Infosys Science Foundation Series in Mathematical Sciences

Debashish Goswami
Jyotishman Bhowmick

Quantum Isometry Groups

Infosys Science Foundation

Springer

Infosys Science Foundation Series

Infosys Science Foundation Series in Mathematical Sciences

The *Infosys Science Foundation Series in Mathematical Sciences* is a sub-series of The *Infosys Science Foundation Series*. This sub-series focuses on high quality content in the domain of mathematical sciences and various disciplines of mathematics, statistics, bio-mathematics, financial mathematics, applied mathematics, operations research, applies statistics and computer science. All content published in the sub-series are written, edited, or vetted by the laureates or jury members of the Infosys Prize. With the Series, Springer and the Infosys Science Foundation hope to provide readers with monographs, handbooks, professional books and textbooks of the highest academic quality on current topics in relevant disciplines. Literature in this sub-series will appeal to a wide audience of researchers, students, educators, and professionals across mathematics, applied mathematics, statistics and computer science disciplines.

More information about this series at http://www.springer.com/series/13817

Debashish Goswami · Jyotishman Bhowmick

Quantum Isometry Groups

Springer

Debashish Goswami
Statistics and Mathematics Unit
Indian Statistical Institute
Kolkata, West Bengal
India

Jyotishman Bhowmick
Statistics and Mathematics Unit
Indian Statistical Institute
Kolkata, West Bengal
India

ISSN 2363-6149 ISSN 2363-6157 (electronic)
Infosys Science Foundation Series
ISSN 2364-4036 ISSN 2364-4044 (electronic)
Infosys Science Foundation Series in Mathematical Sciences
ISBN 978-81-322-3665-8 ISBN 978-81-322-3667-2 (eBook)
DOI 10.1007/978-81-322-3667-2

Library of Congress Control Number: 2016956830

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer (India) Pvt. Ltd.
The registered company address is: 7th Floor, Vijaya Building, 17 Barakhamba Road, New Delhi 110 001, India

Preface

It is not very common to embark upon the assiduous project of writing a book on a research topic within 5 years of its inception. One usually needs somewhat longer time to let the dust settle, allowing a clear picture of the relevant mathematical structures and the simplest forms and elegant proofs of the theorems to emerge. In fact, when we began our ambitious programme of defining an analogue of the group of isometries in the framework of noncommutative geometry and quantum groups around 2008–2009, we did not even dream of a book in near future. However, the enthusiasm and encouragement with which the new-born theory of quantum isometry groups was received among different quarters of researchers from around the globe and the flurry of papers on different aspects of it and several Ph.D. theses being written up within 3–4 years were really overwhelming. It makes us feel that it is perhaps the right time to collect the main results and present them in the form of a book to make it convenient for those who are working in this area. There are two important criteria for deciding the timing of a research monograph. On the one hand, the topic should still be "hot" enough in terms of number of active researchers working on it and recognition of importance of the theme among them. On the other hand, there should be enough developments of the topic, such as some major breakthroughs and some deep and difficult theorems which can pave directions of future research. While there can be no doubt that the theme of the book fulfills the first condition, we opine that the recent proof of the conjecture about nonexistence of genuine isometric (compact) quantum group coaction on classical compact connected manifolds by Goswami and Joardar, along with the new techniques and auxiliary results used in the proof, has brought the theory of quantum isometry to a level of maturity which qualifies for the second criterion.

A unique feature of this book is the emphasis on the interaction of C^* algebraic compact quantum groups with the operator-algebraic (i.e., a la Connes) noncommutative geometry. There are several excellent books on noncommutative geometry as well as quantum groups, but to the best of our knowledge, none of them deals with both these areas together in a significant way. The only notable exception is perhaps the classic treatise [1] by Manin which does present a pioneering formulation of quantum symmetry groups as suitable universal bialgebras or Hopf algebras in the context of noncommutative algebraic geometry, and it is indeed the precursor of the (C^* algebraic) quantum automorphism groups of finite dimensional spaces and algebras defined and studied by Wang, Banica, Bichon et al., serving also as a prime motivation for our theory of quantum isometry groups of more general noncommutative Riemannian manifolds. However, Manin's approach was mostly algebraic and did not deal with the analytic or spectral setup of noncommutative geometry or the setup of compact quantum groups a la Woronowicz. In fact, there does not seem to be any book combining the algebraic noncommutative geometry and compact quantum groups, barring stray discussions or passing references to examples of compact quantum group equivariant spectral triples in some of the recent books on noncommutative geometry. Our book proposes to fill this gap in some sense.

However, there are a few very important topics in the interface of noncommutative geometry and quantum groups which we have not at all touched, as we have mainly dealt with the metric aspects of spectral triples which are relevant for defining and studying isometries given by quantum group actions. These include the theory of Hopf algebra symmetry and the corresponding Hopf-cyclic homology and cohomology [2–6], fusion categories [7], Connes–Kreimer Hopf algebras in the context of renormalization [8], quantum group equivariant KK-theory [9], Baum–Connes conjecture [10, 11], covariant and bi-covariant differential calculus on quantum groups in the sense of Woronowicz, Tsygan [12–15] and others.

We have tried our best to make the exposition as self-contained as possible. However, a background in the basics of operator algebras, differential geometry, and the Peter–Weyl theory of compact groups will be desirable. For the convenience of readers, we have collected the basic concepts and results (mostly without proofs) in the first two chapters. The list of references at the end of the book is by no means exhaustive and we apologize for any inadvertent omission of some relevant reference.

We take this opportunity to express our gratitude to Soumalya Joardar, Adam Skalski, Arnab Mandal, Promit Ghosal, Teodor Banica, Piotr Soltan, Francesco D'Andrea, Ludwik Dabrowski, Biswarup Das, Marc Rieffel, Chelsea Walton, Pavel Etingof, Huichi Huang, Shuzhou Wang, Alexandru Chirvasitu, and Joahan Quaggebear for collaboration or discussion on various topics presented in the book. We are grateful to Alain Connes and Yuri Manin for their encouragement and some valuable advice about the program of understanding quantum symmetries of noncommutative spaces. Let us also thank Mr. Shamim Ahmad and the editorial team of Springer India for their excellent cooperation throughout.

D.G. would like to dedicate this book to his daughters, Samyosree and Srijani, who illuminate and rejuvenate every moment of their parents' lives, without even knowing about it. He is also grateful to his extended family and friends, including his parents, mother-in-law, and Amit da for their invaluable practical and emotional support. It is simply not possible to thank his life-partner Gopa, as she is an integral part of his creative self. This book would not have been written without her constant support, encouragement, and the extraordinary vivacity and efficiency with which

she takes care of day-to-day life and handles any difficult situation. J.B. would like to dedicate this book to his parents who bore a lot of hardship and innumerable sacrifices silently for making him into the human being that he is now.

Kolkata, India

Debashish Goswami
Jyotishman Bhowmick

References

1. Manin, Y: Quantum groups and non-commutative geometry. CRM, Montreal (1988)
2. Kaygun, A., Khalkhali, M.: Bivariant Hopf cyclic cohomology. Comm. Algebra **38**(7), 2513–2537 (2010)
3. Khalkhali, M., Rangipour, B.: Cup products in Hopf-cyclic cohomology. C. R. Math. Acad. Sci. Paris **340**(1), 9–14 (2005)
4. Moscovici, H., Rangipour, B.: Cyclic cohomology of Hopf algebras of transverse symmetries in codimension 1, Adv. Math. **210**(1), 323–374 (2007)
5. Moscovici, H., Rangipour, B.: Hopf cyclic cohomology and transverse characteristic classes, Adv. Math. **227**(1), 654–729 (2011)
6. Hajac, P.M., Khalkhali, M., Rangipour, B., SommerhÃ¤user, Y.: Hopf-cyclic homology and cohomology with coefficients. C. R. Math. Acad. Sci. Paris **338**(9), 667–672 (2004)
7. Etingof, P., Nikshych, D. and Ostrik, V.: On fusion categories, Ann. Math. **162**(2), 581–642 (2005)
8. Connes, A., Kreimer, D.: Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. **199**(1), 203–242 (1998)
9. Vergnioux, R.: KK-theorie Equivariante et Operateur de Julg-Valette pour les groupes quantiques. Ph.D. thesis, Paris (2002)
10. Baum, P., Connes, A., Higson, N.: Classifying space for proper actions and K-theory of group C^*-algebras. C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 240–291. Contemp. Math., **167**. Amer. Math. Soc., Providence, RI, 1994.
11. Meyer, R. and Nest, R.: The Baum-Connes conjecture via localisation of categories. Topology **45**, 209–259 (2006)
12. Woronowicz, S.L.: Differential calculus on compact matrix pseudogroups (quantum groups). Commun. Math. Phys. **122**(1), 125–170 (1989)
13. Tsygan, B.: Notes on differential forms on quantum groups. Selected translations, Selecta Math. Soviet. **12**(1), 75–103 (1993)
14. Heckenberger, I., Schuler, A.: Laplace operator and Hodge decomposition for quantum groups and quantum spaces, Quantum groups and integrable systems (Prague, 2001). Czechoslovak J. Phys. **51**(12), 1342–1347 (2001)
15. Heckenberger, I., Kolb, S.: Differential calculus on quantum homogeneous spaces. Lett. Math. Phys. **63**(3), 255–264 (2003)

Contents

1 Preliminaries . . . 1
1.1 Operator Algebras and Hilbert Modules . . . 1
1.1.1 C^*Algebras . . . 1
1.1.2 Von Neumann Algebras . . . 4
1.1.3 Free Product and Tensor Product . . . 5
1.1.4 Hilbert Modules . . . 6
1.2 Quantum Groups . . . 8
1.2.1 Hopf Algebras . . . 8
1.2.2 Compact Quantum Groups: Basic Definitions and Examples . . . 10
1.2.3 The CQG $U_\mu(2)$ . . . 17
1.2.4 The CQG $SU_\mu(2)$ . . . 18
1.2.5 The Hopf $*$-algebras $\mathcal{O}(SU_\mu(2))$ and $\mathcal{U}_\mu(su(2))$ . . . 19
1.2.6 The CQG $SO_\mu(3)$ . . . 20
1.3 Coaction of Compact Quantum Groups on a C^*Algebra . . . 21
1.3.1 Coactions on Finite Quantum Spaces . . . 22
1.3.2 Free and Half-Liberated Quantum Groups . . . 25
1.3.3 The Coaction of $SO_\mu(3)$ on the Podles' Spheres . . . 27
1.4 Dual of a Compact Quantum Group . . . 29
1.5 Coaction on von Neumann Algebras by Conjugation of Unitary Corepresentation . . . 30
References . . . 33

2 Classical and Noncommutative Geometry . . . 37
2.1 Classical Riemannian Geometry . . . 37
2.1.1 Forms and Connections . . . 37
2.1.2 The Hodge Laplacian of a Riemannian Manifold . . . 38
2.1.3 Spin Groups and Spin Manifolds . . . 39

2.1.4 Dirac Operators . . . 40
2.1.5 Isometry Groups of Classical Manifolds . . . 41
2.2 Noncommutative Geometry . . . 50
2.2.1 Spectral Triples: Definition and Examples . . . 50
2.2.2 The Noncommutative Space of Forms . . . 53
2.2.3 Laplacian in Noncommutative Geometry . . . 56
2.3 Quantum Group Equivariance in Noncommutative Geometry . . . 58
2.3.1 The Example of $SU_\mu(2)$ . . . 58
2.3.2 The Example of the Podles' Spheres . . . 58
2.3.3 Constructions from Coactions by Quantum Isometries . . . 60
2.3.4 R-twisted Volume Form Coming from the Modularity of a Quantum Group . . . 63
References . . . 66

3 Definition and Existence of Quantum Isometry Groups . . . 69
3.1 The Approach Based on Laplacian . . . 69
3.1.1 The Definition and Existence of the Quantum Isometry Group . . . 70
3.1.2 Discussions on the Admissibility Conditions . . . 75
3.2 Definition and Existence of the Quantum Group of Orientation Preserving Isometries . . . 77
3.2.1 Motivation . . . 77
3.2.2 Quantum Group of Orientation-Preserving Isometries of an R-twisted Spectral Triple . . . 78
3.2.3 Stability and C^* Coaction . . . 83
3.2.4 Comparison with the Approach Based on Laplacian . . . 86
3.3 The Case of $\widetilde{J}$ Preserving Quantum Isometries . . . 90
3.4 A Sufficient Condition for Existence of Quantum Isometry Groups Without Fixing the Volume Form . . . 92
References . . . 95

4 Quantum Isometry Groups of Classical and Quantum Spheres . . . 97
4.1 Classical Spheres: No Quantum Isometries . . . 97
4.2 Quantum Isometry Group of a Spectral Triple on Podles' Sphere . . . 99
4.3 Descriptions of the Podles' Spheres . . . 100
4.3.1 The Description as in [3] . . . 100
4.3.2 'Volume Form' on the Podles' Spheres . . . 101
4.4 Computation of the Quantum Isometry Groups . . . 102
4.4.1 Affineness of the Coaction . . . 104
4.4.2 Homomorphism Conditions . . . 108

4.4.3 Relations Coming from the Antipode 109
4.4.4 Identification of $SO_\mu(3)$ as the Quantum Isometry Group 111
4.5 Another Spectral Triple on the Podles' Sphere: A Counterexample 114
4.5.1 The Spectral Triple 115
4.5.2 Computation of the Quantum Isometry Group. 117
References. 127

5 Quantum Isometry Groups of Discrete Quantum Spaces 129
5.1 Quantum Isometry Groups of Finite Metric Spaces and Finite Graphs 130
5.1.1 The Works of Banica [3] and Bichon [2] 130
5.1.2 Noncommutative Geometry on Finite Metric Spaces 131
5.1.3 Quantum Symmetry Groups of Banica and Bichon as Quantum Isometry Groups 133
5.2 Quantum Isometry Groups for Inductive Limits 136
5.2.1 Examples Coming from AF Algebras 138
5.2.2 The Example of the Middle-Third Cantor Set 143
References. 146

6 Nonexistence of Genuine Smooth CQG Coactions on Classical Connected Manifolds 149
6.1 Smooth Coaction of a Compact Quantum Group and the No-Go Conjecture 149
6.1.1 Definition of Smooth Coaction 149
6.1.2 Statement of the Conjecture and Some Positive Evidence 150
6.1.3 Defining the 'Differential' of the Coaction 153
6.2 Brief Sketch of Proof of Nonexistence of Genuine Quantum Isometries 155
6.3 An Example of No-Go Result Without Quadratic Independence 158
References. 161

7 Deformation of Spectral Triples and Their Quantum Isometry Groups 163
7.1 Cocycle Twisting 163
7.1.1 Cocycle Twist of a Compact Quantum Group. 164
7.1.2 Unitary Corepresentations of a Twisted Compact Quantum Group. 166
7.1.3 Deformation of a von Neumann Algebra by Dual Unitary 2-Cocycles 167

7.2 Deformation of Spectral Triples by Unitary Dual Cocycles 168
7.3 Quantum Isometry Groups of Deformed Spectral Triples 169
7.4 Examples and Computations 172
References 177

8 Spectral Triples and Quantum Isometry Groups on Group C^*-Algebras 179
8.1 Connes' Spectral Triple on Group C^*-Algebras and Their Quantum Isometry Groups 180
8.1.1 Quantum Isometry Groups of $(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$ 181
8.2 The Case of Finitely Generated Abelian Groups 183
8.2.1 Computation for the Groups $\mathbb{Z}_n$ and $\mathbb{Z}$ 183
8.2.2 Results for the General Case 188
8.3 The Case of Free Products of Groups 190
8.3.1 Some Quantum Groups 190
8.3.2 Results for the Free Groups $\mathbb{F}_n$ 192
8.3.3 Quantum Isometry Groups of Free Product of Finite Cyclic Groups 192
8.4 Quantum Isometry Groups as Doublings 193
8.4.1 Result for a Generating Set of Transpositions 194
8.4.2 The Case When S Has a Cycle 197
References 197

9 An Example of Physical Interest 199
9.1 Notations and Preliminaries 201
9.1.1 Generalities on Real C^* Algebras 201
9.1.2 Quantum Isometries 202
9.2 The Finite Noncommutative Space F 203
9.2.1 The Elementary Particles and the Hilbert Space of Fermions 203
9.2.2 The Spectral Triple 204
9.2.3 A Hypothesis on the Υ Matrices 205
9.3 Quantum Isometries of F 206
9.3.1 $\widetilde{\mathrm{QISO}}_J^+$ in two special cases 209
9.3.2 Quantum Isometries for the Real C^* Algebra A_F 210
9.4 Quantum Isometries of $M \times F$ 211
9.5 Physical Significance of the Results 211
9.5.1 Analysis of the Result for the Minimal Standard Model 213
9.6 Invariance of the Spectral Action 214
References 217

10 More Examples and Open Questions 221
10.1 Free and Twisted Quantum Spheres and Their Projective Spaces 221
10.1.1 Quantum Isometry Groups of Free and Half-Liberated Quantum Spheres 223
10.1.2 Quantum Group of Affine Isometries a la Banica 225
10.2 Easy Quantum Groups as Quantum Isometry Groups 226
10.3 Quantum Symmetry Groups of Orthogonal Filtrations 227
10.4 Equivariant Spectral Triples on the Drinfeld–Jimbo q-Deformation of Compact Lie Groups and Their Homogeneous Spaces 228
10.5 Quantum Isometry Groups for Metric Spaces 231
References 234

About the Authors

Debashish Goswami is Professor at the Stat-Math Unit of the Indian Statistical Institute, Kolkata, since 2011. Earlier, he held the positions of Associate Professor (2006–2011) and Assistant Professor (2002–2006) at the same department. He received his Ph.D. degree from the Indian Statistical Institute, Kolkata, in 2000. After that, he spent a couple of years in Germany and Italy for his postdoctoral research. His areas of interest include operator algebras, quantum groups, noncommutative geometry, and noncommutative probability. He has published 40 research articles in several premier journals including *Communications in Mathematical Physics* (Springer), *Advances in Mathematics* (Springer), *Journal of Functional Analysis*, *Physics Letters A*, *the Journal of Operator Theory*, *Transactions of the American Mathematical Society*, *Mathematische Annalen* (Springer). In addition, he has published one book, *Quantum Stochastic Processes and Noncommutative Geometry* (Cambridge University Press).

He has received numerous awards and academic recognitions for his work, which include Shanti Swarup Bhatnagar Award (2012), J.C. Bose National Fellowship (2016), Swarnajayanthi Fellowship (2009), B.M. Birla Science Prize (2006), the INSA Medal for Young Scientists (2004), the Junior Associateship of I.C.T.P. (Italy) for 2003–2007 and the Alexander von Humboldt Fellowship for 2000–2001. He has been elected Fellow of the Indian Academy of Sciences (Bangalore) in the year 2015. He has visited several universities and institutes in India and abroad, including Massachusetts Institute of Technology (Cambridge), University of California (Berkeley), Max Planck Institute for Mathematics (Bonn), The Fields Institute (Canada), The Instituto Nacional de Matemática Pura e Aplicada (Brazil), University of Lancaster (England), and The Institute for Research in Fundamental Science (Iran), and delivered invited lectures at many conferences and workshops all over the world.

Jyotishman Bhowmick is Assistant Professor at the Stat-Math Unit of the Indian Statistical Institute, Kolkata. He obtained his Ph.D. from the Indian Statistical Institute, Kolkata, in 2010, followed by postdoctoral positions at the Abdus

Salam International Centre for Theoretical Physics, Trieste, and the University of Oslo. His research interests include noncommutative geometry and compact quantum groups. He has published several articles in several journals of repute including Communications in Mathematical Physics (Springer), *Journal of NoncommutativeGeometry*, *Journal of Functional Analysis*, *Transactions of the American Mathematical Society*, *Proceedings of the American Mathematical Society, and so on*. He has received the INSA medal for Young Scientists in the year 2015. He is an associate of the Indian Academy of Sciences (Bangalore). His academic visits to foreign universities and institutes include Institut des hautes études scientifiques (Paris), The Fields Institute (Canada), University of Glasgow (Scotland), etc.

