimated in Sect. 8.6.

8.2 Statement of the Problem

To describe the domain $G(\varepsilon)$ in $\mathbb{R}^3$ occupied by the waveguide, we first introduce domains G and Ω in $\mathbb{R}^3$ independent of ε. The domain G is the cylinder

$$G = \mathbb{R} \times D = \{(x, y, z) \in \mathbb{R}^3 : x \in \mathbb{R} = (-\infty, +\infty); (y, z) \in D \subset \mathbb{R}^2\}$$

whose cross-section D is a bounded two-dimensional domain with smooth boundary. Let us define Ω. Denote by K a double cone with vertex at the coordinate origin O that contains the axis x and is symmetric about the origin. The set $K \cap S^2$ with S^2 standing for the unit sphere consists of two non-overlapping one-connected domains symmetric about the center of sphere. Assume that the domain Ω contains the cone K together with a neighborhood of its vertex. Moreover, Ω coincides with K outside a sufficiently large ball centered at the origin. The boundary $\partial\Omega$ of Ω is supposed to be smooth.

Let us turn to the waveguide $G(\varepsilon)$. We denote by $\Omega(\varepsilon)$ the domain obtained from Ω by the contraction with center at O and coefficient ε. In other words, $(x, y, z) \in \Omega(\varepsilon)$ if and only if $(x/\varepsilon, y/\varepsilon, z/\varepsilon) \in \Omega$. Let K_j and $\Omega_j(\varepsilon)$ stand for K and $\Omega(\varepsilon)$ shifted by the vector $\mathbf{r}_j = (x_j^0, 0, 0)$, $j = 1, 2$. The value $|x_1^0 - x_2^0|$ is assumed to be sufficiently large so that the distance between $\partial K_1 \cap \partial K_2$ and G is positive. We set

$$G(\varepsilon) = G \cap \Omega_1(\varepsilon) \cap \Omega_2(\varepsilon).$$

The wave function $\boldsymbol{\Psi} = (\Psi_+, \Psi_-)^T$ of an electron with energy $E = k^2\hbar^2/2m$ in a magnetic field $\mathbf{H}_0$ satisfies the Pauli equation

$$(-i\nabla + \mathbf{A})^2\boldsymbol{\Psi} + (\widehat{\sigma}, \mathbf{H})\boldsymbol{\Psi} = k^2\boldsymbol{\Psi} \quad \text{in} \quad G(\varepsilon), \tag{8.2.1}$$

where $\widehat{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\mathbf{H} = -(e/c\hbar\mathbf{H}_0) = \operatorname{rot}\mathbf{A}$. If the magnetic field is directed along the axis z that is $\mathbf{H} = H\mathbf{k}$, H being a scalar function, then (8.2.1) decomposes into two scalar equations

$$(-i\nabla + \mathbf{A})^2\Psi_\pm \pm H\Psi_\pm = k^2\Psi_\pm. \tag{8.2.2}$$

Let the function H depend only on $\rho = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$ with $H(\rho) = 0$ for $\rho > R$, R being a fixed positive number. Such a field is generated by an infinite solenoid with radius R and axis parallel to the axis z. Then $\mathbf{A} = A\mathbf{e}_\psi$, where $\mathbf{e}_\psi = \rho^{-1}(-y + y_0, x - x_0, 0)$ and

$$A(\rho) = \frac{1}{\rho}\begin{cases} \int_0^\rho tH(t)\,dt, & \rho < R; \\ \int_0^R tH(t)\,dt, & \rho > R. \end{cases}$$

The equality $\operatorname{rot}\mathbf{A} = \mathbf{H}$ determines $\mathbf{A}$ up to a term of the form ∇f. We neglect the waveguide boundary permeability to the electrons and consider the Eq. (8.2.2) supplemented by the homogeneous boundary condition

$$\Psi_\pm = 0 \qquad \text{on } \partial G(\varepsilon). \tag{8.2.3}$$

The obtained boundary value problems are self-adjoint with respect to the Green formulas

$$\begin{aligned}((-i\nabla + \mathbf{A})^2u \pm Hu - k^2u, v)_{G(\varepsilon)} - (u, (-i\nabla + \mathbf{A})^2v \pm Hv - k^2v)_{G(\varepsilon)}\\ + (u, (-\partial_n - A_n)v)_{\partial G(\varepsilon)} - ((-\partial_n - A_n)u, v)_{\partial G(\varepsilon)} = 0,\end{aligned}$$

where A_n is the projection of $\mathbf{A}$ onto the outward normal to $\partial G(\varepsilon)$ and $u, v \in C_c^\infty(G(\varepsilon))$ (which means that u and v are smooth functions vanishing outside a bounded set). Besides, $\Psi_\pm$ must satisfy some radiation conditions at infinity. To formulate such conditions, we have to introduce incoming and outgoing waves. From the requirements on $\mathbf{H}$ and the choice of $\mathbf{A}$, it can be seen that the coefficients of Eq. (8.2.2) stabilize at infinity with a power rate. Such a slow stabilization creates difficulties in defining these waves. Therefore, we will modify $\mathbf{A}$ by a gauge transformation so that the coefficients in (8.2.2) become constant for large $|x|$.

Let (ρ, ψ) be polar coordinate on the plane xy centered at (x_0, y_0) and $\psi = 0$ on the ray of the same direction as the axis x. We introduce $f(x, y, z) = c\psi$, where $c = \int_0^R tH(t)\,dt$. For definiteness, assume that $-\pi/2 < \psi < 3\pi/2$. The function f is uniquely determined in the waveguide for $|x - x_0| > 0$, moreover, $\nabla f = \mathbf{A}$ for $|x - x_0| > R$. Let τ be a cut-off function on $\mathbb{R}_+$ equal to 1 for $t > R + 2$ and 0 for $t < R + 1$. We set $\mathbf{A}'(x, y, z) = \mathbf{A}(x, y, z) - \nabla(\tau(|x - x_0|)f(x, y, z))$. Then $\operatorname{rot}\mathbf{A}' = \operatorname{rot}\mathbf{A} = \mathbf{H}$ while $\mathbf{A}' = 0$ for $|x - x_0| > R + 2$. The wave functions $\Psi'_\pm = \Psi_\pm \exp\{i\tau f\}$ satisfy (8.2.2) with $\mathbf{A}$ replaced by $\mathbf{A}'$. For $|x - x_0| > R + 2$, the coefficients of the Eq. (8.2.2) with new vector potential $\mathbf{A}'$ coincide with the coefficients of the Helmholtz equation

$$-\Delta\Psi'_\pm = k^2\Psi'_\pm.$$

In order to formulate the radiation conditions, we consider the problem

$$\begin{aligned} -\Delta v(y,z) - \lambda^2 v(y,z) = 0, &\qquad (y,z) \in D, \\ v(y,z) = 0, &\qquad (y,z) \in \partial D. \end{aligned} \tag{8.2.4}$$

The values of parameter λ^2 that correspond to the nontrivial solutions of this problem form the sequence $\lambda_1^2 < \lambda_2^2 < \dots$ with $\lambda_1^2 > 0$. These numbers are called the thresholds. Assume that k^2 in (8.2.2) coincides with none of the thresholds and take up the equation in (8.2.2) with Ψ_+. For a fixed $k^2 > \lambda_1^2$, there exist finitely many bounded solutions (wave functions) linearly independent modulo $L_2(G(\varepsilon))$; in other words, a linear combination of such solutions belongs to $L_2(G(\varepsilon))$ if and only if all coefficients are equal to zero. The number of wave functions with such properties remains constant for $k^2 \in (\lambda_q^2, \lambda_{q+1}^2)$, $q = 1, 2, \dots$ and step-wise increases at the thresholds.

In the present paper, we discuss only the situation where $k^2 \in (\lambda_1^2, \lambda_2^2)$. In such a case, there exist two independent wave functions. A basis in the space spanned by such functions can be composed of the wave functions u_1^+ and u_2^+ satisfying the radiation conditions

$$\begin{aligned} u_1^+(x,y,z) &= \begin{cases} e^{i\nu_1 x}\Psi_1(y,z) + S_{11}^+(k)\, e^{-i\nu_1 x}\Psi_1(y,z) + O(e^{\delta x}), & x \to -\infty, \\ S_{12}^+(k)\, e^{i\nu_1 x}\Psi_1(y,z) + O(e^{-\delta x}), & x \to +\infty; \end{cases} \\ u_2^+(x,y,z) &= \begin{cases} S_{21}^+(k)\, e^{-i\nu_1 x}\Psi_1(y,z) + O(e^{\delta x}), & x \to -\infty, \\ e^{-i\nu_1 x}\Psi_1(y,z) + S_{22}^+(k)\, e^{i\nu_1 x}\Psi_1(y,z) + O(e^{-\delta x}), & x \to +\infty; \end{cases} \end{aligned} \tag{8.2.5}$$

here $\nu_1 = \sqrt{k^2 - \lambda_1^2}$ and Ψ_1 stands for an eigenfunction of problem (5.1.2) corresponding to λ_1^2 and normalized by the equality

$$2\nu_1 \int_D |\Psi_1(y,z)|^2\, dy\, dz = 1. \tag{8.2.6}$$

The function $U_1(x,y,z) = e^{i\nu_1 x}\Psi_1(y,z)$ in the cylinder G is a wave incoming from $-\infty$ and outgoing to $+\infty$, while $U_2(x,y,z) = e^{-i\nu_1 x}\Psi_1(y,z)$ is a wave going from $+\infty$ to $-\infty$. The matrix

$$S^+ = \|S_{mj}^+\|_{m,j=1,2}$$

with entries determined by (8.2.5) is called the scattering matrix; it is unitary. The quantities

$$R_1^+ := |S_{11}^+|^2, \qquad T_1^+ := |S_{12}^+|^2$$

are called the reflection coefficient and the transition coefficient for the wave U_1 coming in $G(\varepsilon)$ from $-\infty$. (Similar definitions can be given for the wave U_2, incoming from $+\infty$.) In the same manner, we introduce the scattering matrix S^- and the reflection and transition coefficients R_1^- and T_1^- for the equation in (8.2.2) with Ψ_-.

We consider only the scattering of the wave going from $-\infty$ and denote the reflection and transition coefficients by

$$R^{\pm} = R^{\pm}(k, \varepsilon) = |S_{11}^{\pm}(k, \varepsilon)|^2, \qquad T^{\pm} = T^{\pm}(k, \varepsilon) = |S_{12}^{\pm}(k, \varepsilon)|^2. \tag{8.2.7}$$

We intend to find a "resonant" value $k_r^{\pm} = k_r^{\pm}(\varepsilon)$ of the parameter k which corresponds to the maximum of the transition coefficient and to describe the behavior of $T^{\pm}(k, \varepsilon)$ near $k_r^{\pm}(\varepsilon)$ as $\varepsilon \to 0$.

8.3 Limit Problems

To derive the asymptotics of a wave function (i.e., a solution to problem (8.2.2)) as $\varepsilon \to 0$, we make use of the compound asymptotics method. To this end, we introduce the "limit" problems independent of ε. Let the vector potential $\mathbf{A}'$ and, in particular, the magnetic field H differ from zero only in the resonator, which is the part of waveguide between the narrows. Then, outside the resonator and in a neighborhood of the narrows, the wave function under consideration satisfies the Helmholtz equation.

8.3.1 First Kind Limit Problems

We set $G(0) = G \cap K_1 \cap K_2$ (Fig. 5.3), so $G(0)$ consists of three parts: G_0, G_1, and G_2. The boundary value problems

$$\begin{aligned} -\Delta v(x,y,z) - k^2 v(x,y,z) &= f(x,y,z), \qquad (x,y,z) \in G_j, \\ v(x,y,z) &= 0, \qquad (x,y,z) \in \partial G_j, \end{aligned} \tag{8.3.1}$$

where $j = 1, 2$, and

$$\begin{aligned} (-i\nabla + \mathbf{A}')^2 v(x,y,z) \pm H(\rho) v(x,y,z) - k^2 v(x,y,z) &= f(x,y,z), \quad (x,y,z) \in G_0, \\ v(x,y,z) &= 0, \qquad (x,y,z) \in \partial G_0, \end{aligned} \tag{8.3.2}$$

are called the first kind limit problems.

We introduce function spaces for the problem (8.3.2) in G_0. Denote by O_1 and O_2 the conical points of the boundary ∂G_0 and by ϕ_1 and ϕ_2 smooth real functions on the closure $\overline{G_0}$ of G_0 such that $\phi_j = 1$ in a neighborhood of O_j while $\phi_1^2 + \phi_2^2 = 1$. For $l = 0, 1, 2$ and $\gamma \in \mathbb{R}$, we denote by $V_\gamma^l(G_0)$ the completion in the norm

$$\|v; V^l_\gamma(G_0)\| = \left(\int_{G_0} \sum_{|\alpha|=0}^{l} \sum_{j=1}^{2} \phi_j^2(x,y,z) r_j^{2(\gamma-l+|\alpha|)} |\partial^\alpha v(x,y,z)|^2 \, dx\, dy\, dz \right)^{1/2} \tag{8.3.3}$$

of the set of smooth functions on $\overline{G_0}$ vanishing near O_1 and O_2; here r_j is the distance between the points (x, y, z) and O_j, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the multi-index, and $\partial^\alpha = \partial^{|\alpha|}/\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}$.

Let K_j be the tangent cone to ∂G_0 at O_j and $S(K_j)$ the domain that K_j cuts out on the unit sphere centered at O_j. We denote by $\mu_1(\mu_1 + 1)$ and $\mu_2(\mu_2 + 1)$ the first and second eigenvalues of the Dirichlet problem for the Laplace-Beltrami operator in $S(K_1)$, $0 < \mu_1(\mu_1 + 1) < \mu_2(\mu_2 + 1)$. Moreover, we let Φ_1 stand for an eigenfunction corresponding to $\mu_1(\mu_1 + 1)$ and normalized by

$$(2\mu_1 + 1) \int_{S(K_1)} |\Phi_1(\varphi)|^2 d\varphi = 1.$$

The next proposition follows from the general results, e.g., see [37, Chaps. 2 and 4, Sects. 1–3] or [33, Vol. 1,Chap. 1] .

Proposition 8.3.1 *Assume that $|\gamma - 1| < \mu_1 + 1/2$. Then, for $f \in V^0_\gamma(G_0)$ and any k^2 except the positive increasing sequence $\{k_p^2\}_{p=1}^\infty$ of eigenvalues $k_p^2 \to \infty$, there exists a unique solution $v \in V^2_\gamma(G_0)$ to the problem (8.3.2) in G_0. The estimate*

$$\|v; V^2_\gamma(G_0)\| \le c\|f; V^0_\gamma(G_0)\| \tag{8.3.4}$$

holds with a constant c independent of f. If f vanishes in a neighborhood of O_1 and O_2, then v admits the asymptotics

$$v(x,y,z) = \begin{cases} b_1 r_1^{-1/2} \widetilde{J}_{\mu_1+1/2}(kr_1)\Phi_1(\varphi_1) + O\big(r_1^{\mu_2+1/2}\big), & r_1 \to 0; \\ b_2 r_2^{-1/2} \widetilde{J}_{\mu_1+1/2}(kr_2)\Phi_1(-\varphi_2) + O\big(r_2^{\mu_2+1/2}\big), & r_2 \to 0 \end{cases}$$

near O_1 and O_2, where (r_j, φ_j) are "polar coordinates" centered at O_j, $r_j > 0$ and $\varphi_j \in S(K_j)$; b_j are certain constants; $\widetilde{J}_\mu$ denotes the Bessel function multiplied by a constant such that $\widetilde{J}_\mu(kr) = r^\mu + o(r^\mu)$.

Let $k^2 = k_e^2$ be an eigenvalue of problem (8.3.2); then the problem (8.3.2) is solvable if and only if $(f, v_e)_{G_0} = 0$ for any eigenfunction v_e corresponding to k_e^2. Under such conditions, there exists a unique solution v to problem (8.3.2) that is orthogonal to all these eigenfunctions and satisfies (8.3.4).

We turn to problems (8.3.1) for $j = 1, 2$. Let $\chi_{0,j}$ and $\chi_{\infty,j}$ be smooth real functions on the closure $\overline{G}_j$ of G_j such that $\chi_{0,j} = 1$ in a neighborhood of O_j, $\chi_{0,j} = 0$ outside a compact set, and $\chi_{0,j}^2 + \chi_{\infty,j}^2 = 1$. We also assume that the support $\operatorname{supp}\chi_{\infty,j}$ is in the cylindrical part of G_j. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, 2$, the space $V^l_{\gamma,\delta}(G_j)$

is the completion in the norm

$$\|v; V^l_{\gamma,\delta}(G_j)\| = \left(\int_{G_j} \sum_{|\alpha|=0}^{l} \left(\chi^2_{0,j} r_j^{2(\gamma-l+|\alpha|)} + \chi^2_{\infty,j} \exp(2\delta x)\right) |\partial^\alpha v|^2 \, dx\, dy\, dz \right)^{1/2} \tag{8.3.5}$$

of the set of functions with compact support smooth on $\overline{G}_j$ and equal to zero in a neighborhood of O_j.

By assumption, k^2 is between the first and second thresholds, so in every domain G_j there is only one outgoing wave; let $U_1^- = U_2$ be the outgoing wave in G_1 and $U_2^- = U_1$ that in G_2 (the definition of the waves U_j in G see in Sect. 8.2). The next proposition follows from Theorem 5.3.5 in [37].

Proposition 8.3.2 *Let $|\gamma - 1| < \mu_1 + 1/2$ and let the homogeneous problem (8.3.1) (with $f = 0$) have no nontrivial solutions in $V^2_{\gamma,0}(G_j)$. Then, for any right-hand side $f \in V^0_{\gamma,\delta}(G_j)$ there exists a unique solution v to the problem (6.2.1), that admits the representation*

$$v = u + A_j \chi_{\infty,j} U_j^-,$$

where $A_j = const$, $u \in V^2_{\gamma,\delta}(G_j)$ and δ is sufficiently small. Moreover the estimate

$$\|u; V^2_{\gamma,\delta}(G_j)\| + |A_j| \leq c\|f; V^0_{\gamma,\delta}(G_j)\| \tag{8.3.6}$$

holds with a constant c independent of f. If the function f vanishes in a neighborhood of O_j, then the solution v in G_1 admits the decomposition

$$v(x,y,z) = a_1 r_1^{-1/2} \widetilde{J}_{\mu_1+1/2}(kr_1)\Phi_1(-\varphi_1) + O\left(r_1^{\mu_2+1/2}\right), \quad r_1 \to 0,$$

and for the solution in G_2 there holds

$$v(x,y) = a_2 r_2^{-1/2} \widetilde{J}_{\mu_1+1/2}(kr_2)\Phi_1(\varphi_2) + O\left(r_2^{\mu_2+1/2}\right), \quad r_2 \to 0,$$

where a_j are certain constants and μ_l are the same as in the preceding proposition.

8.3.2 Second Kind Limit Problems

In the domains Ω_j, $j = 1, 2$, introduced in Sect. 8.2, we consider the boundary value problems

$$\begin{aligned} -\triangle w(\xi_j, \eta_j, \zeta_j) &= F(\xi_j, \eta_j, \zeta_j), \quad (\xi_j, \eta_j, \zeta_j) \in \Omega_j; \\ w(\xi_j, \eta_j, \zeta_j) &= 0, \quad (\xi_j, \eta_j, \zeta_j) \in \partial\Omega_j, \end{aligned} \tag{8.3.7}$$

which are called second kind limit problems; here (ξ_j, η_j, ζ_j) denote Cartesian coordinates with origin at O_j.

Let $\rho_j = \mathrm{dist}((\xi_j, \eta_j, \zeta_j), O_j)$ and let $\psi_{0,j}$, $\psi_{\infty,j}$ be smooth real functions on $\overline{\Omega}_j$ such that $\psi_{0,j} = 1$ for $\rho_j < N/2$, $\psi_{0,j} = 0$ for $\rho_j > N$, and $\psi_{0,j}^2 + \psi_{\infty,j}^2 = 1$ with sufficiently large positive N. For $\gamma \in \mathbb{R}$ and $l = 0, 1, 2$, the space $V_\gamma^l(\Omega_j)$ is the completion in the norm

$$\|v; V_\gamma^l(\Omega_j)\| = \left(\int_{\Omega_j} \sum_{|\alpha|=0}^{l} \left(\psi_{0,j}^2 + \psi_{\infty,j}^2 \rho_j^{2(\gamma-l+|\alpha|)}\right)|\partial^\alpha v|^2 \, d\xi_j d\eta_j d\zeta_j\right)^{1/2} \tag{8.3.8}$$

of the set $C_c^\infty(\overline{\Omega}_j)$ of smooth functions with compact support in $\overline{\Omega}_j$. The next proposition is a corollary of Theorem 4.3.6 in [37].

Proposition 8.3.3 *Assume that* $|\gamma - 1| < \mu_1 + 1/2$. *Then, for* $F \in V_\gamma^0(\Omega_j)$, *there exists a unique solution* $w \in V_\gamma^2(\Omega_j)$ *of the problem* (8.3.7) *such that*

$$\|w; V_\gamma^2(\Omega_j)\| \le c\|F; V_\gamma^0(\Omega_j)\|, \tag{8.3.9}$$

with a constant c independent of F. If $F \in C_c^\infty(\overline{\Omega}_j)$, *then the function w is smooth on* $\overline{\Omega}_j$ *and admits the representation*

$$w(\xi_j, \eta_j, \zeta_j) = \begin{cases} d_j^l \rho_j^{-\mu_1-1}\Phi_1(-\varphi_j) + O\left(\rho_j^{-\mu_2-1}\right), & \xi_j < 0, \\ d_j^r \rho_j^{-\mu_1-1}\Phi_1(\varphi_j) + O\left(\rho_j^{-\mu_2-1}\right), & \xi_j > 0, \end{cases} \tag{8.3.10}$$

with $\rho_j \to \infty$; *here* (ρ_j, φ_j) *are polar coordinates on* Ω_j *centered at* O_j *while* μ_l *and* Φ_1 *are the same as in Proposition* 8.3.1. *The constants* α_j *and* β_j *are given by*

$$d_j^l = -(F, w_j^l)_\Omega, \qquad d_j^r = -(F, w_j^r)_\Omega,$$

where w_j^l *and* w_j^r *are unique solutions to the homogeneous problem* (8.3.7) *that satisfy, for* $\rho_j \to \infty$, *the conditions*

$$w_j^l = \begin{cases} \left(\rho_j^{\mu_1} + \alpha\rho_j^{-\mu_1-1}\right)\Phi_1(-\varphi_j) + O\left(\rho_j^{-\mu_2-1}\right), & \xi_j < 0; \\ \beta\rho_j^{-\mu_1-1}\Phi_1(\varphi_j) + O\left(\rho_j^{-\mu_2-1}\right), & \xi_j > 0; \end{cases} \tag{8.3.11}$$

$$w_j^r = \begin{cases} \beta\rho_j^{-\mu_1-1}\Phi_1(-\varphi_j) + O\left(\rho_j^{-\mu_2-1}\right), & \xi_j < 0; \\ \left(\rho_j^{\mu_1} + \alpha\rho_j^{-\mu_1-1}\right)\Phi_1(\varphi_j) + O\left(\rho_j^{-\mu_2-1}\right), & \xi_j > 0. \end{cases} \tag{8.3.12}$$

The coefficients α *and* β *depend only on the domain* Ω.

8.4 Special Solutions of Limit Problems

In each domain G_j, $j = 0, 1, 2$, we introduce special solutions to the homogeneous problems (6.2.1). Such solutions will be needed in the next section for constructing the asymptotics of a wave function. The special solutions V_j, $\mathbf{v}_j$ of the limit problems in G_j, $j = 1, 2$, were introduced and studied in Chap. 6. Remember that the following expansions are valid:

$$V_1(x, y, z) = \begin{cases} U_1^+(x, y, z) + S_{11}^0(k)U_1^-(x, y, z) + O(\exp(\delta x)), & x \to -\infty, \\ s_1(k)r_1^{-1/2}\widetilde{J}_{\mu_1+1/2}(kr_1)\Phi_1(-\varphi_1), & r_1 \to 0; \end{cases} \tag{8.4.1}$$

$$V_2(x, y, z) = \begin{cases} U_2^+(x, y, z) + S_{22}^0(k)U_2^-(x, y, z) + O(\exp(-\delta x)), & x \to +\infty, \\ s_2(k)r_2^{-1/2}\widetilde{J}_{\mu_1+1/2}(kr_2)\Phi_1(\varphi_1), & r_2 \to 0. \end{cases} \tag{8.4.2}$$

and

$$\mathbf{v}_1(x, y, z) = \begin{cases} r_1^{-1/2}\big(\widetilde{N}_{\mu_1+1/2}(kr_1) + a_1\widetilde{J}_{\mu_1+1/2}(kr_1)\big)\Phi_1(-\varphi_1) + O(r_1^{\mu_2}), & r_1 \to 0, \\ A_1U_1^-(x, y, z) + O(e^{\delta x}), & x \to -\infty, \end{cases} \tag{8.4.3}$$

$$\mathbf{v}_2(x, y, z) = \begin{cases} r_2^{-1/2}\big(\widetilde{N}_{\mu_1+1/2}(kr_2) + a_2\widetilde{J}_{\mu_1+1/2}(kr_2)\big)\Phi_1(\varphi_2) + O(r_2^{\mu_2}), & r_2 \to 0, \\ A_2U_2^-(x, y, z) + O(e^{-\delta x}), & x \to +\infty, \end{cases} \tag{8.4.4}$$

where $\widetilde{J}_\mu$ is the same function as in Propositions 8.3.1 and 8.3.2 and the constant A_j depends only on the domain G_j.

Lemma 8.4.1 *The equalities* $|A_j|^2 = 2\,\mathrm{Im}\,a_j$, $A_j = i\overline{s_j}S_{jj}^0$ *hold.*

Let $k_{e,\pm}^2$ be a simple eigenvalue of the problem (8.3.2) in the resonator G_0; $v_e^\pm$ is an eigenfunction corresponding to $k_{e,\pm}^2$ and normalized by the condition $\int_{G_0} |v_e^\pm|^2 dx\,dy\,dz = 1$. By virtue of Proposition 8.3.1

$$v_e^\pm(x, y, z) \sim \begin{cases} b_1^\pm r_1^{-1/2}\widetilde{J}_{\mu_1+1/2}(k_{0,\pm}r_1)\Phi_1(\varphi_1), & r_1 \to 0, \\ b_2^\pm r_2^{-1/2}\widetilde{J}_{\mu_1+1/2}(k_{0,\pm}r_2)\Phi_1(-\varphi_2), & r_2 \to 0. \end{cases} \tag{8.4.5}$$

We consider that $b_j^\pm \neq 0$. If $H = 0$, then it is true, for instance, for the eigenfunctions corresponding to the minimal eigenvalue of the resonator. For nonzero H, this condition can be violated. For k^2 in a punctured neighborhood of $k_{e,\pm}^2$ separated from the other eigenvalues, we introduce the solutions $v_{0j}^\pm$ to the homogeneous problem (8.3.2) by the relations

$$\mathbf{v}_{0j}^\pm(x, y, z) = \Theta(r_j)v_j(r_j, \varphi_j) + \widetilde{v}_{0j}^\pm(x, y, z), \qquad j = 1, 2, \tag{8.4.6}$$

where $t \mapsto \Theta(t)$ is a cut-off function on $\mathbb{R}$ equal to 1 for $t < \delta/2$ and to 0 for $t > \delta$ with a small positive δ, v_j are defined by

$$v_1(r_1, \varphi_1) = r_1^{-1/2} \widetilde{N}_{\mu_1+1/2}(kr_1)\Phi_1(\varphi_1), \qquad v_2(r_2, \varphi_2) = r_2^{-1/2} \widetilde{N}_{\mu_1+1/2}(kr_2)\Phi_1(-\varphi_2),$$

$\widetilde{N}_\mu$ is the Neumann function multiplied by such a constant that

$$\widetilde{N}_\mu(kr) = r^{-\mu} + o(r^{-\mu}),$$

and $\widetilde{v}_{0j}^{\pm}$ is a bounded solution to the problem (8.3.2) with $f_j(x, y, z) = -[\triangle, \Theta(r_j)]$ $v_j(r_j, \varphi_j)$.

Lemma 8.4.2 *In a neighborhood $V \subset \mathbb{C}$ of $k_{e,\pm}^2$ containing no eigenvalues of the problem (8.3.2) in G_0 distinct from $k_{e,\pm}^2$, the equalities $\mathbf{v}_{0j}^{\pm} = -\overline{b_j^{\pm}}(k^2 - k_{e,\pm}^2)^{-1} v_e^{\pm} + \widehat{\mathbf{v}}_{0j}^{\pm}$ hold, where $b_j^{\pm}$ are the same as in (8.4.5) and the functions $\widehat{\mathbf{v}}_{0j}^{\pm}$ are analytic in $k^2 \in V$.*

Proof We first verify that $(\mathbf{v}_{0j}^{\pm}, v_e^{\pm})_{G_0} = -\overline{b_j^{\pm}}/(k^2 - k_{e,\pm}^2)$, where $\mathbf{v}_{0j}^{\pm}$ are defined by (8.4.6). We have

$$(\triangle \mathbf{v}_{0j}^{\pm} + k^2 \mathbf{v}_{0j}^{\pm}, v_e^{\pm})_{G_\delta} - (\mathbf{v}_{0j}^{\pm}, \triangle v_e^{\pm} + k^2 v_e^{\pm})_{G_\delta} = -(k^2 - k_{e,\pm}^2)(\mathbf{v}_{0j}^{\pm}, v_e^{\pm})_{G_\delta};$$

the domain G_δ is obtained from G_0 by cutting out the balls of radius δ with centers at O_1 and O_2. Applying the Green formula in the same way as in the proof of Lemma 6.3.1, we arrive at $-(k^2 - k_{e,\pm}^2)(\mathbf{v}_{0j}^{\pm}, v_e^{\pm})_{G_\delta} = \overline{b_j^{\pm}} + o(1)$. It remains to let $\delta \to 0$.

Since $k_{e,\pm}^2$ is a simple eigenvalue, we have

$$\widetilde{v}_{0j}^{\pm} = \frac{B_j^{\pm}(k^2)}{k^2 - k_{e,\pm}^2} v_0^{\pm} + \widehat{v}_{0j}^{\pm}, \tag{8.4.7}$$

where $B_j^{\pm}(k^2)$ does not depend on (x, y, z), and $\widehat{v}_{0j}^{\pm}$ are some functions analytic with respect to k^2 near the point $k^2 = k_{e,\pm}^2$. Multiplying (8.4.6) by $v_e^{\pm}$ and taking into account (8.4.7), the obtained formula for $(\mathbf{v}_{0j}^{\pm}, v_e^{\pm})_{G_0}$, and the condition $(v_e^{\pm}, v_e^{\pm})_{G_0} = 1$, we get the equality $B_j^{\pm}(k^2) = -\overline{b_j^{\pm}} + (k^2 - k_{e,\pm}^2)\widetilde{B}_j^{\pm}(k^2)$, where $\widetilde{B}_j^{\pm}$ are some analytic functions. Together with (8.4.7) that leads to the required statement. □

In view of Proposition 8.3.1,

$$\mathbf{v}_{01}^{\pm}(x, y) \sim \begin{cases} r_1^{-1/2}\big(\widetilde{N}_{\mu_1+1/2}(kr_1) + c_{11}^{\pm}(k)\widetilde{J}_{\mu_1+1/2}(kr_1)\big)\Phi(\varphi_1), & r_1 \to 0, \\ c_{12}^{\pm}(k) r_2^{-1/2} \widetilde{J}_{\mu_1+1/2}(kr_2)\Phi(-\varphi_2), & r_2 \to 0, \end{cases} \tag{8.4.8}$$

$$\mathbf{v}_{02}^{\pm}(x,y) \sim \begin{cases} c_{21}^{\pm}(k) r_1^{-1/2} \widetilde{J}_{\mu_1+1/2}(kr_1)\Phi(\varphi_1), & r_1 \to 0, \\ r_2^{-1/2}\big(\widetilde{N}_{\mu_1+1/2}(kr_2) + c_{22}^{\pm}(k)\widetilde{J}_{\mu_1+1/2}(kr_2)\big)\Phi(-\varphi_2), & r_2 \to 0. \end{cases} \tag{8.4.9}$$

According to Lemma 8.4.2 and relations (8.4.5),

$$c_{pq}^{\pm}(k) = -\frac{\overline{b_p^{\pm}} b_q^{\pm}}{k^2 - k_{e,\pm}^2} + \widehat{c}_{pq}^{\pm}(k), \tag{8.4.10}$$

where $\widehat{c}_{pq}^{\pm}$ analytically depends on k^2 nearby $k_{e,\pm}^2$.

Lemma 8.4.3 *If* $\mathbf{v}_{01}^{\pm}$ *and* $\mathbf{v}_{02}^{\pm}$ *in* (8.4.8) *and* (8.4.9) *make sense for a number k, then* $c_{12}^{\pm}(k) = \overline{c_{21}^{\pm}(k)}$.

Proof It suffices to apply the Green formula to $\mathbf{v}_{01}^{\pm}$ and $\mathbf{v}_{02}^{\pm}$ in the same domain G_δ as in the proof of Lemma 6.4.1, to use (8.4.8) and (8.4.9), and to let δ tend to 0. □

8.5 Asymptotic Formulas

In Sect. 8.5.1, we present an asymptotic formula for a wave function (see (8.5.1)), explain its structure, and describe the solutions of the first kind limit problems involved in the formula. We complete deriving the formula (8.5.1) in Sect. 8.5.2, where we describe the involved solutions of the second kind limit problems and calculate some coefficients in the expressions for the solutions of the first kind problems. In Sect. 8.5.3, when analysing the expression for $\widetilde{S}_{12}$ obtained in Sect. 8.5.2, we derive formal asymptotics of the resonant tunneling characteristics. Note that the remainders in (8.5.24)–(8.5.27) have arisen at the intermediate stage of consideration during simplification of the principal part of the asymptotics; they are not the remainders in the final asymptotic formulas. The "final" remainders are estimated in Sect. 8.6 (see Theorem 8.6.3). For ease of notation, we drop the symbol "±" in this section, meaning that we will deal with any one of the Eq. (8.2.2).

8.5.1 The Asymptotics of a Wave Function

In the waveguide $G(\varepsilon)$, we consider the scattering of the wave $U(x,y,z) = e^{i\nu_1 x}\Psi_1(y,z)$ incoming from $-\infty$ (see (5.1.5)). The corresponding wave function admits the representation

$$\begin{aligned} u(x,y,z;\varepsilon) &= \chi_{1,\varepsilon}(x,y,z)v_1(x,y,z;\varepsilon) \\ &\quad + \Theta(r_1)w_1(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1,\varepsilon^{-1}z_1;\varepsilon) + \chi_{0,\varepsilon}(x,y,z)v_0(x,y,z;\varepsilon) \\ &\quad + \Theta(r_2)w_2(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2,\varepsilon^{-1}z_2;\varepsilon) + \chi_{2,\varepsilon}(x,y,z)v_2(x,y,z;\varepsilon) + R(x,y,z;\varepsilon). \end{aligned} \tag{8.5.1}$$

Let us explain the notation and structure of this formula. When constructing the asymptotics, we first describe the behavior of the wave function u outside the narrows, approximating u by the solutions v_j of the homogeneous problems (6.2.1) and (8.3.2) in G_j. As v_j, we take certain linear combinations of the special solutions introduced in the preceding section; in doing so, we subject v_1 and v_2 to the same radiation conditions at infinity as u:

$$\begin{aligned} v_1(x,y,z;\varepsilon) &= V_1(x,y,z) + \widetilde{C}_{11}\mathbf{v}_1(x,y,z) \\ &\qquad \sim U_1^+(x,y,z) + \widetilde{S}_{11}(\varepsilon)U_1^-(x,y,z), \quad x \to -\infty; \qquad (8.5.2)\\ v_0(x,y,z;\varepsilon) &= C_{12}(\varepsilon)\mathbf{v}_{01}(x,y,z) + C_{13}(\varepsilon)\mathbf{v}_{02}(x,y,z); \qquad (8.5.3)\\ v_2(x,y,z;\varepsilon) &= C_{14}\mathbf{v}_2(x,y,z) \sim \widetilde{S}_{12}(\varepsilon)U_2^-(x,y,z), \quad x \to +\infty; \qquad (8.5.4) \end{aligned}$$

for the time being the approximations $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$ for the entries $S_{11}(\varepsilon)$, $S_{12}(\varepsilon)$ of the scattering matrix and the coefficients $C_{1j}(\varepsilon)$ are unknown. Here $\chi_{j,\varepsilon}$ stand for the cut-off functions defined by the equalities

$$\begin{aligned} &\chi_{1,\varepsilon}(x,y,z) = (1-\Theta(r_1/\varepsilon))\,\mathbf{1}_{G_1}(x,y,z), \qquad \chi_{2,\varepsilon}(x,y,z) = (1-\Theta(r_2/\varepsilon))\,\mathbf{1}_{G_2}(x,y,z),\\ &\chi_{0,\varepsilon}(x,y,z) = (1-\Theta(r_1/\varepsilon)-\Theta(r_2/\varepsilon))\,\mathbf{1}_{G_0}(x,y,z), \qquad (8.5.5) \end{aligned}$$

where $r_j = \sqrt{x_j^2+y_j^2+z_j^2}$ and (x_j, y_j, z_j) are the coordinates of a point (x, y, z) in the system with the origin shifted to O_j; $\mathbf{1}_{G_j}$ is the indicator of the set G_j (equal to 1 in G_j and 0 outside G_j); $\Theta(\rho)$ is the a cut-off function equal to 1 for $0 \leqslant \rho \leqslant \delta/2$ and 0 for $\rho \geqslant \delta$ with a fixed sufficiently small positive δ. Thus $\chi_{j,\varepsilon}$ are defined on the whole waveguide $G(\varepsilon)$ as well as the functions $\chi_{j,\varepsilon}v_j$ in (8.5.1).

When substituting $\sum_{j=0}^{2}\chi_{j,\varepsilon}v_j$ in (8.2.2), we obtain a discrepancy in the right-hand side of the Helmholtz equation supported near the narrows. We compensate the principal part of the discrepancy by making use of the second kind limit problems. In more detail, we rewrite the discrepancy supported near O_j in the coordinates $(\xi_j, \eta_j, \zeta_j) = (\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j)$ in the domain Ω_j and take it as right-hand side for the Laplace equation. Then we rewrite the solution w_j of the corresponding problem (8.3.7) in the coordinates (x_2, y_2, z_2) and multiply it by the cut-off function. As a result, there arises the term $\Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j; \varepsilon)$ in (8.5.1).

The existence of solutions w_j vanishing as $O(\rho_j^{-\mu_1-1})$ at infinity follows from Proposition 8.3.3 (see (8.3.10)). However, choosing such solutions and then substituting (8.5.1) in (8.2.2), we obtain a discrepancy of high order that has to be compensated again. Therefore, we require $w_j = O(\rho_j^{-\mu_2-1})$ as $\rho_j \to \infty$. According to Proposition 8.3.3, such a solution exists if the right-hand side of the problem (8.3.7) satisfies the additional conditions

$$(F, w_j^l)_{\Omega_j} = 0, \qquad (F, w_j^r)_{\Omega_j} = 0.$$

Such conditions (two at each narrow) uniquely define the coefficients $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$, and $C_{11}(\varepsilon), \dots, C_{14}(\varepsilon)$. The remainder $R(x, y, z; \varepsilon)$ is small in comparison with the principal part of (8.5.1) as $\varepsilon \to 0$.

8.5.2 Formulas for $\widetilde{S}_{11}$, $\widetilde{S}_{12}$, and $C_{11}, \dots, C_{14}$

We are now going to define the right-hand side F_j of problem (8.3.7) and to find $\widetilde{S}_{11}(\varepsilon)$, $\widetilde{S}_{12}(\varepsilon)$, and $C_{11}(\varepsilon), \dots, C_{14}(\varepsilon)$. We substitute $\chi_{1,\varepsilon} v_1$ in (8.2.2) and obtain the discrepancy

$$(-\Delta - k^2)\chi_{1,\varepsilon} v_1 = -[\Delta, \chi_{\varepsilon,1}]v_1 + \chi_{\varepsilon,1}(-\Delta - k^2)v_1 = -[\Delta, 1 - \Theta(\varepsilon^{-1} r_1)]v_1,$$

distinct from zero only near the point O_1, where v_1 can be replaced by the asymptotics; the boundary condition (8.2.3) is fulfilled. According to (8.5.2) and (8.4.3),

$$\begin{aligned} v_1(x, y, z; \varepsilon) = r_1^{-1/2}\big(a_1^-(\varepsilon)\widetilde{N}_{\mu_1+1/2}(kr_1) + a_1^+(\varepsilon)\widetilde{J}_{\mu_1+1/2}(kr_1)\big)& \\ \times \Phi_1(-\varphi_1) + O(r_1^{\mu_2}), \quad r_1 \to 0,& \end{aligned}$$

with

$$a_1^-(\varepsilon) = C_{11}, \qquad a_1^+ = s_1 + C_{11} a_1. \tag{8.5.6}$$

We single out the principal part of each term and put $\rho_1 = r_1/\varepsilon$, then

$$\begin{aligned} (-\Delta - k^2)\chi_{\varepsilon,1} v_1 &\sim -[\Delta, 1 - \Theta(\varepsilon^{-1} r_1)]\left(a_1^- r_1^{-\mu_1 - 1} + a_1^+ r_1^{\mu_1}\right)\Phi_1(-\varphi_1) \\ &= -\varepsilon^{-2}[\Delta_{(\rho_1,\varphi_1)}, 1 - \Theta(\rho_1)]\left(a_1^- \varepsilon^{-\mu_1 - 1}\rho_1^{-\mu_1 - 1} + a_1^+ \varepsilon^{\mu_1}\rho_1^{\mu_1}\right)\Phi_1(-\varphi_1). \end{aligned} \tag{8.5.7}$$

In the same way, using (8.5.3) and (8.4.8)–(8.4.9), we obtain the principal part of the discrepancy given by $\chi_{\varepsilon,2} v_2$ supported near O_1:

$$(-\Delta - k^2)\chi_{\varepsilon,1} v_1 \sim -\varepsilon^{-2}[\Delta_{(\rho_1,\varphi_1)}, 1 - \Theta(\rho_1)]\left(b_1^- \varepsilon^{-\mu_1 - 1}\rho_1^{-\mu_1 - 1} + b_1^+ \varepsilon^{\mu_1}\rho_1^{\mu_1}\right)\Phi_1(\varphi_1), \tag{8.5.8}$$

where

$$b_1^- = C_{12}(\varepsilon), \qquad b_1^+ = C_{12}(\varepsilon)c_{11} + C_{13}(\varepsilon)c_{21}. \tag{8.5.9}$$

As right-hand side F_1 of the problem (6.2.6) in Ω_1, we take the function

$$\begin{aligned} F_1(\xi_1, \eta_1, \zeta_1) = [\Delta, \theta^-]&\left(a_1^- \varepsilon^{-\mu_1 - 1}\rho_1^{-\mu_1 - 1} + a_1^+ \varepsilon^{\mu_1}\rho_1^{\mu_1}\right)\Phi_1(-\varphi_1) \\ + [\Delta, \theta^+]&\left(b_1^- \varepsilon^{-\mu_1 - 1}\rho_1^{-\mu_1 - 1} + b_1^+ \varepsilon^{\mu_1}\rho_1^{\mu_1}\right)\Phi_1(\varphi_1), \end{aligned} \tag{8.5.10}$$

where θ^+ (respectively θ^-) stands for the function $1-\Theta$ first restricted to the domain $\xi_1 > 0$ (respectively $\xi_1 < 0$) and then extended by zero to the whole domain Ω_1. Let w_1 be the corresponding solution; then the term $\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1, \varepsilon^{-1}z_1; \varepsilon)$ in (8.5.1) being substituted in (8.2.2) compensates the discrepancies (8.5.7)–(8.5.8).

In a similar manner, making use of (8.5.3)–(8.5.4), (8.4.8)–(8.4.9), and (8.4.4), we find the right-hand side of the problem (8.3.7) for $j = 2$:

$$\begin{aligned} F_2(\xi_2, \eta_2, \zeta_2) &= [\Delta, \theta^-]\left(a_2^-\varepsilon^{-\mu_1-1}\rho_2^{-\mu_1-1} + a_2^+\varepsilon^{\mu_1}\rho_2^{\mu_1}\right)\Phi_1(-\varphi_2)\\ &+ [\Delta, \theta^+]\left(b_2^-\varepsilon^{-\mu_1-1}\rho_2^{-\mu_1-1} + b_2^+\varepsilon^{\mu_1}\rho_2^{\mu_1}\right)\Phi_1(\varphi_2);\\ a_2^-(\varepsilon) &= C_{13}(\varepsilon), \qquad a_2^+(\varepsilon) = C_{12}(\varepsilon)c_{12} + C_{13}(\varepsilon)c_{22},\\ b_2^-(\varepsilon) &= C_{14}(\varepsilon), \qquad b_2^+(\varepsilon) = C_{14}(\varepsilon)a_2. \end{aligned} \tag{8.5.11}$$

Lemma 8.5.1 *If the solution w_j of the problem* (8.3.7) *with right-hand side*

$$\begin{aligned} F_j(\xi_j, \eta_j, \zeta_j) = [\Delta, \theta^-]&\left(a_j^-\varepsilon^{-\mu_1-1}\rho_j^{-\mu_1-1} + a_j^+\varepsilon^{\mu_1}\rho_j^{\mu_1}\right)\Phi_1(-\varphi_j)\\ &+ [\Delta, \theta^+]\left(b_j^-\varepsilon^{-\mu_1-1}\rho_j^{-\mu_1-1} + b_j^+\varepsilon^{\mu_1}\rho_j^{\mu_1}\right)\Phi_1(\varphi_j), \end{aligned}$$

$j = 1, 2$, *admits the estimate $O(\rho_j^{-\mu_2-1})$ as $\rho_j \to \infty$, then*

$$a_j^-\varepsilon^{-\mu_1-1} - \alpha a_j^+\varepsilon^{\mu_1} - \beta b_j^+\varepsilon^{\mu_1} = 0, \qquad b_j^-\varepsilon^{-\mu_1-1} - \alpha b_j^+\varepsilon^{\mu_1} - \beta a_j^+\varepsilon^{\mu_1} = 0, \tag{8.5.12}$$

where α and β are the coefficients in (8.3.11)–(8.3.12).

Remark 8.5.2 The solutions w_j mentioned in Lemma 8.5.1 can be written as linear combinations of certain model functions independent of ε. We present the corresponding expressions, which will be needed in the next section for estimating the remainders of asymptotic formulas. Let w_j^l and w_j^r be the solutions to problem (8.3.7) defined by (8.3.11)–(8.3.12) and θ^+, θ^- the same cut-off functions as in (8.5.10). We set

$$\begin{aligned} \mathbf{w}_j^l &= w_j^l - \theta^-\left(\rho_j^{\mu_1} + \alpha\rho_j^{-\mu_1-1}\right)\Phi_1(-\varphi_j) - \theta^+\beta\rho_j^{-\mu_1-1}\Phi_1(\varphi_j),\\ \mathbf{w}_j^r &= w_j^r - \theta^-\beta\rho_j^{-\mu_1-1}\Phi_1(-\varphi_j) - \zeta^+\left(\rho_j^{\mu_1} + \alpha\rho_j^{-\mu_1-1}\right)\Phi_1(\varphi_j). \end{aligned}$$

A straightforward verification shows that

$$w_j = a_j^+\varepsilon^{\mu_1}\mathbf{w}_j^l + b_j^+\varepsilon^{\mu_1}\mathbf{w}_j^r. \tag{8.5.13}$$

It is convenient to write (8.5.12) in the form

$$(a_j^-, b_j^-) = (a_j^+, b_j^+)\,\Lambda\,\varepsilon^{2\mu_1+1}, \qquad \Lambda = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \tag{8.5.14}$$

We use (8.5.6) and (8.5.9) to transform (8.5.14) with $j = 1$ to the equality

$$(C_{11}, C_{12}) = (s_1 + C_{11}a_1, C_{12}c_{11} + C_{13}c_{21})\,\Lambda\,\varepsilon^{2\mu_1+1}. \tag{8.5.15}$$

For $j = 2$, taking (8.5.11) into account, we reduce (8.5.14) to

$$(C_{13}, C_{14}) = (C_{12}c_{12} + C_{13}c_{22}, C_{14}\,a_2)\,\Lambda\,\varepsilon^{2\mu_1+1}. \tag{8.5.16}$$

Setting $\boldsymbol{\Lambda} = \operatorname{diag}\{\Lambda, \Lambda\}$,

$$a = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \tag{8.5.17}$$

and combining the above relations for C_{1j}, we obtain

$$(C_{11}, C_{12}, C_{13}, C_{14}) = (s_1, 0, 0, 0)\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1} + (C_{11}, C_{12}, C_{13}, C_{14})\,a\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1},$$

hence

$$(C_{11}, C_{12}, C_{13}, C_{14})(I - a\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1}) = (s_1, 0, 0, 0)\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1}. \tag{8.5.18}$$

Let us calculate the inverse matrix for $I - a\boldsymbol{\Lambda}\varepsilon^{2\mu_1+1}$, assuming ε to be sufficiently small. From (8.4.10) it follows that

$$a(k) = -\frac{\mathbf{b}^*\mathbf{b}}{k^2 - k_e^2} + \widehat{a}(k),$$

where $\mathbf{b} = (0,\ b_1,\ b_2,\ 0)$ and the matrix $\widehat{a}$ is analytic near $k = k_e$ and defined by (8.5.17), whereas c_{pq} is replaced for $\widehat{c}_{pq}$. We have

$$\begin{aligned}
&(I - a\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1})^{-1} \\
&\quad = (I - \widehat{a}\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1})^{-1}\left(I - \frac{\mathbf{b}^*\mathbf{b}\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1}(I - \widehat{a}\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1})^{-1}}{k^2 - k_e^2 + \langle \mathbf{b}\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1}(I - \widehat{a}\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1})^{-1}, \mathbf{b}\rangle}\right).
\end{aligned}$$

This leads to

$$\begin{aligned}
(C_{11}, C_{12}, C_{13}, C_{14}) &= (s_1, 0, 0, 0)\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1}(I - a\,\boldsymbol{\Lambda}\,\varepsilon^{2\mu_1+1})^{-1} \\
&= (s_1, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right),
\end{aligned} \tag{8.5.19}$$

where $\mathbf{b} = (0,\ b_1,\ b_2,\ 0)$, $D = \mathbf{\Lambda}\,\varepsilon^{2\mu_1+1}(I - \widehat{a}\,\mathbf{\Lambda}\,\varepsilon^{2\mu_1+1})^{-1}$.

We now seek an approximation to the entries of the first row (S_{11}, S_{12}) of the scattering matrix. By virtue of (8.5.2) and (8.5.4),

$$(\widetilde{S}_{11}, \widetilde{S}_{12}) = (S^0_{11} + C_{11}A_1, C_{14}A_2). \tag{8.5.20}$$

We set

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad s = \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 \end{pmatrix};$$

$S^0 = \operatorname{diag}(S^0_{11}, S^0_{22})$; then, by Lemma 8.4.1, $A = is^*S^0$. In view of (8.5.20) and (8.5.19), we obtain

$$\begin{aligned}(\widetilde{S}_{11}, \widetilde{S}_{12}) &= (S^0_{11}, 0) + (C_{11}, C_{12}, C_{13}, C_{14})A \\ &= (S^0_{11}, 0) + i(s_1, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0.\end{aligned} \tag{8.5.21}$$

An approximation to the second row of the scattering matrix is of the form

$$(\widetilde{S}_{21}, \widetilde{S}_{22}) = (0, S^0_{22}) + i(0, 0, 0, s_2)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^*S^0. \tag{8.5.22}$$

Lemma 8.5.3 *The matrix $\widetilde{S}(\varepsilon)$ is unitary.*

8.5.3 Asymptotics for Resonant Tunneling Characteristics

The solutions of the first limit problems involved in (8.5.1) are defined for the complex k^2 as well. The expressions (8.5.21)–(8.5.22) obtained for $\widetilde{S}(\varepsilon)$ have a pole at k_p^2 in the lower half-plane. To find k_p^2, we equate $k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle$ to zero and solve the equation for $k^2 - k_e^2$:

$$k^2 - k_e^2 = -\langle \mathbf{b}D, \mathbf{b}\rangle = -\varepsilon^{2\pi/\omega}\langle \mathbf{b}\mathbf{\Lambda}\,(I - \widehat{a}\,\mathbf{\Lambda}\,\varepsilon^{2\pi/\omega})^{-1}, \mathbf{b}\rangle. \tag{8.5.23}$$

Since the right-hand side of this equation behaves as $O(\varepsilon^{2\mu_1+1})$ for $\varepsilon \to 0$, its solution can be found by the successive approximation method. Considering the formulas $\operatorname{Im} a_j = |s_j|^2/2$, which follow from the waveguide symmetry and Lemma 8.4.1, and discarding the lower order terms, we get $k_p^2 = k_r^2 - ik_i^2$, where

$$k_r^2 = k_e^2 - \alpha(|b_1|^2 + |b_2|^2)\varepsilon^{2\mu_1+1} + O(\varepsilon^{4\mu_1+2}), \tag{8.5.24}$$

$$k_i^2 = \frac{1}{2}\beta^2(|b_1|^2 + |b_2|^2)|s_1(k_e^2)|^2\varepsilon^{4\mu_1+2} + O(\varepsilon^{6\mu_1+3}).$$

From (8.5.21) and (8.5.22), we obtain

$$\begin{aligned}
\widetilde{S}(k,\varepsilon) &= S^0(k) + is(k)\mathbf{\Lambda}\, s^*(k)S^0(k)\varepsilon^{2\mu_1+1} \\
&\quad - i\frac{s(k)\mathbf{\Lambda}\,\mathbf{b}^*\mathbf{b}\,\mathbf{\Lambda}\, s^*(k)S^0(k)}{k^2-k_p^2}\varepsilon^{4\mu_1+2} + O\left(\frac{\varepsilon^{6\mu_1+3}}{k^2-k_p^2}\right) \\
&= \begin{pmatrix} S_{11}^0(k) & 0 \\ 0 & S_{22}^0(k) \end{pmatrix} + i\begin{pmatrix} |s_1(k)|^2\alpha_1 S_{11}^0(k) & 0 \\ 0 & |s_2(k)|^2\alpha_2 S_{22}^0(k) \end{pmatrix}\varepsilon^{2\mu_1+1} \\
&\quad - \frac{i}{k^2-k_p^2}\begin{pmatrix} |s_1(k)|^2|b_1|^2\beta^2 S_{11}^0(k) & s_1(k)\overline{s_2(k)}\,\overline{b_1}b_2\beta^2 S_{22}^0(k) \\ s_2(k)\overline{s_1(k)}b_1\overline{b_2}\beta^2 S_{11}^0(k) & |s_2(k)|^2|b_2|^2\beta^2 S_{22}^0(k) \end{pmatrix}\varepsilon^{4\mu_1+2} \\
&\quad + O\left(\frac{\varepsilon^{6\mu_1+3}}{k^2-k_p^2}\right).
\end{aligned}$$

Let $k^2 - k_e^2 = O(\varepsilon^{2\mu_1+1})$, then $c\varepsilon^{4\mu_1+2} \le |k^2-k_p^2| \le c\varepsilon^{2\mu_1+1}$, $s_j(k) = s_j(k_e) + O(\varepsilon^{2\mu_1+1})$, $S_{jj}^0(k) = S_{jj}^0(k_e) + O(\varepsilon^{2\mu_1+1})$, and

$$\begin{aligned}
\widetilde{S}_{12}(k,\varepsilon) &= -i\varepsilon^{4\mu_1+2}\frac{s_1(k)\overline{s_2(k)}\,\overline{b_1}b_2\beta^2 S_{22}^0(k)}{k^2-k_p^2}\Big(1+O(\varepsilon^{2\mu_1+1})\Big) \\
&= -\frac{\dfrac{s_1(k_e)}{|s_1(k_e)|}\dfrac{\overline{s_2(k_e)}}{|s_2(k_e)|}\dfrac{\overline{b_1}}{|b_1|}\dfrac{b_2}{|b_2|}S_{22}^0(k_e)}{\dfrac{1}{2}\left(\dfrac{|b_1|}{|b_2|}+\dfrac{|b_2|}{|b_1|}\right) - iP\dfrac{k^2-k_r^2}{\varepsilon^{4\mu_1+2}}}\Big(1+O(\varepsilon^{2\mu_1+1})\Big), \qquad (8.5.25) \\
\widetilde{S}_{21}(k,\varepsilon) &= -i\varepsilon^{4\mu_1+2}\frac{\overline{s_1(k)}s_2(k)b_1\overline{b_2}\beta^2 S_{11}^0(k)}{k^2-k_p^2}\Big(1+O(\varepsilon^{2\mu_1+1})\Big) \\
&= -\frac{\dfrac{\overline{s_1(k_e)}}{|s_1(k_e)|}\dfrac{s_2(k_e)}{|s_2(k_e)|}\dfrac{b_2}{|b_2|}\dfrac{\overline{b_2}}{|b_2|}S_{11}^0(k_e)}{\dfrac{1}{2}\left(\dfrac{|b_1|}{|b_2|}+\dfrac{|b_2|}{|b_1|}\right) - iP\dfrac{k^2-k_r^2}{\varepsilon^{4\mu_1+2}}}\Big(1+O(\varepsilon^{2\mu_1+1})\Big),
\end{aligned}$$

where $P = \big(|b_1||b_2|\beta^2|s_1(k_e)|^2\big)^{-1}$. Thus,

$$\widetilde{T}_1(k,\varepsilon) = \widetilde{T}_2(k,\varepsilon) = |\widetilde{S}_{12}|^2 = \frac{1}{\dfrac{1}{4}\left(\dfrac{|b_1|}{|b_2|}+\dfrac{|b_2|}{|b_1|}\right)^2 + P^2\left(\dfrac{k^2-k_r^2}{\varepsilon^{4\mu_1+2}}\right)^2}(1+O(\varepsilon^{2\mu_1+1})). \qquad (8.5.26)$$

The obtained approximation $\widetilde{T}_j$ to the transition coefficient T_j has a peak at $k^2 = k_r^2$ whose width at its half-height is

$$\widetilde{\Upsilon}(\varepsilon) = \left(\frac{|b_1|}{|b_2|} + \frac{|b_2|}{|b_1|}\right) \frac{1}{P} \varepsilon^{4\mu_1+2}. \tag{8.5.27}$$

8.6 Justification of the Asymtotics

As in the previous section, here we omit "±" in the notations and do not specify which equation of (8.2.2) is considered. We return to the full notations in Theorem 8.6.3.

Let us introduce functional spaces for the problem

$$(-i\nabla + \mathbf{A})^2 u \pm Hu - k^2 u = f \quad \text{in } G(\varepsilon), \qquad u = 0 \quad \text{on } \partial G(\varepsilon). \tag{8.6.1}$$

Recall that the functions $\mathbf{A}$ and H are compactly supported, and, besides, they are nonzero only in the resonator at some distance from the narrows. Let Θ be the same function as in (8.5.5) and let the cut-off functions η_j, $j = 0, 1, 2$, be nonzero in G_j and satisfy the relation $\eta_1(x, y) + \Theta(r_1) + \eta_0(x, y) + \Theta(r_2) + \eta_2(x, y) = 1$ in $G(\varepsilon)$. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G(\varepsilon))$ is the completion in the norm

$$\begin{aligned} &\|u; V^l_{\gamma,\delta}(G(\varepsilon))\| \\ &\quad = \Bigg(\int_{G(\varepsilon)} \sum_{|\alpha|=0}^{l} \Bigg(\sum_{j=1}^{2} \Theta^2(r_j)\, (r_j^2 + \varepsilon_j^2)^{\gamma-l+|\alpha|} + \eta_1^2 e^{2\delta|x|} \\ &\qquad\qquad + \eta_0 + \eta_2^2 e^{2\delta|x|}\Bigg) |\partial^\alpha v|^2\, dx\, dy\, dz\Bigg)^{1/2} \end{aligned} \tag{8.6.2}$$

of the set of smooth functions compactly supported on $\overline{G(\varepsilon)}$. Denote by $V^{0,\perp}_{\gamma,\delta}$ the space of function f, analytic in k^2, with values in $V^0_{\gamma,\delta}(G(\varepsilon))$ that satisfy, at $k^2 = k_e^2$, the condition $(\chi_{0,\varepsilon^\sigma} f, v_e)_{G_0} = 0$ with a small $\sigma > 0$.