Notations

$I\!N$	The set of natural numbers
$I\!N_0$	$I\!N \cup \{0\}$
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathcal{M}_n(\mathbb{C})$	The set of all $n \times n$ complex matrices
S^1	The circle group
$\mathbb{T}^n$	The n-torus
ev	Evaluation map
id	The identity map
$\mathrm{Span}\{X\}$	The span of elements of a subset X of a vector space
$l^2(I\!N)$	The Hilbert space of square summable sequences
$C^\infty(M)$	The space of smooth functions on a smooth manifold M
$C_c^\infty(M)$	The space of compactly supported smooth functions on M
$V_1 \otimes_{\mathrm{alg}} V_2$	Algebraic tensor product of two vector spaces V_1 and V_2
$\mathcal{A} \otimes \mathcal{B}$	Minimal tensor product of two C^* algebras $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{M}\overline{\otimes}\mathcal{N}$	von Neumann algebra tensor product of two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$
$\mathcal{M}(\mathcal{A})$	The multiplier algebra of a C^* algebra $\mathcal{A}$
$\mathcal{L}(E,F)$	The space of adjointable maps from Hilbert modules E to F
$\mathcal{L}(E)$	The space of adjointable maps from a Hilbert module E to itself
$\mathcal{K}(E,F)$	The space of compact operators from Hilbert modules E to F
$\mathcal{K}(E)$	The space of compact operators from a Hilbert module E to itself
$\mathcal{B}(\mathcal{H})$	The set of all bounded linear operators on a Hilbert space $\mathcal{H}$
$\mathcal{A} * \mathcal{B}$	Free product of two C^* algebras $\mathcal{A}$ and $\mathcal{B}$
$G * H$	Free product of two groups G and H
$G \rtimes H$	Semi direct product of two groups G and H

Introduction

The theme of this book lies on the interface of two areas of the so-called "noncommutative mathematics," namely noncommutative geometry (NCG) a la Connes, cf [1] and the theory of (C^* -algebraic) compact quantum groups (CQG) a la Woronowicz, cf [2] which are generalizations of classical Riemannian spin geometry and that of compact topological groups respectively.

The root of NCG can be traced back to the Gelfand Theorem which says that there is an anti-equivalence between the category of (locally) compact Hausdorff spaces and (proper, vanishing at infinity) continuous maps and the category of (not necessarily) commutative unital C^* algebras and $*$-homomorphisms. This means that the entire topological information of a locally compact Hausdorff space is encoded in the commutative C^* algebra of continuous functions vanishing at infinity. This motivates one to view a possibly noncommutative C^* algebra as the algebra of "functions on some noncommutative space".

In classical Riemannian geometry on spin manifolds, the Dirac operator on the Hilbert space $L^2(S)$ of square integrable sections of the spinor bundle contains a lot of geometric information. For example, the metric, the volume form and the dimension of the manifold can be captured from the Dirac operator. This motivated Alain Connes to define his noncommutative geometry with the central object as the spectral triple which is a triplet $(A^\infty, \mathcal{H}, D)$ where $\mathcal{H}$ is a separable Hilbert space, A^∞ is a (not necessarily closed) -algebra of $\mathcal{B}(\mathcal{H}); D$ is a self-adjoint (typically unbounded) operator (sometimes called the Dirac operator of the spectral triple) such that $[D, a]$ admits a bounded extension. This generalizes the classical spectral triple $(C^\infty(M), L^2(S), D)$ on any Riemannian spin manifold M, where D denotes the usual Dirac operator.

On the other hand, quantum groups have their origin in different problems in mathematical physics as well as the theory of classical locally compact groups. On the algebraic level, quantum groups can be viewed as Hopf algebras typically arising as deformations of semisimple Lie algebras. This was the viewpoint taken by Drinfeld and Jimbo [3–6] leading to a deep and successful theory having connections with physics, knot theory, number theory, representation theory etc. It was natural to look for an analytic counterpart of the theory of quantum groups with the aim of building a theory of quantum harmonic analysis. It was achieved by S.L. Woronowicz, who in [2, 7] was able to pinpoint a set of axioms for defining compact quantum groups (CQG for short) as the correct generalization of compact topological groups. The formulation of locally compact quantum groups remained elusive for quite some time and there were a number of different approaches to this problem. Finally, a satisfactory theory of locally compact quantum groups was proposed in [8–10].

Groups are most often viewed as "symmetry objects," and in a similar way, quantum groups correspond to some kind of "generalized symmetry" of physical systems and mathematical structures. Indeed, the idea of a group acting on a space was extended to the idea of a CQG coacting on a noncommutative space (that is, a possibly noncommutative C^* algebra). The question of defining and finding "all quantum symmetries" arises naturally in this context. Such an approach was taken in the pioneering work by Manin [11], though in a purely algebraic framework. Indeed, Manin's quantum semigroups such as $M_q(2)$ [11, 12] were constructed as universal symmetry objects of suitable mathematical entities. It was Shuzhou Wang who began the study of the formulation of such universal symmetry objects in the analytic framework of CQG. In [13] he defined and proved the existence of quantum automorphism groups on finite dimensional C^* algebras. Since then, many interesting examples of such quantum groups, particularly the quantum permutation groups of finite sets and finite graphs, have been extensively studied by a number of mathematicians (see for example [14–21] and references therein). The underlying basic principle of defining a quantum automorphism group corresponding to some given mathematical structure (for example a finite set, a graph, a C^* or von Neumann algebra) consists of two steps: first, to identify (if possible) the group of automorphism of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category, replacing groups by quantum groups of appropriate type.

However, most of the work done by the above mathematicians concerned some kind of quantum automorphism group of a "finite" structure, for example, of finite sets or finite dimensional matrix algebras. It was thus quite natural to try to extend these ideas to the "infinite" or "continuous" mathematical structures, for example classical and noncommutative manifolds. With this motivation, one of the authors of this book [22] formulated and studied the quantum analogues of the group of Riemannian isometries called the quantum isometry group. Classically, an isometry is characterized by the fact that its action commutes with the Laplacian. Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which coact on the manifold (or more generally on a noncommutative manifold is given by a spectral triple) in such a way that the coaction commutes with the Laplacian, say $\mathcal{L}$, coming from the spectral triple. It is proven in [22] that a universal object in the category (denoted by $Q^{'}_{\mathcal{L}}$) of such quantum groups does exist (denoted by $QISO^{\mathcal{L}}$) if one makes some mild assumptions on the spectral triple all of which are valid for a compact Riemannian spin manifold. However, the formulation of quantum isometry groups in [22] had a major drawback from the view point of noncommutative geometry, since it needed a "good" Laplacian to exist. In noncommutative geometry, it is not always easy to verify such an assumption about the Laplacian, and thus it would be more appropriate to have a formulation in terms of the Dirac operator directly. This is what is done by the authors of this book in their joint work [23] where the notion of a quantum group analogue of the group of orientation preserving isometries was given and its existence as the universal object in a suitable category was proved.

Then, a number of computations for quantum isometry groups were done by the authors of the present book and other mathematicians including Banica, Skalski, Soltan, just to name a few [23–30]. Some of the main tools for making explicit computations were provided by the results about the effect of deformation and taking a suitable inductive limit on quantum isometry group. In particular, many interesting noncommutative manifolds were obtained by deforming classical Riemannian manifolds in a suitable sense and it was proved that the quantum isometry group of such a deformed (noncommutative) manifold is nothing but a similar deformation or twist of the quantum isometry group of the original, undeformed classical manifold. This led to the problem of computing quantum isometry groups of classical Riemannian manifolds.

However, initial attempts of computing quantum isometry groups of connected classical manifolds including the spheres and the tori (with the usual Riemannian metrics) did not give any genuine quantum group, i.e., the quantum isometry groups for all these manifolds turned out to be the same as the classical isometry groups. It may be mentioned here that it is easy to have faithful isometric coaction of the genuine compact quantum group on disconnected Riemannian manifolds with at least four components. However, no examples of even faithful continuous coaction by genuine compact quantum groups on $C(X)$ with X being connected compact space were known until H. Huang [31] constructed examples of such coactions on topological spaces which are typically obtained by topologically connected sums of copies of some given compact metric space. Another example of a genuine finite dimensional compact quantum group on a compact connected space (which is a subset of a nonsmooth variety) is essentially contained in Remark 4.3 of [32]. On the other hand, it follows from the work of Banica et al. [33] that most of the known compact quantum groups, including the quantum permutation groups of Wang, can never coact faithfully and isometrically on a connected compact Riemannian manifold. All these led the second author to conjecture that it is not possible to have smooth faithful coactions of genuine compact quantum groups on $C(M)$ when M is a compact connected smooth manifold.

After a series of initial attempts, the above conjecture has finally been proved by the second author in collaboration with Joardar [34] for two important cases: (i) when the coaction is isometric for some Riemannian metric on the manifold, and (ii) when the quantum group is finite dimensional. In particular, the quantum isometry group of an arbitrary compact, connected Riemannian manifold M is classical, i.e., same as $C(ISO(M))$. This also implies that the quantum isometry group of a noncommutative manifold obtained by cocycle twisting of a classical, connected, compact manifold is a similar cocycle twisted version of the isometry group of the classical manifold, thus allowing us to explicitly compute quantum isometry groups of a large class of noncommutative manifolds.

It is interesting to note that Etingof and Walton obtained a somewhat similar nonexistence result in [32] in a purely algebraic setup of coactions of finite dimensional semisimple Hopf algebras on commutative domains. However, the proof by Etingof and Walton depends crucially on the finite dimensionality as well as semisimplicity of the Hopf algebras considered by them and there is no obvious way to extend the scope of their arguments to possibly infinite dimensional Hopf algebras.

The above results are quite interesting from a physical point of view. Firstly, it follows that for a classical mechanical system with phase space modeled by a compact connected manifold, the generalized notion of symmetries in terms of quantum groups coincides with the conventional notion, i.e., symmetries coming from group actions. This gives some kind of consistency of the philosophy of thinking quantum group coactions as symmetries. Secondly, it also allows us to describe all the (quantum) symmetries of a physical model obtained by suitable deformation of a classical model with connected compact phase space, showing that such quantum symmetries are indeed deformations of the classical (group) symmetries of the original classical model.

The emerging theory of quantum isometry groups should have potential applications to a number of problems in mathematics and physics. We try to list a few of them here.

One of the main motivations of noncommutative geometry was to treat “bad quotients” arising in classical topology, geometry, dynamical systems, number theory and many other areas of mathematics. Non-Hausdorff leaf spaces of foliations, orbit spaces for nonproper group actions, etc., are some of the standard examples of this kind which are usually substituted by their nice noncommutative (C^* -algebraic) analogues which are typically some sort of generalized crossed product algebras. It often happens that the process of taking quotient destroys group symmetries of a space, i.e., the noncommutative space modeling a quotient of some nice space having a large symmetry group (e.g., homogeneous space of a Lie group) does not admit too much group symmetries. However, it may have a large class of natural and interesting quantum group symmetries, often (but not necessarily) those coming from some kind of deformation or twist of the group symmetries of the original space (before quotienting). For example, one may get an ergodic coaction by the quantum isometry group for some natural spectral triple on such a noncommutative quotient space. In a sense, this equips the noncommutative manifold modeling the bad classical quotient with a nice quantum homogeneous space structure.

The theory of quantum isometry groups should also play an important role in the theory of compact quantum groups. Besides being a rich source of possibly new examples of compact quantum groups, they should be thought of the prototypes of compact Lie groups among compact quantum groups. It will be a very interesting program to build a natural equivariant noncommutative Riemannian geometry on the quantum isometry group of a noncommutative Riemannian manifold (given by a spectral triple). In this context, it is perhaps necessary to develop a noncommutative counterpart of the classical theory of Killing forms in the framework of noncommutative geometry. Another important program is to study the dual discrete quantum groups of quantum isometry groups and try to prove the quantum Baum–Connes property [35] for them. Let us remark that for duals of compact Lie groups this has already been verified.

Let us now speculate some possible roles of quantum isometry groups in physics. Symmetry always plays a very important role in any physical theory, and there are already strong indications that a noncommutative model of space-time is called for to build a satisfactory theory of still-elusive quantum gravity. For example, there are interesting models given by Connes, Chamseddine, Marcolli and others [36–40] and the natural symmetry for such noncommutative space-time should be given by quantum groups. A rather ambitious program in this context will be to define the (locally compact) quantum symmetry group of the noncommutative space-time manifold and study its representation theory in the spirit of Wigner's classical work on prediction of fundamental particles of nature from the representation theory of the Poincare group [41]. In fact, Connes–Chamseddine model is based on the algebra $C^\infty(M) \otimes A$ where M is a smooth manifold and A is a finite dimensional C^* algebra. The spectral triples chosen are typically product type. A first step towards understanding the quantum symmetries of such spectral triples has been taken in [42] and [43].

It may be interesting to point out that in some physics literature on possible theories of quantum gravity see, e.g. [44], such deformed versions of the classical isometry groups of the space-time manifold (which is not compact though) have been extensively used, and even termed as the "quantum isometry groups". It is perhaps intuitively assumed there that such a deformed quantum group is the maximal quantum group coacting isometrically on the deformed space-time, but we have not found a rigorous statement or proof of such assumption. Such a rigorous proof can be given if we can extend our framework of quantum isometry including the results about nonexistence of genuine quantum isometric coactions on connected manifolds to the setup of locally compact quantum groups coacting isometrically on locally compact noncommutative manifolds.

Besides quantum gravity, we speculate that there may be interesting applications of quantum isometry groups to mathematical theories of quasi-crystals, nonintegral quantum Hall effect, topological order, etc.

Now we try to give an idea of the contents of each of the chapters. In Chap. 1, we discuss the concepts and results needed in the later chapters of the book. For the sake of completeness, we begin with a glimpse of operator algebras and Hilbert modules, free product and tensor products of C^* algebras and some examples. The next section is on quantum groups which we start with the basics of Hopf algebras and then define compact quantum groups (CQG) and give relevant definitions and properties including a brief review of Peter Weyl theory. After that, we introduce the quantum groups $U_\mu(2), SU_\mu(2), \mathcal{U}_\mu(su(2))$ and $SO_\mu(3)$. We begin the next section by introducing the notion of a C^* coaction of a compact quantum group on a C^* algebra and giving an account of Shuzhou Wang's work in [13] and [45]. We discuss the example of the coaction of $SO_\mu(3)$ on Podles' spheres in details. The chapter ends with two short sections on the notion of the dual of a CQG and coactions on von Neumann algebras.

The second chapter deals with classical Riemannian geometry and its noncommutative geometric counterparts. In particular, the definition and properties of the Hodge Laplacian and the Dirac operator are discussed. We also derive the characterizations of isometries (resp. orientation preserving isometries) in terms of the Laplacian (resp. Dirac operator). In the section on noncommutative geometry, spectral triples are introduced followed by the definitions of noncommutative space of forms and the Laplacian in this setup. The last section of this chapter is on the quantum group equivariance in Noncommutative Geometry where we discuss the particular case of Podles' spheres, the construction of equivariant spectral triples from coactions by quantum isometries and the twisted volume form coming from a coaction of a CQG with possibly nontrivial modularity.

The characterizations of the isometry group (resp. the group of orientation preserving isometries) on a compact Riemannian (resp. spin) manifolds in Chap. 2 motivate the definition of the quantum isometry group (resp. quantum group of orientation preserving isometries). This is the content of Chap. 3. In the first section of this chapter, under some reasonable assumption on a spectral triple $(A^\infty, \mathcal{H}, D)$ (which we call admissibility), the quantum isometry group $QISO^{\mathcal{L}}$ is defined to be the universal object in the category of CQG's admitting a smooth coaction on the closure of A^∞ such that the coaction commutes with the Laplacian. The case of a quantum group of orientation preserving isometries is taken up in the next section. The quantum group of orientation-preserving isometries of an R-twisted spectral triple is defined (see [46] for the definition of an R-twisted spectral triple) and its existence is proven. Given an R-twisted spectral triple $(A^\infty, \mathcal{H}, D)$ of compact type, we consider a category $\mathbf{Q'}$ of pairs $(\mathcal{Q}, U)$ where $\mathcal{Q}'$ is a compact quantum group which has a unitary (co)-representation U on $\mathcal{H}$ commuting with D, and such that for every state ϕ on $\mathcal{Q}$, $(\mathrm{id} \otimes \phi) ad_U$ maps A^∞ inside A''_∞. Moreover, let $\mathbf{Q'_R}$ be a subcategory of $\mathbf{Q'}$ consisting of those $(\mathcal{Q}, U)$ for which ad_U preserves the R-twisted volume form. It is proven that $\mathbf{Q'_R}$ has a universal object to be denoted by $\widetilde{QISO_R^+}(D)$. The Woronowicz C^* subalgebra of $\widetilde{QISO_R^+}(D)$ generated by elements of the form $\langle ad_U(a)(\eta \otimes 1), \eta' \otimes 1\rangle_{\widetilde{QISO_R^+}(D)}$ where η, η' are in $\mathcal{H}, a$ is in A^∞ and $\langle ., . \rangle_{\widetilde{QISO_R^+}(D)}$ denotes the $\widetilde{QISO_R^+}(D)$ valued inner product of $\mathcal{H} \otimes \widetilde{QISO_R^+}(D)$, is defined to be the quantum group of orientation and volume preserving isometries of the spectral triple $(A^\infty, \mathcal{H}, D)$ and is denoted by $QISO_R^+(D)$. The next subsection explores the conditions under which the coaction of this compact quantum group keeps the C^* algebra invariant and is a C^* coaction. After this, we compare this approach with the Laplacian-based approach of [22]. The third section illustrates the case when the spectral triple is endowed with a real structure J. Here, an adaptation of the above-mentioned definition forces a canonical choice of R and a similar technique as above helps us to prove the existence of the universal object. In the last section, we have given some sufficient conditions under which the universal object in the bigger category $\mathbf{Q}'$ exists which is denoted by $\widetilde{QISO^+}(D)$ and the corresponding Woronowicz C^* subalgebra as above is denoted by $QISO^+(D)$.

In Chap. 4, we compute $QISO^{\mathcal{L}}$ for the classical spheres and show that these coincide with the commutative C^* algebra of continuous functions on their classical isometry groups. The rest of the chapter is devoted to the computation of the quantum group of orientation preserving isometries for two different families of spectral triples on the Podles' spheres, one constructed by Dabrowski et al. in [47] and the other by Chakraborty and Pal in [48]. We start by giving different descriptions of the Podles' spheres and the formula for the Haar functional on it. Then we show that the spectral triple on the Podles' sphere constructed in [47] is $SU_\mu(2)$ equivariant and R-twisted (for a suitable R). In the fourth section, the computation for identifying $SO_\mu(3)$ as $QISO_R^+$ for this spectral triple is given. In the fifth section, the spectral triple defined in [48] is introduced and then the corresponding $QISO^+$ is computed. In particular, it follows that $QISO^+$ in general may not be a matrix quantum group and that it may not have a C^* coaction.

In Chap. 5, we make contact with the works of Banica, Bichon et al. by showing that their definition of quantum symmetry groups for finite metric spaces and graphs can be viewed as quantum isometry groups in our sense. Next, we show that $QISO^+$ of spectral triples associated with some approximately finite dimensional C^* algebras arise as the inductive limit of $QISO^+$ of the constituent finite dimensional algebras. The results of this chapter are taken from [24] and [25].

In Chap. 6, the conjecture about nonexistence of a smooth faithful, coaction by a genuine CQG on a compact connected Riemannian manifold is discussed. We give a very brief outline of the long and technically involved proof of some important cases of the conjecture as well as a number of examples supporting the conjecture.

Chapter 7 is about the $QISO^{\mathcal{L}}$ and $QISO_R^+$ of a cocycle twisted noncommutative manifold. We first discuss the cocycle deformation of compact quantum groups, von Neumann algebras, and spectral triples, followed by the proof of the fact that $QISO_R^+$ and $QISO^{\mathcal{L}}$ of a cocycle twist of a (noncommutative) manifold is a cocycle twist of the $QISO_R^+$ and $QISO^{\mathcal{L}}$ (respectively) of the original (undeformed) manifold.

In Chap. 8 of the book, we discuss Connes' spectral triples on $C_r^*(\Gamma)$ coming from length functions, where Γ is a finitely generated discrete group. We prove the existence of quantum isometry groups for such spectral triples using results of Sect. 3.4 of Chap. 3 and then present detailed computation for a number of interesting examples.

Chapter 9 is devoted to a specific example which is very interesting and important from a physical point of view, namely the quantum isometry group of the finite geometry of the Connes–Chamseddine picture of the Standard Model. We begin with some generalities on real C^* algebras, followed by a brief discussion in the finite noncommutative space of the Connes–Chamseddine model. Then we compute the quantum isometry group of the corresponding spectral triple and also discuss some physical significance of our results.

In the last chapter, we briefly discuss quantum isometry groups of few more interesting examples, including the free and half-liberated spheres, examples due to Raum and Weber as well as some Drinfeld-Jimbo q-deformed examples. We also give the outlines of other approaches to quantum isometry groups, such as the framework of orthogonal filtrations due to Banica, Skalski and de Chanvalon, affine quantum isometry groups in the sense of Banica and quantum isometry groups of compact metric spaces due to Banica, Goswami, Sabbe and Quaegebeur. We mention several open questions in this context.

References

1. Connes, A.: Noncommutative Geometry. Academic Press, London-New York (1994)
2. Woronowicz, S.L.: Compact quantum groups. In: Connes, A. et al. (eds.) Syme'tries quantiques (Quantum symmetries) (Les Houches, 1995), pp. 845–884. Elsevier, Amsterdam (1998)
3. Drinfeld, V.G.: Quasi-Hopf algebras. Leningrad Math. J. **1**, 1419–1457 (1990)
4. Drinfeld, V.G.: Quantum groups. In: Proceedings of the International Congress of Mathematicians, Berkeley, 1986.
5. Jimbo, M.: Quantum R-matrix for the generalized Toda system. Commun. Math. Phys. **10**, 63–69 (1985)
6. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang-Baxter equation, Lett. Math. Phys. **10**, no. 1, 63–69 (1985)
7. Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**(4), 613–665 (1987)
8. Kustermans, J., Vaes, S.: Locally compact quantum groups. Ann. Sci. Ecole Norm. Sup. (4) **33**(6), 837–934 (2000)
9. Kustermans, J., Vaes, S.: Locally compact quantum groups in the von Neumann algebraic setting, Math. Scand. **92**(1), 68–92 (2003)
10. Masuda, T., Nakagami, Y., Woronowicz, S.L.: A C^*-algebraic framework for quantum groups. Internat. J. Math. **14**(9), 903–1001 (2003)
11. Manin, Y: Quantum Groups and Non-Commutative Geometry. CRM, Montreal (1988)
12. Manin, Y: Some remarks on Koszul algebras and quantum groups. Ann. Inst. Fourier Grenoble **37**(4), 191–205 (1987)
13. Wang, S.: Quantum symmetry groups of finite spaces. Comm. Math. Phys. **195**, 195–211 (1998)
14. Banica, T.: Quantum automorphism groups of small metric spaces. Pacific J. Math. **219**(1), 27–51 (2005)
15. Banica, T.: Quantum automorphism groups of homogeneous graphs. J. Funct. Anal. **224**(2), 243–280 (2005)
16. Banica, T., Bichon, J.: Quantum groups acting on 4 points. J. Reine Angew. Math. **626**, 75–114 (2009)
17. Banica, T., Bichon, J., Collins, B.: Quantum permutation groups: a survey. Noncommutative harmonic analysis with applications to probability, 13–34, Banach Center Publ., **78**, Polish Acad. Sci. Inst. Math., Warsaw (2007)
18. Banica, T., Moroianu, S.: On the structure of quantum permutation groups. Proc. Amer. Math. Soc. **135**, no. 1, 21–29 (electronic) (2007)

19. Bichon, J.: Quantum automorphism groups of finite graphs. Proc. Amer. Math. Soc. **131**(3), 665–673 (2003)
20. Bichon, J.: Algebraic quantum permutation groups, Asian-Eur. J. Math. **1**(1), 1–13 (2008)
21. Wang, S.: Structure and isomorphism classification of compact quantum groups $A_u(Q)$ and $B_u(Q)$. J. Operator Theory. **48**, 573–583 (2002)
22. Goswami, D.: Quantum group of isometries in classical and non commutative geometry, Comm. Math. Phys. **285**, no. 1, 141–160 (2009)
23. Bhowmick, J., Goswami, D.: Quantum group of orientation preserving Riemannian Isometries, J. Funct. Anal. **257**(8), 2530–2572 (2009)
24. Bhowmick, J.: Quantum isometry group of the n tori. Proc. Am. Math. Soc. 137(9), 3155–3161 (2009)
25. Bhowmick, J., Goswami, D., Skalski, A.: Quantum isometry groups of 0-dimensional manifolds. Trans. Amer. Math. Soc. **363**, 901–921 (2011)
26. Bhowmick, J., Goswami, D.: Quantum isometry groups of the Podles' Spheres. J. Funct. Anal. **258**, 2937–2960 (2010)
27. Banica, T., Skalski, A.: Two-parameter families of quantum symmetry groups. J. Funct. Anal. **260**(11), 3252–3282 (2011)
28. Banica, T., Skalski, A.: Quantum isometry groups of duals of free powers of cyclic groups. Int. Math. Res. Not. **9**, 2094–2122 (2012)
29. Banica, T., Skalski, A.: Quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations, Proc. Lon. Math. Soc. **3**(106, 5), 980–1004 (2013)
30. Liszka-Dalecki, J., SoÅtan, P.M.: Quantum isometry groups of symmetric groups. Int. J. Math. **23**(7) (2012)
31. Huang, H.: Faithful compact quantum group actions on connected compact metrizable spaces. J. Geom. Phys. **70**, 232–236 (2013)
32. Etingof, P., Walton, C.: Semisimple Hopf actions on commutative domains. Adv. Math. **251**, 47–61 (2014)
33. Banica, T., Bhowmick, J., De Commer, K.: Quantum isometries and group dual subgroups, Ann. Math. Blaise Pascal **19**(1), 1–27 (2012)
34. Goswami, D., Joardar S.: Rigidity of action of compact quantum groups on compact connected manifolds, arxiv:1309.1294v4
35. Meyer, R., Nest, R.: The results of this chapter are taken of categories. Topology **45**, 209–259 (2006)
36. Chamseddine, A.H., Connes, A.: Noncommutative geometry as a framework for unification of all fundamental interactions including gravity, Part I. Fortschr. Phys. **58**(6), 553–600 (2010)
37. Chamseddine, A.H., Connes, A., van Suijlekom, W.D.: Inner fluctuations in noncommutative geometry without the first order condition, J. Geom. Phys. **73**, 222–234 (2013)
38. Chamseddine, A.H., Connes, A., van Suijlekom, W.D.: Beyond the spectral standard model: emergence of pati-salam unification. J. High. Energ. Phys. 132 (2013)
39. Chamseddine, A.H., Connes, A.: Why the standard model. J. Geom. Phys. **58**(1), 38–47 (2008)
40. Chamseddine, A.H., Connes, A., Marcolli, M.: Gravity and the standard model with neutrino mixing. Adv. Theor. Math. Phys, **11**(6), 991–1089 (2007)
41. Wigner, E.: On Unitary Representations of the Inhomogeneous Lorentz group. Ann. Math. Second Series **40**(1), 149–204 (1939)
42. Bhowmick, J., D'Andrea, F., Dabrowski, L.: Quantum Isometries of the finite noncommutative geometry of the standard model. Commun. Math. Phys. **307**, 101–131 (2011)
43. Bhowmick, J., D'Andrea, F., Das, B. and Dabrowski, L.: Quantum gauge symmetries in noncommutative geometry. J. Noncommut. Geoemtry **8**(2), 433–471 (2014)
44. Schroers, B.J.: Quantum Gravity and non-commutative space-times in three dimensions: a unified approach. Acta Physica Polonica B Proc. Sup No 3 **4** (2011)
45. Wang, S.: Ergodic actions of universal quantum groups on operator algebras. Comm. Math. Phys. **203**, no. 2, 481–498 (1999)

46. Goswami, D.: Twisted entire cyclic cohomology, JLO cocycles and equivariant spectral triples. Rev. Math. Phys. **16**(5), 583–602 (2004)
47. Dabrowski, L., D'Andrea, F., Landi, G., Wagner, E.: Dirac operators on all Podles' quantum spheres. J. Noncomm. Geom. **1**, 213–239 (2007)
48. Chakraborty, P.S., Pal, A.: Spectral triples and associated Connes-de Rham complex for the quantum SU(2) and the quantum sphere, Comm. Math. Phys. **240**(3), 447–456 (2003)

Chapter 1
Preliminaries

Abstract In this chapter, we discuss the basic concepts and results needed in the later chapters of the book. Beginning with a glimpse of operator algebras and Hilbert modules, free product and tensor products of C^* algebras and some examples, we proceed to the generalities on quantum groups including the basics of Hopf algebras and compact quantum groups as well as some concrete examples such as $U_\mu(2), SU_\mu(2), \mathcal{U}_\mu(su(2))$ and $SO_\mu(3)$. We also introduce the notion of a C^* coaction of a compact quantum group on a C^* algebra and give an account of Shuzhou Wang's work in [35] and [42]. We discuss the example of the action of $SO_\mu(3)$ on Podles' spheres in details.

1.1 Operator Algebras and Hilbert Modules

We presume the reader's familiarity with the theory of operator algebras and Hilbert modules. However, for the sake of completeness, we give a sketchy review of some basic definitions and facts and refer to [1–5] for the details. Throughout this book, all algebras will be over $\mathbb{C}$ unless otherwise mentioned.

1.1.1 C Algebras*

A C^* algebra $\mathcal{A}$ is a Banach $*$-algebra satisfying the C^* property: $\|x^*x\| = \|x\|^2$ for all x in $\mathcal{A}$. The algebra $\mathcal{A}$ is said to be unital or non-unital depending on whether it has an identity or not. Every commutative C^* algebra $\mathcal{A}$ is isometrically isomorphic to the C^* algebra $C_0(X)$ consisting of complex valued functions on a locally compact Hausdorff space X vanishing at infinity (Gelfand's Theorem). An arbitrary (possibly noncommutative) C^* algebra is isometrically isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, the set of all bounded operators on a Hilbert space $\mathcal{H}$.

For x in $\mathcal{A}$, the **spectrum** of x, denoted by $\sigma(x)$, is defined as the complement of the set $\{z \in \mathbb{C} : (z1 - x)^{-1} \in \mathcal{A}\}$. An element x in $\mathcal{A}$ is called **self-adjoint** if $x = x^*$, **normal** if $x^*x = xx^*$, **unitary** if $x^* = x^{-1}$, **projection** if $x = x^* = x^2$ and **positive** if $x = y^*y$ for some y in $\mathcal{A}$. When x is normal, there is a continuous

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_1

functional calculus sending f in $C(\sigma(x))$ to $f(x)$ in $\mathcal{A}$, where $f \mapsto f(x)$ is a $*$ isometric isomorphism from $C(\sigma(x))$ onto $C^*(x)$.

A linear map between two C^* algebras is said to be **positive** if it maps positive elements to positive elements. A positive linear functional ϕ such that $\phi(1) = 1$ is called a **state** on $\mathcal{A}$. A state ϕ is called a **trace** if $\phi(ab) = \phi(ba)$ for all a, b in $\mathcal{A}$ and **faithful** if $\phi(x^*x) = 0$ implies $x = 0$. Given a state ϕ on a C^* algebra $\mathcal{A}$, there exists a triple (called the GNS triple) $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ consisting of a Hilbert space $\mathcal{H}_\phi$, a $*$ representation π_ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\phi)$ and a vector ξ_ϕ in $\mathcal{H}_\phi$ which is cyclic in the sense that $\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is total in $\mathcal{H}_\phi$ satisfying

$$\phi(x) = \left\langle \xi_\phi,\ \pi_\phi(x)\xi_\phi \right\rangle.$$

For a two-sided norm closed ideal $\mathcal{I}$ of a C^* algebra $\mathcal{A}$, the canonical quotient norm on the Banach space $\mathcal{A}/\mathcal{I}$ is in fact the unique C^* norm making $\mathcal{A}/\mathcal{I}$ into a C^* algebra. Here we prove two results which we are going to need later on.

Lemma 1.1.1 *[6] Let $\mathcal{C}$ be a C^* algebra and $\mathcal{F}$ be a nonempty collection of C^*-ideals (closed two-sided ideals) of $\mathcal{C}$. Then for any x in $\mathcal{C}$, we have*

$$\sup_{I \in \mathcal{F}} \|x + I\| = \|x + I_0\|,$$

where I_0 denotes the intersection of all I in $\mathcal{F}$ and $\|x + I\| = \inf\{\|x - y\| : y \in I\}$ denotes the norm in $\mathcal{C}/I$.

Proof It is clear that $\sup_{I \in \mathcal{F}} \|x + I\|$ defines a norm on $\mathcal{C}/I_0$, which is in fact a C^* norm since each of the quotient norms $\|. + I\|$ is so. Thus the lemma follows from the uniqueness of C^* norm on the C^* algebra $\mathcal{C}/I_0$. $\square$

Lemma 1.1.2 *Let $\mathcal{C}$ be a unital C^* algebra and $\mathcal{F}$ be a nonempty collection of C^*-ideals (closed two-sided ideals) of $\mathcal{C}$. Let $\mathcal{I}_0$ denote the intersection of all $\mathcal{I}$ in $\mathcal{F}$, and let $\rho_\mathcal{I}$ denote the map $\mathcal{C}/\mathcal{I}_0 \ni x + \mathcal{I}_0 \mapsto x + I \in \mathcal{C}/\mathcal{I}$ for $\mathcal{I}$ in $\mathcal{F}$. Denote by Ω the set $\{\omega \circ \rho_\mathcal{I}, \mathcal{I} \in \mathcal{F},\ \omega \text{ state on } \mathcal{C}/\mathcal{I}\}$, and let K be the weak-$*$ closure of the convex hull of $\Omega \bigcup (-\Omega)$. Then K coincides with the set of bounded linear functionals ω on $\mathcal{C}/\mathcal{I}_0$ satisfying $\|\omega\| = 1$ and $\omega(x^* + \mathcal{I}_0) = \overline{\omega(x + \mathcal{I}_0)}$.*

Proof We will use Lemma 1.1.1. Clearly, K is a weak-$*$ compact, convex subset of the unit ball $(\mathcal{C}/\mathcal{I}_0)^*_1$ of the dual of $\mathcal{C}/\mathcal{I}_0$, satisfying $-K = K$. If K is strictly smaller than the self-adjoint part of unit ball of the dual of $\mathcal{C}/\mathcal{I}_0$, we can find a state ω on $\mathcal{C}/\mathcal{I}_0$ which is not in K. Considering the real Banach space $X = (\mathcal{C}/\mathcal{I}_0)^*_{\text{s.a.}}$ and using standard separation theorems for real Banach spaces (for example, Theorem 3.4 of [7], page 58), we can find a self-adjoint element x of $\mathcal{C}$ such that $\|x + \mathcal{I}_0\| = 1$, and

$$\sup_{\omega' \in K} \omega'(x + \mathcal{I}_0) < \omega(x + \mathcal{I}_0).$$

Let γ belonging to $\mathbb{R}$ be such that $\sup_{\omega' \in K} \omega'(x + \mathcal{I}_0) < \gamma < \omega(x + \mathcal{I}_0)$. Fix $0 < \epsilon < \omega(x + \mathcal{I}_0) - \gamma$, and let $\mathcal{I}$ be an element of $\mathcal{F}$ be such that $\|x + \mathcal{I}_0\| - \frac{\epsilon}{2} \leq$

$\|x+\mathcal{I}\| \leq \|x+\mathcal{I}_0\|$. Let ϕ be a state on $\mathcal{C}/\mathcal{I}$ such that $\|x+\mathcal{I}\| = |\phi(x+\mathcal{I})|$. Since x is self-adjoint, either $\phi(x+\mathcal{I})$ or $-\phi(x+\mathcal{I})$ equals $\|x+\mathcal{I}\|$, and $\phi' := \pm\phi \circ \rho_{\mathcal{I}}$, where the sign is chosen so that $\phi'(x+\mathcal{I}_0) = \|x+\mathcal{I}\|$. Thus, ϕ' is in K, so $\|x+\mathcal{I}_0\| = \phi'(x+\mathcal{I}) \leq \gamma < \omega(x+\mathcal{I}_0) - \epsilon$. But this implies $\|x+\mathcal{I}_0\| \leq \|x+\mathcal{I}\| + \frac{\epsilon}{2} < \omega(x+\mathcal{I}_0) - \frac{\epsilon}{2} \leq \|x+\mathcal{I}_0\| - \epsilon$ (since ω is a state), which is a contradiction completing the proof. □

For a C^* algebra $\mathcal{A}$ (possibly non-unital), its multiplier algebra, denoted by $\mathcal{M}(\mathcal{A})$, is defined as the maximal C^* algebra which contains $\mathcal{A}$ as an essential two-sided ideal, that is, $\mathcal{A}$ is an ideal in $\mathcal{M}(\mathcal{A})$ and for y in $\mathcal{M}(\mathcal{A})$, $ya = 0$ for all a in $\mathcal{A}$ implies $y = 0$. The norm of $\mathcal{M}(\mathcal{A})$ is given by $\|x\| = \sup_{a\in\mathcal{A}, \|a\|\leq 1}\{\|xa\|, \|ax\|\}$. There is a locally convex topology called the strict topology on $\mathcal{M}(\mathcal{A})$, which is given by the family of seminorms $\{\|.\|_a, a \in \mathcal{A}\}$, where $\|x\|_a = \mathrm{Max}(\|xa\|, \|ax\|)$, for x in $\mathcal{M}(\mathcal{A})$. $\mathcal{M}(\mathcal{A})$ is the completion of $\mathcal{A}$ in the strict topology.

We now come to the **inductive limit of C^* algebras**. Let I be a directed set and $\{\mathcal{A}_i\}_{i\in I}$ be a family of C^* algebras equipped with a family of C^* homomorphisms $\Phi_{ij} : \mathcal{A}_j \to \mathcal{A}_i$ (when $j < i$) such that $\Phi_{ij} = \Phi_{ik}\Phi_{kj}$ when $j < k < i$. Then there exists a unique C^* algebra denoted by $\lim_i \mathcal{A}_i$ and C^* homomorphisms $\phi_i : \mathcal{A}_i \to \lim_i \mathcal{A}_i$ with the universal property that given any other C^* algebra $\mathcal{A}'$ and C^* homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}'$ satisfying $\psi_j = \psi_i\Phi_{ij}$ for $j < i$, then there exists unique C^* homomorphism $\chi : \lim_i \mathcal{A}_i \to \mathcal{A}'$ satisfying $\chi\phi_i = \psi_i$. $\lim_i \mathcal{A}_i$ is called the inductive limit C^* algebra corresponding to the inductive system $(\mathcal{A}_i, \Phi_{ij})$. The inductive limit of a sequence of finite dimensional C^* algebras is called an approximately finite dimensional C^* algebra or AF algebra.