Proposition 8.6.1 *Let k_r^2 be a resonance, $k_r^2 \to k_e^2$ as $\varepsilon \to 0$, and let $|k^2 - k_r^2| = O(\varepsilon^{2\mu_1+1})$. Let γ satisfy the condition $\mu_1 - 3/2 < \gamma - 1 < \mu_1 + 1/2$, $f \in V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$, and let u be a solution to problem (8.6.1) that admits the representation*

$$u = \widetilde{u} + \eta_1 A_1^- U_1^- + \eta_2 A_2^- U_2^-;$$

here $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma,\delta}(G(\varepsilon))$ for small $\delta > 0$. Then

$$\|\widetilde{u}; V^2_{\gamma,\delta}(G(\varepsilon))\| + |A_1^-| + |A_2^-| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \tag{8.6.3}$$

where c is a constant independent of f and ε.

Proof Step A. First we construct an auxiliary function u_p. As mentioned above, $\widetilde{S}$ has a pole $k_p^2 = k_r^2 - ik_i^2$ (see (8.5.24)). Let us multiply the solutions to the limit problems, involved in (8.5.1), by $g := -(k^2 - k_e^2 + \langle \mathbf{b}D(k), \mathbf{b}\rangle)/\langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle$, put $k = k_p$, and denote the resulting functions by adding the subscript p. In view of (8.5.19) and the equality $(s_1, 0, 0, 0)D\mathbf{b}^* = \langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle$, we get

$$g\,(C_{11}, C_{12}, C_{13}, C_{14})|_{k=k_p} = \mathbf{b}D(k_p) = (b_1\beta, b_1\alpha, b_2\alpha, b_2\beta)\varepsilon^{2\mu_1+1} + O(\varepsilon^{4\mu_1+2}). \tag{8.6.4}$$

This and (8.5.2), (8.5.4) lead to

$$v_{1p}(x, y, z; \varepsilon) = g\,C_{11}|_{k=k_p}\mathbf{v}_1(x, y, z; k_p) = \varepsilon^{2\mu_1+1}\left(b_1\beta + O\left(\varepsilon^{2\mu_1+1}\right)\right)\mathbf{v}_1(x, y, z; k_p), \tag{8.6.5}$$

$$v_{2p}(x, y, z; \varepsilon) = g\,C_{14}|_{k=k_p}\mathbf{v}_2(x, y, z; k_p) = \varepsilon^{2\mu_1+1}\left(b_2\beta + O\left(\varepsilon^{2\mu_1+1}\right)\right)\mathbf{v}_2(x, y, z; k_p);$$

the dependence of k_p on ε is not shown. According to (8.5.3) and Lemma 8.4.2,

$$\begin{aligned} v_{0p}(x, y, z; \varepsilon) = -\frac{(g\,C_{12}\overline{b_1} + g\,C_{13}\overline{b_2})|_{k=k_p}}{k_p^2 - k_e^2}v_e(x, y, z) + g\,C_{12}|_{k=k_p}\widehat{\mathbf{v}}_{01}(x, y, z) \\ + g\,C_{13}|_{k=k_p}\widehat{\mathbf{v}}_{02}(x, y, z). \end{aligned}$$

Taking into account (8.5.19), we obtain

$$\begin{aligned} C_{12}b_1 + C_{13}b_2 &= (C_{11}, C_{12}, C_{13}, C_{14})\mathbf{b}^* \\ &= (s_1, 0, 0, 0)D\mathbf{b}^*\left(1 - \frac{\langle \mathbf{b}D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right) \\ &= (k^2 - k_e^2)\frac{\langle (s_1, 0, 0, 0)D, \mathbf{b}\rangle}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}. \end{aligned} \tag{8.6.6}$$

Hence,

$$\begin{aligned} v_{0p}(x, y, z; \varepsilon) = v_e(x, y, z) + \varepsilon^{2\mu_1+1}(b_1\alpha + O(\varepsilon^{2\mu_1+1}))\widehat{\mathbf{v}}_{01}(x, y, z) \\ + \varepsilon^{2\mu_1+1}(b_2\alpha + O(\varepsilon^{2\mu_1+1}))\widehat{\mathbf{v}}_{02}(x, y, z). \end{aligned}$$

Finally, using (8.5.13) and formulas (8.5.6), (8.5.9) and (8.5.11) for a_j^+ and b_j^+, we find

$$\begin{aligned} w_{1p}(\xi_1, \eta_1, \zeta_1; \varepsilon) &= (gC_{11})|_{k=k_p}a_1\varepsilon^{\mu_1}\mathbf{w}_1^l(\xi_1, \eta_1, \zeta_1) + (gC_{12}c_{11} \\ &\quad + gC_{13}c_{21})|_{k=k_p}\varepsilon^{\mu_1}\mathbf{w}_1^r(\xi_1, \eta_1, \zeta_1), \\ w_{2p}(\xi_2, \eta_2, \zeta_2; \varepsilon) &= (gC_{22}c_{11} + gC_{23}c_{21})|_{k=k_p}\varepsilon^{\mu_1}\mathbf{w}_2^l(\xi_2, \eta_2, \zeta_2) \\ &\quad + (gC_{14})|_{k=k_p}a_2\varepsilon^{\mu_1}\mathbf{w}_2^r(\xi_2, \eta_2, \zeta_2). \end{aligned}$$

Compare the equalities (8.4.10), (8.6.6) and (8.6.4), then

$$(gC_{12}c_{1j}+gC_{j3}c_{2j})|_{k=k_p}=-\,b_j\frac{(g\,C_{12}\overline{b_1}+g\,C_{13}\overline{b_2})|_{k=k_p}}{k_p^2-k_e^2}$$
$$+\,(gC_{12}\widehat{c}_{1j}+gC_{j3}\widehat{c}_{2j})|_{k=k_p}=b_j+O(\varepsilon^{2\mu_1+1}),$$

where $j=1,2$. Thus

$$\begin{aligned} w_{1p}(\xi_1,\eta_1,\zeta_1;\varepsilon)=&\,\varepsilon^{3\mu_1+1}(a_1b_1\beta+O(\varepsilon^{2\mu_1+1}))\mathbf{w}_1^l(\xi_1,\eta_1,\zeta_1)\\ &+\varepsilon^{\mu_1}(b_1+O(\varepsilon^{2\mu_1+1}))\mathbf{w}_1^r(\xi_1,\eta_1,\zeta_1), \qquad (8.6.7)\\ w_{2p}(\xi_2,\eta_2,\zeta_2;\varepsilon)=&\,\varepsilon^{\mu_1}(b_2+O(\varepsilon^{2\mu_1+1}))\mathbf{w}_2^l(\xi_2,\eta_2,\zeta_2)\\ &+\varepsilon^{3\mu_1+1}(a_2b_2\beta+O(\varepsilon^{2\mu_1+1}))\mathbf{w}_2^r(\xi_2,\eta_2,\zeta_2). \qquad (8.6.8)\end{aligned}$$

We set

$$\begin{aligned} u_p(x,y,z;\varepsilon)=\Xi(x,y,z)\big[&\chi_{1,\varepsilon}(x,y,z)v_{1p}(x,y,z;\varepsilon)\\ &+\Theta(\varepsilon^{-2\sigma}r_1)w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1,\varepsilon^{-1}z_1;\varepsilon)\\ &+\chi_{0,\varepsilon}(x,y,z)v_{0p}(x,y,z;\varepsilon)\\ &+\Theta(\varepsilon^{-2\sigma}r_2)w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2,\varepsilon^{-1}z_2;k,\varepsilon)\\ &+\chi_{2,\varepsilon}(x,y,z)v_{2p}(x,y,z;k,\varepsilon)\big], \qquad (8.6.9)\end{aligned}$$

where Ξ is a cut-off function in $G(\varepsilon)$ that is equal to 1 on the set $G(\varepsilon)\cap\{|x|<R\}$ and to 0 on $G(\varepsilon)\cap\{|x|>R+1\}$ for a large $R>0$; σ is such that $2\sigma<1$. The principal part of the norm of u_p is given by $\chi_{0,\varepsilon}v_{0p}$. Considering the definitions of v_{0p} and $\widehat{\mathbf{v}}_{0j}$ (see Sect. 8.3) and Lemma 8.4.2, we obtain $\|\chi_{0,\varepsilon}v_{0p}\|=\|v_e\|+o(1)$.

Step B. We show that

$$\|((-i\nabla+\mathbf{A})^2\pm H-k_p^2)u_p;\,V^0_{\gamma,\,\delta}(G(\varepsilon))\|\le c\varepsilon^{\mu_1+\kappa}, \qquad (8.6.10)$$

where $\kappa=\min\{\mu_1+1,\ \mu_2+1-\sigma_1,\ \gamma+3/2\}$, $\sigma_1=2\sigma(\mu_2-\gamma+3/2)$. If $\mu_1-3/2<\gamma-1$ and σ is sufficiently small so that $\mu_2-\mu_1>\sigma_1$, then $\kappa=\mu_1+1$.

By virtue of (8.6.9)

$$\begin{aligned} &((-i\nabla+\mathbf{A})^2\pm H-k_p^2)u_p(x,y,z;\varepsilon)\\ &=[\triangle,\chi_{1,\varepsilon}]\Big(v_1(x,y,z;\varepsilon)-b_1\beta\varepsilon^{2\mu_1+1}(r_1^{-\mu_1-1}+a(k_p)r_1^{\mu_1})\Phi_1(-\varphi_1)\Big)\\ &\quad+[\triangle,\Theta]w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1,\varepsilon^{-1}z_1;\varepsilon)-k^2\Theta(\varepsilon^{-2\sigma}r_1)w_{1p}(\varepsilon^{-1}x_1,\varepsilon^{-1}y_1,\varepsilon^{-1}z_1;\varepsilon)\\ &\quad+[\triangle,\chi_{0,\varepsilon}]\Big(v_0(x,y,z;\varepsilon)-\Theta(r_1)\big(b_{1p}^-(\varepsilon)r_1^{-\mu_1-1}+b_{1p}^+(\varepsilon)r_1^{\mu_1}\big)\Phi_1(-\varphi_1)\\ &\qquad\qquad-\Theta(r_2)\big(a_{2p}^-(\varepsilon)r_2^{-\mu_1-1}+a_{2p}^+(\varepsilon)r_2^{\mu_1}\big)\Phi_1(\varphi_2)\Big)\\ &\quad+[\triangle,\Theta]w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2,\varepsilon^{-1}z_2;\varepsilon)-k^2\Theta(\varepsilon^{-2\sigma}r_2)w_{2p}(\varepsilon^{-1}x_2,\varepsilon^{-1}y_2,\varepsilon^{-1}z_2;\varepsilon)\\ &\quad+[\triangle,\chi_{2,\varepsilon}]\Big(v_2(x,y,z;\varepsilon)-b_2\beta\varepsilon^{2\mu_1+1}(r_2^{-\mu_1-1}+a(k_p)r_2^{\mu_1})\Phi_1(\varphi_2)\Big)\\ &\quad+[\triangle,\Xi]v_1(x,y,z;\varepsilon)+[\triangle,\Xi]v_3(x,y,z;\varepsilon),\end{aligned}$$

where $b^-_{1p} = O(\varepsilon^{2\mu_1+1})$, $b^+_{1p} = b_1 + O(\varepsilon^{2\mu_1+1})$, $a^-_{2p} = O(\varepsilon^{2\mu_1+1})$, $a^+_{2p} = b_2 + O(\varepsilon^{2\mu_1+1})$. Taking account of the asymptotics $\mathbf{v}_1$ as $r_1 \to 0$ and going to the variables $(\xi_1, \eta_1, \zeta_1) = (\varepsilon^{-1}x_1, \varepsilon^{-1}y_1, \varepsilon^{-1}z_1)$, we arrive at

$$\left\|(x, y, z) \mapsto [\triangle, \chi_{1,\varepsilon}]\left(\mathbf{v}_1(x, y, z) - (r_1^{-\mu_1-1} + a(k_p)r_1^{\mu_1})\Phi_1(-\varphi_1)\right); V^0_{\gamma,\delta}(G(\varepsilon))\right\|^2$$
$$\leq c\int_{G(\varepsilon)} (r_1^2 + \varepsilon^2)^\gamma \left|[\triangle, \chi_{1,\varepsilon}]r_1^{-\mu_1+1}\Phi(-\varphi_1)\right|^2 dx\, dy\, dz \leq c\varepsilon^{2(\gamma-\mu_1+1/2)}.$$

This and (8.5.2) imply that

$$\left\|(x, y, z) \mapsto [\triangle, \chi_{1,\varepsilon}]\left(v_1(x, y, z) - (r_1^{-\mu_1-1} + a(k_p)r_1^{\mu_1})\Phi(-\varphi_1)\right); V^0_{\gamma,\delta}(G(\varepsilon))\right\|$$
$$\leq c\varepsilon^{\gamma+\mu_1+3/2}.$$

Similarly,

$$\Big\|(x, y, z) \mapsto [\triangle, \chi_{0,\varepsilon}]\Big(v_0(x, y, z) - \Theta(r_1)\big(b^-_{1p}(\varepsilon)r_1^{-\mu_1-1} + b^+_{1p}(\varepsilon)r_1^{\mu_1}\big)\Phi_1(-\varphi_1)$$
$$- \Theta(r_2)\big(a^-_{2p}(\varepsilon)r_2^{-\mu_1-1} + a^+_{2p}(\varepsilon)r_2^{\mu_1}\big)\Phi_1(\varphi_2)\Big)\Big\| \leq c\varepsilon^{\gamma+\mu_1+3/2},$$
$$\left\|(x, y, z) \mapsto [\triangle, \chi_{2,\varepsilon}]\left(v_2(x, y, z) - (r_2^{-\mu_1-1} + a(k_p)r_2^{\mu_1})\Phi_1(\varphi_2)\right); V^0_{\gamma,\delta}(G(\varepsilon))\right\|$$
$$\leq c\varepsilon^{\gamma+\mu_1+3/2}.$$

It is clear that
$$\left\|[\triangle, \Xi]v_l; V^0_{\gamma,\delta}(G(\varepsilon))\right\| \leq c\varepsilon^{2\mu_1+1}, \quad l = 1, 3.$$

Further, since $\mathbf{w}^l_j$ behaves as $O(\rho_j^{-\mu_2-1})$ at infinity, we have

$$\int_{G(\varepsilon)} (r_j^2 + \varepsilon^2)^\gamma \left|[\triangle, \Theta]\mathbf{w}^l_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j)\right|^2 dx_j dy_j dz_j$$
$$\leq c\int_{K_j} (r_j^2 + \varepsilon^2)^\gamma \left|[\triangle, \Theta](\varepsilon^{-1}r_j)^{-\mu_2-1}\Phi_2(\varphi_j)\right|^2 dx_j dy_j dz_j \leq c\varepsilon^{2(\mu_2+1-\sigma_1)},$$

where $\sigma_1 = 2\sigma(\mu_2 - \gamma + 3/2)$. A similar inequality holds with $\mathbf{w}^l_j$ changed for $\mathbf{w}^r_j$. In view of (8.6.7) and (8.6.8), we obtain

$$\left\|[\triangle, \Theta]w_{jp}; V^0_{\gamma,\delta}(G(\varepsilon))\right\| \leq c\varepsilon^{\mu_1+\mu_2+1-\sigma_1}.$$

Finally, using (8.6.7) and (8.6.8) once more and taking into account the estimate

$$\int_{G(\varepsilon)} (r_j^2+\varepsilon^2)^\gamma \left|\Theta(\varepsilon^{-2\sigma} r_j)\mathbf{w}_j^l(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j)\right|^2 dx_j dy_j dz_j$$
$$= \varepsilon^{2\gamma+3} \int_\Omega (\rho_j^2+1)^\gamma \left|\Theta(\varepsilon^{1-2\sigma}\rho_j)\mathbf{w}_j^l(\xi_j, \eta_j, \zeta_j)\right|^2 d\xi_j d\eta_j d\zeta_j \le c\varepsilon^{2\gamma+3},$$

and a similar estimate for $\mathbf{w}_j^r$, we derive

$$\left\|(x, y, z) \mapsto \Theta(\varepsilon^{-2\sigma} r_j) w_{jp}(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j);\ V^0_{\gamma,\delta}(G(\varepsilon))\right\| \le c\varepsilon^{\mu_1+\gamma+3/2}.$$

Combining the obtained inequalities, we arrive at (8.6.10).

Step C. Let us write the right-hand side of problem (8.6.1) in the form

$$\begin{aligned} f(x, y, z) = {} & f_1(x, y, z; \varepsilon) + f_0(x, y, z; \varepsilon) + f_2(x, y, z; \varepsilon) \\ & + \varepsilon^{-\gamma-3/2} F_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1, \varepsilon^{-1}z_1; \varepsilon_1) \\ & + \varepsilon^{-\gamma-3/2} F_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2, \varepsilon^{-1}z_2; \varepsilon), \end{aligned}$$

where

$$\begin{aligned} & f_l(x, y, z; \varepsilon) = \chi_{l,\varepsilon^\sigma}(x, y, z) f(x, y, z), \\ & F_j(\xi_j, \eta_j, \zeta_j; \varepsilon) = \varepsilon^{\gamma+3/2}\Theta(\varepsilon^{1-\sigma}\rho_j) f(x_{O_j} + \varepsilon\xi_j, y_{O_j} + \varepsilon\eta_j, z_{O_j} + \varepsilon\zeta_j); \end{aligned}$$

(x, y, z) are arbitrary Cartesian coordinates; $(x_{O_j}, y_{O_j}, z_{O_j})$ denote the coordinates of the point O_j in the system (x, y, z); x_j, y_j, z_j were introduced in Sect. 8.2. From the definition of the norms, it follows that

$$\|f_1; V^0_{\gamma,\delta}(G_1)\| + \|f_0; V^0_\gamma(G_0)\| + \|f_2; V^0_{\gamma,\delta}(G_2)\| + \|F_j; V^0_\gamma(\Omega_j)\| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{8.6.11}$$

We consider solutions v_l and w_j to the limit problems

$$\begin{aligned} (-i\nabla + \mathbf{A})^2 v_0 \pm H v_0 - k^2 v_0 = f_0 \text{ in } G_0, &\qquad v_0 = 0 \text{ on } \partial G_0, \\ \Delta v_l + k^2 v_l = f_l \text{ in } G_l, &\qquad v_l = 0 \text{ on } \partial G_l, \\ \Delta w_j = F_j \text{ in } \Omega_j, &\qquad w_j = 0 \text{ on } \partial\Omega_j, \end{aligned}$$

respectively; moreover, the v_l with $l = 1, 2$ satisfy the intrinsic radiation conditions at infinity, and the v_0 is subject to the condition $(v_0, v_e)_{G_0} = 0$. According to Propositions 8.3.1, 8.3.2, and 8.3.3, the problems in G_l and Ω_j are uniquely solvable and

$$\begin{aligned} \|v_0; V^2_\gamma(G_0)\| &\le c_0\|f_0; V^0_\gamma(G_0)\|, \\ \|v_l; V^2_{\gamma,\delta,-}(G_l)\| &\le c_l\|f_l; V^0_{\gamma,\delta}(G_l)\|,\ l = 1, 2, \\ \|w_j; V^2_\gamma(\Omega_j)\| &\le C_j\|F_j; V^0_\gamma(\Omega_j)\|,\ j = 1, 2, \end{aligned} \tag{8.6.12}$$

where c_l and C_j are independent of ε. We set

$$
\begin{aligned}
U(x, y, z; \varepsilon) = {} & \chi_{1,\varepsilon}(x, y, z)v_1(x, y, z; \varepsilon) + \varepsilon^{-\gamma+3/2}\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1, \varepsilon^{-1}z_1; \varepsilon) \\
& + \chi_{0,\varepsilon}(x, y, z)v_0(x, y, z; \varepsilon) + \varepsilon^{-\gamma+3/2}\Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2, \varepsilon^{-1}z_2; \varepsilon) \\
& + \chi_{2,\varepsilon}(x, y, z)v_2(x, y, z; \varepsilon).
\end{aligned}
$$

Estimates (8.6.11) and (8.6.12) lead to

$$
\|U; V^2_{\gamma,\delta,-}(G(\varepsilon))\| \le c\|f; V^0_{\gamma,\delta}(G(\varepsilon))\| \tag{8.6.13}
$$

with c independent of ε. Let R_ε denote the mapping $f \mapsto U$.

Let us show that $((-i\nabla + \mathbf{A})^2 \pm H - k^2)R_\varepsilon = I + S_\varepsilon$, where S_ε is an operator in $V^0_{\gamma,\delta}(G(\varepsilon))$ of small norm. We have

$$
\begin{aligned}
& ((-i\nabla + \mathbf{A})^2 \pm H - k^2)R_\varepsilon f(x, y, z) \\
& \quad = f(x, y, z) + [\Delta, \chi_{1,\varepsilon}]v_1(x, y, z; \varepsilon) + [\Delta, \chi_{0,\varepsilon}]v_0(x, y, z; \varepsilon) + [\Delta, \chi_{2,\varepsilon}]v_2(x, y, z; \varepsilon) \\
& \quad + \varepsilon^{-\gamma+1/2}[\Delta, \Theta]w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1, \varepsilon^{-1}z_1; \varepsilon) + k^2\varepsilon^{-\gamma+1/2}\Theta(r_1)w_1(\varepsilon^{-1}x_1, \varepsilon^{-1}y_1, \varepsilon^{-1}z_1; \varepsilon) \\
& \quad + \varepsilon^{-\gamma+1/2}[\Delta, \Theta]w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2, \varepsilon^{-1}z_2; \varepsilon) + k^2\varepsilon^{-\gamma+1/2}\Theta(r_2)w_2(\varepsilon^{-1}x_2, \varepsilon^{-1}y_2, \varepsilon^{-1}z_2; \varepsilon).
\end{aligned} \tag{8.6.14}
$$

Let d be a positive number such that $\gamma - 1 > \mu_1 + 1/2 - d$. On the support of the function $[\Delta, \chi_{1,\varepsilon}]v_1$ the estimate $r_1 = O(\varepsilon)$ holds, therefore,

$$
\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^d\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma-d,\delta}(G_1)\| \le c\varepsilon^d\|v_1; V^2_{\gamma-d,\delta}(G_1)\|.
$$

This and (8.6.12) lead to

$$
\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon_1^d\|f_1; V^0_{\gamma-d,\delta}(G_1)\|.
$$

Moreover, $f_1 = 0$ outside the zone $c\varepsilon^\sigma \le r_1 \le C\varepsilon^\sigma$, therefore,

$$
\|f_1; V^0_{\gamma-d,\delta}(G_1)\| \le c\varepsilon^{-d\sigma}\|f_1; V^0_{\gamma,\delta}(G_1)\|.
$$

The two last estimates together with (8.6.11) show that

$$
\|[\Delta, \chi_{1,\varepsilon}]v_1; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{8.6.15}
$$

In a similar way, we obtain

$$
\|[\Delta, \chi_{0,\varepsilon}]v_0; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|, \tag{8.6.16}
$$

$$
\|[\Delta, \chi_{2,\varepsilon}]v_2; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|. \tag{8.6.17}
$$

We now assume in addition that the d satisfies $-\mu_1 - 1/2 + d < \gamma - 1$. Because the support of the function $[\Delta, \Theta(\varepsilon\rho_j)]w_j(\xi_j, \eta_j, \zeta_j; \varepsilon)$, $j = 1, 2$, belongs to the domain $c\varepsilon^{-1} \le \rho_j \le C\varepsilon^{-1}$,

$$\begin{aligned}&\|(\xi_j, \eta_j, \zeta_j) \mapsto [\Delta_{\xi_j,\eta_j,\zeta_j}, \Theta(\varepsilon_j\rho_j)]w_j(\xi_j, \eta_j, \zeta_j; \varepsilon); V^0_\gamma(\Omega_j)\|\\ &\le c\varepsilon^d\|(\xi_j, \eta_j, \zeta_j) \mapsto [\Delta_{\xi_j,\eta_j,\zeta_j}, \Theta(\varepsilon_j\rho_j)]w_j(\xi_j, \eta_j, \zeta_j; \varepsilon); V^0_{\gamma+d}(\Omega_j)\|\\ &\le c\varepsilon^d\|w_j; V^2_{\gamma+d}(\Omega_j)\|.\end{aligned}$$

Now, taking into account (8.6.12), we obtain

$$\begin{aligned}&\varepsilon^{-\gamma+1/2}\|(x_j, y_j, z_j) \mapsto [\Delta, \Theta(r_j)]w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\|\\ &\quad\le c\varepsilon^d\|F_j; V^0_{\gamma+d}(\Omega_j)\|.\end{aligned}$$

Since $F_j = 0$ for $\rho_j > c\varepsilon^{-\sigma}$,

$$\|F_j; V^0_{\gamma+d}(\Omega_j)\| \le c\varepsilon^{-d\sigma}\|F_j; V^0_\gamma(\Omega_j)\|. \tag{8.6.18}$$

Consequently,

$$\begin{aligned}\varepsilon^{-\gamma+1/2}\|(x_j, y_j, z_j) \mapsto [\Delta, \Theta(r_j)]w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\|\\ \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|.\end{aligned} \tag{8.6.19}$$

It remains to estimate the middle terms of the two last lines in (8.6.14). We have

$$\begin{aligned}&\varepsilon^{-\gamma+1/2}\|(x_j, y_j, z_j) \mapsto \Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\|\\ &= \varepsilon^2\|(\xi_j, \eta_j, \zeta_j) \mapsto \Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j, \zeta_j; \varepsilon); V^0_\gamma(\Omega_j)\|\\ &\le \varepsilon^2\|(\xi_j, \eta_j, \zeta_j) \mapsto \Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j, \zeta_j; \varepsilon); V^2_{\gamma+2}(\Omega_j)\| \le c\varepsilon^d\|w_j; V^2_{\gamma+d}(\Omega_j)\|;\end{aligned}$$

in the last inequality, we took into account that $\Theta(\varepsilon\rho_j)w_j(\xi_j, \eta_j, \zeta_j; \varepsilon) = 0$ for $\rho_j \ge c\varepsilon^{-1}$; besides, we assume that $2 - d > 0$. In view of (8.6.12), (8.6.18), and (8.6.11), we obtain

$$\begin{aligned}\varepsilon^{-\gamma+1/2}\|(x_j, y_j, z_j) \mapsto \Theta(r_j)w_j(\varepsilon^{-1}x_j, \varepsilon^{-1}y_j, \varepsilon^{-1}z_j; \varepsilon); V^0_{\gamma,\delta}(G(\varepsilon))\|\\ \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|.\end{aligned} \tag{8.6.20}$$

Thus, (8.6.14)–(8.6.17) and (8.6.19)–(8.6.20) lead to the inequality

$$\|((-i\nabla + \mathbf{A})^2 \pm H - k^2)R_\varepsilon f - f; V^0_{\gamma,\delta}(G(\varepsilon))\| \le c\varepsilon^{d(1-\sigma)}\|f; V^0_{\gamma,\delta}(G(\varepsilon))\|,$$

which means that $((-i\nabla + \mathbf{A})^2 \pm H - k^2)R_\varepsilon = I + S_\varepsilon$ and the norm of the operator S_ε in the space $V^0_{\gamma,\delta}(G(\varepsilon))$ admits the estimate $\|S_\varepsilon\| \le c\varepsilon^{d(1-\sigma)}$.

Step D. Let us recall that the operator S_ε is defined on the subspace $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. We also need the range of the operator S_ε be included in $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. To this end, we replace the mapping R_ε by $\widetilde{R}_\varepsilon : f \mapsto U(f)+a(f)u_p$, the u_p was constructed in Step A, and $a(f)$ is a constant. Then $((-i\nabla+\mathbf{A})^2 \pm H - k^2)\widetilde{R}_\varepsilon = I + \widetilde{S}_\varepsilon$ with $\widetilde{S}_\varepsilon = S_\varepsilon + a(\cdot)((-i\nabla+\mathbf{A})^2 \pm H - k^2)u_p$. As $k = k_e$, the condition $(\chi_{0,\varepsilon^\sigma}\widetilde{S}_\varepsilon f, v_e)_{G_0} = 0$ implies

$$a(f) = -(\chi_{0,\varepsilon^\sigma} S_\varepsilon f, v_e)_{G_0}/(\chi_{0,\varepsilon^\sigma}((-i\nabla+\mathbf{A})^2 \pm H - k_e^2)u_p, v_e)_{G_0}.$$

Now, we prove that $\|\widetilde{S}_\varepsilon\| \le c\|S_\varepsilon\|$, where c is independent of ε and k. We have

$$\|\widetilde{S}_\varepsilon f\| \le \|S_\varepsilon f\| + |a(f)|\,\|((-i\nabla+\mathbf{A})^2 \pm H - k^2)u_p\|.$$

Estimate (7.5.10) (with $\gamma > \mu_1 - 1/2$ and $\mu_2 - \mu_1 > \sigma_1$), the formula for k_p, and the condition $k^2 - k_e^2 = O\big(\varepsilon^{2\mu_1+1}\big)$ imply the inequalities

$$\begin{aligned}\|((-i\nabla+\mathbf{A})^2 \pm H - k^2)u_p; V^0_{\gamma,\delta}\| &\le |k^2 - k_p^2|\,\|u_p; V^0_{\gamma,\delta}\| \\ &\quad + \|((-i\nabla+\mathbf{A})^2 \pm H - k_p^2)u_p; V^0_{\gamma,\delta}\| \le c\varepsilon^{2\mu_1+1}.\end{aligned}$$

Since the supports of the functions $((-i\nabla+\mathbf{A})^2 \pm H - k_p^2)u_p$ and $\chi_{0,\varepsilon^\sigma}$ are disjoint, we obtain

$$|(\chi_{0,\varepsilon^\sigma}((-i\nabla+\mathbf{A})^2 \pm H - k_e^2)u_p, v_e)_{G_0}| = |(k_e^2 - k_p^2)(u_p, v_e)_{G_0}| \ge c\varepsilon^{2\mu_1+1}.$$

Further, $\gamma - 1 < \mu_1 + 1/2$, therefore,

$$|(\chi_{0,\varepsilon^\sigma} S_\varepsilon f, v_e)_{G_0}| \le \|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\|\,\|v_e; V^0_{-\gamma}(G_0)\| \le c\|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\|.$$

Hence,

$$|a(f)| \le c\varepsilon^{-2\mu_1-1}\|S_\varepsilon f; V^0_{\gamma,\delta}(G(\varepsilon))\|$$

and $\|\widetilde{S}_\varepsilon f\| \le c\|S_\varepsilon f\|$. Thus, the operator $I + \widetilde{S}_\varepsilon$ in $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$ is invertible, which is also true for the operator of problem (8.6.1):

$$A_\varepsilon : u \mapsto (-i\nabla+\mathbf{A})^2 u \pm Hu - k^2 u : \overset{\circ}{V}{}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon)) \mapsto V^{0,\perp}_{\gamma,\delta}(G(\varepsilon));$$

the $\overset{\circ}{V}{}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon))$ consists of the elements in $V^2_{\gamma,\delta,-}(G(\varepsilon))$ that vanish on $\partial G(\varepsilon)$, and the operator $(-i\nabla+\mathbf{A})^2 \pm H - k^2$ takes $\overset{\circ}{V}{}^{2,\perp}_{\gamma,\delta,-}(G(\varepsilon))$ to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$ to $V^{0,\perp}_{\gamma,\delta}(G(\varepsilon))$. The inverse operator $A_\varepsilon^{-1} = \widetilde{R}_\varepsilon(I + \widetilde{S}_\varepsilon)^{-1}$ is bounded uniformly with respect to ε and k. Therefore, the inequality (8.6.3) holds with c independent of ε and k. □

We consider solution u_1 and u_2 to the homogeneous problem (8.2.2)–(8.2.3) defined by

$$u_1(x,y,z)=\begin{cases}U_1^+(x,y,z)+S_{11}\,U_1^-(x,y,z)+O(\exp(\delta x)), & x\to-\infty,\\ S_{12}\,U_2^-(x,y,z)+O(\exp(-\delta x)), & x\to+\infty;\end{cases}$$

$$u_2(x,y,z)=\begin{cases}S_{21}\,U_1^-(x,y,z)+O(\exp(\delta x)), & x\to-\infty,\\ U_2^+(x,y,z)+S_{22}\,U_2^-(x,y,z)+O(\exp(-\delta x)), & x\to+\infty.\end{cases}$$

Let S_{lm} be the elements of the scattering matrix determined by these solutions; $\widetilde{S}_{11}$, $\widetilde{S}_{12}$ are the same as in (8.5.21)–(8.5.22).

Theorem 8.6.2 *Let the hypotheses of Proposition 8.6.1 be fulfilled. Then the inequalities*

$$|S_{11}-\widetilde{S}_{11}|+|S_{12}-\widetilde{S}_{12}|\le c|\widetilde{S}_{12}|\varepsilon^{2-\delta},$$
$$|S_{21}-\widetilde{S}_{21}|+|S_{22}-\widetilde{S}_{22}|\le c|\widetilde{S}_{22}|\varepsilon^{2-\delta}$$

hold with a constant c, independent of ε and k, δ being an arbitrarily small positive number.

Now we return to the detailed notations introduced in the first three sections. We denote by $k_{e,\pm}^2$ an eigenvalue of problem (8.3.1) in the resonator G_0 and by $k_{r,\pm}^2(\varepsilon)$ a resonance frequency such that $k_{r,\pm}^2(\varepsilon)\to k_{e,\pm}^2$ as $\varepsilon\to 0$. Moreover, let $b_j^\pm$ be the constants in asymptotics (8.4.5) of an eigenfunction corresponding to the eigenvalue $k_{e,\pm}^2$ and $s_j(k)$ the constant in asymptotics (8.4.1) of the special solution V_j for $r_j\to 0$, $j=1,2$. Finally, the constants α and β are defined by (8.3.11) and (8.3.12). We set $P_\pm=(|b_1||b_2|\beta^2|s_1(k_e)|^2)^{-1}$; this is the same constant as in (8.5.25)–(8.5.27). Theorem 8.6.2 and formulas (8.5.26)–(8.5.27) lead to the next statement.

Theorem 8.6.3 *For $|k^2-k_{r,\pm}^2|=O(\varepsilon^{2\mu_1+1})$ the asymptotic expansions*

$$T^\pm(k,\varepsilon)=\frac{1}{\frac14\left(\frac{|b_1^\pm|}{|b_2^\pm|}+\frac{|b_2^\pm|}{|b_1^\pm|}\right)^2+P_\pm^2\left(\frac{k^2-k_{r,\pm}^2}{\varepsilon^{4\mu_1+2}}\right)^2}(1+O(\varepsilon^{2-\delta})),$$
$$k_{r,\pm}^2=k_{0,\pm}^2-\alpha(|b_1^\pm|^2+|b_2^\pm|^2)\varepsilon^{2\mu_1+1}+O\big(\varepsilon^{2\mu_1+1+2-\delta}\big),$$
$$\Upsilon^\pm(\varepsilon)=\left(\frac{|b_1^\pm|}{|b_2^\pm|}+\frac{|b_2^\pm|}{|b_1^\pm|}\right)P_\pm^{-1}\varepsilon^{4\mu_1+2}\big(1+O(\varepsilon^{2-\delta})\big),$$

hold, where $\Upsilon^\pm(\varepsilon)$ is the width of the resonant peak at its half-height and δ is an arbitrarily small positive number.

Chapter 9
Numerical Simulation of High Energy Electron Transport

The chapter is devoted to the numerical simulation of resonant tunneling for electrons with energy E between the first and the fifth thresholds. We approximately calculate the electron transmission probability $T_{nk}(E) = |S_{nk}(E)|^2$, where $S_{nk}(E)$ is the entry of the scattering matrix $S(E)$. Generally, the dependence $E \to T_{nk}(E)$ turns out to be rather complicated. Let us denote by $\mathrm{Re}E_1, \mathrm{Re}E_2, \ldots$ all the waveguide resonant energies. To interpret $E \to T_{nk}(E)$, we introduce the probability amplitude $A_{nsk}(E)$ of the electron resonant tunneling with resonant energy $\mathrm{Re}E_s$; thus, we have $S_{nk}(E) = \sum_s A_{nsk}(E)$. We consider the function $E \to |\sum_s A_{nsk}(E)|^2$ in a small neighborhood of $\mathrm{Re}E_r$ and obtain an approximate relation

$$\mathcal{T}_{nk}(E) \approx |\sum_s A_{nsk}(E)|^2,$$

where $\mathcal{T}_{nk}$ is a sufficiently simple function containing several unknown parameters of the tunneling with resonant energy $\mathrm{Re}E_r$. Comparing the functions $E \to T_{nk}(E)$ and $E \to \mathcal{T}_{nk}(E)$, we find the mentioned parameters by the method of the least squares.

9.1 Numerical Simulation of Multichannel Resonant Tunneling

9.1.1 Closed Resonator

A necessary condition of electron resonant tunneling consists in proximity of the incident electron energy E to one of the eigenenergies k_{ev}^2 of the closed resonator (Fig. 9.1). Table 9.1 shows the calculated values of k_{ev}^2 and the figures of the corresponding eigenfunctions.

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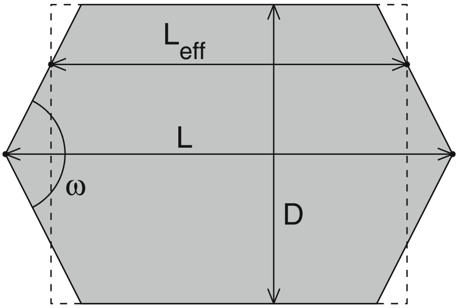
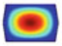
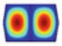
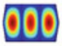
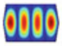
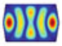
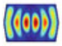
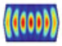
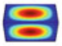
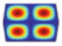
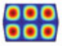
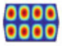
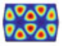
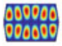
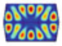
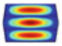
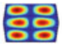
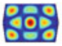
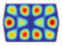
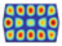
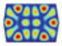
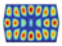
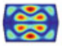
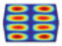
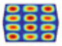
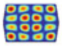
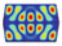
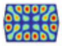


Fig. 9.1 The resonator

Table 9.1 Eigenvalues and eigenfunctions for the closed resonator

$n \backslash m$	1	2	3	4	5	6	7
1	$k_{ev}^2 = 14.5765$ 	$k_{ev}^2 = 28.6845$ 	$k_{ev}^2 = 52.1479$ 	$k_{ev}^2 = 84.8217$ 	$k_{ev}^2 = 125.6741$ 	$k_{ev}^2 = 180.1483$ 	$k_{ev}^2 = 240.3497$ 
2	$k_{ev}^2 = 44.3978$ 	$k_{ev}^2 = 59.1481$ 	$k_{ev}^2 = 83.7015$ 	$k_{ev}^2 = 117.9935$ 	$k_{ev}^2 = 161.2690$ 	$k_{ev}^2 = 214.9305$ 	$k_{ev}^2 = 273.2698$ 
3	$k_{ev}^2 = 93.7270$ 	$k_{ev}^2 = 108.5681$ 	$k_{ev}^2 = 134.3437$ 	$k_{ev}^2 = 165.5023$ 	$k_{ev}^2 = 209.7926$ 	$k_{ev}^2 = 261.8551$ 	$k_{ev}^2 = 326.4400$ 
4	$k_{ev}^2 = 163.4579$ 	$k_{ev}^2 = 177.5804$ 	$k_{ev}^2 = 202.4174$ 	$k_{ev}^2 = 237.7302$ 	$k_{ev}^2 = 287.1038$ 	$k_{ev}^2 = 330.8411$ 	$k_{ev}^2 > 370$

For the rectangular resonator with unit width (i.e., $D = 1$) and length L,

$$k_{\text{ev}}^2 = \pi^2 n^2 + \pi^2 m^2 / L^2, \tag{9.1.1}$$

where n and m are transversal and longitudinal quantum numbers. Since the shape of the resonator is close to rectangular, the eigenvalues are well approximated by the expression (9.1.1) with L replaced by L_{eff}. For the resonator with angle $\omega = 0.9\pi$ at the vertex and with length $L = 1.5$, the value of L_{eff} is approximately equal to 1.45 for $n = 1$ and to 1.42 for $n > 1$.

The disparity between the calculated eigenvalues and approximations by formula (9.1.1) is less than 0.5 %. Note that such an accuracy is achieved in spite of the significant difference between the considered eigenfunctions and those for the rectangular resonator (see the figures in Table 9.1).

9.1.2 The Method for Computing Scattering Matrix

We now describe a calculation scheme for a scattering matrix based on the method presented in Chap. 4. The energy E of an electron moving in a cylindrical waveguide can be represented in the form $E = E_\perp + E_\parallel$, where $E_\perp$ and $E_\parallel$ are transversal and longitudinal components, respectively. The values of $E_\perp$ are quantized. In the simplest two-dimensional case, where the waveguide is a strip of width D, we have $E = k^2$, $E_\perp(n) = \pi^2 n^2/D^2$, and $E_\parallel(n) = k^2 - \pi^2 n^2/D^2$. We consider scattering of electrons incident on the resonator from $-\infty$ with energy k^2 between the first and the fifth thresholds, that is, $\pi^2 < k^2 < 5^2\pi^@$. Thus, we consider solutions of the form

$$u_n(x, y) = U_n^{\text{in}}(x, y) + \sum_{j=1}^{2n_{\max}} S_{nj} U_j^{\text{out}}(x, y) + O(e^{\delta|x|}), \quad |x| \to \infty,$$

where

$$\begin{aligned}
U_n^{\text{in}}(x, y) &:= \mathbf{1}_{\text{left}} e^{i\sqrt{E_{||}(n)}x} \chi_n(y), & n &= 1, 2, \dots, n_{\max}, \\
U_j^{\text{out}}(x, y) &:= \mathbf{1}_{\text{left}} e^{-i\sqrt{E_{||}(j)}x} \chi_j(y), & j &= 1, 2, \dots, n_{\max}, \\
U_j^{\text{out}}(x, y) &:= \mathbf{1}_{\text{right}} e^{i\sqrt{E_{||}(j-n_{\max})}x} \chi_{j-n_{\max}x}(y), & j &= n_{\max}+1, n_{\max}+2, \dots, 2n_{\max},
\end{aligned}$$

$E_{||}(j) = E - E_\perp(j)$, $\mathbf{1}_{\text{left}}$ and $\mathbf{1}_{\text{right}}$ are the indicators of the left and the right outlets of the waveguide, and

$$\chi_n(y) = \frac{(1-(-1)^n)}{2} \cos\frac{\pi n y}{D} + \frac{(1+(-1)^n)}{2} \sin\frac{\pi n y}{D}.$$

The matrix $\mathbb{S} = \{S_{nj}\}$, $n = 1, 2, \dots, n_{\max}$, $j = 1, 2, \dots, 2n_{\max}$, is the upper half of the waveguide scattering matrix. We denote the domain occupied by the waveguide by G and introduce the notations:

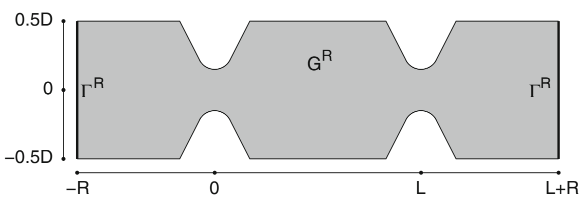
$$\begin{aligned}
G^R &:= G \cap \{(x, y) : |x - L/2| < L/2 + R\}, \\
\Gamma^R &:= \partial G^R \cap \{(x, y) : |x - L/2| = L/2 + R\},
\end{aligned}$$

where R is a sufficiently large positive constant (see Fig. 9.2).

As an approximation for the nth row of the scattering matrix, we take the minimizer $a_n^0 = (a_{n1}^0, a_{n2}^0, \dots, a_{n,2n_{\max}}^0)$ of the functional

$$J_n^R = \|\mathcal{X}_n^R - U_n^{\text{in}} - \sum_{j=1}^{2n_{\max}} a_{nj} U_j^{\text{out}}\|^2_{L_2(\Gamma^R)}.$$

Fig. 9.2 The truncated domain G^R



Here $\mathcal{X}_n^R$ is a solution to the problem

$$\begin{aligned} -(\Delta + E)\mathcal{X}_n^R &= 0 && \text{in } G^R, \\ \mathcal{X}_n^R &= 0 && \text{on } \partial G^R \setminus \Gamma^R, \\ (\partial_\nu + i\zeta)\mathcal{X}_n^R &= (\partial_\nu + i\zeta)\Big(U_n^{\text{in}} + \sum_{j=1}^{2n_{\max}} a_{nj} U_j^{\text{out}}\Big) && \text{on } \Gamma^R, \end{aligned}$$

where $\zeta \in \mathbb{R} \setminus \{0\}$ is an arbitrary fixed number and ν is the outward normal. From the results of Chap. 4, it follows that $a_{nj}^0(R,k) \to S_{nj}(k)$ with exponential rate as $R \to \infty$. More precisely, there exist positive constants Λ and C such that

$$|a_{nj}^0(R,k) - S_{nj}(k)| \leqslant C \exp(-\Lambda R), \quad j = 1, 2, \ldots, 2n_{\max},$$

for all $k^2 \in [\mu_1, \mu_2]$ and sufficiently large R; the interval $[\mu_1, \mu_2]$ of the continuous spectrum lies between two neighboring thresholds and does not contain the thresholds. (Note that application of the method is not hindered by possible presence, on the interval $[\mu_1, \mu_2]$, of eigenvalues of the problem).

We can put $\mathcal{X}_n^R = v_n^{\text{in}} + \sum_{j=1}^{2n_{\max}} a_{nj} v_j^{\text{out}}$, where v_n^{in}, v_n^{out} are solutions to the problems

$$\begin{aligned} -(\Delta + E)v_n^{\text{in}} &= 0 && \text{in } G^R, \\ v_n^{\text{in}} &= 0 && \text{on } \partial G^R \setminus \Gamma^R, \\ (\partial_\nu + i\zeta)v_n^{\text{in}} &= (\partial_\nu + i\zeta)U_n^{\text{in}} && \text{on } \Gamma^R \end{aligned}$$

and

$$\begin{aligned} (\Delta + E)v_j^{\text{out}} &= 0 && \text{in } G^R, \\ v_j^{\text{out}} &= 0 && \text{on } \partial G^R \setminus \Gamma^R, \\ (\partial_\nu + i\zeta)v_j^{\text{out}} &= (\partial_\nu + i\zeta)U_j^{\text{out}} && \text{on } \Gamma^R, \quad j = 1, 2, \ldots, 2n_{\max}. \end{aligned}$$

Now we can rewrite the functional J_n^R in the form

$$J_n^R = \langle a_n \mathcal{E}^R, a_n \rangle + 2\text{Re}\, \langle \mathcal{F}_n^R, a_n \rangle + \mathcal{G}_n^R,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{C}^{2n_{\max}}$, $\mathcal{E}^R$ denotes the matrix with entries

$$\mathcal{E}_{pq}^R = \Big(v_p^{\text{out}} - U_p^{\text{out}}, v_q^{\text{out}} - U_q^{\text{out}}\Big)_{L_2(\Gamma^R)}, \quad p, q = 1, 2, \ldots, 2n_{\max},$$

the row $\mathcal{F}_n^R$ consists of the elements

$$\mathcal{F}_{nq}^R = \Big(v_n^{\text{in}} - U_n^{\text{in}}, v_q^{\text{out}} - U_q^{\text{out}}\Big)_{L_2(\Gamma^R)}, \quad q = 1, 2, \ldots, 2n_{\max},$$

and the number $\mathcal{G}_n^R$ is defined by

$$\mathcal{G}_n^R = \left\| v_n^{\mathrm{in}} - U_n^{\mathrm{in}} \right\|_{L_2(\Gamma^R)}^2 .$$

The minimizer of J_n^R satisfies $a_n \mathcal{E}^R + \mathcal{F}_n^R = 0$. We take the solution of this equation as an approximation to the nth row of the scattering matrix.

9.1.3 Discussion of Numerical Results

If the resonator is symmetric about the x-axis, then only scattering with preserved transverse quantum number evenness is possible (the incident and scattered waves have the same evenness). Let us explain the fact in more detail. The original problem reads

$$\begin{aligned} -\Delta u_n - k^2 u_n &= 0 \quad \text{in} \quad G, \\ u_n &= 0 \quad \text{on} \quad \partial G, \end{aligned} \tag{9.1.2}$$

$$u_n = U_n^{\mathrm{in}} + \sum_{j=1}^{2n_{\max}} S_{nj} U_j^{\mathrm{out}} + O(e^{-\delta|x|}) \quad \text{as} \quad |x| \to \infty.$$

Let $v_n(x, y) = u_n(x, -y)$ and $n \le n_{\max}$ (we consider only the upper half of the scattering matrix). The function v_n satisfies

$$\begin{aligned} -\Delta v_n - k^2 v_n &= 0 \quad \text{in} \quad G, \\ v_n &= 0 \quad \text{on} \quad \partial G, \end{aligned} \tag{9.1.3}$$

and

$$v_n = (-1)^{n+1} U_n^{\mathrm{in}} + \sum_{j=1}^{n_{\max}} S_{nj} (-1)^{j+1} U_j^{\mathrm{out}} + \sum_{j=n_{\max}+1}^{2n_{\max}} S_{nj} (-1)^{j-n_{\max}+1} U_j^{\mathrm{out}} + O(e^{-\delta|x|})$$

as $|x| \to \infty$. Let us assume that n is even. Then $w_n = (u_n + v_n)/2$ satisfies

$$\begin{aligned} -\Delta w_n - k^2 w_n &= 0 \quad \text{in} \quad G, \\ w_n &= 0 \quad \text{on} \quad \partial G, \end{aligned} \tag{9.1.4}$$

and

$$w_n = \sum_{j=1}^{n_{\max}} S_{nj} U_j^{\mathrm{out}} + \sum_{j=n_{\max}+1}^{2n_{\max}} S_{nj} U_j^{\mathrm{out}} + O(e^{-\delta|x|}) \quad \text{as} \quad |x| \to \infty,$$

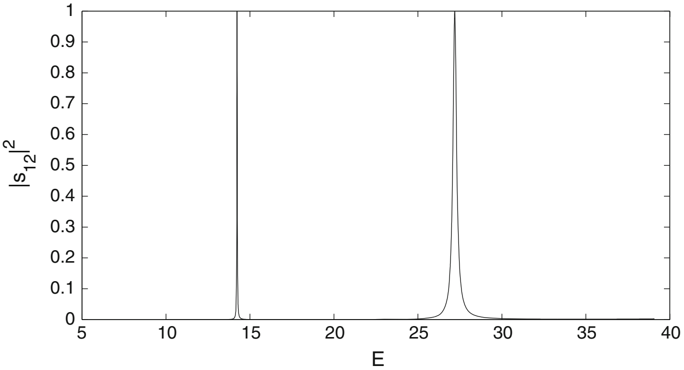


Fig. 9.3 Transmission probability for the wave U_1^{in} ($n_{max} = 1$, the transversal quantum number $n = 1$ for the scattered wave)

where the sums do not contain the terms with even j and $j - n_{\max}$, respectively. The problem has the trivial solution $w_n = 0$ only, therefore, S_{nj} with odd j (for $j \leq n_{\max}$) and $j - n_{\max}$ (for $j > n_{\max}$) are zero. If n is odd, we consider $w_n = (u_n - v_n)/2$ and conclude that S_{nj} with even j (for $j \leq n_{\max}$) and $j - n_{\max}$ (for $j > n_{\max}$) are zero.

For example, if the electron energy is between the second and the third thresholds ($4\pi^2 < E < 9\pi^2$), we have $S_{12} = S_{14} = 0$. This means that there are no transmissions between the transverse states.

The energies $E \approx 14.58$ and $E \approx 28.68$ are resonant and correspond to $n = 1$ and $m = 1, 2$ (Fig. 9.3). For $4\pi^2 < E < 9\pi^2$ (the electron energy between the second and the third thresholds), there are no changes of the transverse states, due to the evenness invariance. For the incident wave with $n = 1$, the resonant tunneling occurs at $E \approx 48$ and $E \approx 76$ (Fig. 9.4), which correspond to the closed resonator eigenvalues $E \approx 52.15$ and $E \approx 84.8$ with $n = 1$, $m = 3, 4$.

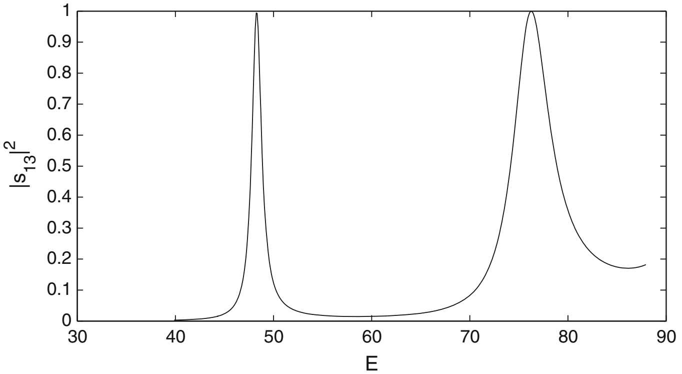


Fig. 9.4 Transmission probability for the wave U_1^{in} ($n_{max} = 2$, the transversal quantum number $n = 1$ for the scattered wave)

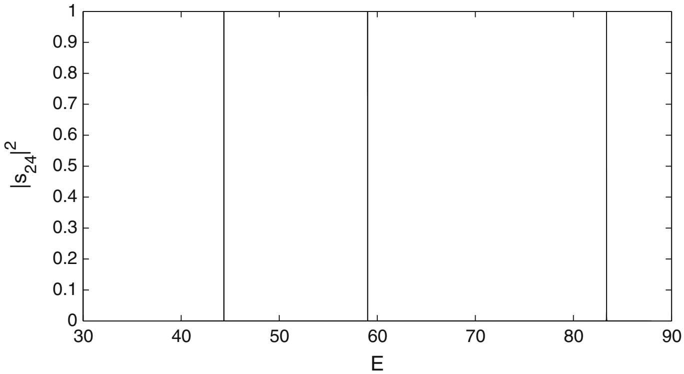


Fig. 9.5 Transmission probability for the wave U_2^{in} ($n_{max} = 2$, the transversal quantum number $n = 2$ for the scattered wave)

For incident wave U_2^{in}, resonant tunneling occurs at energies $E \approx 44.4$, 59.1, and 83.7 (Fig. 9.5), which correspond to $n = 2$, $m = 1, 2, 3$. The width of the resonant peaks for this case is significantly less than that for the wave U_1^{in}.

The explanation is that the height of the effective potential barrier created by a narrow is proportional to n^2/d^2, where n is the transversal quantum number of an incident wave and d is the narrow diameter; therefore, the barrier for U_2^{in} is four times greater than that for U_1^{in}. This also explains the smaller distance between the resonant peaks and the corresponding eigenvalues of the closed resonator.

For a waveguide symmetric about x-axis, transmissions between channels become possible only when $9\pi^2 < E < 16\pi^2$, i.e., when the electron energy is between the third and the fourth thresholds. We denote by $E_{\text{cr},n}$ (where "cr" means "critical") the height of an effective potential barrier for an electron with transversal quantum number n. Since for the wave U_1^{in} the longitudinal energy $E_{||}$ is large (it is above the barrier height for $E > E_{\text{cr},1} = 109, 7$), the probability of electron transmission without change of a transversal quantum number is fairly high (Fig. 9.6). The probability of electron transmission with change of transversal state $(n = 1) \to (n = 3)$ is high as well (Fig. 9.7). Both $|S_{14}|^2$ and $|S_{16}|^2$ have sharp resonance at $E \approx 93$, which corresponds to the eigenenergy of the closed resonator with $n = 3, m = 1$. For $|S_{16}|^2$, the peak is natural, since, for the longitudinal component, the resonance conditions hold. But, for $|S_{14}|^2$, the resonance is caused by the transmission $(n = 1) \to (n = 3)$, the resonant amplification of the wave, and the transmission to the initial state $(n = 3) \to (n = 1)$. The wave U_3^{in} behaves similarly: with strong direct $(n = 3) \to (n = 1)$ and reverse $(n = 1) \to (n = 3)$ transmissions with a change of the transversal quantum number and with the resonance at $E \approx 93$.

For incident electrons with $n = 2$, no change of transverse state is possible and there is a sharp resonance of unit height at $E \approx 117$ (Fig. 9.8). For an incident electron with energy $16\pi^2 < E < 25\pi^2$ the situation is even more complicated, since transitions with change of transverse quantum number become possible for

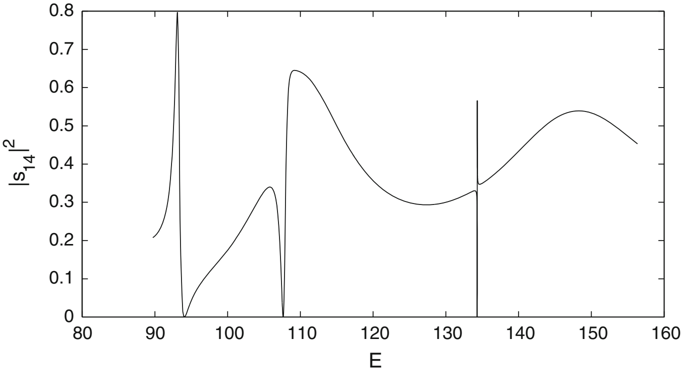


Fig. 9.6 Transmission probability for the wave U_1^{in} ($n_{max} = 3$, the transversal quantum number $n = 1$ for the scattered wave)

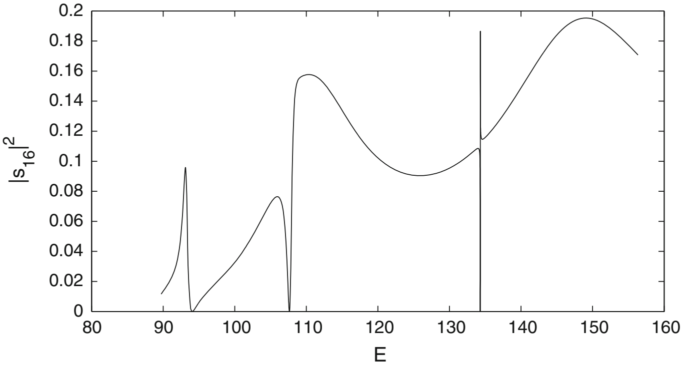


Fig. 9.7 Transmission probability for the wave U_1^{in} ($n_{max} = 3$, the transversal quantum number $n = 3$ for the scattered wave)

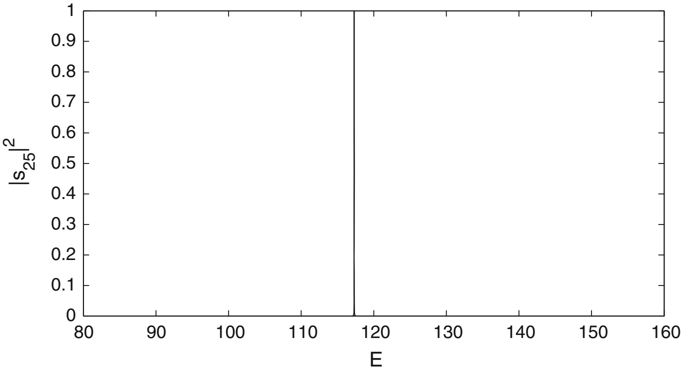


Fig. 9.8 Transmission probability for the wave U_2^{in} ($n_{max} = 3$, the transversal quantum number $n = 2$ for the scattered wave)

$n = 2$, too. We do not analyze the obtained results here, because qualitatively the effects are similar to those for $9\pi^2 < E < 16\pi^2$.

The wave U_3^{in} with energy greater than $E_{cr,1}$ approaches the narrows and partially goes in the state with $n = 1$, so the probability to pass through the resonator is large even for non-resonant energies (due to passing over the barrier) (see Fig. 9.11). The critical energy $E_{cr,3}$ for the original mode is greater than the electron energy, and the transmission probability without changing the state has peaks at the energies which are resonant for the state with $n = 3$. An electron with $n = 1$ whose energy exceeds the third threshold partially goes in the state with $n = 3$, having resonances at the same energies as an electron in the state with $n = 3$.