A large class of C^* algebras is obtained by the following construction. Let $\mathcal{A}_0$ be an associative $*$-algebra without any a priori norm such that the set $\mathcal{F} = \{\pi : \mathcal{A}_0 \to \mathcal{B}(\mathcal{H}_\pi)\ * -\text{homomorphism}, \mathcal{H}_\pi \text{ a Hilbert space}\}$ is non empty and $\|.\|_u$ defined by $\|a\|_u = \sup\{\|\pi(a)\| : \pi \in \mathcal{F}\}$ is finite for all a. Then the completion of $\mathcal{A}_0$ in $\|.\|_u$ is a C^* algebra known as the universal C^* algebra corresponding to $\mathcal{A}_0$.

Example 1 **Noncommutative two-torus** Let θ belong to $[0, 1]$. Consider the $*$ algebra $\mathcal{A}_0$ generated by two unitary symbols U and V satisfying the relation $UV = e^{2\pi i\theta}VU$. It has a representation π on the Hilbert space $L^2(S^1)$ defined by $\pi(U)(f)(z) = f(e^{2\pi i\theta}z)$, $\pi(V)(f)(z) = zf(z)$ where f is in $L^2(S^1)$, z is in S^1. Then $\|a\|_u$ is finite for all a in $\mathcal{A}_0$. The resulting C^* algebra is called the noncommutative two-torus and denoted by $\mathcal{A}_\theta$.

Example 2 **Group C^* algebra** Let G be a locally compact group with left Haar measure μ. One can make $L^1(G)$ into a Banach $*$-algebra by defining

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)d\mu(s),$$

$$f^*(t) = \Delta(t)^{-1}\overline{f(t^{-1})}.$$

Here $f,\ g$ are in $L^1(G),\ \Delta$ is the modular homomorphism of G.

$L^1(G)$ has a distinguished representation π_{reg} on $L^2(G)$ defined by $\pi_{reg}(f) = \int f(t)\pi(t)d\mu(t)$ where $\pi(t)$ is a unitary operator on $L^2(G)$ defined by $(\pi(t)f)(g) = f(t^{-1}g)\ (f \in L^2(G),\ t, g \in G)$. The reduced group C^* algebra C_r^*G of G is defined to be the completion of $\pi_{reg}(L^1(G))$ in the norm of $\mathcal{B}(L^2(G))$.

Remark 1.1.3 For G abelian, we have $C_r^*(G) \cong C_0(\widehat{G})$ where $\widehat{G}$ is the group of characters on G.

One can also consider the universal C^* algebra described before corresponding to the Banach $*$-algebra $L^1(G)$. This is called the free or full group C^* algebra and denoted by $C^*(G)$.

Remark 1.1.4 For the so-called amenable groups (which include compact and abelian groups) we have $C^*(G) \cong C_r^*(G)$.

1.1.2 Von Neumann Algebras

We recall that for a Hilbert space $\mathcal{H}$, the **strong operator topology (SOT)**, the **weak operator topology (WOT)** and the **ultraweak topology** are the locally convex topologies on $\mathcal{B}(\mathcal{H})$ given by families of seminorms $\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3$ respectively where $\mathcal{F}_1 = \{p_\xi : \xi \in \mathcal{H}\},\ \mathcal{F}_2 = \{p_{\xi,\eta} : \xi, \eta \in \mathcal{H}\},\ \mathcal{F}_3 = \{p_\rho : \rho$ is a trace class operator on $\mathcal{H}\}$ and $p_\xi(x) = \|x\xi\|\ ,\ p_{\xi,\eta}(x) = |\langle x\xi, \eta\rangle|\ ,\ p_\rho(x) = |\mathrm{Tr}(x\rho)|$ (where Tr denotes the usual trace on $\mathcal{B}(\mathcal{H})$).

Now we state a well-known fact.

Lemma 1.1.5 *If T_n is a sequence of bounded operators converging to zero in SOT, then for any trace class operator $W,\ Tr(T_nW) \to 0$ as $n \to \infty$.*

For any subset $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$, we denote by $\mathcal{B}'$ the commutant of $\mathcal{B}$, that is, $\mathcal{B}' = \{x \in \mathcal{B}(\mathcal{H}) : xb = bx$ for all $b \in \mathcal{B}\}$. A unital C^* subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a **von Neumann algebra** if $\mathcal{A} = \mathcal{A}''$ which is equivalent to being closed in any of the three topologies mentioned above.

A state ϕ on a von Neumann algebra $\mathcal{A}$ is called normal if $\phi(x_\alpha)$ increases to $\phi(x)$ whenever x_α increases to x. A state ϕ on $\mathcal{A}$ is normal if and only if there is a trace class operator ρ on $\mathcal{H}$ such that $\phi(x) = \mathrm{Tr}(\rho x)$ for all x in $\mathcal{A}$. More generally, we call a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ (where $\mathcal{B}$ is a von Neumann algebra) normal if whenever x_α increases to x for a net x_α of positive elements from $\mathcal{A}$, the net $\Phi(x_\alpha)$ also increases to $\Phi(x)$ in $\mathcal{B}$. It is known that a positive linear map is normal if and only if it is continuous with respect to the ultraweak topology. In view of this fact, we shall say that a bounded linear map between two von Neumann algebras is normal if it is continuous with respect to the respective ultraweak topologies.

1.1.3 Free Product and Tensor Product

For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we will denote the Hilbert space tensor product by $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $(\mathcal{A}_i)_{i\in I}$ is a family of unital C^* algebras, then their unital C^* algebra free product $*_{i\in I} A_i$ is the unique C^* algebra $\mathcal{A}$ together with unital $*$-homomorphisms $\psi_i : \mathcal{A}_i \to \mathcal{A}$ such that given any unital C^* algebra $\mathcal{B}$ and unital $*$-homomorphisms $\phi_i : \mathcal{A}_i \to \mathcal{B}$ there exists a unique unital $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\phi_i = \Phi \circ \psi_i$.

Remark 1.1.6 It is a direct consequence of the above definition that given a family of C^* homomorphisms ϕ_i from $\mathcal{A}_i$ to $\mathcal{B}$, there exists a C^* homomorphism $*_i\phi_i$ such that $(*_i\phi_i) \circ \psi_i = \phi_i$ for all i.

Remark 1.1.7 We recall that for discrete groups $\{G_i\}_{i\in I}$, $C^*(*_{i\in I} G_i) \cong *_{i\in I} C^*(G_i)$.

For two algebras $\mathcal{A}$ and $\mathcal{B}$, we will denote the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$ by the symbol $\mathcal{A} \otimes_{alg} \mathcal{B}$. When $\mathcal{A}$ and $\mathcal{B}$ are C^* algebras, there is usually more than one norm on $\mathcal{A} \otimes_{alg} \mathcal{B}$ so that the completion with respect to that norm is a C^* algebra. Throughout this book, we will mainly work with the so called injective tensor product, that is, the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with respect to the norm given on $\mathcal{A} \otimes_{alg} \mathcal{B}$ by $\| \sum_{i=1}^n a_i \otimes b_i \| = \sup \| \sum_{i=1}^n \pi_1(a_i) \otimes \pi_2(b_i) \|_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$ where a_i is in $\mathcal{A}$, b_i is in $\mathcal{B}$ and the supremum runs over all possible choices of $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ where $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, $\pi_1 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1)$ and $\pi_2 : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_2)$ are $*$-homomorphisms and $\|\cdot\|_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$ denotes the operator norm of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For a treatment of maximal tensor products of C^* algebras, to be denoted as $\otimes^{\max}$, we refer to the book [1]. We will denote the minimal tensor product of two C^* algebras $\mathcal{A}$ and $\mathcal{B}$ by the symbol $\mathcal{A} \otimes \mathcal{B}$. When $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1)$, $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$ are von Neumann algebras, then $\mathcal{A} \overline{\otimes} \mathcal{B}$ will stand for the von Neumann algebra tensor product, that is, the WOT closure of $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We refer to [1] for more details.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital $*$-algebras. Then a linear map T from $\mathcal{A}$ to $\mathcal{B}$ is called n-positive if $T \otimes \mathrm{Id_n} : \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \to \mathcal{B} \otimes \mathcal{M}_n(\mathbb{C})$ is positive for all $k \leq n$ but not necessarily for $k = n + 1$. T is said to be completely positive (CP for short) if it is n-positive for all n. It is a well-known result that for a CP map $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, one has the following operator inequality for all a in $\mathcal{A}$:

$$T(a)^* T(a) \leq \|T(1)\| \, T(a^* a). \tag{1.1.1}$$

Clearly, the composition of two CP maps is again CP. Moreover, given two CP maps T_i from $\mathcal{A}_i$ to $\mathcal{B}_i$, $i = 1, 2$, where $\mathcal{A}_i, \mathcal{B}_i$ are C^* algebras, $T_1 \otimes_{\mathrm{alg}} T_2$ extends to a unique CP map denoted by $T_1 \otimes T_2$ from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{B}_1 \otimes \mathcal{B}_2$. When $\mathcal{A}_i, \mathcal{B}_i$ are von Neumann algebras, there is a similar CP extension of $T_1 \otimes_{\mathrm{alg}} T_2$ from $\mathcal{A}_1 \overline{\otimes} \mathcal{A}_2$ to $\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$ which we will continue to denote (by a slight abuse of notation) by the symbol $T_1 \otimes T_2$. The following is an useful result about CP maps.

Proposition 1.1.8 *If A and B are C^* algebras with A commutative, ϕ is a positive map from A to B, then ϕ is CP. The same holds if ϕ is from B to A.*

We end this subsection with a couple of results.

Lemma 1.1.9 *Let $\mathcal{A}, \mathcal{B}$ be C^* algebras and $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$. Let $\mathcal{F}$ be a nonempty collection of C^*-ideals of $\mathcal{B}$ and I_0 be the intersection of all ideals in $\mathcal{F}$. If π_I and π_{I_0} are the canonical quotient maps from $\mathcal{B}$ to $\mathcal{B}/I$ and $\mathcal{B}/I_0$ respectively, then* $\cap_{I \in \mathcal{F}} \mathrm{Ker}(\mathrm{id} \otimes \pi_I) = \mathrm{Ker}(\mathrm{id} \otimes \pi_{I_0})$.

Proof We first observe that $\mathrm{Ker}(\mathrm{id} \otimes \pi_I) = \{X \in \mathcal{A} \otimes \mathcal{B} : \forall \omega \in \mathcal{A}^* : (\omega \otimes \mathrm{id})(X) \in I\}$. Thus,
$\mathrm{Ker}(\mathrm{id} \otimes \pi_{I_0}) = \{X \in \mathcal{A} \otimes \mathcal{B} : (\omega \otimes \mathrm{id})(X) \in I_0\}$
$= \{X \in \mathcal{A} \otimes \mathcal{B} : (\omega \otimes \mathrm{id})(X) \in I \ \forall \ I \in \mathcal{F}\}$
$= \cap_{I \in \mathcal{F}} \{X \in \mathcal{A} \otimes \mathcal{B} : (\omega \otimes \mathrm{id})(X) \in I\}$
$= \cap_{I \in \mathcal{F}} \mathrm{Ker}(\mathrm{id} \otimes \pi_I)$. □

Lemma 1.1.10 *Let $\mathcal{A}$ be a C^* algebra and ω, ω_j ($j = 1, 2, \ldots$) be states on $\mathcal{A}$ such that $\omega_j \to \omega$ in the weak-$*$ topology of $\mathcal{A}^*$. Then for any separable Hilbert space $\mathcal{H}$ and for all Y in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, we have $(\mathrm{id} \otimes \omega_j)(Y) \to (\mathrm{id} \otimes \omega)(Y)$ in the strong operator topology.*

Proof Clearly, $(\mathrm{id} \otimes \omega_j)(Y) \to (\mathrm{id} \otimes \omega)(Y)$ (in the strong operator topology) for all Y in $\mathrm{Fin}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{A}$, where $\mathrm{Fin}(\mathcal{H})$ denotes the set of finite rank operators on $\mathcal{H}$. Using the strict density of $\mathrm{Fin}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{A}$ in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, we choose, for a given Y in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, ξ in $\mathcal{H}$ with $\|\xi\| = 1$, and $\delta > 0$, an element Y_0 in $\mathrm{Fin}(\mathcal{H}) \otimes_{\mathrm{alg}} \mathcal{A}$ such that $\|(Y - Y_0)(|\xi >< \xi| \otimes 1)\| < \delta$. Thus,

$$\begin{aligned}
&\|(\mathrm{id} \otimes \omega_j)(Y)\xi - (\mathrm{id} \otimes \omega)(Y)\xi\| \\
&\quad = \|(\mathrm{id} \otimes \omega_j)(Y(|\xi >< \xi| \otimes 1))\xi - (\mathrm{id} \otimes \omega)(Y(|\xi >< \xi| \otimes 1))\xi\| \\
&\quad \leq \|(\mathrm{id} \otimes \omega_j)(Y_0(|\xi >< \xi| \otimes 1))\xi - (\mathrm{id} \otimes \omega)(Y_0(|\xi >< \xi| \otimes 1))\xi\| \\
&\quad + 2\|(Y - Y_0)(|\xi >< \xi| \otimes 1)\| \\
&\quad \leq \|(\mathrm{id} \otimes \omega_j)(Y_0(|\xi >< \xi| \otimes 1))\xi - (\mathrm{id} \otimes \omega)(Y_0(|\xi >< \xi| \otimes 1))\xi\| + 2\delta,
\end{aligned}$$

from which it follows that $(\mathrm{id} \otimes \omega_j)(Y) \to (\mathrm{id} \otimes \omega)(Y)$ in the strong operator topology. □

1.1.4 Hilbert Modules

Given a $*$-subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ (where $\mathcal{H}$ is a Hilbert space), a semi-Hilbert $\mathcal{A}$ module E is a right $\mathcal{A}$-module equipped with a sesquilinear map $\langle . , . \rangle : E \times E \to \mathcal{A}$ satisfying $\langle x, y \rangle^* = \langle y, x \rangle$, $\langle x, ya \rangle = \langle x, y \rangle a$ and $\langle x, x \rangle \geq 0$ for x, y in E and a in $\mathcal{A}$. A semi-Hilbert module is called a pre-Hilbert module if $\langle x, x \rangle = 0$ if and only if $x = 0$. It is called a Hilbert module if furthermore, $\mathcal{A}$ is a C^* algebra and E is complete in the norm $x \mapsto \|\langle x, x \rangle\|^{\frac{1}{2}}$ where $\|.\|$ is the C^* norm of $\mathcal{A}$.

The simplest examples of Hilbert $\mathcal{A}$ modules are the so-called trivial $\mathcal{A}$ modules of the form $\mathcal{H} \otimes \mathcal{A}$ where $\mathcal{H}$ is a Hilbert space with an $\mathcal{A}$ valued sesquilinear form defined on $\mathcal{H} \otimes_{\text{alg}} \mathcal{A}$ by: $\left\langle \xi \otimes a,\ \xi' \otimes a' \right\rangle = \left\langle \xi, \xi' \right\rangle a^* a'$. The completion of $\mathcal{H} \otimes_{\text{alg}} \mathcal{A}$ with respect to this pre-Hilbert module structure is a Hilbert $\mathcal{A}$ module and is denoted by $\mathcal{H} \otimes \mathcal{A}$.

We recall that for a pre-Hilbert $\mathcal{A}$ module E ($\mathcal{A}$ is a C^* algebra), the Cauchy Schwarz inequality holds in the following form: $0 \leq \langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle \| \langle y, y \rangle \|$.

Let E and F be two Hilbert $\mathcal{A}$ modules. We say that a $\mathbb{C}$ linear map L from E to F is adjointable if there exists a $\mathbb{C}$ linear map L^* from F to E such that $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$ for all x in $E,\ y$ in F. We call L^* the adjoint of L. The set of all adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$. In case $E = F$, we write $\mathcal{L}(E)$ for $\mathcal{L}(E, E)$. For an adjointable map L, both L and L^* are automatically $\mathcal{A}$-linear and norm bounded maps between Banach spaces. We say that an element L in $\mathcal{L}(E, F)$ is an isometry if $\langle Lx, Ly \rangle = \langle x, y \rangle$ for all x, y in E. L is said to be a unitary if L is isometry and its range is the whole of F. One defines a norm on $\mathcal{L}(E, F)$ by $\|L\| = \sup_{x \in R,\ \|x\| \leq 1} \|L(x)\|$. $\mathcal{L}(E)$ is a C^* algebra with this norm.

There is a topology on $\mathcal{L}(E, F)$ given by a family of seminorms $\{\|.\|_x, \|.\|_y : x \in E,\ y \in F\}$ (where $\|t\|_x = \left\| \langle tx, tx \rangle^{\frac{1}{2}} \right\|$ and $\|t\|_y = \left\| \langle t^* y, t^* y \rangle^{\frac{1}{2}} \right\|$) known as the strict topology. For x in $E,\ y$ in F, we denote by $\theta_{x,y}$ the element of $\mathcal{L}(E, F)$ defined by $\theta_{x,y}(z) = y \langle x, z \rangle$ (where z is in E). The norm closure (in $\mathcal{L}(E, F)$) of the $\mathcal{A}$ linear span of $\{\theta_{x,y} : x \in E,\ y \in F\}$ is called the set of compact operators and denoted by $\mathcal{K}(E, F)$ and we denote $\mathcal{K}(E, E)$ by $\mathcal{K}(E)$. These are not necessarily compact in the sense of compact operators between two Banach spaces. One has the following important result:

Proposition 1.1.11 *The multiplier algebra $\mathcal{M}(\mathcal{K}(E))$ of $\mathcal{K}(E)$ is isomorphic with $\mathcal{L}(E)$ for any Hilbert module E.*

Using this, for a possibly non-unital C^* algebra $\mathcal{B}$, we often identify an element V of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{B})$ with the map from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{B}$ which sends a vector ξ of $\mathcal{H}$ to $V(\xi \otimes 1_{\mathcal{B}})$ in $\mathcal{H} \otimes \mathcal{B}$.

Given a Hilbert space $\mathcal{H}$ and a C^* algebra $\mathcal{A}$, and a unitary element U of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$, we shall denote by ad_U the $*$-homomorphism $\text{ad}_U(X) = \widetilde{U}(X \otimes 1)\widetilde{U}^*$ for X belonging to $\mathcal{B}(\mathcal{H})$. For a not necessarily bounded, densely defined (in the weak operator topology) linear functional τ on $\mathcal{B}(\mathcal{H})$, we say that ad_U preserves τ if ad_U maps a suitable (weakly) dense $*$-subalgebra (say $\mathcal{D}$) in the domain of τ into $\mathcal{D} \otimes_{\text{alg}} \mathcal{A}$ and $(\tau \otimes \text{id})(\text{ad}_U(x)) = \tau(x).1_{\mathcal{A}}$ for all x in $\mathcal{D}$. When τ is bounded and normal, this is equivalent to $(\tau \otimes \text{id})(\text{ad}_U(x)) = \tau(x)1_{\mathcal{A}}$ for all x belonging to $\mathcal{B}(\mathcal{H})$.

We say that a (possibly unbounded) operator T on $\mathcal{H}$ commutes with U if $T \otimes I$ (with the natural domain) commutes with $\widetilde{U}$. Sometimes such an operator will be called U-equivariant.

1.2 Quantum Groups

In this section, we will recall the basics of Hopf algebras and then define compact quantum groups (following [8, 9]). After that, we will discuss a few examples of quantum groups and the concept of a coaction of a compact quantum group on a C^* algebra. For more detailed discussion, we refer to [10–14] and the references therein. In this book, we will be concerned about compact quantum groups only. For other types of quantum groups, we refer to [11, 15–20], etc.