Figures 9.9, 9.10, 9.11 and 9.12 show the E-dependence of T_1–T_4; here T_n stands for the full transmission probability of U_n^{in}. The numbers (n, m) are the transversal and longitudinal quantum numbers of the respective eigenvalues of the closed resonator. Similarly to U_3^{in}, the wave U_4^{in} partially changes its transversal state (goes in the state with $n = 2$) and has resonances corresponding to the states $n = 4$ and $n = 2$. However, the fourth threshold is less than $E_{cr,2}$ and, a fortiori, than $E_{cr,4}$ (the

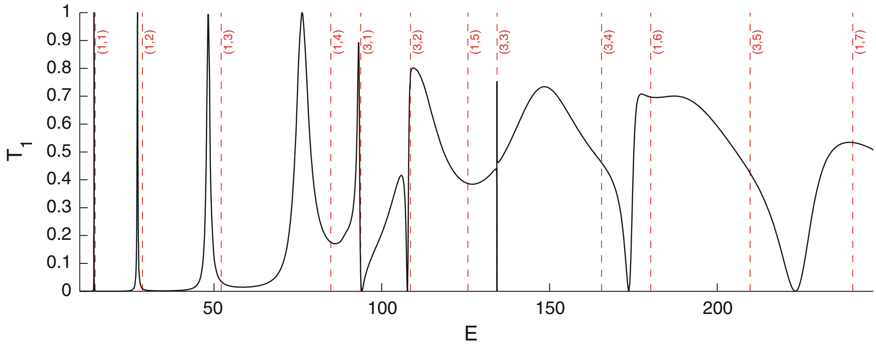


Fig. 9.9 Total transition probability for U_1^{in}

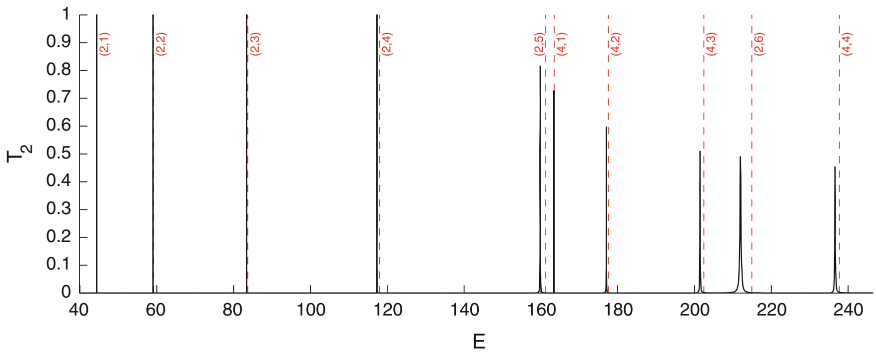


Fig. 9.10 Total transition probability for U_2^{in}

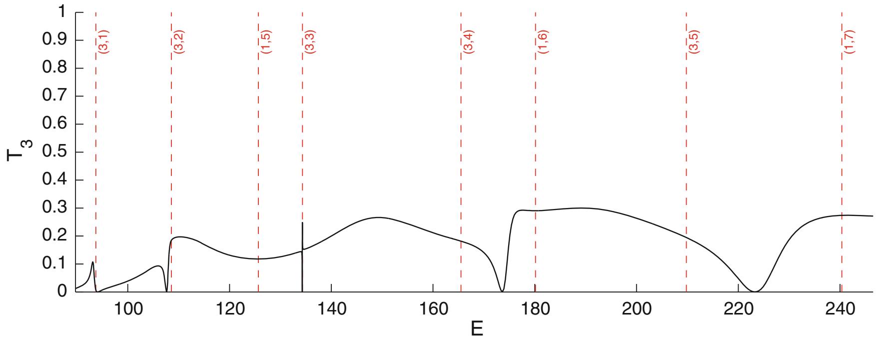


Fig. 9.11 Total transition probability for U_3^{in}

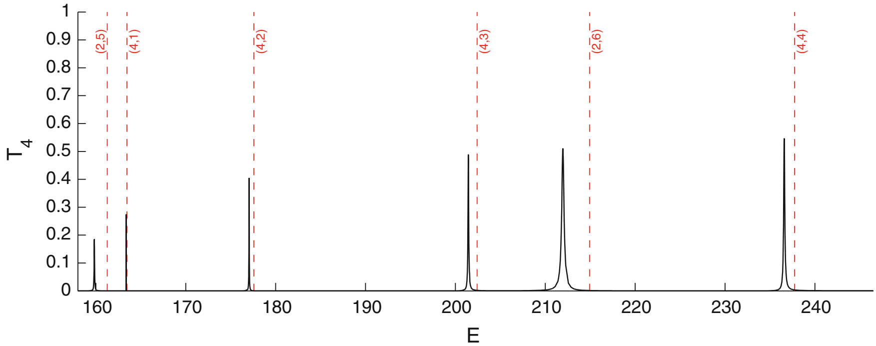


Fig. 9.12 Total transition probability for U_4^{in}

barriers are very high), so the free path through the resonator is impossible for $n = 2$ and $n = 4$. Therefore, the peaks in Figs. 9.10 and 9.12 are very narrow regardless of the interference of the modes with $n = 2$ and $n = 4$ (causing the slight asymmetry of the peaks).

The peaks in Fig. 9.12 corresponding to the resonant energies for $n = 2$ ($E \approx 160$ and $E \approx 212$) are wider than the nearest peaks with $n = 4$ because the barrier height for the state with $n = 2$ is notably lower than with $n = 4$.

Evidently, the sharp resonances with transmission probability T close to unit exist only below the third threshold. Therefore, in designing electronic devices based on the resonant tunneling in quantum waveguides of variable cross-section, the parameters of the system (the cross-section area, the waveguide material, the operation voltage) should be chosen so that the energy of an electron in the system would not exceed the third threshold.

9.2 Fano Resonances

We are now going to interpret the above numerical results from another point of view. We consider the entry $S_{nk}(E)$ of the scattering matrix $S(E)$ as the probability amplitude of electron transmission from n-state before the resonator to k-state after passing through the resonator. The amplitude $S_{nk}(E)$ can be represented as the sum $S_{nk}(E) = \sum_s A_{nsk}(E)$, where $A_{nsk}(E)$ is the probability amplitude of electron transmission from n to k through an intermediate state s.

Let us explain the origination of A_{nsk}. As before, we denote by G_0 the bounded part of the limit resonator $G(0, 0)$ and let $k_1^2 \leq k_2^2 \leq \dots$ be the eigenvalues of the problem

$$\begin{aligned} -\Delta v(x, y) - k^2 v(x, y) = 0, \quad (x, y) \in G_0, \\ v(x, y) = 0, \quad (x, y) \in \partial G_0, \end{aligned}$$

numbered according to their multiplicities. The resonant energies of the waveguide $G(\varepsilon, \varepsilon)$ form the sequence $\mathrm{Re}E_1, \mathrm{Re}E_2, \dots$, where $E_1, E_2, \dots$ can be viewed as the "perturbed" $k_1^2, k_2^2, \dots$, while $\mathrm{Im}E_j < 0$ for all $j = 1, 2, \dots$. The amplitude A_{nsk} admits the representation

$$A_{nsk}(E) = H_{nk}^{(s)}(E) + \frac{R_{nk}^{(s)}(E)}{E - E_s}$$

with continuous functions $E \to H_{nk}^{(s)}(E)$ and $E \to R_{nk}^{(s)}(E)$. Thus, $A_{nsk}(E)$ is the probability amplitude of electron transmission from n-state to k-state with resonant energy $\mathrm{Re}E_s$. The transmission probability $T_{nk}(E)$ from the n-state on the left of the resonator to the k-state on the right of the resonator is equal to $|S_{nk}(E)|^2$. When considering T_{nk} in a small energy interval, we ignore the weak dependence of the $H_{nk}^{(s)}$ and $R_{nk}^{(s)}$ on the electron energy E. Then

$$T_{nk}(E) \approx \mathcal{T}_{nk}(E) \equiv |H_{nk} + \frac{R_{nk}^{(1)}}{E - E_1} + \dots + \frac{R_{nk}^{(q)}}{E - E_q}|^2, \tag{9.2.1}$$

while $\mathrm{Re}E_1, \dots, \mathrm{Re}E_q$ are all resonant energies in the mentioned small interval and H_{nk}, $R_{nk}^{(1)}$, ..., $R_{nk}^{(q)}$ are some constants.

Let us consider $\mathcal{T}_{nk}(E)$ of the form

$$\mathcal{T}_{nk}(E) = |H_{nk} + \frac{R_{nk}^{(r)}}{E - E_r}|^2, \tag{9.2.2}$$

where $H_{nk} = \sum_s H_{nk}^{(s)}$. Since the values H_{nk}, $R_{nk}^{(r)}$, and E_r are complex, the expression (9.2.2) contains the five independent parameters $|H_{nk}|$, $|R_{nk}^{(r)}|$, $|E_r|$, $\psi = \arg E_r$, and $\varphi = \arg(H_{nk} - R_{nk}^{(r)})$:

$$\mathcal{T}_{nk}(E) = |e^{i\varphi}|H_{nk}| + \frac{|R_{nk}^{(r)}|}{E - |E_r|e^{i\psi}}|^2.$$

This equality can be rearranged to

$$\mathcal{T}_{nk}(E) = |H_{nk}|^2 \left(1 + 2Q\frac{E\cos\varphi - |E_r|\cos(\varphi+\psi)}{E^2 - 2E|E_r|\cos\psi + |E_r|^2} + Q^2\frac{1}{E^2 - 2E|E_r|\cos\psi + |E_r|^2}\right), \tag{9.2.3}$$

where $Q = |R_{nk}^{(r)}/|H_{nk}|$. The expression in brackets depends on the four parameters Q, $|E_r|$, φ, and ψ.

If $H_{nk} = 0$, we obtain

$$\mathcal{T}_{nk}(E) = \frac{|R_{nk}^{(r)}|^2}{E^2 - 2E|E_r|\cos\psi + |E_r|^2} = \frac{|R_{nk}^{(r)}|^2}{(E - |E_r|\cos\psi)^2 + (|E_r|\sin\psi)^2}.$$

Thus, in such a case, the resonant curve takes the form of a standard Breit-Wigner's resonant curve with frequency $|E_r|\cos\psi$ and with half-width of the resonant peak equal to $|E_r|\sin\psi$ at the peak half-height.

Figures 9.13 and 9.14 show typical dependences of $\mathcal{T}_{nk}$ on E for $H_{nk} \neq 0$. The values of H_{nk} have been chosen to provide $\max \mathcal{T}_{nk} = 1$.

Figure 9.13 depicts a resonant curve resulting from the interference of a resonant mode and a non-resonant mode. Such a situation is typical for Fano resonances (e.g., see [36]). In what follows, any resonant curve of the form (9.2.1) with $H_{nk} \neq 0$ is called a Fano resonant curve.

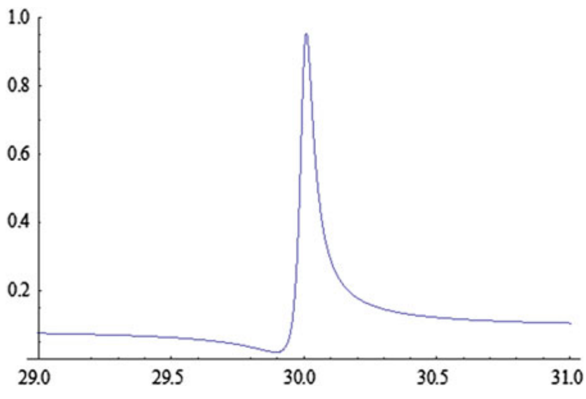


Fig. 9.13 E-dependence of $\mathcal{T}_{nk}$ for $|E_r| = 30$, $Q = 10$, $\varphi = 0.2$ and $\psi = 0.001$

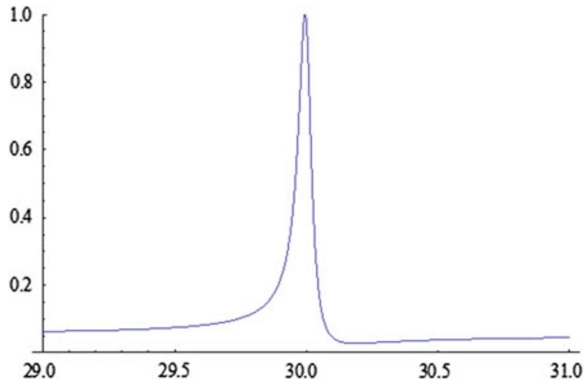


Fig. 9.14 E-dependence of $\mathcal{T}_{nk}$ for $|E_r| = 30$, $Q = 10$, $\varphi = 2.5$ and $\psi = 0.001$

The extrema of function (9.2.3) coincide with the roots of the quadratic equation

$$E^2 \cos\varphi + (Q - 2|E_r|\cos(\varphi+\psi))E + |E_r|(|E_r| - Q\cos\psi) = 0.$$

The roots are given by

$$\frac{1}{2}(\sec\varphi)\left(-Q + 2|E_r|\cos(\varphi+\psi) \pm (Q^2 + 4Q|E_r|\sin\varphi\sin\psi + 4|E_r|^2\sin^2\psi)^{1/2}\right);$$

one of the roots relates to the maximum of function (9.2.3) and the other one corresponds to its minimum.

Let us assume that we know the calculated T_{nk} obtained, for instance, by computing the scattering matrix. Then we can employ the method of the least squares to approximate the obtained T_{nk} by expression (9.2.1); in doing this, we find the unknown parameters E_r, $|H_{nk}|$ etc., in (9.2.1). Figure 9.15 shows the results of such kind for a wave incident on the resonator in the transverse state $n = 1$ and scattered into various states in the resonator that arises from the resonator in the state $k = 1$ (see Fig. 9.6); the resonance occurred for the mode in the state $r = 3$ inside the resonator. Thus, the approximating curve is of the form

$$\mathcal{T}_{11}(E) = |H_{11} + \frac{R_{11}^{(3)}}{E - E_3}|^2.$$

Figure 9.16 depicts the approximating

$$\mathcal{T}_{13}(E) = |H_{13} + \frac{R_{13}^{(3)}}{E - E_3}|^2.$$

for the calculated $T_{13}(E)$ (see Fig. 9.7).

Figure 9.17 relates to the passage $n = 3 \to k = 3$. The $|H_{33}| = 0.1921$ and $|R_{33}^{(3)}| = 0.0005673$ are significantly less than those for the passages $n = 1 \to k = 1$

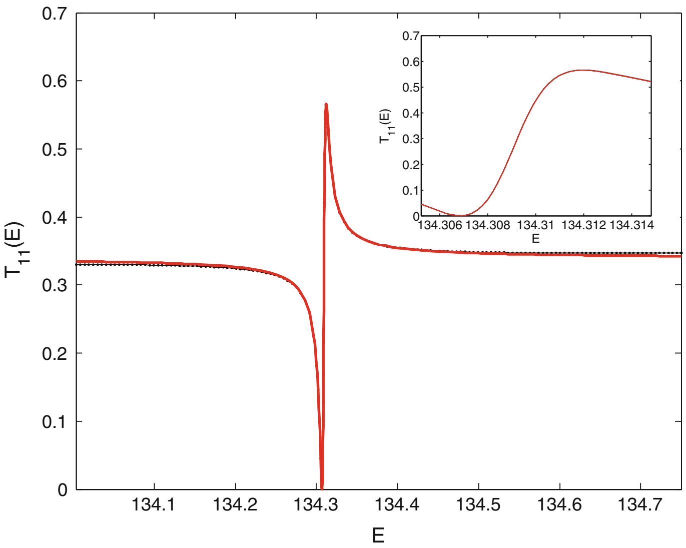


Fig. 9.15 The calculated curve T_{11} obtained by computing the scattering matrix (*solid line*) and the approximating curve $\mathcal{T}_{11}$ obtained by the method of the least squares (*dashed line*) practically coincide; $|H_{11}| = 0.5827$; $|R_{11}^{(3)}| = 0.00194$; $\mathrm{Re}\,E_3 = 134.309$; $\mathrm{Im}\,E_3 = 0.00252$. *Inset* the domain of rapid varying of $T_{11}(E)$; minimum at 134.311, maximum at 134.37

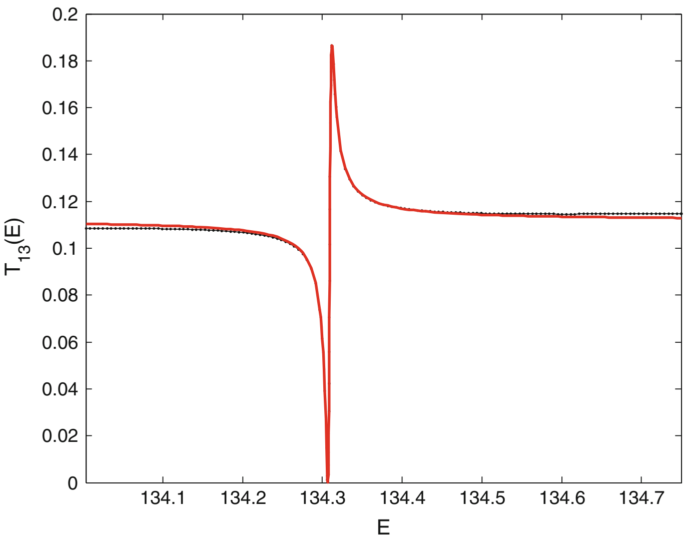


Fig. 9.16 The calculated curve T_{13} obtained by computing the scattering matrix (*solid line*) and the approximating curve $\mathcal{T}_{13}$ obtained by the method of the least squares (*dashed line*) practically coincide; $|H_{13}| = 0.3345$; $|R_{13}^{(3)}| = 0.00111$; $\mathrm{Re}\,E_3 = 134.309$; $\mathrm{Im}\,E_3 = 0.00252$

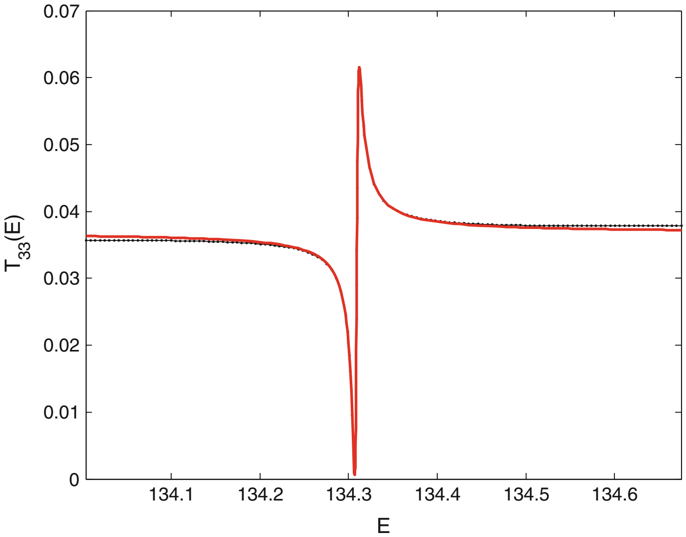


Fig. 9.17 Transmission $n = 3 \to k = 3$, $|H_{33}| = 0.1921$ and $|R^3_{33}| = 0.0005673$

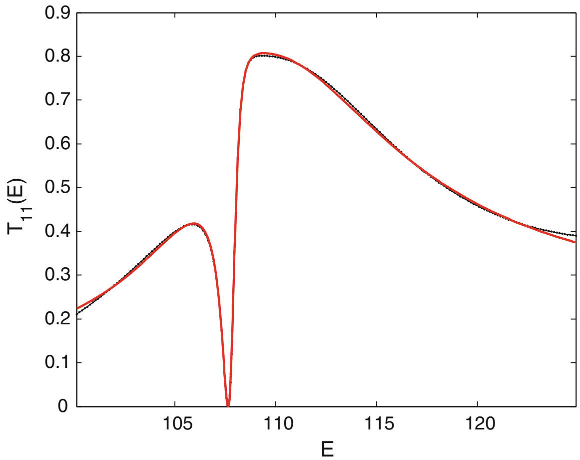


Fig. 9.18 Approximating function (9.2.4) with $\mathrm{Re}E_1 = 107.79$, $\mathrm{Re}E_2 = 108.83$, $\mathrm{Im}E_1 = 0.394$, and $\mathrm{Im}E_2 = 6.83$

and $n = 1 \to k = 3$. However, the $\mathrm{Re}E_3$ and $\mathrm{Im}E_3$ are the same for all the passages shown by Figs. 9.15, 9.16 and 9.17. Note that the $\mathrm{Re}E_3$ is about 0.001 less than the corresponding eigenvalue of the closed resonator (see Table 9.1 in Sect. 9.1.1).

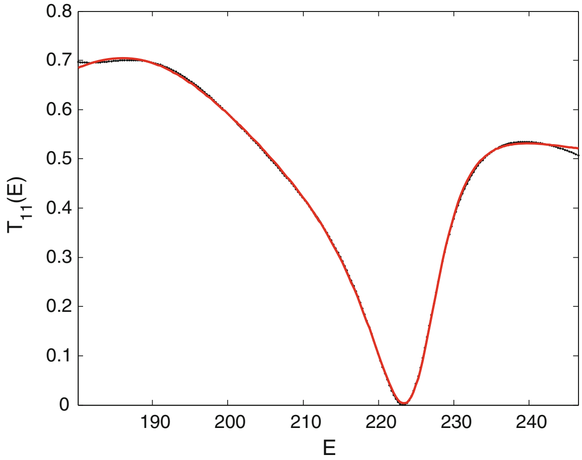


Fig. 9.19 Approximation shows two resonances $\mathrm{Re}E_1 = 188.26$ and $\mathrm{Re}E_2 = 225.35$ with $\mathrm{Im}E_1 = 6.04$ and $\mathrm{Im}E_2 = 24.38$. Due to the interference of waves, the second resonance energy is close not to the maximum of the transmission coefficient but to its minimum

When considering the resonances near $E = 110$ (see Fig. 9.6), we use instead of (9.2.1) an approximating function of the form

$$\mathcal{T}_{nk}(E) = |H_{nk} + \frac{R_{nk}^{(1)}}{E - E_1} + \frac{R_{nk}^{(2)}}{E - E_2}|^2 \tag{9.2.4}$$

Figure 9.18 it turns out that, near $E = 110$, there are two resonances of a low quality factor and the approximation (9.2.1) is not proper. In analogous way, one can consider the resonance near $E = 220$, see Figs. 9.9 and 9.19.

Chapter 10
Asymptotic Analysis of Multichannel Resonant Tunneling

In the chapter, we generalize, for electrons of high energy, the asymptotic theory exposed in Chap. 6. We present and justify the asymptotics of tunneling characteristics as the narrow diameters tend to zero.

10.1 Statement of the Problem and Limit Problems

Let a waveguide $G(\varepsilon_1, \varepsilon_2)$ be the same as in Chap. 6. Outside a large ball, it coincides with the union of two semicylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ and has two narrows of small diameters ε_1 and ε_2. The wave function of a free electron of energy $E = \hbar^2 k^2/2m$ satisfies the boundary value problem

$$-\Delta u - k^2 u = 0 \text{ in } G(\varepsilon_1, \varepsilon_2), \quad u = 0 \text{ on } \partial G(\varepsilon_1, \varepsilon_2). \tag{10.1.1}$$

To formulate radiation conditions, we need the boundary value problem on the cross-section D_j of the semicylinder $\mathcal{C}_j$, $j = 1, 2$:

$$-\Delta v - \lambda^2 v = 0 \text{ in } D_j, \quad v = 0 \text{ on } \partial D_j. \tag{10.1.2}$$

The eigenvalues λ_{jm}^2 of this problem, where $m = 1, 2, \dots$, are called the thresholds; they form an increasing sequence of positive numbers tending to $+\infty$. We assume that k^2 is not a threshold and denote by M_j the number of thresholds satisfying $\lambda_{jm}^2 < k^2$. For such an eigenvalue λ_{jm}^2, let Ψ_{jm} be an eigenfunction of problem (10.1.2) that corresponds to λ_{jm}^2 and is normalized by

$$2\nu_{jm} \int_{D_j} |\Psi_{jm}(x_2, x_3)|^2 dx_2\, dx_3 = 1 \tag{10.1.3}$$

L. Baskin et al., *Resonant Tunneling*, Lecture Notes on Numerical Methods in Engineering and Sciences, DOI 10.1007/978-3-319-15105-2_10

with $\nu_{jm} = \sqrt{k^2 - \lambda_{jm}^2}$. The function U_m^+, $m = 1, \dots, M_1$, defined in the semicylinder $\mathcal{C}_1$ by $U_m^+(x^1) = \exp(-i\nu_{1m}x_1^1)\Psi_{1m}(x_2^1, x_3^1)$, is a wave coming in $\mathcal{C}_1$ from infinity (the positive half-axis x_1^1 lies in $\mathcal{C}_1$). The function $U_{M_1+m}^+(x^2) = \exp(-i\nu_{2m}x_1^2)\Psi_{2m}(x_2^2, x_3^2)$, $m = 1, \dots, M_2$, is a wave coming from infinity in $\mathcal{C}_2$. The outgoing waves U_m^-, $m = 1, \dots, M_1 + M_2$, are obtained from the incoming ones by complex conjugation: $U_m^- = \overline{U_m^+}$.

There exist (smooth) solutions u_m, $m = 1, \dots, M_1 + M_2$, to problem (10.1.1) satisfying the radiation conditions

$$u_m(x) = \begin{cases} U_m^+(x^1) + \sum_{p=1}^{M_1} S_{mp}\, U_p^-(x^1) + O(e^{-\delta x_1^1}), & x_1^1 \to +\infty, \\ \sum_{p=1}^{M_2} S_{m,M_1+p}\, U_{M_1+p}^-(x^2) + O(e^{-\delta x_1^2}), & x_1^2 \to +\infty, \end{cases} \quad m = 1, \dots, M_1; \tag{10.1.4}$$

$$u_{M_1+m}(x) = \begin{cases} \sum_{p=1}^{M_1} S_{M_1+m,p}\, U_p^-(x^1) + O(e^{-\delta x_1^1}), & x_1^1 \to +\infty, \\ U_{M_1+m}^+(x^2) & \\ + \sum_{p=1}^{M_2} S_{M_1+m,M_1+p}\, U_{M_1+p}^-(x^2) + O(e^{-\delta x_1^2}), & x_1^2 \to +\infty, \end{cases} \quad m = 1, \dots, M_2. \tag{10.1.5}$$

with sufficiently small positive δ. The scattering matrix $S = \|S_{pq}\|_{p,q=1,\dots,M_1+M_2}$ is unitary. The value

$$R_m = \sum_{p=1}^{M_1} |S_{mp}|^2, \quad m = 1, \dots, M_1, \tag{10.1.6}$$

is called the reflection coefficient for the wave U_m coming to $G(\varepsilon_1, \varepsilon_2)$ from $\mathcal{C}_1$. The transition coefficient for this wave is defined by

$$T_m = \sum_{p=1}^{M_2} |S_{m,M_1+p}|^2. \tag{10.1.7}$$

One can give similar definitions for the wave U_{M_1+m} coming from $\mathcal{C}_2$.

We seek the resonant values $k_r = k_r(\varepsilon_1, \varepsilon_2)$ of the parameter k, where the transition coefficient $T_m = T_m(k, \varepsilon_1, \varepsilon_2)$ takes the maximal values. Moreover, we are interested in the behavior of $k_r(\varepsilon_1, \varepsilon_2)$, $T_m(k, \varepsilon_1, \varepsilon_2)$ and that of the reflection coefficient $R_m = R_m(k, \varepsilon_1, \varepsilon_2)$ as ε_1 and ε_2 tend to 0.

To derive an asymptotics of a wave function (i.e., a solution to problem (10.1.1)) as $\varepsilon_1, \varepsilon_2 \to 0$, we use the method of compound asymptotic expansions. To this end, we introduce "limit" boundary value problems independent of the parameters ε_1 and ε_2. Recall that the limit domain $G(0, 0)$ consists of the unbounded parts G_1, G_2, and the bounded resonator G_0. The domain G_j, $j = 1, 2$, has a conical point O_j and a cylindrical end $\mathcal{C}_j$, and the resonator G_0 has two conical points O_1 and O_2. The problems

$$-\Delta v(x) - k^2 v(x) = f, \quad x \in G_j; \quad v(x) = 0, \quad x \in \partial G_j, \tag{10.1.8}$$

are called first kind limit problems, where $j = 0, 1, 2$. In the domains Ω_j, $j = 1, 2$, introduced in Sect. 6.1, we consider the boundary value problems

$$\triangle w(\xi^j) = F(\xi^j) \text{ in } \Omega_j, \quad w(\xi^j) = 0 \text{ on } \partial\Omega_j, \tag{10.1.9}$$

which are called second kind limit problems; by $\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j)$ we mean Cartesian coordinates with origin at O_j. The limit problems in G_0 and Ω_j are the same as in Chap. 6; in particular, Propositions 6.2.1 and 6.2.3 remain valid. Now, problems in G_1 and G_2 have some new features in comparison with those in Chap. 6, because we consider multichannel scattering. The next proposition has to replace Proposition 6.2.2. For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \ldots$, the space $V^l_{\gamma,\delta}(G_j)$ is the completion, in the norm defined by (6.2.7), of the set $C_c^\infty(\overline{\Omega}_j)$ of compactly supported smooth functions in $\overline{\Omega}_j$. Let the cut-off functions $\chi_{\infty,j}$ be the same as in (6.2.7).

Proposition 10.1.1 *Assume that* $|\gamma - 1| < \mu_{j1} + 1/2$ *and, moreover, there is no nontrivial solution to the homogeneous problem* (10.1.8) *(where* $f = 0$*) in* $V^2_{\gamma,\delta}(G_j)$ *with arbitrary small positive* δ. *Then, for any* $f_j \in V^0_{\gamma,\delta}(G_j)$*, there exists a unique solution* v_j *to problem* (10.1.8) *such that the representations*

$$v_1 = u_1 + \sum_{m=1}^{M_1} A_m \chi_{\infty,1} U_m^- \quad \text{in} \quad G_1,$$
$$v_2 = u_2 + \sum_{m=1}^{M_2} A_{M_1+m} \chi_{\infty,2} U_{M_1+m}^- \quad \text{in} \quad G_2$$

are valid; $A_m = const$, $u_j \in V^2_{\gamma,\delta}(G_j)$ *with a sufficiently small* δ, *and the estimates*

$$\|u_1; V^2_{\gamma,\delta}(G_1)\| + \sum_{m=1}^{M_1} |A_m| \leq c\|f_1; V^0_{\gamma,\delta}(G_1)\|, \tag{10.1.10}$$

$$\|u_2; V^2_{\gamma,\delta}(G_2)\| + \sum_{m=1}^{M_2} |A_{M_1+m}| \leq c\|f_2; V^0_{\gamma,\delta}(G_2)\| \tag{10.1.11}$$

hold with constants c *independent of* f_j. *If, in addition,* f_j *is smooth and vanishes near* O_j, *the solution* v_j *satisfies*

$$v_j(x^j) = a_j \frac{1}{\sqrt{r_j}} \widetilde{J}_{\mu_{j1}+1/2}(kr_j)\Phi^K_{j1}(\varphi_j) + O\big(r_j^{\mu_{j2}}\big), \quad r_j \to 0,$$

where a_j is a constant and Φ_{j1}^K denotes an eigenfunction to the Beltrami operator corresponding to $\mu_{j1}(\mu_{j1}+1)$ and normalized by

$$(2\mu_{j1}+1)\int_{S(K_j)} |\Phi_{j1}^K(\varphi)|^2 d\varphi = 1.$$

10.2 Tunneling in a Waveguide with One Narrow

In this section, we consider the electron motion in a waveguide $G(\varepsilon)$ with one narrow. To describe $G(\varepsilon)$, we assume that $G = D \times \mathbb{R}$, where D is a bounded domain in $\mathbb{R}^2$ and ∂D is a smooth simple contour. A double cone $K \cup L$ with vertex $O \in G$ and domains Ω and $\Omega(\varepsilon)$ are defined like $K_1 \cup L_1$, Ω_1, and $\Omega_1(\varepsilon)$ in Sect. 6.1. We set $G(\varepsilon) = G \cap \Omega(\varepsilon)$. The limit waveguide $G(0)$ consists of two components; either of them has one conical point and one cylindrical end at infinity. We denote the components by G_1 and G_2.