1.2.1 Hopf Algebras

We recall that an associative algebra with an unit is a vector space A over $\mathbb{C}$ together with two linear maps $m : A \otimes_{\text{alg}} A \to A$ called the multiplication or the product and $\eta : \mathbb{C} \to A$ called the unit such that $m(m \otimes \text{id}) = m(\text{id} \otimes m)$ and $m(\eta \otimes \text{id}) = \text{id} = m(\text{id} \otimes \eta)$. Dualizing this, we get the following definition.

A **coalgebra** A is a vector space over $\mathbb{C}$ equipped with two linear maps $\Delta : A \to A \otimes A$ called the comultiplication or coproduct and $\epsilon : A \to \mathbb{C}$ such that

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta,$$

$$(\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta.$$

Let $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ be co algebras. A $\mathbb{C}$ linear mapping $\phi : A \to B$ is said to be a **cohomomorphism** if

$$\Delta_B \circ \phi = (\phi \otimes \phi)\Delta_A, \ \epsilon_A = \epsilon_B \circ \phi.$$

Let σ denote the flip map: $A \otimes_{\text{alg}} A \to A_{\text{alg}} \otimes A$ given by $\sigma(a \otimes b) = b \otimes a$. A coalgebra is said to be **cocommutative** if $\sigma \circ \Delta = \Delta$. A linear subspace B of A is a **subcoalgebra** of A if $\Delta(B) \subseteq B \otimes_{\text{alg}} B$. A $\mathbb{C}$ linear subspace $\mathcal{I}$ of A is called a **coideal** if $\Delta(\mathcal{I}) \subseteq A \otimes_{\text{alg}} \mathcal{I} + \mathcal{I} \otimes_{\text{alg}} A$ and $\epsilon(\mathcal{I}) = \{0\}$. If $\mathcal{I}$ is a coideal of A, the quotient vector space $A/\mathcal{I}$ becomes a coalgebra with comultiplication and counit induced from A.

Sweedler notation We introduce the so-called Sweedler notation for comultiplication. If a is an element of a coalgebra $\mathcal{A}$, the element $\Delta(a)$ in $\mathcal{A} \otimes \mathcal{A}$ is a finite sum $\Delta(a) = \sum_i a_{1i} \otimes a_{2i}$ where $a_{1i}, \ a_{2i}$ belongs to $\mathcal{A}$. Moreover, the representation of $\Delta(a)$ is not unique. For notational simplicity we shall suppress the index i and write the above sum symbolically as $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Here the subscripts (1) and (2) refer to the corresponding tensor factors.

Definition 1.2.1 A vector space $\mathcal{A}$ is called a bialgebra if it is an algebra and a coalgebra along with the condition that $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\text{alg}} \mathcal{A}$ and $\epsilon : \mathcal{A} \to \mathbb{C}$ are algebra homomorphisms (equivalently, $m : \mathcal{A} \otimes_{\text{alg}} \mathcal{A} \to \mathcal{A}$ and $\eta : \mathbb{C} \to \mathcal{A}$ are coalgebra cohomomorphisms).

Definition 1.2.2 A bialgebra $\mathcal{A}$ is called a Hopf algebra if there exists a linear map $\kappa : \mathcal{A} \to \mathcal{A}$ called the antipode or the coinverse of $\mathcal{A}$, such that $m \circ (\kappa \otimes \mathrm{id})\Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes \kappa) \circ \Delta$.

Dual Hopf algebra

Let us consider a finite dimensional Hopf algebra $\mathcal{A}$. Then the dual vector space $\mathcal{A}'$ is an algebra with respect to the multiplication $fg(a) = (f \otimes g)\Delta(a)$. Moreover, for f in $\mathcal{A}'$, one defines the functional $\Delta(f) \in (\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A})'$ by $\Delta(f)(a \otimes b) = f(ab),\ a,\ b$ in $\mathcal{A}$. Since $\mathcal{A}$ is finite dimensional, $(\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A})' = \mathcal{A}' \otimes_{\mathrm{alg}} \mathcal{A}'$ and so $\Delta(f)$ belongs to $\mathcal{A}' \otimes \mathcal{A}'$. Then the algebra $\mathcal{A}'$ equipped with the comultiplication Δ, antipode κ defined by $(\kappa f)(a) = f(\kappa(a))$, counit $\epsilon_{\mathcal{A}'}$ defined by $\epsilon_{\mathcal{A}'}(f) = f(1)$ and $1_{\mathcal{A}'}(a) = \epsilon(a)$ gives a Hopf algebra. This is called the dual Hopf algebra of $\mathcal{A}$.

Definition 1.2.3 A Hopf $*$-algebra is a Hopf algebra $(\mathcal{A}, \Delta, \kappa, \epsilon)$ endowed with an involution $*$ which maps a to an element denoted by a^* satisfying the following properties:

1. For all a, b in $\mathcal{A}$, α, β in $\mathbb{C}$, $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*,\ (a^*)^* = a,\ (a.b)^* = b^* a^*$.
2. $\Delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ is a $*$-homomorphism which means that $\Delta(a^*) = \Delta(a)^*$ where the involution on $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ is defined by $(a \otimes b)^* = a^* \otimes b^*$.
3. $\kappa(\kappa(a^*)^*) = a$ for all a in $\mathcal{A}$.

Proposition 1.2.4 *In any Hopf $*$-algebra $(\mathcal{A}, \Delta, \kappa, \epsilon)$, we have $\epsilon(a^*) = \overline{\epsilon(a)}$ for all a in $\mathcal{A}$.*

We recall that the dual algebra $\mathcal{A}'$ of a Hopf $*$-algebra $\mathcal{A}$ is a $*$-algebra with involution defined by

$$f^*(a) = \overline{f(\kappa(a)^*)}, \text{ for } f \text{ in } \mathcal{A}'.$$

Dual Pairing

A left action of a Hopf $*$-algebra $(\mathcal{U}, \Delta_{\mathcal{U}}, \kappa_{\mathcal{U}}, \epsilon_{\mathcal{U}})$ on another Hopf $*$-algebra $(\mathcal{A}, \Delta_{\mathcal{A}}, \kappa_{\mathcal{A}}, \epsilon_{\mathcal{A}})$ is a bilinear form $\triangleright : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ if the following conditions hold:

$$(1)\, f \triangleright (a_1 a_2) = \Delta_{\mathcal{U}}(f) \triangleright (a_1 \otimes a_2),\ (f_1 f_2) \triangleright a = (f_1 \otimes f_2) \triangleright \Delta_{\mathcal{A}}(a);$$

$$(2)\, f \triangleright 1_{\mathcal{A}} = \epsilon_{\mathcal{U}}(f),\ 1_{\mathcal{U}} \triangleright a = \epsilon_{\mathcal{A}}(a);$$

$$(3)\, f^* \triangleright a = \overline{f \triangleright \kappa_{\mathcal{A}}(a)^*} \text{ (equivalently } f \triangleright a^* = \overline{\kappa_{\mathcal{U}}(f)^* \triangleright a})$$

for all f, f_1, f_2 in $\mathcal{U}$ and a, a_1, a_2 in $\mathcal{A}$.

Similarly, a right action of a Hopf $*$-algebra $(\mathcal{U}, \Delta_{\mathcal{U}}, \kappa_{\mathcal{U}}, \epsilon_{\mathcal{U}})$ on another Hopf $*$-algebra $(\mathcal{A}, \Delta_{\mathcal{A}}, \kappa_{\mathcal{A}}, \epsilon_{\mathcal{A}})$ is a bilinear form $\triangleleft : \mathcal{A} \times \mathcal{U} \to \mathbb{C}$ if the following conditions hold: $a_1 a_2 \triangleleft f = (a_1 \triangleleft f_{(1)})(a_2 \triangleleft f_{(2)}),\ a \triangleleft (f_1 f_2) = \Delta_{\mathcal{A}}(a) \triangleleft (f_1 \otimes f_2), 1_{\mathcal{A}} \triangleleft f = \epsilon_{\mathcal{U}}(f),\ a \triangleleft 1_{\mathcal{U}} = \epsilon_{\mathcal{A}}(a), a \triangleleft f^* = \overline{\kappa_{\mathcal{A}}(a)^* \triangleleft f}$ (equivalently $a^* \triangleleft f = \overline{a \triangleleft \kappa_{\mathcal{U}}(f)^*}$) for all f, f_1, f_2 in $\mathcal{U}$ and a, a_1, a_2 in $\mathcal{A}$.

$\mathcal{U} = \mathcal{A}'$ gives a particular case of this duality pairing.

1.2.2 Compact Quantum Groups: Basic Definitions and Examples

The aim of this subsection is to introduce compact quantum groups developed by Woronowicz in [8, 9, 21]. For more recent expositions, we refer to [22, 23].

Definition 1.2.5 A **compact quantum group** (to be abbreviated as CQG from now on) is given by a pair $(\mathcal{S}, \Delta)$, where $\mathcal{S}$ is a unital separable C^* algebra equipped with a unital C^*-homomorphism $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ (where $\otimes$ denotes the injective tensor product) satisfying

(ai) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (coassociativity), and
(aii) each of the linear spans of $\Delta(\mathcal{S})(\mathcal{S} \otimes 1)$ and of $\Delta(\mathcal{S})(1 \otimes \mathcal{S})$ are norm dense in $\mathcal{S} \otimes \mathcal{S}$.

It is well known (see [8, 9]) that there is a canonical dense $*$-subalgebra $\mathcal{S}_0$ of $\mathcal{S}$, consisting of the matrix elements of the finite dimensional unitary corepresentations (to be defined shortly) of $\mathcal{S}$, and maps $\epsilon : \mathcal{S}_0 \to \mathbb{C}$ (counit) and $\kappa : \mathcal{S}_0 \to \mathcal{S}_0$ (antipode) defined on $\mathcal{S}_0$ which make $\mathcal{S}_0$ a Hopf $*$-algebra.

The following theorem is the analogue of Gelfand Naimark duality for commutative CQGs.

Proposition 1.2.6 *Let G be a compact group. Let $C(G)$ be the algebra of continuous functions on G. If we define Δ by $\Delta(f)(g, h) = f(g.h)$ for f in $C(G)$, g, h in G, then this defines a CQG structure on $C(G)$.*

Conversely, let $(\mathcal{S}, \Delta)$ be a CQG such that S is a commutative C^* algebra. Let $H(\mathcal{S})$ denote the Gelfand spectrum of $\mathcal{S}$ and endow it with the product structure given by $\chi\chi' = (\chi \otimes \chi')\Delta$ where $\chi, \ \chi'$ are in $H(\mathcal{S})$. Then $H(\mathcal{S})$ is a compact group.

Remark 1.2.7 In [24], A Van Daele removed Woronowicz's separability assumption (in [8]) for the C^* algebra of the underlying compact quantum group. We remark that although we assume that CQG's are separable, most of the results in this book go through in the non separable case also.

Definition 1.2.8 Let $(\mathcal{S}, \Delta_{\mathcal{S}})$ be a compact quantum group. A vector space M is said to be an algebraic $\mathcal{S}$ comodule (or $\mathcal{S}$ comodule) if there exists a linear map $\alpha : M \to M \otimes_{\mathrm{alg}} \mathcal{S}_0$ such that

1. $\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta_{\mathcal{S}})\alpha$,
2. $(\mathrm{id} \otimes \epsilon)\alpha(m) = m$ for all m in M.

In the notations as above, let us define $\widetilde{\alpha} : M \otimes_{\mathrm{alg}} \mathcal{S} \to M \otimes_{\mathrm{alg}} \mathcal{S}$ by $\widetilde{\alpha} = (\mathrm{id} \otimes m)(\alpha \otimes \mathrm{id})$. Then we claim that $\widetilde{\alpha}$ is invertible with the inverse given by $T(m \otimes q) = (\mathrm{id} \otimes \kappa)\alpha(m)(1 \otimes q)$, where m is in $M, \ q$ is in $\mathcal{S}$. As T is defined to be $\mathcal{S}_0$ linear, it is enough to check that $\widetilde{\alpha}T(m \otimes 1) = m \otimes 1$.

$$\begin{aligned}
&\widetilde{\alpha}T(m \otimes 1)\\
&\quad = \widetilde{\alpha}(m_{(1)} \otimes \kappa(m_{(2)})1)\\
&\quad = m_{(1)(1)} \otimes m_{(1)(2)}\kappa(m_{(2)})\\
&\quad = (\mathrm{id} \otimes m(\mathrm{id} \otimes \kappa)\Delta)\alpha(m)\\
&\quad = (\mathrm{id} \otimes \epsilon().1)\alpha(m)\\
&\quad = m \otimes 1.
\end{aligned}$$

Similarly, $T\widetilde{\alpha} = \mathrm{id}$. Thus,

$$T = \widetilde{\alpha}^{-1}. \tag{1.2.1}$$

Definition 1.2.9 A morphism from a CQG $(\mathcal{S}_1, \Delta_1)$ to another CQG $(\mathcal{S}_2, \Delta_2)$ is a unital C^* homomorphism $\pi : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$(\pi \otimes \pi)\Delta_1 = \Delta_2\pi.$$

It follows that in such a case, π preserves the Hopf $*$-algebra structures, that is, we have

$$\pi((\mathcal{S}_1)_0) \subseteq (\mathcal{S}_2)_0,\ \pi\kappa_1 = \kappa_2\pi,\ \epsilon_2\pi = \epsilon_1,$$

where κ_1, ϵ_1 denote the antipode and counit of $\mathcal{S}_1$ respectively, while κ_2, ϵ_2 denote those of $\mathcal{S}_2$.

Definition 1.2.10 A **Woronowicz C^*-subalgebra** of a CQG $(\mathcal{S}_1, \Delta)$ is a C^* subalgebra $\mathcal{S}_2$ of $\mathcal{S}_1$ such that $(\mathcal{S}_2, \Delta|_{\mathcal{S}_2})$ is a CQG and the inclusion map from $\mathcal{S}_2 \to \mathcal{S}_1$ is a morphism of CQG s.

Definition 1.2.11 A **Woronowicz C^*-ideal** of a CQG $(\mathcal{S}, \Delta)$ is a C^* ideal J of $\mathcal{S}$ such that $\Delta(J) \subseteq \mathrm{Ker}(\pi \otimes \pi)$, where π is the quotient map from $\mathcal{S}$ to $\mathcal{S}/J$.

It can be easily seen that a kernel of a CQG morphism is a Woronowicz C^*-ideal. We recall the following isomorphism theorem:

Proposition 1.2.12 *The quotient of a CQG $(\mathcal{S}, \Delta)$ by a Woronowicz C^*-ideal $\mathcal{I}$ has a unique CQG structure such that the quotient map π is a morphism of CQG s. More precisely, the coproduct $\widetilde{\Delta}$ on $\mathcal{S}/\mathcal{I}$ is given by $\widetilde{\Delta}(s + \mathcal{I}) = (\pi \otimes \pi)\Delta(s)$.*

Definition 1.2.13 A CQG $(\mathcal{S}', \Delta')$ is called a quantum subgroup of another CQG $(\mathcal{S}, \Delta)$ if there is a Woronowicz C^*-ideal J of $\mathcal{S}$ such that $(\mathcal{S}', \Delta') \cong (\mathcal{S}, \Delta)/J$.

Let us mention a convention which we are going to follow.

Remark 1.2.14 We shall use most of the terminologies of [25], for example, Woronowicz C^* -subalgebra, Woronowicz C^*-ideal etc., however with the exception that we call the Woronowicz C^* algebras just compact quantum groups, and not use the term compact quantum groups for the dual objects as done in [25].

Let $(\mathcal{S}, \Delta)$ be a compact quantum group. Then there exists a state h on $\mathcal{S}$, to be called a **Haar state** on $\mathcal{S}$ such that $(h \otimes \mathrm{id})\Delta(s) = (\mathrm{id} \otimes h)\Delta(s) = h(s).1$. We recall that unlike the group case, h may not be faithful. But on the dense Hopf $*$-algebra $\mathcal{S}_0$ mentioned above, it is faithful. We have the following:

Proposition 1.2.15 *Let $i : \mathcal{S}_1 \to \mathcal{S}_2$ be an injective morphism of CQG s. Then the Haar state on $\mathcal{S}_1$ is the restriction of that of $\mathcal{S}_2$ on $\mathcal{S}_1$.*

Remark 1.2.16 In general, the Haar state may not be tracial. In fact, there exists a multiplicative linear functional denoted by f_1 in [9] such that $h(ab) = h(b(f_1 \triangleleft a \triangleright f_1))$. Moreover, from Theorem 1.5 of [8], it follows that the Haar state of a CQG is tracial if and only if $\kappa^2 = \mathrm{id}$.

Corepresentations of a compact quantum group

Definition 1.2.17 A corepresentation of a compact quantum group (S, Δ) on a Hilbert space $\mathcal{H}$ is a map U from $\mathcal{H}$ to the Hilbert $\mathcal{S}$ module $\mathcal{H} \otimes \mathcal{S}$ such that the element $\widetilde{U} \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S})$ given by $\widetilde{U}(\xi \otimes b) = U(\xi)b$ (ξ in $\mathcal{H}, b$ in $\mathcal{S}$) satisfies

$$(\mathrm{id} \otimes \Delta)\widetilde{U} = \widetilde{U}_{(12)}\widetilde{U}_{(13)},$$

where for an operator X in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ we have denoted by $X_{(12)}$ and $X_{(13)}$ the operators $X \otimes I_{\mathcal{H}_2}$ and $\Sigma_{23} X_{(12)} \Sigma_{23}$ respectively and Σ_{23} is the unitary on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ which flips the two copies of $\mathcal{H}_2$.

If $\widetilde{U}$ is an unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S})$, then U is called a unitary corepresentation.

Remark 1.2.18 Let ϕ be a CQG morphism from a CQG $(\mathcal{S}_1, \Delta_1)$ to another CQG $(\mathcal{S}_2, \Delta_2)$. Then for every unitary corepresentation U of $\mathcal{S}_1$, $(\mathrm{id} \otimes \phi)U$ is a unitary corepresentation of $\mathcal{S}_2$.

Following the definitions given in the last part of Sect. 1.1.4 and a unitary corepresentation U of a CQG on a Hilbert space $\mathcal{H}$, and a not necessarily bounded, densely defined (in the weak operator topology) linear functional τ on $\mathcal{B}(\mathcal{H})$, we will use the notation ad_U and the terms "ad_U preserves τ" and "U equivariant" throughout this book.

A CQG $(\mathcal{S}, \Delta)$ has a distinguished corepresentation which generalizes the right regular representation of groups. Let $\mathcal{H}$ be the GNS space of $\mathcal{S}$ associated with the Haar state h, ξ_0 be the associated cyclic vector and $\mathcal{K}$ be a Hilbert space on which $\mathcal{S}$ acts faithfully and nondegenerately. There is a unitary operator u on $\mathcal{H} \otimes \mathcal{K}$ defined by $u(a\xi_0 \otimes \eta) = \Delta(a)(\xi_0 \otimes \eta)$ where a is in $\mathcal{S}$, η is in $\mathcal{K}$. Then u can be shown to be an element of the multiplier of $\mathcal{K}(\mathcal{H}) \otimes \mathcal{S}$ and called the right regular corepresentation of $\mathcal{S}$.

Let v be a corepresentation of a CQG $(\mathcal{S}, \Delta)$ on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ is said to be invariant if $(e \otimes 1)v(e \otimes 1) = v(e \otimes 1)$, where e is the orthogonal projection onto this subspace. The corepresentation v is called irreducible if the only invariant subspaces are $\{0\}$ and $\mathcal{H}$. It is clear that one can make sense of direct sum of corepresentations in this case also. Moreover, for two corepresentations v and w of a CQG $(\mathcal{S}, \Delta)$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the tensor product of v and w is given by the element $v_{(13)}w_{(23)}$. The intertwiner between v and w is an element x in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $(x \otimes 1)v = w(x \otimes 1)$. The set of intertwiners between v and w is denoted by $\mathrm{Mor}(v, w)$. Two corepresentations are said to be equivalent if there is an invertible intertwiner. They are unitarily equivalent if the intertwiner can be chosen to be unitary. We denote by $\mathrm{Rep}(\mathcal{S})$ the set of inequivalent irreducible unitary corepresentations of $\mathcal{S}$.