10.2.1 Special Solutions to the First Kind Homogeneous Problems

In the domain G_1, a basis in the space of bounded solutions is formed by V_m satisfying the radiation condition

$$V_m(x) = U_m^+(x) + \sum_{p=1}^{M} S_{mp}^0 U_p^-(x) + O(\exp(-\delta x_1)), \quad x_1 \to +\infty, \tag{10.2.1}$$

with arbitrary small positive δ, where $m = 1, \dots, M$. The $S_1^0 = \|S_{mp}^0\|_{m,p=1}^M$ is the scattering matrix in G_1, and it is unitary. In a neighborhood of O, there is the asymptotics

$$V_m(x) = s_{m1}\frac{1}{\sqrt{r}}\widetilde{J}_{\mu_1+1/2}(kr)\Phi_1^K(\varphi) + O\big(r^{\mu_2}\big), \quad r \to 0. \tag{10.2.2}$$

In G_2, we consider analogous solutions admitting the expansions

$$V_{M+m}(x) = \begin{cases} U_{M+m}^-(x) + \sum_{p=1}^M S_{M+m,M+p}^0 U_{M+p}^-(x) + O(e^{\delta x_1}), & x_1 \to -\infty, \\ s_{M+m,2}\dfrac{1}{\sqrt{r}}\widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^L(\varphi) + O(r^{\mu_2}), & r \to 0, \end{cases} \tag{10.2.3}$$

$m = 1, \dots, M$. The scattering matrix $S_2^0 = \|S^0_{M+m,M+p}\|^M_{m,p=1}$ in G_2 is unitary. In either of the domains G_1 and G_2, we assume that the homogeneous problem (10.1.8) (with $f = 0$) has no nontrivial bounded solutions exponentially decaying at infinity. In what follows, to construct an asymptotics of a wave function, we will use special solutions to the problem unbounded near the point O.

Let v_1^K be the function defined by (6.3.5) and introduce a solution

$$\mathbf{v}_1(x) = \Theta(r) v_1^K(x) + \widetilde{v}_1(x) \tag{10.2.4}$$

of homogeneous problem (10.1.8) in G_1, where $\widetilde{v}_1$ is the solution provided by Proposition 10.1.1 for problem (10.1.8) with $f = -[\triangle, \Theta] v_1^K$. Thus,

$$\mathbf{v}_1(x) = \begin{cases} \dfrac{1}{\sqrt{r}}\big(\widetilde{N}_{\mu_1+1/2}(kr) + a_1 \widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^K(\varphi) + O(r^{\mu_2}), & r \to 0, \\ \sum_{m=1}^M A_{1m} U_m^-(x) + O(e^{-\delta x_1}), & x_1 \to +\infty, \end{cases} \tag{10.2.5}$$

where $\widetilde{J}_\mu$ is the same as in Propositions 6.2.1 and 10.1.1. In G_2, a similar solution $\mathbf{v}_2$ admits the representation

$$\mathbf{v}_2(x) = \begin{cases} \dfrac{1}{\sqrt{r}}\big(\widetilde{N}_{\mu_1+1/2}(kr) + a_2 \widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^L(\varphi) + O(r^{\mu_2}), & r \to 0, \\ \sum_{m=1}^M A_{2,M+m} U_{M+m}^-(x) + O(e^{\delta x_1}), & x_1 \to -\infty. \end{cases} \tag{10.2.6}$$

We define $(M \times 1)$-matrices s_1, s_2 and $(1 \times M)$-matrices A_1, A_2 by

$$s_1 = \|s_{m1}\|^M_{m=1}, \quad s_2 = \|s_{M+m,2}\|^M_{m=1}, \quad A_1 = \|A_{1m}\|^M_{m=1}, \quad A_2 = \|A_{2,M+m}\|^M_{m=1}.$$

Lemma 10.2.1 *The equalities $A_j A_j^* = 2\,\mathrm{Im}\, a_j$ and $A_j = i s_j^* S_j^0$ hold.*

Proof We restrict ourselves to considering $j = 1$; the case $j = 2$ can be treated in a similar way. Let $(u, v)_Q$ denote the integral $\int_Q u(x)\overline{v(x)}\,dx$ and let $G_{N,\delta}$ stand for the truncated domain $G_1 \cap \{x_1 < N\} \cap \{r > \delta\}$. By the Green formula,

$$\begin{aligned} 0 = (\triangle \mathbf{v}_1 + k^2 \mathbf{v}_1, \mathbf{v}_1)_{G_{N,\delta}} - (\mathbf{v}_1, \triangle \mathbf{v}_1 + k^2 \mathbf{v}_1)_{G_{N,\delta}} = (\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta}} \\ - (\mathbf{v}_1, \partial \mathbf{v}_1/\partial n)_{\partial G_{N,\delta}} = 2i\,\mathrm{Im}\,(\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_E \end{aligned}$$

with $E = (\partial G_{N,\delta} \cap \{x_1 = N\}) \cup (\partial G_{N,\delta} \cap \{r = \delta\})$. Taking into account (10.2.5) as $x_1 \to +\infty$ and (10.1.3), we have

$$\begin{aligned} \mathrm{Im}\,(\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta} \cap \{x_1 = N\}} &= \mathrm{Im} \sum_{m=1}^M \int_{D_1} A_{1m} \frac{\partial U_m^-}{\partial x_1}(x) \overline{A_{1m} U_m^-(x)}\Big|_{x_1 = N} dx_2\,dx_3 + o(1) \\ &= A_1 A_1^* \nu_1 \int_{D_1} |\Psi_1(x_2, x_3)|^2 dx_2\,dx_3 + o(1) = A_1 A_1^*/2 + o(1). \end{aligned}$$

Using (10.2.5) as $r \to 0$ and the normalization of Φ_1^K (see Proposition 10.1.1), we obtain

$$\begin{aligned}\operatorname{Im}(\partial \mathbf{v}_1/\partial n, \mathbf{v}_1)_{\partial G_{N,\delta}\cap\{r=\delta\}} &= \operatorname{Im}\int_{S(K)}\left[-\frac{\partial}{\partial r}\frac{1}{\sqrt{r}}\big(\widetilde{N}_{\mu_1+1/2}(kr) + a_1\widetilde{J}_{\mu_1+1/2}(kr)\big)\right]\\ &\quad\times\frac{1}{\sqrt{r}}\big(\widetilde{N}_{\mu_1+1/2}(kr)+\overline{a}_1\widetilde{J}_{\mu_1+1/2}(kr)\big)|\Phi_1^K(\varphi)|^2 r^2\Big|_{r=\delta} d\varphi + o(1)\\ &= -(\operatorname{Im} a_1)(2\mu_1+1)\int_{G_{N,\delta}}|\Phi_1^K(\varphi)|^2 d\varphi + o(1) = -\operatorname{Im} a_1 + o(1).\end{aligned}$$

Thus, $A_1A_1^*/2 - \operatorname{Im} a_1 + o(1) = 0$ as $N \to \infty$ and $\delta \to 0$, which implies the first equality of this lemma. To obtain the second one, we apply the Green formula in the domain $G_{N,\delta}$ to the functions $\mathbf{v}_1$ and V_m and arrive at $i\sum_{p=1}^M A_{1p}\overline{S_{mp}^0} + \overline{s_{m1}} + o(1) = 0$ with $N \to \infty$ and $\delta \to 0$. It remains to take into account that S_1^0 is unitary. $\square$

10.2.2 Asymptotic Formulas

Here we obtain the asymptotics of the amplitudes of the reflected and transited waves as $\varepsilon \to 0$. Let the wave function u_m, defined by asymptotics (10.1.4), be approximated in G_1 by the solution $v_1 = V_m + C_{m1}\mathbf{v}_1$ and in G_2 by the solution $v_2 = C_{m2}\mathbf{v}_2$ of the homogeneous limit problem. The special solutions V_m, $\mathbf{v}_1$, and $\mathbf{v}_2$ were defined in 10.2.1. For the time being, the constants C_{m1} and C_{m2} are unknown; we will find them when compensating the discrepancy principle terms. According to (10.2.2) and (10.2.5), we have, as $r \to 0$,

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{r}}\big(C_{m1}\widetilde{N}_{\mu_1+1/2}(kr) + (s_{m1}+C_{m1}a_1)\widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^K(\varphi) + O(r^{\mu_2}), \quad r \to 0,\\ v_2 &= \frac{1}{\sqrt{r}}\big(C_{m2}\widetilde{N}_{\mu_1+1/2}(kr) + C_{m2}\,a_2\widetilde{J}_{\mu_1+1/2}(kr)\big)\Phi_1^L(\varphi) + O(r^{\mu_2}), \quad r \to 0,\end{aligned}$$

that is, v_1 and v_2 admit expansions (6.3.9) and (6.3.10) with the constants

$$(a_1^-, a_2^-) = (C_{m1}, C_{m2}), \quad (a_1^+, a_2^+) = (s_{m1} + C_{m1}a_1, C_{m2}\,a_2). \tag{10.2.7}$$

As was shown in 6.3.2, the constants must satisfy the relation

$$(C_{m1}, C_{m2}) = (s_{m1} + C_{m1}a_1, C_{m2}\,a_2)\,\Lambda\,\varepsilon^{2\mu_1+1}.$$

We introduce the matrix $a = \operatorname{diag}(a_1, a_2)$ and obtain

$$(C_{m1}(\varepsilon), C_{m2}(\varepsilon)) = (s_{m1}, 0)(I - \Lambda\, a\,\varepsilon^{2\mu_1+1})^{-1}\Lambda\varepsilon^{2\mu_1+1}. \tag{10.2.8}$$

By virtue of (10.2.1) and (10.2.5) for $x_1^1 \to +\infty$,

$$v_1(x^1) = U_m^+(x^1) + \sum_{p=1}^{M} (S_{mp}^0 + C_{m1}(\varepsilon) A_{1p}) U_p^-(x^1) + O(\exp(-\delta x_1^1)), \quad x_1^1 \to +\infty,$$

$$v_2(x^2) = \sum_{p=1}^{M} C_{m2}(\varepsilon) A_{2p} U_{M+p}^-(x^2) + O(\exp(-\delta x_1^2)), \quad x_1^2 \to +\infty.$$

This provides an approximation $(\widetilde{S}_{m1}, \dots, \widetilde{S}_{m,2M})$ to the m-th line of the scattering matrix:

$$\widetilde{S}_{mp}(\varepsilon) = S_{mp}^0 + C_{m1}(\varepsilon) A_{1p}, \quad \widetilde{S}_{m,M+p}(\varepsilon) = C_{m2}(\varepsilon) A_{2,M+p}, \quad m, p = 1, \dots, M.$$

We introduce the $(M \times M)$-matrices $\widetilde{S}_{11} = \|\widetilde{S}_{mp}\|_{m,p=1}^M$, $\widetilde{S}_{12} = \|\widetilde{S}_{m,M+p}\|_{m,p=1}^M$, and temporarily denote the columns $\|C_{mq}\|_{m=1}^M$ by $\mathbf{C}_q$, $q = 1, 2$, then

$$(\widetilde{S}_{11}(\varepsilon), \widetilde{S}_{12}(\varepsilon)) = (S_1^0 + \mathbf{C}_1(\varepsilon) A_1, \mathbf{C}_2(\varepsilon) A_2) = (S_1^0, 0_{M\times M}) + (\mathbf{C}_1(\varepsilon), \mathbf{C}_2(\varepsilon)) A,$$

where $A = \operatorname{diag}(A_1, A_2)$. We set $s = \operatorname{diag}(s_1, s_2)$ and $S^0 = \operatorname{diag}(S_1^0, S_2^0)$, then by Lemma 10.2.1

$$A = i s^* S^0. \tag{10.2.9}$$

In view of (10.2.8) and (10.2.9), we obtain

$$(\widetilde{S}_{11}(\varepsilon), \widetilde{S}_{12}(\varepsilon)) = (S_1^0, 0_{M\times M}) + i(s_1, 0_{M\times 1})(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1} \Lambda\, s^* S^0 \varepsilon^{2\mu_1+1}. \tag{10.2.10}$$

An approximation to the wave function is of the form

$$\widetilde{u}_m(x; \varepsilon) = \chi_{\varepsilon,1}(x) v_1(x; \varepsilon) + \Theta(r) w(\varepsilon^{-1} x; \varepsilon) + \chi_{\varepsilon,2}(x) v_2(x; \varepsilon), \tag{10.2.11}$$

where, owing to (6.3.14),

$$v_1(x; \varepsilon) = V_m(x) + C_{m1}(\varepsilon) \mathbf{v}_1(x), \tag{10.2.12}$$

$$w(\xi; \varepsilon) = a_1^+(\varepsilon) \varepsilon^{\mu_1} \mathbf{w}^K(\xi) + a_2^+(\varepsilon) \varepsilon^{\mu_1} \mathbf{w}^L(\xi), \tag{10.2.13}$$

$$v_2(x; \varepsilon) = C_{m2}(\varepsilon) \mathbf{v}_2(x). \tag{10.2.14}$$

From (10.2.7) and (10.2.8) it follows that

$$\begin{aligned}(a_1^+(\varepsilon), a_2^+(\varepsilon)) &= (s_1, 0_{M\times 1}) + (s_1, 0_{M\times 1})(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1} \Lambda\, a\, \varepsilon^{2\mu_1+1} \\ &= (s_1, 0_{M\times 1})\, (I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}.\end{aligned} \tag{10.2.15}$$

An approximation $\widetilde{u}_{M+m}$ to wave function (10.1.5) is derived in the same way. It takes the form of (10.2.11), where

$$
\begin{aligned}
v_1(x;\varepsilon) &= C_{M+m,1}(\varepsilon)\mathbf{v}_1(x),\\
w(\xi;\varepsilon) &= a_1^+(\varepsilon)\varepsilon^{\mu_1}\mathbf{w}^K(\xi) + a_2^+(\varepsilon)\varepsilon^{\mu_1}\mathbf{w}^L(\xi),\\
v_2(x;\varepsilon) &= V_{M+m}(x) + C_{M+m,2}(\varepsilon)\mathbf{v}_2(x).
\end{aligned}
$$

The functions v_1 and v_2 admit, near the point O, expansions of the form (6.3.9) and (6.3.10) with constants

$$
(a_1^-, a_2^-) = (C_{M+m,1}, C_{M+m,2}), \quad (a_1^+, a_2^+) = (C_{M+m,1}a_1, s_{M+m,2} + C_{M+m,2}\, a_2),
$$

related by the equality

$$
(C_{M+m,1}, C_{M+m,2}) = (C_{M+m,1}a_1, s_{M+m,2} + C_{M+m,2}\, a_2)\, \Lambda\, \varepsilon^{2\mu_1+1},
$$

see (6.3.15). It follows that

$$
\begin{aligned}
(C_{M+m,1}(\varepsilon), C_{M+m,2}(\varepsilon)) &= (0, s_{M+m,2})(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}\, \Lambda\, \varepsilon^{2\mu_1+1},\\
(a_1^+(\varepsilon), a_2^+(\varepsilon)) &= (0, s_{M+m,2})\,(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}.
\end{aligned}
$$

Using expansions (10.2.1) and (10.2.5) for $x_1^1 \to +\infty$ and the formulas for $C_{M+m,1}$ and $C_{M+m,2}$, we obtain an approximation to the remaining lines of the scattering matrix. Put $\widetilde{S}_{21} = \|\widetilde{S}_{M+m,p}\|_{m,p=1}^M$ and $\widetilde{S}_{22} = \|\widetilde{S}_{M+m,M+p}\|_{m,p=1}^M$, then

$$
(\widetilde{S}_{21}(\varepsilon), \widetilde{S}_{22}(\varepsilon)) = (0_{M\times M}, S_2^0) + i(0_{M\times 1}, s_2)(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}\Lambda s^* S^0 \varepsilon^{2\mu_1+1}. \tag{10.2.16}
$$

We set $\widetilde{S} = \|\widetilde{S}_{pq}\|_{p,q=1,2}$ and unite (10.2.10) and (10.2.16) into the matrix equality

$$
\widetilde{S}(\varepsilon) = S^0 + is(I - \Lambda\, a\, \varepsilon^{2\mu_1+1})^{-1}\Lambda s^* S^0 \varepsilon^{2\mu_1+1}. \tag{10.2.17}
$$

The next lemma can be proved along the same way as Lemma 6.3.4 with the use of Lemma 10.2.1 instead of 6.3.1.

Lemma 10.2.2 *The matrix $\widetilde{S}(\varepsilon)$ is unitary.*

We set $\widetilde{T}_m(\varepsilon) = \sum_{p=1}^M |\widetilde{S}_{m,M+p}(\varepsilon)|^2$ and $\widetilde{T}_{M+m}(\varepsilon) = \sum_{p=1}^M |\widetilde{S}_{M+m,p}(\varepsilon)|^2$ for $m = 1, \dots, M$. According to (10.2.17),

$$
\begin{aligned}
\widetilde{S}(\varepsilon) &= S^0 + is\Lambda s^* S^0 \varepsilon^{2\mu_1+1} + O(\varepsilon^{4\mu_1+2})\\
&= \begin{pmatrix} S_1^0 & 0 \\ 0 & S_2^0 \end{pmatrix} + \begin{pmatrix} \alpha s_1 s_1^* S_1^0 & \beta s_1 s_2^* S_2^0 \\ \beta s_2 s_1^* S_1^0 & \alpha s_2 s_2^* S_2^0 \end{pmatrix} \varepsilon^{2\mu_1+1} + O(\varepsilon^{4\mu_1+2}).
\end{aligned}
$$

Therefore,

$$\widetilde{T}_m(\varepsilon) = |s_{m1}|^2 \sum_{p=1}^{M} |s_{M+p,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O(\varepsilon^{6\mu_1+3}), \tag{10.2.18}$$

$$\widetilde{T}_{M+m}(\varepsilon) = \sum_{p=1}^{M} |s_{p1}|^2 |s_{M+m,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O(\varepsilon^{6\mu_1+3}). \tag{10.2.19}$$

By Lemma 10.2.2, $\widetilde{R}_q(\varepsilon) + \widetilde{T}_q(\varepsilon) = 1$, $q = 1, \dots, 2M$, with $\widetilde{R}_m(\varepsilon) = \sum_{p=1}^{M} |\widetilde{S}^0_{mp}(\varepsilon)|^2$ and $\widetilde{R}_{M+m}(\varepsilon) = \sum_{p=1}^{M} |\widetilde{S}^0_{M+m,M+p}(\varepsilon)|^2$, $m = 1, \dots, M$. Hence,

$$\widetilde{R}_m(\varepsilon) = 1 - |s_{m1}|^2 \sum_{p=1}^{M} |s_{M+p,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O(\varepsilon^{6\mu_1+3}),$$

$$\widetilde{R}_{M+m}(\varepsilon) = 1 - \sum_{p=1}^{M} |s_{p1}|^2 |s_{M+m,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O(\varepsilon^{6\mu_1+3}).$$

Consider the problem

$$\triangle u + k^2 u = f \text{ in } G(\varepsilon), \quad u = 0 \text{ on } \partial G(\varepsilon). \tag{10.2.20}$$

For $\gamma \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma,\delta}(G(\varepsilon))$ is the completion in the norm (6.3.29) of the set of smooth functions in $\overline{G(\varepsilon)}$ having compact supports. The cut-off functions η_j are the same as in (6.3.29).

Proposition 10.2.3 *Let $|\gamma - 1| < \mu_1 + 1/2$, $f \in V^0_{\gamma,\delta}(G(\varepsilon))$, and let u be a solution to (10.2.20) that admits the representation*

$$u = \widetilde{u} + \eta_1 \sum_{m=1}^{M} A^-_m U^-_m + \eta_2 \sum_{m=1}^{M} A^-_{M+m} U^-_{M+m},$$

where $A^-_q = const$ and $\widetilde{u} \in V^2_{\gamma,\delta}(G(\varepsilon))$, δ being a small positive number. Then

$$\|\widetilde{u}; V^2_{\gamma,\delta}(G(\varepsilon))\| + \sum_{q=1}^{2M} |A^-_q| \le c \|f; V^0_{\gamma,\delta}(G(\varepsilon))\|$$

holds with a constant c independent of f and ε.

Theorem 10.2.4 *Under the hypotheses of Proposition 10.1.1, the inequality*

$$\sum_{q=1}^{2M} |S_{mq}(\varepsilon) - \widetilde{S}_{mq}(\varepsilon)| \le c(\varepsilon^{\mu_2+1} + \varepsilon^{\gamma+3/2})\varepsilon^{\mu_1} \tag{10.2.21}$$

holds, where $m = 1, \dots, 2M$, $\gamma > 0$, $|\gamma - 1| < \mu_1 + 1/2$, *and the constant* c *is independent of* ε.

Corollary 10.2.5 *The asymptotic formulas*

$$\widetilde{T}_m(\varepsilon) = |s_{m1}|^2 \sum_{p=1}^{M} |s_{M+p,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O\left(\varepsilon^{4\mu_1+2+\tau}\right),$$

$$\widetilde{T}_{M+m}(\varepsilon) = \sum_{p=1}^{M} |s_{p1}|^2 |s_{M+m,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O\left(\varepsilon^{4\mu_1+2+\tau}\right),$$

$$\widetilde{R}_m(\varepsilon) = 1 - |s_{m1}|^2 \sum_{p=1}^{M} |s_{M+p,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O\left(\varepsilon^{4\mu_1+2+\tau}\right),$$

$$\widetilde{R}_{M+m}(\varepsilon) = 1 - \sum_{p=1}^{M} |s_{p1}|^2 |s_{M+m,2}|^2 \beta^2 \varepsilon^{4\mu_1+2} + O\left(\varepsilon^{4\mu_1+2+\tau}\right).$$

hold with $m = 1, \dots, M$, *and* $\tau = \min\{\mu_2 - \mu_1, 2 - \sigma\}$, *where* σ *is a small positive number.*

10.3 Tunneling in a Waveguide with Two Narrows

We turn to the waveguide $G(\varepsilon_1, \varepsilon_2)$ with two narrows. The limit domain $G(0, 0)$ consists of the infinite domains G_1, G_2, and the bounded "resonator" G_0. We assume that k^2 varies in a neighborhood of an eigenvalue k_e^2 of limit problem (10.1.8) in G_0. For the sake of simplicity, the eigenvalue is assumed to be simple.

10.3.1 Formal Asymptotics

Let us consider the wave function u_m in $G(\varepsilon_1, \varepsilon_2)$ satisfying

$$u_m(x; k, \varepsilon_1, \varepsilon_2) \sim \begin{cases} U_m^+(x^1; k) + \sum_{p=1}^{M} S_{mp}(k, \varepsilon_1, \varepsilon_2)\, U_p^-(x^1; k), & x_1^1 \to +\infty, \\ \sum_{p=1}^{M} S_{m,M+p}(k, \varepsilon_1, \varepsilon_2)\, U_{M+p}^-(x^2; k), & x_1^2 \to +\infty. \end{cases}$$

In G_j, $j = 0, 1, 2$, u_1 is approximated by the solutions v_j to (6.2.1) such that

$$v_1 = V_m + C_{m1}\mathbf{v}_1, \quad v_0 = C_{m2}\mathbf{v}_{01} + C_{m3}\mathbf{v}_{02}, \quad v_2 = C_{m4}\mathbf{v}_2, \tag{10.3.1}$$

the V_m and $\mathbf{v}_1$ are defined in 10.2.1, the $\mathbf{v}_{0j}$ is defined in 6.4.1, and the $\mathbf{v}_2$ is an analog of the $\mathbf{v}_1$ in the domain G_2. The constants C_{1j} depend on ε_1, ε_2 and k. According to (10.2.2), (6.4.4), (6.4.5), and (10.2.5) for $r \to 0$,

$$v_1 = \frac{1}{\sqrt{r_1}}\big(C_{m1}\widetilde{N}_{\mu_{11}+1/2}(kr_1) + (s_{m1} + C_{m1}a_1)\widetilde{J}_{\mu_{11}+1/2}(kr_1)\big)\Phi^K_{11}(\varphi) + O(r_1^{\mu_{12}}), \quad r_1 \to 0,$$

$$v_0 = \begin{cases} \frac{1}{\sqrt{r_1}}\big(C_{m2}\widetilde{N}_{\mu_{11}+1/2}(kr_1) + (C_{m2}c_{11} + C_{m3}c_{21})\widetilde{J}_{\mu_{11}+1/2}(kr_1)\big)\Phi^L_{11}(\varphi_1) + O(r_1^{\mu_{12}}), & r_1 \to 0, \\ \frac{1}{\sqrt{r_2}}\big(C_{m3}\widetilde{N}_{\mu_{21}+1/2}(kr_2) + (C_{m2}c_{12} + C_{m3}c_{22})\widetilde{J}_{\mu_{21}+1/2}(kr_2)\big)\Phi^L_{21}(\varphi_2) + O(r_2^{\mu_{22}}), & r_2 \to 0. \end{cases}$$

$$v_2 = \frac{1}{\sqrt{r_2}}\big(C_{m4}\widetilde{N}_{\mu_1+1/2}(kr_2) + C_{m4}\, a_2\widetilde{J}_{\mu_1+1/2}(kr_2)\big)\Phi^L_{21}(\varphi) + O(r_2^{\mu_{22}}), \quad r_2 \to 0.$$

For every narrow, we introduce a matrix Λ_j (like the matrix Λ in (6.3.15)). Applying Lemma 6.3.2, we obtain

$$(C_{m1}, C_{m2}) = (s_{m1} + C_{m1}a_1, C_{m2}c_{11} + C_{m3}c_{21})\,\Lambda_1\,\varepsilon_1^{2\mu_{11}+1}$$

for the first narrow and

$$(C_{m3}, C_{m4}) = (C_{m2}c_{12} + C_{m3}c_{22}, C_{m4}\, a_2)\,\Lambda_2\,\varepsilon_2^{2\mu_{21}+1}$$

for the second narrow. The corresponding solutions of the second kind limit problems are of the form (see (6.3.14))

$$w_1(\xi^1) = (s_{m1} + C_{m1}a_1)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^K(\xi^1) + (C_{m2}c_{11} + C_{m3}c_{21})\varepsilon_1^{\mu_{11}}\mathbf{w}_1^L(\xi^1), \tag{10.3.2}$$

$$w_2(\xi^2) = C_{m4}a_2\varepsilon_2^{\mu_{21}}\mathbf{w}_2^K(\xi^2) + (C_{m2}c_{12} + C_{m3}c_{22})\varepsilon_2^{\mu_{21}}\mathbf{w}_2^L(\xi^2), \tag{10.3.3}$$

where $\mathbf{w}_j^K$ and $\mathbf{w}_j^L$ are analogs for the domains Ω_j, $j = 1, 2$, of the functions defined in Remark 6.3.3. We set $\Lambda = \operatorname{diag}\{\Lambda_1, \Lambda_2\}$, $\mathcal{E} = \operatorname{diag}\{\varepsilon_1^{2\mu_{11}+1},\ \varepsilon_1^{2\mu_{11}+1},\ \varepsilon_2^{2\mu_{21}+1},\ \varepsilon_2^{2\mu_{21}+1}\}$, and

$$a = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \tag{10.3.4}$$

and, combining the relations obtained for C_{mj}, we arrive at the equality

$$(C_{m1}, C_{m2}, C_{m3}, C_{m4}) = (s_{m1}, 0, 0, 0)\,\Lambda\,\mathcal{E} + (C_{m1}, C_{m2}, C_{m3}, C_{m4})\,a\,\Lambda\,\mathcal{E}.$$

Thus,

$$(C_{m1}, C_{m2}, C_{m3}, C_{m4})(I - a\,\Lambda\,\mathcal{E}) = (s_{m1}, 0, 0, 0)\,\Lambda\,\mathcal{E}. \tag{10.3.5}$$

As was shown in Sect. 6.4.2, there exists the inverse matrix of $I - a\Lambda\mathcal{E}$ for sufficiently small ε_1, ε_2 and

$$(I - a\Lambda\mathcal{E})^{-1} = (I - \widehat{a}\,\Lambda\,\mathcal{E})^{-1}\left(I - \frac{\mathbf{b}^*\mathbf{b}\,\Lambda\,\mathcal{E}(I - \widehat{a}\,\Lambda\,\mathcal{E})^{-1}}{k^2 - k_e^2 + \langle \mathbf{b}\,\Lambda\,\mathcal{E}(I - \widehat{a}\,\Lambda\,\mathcal{E})^{-1}, \mathbf{b}\rangle}\right),$$

where $\mathbf{b} = (0,\ b_1,\ b_2,\ 0)$ and the matrix $\widehat{a}$ is analytic near $k = k_e$ and is defined by (10.3.4) with c_{pq} changed for $\widehat{c}_{pq}$ (cf. (6.4.6)). Using this equality and (10.3.5), we find the constants C_{mj}:

$$\begin{aligned}(C_{m1}, C_{m2}, C_{m3}, C_{m4}) &= (s_{m1}, 0, 0, 0)\,\Lambda\,\mathcal{E}(I - a\,\Lambda\,\mathcal{E})^{-1}\\ &= (s_{m1}, 0, 0, 0)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right),\end{aligned} \tag{10.3.6}$$

where $D = \Lambda\,\mathcal{E}(I - \widehat{a}\,\Lambda\,\mathcal{E})^{-1}$; thereby, we have constructed an approximation to the wave function u_1. Arguing as in Sect. 6.4.2, one can see, that the solution v_0 of the limit problem in the resonator and the solutions w_j of the second kind limit problems do not have a pole at $k^2 = k_e^2$. The following formulas are valid:

$$v_0(x) = C_{m2}\widehat{\mathbf{v}}_{01}(x) + C_{m3}\widehat{\mathbf{v}}_{02}(x) - \frac{(s_{m1}, 0, 0, 0)D\,\mathbf{b}^*}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle} v_e(x). \tag{10.3.7}$$

$$\begin{aligned}w_1(\xi^1) = (s_{m1} + C_{m1}a_1)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^K(\xi^1) + \left(C_{m2}\widehat{c}_{11} + C_{m3}\widehat{c}_{21} - \frac{b_1(s_{m1}, 0, 0, 0)D\,\mathbf{b}^*}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)&\\ \times\, \varepsilon_1^{\mu_{11}}\mathbf{w}_1^L(\xi^1),&\end{aligned} \tag{10.3.8}$$

$$\begin{aligned}w_2(\xi^2) = C_{m4}a_2\varepsilon_2^{\mu_{21}}\mathbf{w}_2^K(\xi^2) + \left(C_{m2}\widehat{c}_{12} + C_{m3}\widehat{c}_{22} - \frac{b_2(s_{m1}, 0, 0, 0)D\,\mathbf{b}^*}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)&\\ \times\, \varepsilon_2^{\mu_{21}}\mathbf{w}_2^L(\xi^2).&\end{aligned} \tag{10.3.9}$$

Finally, we present the asymptotics of the wave function. Let $t \mapsto \Theta(t)$ be a cut-off function on $\mathbb{R}$ equal to 1 for $t < \delta/2$ and to 0 for $t > \delta$ with a small positive δ. We introduce $x \mapsto \chi_{\varepsilon_1,\varepsilon_2}(x) = \mathbf{1}_{G_0}(x)\,(1 - \Theta(r_1/\varepsilon_1))\,(1 - \Theta(r_2/\varepsilon_2))$, where $\mathbf{1}_{G_0}$ is the characteristic function of G_0. The principal term $\widetilde{u}_1$ of the asymptotics of the wave function u_1 is of the form

$$\begin{aligned}\widetilde{u}_1(x; k, \varepsilon_1, \varepsilon_2) = {}&\chi_{1,\varepsilon_1}(x^1)v_1(x^1; k, \varepsilon_1, \varepsilon_2) + \Theta(r_1)w_1(\varepsilon_1^{-1}x^1; k, \varepsilon_1, \varepsilon_2)\\ &+ \chi_{\varepsilon_1,\varepsilon_2}(x)v_0(x; k, \varepsilon_1, \varepsilon_2) + \Theta(r_2)w_2(\varepsilon_2^{-1}x^2; k, \varepsilon_1, \varepsilon_2)\\ &+ \chi_{2,\varepsilon_2}(x^2)v_2(x^2; k, \varepsilon_1, \varepsilon_2),\end{aligned} \tag{10.3.10}$$

where the solutions v_1 and v_2 of the first kind limit problems are defined by relations (6.4.7), v_0 is given by (10.3.7), and the solutions w_1 and w_2 of the second kind limit problems are defined by (10.3.8) and (10.3.9).