Just like the case of compact groups, CQGs have an analogous Peter Weyl theory which corresponds to the usual Peter Weyl theory in the group case. We will give a sketch of it by mentioning the main results and refer to [8, 9, 26] for the details.

Let v be a unitary corepresentation of $(\mathcal{S}, \Delta)$ on $\mathcal{H}$. If $\mathcal{H}_1$ is an invariant subspace, then the orthogonal complement of $\mathcal{H}_1$ is also invariant. Any nondegenerate finite dimensional corepresentation is equivalent to a unitary corepresentation.

Every irreducible unitary corepresentation of a CQG is contained in the right regular corepresentation. Let v be a corepresentation on a finite dimensional Hilbert space $\mathcal{H}$. If we denote the matrix units in $\mathcal{B}(\mathcal{H})$ by (e_{pq}), we can write $v = \sum e_{pq} \otimes v_{pq}$. v_{pq} are called the matrix elements of the finite dimensional corepresentation v. Define $\overline{v} = \sum e_{pq} \otimes v^*_{pq}$. Then $\overline{v}$ is a corepresentation and is called the adjoint of v. It can be shown that if v is a finite dimensional irreducible corepresentation, then $\overline{v}$ is also irreducible. Moreover, for an irreducible unitary corepresentation, its adjoint is equivalent to a unitary corepresentation.

The subspace spanned by the matrix elements of finite dimensional unitary corepresentations is denoted by $\mathcal{S}_0$. Firstly, $\mathcal{S}_0$ is a subalgebra as the product of two matrix elements of finite dimensional unitary corepresentations is a matrix element of the tensor product of these corepresentations. Moreover, as the adjoint of a finite dimensional unitary corepresentation is equivalent to a unitary corepresentation, $\mathcal{S}_0$ is $*$ invariant. We note that 1 is in $\mathcal{S}_0$ as 1 is a corepresentation. Now, we will recall some basic facts about the subalgebra $\mathcal{S}_0$. We will denote the Haar state of $\mathcal{S}$ by h. We refer to [9] for the proof of the following statement.

Proposition 1.2.19 *(1) $\mathcal{S}_0$ is a dense $*$-subalgebra of $\mathcal{S}$.*

(2) Let $\{u^\pi : \pi \in \mathrm{Rep}(S)\}$ be a complete set of mutually inequivalent, irreducible unitary corepresentations. We will denote the corepresentation space and dimension of u^π by $\mathcal{H}_\pi$ and d_π respectively. Then the Schur's orthogonality relation takes the following form:

For any π in $\mathrm{Rep}(S)$, there is a unique positive invertible operator $F^\pi = ((F^\pi(i, j)))$ acting on $\mathcal{H}_\pi$ such that for any π, β in $\mathrm{Rep}(S)$ and $1 \le j, q \le d_\pi$, $1 \le i, p \le d_\beta$,

$$\mathrm{Tr}(F^\pi) = \mathrm{Tr}((F^\pi)^{-1}) = M_\pi(\mathrm{say}),\ h((u^\beta_{ip})^* u^\pi_{jq}) = \frac{1}{M_\alpha}\delta_{\alpha\beta}\delta_{pq}(F^\pi)^{-1}(i, j),$$

$$h(u^\beta_{pi}(u^\pi_{qj})^*) = \frac{1}{M_\pi}\delta_{\pi\beta}\delta_{pq}F^\pi(i, j).$$

M_π is called the quantum dimension of the corepresentation π, often denoted by d_π^q.

(3) For any complex number z, let ϕ_z be the functional on $\mathcal{S}_0$ defined by $\phi_z(u_{ij}^\pi) = (F^\pi)^z(j, i)$. Then each ϕ_z is multiplicative, $(\phi_z * 1) = 1$, $\phi_z * \phi_w = \phi_{(z+w)}$, and for any fixed element $a \in \mathcal{S}_0$, $z \mapsto \phi_z(a)$ is a complex analytic map.

(4) $\{u_{pq}^\pi : \pi \in \mathrm{Rep}(S), 1 \le p, q \le d_\pi\}$ form a basis for $\mathcal{S}_0$.

(5) Moreover, Δ maps $\mathcal{S}_0$ into $\mathcal{S}_0 \otimes_{\mathrm{alg}} \mathcal{S}_0$. In fact, Δ is given by $\Delta(u_{pq}^\alpha) = \sum_{k=1}^{n_\alpha} u_{pk}^\alpha \otimes u_{kq}^\alpha$. A counit and an antipode are defined on $\mathcal{S}_0$ respectively by the formulas:

$$\epsilon(u_{pq}^\alpha) = \delta_{pq},\ \kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*.$$

With this, $\mathcal{S}_0$ becomes a Hopf $*$-algebra.

Corresponding to $\pi \in \mathrm{Rep}(\mathcal{S})$, let ρ_{sm}^π be the linear functional on $\mathcal{S}$ given by $\rho_{sm}^\pi(x) = h(x_{sm}^\pi x)$, $s, m = 1, \ldots, d_\pi$ for $x \in \mathcal{S}$, where $x_{sm}^\pi = (d_\pi^q) u_{km}^{\pi *}(F^\pi(k, s))$. Also let $\rho^\pi = \sum_{s=1}^{d_\pi} \rho_{ss}^\pi$. Given a unitary corepresentation V of $\mathcal{S}$ on a Hilbert space $\mathcal{H}$, we get a decomposition of $\mathcal{H}$ (called the spectral decomposition) as

$$\mathcal{H} = \oplus_{\pi \in Rep(\mathcal{S}), 1 \le i \le m_\pi} \mathcal{H}_i^\pi,$$

where m_π is the multiplicity of π in the corepresentation V and $V|_{\mathcal{H}_i^\pi}$ is unitarily equivalent to a unitary irreducible corepresentation u^π. The subspace $\mathcal{H}^\pi = \oplus_i \mathcal{H}_i^\pi$ is called the spectral subspace of type π corresponding to the irreducible corepresentation π. It is nothing but the image of the spectral projection given by $(id \otimes \rho^\pi)V$. Then we have from Lemma 8.1 of [26], $\rho_{pr}^\pi(u_{pr}^\pi) = 1$ and ρ_{pr}^π is zero on all other matrix elements.

Recall from [27], the modular operator $\Phi = S^* S$, where S is the conjugate-linear acting on the $L^2(h)$ given by $S(a.1) := a^*.1$ for $a \in \mathcal{S}$. The one parameter modular automorphism group (see [27]) Θ_t (say), corresponding to the state h is given by $\Theta_t(a) = \Phi^{it} a \Phi^{-it}$. Note that here we have used the symbol Φ for the modular operator as Δ has been used for the coproduct. From part (2) of Proposition 1.2.19, we see that

$$\Phi|_{L^2(h)_i^\pi} = F^\pi,\ \ for\ all\ \pi\ and\ i. \tag{1.2.2}$$

In particular Φ maps $L^2(h)_i^\pi$ into $L^2(h)_i^\pi$ for all i.

Given the Hopf $*$-algebra $\mathcal{S}_0$, there can be several CQG's which have this $*$-algebra as the Hopf $*$-algebra generated by the matrix elements of finite dimensional corepresentations. However, there exists a largest such CQG $\mathcal{S}^u$, called the universal CQG corresponding to $\mathcal{S}_0$. It is obtained as the universal enveloping C^* algebra of $\mathcal{S}_0$. We also say that a CQG $\mathcal{S}$ is universal if $\mathcal{S} = \mathcal{S}^u$. For details the reader is referred to [28]. The C^*-completion $\mathcal{S}_r$ of $\mathcal{S}_0$ in the norm of $\mathcal{B}(L^2(h))$ ($L^2(h)$ denotes the GNS space associated to h) is a CQG and called the reduced quantum group corresponding to $\mathcal{S}$. If h is faithful then $\mathcal{S}$ and $\mathcal{S}_r$ coincide. In general there will be a surjective CQG morphism from $\mathcal{S}$ to $\mathcal{S}_r$ identifying the latter as a quantum subgroup of the former. Clearly, $\mathcal{S}_0$ sits as a common subalgebra in both $\mathcal{S}$ and $\mathcal{S}^u$, dense in their respective norms. Thus, for π in Rep(S), we have $\pi : \mathbb{C}^{d_\pi} \to \mathbb{C}^{d_\pi} \otimes_{\mathrm{alg}} \mathcal{S}_0 \subseteq \mathbb{C}^{d_\pi} \otimes_{\mathrm{alg}} \mathcal{S}^u$ is also an irreducible unitary corepresentation of $\mathcal{S}^u$, to be denoted by π_u. In fact, $\pi \mapsto \pi_u$ is a bijective correspondence between Rep$(\mathcal{S})$ and Rep$(\mathcal{S}^u)$. Moreover, as any unitary corepresentation decomposes into a direct sum of irreducible ones, the above correspondence extends to a bijective correspondence $v \mapsto v_u$ between unitary corepresentations of $\mathcal{S}$ and those of $\mathcal{S}^u$.

There is also a von Neumann algebraic framework of quantum groups suitable for development of the theory of locally compact quantum groups (see [16, 29, 30] and the references therein). In this theory, the von Neumann algebraic version of the CQG is a von Neumann algebra $\mathcal{M}$ with a coassociative, normal, injective coproduct map Δ from $\mathcal{M}$ to $\mathcal{M}\bar{\otimes}\mathcal{M}$ and a faithful, normal, bi-invariant state ψ on $\mathcal{M}$. Indeed, given a CQG $\mathcal{S}$, the weak closure $\mathcal{S}_r{}''$ of the reduced quantum group in the GNS space of the Haar state is a von Neumann algebraic compact quantum group.

A **compact matrix quantum group** is a CQG such that there exists a distinguished unitary irreducible corepresentation called the fundamental corepresentation such that the $*$-algebra spanned by its matrix elements is a dense Hopf $*$-subalgebra of the CQG.

We now discuss the **free product and tensor product of CQG** s which were developed in [25, 31]. Let $(\mathcal{S}_1, \Delta_1)$ and $(\mathcal{S}_2, \Delta_2)$ be two CQG s. Let i_1 and i_2 denote the canonical injections of $\mathcal{S}_1$ and $\mathcal{S}_2$ into the C^* algebra $\mathcal{S}_1 * \mathcal{S}_2$. Put $\rho_1 = (i_1 \otimes i_1)\Delta_1$ and $\rho_2 = (i_2 \otimes i_2)\Delta_2$. By the universal property of $\mathcal{S}_1 * \mathcal{S}_2$, there exists a map $\Delta : \mathcal{S}_1 * \mathcal{S}_2 \to (\mathcal{S}_1 * \mathcal{S}_2) \otimes (\mathcal{S}_1 * \mathcal{S}_2)$ such that $\Delta i_1 = \rho_1$ and $\Delta i_2 = \rho_2$. It can be shown that Δ indeed has the required properties so that $(\mathcal{S}, \Delta)$ is a CQG.

Let $\{\mathcal{S}_n\}_{n \in \mathbb{N}}$ be an inductive sequence of CQG's, where the connecting morphisms π_{mn} from $\mathcal{S}_n$ to $\mathcal{S}_m$ $(n < m)$ are injective morphisms of CQG. It follows from Proposition 3.1 of [25] that the inductive limit $\mathcal{S}_0$ of $\mathcal{S}_n$ s has a unique CQG structure. More generally, the inductive limit of an arbitrary sequence of CQG has the structure of a CQG. The following lemma is probably known, but we include the proof (taken from [32]) for the sake of completeness.

Lemma 1.2.20 *Suppose that $(\mathcal{S}_n)_{n \in \mathbb{N}}$ is a sequence of CQG and for each $n \le m$ in $\mathbb{N}$, there is a CQG morphism $\pi_{n,m} : \mathcal{S}_n \to \mathcal{S}_m$ with the compatibility property*

$$\pi_{m,k} \circ \pi_{n,m} = \pi_{n,k}, \quad n \le m \le k.$$

Then the inductive limit of C^*-algebras $(\mathcal{S}_n)_{n \in \mathbb{N}}$ has a canonical structure of a CQG. It will be denoted $\mathcal{S}_\infty$ or $\lim_{n \in \mathbb{N}} \mathcal{S}_n$. It has the following universality property: for any CQG $(\mathcal{S}, \Delta)$ such that there are CQG morphisms $\pi_n : \mathcal{S}_n \to \mathcal{S}$ $(n \in \mathbb{N})$ satisfying $\pi_m \circ \pi_{n,m} = \pi_n$ for all $m \ge n$, there exists a unique CQG morphism $\pi_\infty : \mathcal{S}_\infty \to \mathcal{S}$ such that $\pi_n = \pi_\infty \circ \pi_{n,\infty}$ for all $n \in \mathbb{N}$, where we have denoted by $\pi_{n,\infty}$ the canonical unital C^*-homomorphism from $\mathcal{S}_n$ into $\mathcal{S}_\infty$.

Proof Let us denote the coproduct on $\mathcal{S}_n$ by Δ_n. We consider the unital C^*-homomorphism $\rho_n : \mathcal{S}_n \to \mathcal{S}_\infty \otimes \mathcal{S}_\infty$ given by $\rho_n = (\pi_{n,\infty} \otimes \pi_{n,\infty}) \circ \Delta_n$, and observe that these maps do satisfy the compatibility property:

$$\rho_m \circ \pi_{n,m} = \rho_n \quad \forall n \le m.$$

Thus, by the general properties of the C^*-algebraic inductive limit, we have a unique unital C^*-homomorphism $\Delta_\infty : \mathcal{S}_\infty \to \mathcal{S}_\infty \otimes \mathcal{S}_\infty$ satisfying $\Delta_\infty \circ \pi_{n,\infty} = \rho_n$ for all n. We claim that $(\mathcal{S}_\infty, \Delta_\infty)$ is a CQG.

We first check that Δ_∞ is coassociative. It is enough to verify the coassociativity on the dense set $\cup_n \pi_{n,\infty}(\mathcal{S}_n)$. Indeed, for $s = \pi_{n,\infty}(a)$ $(a \in \mathcal{S}_n)$, by using $\Delta_\infty \circ \pi_{n,\infty} = (\pi_{n,\infty} \otimes \pi_{n,\infty}) \circ \Delta_n$, we have the following:

$$\begin{aligned}
&(\Delta_\infty \otimes \mathrm{id})\Delta_\infty(\pi_{n,\infty}(a)) \\
&\quad = (\Delta_\infty \otimes \mathrm{id})(\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(a)) \\
&\quad = (\pi_{n,\infty} \otimes \pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n \otimes \mathrm{id})(\Delta_n(a)) \\
&\quad = (\pi_{n,\infty} \otimes \pi_{n,\infty} \otimes \pi_{n,\infty})(\mathrm{id} \otimes \Delta_n)(\Delta_n(a)) \\
&\quad = (\pi_{n,\infty} \otimes (\pi_{n,\infty} \otimes \pi_{n,\infty}) \circ \Delta_n)(\Delta_n(a)) \\
&\quad = (\pi_{n,\infty} \otimes \Delta_\infty \circ \pi_{n,\infty})(\Delta_n(a)) \\
&\quad = (\mathrm{id} \otimes \Delta_\infty)((\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(a))) \\
&\quad = (\mathrm{id} \otimes \Delta_\infty)(\Delta_\infty(\pi_{n,\infty}(a)))
\end{aligned}$$

which proves the coassociativity.

Finally, we need to verify the density conditions aii. of Definition 1.2.5. Note that to show that $\mathrm{Span}\{\Delta_\infty(\mathcal{S}_\infty)(1 \otimes \mathcal{S}_\infty)\}$ is dense in $\mathcal{S}_\infty \otimes \mathcal{S}_\infty$ it is enough to show that the above assertion is true with $\mathcal{S}_\infty$ replaced by the dense subalgebra $\bigcup_n \pi_{n,\infty}(\mathcal{S}_n)$.

Using the density of $\mathrm{Span}\{\Delta_n(\mathcal{S}_n)(1 \otimes \mathcal{S}_n)\}$ in $\mathcal{S}_n \otimes \mathcal{S}_n$ and the contractivity of the map $\pi_{n,\infty}$ we note that the span of $(\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(S_n)(1 \otimes \mathcal{S}_n))$ is dense in $\mathrm{Span}((\pi_{n,\infty} \otimes \pi_{n,\infty})(\mathcal{S}_n \otimes \mathcal{S}_n))$. This implies that $\mathrm{Span}\{(\pi_{n,\infty} \otimes \pi_{n,\infty})(\Delta_n(\mathcal{S}_n))(1 \otimes \pi_{n,\infty}(\mathcal{S}_n))\}$ is dense in $\pi_{n,\infty}(\mathcal{S}_n) \otimes \pi_{n,\infty}(\mathcal{S}_n)$ and hence $\mathrm{Span}\{\Delta_\infty(\pi_{n,\infty}(\mathcal{S}_n))(1 \otimes \pi_{n,\infty}(\mathcal{S}_n))\}$ is dense in $\pi_{n,\infty}(\mathcal{S}_n) \otimes \pi_{n,\infty}(\mathcal{S}_n)$. The proof of the claim now follows by noting that $\pi_{n,\infty}(\mathcal{S}_n) = \pi_{m,\infty}\pi_{n,m}(\mathcal{S}_n) \subseteq \pi_{m,\infty}(\mathcal{S}_m)$ for any $m \ge n$, along with the above observations. In a similar way, the density of $\mathrm{Span}\{\Delta_\infty(\mathcal{S}_\infty)(\mathcal{S}_\infty \otimes 1)\}$ in $\mathcal{S}_\infty \otimes \mathcal{S}_\infty$ is proved.

The proof of the universality property is routine and hence omitted. □

We note that the proof remains valid for any other indexing set for the net, not necessarily $\mathbb{N}$.

Combining the above two results, it follows that the free product C^* algebra of an arbitrary sequence of CQG s has a natural CQG structure.

The following result was derived in [25].

Proposition 1.2.21 *Let Γ_1, Γ_2 be a discrete abelian groups. Then the natural isomorphisms $C^*(\Gamma_1) \cong C(\widehat{\Gamma_1})$ and $C^*(\Gamma_1) * C^*(\Gamma_2) \cong C^*(\Gamma_1 * \Gamma_2)$ are isomorphism of CQG s.*

Let i_1 and i_2 be the inclusion of CQG's $\mathcal{S}_1$ and $\mathcal{S}_2$ into $\mathcal{S}_1 * \mathcal{S}_2$. If U_1 and U_2 are unitary corepresentations of CQG's $\mathcal{S}_1$ and $\mathcal{S}_2$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then the **free product corepresentation** of U_1 and U_2 is a corepresentation of the CQG $\mathcal{S}_1 * \mathcal{S}_2$ on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by the $\mathcal{S}_1 * \mathcal{S}_2$ valued matrix $\begin{pmatrix} (\mathrm{id} \otimes i_1)U_1 & 0 \\ 0 & (\mathrm{id} \otimes i_2)U_2 \end{pmatrix}$.

Similarly, the free product corepresentation of an arbitrary family of CQG corepresentations are defined.

The minimal and maximal tensor product of two CQG's also admit natural CQG structure [31]. The free product and maximal tensor product have the following universal properties (see [25, 31]).