We now find an approximation $\widetilde{S}_{ij}$ to the entries of the scattering matrix $S = (S_{ij})_{i,j=1}^{M_1+M_2}$. By $\widetilde{S}_{rr}$, $r = 1, 2$, we denote $(M_r \times M_r)$-matrices such that

$$\widetilde{S} = \begin{pmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{pmatrix}.$$

By virtue of (10.2.1) and (6.3.7) for $x_1^1 \to +\infty$,

$$v_1(x^1) = U_m^+(x^1) + \sum_{p=1}^{M_1}(S_{mp}^0 + C_{m1}(\varepsilon)A_{1p})U_p^-(x^1) + O(\exp(-\delta x_1^1)), \quad x_1^1 \to +\infty,$$

$$v_2(x^2) = C_{m4}(\varepsilon)\sum_{p=1}^{M_2} A_{2,M_1+p}U_{M_1+p}^-(x^2) + O(\exp(-\delta x_1^2)), \quad x_1^2 \to +\infty,$$

whence

$$(\widetilde{S}_{11}, \widetilde{S}_{12}) = (S_1^0 + \mathbf{C}_1 A_1, \mathbf{C}_4 A_2), \tag{10.3.11}$$

where, as in the previous subsection,

$$\mathbf{C}_j = (C_{1j}, \dots, C_{M_1 j})^T, \quad A_1 = (A_{11}, \dots, A_{1M_1}), \quad A_2 = (A_{2,M_1+1}, \dots, A_{2,M_1+M_2}).$$

We set

$$A = \begin{pmatrix} A_1 & 0_{1\times M_2} \\ 0_{1\times M_1} & 0_{1\times M_2} \\ 0_{1\times M_1} & 0_{1\times M_2} \\ 0_{1\times M_1} & A_2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 & 0_{M_1\times 1} & 0_{M_1\times 1} & 0_{M_1\times 1} \\ 0_{M_2\times 1} & 0_{M_2\times 1} & 0_{M_2\times 1} & s_2 \end{pmatrix};$$

as before, let $S^0 = \operatorname{diag}(S_1^0, S_2^0)$; then, by Lemma 10.2.1, equality (10.2.9) remains valid. Taking into account (10.3.11), (10.3.6), and (10.2.9), we obtain

$$\begin{aligned}(\widetilde{S}_{11}, \widetilde{S}_{12}) &= (S_1^0, 0_{M_1\times M_2}) + (\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4)A \\ &= (S_1^0, 0_{M_1\times M_2}) + i(s_1, 0_{M_1\times 1}, 0_{M_1\times 1}, 0_{M_1\times 1})\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^* S^0,\end{aligned} \tag{10.3.12}$$

where $D = \Lambda\,\mathcal{E}(I - \widehat{a}\,\Lambda\,\mathcal{E})^{-1}$. The approximation

$$\begin{aligned}(\widetilde{S}_{21}, \widetilde{S}_{22}) &= (0_{M_2\times M_1}, S_2^0) \\ &\quad + i(0_{M_2\times 1}, 0_{M_2\times 1}, 0_{M_2\times 1}, s_2)\left(D - \frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2 - k_e^2 + \langle \mathbf{b}D, \mathbf{b}\rangle}\right)s^* S^0.\end{aligned} \tag{10.3.13}$$

to the remaining rows of the scattering matrix S is derived from the asymptotics of the wave functions u_{M_1+m},

$$u_{M_1+m}(x;k,\varepsilon_1,\varepsilon_2)\sim\begin{cases}\sum_{p=1}^{M_1} S_{M_1+m,p}(k,\varepsilon_1,\varepsilon_2)\,U_p^-(x^1;k), & x_1^1\to+\infty,\\ U_{M_1+m}^+(x^1;k)+\sum_{p=1}^{M_2} S_{M_1+m,M_1+p}(k,\varepsilon_1,\varepsilon_2)\,U_{M_1+p}^-(x^2;k), & x_1^2\to+\infty.\end{cases}$$

The principal term $\widetilde{u}_{M_1+m}$ of the asymptotics takes the form of (10.3.10), where

$$\begin{aligned}
v_1(x^1) &= C_{M_1+m,1}\mathbf{v}_1(x^1),\\
w_1(\xi^1) &= C_{M_1+m,1}a_1\varepsilon_1^{\mu_{11}}\mathbf{w}_1^K(\xi^1)\\
&\quad+\left(C_{M_1+m,2}\widehat{c}_{11}+C_{M_1+m,3}\widehat{c}_{M_1+m,1}-\frac{b_1(0,0,0,s_{M_1+m,2})D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right)\varepsilon_1^{\mu_{11}}\mathbf{w}_1^L(\xi^1),\\
v_0(x) &= C_{M_1+m,2}\widehat{\mathbf{v}}_{01}(x)+C_{M_1+m,3}\widehat{\mathbf{v}}_{02}(x)-\frac{(0,0,0,s_{M_1+m,2})D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}v_e(x),\\
w_2(\xi^2) &= (s_{M_1+m,2}+C_{M_1+m,4}a_2)\varepsilon_2^{\mu_{21}}\mathbf{w}_2^K(\xi^2)\\
&\quad+\left(C_{M_1+m,2}\widehat{c}_{12}+C_{M_1+m,3}\widehat{c}_{22}-\frac{b_2(0,0,0,s_{M_1+m,2})D\,\mathbf{b}^*}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right)\varepsilon_2^{\mu_{21}}\mathbf{w}_2^L(\xi^2),\\
v_2(x^2) &= V_2(x^2)+C_{M_1+m,4}\mathbf{v}_2(x^2),
\end{aligned}$$

the constants $C_{M_1+m,j}$ are given by

$$(C_{M_1+m,1},C_{M_1+m,2},C_{M_1+m,3},C_{M_1+m,4})=(0,0,0,s_{M_1+m,2})\left(D-\frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right).$$

Combining relations (10.3.12) and (10.3.13), we obtain the approximation $\widetilde{S}$ to the scattering matrix S:

$$\widetilde{S}(k,\varepsilon_1,\varepsilon_2)=S^0(k)+is(k)\left(D-\frac{D\,\mathbf{b}^*\mathbf{b}\,D}{k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle}\right)s^*(k)S^0(k), \quad (10.3.14)$$

where $D=D(k,\varepsilon_1,\varepsilon_2)=\Lambda\,\mathcal{E}(\varepsilon_1,\varepsilon_2)(I-\widehat{a}(k)\,\Lambda\,\mathcal{E}(\varepsilon_1,\varepsilon_2))^{-1}$; the k_e^2 and $\mathbf{b}$ are independent of k, ε_1, ε_2. Arguing as in the proof of Lemma 6.3.4 and taking into account $(\Lambda\,\mathcal{E})^*=\Lambda\,\mathcal{E}$ and $a-a^*=is^*s$ (Lemma 10.2.1), one can verify that the matrix $\widetilde{S}$ is unitary.

We denote by k_p the pole of the matrix $\widetilde{S}$, that is, k_p satisfies the equation $k^2-k_e^2+\langle\mathbf{b}D,\mathbf{b}\rangle=0$. We substitute the expression for D and obtain

$$k^2-k_e^2=\langle\mathbf{b}\Lambda\,\mathcal{E}(I-\widehat{a}(k)\,\Lambda\,\mathcal{E})^{-1},\mathbf{b}\rangle; \quad (10.3.15)$$

here $\mathcal{E}=\operatorname{diag}(\varepsilon_1^{2\mu_{11}+1},\varepsilon_1^{2\mu_{11}+1},\varepsilon_2^{2\mu_{21}+1},\varepsilon_2^{2\mu_{21}+1})$ and $\Lambda=\operatorname{diag}(\Lambda_1,\Lambda_2)$ with

$$\Lambda_j=\begin{pmatrix}\alpha_j & \beta_j\\ \beta_j & \alpha_j\end{pmatrix},\quad j=1,\,2.$$

Since ε_1 and ε_2 are small, a solution to equation (10.3.15) can be found by the successive approximation method. We have $k_p^2 = k_r^2 - ik_i^2$, where

$$\begin{aligned}
k_r^2 &= k_e^2 - \langle \mathbf{b}\,\Lambda\,\mathcal{E}, \mathbf{b}\rangle + O\left(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}\right) \\
&= k_e^2 - \alpha_1 b_1^2 \varepsilon_1^{2\mu_{11}+1} - \alpha_2 b_2^2 \varepsilon_2^{2\mu_{21}+1} + O\left(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}\right), \qquad (10.3.16)
\end{aligned}$$

$$\begin{aligned}
k_i^2 &= \operatorname{Im} \langle \mathbf{b}\,\Lambda\,\mathcal{E}\,\widehat{a}(k_e)\,\Lambda\,\mathcal{E}, \mathbf{b}\rangle + O\left(\varepsilon_1^{6\mu_{11}+3} + \varepsilon_2^{6\mu_{21}+3}\right) \\
&= \frac{1}{2} s_1(k_e)^* s_1(k_e) b_1^2 \beta_1^2 \varepsilon_1^{4\mu_{11}+2} + \frac{1}{2} s_2(k_e)^* s_2(k_e) b_2^2 \beta_2^2 \varepsilon_2^{4\mu_{21}+2} \\
&\quad + O\left(\varepsilon_1^{6\mu_{11}+3} + \varepsilon_2^{6\mu_{21}+3}\right); \qquad (10.3.17)
\end{aligned}$$

in the last equality, we used the relation $\operatorname{Im} a_j = s_j^* s_j/2$, $j = 1, 2$, which follows from Lemma 10.2.1. We suppose the constants b_j and the columns s_j to be distinct from zero. Then, by virtue of (10.3.17), $|k^2 - k_p^2| \geqslant c(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})$ for all real k.

Let us find the principal terms of the power series in ε_1 and ε_2 for the matrix $\widetilde{S}$. As was proved in Chap. 6 (cf. (6.4.26)),

$$\frac{1}{k^2 - k_e^2 + \langle \mathbf{b}D(k), \mathbf{b}\rangle} = \frac{1 + O(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})}{k^2 - k_p^2} \qquad (10.3.18)$$

uniformly with respect to k in any interval that contains no thresholds and no eigenvalues of the resonator except k_e. We substitute (10.3.18) into (10.3.14) and take into account that $D = \Lambda\,\mathcal{E} + O(\varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2})$. Then

$$\begin{aligned}
&\widetilde{S}(k, \varepsilon_1, \varepsilon_2) \sim S^0(k) + is(k)\Lambda\,\mathcal{E}s^*(k)S^0(k) - i\,\frac{s(k)\Lambda\,\mathcal{E}\,\mathbf{b}^*\mathbf{b}\,\Lambda\,\mathcal{E}s^*(k)S^0(k)}{k^2 - k_p^2} \\
&= \begin{pmatrix} S_{11}^0(k) & 0 \\ 0 & S_{22}^0(k) \end{pmatrix} + i \begin{pmatrix} s_1(k)^* s_1(k) \alpha_1 S_1^0(k) \varepsilon_1^{2\mu_{11}+1} & 0 \\ 0 & s_2(k)^* s_2(k) \alpha_2 S_2^0(k) \varepsilon_2^{2\mu_{21}+1} \end{pmatrix} \\
&- \frac{i}{k^2 - k_p^2} \begin{pmatrix} s_1(k)^* s_1(k) b_1^2 \beta_1^2 S_1^0(k) \varepsilon_1^{4\mu_{11}+2} & s_2(k)^* s_1(k) b_1 b_2 \beta_1 \beta_2 S_2^0(k) \varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{2\mu_{21}+1} \\ s_1(k)^* s_2(k) b_1 b_2 \beta_1 \beta_2 S_1^0(k) \varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{2\mu_{21}+1} & s_2(k)^* s_2(k) b_2^2 \beta_2^2 S_2^0(k) \varepsilon_2^{4\mu_{21}+2} \end{pmatrix},
\end{aligned}$$

where we dropped the terms that admit the estimate $O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$ uniformly with respect to k. For $(k^2 - k_p^2)^{-1} = O(1)$, the third term can be neglected as well; however, it must be taken into account for small $k^2 - k_p^2$.

Let us choose a more narrow interval for k^2, assuming $k^2 - k_r^2 = O\big(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1}\big)$. Using relations (10.3.16) and (10.3.17), $S_j^0(k) = S_j^0(k_e) + O(k^2 - k_e^2)$, and $s_j(k) = s_j(k_e) + O(k^2 - k_e^2)$, we obtain

$$\widetilde{S}_{12}(k,\varepsilon_1,\varepsilon_2) = \frac{i\dfrac{s_1(k_e)}{(s_1(k_e)^*s_1(k_e))^{1/2}}\dfrac{s_2(k_e)^*}{(s_2(k_e)^*s_2(k_e))^{1/2}}}{\dfrac{i}{2}\left(z+\dfrac{1}{z}\right)+P\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}} \times\left(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\right), \tag{10.3.19}$$

$$\widetilde{S}_{21}(k,\varepsilon_1,\varepsilon_2) = \frac{i\dfrac{s_2(k_e)}{(s_2(k_e)^*s_2(k_e))^{1/2}}\dfrac{s_1(k_e)^*}{(s_1(k_e)^*s_1(k_e))^{1/2}}}{\dfrac{i}{2}\left(z+\dfrac{1}{z}\right)+P\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}} \times\left(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\right), \tag{10.3.20}$$

where

$$z=\frac{b_1\beta_1(s_1(k_e)^*s_1(k_e))^{1/2}\varepsilon_1^{2\mu_{11}+1}}{b_2\beta_2(s_2(k_e)^*s_2(k_e))^{1/2}\varepsilon_2^{2\mu_{21}+1}},\quad P=\frac{1}{b_1b_2\beta_1\beta_2(s_1(k_e)^*s_1(k_e))^{1/2}(s_2(k_e)^*s_2(k_e))^{1/2}}.$$

Now, we find approximations to the transmission coefficients:

$$\widetilde{T}_m(k,\varepsilon_1,\varepsilon_2)=\frac{\dfrac{|s_{m1}|^2}{(s_1(k_e)^*s_1(k_e))^{1/2}}}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\left(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\right),$$

$$\widetilde{T}_{M+m}(k,\varepsilon_1,\varepsilon_2)=\frac{\dfrac{|s_{M+m,2}|^2}{(s_2(k_e)^*s_2(k_e))^{1/2}}}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2+P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\left(1+O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})\right).$$

It is easy to see that $\widetilde{T}_j$ has a peak at $k^2 = k_r^2$, and the width of that peak at its half-height is

$$\widetilde{\Upsilon}(\varepsilon_1,\varepsilon_2)=|(z+z^{-1})/P|\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}. \tag{10.3.21}$$

10.3.2 The Estimate of the Remainder

We consider the problem

$$\triangle u+k^2u=f \quad \text{in } G(\varepsilon_1,\varepsilon_2),\quad u=0 \quad \text{on } \partial G(\varepsilon_1,\varepsilon_2). \tag{10.3.22}$$

For $\gamma_1, \gamma_2 \in \mathbb{R}$, $\delta > 0$, and $l = 0, 1, \dots$, the space $V^l_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ is the completion in the norm (6.4.31) of the set of smooth functions in $\overline{G(\varepsilon_1, \varepsilon_2)}$ with compact supports. As before, we denote by $V^{0,\perp}_{\gamma_1,\gamma_2,\delta}$ the space of functions f that are analytic in k^2, take values in $V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$, and, for $k^2 = k_e^2$, satisfy $(\chi_{\varepsilon_1^\sigma,\varepsilon_2^\sigma} f, v_e)_{G_0} = 0$ with small $\sigma > 0$.

Proposition 10.3.1 *Let k_r^2 be a resonance, $k_r^2 \to k_e^2$ as $\varepsilon_1, \varepsilon_2 \to 0$, and let $|k^2 - k_r^2| = O(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})$. Assume that γ_1, γ_2 satisfy $\mu_{j1} - 3/2 < \gamma_j - 1 < \mu_{j1} + 1/2$, $f \in V^{0,\perp}_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$, and u is a solution to (10.3.22) that admits the representation*

$$u = \widetilde{u} + \eta_1 \sum_{m=1}^{M_1} A_m^- U_m^- + \eta_2 \sum_{m=1}^{M_2} A^-_{M_1+m} U^-_{M_1+m};$$

here $A_j^- = const$ and $\widetilde{u} \in V^2_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))$ with small $\delta > 0$. Then

$$\|\widetilde{u}; V^2_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\| + \sum_{q=1}^{M_1+M_2} |A_q^-| \le c\|f; V^0_{\gamma_1,\gamma_2,\delta}(G(\varepsilon_1, \varepsilon_2))\|,$$

where c is a constant independent of f and ε_1, ε_2.

Consider the solution u_m of the homogeneous problem (6.1.1) defined by (10.1.4). Let $S_{m1}, \dots, S_{m,M_1+M_2}$ be the entries of the scattering matrix determined by this solution. Denote by $\widetilde{u}_{m,\sigma}$ the function defined by (10.3.10) with $\Theta(r_j)$ replaced by $\Theta(\varepsilon_j^{-2\sigma} r_j)$. The $\widetilde{S}_{m1}, \dots, \widetilde{S}_{m,M_1+M_2}$ are the same as in the previous subsection.

Theorem 10.3.2 *Let the hypotheses of Propositions 10.1.1 and 10.3.1 be fulfilled and assume that the coefficients s_{qj} in (10.2.2), (10.2.3) and the coefficients b_j in (6.4.1) are nonzero. Then*

$$\sum_{q=1}^{M_1+M_2} |S_{mq} - \widetilde{S}_{mq}| \le c \frac{(\varepsilon_1^{2\mu_{11}+1} + \varepsilon_2^{2\mu_{21}+1})\varepsilon_1^{\mu_{11}} \left(\varepsilon_1^{\mu_{12}+1} + \varepsilon_1^{\gamma_1+3/2}\right) + \varepsilon_1^{2\mu_{11}+1} \varepsilon_2^{\mu_{21}} \left(\varepsilon_2^{\mu_{22}+1} + \varepsilon_2^{\gamma_2+3/2}\right)}{|k^2 - k_r^2| + \varepsilon_1^{4\mu_{11}+2} + \varepsilon_2^{4\mu_{21}+2}}$$

with c independent of $\varepsilon_1, \varepsilon_2, k$; $p = 1, 2$.

We denote by k_e^2 an eigenvalue of problem (6.2.1) in the resonator G_0 and by $k_r^2(\varepsilon)$ a resonance frequency such that $k_r^2(\varepsilon) \to k_e^2$ as $\varepsilon \to 0$. Moreover, let b_j be the constants in asymptotics (6.4.1) of an eigenfunction corresponding to the eigenvalue k_e^2 and $s_{qj}(k)$ the constant in asymptotics (10.2.2) and (10.2.3) of the special solution V_j for $r_j \to 0$, $j = 1, 2$. Finally, the constants α and β are defined by (6.2.10) and (6.2.11). We set

$$z = \frac{b_1\beta_1(\sum_{p=1}^{M_1}|s_{p1}(k_e)|^2)^{1/2}\varepsilon_1^{2\mu_{11}+1}}{b_2\beta_2(\sum_{p=1}^{M_2}|s_{M_1+p,2}(k_e)|^2)^{1/2}\varepsilon_2^{2\mu_{21}+1}},$$
$$P = \frac{1}{b_1b_2\beta_1\beta_2(\sum_{p=1}^{M_1}|s_{p1}(k_e)|^2)^{1/2}(\sum_{p=1}^{M_2}|s_{M_1+p,2}(k_e)|^2)^{1/2}};$$

these are the same values as in (10.3.19) and (10.3.21).

Theorem 10.3.3 *For $|k^2-k_r^2| = O(\varepsilon_1^{2\mu_{11}+1}+\varepsilon_2^{2\mu_{21}+1})$, the asymptotic expansions*

$$T_m(k,\varepsilon_1,\varepsilon_2) = \frac{\dfrac{|s_{m1}|^2}{(\sum_{p=1}^{M_1}|s_{p1}(k_e)|^2)^{1/2}}}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2 + P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\big(1+O(\varepsilon_1^{\tau_1}+\varepsilon_2^{\tau_2})\big),$$
$$m=1,\dots,M_1,$$
$$T_{M_1+m}(k,\varepsilon_1,\varepsilon_2) = \frac{\dfrac{|s_{M_1+m,21}|^2}{(\sum_{p=1}^{M_1}|s_{M_1+p,1}(k_e)|^2)^{1/2}}}{\dfrac{1}{4}\left(z+\dfrac{1}{z}\right)^2 + P^2\left(\dfrac{k^2-k_r^2}{\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}}\right)^2}\big(1+O(\varepsilon_1^{\tau_1}+\varepsilon_2^{\tau_2})\big),$$
$$m=1,\dots,M_2,$$
$$k_r^2(\varepsilon_1,\varepsilon_2) = k_e^2 - \alpha_1 b_1^2\varepsilon_1^{2\mu_{11}+1} - \alpha_2 b_2^2\varepsilon_2^{2\mu_{21}+1} + O\big(\varepsilon_1^{2\mu_{11}+1+\tau_1}+\varepsilon_2^{2\mu_{21}+1+\tau_2}\big),$$
$$\Upsilon(\varepsilon_1,\varepsilon_2) = \left|\frac{z+z^{-1}}{P}\right|\varepsilon_1^{2\mu_{11}+1}\varepsilon_2^{2\mu_{21}+1}\big(1+O(\varepsilon_1^{\tau_1}+\varepsilon_2^{\tau_2})\big)$$

hold, where $\tau_j = \min\{\mu_{j2}-\mu_{j1}, 2-\sigma_j\}$ and σ_j are small positive numbers.

Proof From Theorem 10.3.2 and (10.3.19) we obtain

$$\big||S_{mp}|^2 - |\widetilde{S}_{mp}|^2\big| \le c\left|\frac{S_{mp}-\widetilde{S}_{mp}}{\widetilde{S}_{mp}}\right||\widetilde{S}_{mp}|^2 \le c|\widetilde{S}_{mp}|^2\big(\varepsilon_1^{\tau_1}+\varepsilon_2^{\tau_2}+\varepsilon_2^{\tau_2+2\mu_{21}+1}/\varepsilon_1^{2\mu_{11}+1}\big)$$

with $\tau_j = \min\{\mu_{j2}-\mu_{j1}, 2-\sigma_j\}$, $\sigma_j = \mu_{j1}+3/2-\gamma_j$, $j=1,2$. Hence,

$$|T_m - \widetilde{T}_m| \le c\widetilde{T}_m\big(\varepsilon_1^{\tau_1}+\varepsilon_2^{\tau_2}+\varepsilon_2^{\tau_2+2\mu_{21}+1}/\varepsilon_1^{2\mu_{11}+1}\big).$$

When $\varepsilon_1^{2\mu_{11}+1} \ge \varepsilon_2^{2\mu_{12}+1}$, we get the desired expansion for T_m. Assume that $\varepsilon_1^{2\mu_{11}+1} \le \varepsilon_2^{2\mu_{21}+1}$. Similarly, we can obtain

$$\big||S_{M_1+m,p}|^2 - |\widetilde{S}_{M_1+m,p}|^2\big| \le c|\widetilde{S}_{M_1+m,p}|^2\big(\varepsilon_1^{\tau_1}+\varepsilon_2^{\tau_2}+\varepsilon_1^{\tau_1+2\mu_{11}+1}/\varepsilon_2^{2\mu_{21}+1}\big)$$

with the same τ_j. As is known, $|S_{M_1+m,p}| = |S_{M_1+p,m}|$, and it is easy to see that $|\widetilde{S}_{M_1+m,p}| = |\widetilde{S}_{M_1+p,m}|$ (indeed, all characteristics of both narrows are interchangeably included in the formulas for $\widetilde{S}_{mp}$). This leads to the required expansion for T_m as $\varepsilon_1^{2\mu_{11}+1} \le \varepsilon_2^{2\mu_{21}+1}$. The formulas for k_r^2 and Υ follow from that for T_p. $\square$

Chapter 11
Electronics Devices Based on Resonant Tunneling

Devices based on the phenomenon of electron resonant tunneling are widely used in electronics. Most of these devices are multilayer planar structures having large transverse and small longitudinal sizes. In classic two-barrier resonant tunneling diodes, the electrodes and the well are made of GaAs and the barriers of GaAlAs. Due to the structure sizes, there is no electron energy quantization along the layers and, therefore, the process of one-dimensional resonant tunneling is implemented there. Resonant tunneling diodes enable the creation of solid-state microwave generators and transistor amplifier devices.

Theoretically, the limit of the ideal diode operation speed is about 0.1 ps. Resonant tunneling phenomenon allows the building of diodes with extremely high switching speed close to the theoretical limit, that is, in the frequency range up to several THz (10^{12} Hz), [15, 16, 44].

One of the advantages of the planar resonant structure is a high working current due to the large transverse size of the structure and a large current density. To reach a high working current density, it is necessary that the barriers be thin (several monoatomic layers) and the interfaces between the well and the barriers and those between the barriers and the electrodes flat and distinct (to avoid incoherent electron scattering by the interfaces). However, as is experimentally shown, the interfaces are not flat and sharp. For instance, the transition from GaAs to GaAlAs involves 1–4 monoatomic layers, and the potential barrier is vague. As to the resonant quantum dot systems, their properties also heavily depend on inhomogeneities of the interfaces between the electrodes and vacuum and those between the quantum dot and vacuum.

In the aforementioned resonant systems, one can change the barrier thickness only. In the planar multilayer systems, there is no way to adjust a resonance frequency once the system is produced because there is no possibility to change the barrier thickness.

Resonant electronics devices can be based on quantum waveguides of variable cross-section. In these, the potential barriers are formed without phase boundaries, therefore, incoherent electron scattering does not occur. Changing the narrow forms, one can change not only the effective barrier thickness but also the barrier form. It is possible to adjust a resonance frequency by cutting out small fragments of the resonator (for example, with a focus electron beam). Due to the smallness of the

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waveguide cross-section diameter, it is easy to apply external electric and magnetic fields localized in the resonator. This enables the creation of devices controlled by an external electric potentials and magnetic fields alike. Small sizes of such systems lead to their small capacity, which increases their operation speed. However, let us emphasize that, because of a small waveguide cross-section, the devices based on a quantum waveguide are designed in principle for small currents ($<10^{-6}$ A) and sufficiently small working voltages (<1 V).

In this chapter, we present examples of electronics devices based on quantum waveguides with narrows: transistors controlled by external electric field and magnetic field sensors controlled by external magnetic field. Moreover, we also consider an electron flow switch for quantum nets. Unlike the transistors and the sensors, the switch has no relation to the phenomenon of resonant tunneling. However, the scattering matrix needed for analyzing the switch operation has been calculated by the method presenteded in Chap. 4.

11.1 Magnetic Field Sensors Based on Quantum Waveguides

Let us consider resonance structures like those in Chaps. 7 and 8. Being supplied in the resonator, a magnetic field B, orthogonal to the resonator axis, splits any resonant energy level into two levels. We make use of this phenomenon to detect magnetic fields localized in domains of about 1 nm in diameter. Detection of the vector B in the classical resonance systems is onerous because of large transverse sizes of such systems. We will describe one-resonator and two-resonator magnetic field sensors; first we discuss the one with a single resonator (Fig. 11.1).