Proposition 1.2.22 *1. The canonical injections, say $i_1,\ i_2$ ($j_1,\ j_2$ respectively) from $\mathcal{S}_1$ and $\mathcal{S}_2$ to $\mathcal{S}_1 \star \mathcal{S}_2$ ($\mathcal{S}_1 \otimes^{\max} \mathcal{S}_2$ respectively) are CQG morphisms.*
2. *Given any CQG $\mathcal{C}$ and morphisms $\pi_1 : \mathcal{S}_1 \mapsto \mathcal{C}$ and $\pi_2 : \mathcal{S}_2 \mapsto \mathcal{C}$ there always exists a unique morphism denoted by $\pi := \pi_1 * \pi_2$ from $\mathcal{S}_1 \star \mathcal{S}_2$ to $\mathcal{C}$ satisfying $\pi \circ i_k = \pi_k$ for $k = 1, 2$.*
3. *Furthermore, if the ranges of π_1 and π_2 commute, i.e., $\pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ $\forall\, a \in \mathcal{S}_1,\ b \in \mathcal{S}_2$, we have a unique morphism π' from $\mathcal{S}_1 \otimes^{\max} \mathcal{S}_2$ to $\mathcal{C}$ satisfying $\pi' \circ j_k = \pi_k$ for $k = 1, 2$.*
4. *The above conclusions hold for free or maximal tensor product of any finite number of CQG's as well.*

1.2.3 The CQG $U_\mu(2)$

We now introduce the compact quantum group $U_\mu(2)$. We refer to [11] for more details.

As a unital C^* algebra, $U_\mu(2)$ is generated by 4 elements $u_{11}, u_{12}, u_{21}, u_{22}$ satisfying:

$$u_{11}u_{12} = \mu u_{12}u_{11}, \tag{1.2.3}$$

$$u_{11}u_{21} = \mu u_{21}u_{11}, \tag{1.2.4}$$

$$u_{12}u_{22} = \mu u_{22}u_{12}, \tag{1.2.5}$$

$$u_{21}u_{22} = \mu u_{22}u_{21}, \tag{1.2.6}$$

$$u_{12}u_{21} = u_{21}u_{12}, \tag{1.2.7}$$

$$u_{11}u_{22} - u_{22}u_{11} = (\mu - \mu^{-1})u_{12}u_{21}. \tag{1.2.8}$$

and the condition that the matrix $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ is a unitary. Thus, the above matrix u is the fundamental unitary for $U_\mu(2)$.

The CQG structure is given by

$$\Delta(u_{ij}) = \sum_{k=1,2} u_{ik} \otimes u_{kj},\ \kappa(u_{ij}) = {u_{ji}}^*, \epsilon(u_{ij}) = \delta_{ij}. \tag{1.2.9}$$

The quantum determinant D_μ is defined by

$$D_\mu = u_{11}u_{22} - \mu u_{12}u_{21} = u_{22}u_{11} - \mu^{-1}u_{12}u_{21}. \tag{1.2.10}$$

Then, ${D_\mu}^* D_\mu = D_\mu {D_\mu}^* = 1$. Moreover, D_μ belongs to the center of $U_\mu(2)$.

1.2.4 The CQG $SU_\mu(2)$

Let μ belongs to $[-1, 1]$. The C^* algebra $SU_\mu(2)$ is defined as the universal unital C^* algebra generated by α, γ satisfying:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \tag{1.2.11}$$

$$\alpha\alpha^* + \mu^2\gamma\gamma^* = 1, \tag{1.2.12}$$

$$\gamma\gamma^* = \gamma^*\gamma, \tag{1.2.13}$$

$$\mu\gamma\alpha = \alpha\gamma, \tag{1.2.14}$$

$$\mu\gamma^*\alpha = \alpha\gamma^*. \tag{1.2.15}$$

The fundamental corepresentation of $SU_\mu(2)$ is given by: $\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$.

There is a coproduct Δ of $SU_\mu(2)$ given by:

$$\Delta(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma,\ \ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$

which makes it into a CQG. Let h denote the Haar state and $\mathcal{H} = L^2(SU_\mu(2))$ be the corresponding G.N.S space.

Haar state on $SU_\mu(2)$

We restate the content of Theorem 14, Chap. 4 (page 113) of [11] in a convenient form below. For all $m \geq 1, n, l, k \geq 0, k' \neq k''$,

$$h((\gamma^*\gamma)^k) = \frac{1-\mu^2}{1-\mu^{2k+2}},\ h(\alpha^m\gamma^{*n}\gamma^l) = 0,\ h(\alpha^{*m}\gamma^{*n}\gamma^l) = 0,\ h(\gamma^{*k'}\gamma^{*k''}) = 0. \tag{1.2.16}$$

Corepresentations of $SU_\mu(2)$

For each n in $\{0, 1/2, 1, \ldots\}$, there is a unique irreducible corepresentation T^n of dimension $2n+1$. Denote by t^n_{ij} the ij th entry of T^n. They form an orthogonal basis of $\mathcal{H}$. Denote by e^n_{ij} the normalized t^n_{ij} s so that $\{e^n_{ij} : n = 0, 1/2, 1, \dots, i, j = -n, -n+1, \dots n\}$ is an orthonormal basis.

We recall from [11] that

$$t^{1/2}_{-1/2,-1/2} = \alpha,\ t^{1/2}_{-1/2,1/2} = -\mu\gamma^*,\ t^{1/2}_{1/2,-1/2} = \gamma,\ t^{1/2}_{1/2,1/2} = \alpha^*. \tag{1.2.17}$$

1.2.5 The Hopf ∗-algebras $\mathcal{O}(SU_\mu(2))$ and $\mathcal{U}_\mu(su(2))$

We define the Hopf ∗-algebra $\mathcal{O}(SU_\mu(2))$ following the notations of [11].

$\mathcal{O}(SL_\mu(2))$ is the complex associative algebra with generators a, b, c, d such that

$$ab = \mu ba,\ ac = \mu ca,\ bd = \mu db,\ cd = \mu dc,\ bc = cb,\ ad - \mu bc = da - \mu^{-1}bc = 1. \tag{1.2.18}$$

The coproduct is given by

$$\Delta(a) = a \otimes a + b \otimes c,\ \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c,\ \Delta(d) = c \otimes b + d \otimes d.$$

The antipode is given by:

$$\kappa(a) = d,\ \kappa(b) = -b,\ \kappa(c) = -c,\ \kappa(d) = a$$

Finally, the counit is given by:

$$\epsilon(a) = \epsilon(d) = 1,\ \epsilon(b) = \epsilon(c) = 0.$$

For all μ in $\mathbb{R}$, there is an involution of the algebra $\mathcal{O}(SL_\mu(2))$ determined by

$$a^* = d,\ \ b^* = -\mu c. \tag{1.2.19}$$

The corresponding Hopf ∗-algebra is denoted by $\mathcal{O}(SU_\mu(2))$.

Proposition 1.2.23 *$\mathcal{O}(SU_\mu(2))$ can be identified with $(SU_\mu(2))_0$, i.e. the Hopf ∗-algebra generated by the matrix elements of irreducible unitary corepresentations of $SU_\mu(2)$, via the isomorphism given on the generators by*

$$\alpha \mapsto a,\ \gamma \mapsto c,\ \alpha^* \mapsto d,\ \gamma^* \mapsto -\mu^{-1}b. \tag{1.2.20}$$

Proof $(SU_\mu(2))_0$ is generated by the matrix elements of the fundamental unitary of $SU_\mu(2)$, that is, the ∗-algebra generated by α and γ. On the other hand, inserting (1.2.19) in (1.2.18), we see that $\mathcal{O}(SU_\mu(2))$ is generated by 4 elements a, b, c, d

such that $ac = \mu ca,\ ac^* = \mu c^* a,\ cc^* = c^* c,\ a^* a + c^* c = 1,\ aa^* + \mu^2 c^* c = 1$. Comparing with the defining equations of $SU_\mu(2)$, that is, (1.2.11)–(1.2.15), it is clear that the above correspondence gives the required isomorphism. □

Next, we recall from [33] the Hopf $*$ algebra $\mathcal{U}_\mu(su(2))$ which is the dual Hopf $*$-algebra of $\mathcal{O}(SU_\mu(2))$. It is generated by elements F, E, K, K^{-1} with the defining relations:

$$KK^{-1} = K^{-1}K = 1,\ KE = \mu EK,\ FK = \mu KF,\ EF - FE = (\mu - \mu^{-1})^{-1}(K^2 - K^{-2}),$$

with involution given by $E^* = F,\ K^* = K$ and comultiplication given by:

$$\Delta(E) = E \otimes K + K^{-1} \otimes E,\ \Delta(F) = F \otimes K + K^{-1} \otimes F,\ \Delta(K) = K \otimes K.$$

The counit is given by $\epsilon(E) = \epsilon(F) = \epsilon(K - 1) = 0$ and antipode $\kappa(K) = K^{-1},\ \kappa(E) = -\mu E,\ \kappa(F) = -\mu^{-1} F$.

There is a dual pairing $\langle ., . \rangle$ of $\mathcal{U}_\mu(su(2))$ and $\mathcal{O}(SU_\mu(2))$ given on the generators by: $\left\langle K^{\pm 1}, \alpha^* \right\rangle = \left\langle K^{\mp 1}, \alpha \right\rangle = \mu^{\pm \frac{1}{2}},\ \langle E, \gamma \rangle = \langle F, -\mu\gamma^* \rangle = 1$

and zero otherwise.

The left action $\triangleright$ and right action $\triangleleft$ of $\mathcal{U}_\mu(su(2))$ on $SU_\mu(2)$ are given by:

$f \triangleright x = \left\langle f, x_{(2)} \right\rangle x_{(1)},\ x \triangleleft f = \left\langle f, x_{(1)} \right\rangle x_{(2)},\ x \in \mathcal{O}(SU_\mu(2)),\ f \in \mathcal{U}_\mu(su(2))$, where we use the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

The actions satisfy:

$(f \triangleright x)^* = \kappa(f)^* \triangleright x^*,\ (x \triangleleft f)^* = x^* \triangleleft \kappa(f)^*,\ f \triangleright xy = (f_{(1)} \triangleright x)(f_{(2)} \triangleright y),\ xy \triangleleft f = (x \triangleleft f_{(1)})(y \triangleleft f_{(2)})$.

The action on the generators is given by:

$$\left\{ \begin{array}{llll} E \triangleright \alpha = -\mu\gamma^* & E \triangleright \gamma = \alpha^*, & E \triangleright \gamma^* = 0, & E \triangleright \alpha^* = 0, \\ F \triangleright (-\mu\gamma^*) = \alpha & F \triangleright \alpha^* = \gamma, & F \triangleright \alpha = 0, & F \triangleright \gamma = 0, \\ K \triangleright \alpha = \mu^{-\frac{1}{2}}\alpha, & K \triangleright (\gamma^*) = \mu^{\frac{1}{2}}\gamma^*, & K \triangleright \gamma = \mu^{-\frac{1}{2}}\gamma, & K \triangleright \alpha^* = \mu^{\frac{1}{2}}\alpha^*. \end{array} \right\} \tag{1.2.21}$$

$$\left\{ \begin{array}{llll} \gamma \triangleleft E = \alpha, & \alpha^* \triangleleft E = -\mu\gamma^*, & \alpha \triangleleft E = 0, & \gamma^* \triangleleft E = 0 \\ \alpha \triangleleft F = \gamma, & -\mu\gamma^* \triangleleft F = \alpha^*, & \gamma \triangleleft F = 0, & \alpha^* \triangleleft F = 0, \\ \alpha \triangleleft K = \mu^{-\frac{1}{2}}\alpha, & \gamma^* \triangleleft K = \mu^{-\frac{1}{2}}\gamma^*, & \gamma \triangleleft K = \mu^{\frac{1}{2}}\gamma, & \alpha^* \triangleleft K = \mu^{\frac{1}{2}}\alpha^*. \end{array} \right\} \tag{1.2.22}$$

1.2.6 The CQG $SO_\mu(3)$

Here we recall the CQG $SO_\mu(3)$ as described in [34].

$SO_\mu(3)$ is the universal unital C^* algebra generated by elements M, N, G, C, L satisfying:

$$\left\{\begin{array}{c} L^*L = (I-N)(I-\mu^{-2}N), LL^* = (I-\mu^2 N)(I-\mu^4 N), G^*G = GG^* = N^2, \\ M^*M = N - N^2,\ MM^* = \mu^2 N - \mu^4 N^2,\ C^*C = N - N^2, \\ CC^* = \mu^2 N - \mu^4 N^2,\ LN = \mu^4 NL,\ GN = NG, \\ MN = \mu^2 NM,\ CN = \mu^2 NC,\ LG = \mu^4 GL, \\ LM = \mu^2 ML,\ MG = \mu^2 GM,\ CM = MC, \\ LG^* = \mu^4 G^* L,\ M^2 = \mu^{-1} LG,\ M^*L = \mu^{-1}(I-N)C,\ N^* = N. \end{array}\right\} \tag{1.2.23}$$

This CQG can be identified with a Woronowicz subalgebra of $SU_\mu(2)$ by taking:

$$N = \gamma^*\gamma,\ M = \alpha\gamma,\ C = \alpha\gamma^*,\ G = \gamma^2,\ L = \alpha^2,$$

where α, γ are as in Sect. 1.2.4.

1.3 Coaction of Compact Quantum Groups on a C^* Algebra

In this section, we discuss the notion of coaction of compact quantum groups on C^* algebra, as defined in [34].

Definition 1.3.1 We say that the compact quantum group $(\mathcal{S}, \Delta)$ coacts on a unital C^* algebra $\mathcal{B}$, if there is a unital C^*-homomorphism (called a coaction) $\alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{S}$ satisfying the following:
(bi) $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$, and
(bii) the linear span of $\alpha(\mathcal{B})(1 \otimes \mathcal{S})$ is norm dense in $\mathcal{B} \otimes \mathcal{S}$.

It is known (see, for example, [34, 35]) that (bii) is equivalent to the existence of a norm dense, unital $*$-subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ such that $\alpha(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes_{\mathrm{alg}} \mathcal{S}_0$ and on $\mathcal{B}_0$, $(\mathrm{id} \otimes \epsilon) \circ \alpha = \mathrm{id}$.

We shall sometimes say that α is a ‘topological’ or C^* coaction to distinguish it from a normal coaction of von Neumann algebraic quantum group.

Definition 1.3.2 Let α be a C^* coaction of $(\mathcal{S}, \Delta)$ on the C^* algebra $\mathcal{B}$. We say that the coaction α is **faithful** if there is no proper Woronowicz C^*-subalgebra $\mathcal{S}_1$ of $\mathcal{S}$ such that α is a C^* coaction of $\mathcal{S}_1$ on $\mathcal{B}$. A continuous linear functional ϕ on $\mathcal{B}$ is said to be **invariant under** α if

$$(\phi \otimes \mathrm{id})\alpha(b) = \phi(b).1_{\mathcal{S}}.$$

Remark 1.3.3 A coaction α of a CQG $\mathcal{S}$ on a C^* algebra $\mathcal{A}$ is faithful if and only if the C^* algebra generated by $\{(\omega \otimes \mathrm{id})\alpha(a) : a \in \mathcal{A},\ \omega \text{ is a state on } \mathcal{A}\}$, coincides with $\mathcal{A}$.

For the study of ergodic coactions on compact quantum groups, we refer to [36]. We record the following simple fact for future use.

Lemma 1.3.4 *Let $\mathcal{B}, \mathcal{S}$ be a unital C^*-algebras and ϕ a state on $\mathcal{B}$. Moreover, assume that $\alpha : \mathcal{B} \to \mathcal{B} \otimes \mathcal{S}$ is a unital C^*-homomorphism such that the following two conditions hold:*

(1) $(\tau \otimes \mathrm{id})\alpha(a) = \tau(a).1_{\mathcal{S}}$,

(2) $\alpha(\mathcal{B})(1 \otimes \mathcal{S})$ *is norm-total in* $\mathcal{B} \otimes \mathcal{S}$.

Then $\tilde{\alpha} : \mathcal{B} \otimes \mathcal{S} \to \mathcal{B} \otimes \mathcal{S}$ defined by $\widetilde{\alpha}(b \otimes s) = \alpha(b)(1 \otimes s)$ extends to an $\mathcal{S}$ linear unitary of the Hilbert $\mathcal{S}$ module $L^2(\mathcal{B}, \phi) \otimes \mathcal{S}$. In particular, if $\mathcal{S}$ is a CQG and α a coaction of $\mathcal{S}$ on $\mathcal{B}$, then $\tilde{\alpha}$ is a unitary corepresentation of $\mathcal{S}$ on $L^2(\mathcal{B}, \phi)$.

Proof By condition (1), we have $< \alpha(x), \alpha(y) >_{\mathcal{S}} = < x, y > 1_{\mathcal{S}}$, where $< \cdot, \cdot >_{\mathcal{S}}$ denotes the $\mathcal{S}$-valued inner product of the Hilbert module $L^2(\mathcal{B}, \phi) \otimes \mathcal{S}$. This proves that $\tilde{\alpha}$ defined as above extends to an $\mathcal{S}$-linear isometry on the Hilbert $\mathcal{S}$-module $L^2(\mathcal{B}, \phi) \otimes \mathcal{S}$. Moreover, by condition (2), the $\mathcal{S}$-linear span of the range of $\alpha(\mathcal{B})$ is dense in the Hilbert module $L^2(\mathcal{B}, \phi) \otimes \mathcal{S}$. Thus, the isometry $\tilde{\alpha}$ has a dense range and so it is a unitary. □

1.3.1 Coactions on Finite Quantum Spaces

Now, we recall the work of Shuzhou Wang done in [35]. We refer to [37, 38] for more details.

The **quantum permutation group** $A_s(n)$ is defined to be the universal C^* algebra generated by a_{ij} $(i, j = 1, 2, \ldots n)$ satisfying the following relations:

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \ i, j = 1, 2, \ldots n,$$

$$\sum_{j=1}^{n} a_{ij} = 1, \ i = 1, 2, \ldots n,$$

$$\sum_{i=1}^{n} a_{ij} = 1, \ i = 1, 2, \ldots n.$$

The name comes from the fact that the universal commutative C^* algebra generated by the above set of relations is isomorphic to $C(S_n)$ where S_n denotes the permutation group on n symbols.

Let us consider the category with objects as compact groups acting on a n-point set $X_n = \{x_1, x_2, \ldots, x_n\}$. If two groups G_1 and G_2 have actions α_1 and α_2 respectively, then a morphism from G_1 to G_2 is a group homomorphism ϕ such that $\alpha_2(\phi \times \mathrm{id}) = \alpha_1$. Then $C(S_n)$ is the universal object in this category. It is proved in [35] that the quantum permutation group enjoys a similar property.

We have that $C(X_n) = C^*\{e_i : e_i^2 = e_i = e_i^*, \ \sum_{r=1}^{n} e_r = 1, i = 1, 2, \ldots, n\}$. Then $A_s(n)$ has a C^* coaction on $C(X_n)$ given by:

$$\alpha(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij},\ j = 1, 2, \ldots n.$$

Proposition 1.3.5 *Consider the category with objects being CQGs having C^* coactions on $C(X_n)$ and morphisms being CQG morphisms intertwining the coactions. Then $A_s(n)$ is the universal object in this category.*

We note the following facts for future use.

Lemma 1.3.6 *Let α be a coaction of a CQG $\mathcal{S}$ on $C(X)$ where X is a finite set and $D_{X,X} = \{(x, x) : x \in X\}$. Then α automatically preserves the functional τ corresponding to the counting measure:*

$$(\tau \otimes \mathrm{id})(\alpha(f)) = \tau(f)1_{\mathcal{S}}.$$

Thus α induces a unitary $\widetilde{\alpha} \in \mathcal{B}(l^2(X)) \otimes \mathcal{S}$ given by $\widetilde{\alpha}(f \otimes q) = \alpha(f)q$. If we define $\alpha^{(2)} : C(X) \otimes C(X) \to C(X) \otimes C(X) \otimes \mathcal{S}$ by $\alpha^{(2)} = (\mathrm{id}_2 \otimes m_{\mathcal{S}})\sigma_{23}(\alpha \otimes \alpha)$, where $m_{\mathcal{S}}$ denotes the multiplication map from $\mathcal{S} \otimes \mathcal{S}$ to $\mathcal{S}$ and id_2 *denotes the identity map on $C(X) \otimes C(X)$, then $\alpha^{(2)}$ leaves the diagonal algebra $C(D_{X\times X})$ invariant.*

Proof Let $X = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and denote by δ_i the characteristic function of the point i. Let $\alpha(\delta_i) = \sum_j \delta_j \otimes q_{ji}$ where $\{q_{ij} : i, j = 1 \ldots n\}$ are the images of the canonical generators of the quantum permutation group as above. Then τ-preservation of α follows from the properties of the generators of the quantum permutation group, which in particular imply that $\sum_j q_{ij} = 1 = \sum_i q_{ij}$.