The system consists of a quantum waveguide (1) with metallic contacts at its ends (a source (2) and a drain (3)). The sizes of the waveguide and its material are chosen to provide ballistic (collisionless) electron transport from the source to the drain. Between the contacts, an accelerating voltage U is supplied. Therefore, the Fermi level at the drain is eU below that at the source. The waveguide has two narrows (4) and (5); the domain (6) between the narrows is a quantum resonator. Such narrows can be produced by electron lithography or X-ray lithography, for example. The waveguide cross-section has been chosen so that $E - E_F \geqslant k_B T$ for all the energies

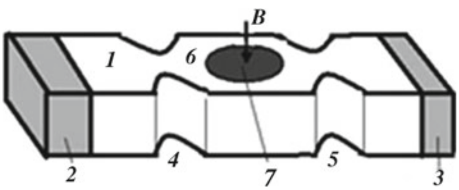


Fig. 11.1 Scheme of one-resonator device for registering magnetic fields: *1* quantum waveguide; *2* and *3* source and drain; *4* and *5* waveguide's narrows forming the resonator *6*; *7* magnetic field domain

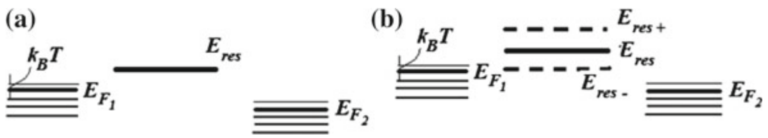


Fig. 11.2 Energy diagram of a one-resonator device for registering magnetic fields: In the absence of magnetic field the resonant level over the Fermi level at the source; **a** $B = 0$; **b** $B \neq 0$

E of electrons with transverse quantum number greater than 1 and for the Fermi levels E_F at the source and the drain, the $k_{\mathcal{B}}$ being Bolzmann constant and T a temperature. Therefore, the electrons with energy exceeding the second threshold practically do not occur in the waveguide.

The current density J through the system is defined by

$$J = \int_E g(E)\nu(E)T(E)f_S(E)(1 - f_D(E - eU))\,dE,$$

the integration over the electrons moving from the source, $g(E)$ being the state density in the non-deformed waveguide, $\nu(E)$ the electron velocity along the waveguide axis, $T(E)$ the probability for an electron with energy E to pass through the resonator. Finally, $f_S(E)$ is the Fermi function for the electrons of the source (that is, the filling probability of the level E with electrons), and $f_D(E)$ is that for the electrons of the drain. Therefore, there is a current in the systems only if the resonant level is below the Fermi level at the source and above that at the drain.

A resonant level is determined by the resonator length. We choose the latter, so that the resonant level E_{res} would satisfy $E_{res} - E_{F_1} > k_{\mathcal{B}}T$ (Fig. 11.2a). Since $f_S(E)$ is quite small for $E > E_{F_1} + k_{\mathcal{B}}T$, there is no current, practically, in the system because the incident flow contains few electrons that could pass through the resonator.

When a magnetic field is supplied in the resonator, the resonant level E_{res} splits into two levels E^+_{res} and E^-_{res} so that $E^-_{res} < E_{res} < E^+_{res}$. The E^+_{res} corresponds to the electrons whose spin direction coincides with the direction of the magnetic field vector B and E^-_{res} corresponds to the electrons whose spin in direction is opposite to B. For a certain magnetic field strength, the resonant level E^-_{res} gets to the energy interval $E < E_{F_1} + k_{\mathcal{B}}T$ and a current through the resonator arises (Fig. 11.2b). When the magnetic field strength B further increases, the resonant level E^-_{res} becomes lower than the Fermi level E_{F_2} at the drain and the current sharply decreases at $E^-_{res} = E_{F_2} - k_{\mathcal{B}}T$ because the final electron states in the drain turn out to be occupied. Thus, a current through the resonator exists only on the conditions

$$E_{F_2} - k_{\mathcal{B}}T < E^-_{res} < E_{F_1} + k_{\mathcal{B}}T.$$

The B-dependence of the current through the system is depicted by Fig. 11.3.

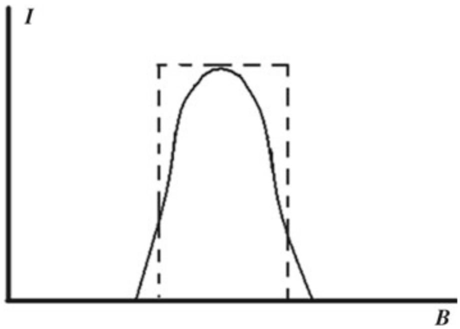


Fig. 11.3 B-dependence of the current through the resonance system: $T = 0$ (*dashed line*); $T > 0$ (*solid line*)

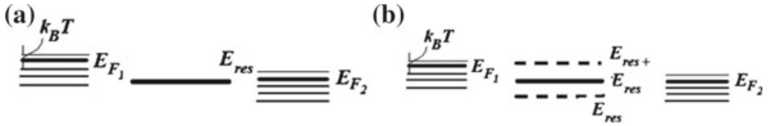


Fig. 11.4 Energy diagram of one-resonator device for registering magnetic fields: In the absence of magnetic field the resonant level below the Fermi level at the drain; **a** $B = 0$; **b** $B \neq 0$

A one-resonator device can be implemented also in a slightly different way. If the resonator length has been chosen so that $E_{F_2} - E_{res} > k_{\mathcal{B}}T$, there is no current because all finite electron states of the drain have been occupied (Figs. 11.4 and 11.5).

When a magnetic field is supplied in the resonator and increases, the E^+_{res} increases as well. A current through the resonator exists under the conditions

$$E_{F_2} - k_{\mathcal{B}}T < E^+_{res} < E_{F_1} + k_{\mathcal{B}}T.$$

The one-resonator sensors based on the above schemes possess several disadvantages. Devices of this kind are of high thermal sensitivity, that is, the magnetic field strength, needed for a through current, depends on temperature. The strength to initiate such a current is sufficiently large: for $T \approx 10\,\mathrm{K}$, the E^+_{res} and E^-_{res} must satisfy $|E^+_{res} - E^-_{res}| \geq k_{\mathcal{B}}T \approx 10^{-3}\,\mathrm{eV}$. Thus, the devices are of comparatively low sensitivity to magnetic field.

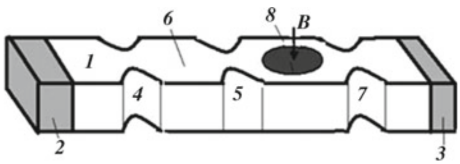


Fig. 11.5 The scheme of a two-resonator device for registering magnetic fields: *1* quantum waveguide; *2* and *3* source and drain; *4*, *5*, and *7* waveguide narrows forming the resonators *6* and *8*; *B* magnetic field domain

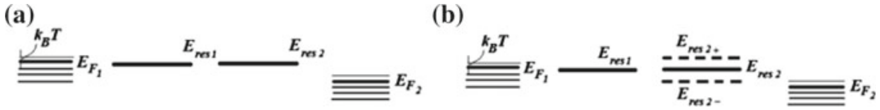


Fig. 11.6 Energy diagram of a two-resonator device for registering magnetic fields (type 1): **a** In the absence of magnetic field, there is current through both resonators; **b** no current in the presence of magnetic field

The two-resonator systems have none of the disadvantages described above. Such devices are similar to those of the first type; however, the waveguide has three narrows forming two quantum resonators in series. A magnetic field is located in one of the resonators.

The resonators lengths have been chosen so that the resonant levels E_{res1} and E_{res2} of the first and the second resonators coincide being below the Fermi level E_{F_1} at the source and above the Fermi level E_{F_2} at the drain. Then, in the absence of magnetic field, there is a through current in the system (Fig. 11.6a).

After turning on a magnetic field, the resonant level E_{res2} splits into the levels E^{-}_{res2} and E^{+}_{res2}. If the distance between the E^{-}_{res2} and E^{+}_{res2} is greater than the resonance width (that is, the resonant peak width) corresponding to E_{res1}, the through current vanishes. A decrease in the narrow diameter leads to a decrease in the resonance width, which raises the device sensitivity to magnetic field. However, let us note that an effective narrow diameter cannot be made arbitrary small because of the impact of the waveguide work function (see Sect. 5.7). This confines the possibility to improve the resonator quality factor by diminishing a narrow diameter. On the other hand, a too strong decrease in the resonance width (or, equivalently, increasing the resonator quality factor) will worsen the device operation speed (the time, a tunneling electron spends in the resonator, is proportional to the resonator quality factor).

Now, we consider another two-resonator device; unlike the first device, there is no through electron current in the absence of magnetic field (Fig. 11.7).

The resonator lengths have been chosen so that the distance between the resonant levels E_{res1} and E_{res2} is greater than the width of the corresponding resonances; moreover,

$$E_{F_2} < E_{resJ} < E_{F_1}, \quad J = 1, 2.$$

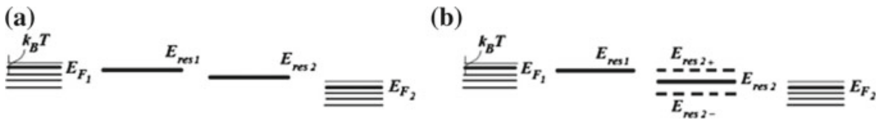


Fig. 11.7 Energy diagram of a two-resonator device for registering magnetic fields (type 2): **a** In the presence of magnetic field, there is current through both resonators; **b** through current arises for a certain value of magnetic field

Then, in the absence of magnetic field, there is no current in the system. When a magnetic field is turned on, the E_{res2} splits into E^{-}_{res2} and E^{+}_{res2}. For a certain magnetic field strength, one of the $E^{\pm}_{res2}$ coincides with E_{res1} and a through current arises. On further increasing magnetic field strength, the resonant levels again become different and the current vanishes. By choosing the geometry of the two resonators, one could tune the system to turning on for a given magnetic flow value.

11.2 Transistors Based on Quantum Waveguides

Here we describe one- and two-resonator transistors schemes. Let us begin with one-resonator devices (Fig. 11.8).

A transistor with one resonator comprises a quantum wire of a high-ohmic semiconductor with two narrows and injection contacts at its opposite ends (a source and a drain). The sizes of the waveguide and its material are chosen to provide ballistic (collisionless) electron transport from the source to the drain. The domain between the narrows is a quantum resonator. Near the resonator, there is a metallic electrode with control voltage U_C. The diameter of the waveguide cross-section has to be less than the Debye length of the semiconductor; then a change in U_C causes a change in the resonant level. Between the source and the drain, there is a small voltage U providing a current along the wire. Therefore, the Fermi level E_{F_1} at the source is eU above the Fermi level E_{F_2} at the drain. The resonator has been chosen so that the minimal resonant level is greater than $E_{F_1} + k_{\mathcal{B}}T$, which is the maximal electron energy at the contacts.

If $U_C = 0$, the current is negligible because the incident electron flow contains few electrons of energy close to the resonant level (Fig. 11.9a). On increasing U_C, the resonant level diminishes and, finally, reaches E_{F_1}, where the current sharply increases (Fig. 11.9b).

On further increasing U_C, the current intensity practically remains constant because, in the conductivity, electrons of the same energy interval determined by the resonance width act as participants there. When the resonant level closely approaches

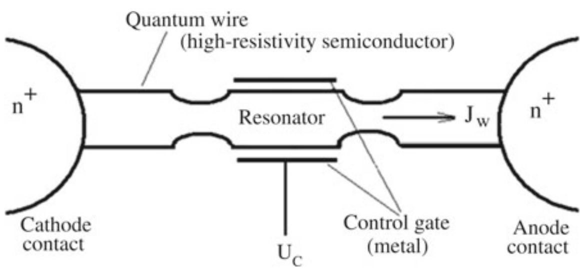


Fig. 11.8 Scheme of one-resonator transistor

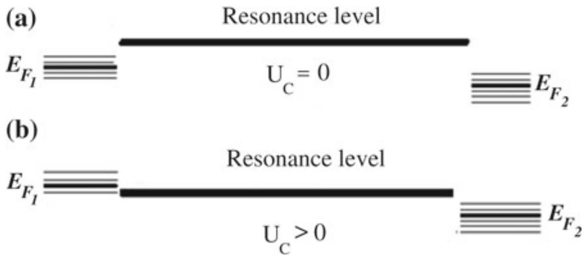


Fig. 11.9 Energy diagram of one-resonator transistor (type 1)

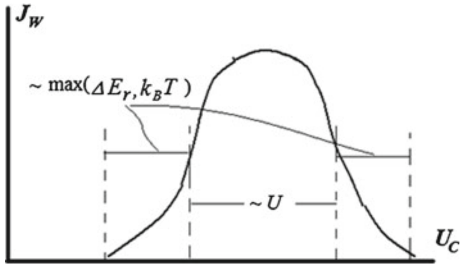


Fig. 11.10 Current J_W versus control voltage U_C in a waveguide with one resonator

the Fermi level E_{F_2} at the drain, the current intensity reduces because all final electron states at the drain have been occupied. The U_C-dependence of the current J_w through the quantum waveguide is shown by Fig. 11.10.

The "steepness" of the curve (that is, dJ_W/dU_C) is determined by the value $\max\{\Delta E_r, k_{\mathcal{B}}T\}$, where ΔE_r denotes the resonance width with respect to electron energy and $k_{\mathcal{B}}T$ is the spread of electron energy around a Fermi level. The system is of high thermal sensitivity: on increasing temperature, the curve steepness diminishes.

Let us consider also another version of the one-resonator transistor. We suppose that the resonator has been chosen so that the resonant level is below the Fermi level E_{F_2} at the drain (Fig. 11.11). Then there is no current for $U_C = 0$ because the final electron states at the drain have been occupied. On supplying a control voltage $U_c < 0$, the resonant level increases. For a certain strength of U_C, there appear free states at the drain and a through current arises. On further increasing $|U_C|$, the resonant level turns out to be above the occupied states at the source and the current sharply decreases. The U_C-dependence of the current J_w through the waveguide is similar to that in Fig. 11.10 (up to a sign of U_C).

Let us consider two-resonator transistors that have no disadvantage of thermal sensitivity.

The waveguide has three narrows that form the two resonators in series. The lengths of the resonators are different so that their resonant levels E_{res1} and E_{res2} are distinct. Moreover,

$$E_{F_2} < E_{resJ} < E_{F_1}, \quad J = 1, 2.$$

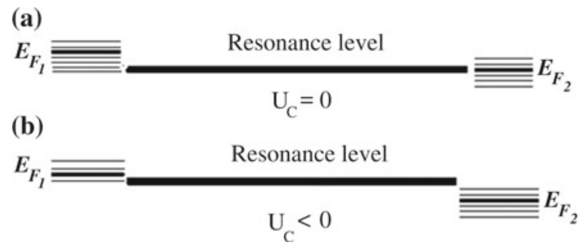


Fig. 11.11 Energy diagram of one-resonator transistor (type 2)

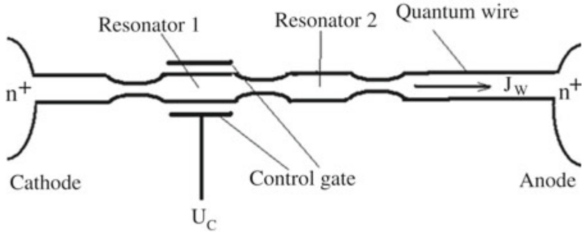


Fig. 11.12 A transistor scheme with two resonators

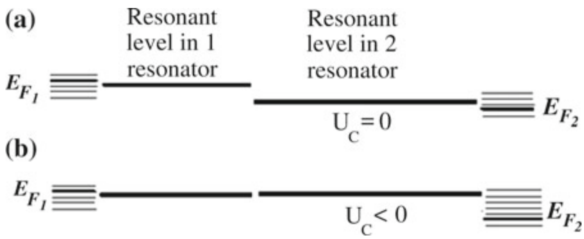


Fig. 11.13 Energy diagram of two-resonator transistor

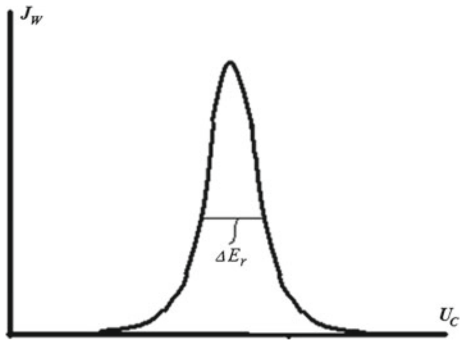
A small voltage U is supplied between the source and the drain. A metallic electrode with control voltage U_C is located near one of the resonators (Fig. 11.12).

Figure 11.13 shows the energy diagram for the case where the resonant level of the resonator with control electrode is below the one of the other resonator. In the absence of the control voltage, the current in the system is negligible because the E_{res1} and E_{res2} are different. On supplying control voltage U_C, one of the levels shifts. When the levels coincide, the through current sharply increases. The needed variation of the control voltage is independent of temperature and determined by the resonance width only (Fig. 11.14).

We obtain another implementation of the two-resonator transistor by choosing $E_{res1} = E_{res2}$. A through current exists for $U_C = 0$. On supplying control voltage (of any sign), the current sharply decreases.

If the resonator quality factors are too large, the U_C-operation band turns out to be too narrow. Therefore, it is reasonable to choose resonators with a minimal quality factor. This improves the device operation speed because the current stabilization time is proportional to the system quality factor.

Fig. 11.14 Current J_W versus control voltage U_C in a waveguide with two resonators



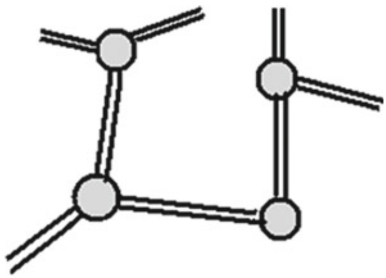
11.3 Electron Flow Switch for Quantum Nets

We now consider a quantum net consisting of quantum waveguides and electron flow switches at the nodes of a net. An electron flow comes in a node through one of the attached waveguides, and a switch controlled by an electric field chooses a waveguide for the flow to go out (Fig. 11.15).

If the switch size were macroscopic and the electron motion were classic, then a control electric field could be chosen to direct all electron trajectories from an inlet waveguide to any given outlet. However, we deal with a switch whose size is comparable to the electron wavelength. Therefore, only probability makes sense for an electron to pass from an inlet to a given outlet. Figure 11.16 shows the scheme of a device where the electron probability to get in a given outlet is greater than 0.99.

Let us consider a two-dimensional model of a quantum control system comprising a cylindrical resonator and three attached waveguides. An electron flow, in the collisionless regime, is supplied to the resonator through one of the waveguides. The two other waveguides are outlets. The waveguide's and the resonator's walls are of a sufficiently large work function so that the electron penetration through the surface potential barrier is negligible. Three control electrodes are adjacent to the resonator that is separated from the electrodes by a thin dielectric film. Constant voltages V_1, V_2, and V_3, whose values can be independently varied are applied to the electrodes. The sizes of the system are small ($10 \div 100\,\mathrm{nm}$), therefore, it's capacity is low

Fig. 11.15 A quantum net example. The nodes with switches are shown by gray color



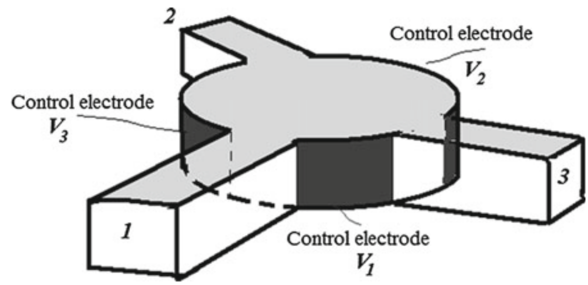


Fig. 11.16 The scheme of a device with one inlet waveguide (*1*) and two outlet waveguides (*2*) and (*3*). The three control electrodes are marked out by dark color

enough to provide a high operation speed. Such a device can be used as a switch and a signal generator of millimeter-wavelength range.

We now describe a mathematical model of the system [4]. Let G be a two-dimensional domain consisting of a resonator (a disk of radius ρ) and three attached waveguides (half-strips of the same width). We assume that the waveguide width is

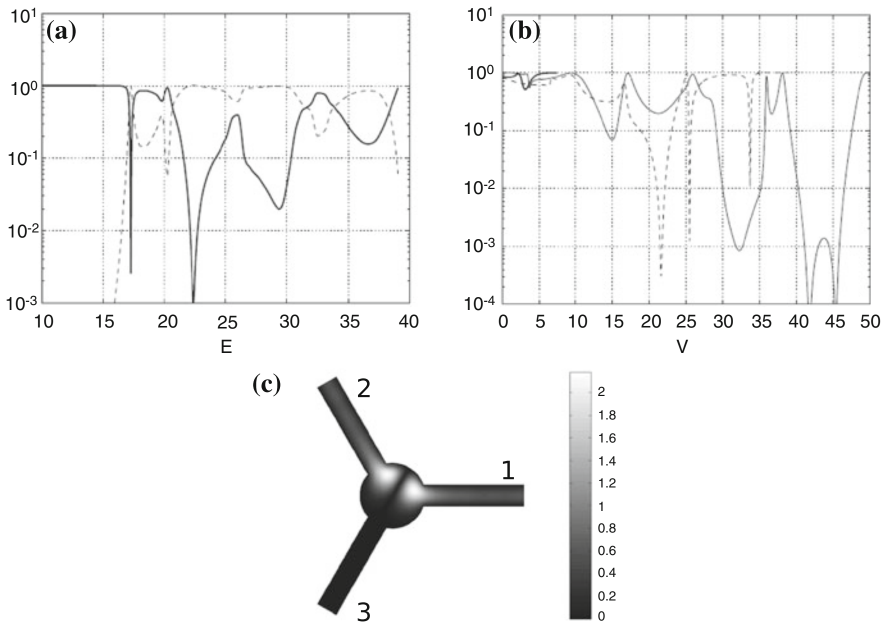


Fig. 11.17 The resonator diameter is equal to 3. **a** The loss probability $|S_{11}|^2 + |S_{13}|^2$ (*solid line*) and the electron transmission probability $|S_{12}|^2$ (*dashed line*) in relation to electron energy E for control voltage $V = (1.5\pi)^2 \approx 22.2$. **b** The loss probability $|S_{11}|^2 + |S_{13}|^2$ in relation to control voltage V for energy $E = (1.9\pi)^2$ (*solid line*); $E = (1.5\pi)^2$ (*dashed line*); $E = (1.1\pi)^2$ (*thick solid line*, only for $V < 10$). **c** The distribution of probability density $|\Psi|^2$, Ψ being an electron wave function, inside the resonator for $E = 17.26$ and $V = (1.5\pi)^2 \approx 22.2$, which corresponds to the first minimum of $|S_{11}|^2 + |S_{13}|^2$

equal to 1 and the angles between the waveguide axes are equal to $2\pi/3$. An electron wave function satisfies the boundary value problem

$$\begin{aligned} -\Delta\Psi(x) + U(x)\Psi(x) - E\Psi(x) = 0, &\quad x \in G, \\ \Psi(x) = 0, &\quad x \in \partial G. \end{aligned} \tag{11.3.1}$$

Control voltages V_1, V_2, and V_3 produce the potential U in equation (11.3.1).

We next consider electrons of energy E between the first and the second thresholds, which means that $\pi^2 < E < (2\pi)^2$; here the length unit is the waveguide width d and the energy unit is $\hbar^2/(2m^*d^2)$, m^* being an effective electron mass. Scattering an electron wave coming in the resonator through waveguide 1 is described by the row (S_{11}, S_{12}, S_{13}) of the scattering matrix. Choosing V_1, V_2, V_3, and E, we would like to provide $|S_{12}|^2$ close to 1. (Recall that $|S_{12}|^2$ is the probability of an electron passing from inlet 1 to outlet 2.) The electron flow switching from outlet 2 to outlet 3 can be performed by interchanging the V_1 and V_3.

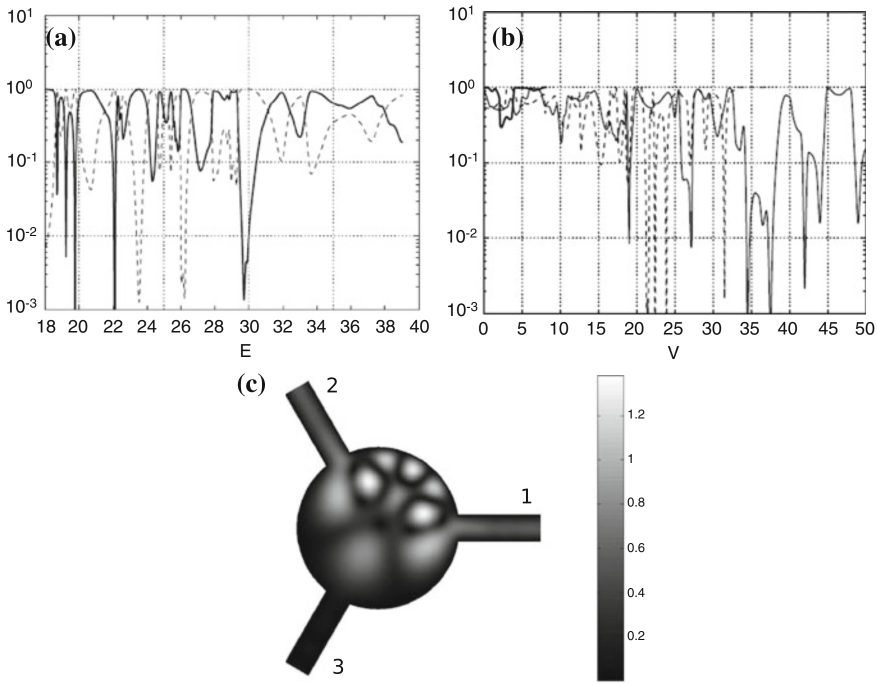


Fig. 11.18 The resonator diameter is equal to 6. **a** The loss probability $|S_{11}|^2 + |S_{13}|^2$ (*solid line*) and the electron transmission probability $|S_{12}|^2$ (*dashed line*)in relation to electron energy E for control voltage $V = (1.5\pi)^2 \approx 22.2$. **b** The loss probability $|S_{11}|^2 + |S_{13}|^2$ in relation to control voltage V for energy $E = (1.9\pi)^2$ (*solid line*); $E = (1.5\pi)^2$ (*dashed line*); $E = (1.1\pi)^2$ (*thick solid line*, only for $V < 10$). **c** The distribution of probability density $|\Psi|^2$, Ψ being the electron wave function, inside the resonator for $E = 22.09$ and $V = (1.5\pi)^2 \approx 22.2$, which corresponds to the fourth minimum of $|S_{11}|^2 + |S_{13}|^2$

We assume that $V_1 = 0$ because only the voltage differences play a role. Moreover, test calculations show that the impact of $V_2 - V_3$ on the results is comparatively small. Therefore, in what follows we set $V_2 = V_3 \equiv V$. The entries (S_{11}, S_{12}, S_{13}) of the scattering matrix have been approximately calculated by the method introduced in Chap. 4. Some results are presented in Figs. 11.17 and 11.18.

For small electron energy $(10 < \mathrm{E} < 15)$, the loss probability is close to 1. For large energy, this probability can be reduced below 10^{-3} by varying one of the parameter E and V; small losses occur if E and V are comparable. On increasing the resonator diameter, the number of loss minima increases (compare Figs. 11.17 and 11.18). For the loss probability close to 10^{-3}, the range of control voltage is quite narrow; on varying V within 0.01, the probability sharply increases. However, the probability of around 10^{-2} is much more stable relative to control voltage. It is seen from Fig. 11.17 that the loss probability remains below 10^{-2} for $E = (1.9\pi)^2$ and $30 < V < 35$. Thus, by varying potentials V_j, we could direct an electron flow to a given outlet waveguide with probability greater than 0.99 and low requirements to the control voltage stability.

Bibliographical Sketch

In the book we use known results of the theory of elliptic boundary value problems in domains with piece-wise smooth boundary exposed in [25, 28, 31, 37]. We mainly refer [37].

Chapter 2 in essence presents a special version, for the Helmholtz equation, of the theory developed in [37] (Chap. 5) for the general self-adjoint elliptic boundary value problem in domains with cylindrical outlets to infinity (the statement and solvability of the problem with intrinsic radiation conditions at infinity, the definition of the scattering matrix).

Chapter 3 is based on Plamenevskii et al. [41]. Augmented scattering matrices were considered in various geometric situations in Nazarov and Plamenevskii [37, 38], and Kamotskii and Nazarov [27] for general elliptic problems self-adjoint with respect to the Green formula. The use of "stable bases" is quite traditional in asymptotics studies. In this connection we mention Costabel and Dauge [17] and Maz'ya and Rossmann [34] dealing with asymptotics of solutions to elliptic boundary value problems near a corner point at the boundary. In Nazarov and Kamotskii [26], the asymptotics of the scattering matrix near a threshold for a two-dimensional diffraction grating was justified, in essence, by using a stable basis of waves.

The method for computing scattering matrices, presented in Chap. 4, was suggested for a close situation in Grikurov et al. [24]. The justification of the method in [24] was based on Proposition 3 (given without proof) valid only under an additional condition not presented in Proposition 3. The condition requires that the value of spectral parameter μ, for which the method is applying, is not an eigenvalue of the original boundary value problem. The method was justified for two-dimensional waveguides without the aforementioned additional condition in Plamenevskii and Sarafanov [39]. Chapter 4 exposes a new proof of the method, which is much simpler than that in [39]; in this connection we mention also Plamenevskii and Sarafanov [40] and Plamenevskii et al. [42].

Chapter 5 (Sects. 5.1–5.6) contains results from Baskin et al. [9]. The results of Sect. 5.7 are published for the first time.

Chapter 6 is based on Baskin et al. [6].

Chapter 7 exposes the results of Baskin et al. [12].

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Chapter 8 is based on Baskin, Plamenevskii, and Sarafanov [11].

The results of Chapter 9 are obtained by L.M. Baskin, M.M. Kabardov, and N.M. Sharkova and taken from their forthcoming paper. In this connection we mention also Racec et al. [43], where electron transport in a waveguide was studied by approximate computing the waveguide R-matrix.

Chapter 10 presents results obtained by O.V. Sarafanov; they are published for the first time.

Chapter 11 contains Sect. 11.1 based on Baskin et al. [10]; Sect. 11.2 based on Baskin et al. [7]; Sect. 11.3 based on Grikurov et al. [4].

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