Using the fact that q_{ji} and q_{ki} are orthogonal for $i, j, k \in \{1, \ldots, n\}$, $j \neq k$, we obtain

$$\alpha^{(2)}(\delta_i \otimes \delta_i) = \sum_{j,k} \delta_j \otimes \delta_k \otimes q_{ji}q_{ki} = \sum_j \delta_j \otimes \delta_j \otimes q_{ji},$$

from which the invariance of the diagonal under $\alpha^{(2)}$ is immediate.

The other statements follow easily. □

Corollary 1.3.7 *We observe that since $\widetilde{\alpha^{(2)}}$ is a unitary, it leaves the τ-orthogonal complement of $C(D_{X\times X})$ in $l^2(X \times X)$, i.e., the space of functions on the set $Y = \{(x, y) : x, y \in X, x \neq y\}$ invariant as well.*

It is easy to note that if M_f denotes the multiplication operator on $l^2(X)$ for a function f on a finite set X, then

$$\alpha(f) = \widetilde{\alpha}(M_f \otimes 1)\widetilde{\alpha}^{-1},\ \ \alpha^{(2)} = \widetilde{\alpha}_{(13)}\widetilde{\alpha}_{(23)}, \tag{1.3.1}$$

where we have used the leg numbering notations explained before.

We also observe the following:

Lemma 1.3.8 *Let α be a coaction of a CQG $\mathcal{S}$ on $C(X)$ where X is a finite set. For all functions f on X, we have*

$$\widetilde{\alpha}^{-1}(M_f \otimes 1)\widetilde{\alpha} = (\mathrm{id} \otimes \kappa)\alpha(f),$$

where κ denotes the antipode on $\mathcal{S}$.

Proof For g in $l^2(X)$ and q in $\mathcal{S}$, using the commutativity of $C(X)$, we have

$$\begin{aligned}
&\widetilde{\alpha}^{-1}(M_f \otimes 1)\widetilde{\alpha}(g \otimes q)\\
&= \widetilde{\alpha}^{-1}(g_{(0)}f \otimes g_{(1)}q) = g_{(0)(0)}f_{(0)} \otimes \kappa(g_{(0)(1)}f_{(1)})g_{(1)}q\\
&= f_{(0)}g_{(0)(0)} \otimes \kappa(f_{(1)})\kappa(g_{(0)(1)})g_{(1)}q = f_{(0)}g_{(0)} \otimes \kappa(f_{(1)})\kappa(g_{(1)(1)})g_{(1)(2)}q\\
&= f_{(0)}g_{(0)} \otimes \kappa(f_{(1)})\epsilon(g_{(1)})q = f_{(0)}g \otimes \kappa(f_{(1)})q\\
&= (\mathrm{id} \otimes \kappa)\alpha(f)(g \otimes q).
\end{aligned}$$

Wang also identified the universal object in the category of all CQG's having C^* coactions on $M_n(\mathbb{C})$ (with the obvious morphisms) such that the functional $\frac{1}{n}\mathrm{Tr}$ is kept invariant by the coaction. However, no universal object exists if the invariance of the functional is not assumed. The precise statement is contained in the following theorem.

Before that, we recall that $M_n(\mathbb{C}) = C^*\{e_{ij} : e_{ij}e_{kl} = \delta_{jk}e_{il},\ e_{ij}^* = e_{ji},\ \sum_{r=1}^n e_{rr} = 1, i, j, k, l = 1, 2, \ldots n\}$.

Proposition 1.3.9 *Let $A_{\mathrm{aut}}(M_n(\mathbb{C}), \frac{1}{\mathrm{n}}\mathrm{Tr})$ be the C^* algebra with generators a_{ij}^{kl} and the following defining relations:*

$$\sum_{v=1}^{n} a_{ij}^{kv}a_{rs}^{vl} = \delta_{jr}a_{is}^{kl},\ i, j, k, l, r, s = 1, 2, \ldots, n,$$

$$\sum_{v=1}^{n} a_{lv}^{sr}a_{vk}^{ji} = \delta_{jr}a_{lk}^{si}, i, j, k, l, r, s = 1, 2, \ldots, n,$$

$$a_{ij}^{kl*} = a_{ji}^{lk},\ i, j, k, l = 1, 2, \ldots, n,$$

$$\sum_{r=1}^{n} a_{rr}^{kl} = \delta_{kl},\ k, l = 1, 2, \ldots, n,$$

$$\sum_{r=1}^{n} a_{kl}^{rr} = \delta_{kl},\ k, l = 1, \ldots, n.$$

Then,

(1) $A_{\mathrm{aut}}(M_n(\mathbb{C}), \frac{1}{\mathrm{n}}\mathrm{Tr})$ is a CQG with coproduct Δ defined by $\Delta(a_{ij}^{kl}) = \sum_{r,s=1}^n a_{rs}^{kl} \otimes a_{ij}^{rs},\ i, j, k, l = 1, 2, \ldots, n$.

(2) $A_{\mathrm{aut}}(M_n(\mathbb{C}), \frac{1}{\mathrm{n}}\mathrm{Tr})$ has a C^* coaction α_1 on $M_n(\mathbb{C})$ given by $\alpha_1(e_{ij}) = \sum_{k,l=1}^{n} e_{kl} \otimes a_{ij}^{kl}$, $i, j = 1, 2, \ldots, n$. Moreover, $A_{\mathrm{aut}}(M_n(\mathbb{C}), \frac{1}{\mathrm{n}}\mathrm{Tr})$ is the universal object in the category of all CQG s having C^* coaction on $M_n(\mathbb{C})$ such that the functional $\frac{1}{n}\mathrm{Tr}$ is kept invariant under the coaction.

(3) There does not exist any universal object in the category of all CQG s having C^* coaction on $M_n(\mathbb{C})$.

Proposition 1.3.10 *Since any faithful state on a finite dimensional C^* algebra $\mathcal{A}$ is of the form $\mathrm{Tr}(Rx)$ for some operator R, it follows from Theorem 6.1, (2) of [35] that the universal CQG acting on any finite dimensional C^* algebra preserving a faithful state ϕ exists which will be denoted by $A_{\mathrm{aut}}(\mathcal{A}, \phi)$.*

1.3.2 Free and Half-Liberated Quantum Groups

Let us now recall the universal quantum groups from [25, 39, 40] and the references therein. For an $n \times n$ positive invertible matrix $Q = (Q_{ij})$. let $A_{u,n}(Q)$ be the compact quantum group defined and studied in [35, 40], which is the universal C^*-algebra generated by $\{u_{kj}^{Q}, k, j = 1, \ldots, n\}$ such that $u := ((u_{kj}^{Q}))$ satisfies

$$uu^* = I_n = u^*u, \quad u'Q\overline{u}Q^{-1} = I_n = Q\overline{u}Q^{-1}u'. \tag{1.3.2}$$

Here $u' = ((u_{ji}))$ and $\overline{u} = ((u_{ij}^*))$. The coproduct, say $\tilde{\Delta}$, is given by:

$$\tilde{\Delta}(u_{ij}) = \sum_{k=1}^{n} u_{ik}^{Q} \otimes u_{kj}^{Q}.$$

We refer the reader to [40] for a detailed discussion on the structure and classification of them.

Remark 1.3.11 It was proved in [25] that in the case where $Q = I$, $\kappa(u_{ij}^{I}) = u_{ji}^{I*}$ and hence $\kappa^2 = \mathrm{id}$ holds for $A_{u,n}(I)$.

Let us also observe the following fact about the Wang algebras, which will be needed in Chap. 3.

Lemma 1.3.12 *Let $\{e_i\}_i$ be the standard basis of $\mathbb{C}^n$. Let $\mathcal{S}$ be a unital C^*-algebra generated by $\{s_{ij} : 1 \leq i, j \leq n\}$ such that there exists a linear map $V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathcal{S}$ given by $V(e_i) = \sum_{k=1}^{n} e_k \otimes s_{ki}$ satisfying the following conditions:*

(1) $\widetilde{V} \in M_n(\mathbb{C}) \otimes \mathcal{S} = \mathcal{L}(\mathbb{C}^n \otimes \mathcal{S})$ defined by $\widetilde{V}(e_i \otimes s) = \sum_k e_k \otimes s_{ki}s$ is a unitary,

(2) $(\tau \otimes \mathrm{id})(\widetilde{V}(x \otimes 1)(\widetilde{V})^*) = \tau(x)1_{\mathcal{S}}$,

(3) $(\tau' \otimes \mathrm{id})((\widetilde{V})^*(x \otimes \mathrm{id})\widetilde{V}) = \tau'(x)1_{\mathcal{S}}$,

for all $x \in M_n(\mathbb{C})$, and where $\tau(x) = \mathrm{Tr}(Q^T x)$, and $\tau'(x) = \mathrm{Tr}((Q^{-1})^T x)$.

Then, there is a surjective unital $*$-homomorphism $\pi : A_{u,n}(Q) \to \mathcal{S}$ such that $\pi(u^Q_{ij}) = s_{ij}$.

Proof For the proof, we refer to Proposition 3.4 of [41] which is a reformulation of Proposition 3.1 of [42] in the context of quantum families. □

Remark 1.3.13 It has been noted in Remark 3.3 of [41] that when $\mathcal{S}$ is a CQG and V a unitary corepresentation of $\mathcal{S}$ on $\mathbb{C}^n$, each of the conditions (2) and (3) implies the other.

We will sometimes denote the quantum group $A_u, n(I)$ simply by the symbol $A_u(n)$. In the literature, it is sometimes denoted by the symbol U_n^+. It is easy to see that the quotient C^* algebra of $A_u(n)$ by the commutator ideal is the space of continuous functions on the compact group $U(n)$. It is due to this reason that $A_u(n)$ is also called the free unitary group.

The free version of the orthogonal group is given by the following:

Definition 1.3.14 The quantum subgroup of $A_u(n)$ defined by the ideal generated by the relation $u_{ij} = u^*_{ij}$ for all $i, j = 1, 2, \ldots, n$ is called the free orthogonal quantum group and denoted by $A_o(n)$.

$A_o(n)$ is also denoted by the symbol $O^+(n)$. We refer to [43, 44] for the corepresentation theory of the quantum groups $A_o(n)$ and $A_u(n)$ respectively.

The orthogonal half-liberated quantum group $A^*_o(n)$ was discovered by Banica and Speicher in [45]. Its corepresentation theory was studied in [46].

Definition 1.3.15 The orthogonal half-liberated quantum group $A^*_o(n)$ is the quantum subgroup of the free orthogonal quantum group $A_o(n)$ defined by the ideal generated by the relation $ab^*c = cb^*a$, $\forall\ a, b, c \in \{u_{ij},\ i, j = 1, \ldots, n\}$ where u'_{ij}s are as in the definition of $A_o(n)$.

There is more than one way to half-liberate the free unitary group. The following version will be of use to us later on.

Definition 1.3.16 [47] $A^*_u(n)$ is the quantum subgroup of $A_u(n)$ defined by the ideal generated by the relations

$$ab^*c = cb^*a\ , \qquad \forall\, a, b, c \in \{u_{ij},\ i, j = 1, \ldots, n\}\ . \tag{1.3.3}$$

There are two other possible ways to "half-liberate" the free unitary group. Instead of $ab^*c = cb^*a$ (equivalent to $a^*bc^* = c^*ba^*$), one can consider respectively the relation $a^*bc = cba^*$ (equivalent to $abc^* = c^*ba$ and to the adjoints $ab^*c^* = c^*b^*a$ and $a^*b^*c = cb^*a^*$) or $abc = cba$ (equivalent to $a^*b^*c^* = c^*b^*a^*$) for any triple $a, b, c \in \{u_{ij},\ i, j = 1, \ldots, n\}$. In fact, the half-liberated quantum group obtained from the relation $abc = cba$ was considered in [48] and was denoted by $A^{**}_u(n)$.

The analogue of projective unitary groups was introduced in [44] (see also Sect. 3 of [46]).

Definition 1.3.17 We denote by $PA_u(n)$ the C^*-subalgebra of $A_u(n)$ generated by $\{(u_{ij})^* u_{kl} : i, j, k, l = 1, \ldots, n\}$. This is a CQG with the coproduct induced from $A_u(n)$.

The projective version of any quantum subgroup of $A_u(n)$ can be defined similarly.

The following result connects the projective unitary group with the quantum automorphism group of $M_n(\mathbb{C})$.

Proposition 1.3.18 *([44, 46]) We have* $PA_u(n) \simeq A_{\mathrm{aut}}(M_n(\mathbb{C}), \frac{1}{n}\,\mathrm{Tr})$.

Like $A_o^*(n)$, the projective version of $A_u^*(n)$ is also commutative.

Proposition 1.3.19 *([47]) The CQG* $PA_u^*(n)$ *is isomorphic to* $C(PU(n))$.

For combinatorial and free probabilistic aspects of many of the above quantum groups, we refer to the papers [45, 49–52] and the references therein. The quantum permutation group $A_s(n)$ as well as $A_o(n)$ and $A_o^*(n)$ are examples of orthogonal easy quantum groups, the theory of which first appeared in [45]. Subsequently, the easy quantum groups and its classification program gained a lot of attention, for which we refer to [49, 53–57] and the references therein. More recently, the unitary analogues of orthogonal easy quantum groups and their classifications are being studied. For this, we refer to [58] and the references therein.

1.3.3 The Coaction of $SO_\mu(3)$ on the Podles' Spheres

We have already defined the CQG $SO_\mu(3)$ in Sect. 1.2.6 and discussed its realization as a Woronowicz subalgebra of $SU_\mu(2)$. In this subsection, we will define the Podles' sphere discovered in [59] and discuss the coaction of $SO_\mu(3)$ on it. In fact, we will start with two equivalent descriptions of the Podles' spheres.

The original definition by Podles'

Let $c \in \mathbb{R}$. The Podles' sphere $S^2_{\mu,c}$ is the universal C^* algebra generated by elements e_{-1}, e_0, e_1 such that:

$$e_i^* = e_{-i}, \ i = -1, 0, 1,$$

$$(1+\mu^2)(e_{-1}e_1 + \mu^{-2}e_1e_{-1}) + {e_0}^2 = ((1+\mu^2)^2\mu^{-2}c + 1)1,$$

$$e_0e_{-1} - \mu^2 e_{-1}e_0 = (1-\mu^2)e_{-1},$$

$$(1+\mu^2)(e_{-1}e_1 - e_1e_{-1}) + (1-\mu^2){e_0}^2 = (1-\mu^2)e_0,$$

$$e_1e_0 - \mu^2 e_0e_1 = (1-\mu^2)e_1.$$

Let

$$B = e_1,\ A = (1+\mu^2)^{-1}(1-e_0). \tag{1.3.4}$$

Then we have an alternate description of the Podles' spheres, that is, the universal C^* algebra generated by elements A and B satisfying the relations:

$$A^* = A,\ AB = \mu^{-2}BA,$$

$$B^*B = A - A^2 + cI,\ BB^* = \mu^2 A - \mu^4 A^2 + cI.$$

The Podles' spheres as in [60]

We take μ in (0, 1) and t in (0, 1]. For n belonging to $\mathbb{R}$, let $[n]_\mu = \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}}$.

Then $S^2_{\mu,c}$ be the universal C^* algebra generated by elements x_{-1}, x_0, x_1 satisfying the relations:

$$x_{-1}(x_0 - t) = \mu^2(x_0 - t)x_{-1}, \tag{1.3.5}$$

$$x_1(x_0 - t) = \mu^{-2}(x_0 - t)x_1, \tag{1.3.6}$$

$$-[2]x_{-1}x_1 + (\mu^2 x_0 + t)(x_0 - t) = [2]^2(1-t), \tag{1.3.7}$$

$$-[2]x_1x_{-1} + (\mu^{-2}x_0 + t)(x_0 - t) = [2]^2(1-t), \tag{1.3.8}$$

where $c = t^{-1} - t, t > 0$.

The involution on $S^2_{\mu,c}$ is given by

$$x^*_{-1} = -\mu^{-1}x_1,\ \ x^*_0 = x_0. \tag{1.3.9}$$

Setting

$$A = \frac{1 - t^{-1}x_0}{1+\mu^2},\ \ B = \mu(1+\mu^2)^{-\frac{1}{2}}t^{-1}x_{-1}, \tag{1.3.10}$$

one obtains [60] that $S^2_{\mu,c}$ is the same as the Podles' sphere defined in [59].

It follows (Sect. 5.1.3 of [61]) that we have the following expressions of $x_{-1},\ x_0,\ x_1$ in terms of $SU_\mu(2)$ elements:

$$x_{-1} = \frac{\mu\alpha^2 + \rho(1+\mu^2)\alpha\gamma - \mu^2\gamma^2}{\mu(1+\mu^2)^{\frac{1}{2}}}, \tag{1.3.11}$$

$$x_0 = -\mu\gamma^*\alpha + \rho(1 - (1+\mu^2)\gamma^*\gamma) - \gamma\alpha^*, \tag{1.3.12}$$

$$x_1 = \frac{\mu^2\gamma^{*2} - \rho\mu(1+\mu^2)\alpha^*\gamma^* - \mu\alpha^{*2}}{(1+\mu^2)^{\frac{1}{2}}}, \tag{1.3.13}$$

where $\rho^2 = \frac{\mu^2 t^2}{(\mu^2+1)^2(1-t)}$.

The coaction of $SO_\mu(3)$

The canonical coaction of $SU_\mu(2)$ on $S^2_{\mu,c}$, that is the coaction obtained by restricting the coproduct of $SU_\mu(2)$ to the subalgebra $S^2_{\mu,c}$, is actually a faithful coaction of $SO_\mu(3)$.

With respect to the ordered basis $\{x_{-1}, \ x_0, \ x_1\}$, this coaction on the subspace generated by them is given by the following $SO_\mu(3)$-valued 3×3-matrix:

$$Z_1 = \begin{pmatrix} L & -\mu(1+\mu^{-2})^{\frac{1}{2}} C & \mu^2 G^* \\ (1+\mu^{-2})^{\frac{1}{2}} M & I - \mu(\mu+\mu^{-1}) N & -\mu(1+\mu^{-2})^{\frac{1}{2}} M^* \\ G & (1+\mu^{-2})^{\frac{1}{2}} C^* & L^* \end{pmatrix}. \qquad (1.3.14)$$

1.4 Dual of a Compact Quantum Group

Let $(\mathcal{S}, \Delta)$ be a compact quantum group and $\mathcal{S}_0$ be its canonical dense Hopf $*$-algebra. We will use the notations introduced after Proposition 1.2.19. We define $\widehat{\mathcal{S}_0}$ to be the space of linear functionals on $\mathcal{S}$ defined by $q \to h(aq)$, for $a \in \mathcal{S}_0$, where h is the Haar state on $\mathcal{S}$. Then $\widehat{\mathcal{S}_0}$ is a subspace of the dual $\mathcal{S}'$.

Note that (see [26]) $\widehat{\mathcal{S}_0}$ is a $*$-subalgebra of $\mathcal{S}'$ and as a $*$-algebra it is the algebraic direct sum of matrix algebras of the form $\oplus_{\pi \in Rep(\mathcal{S})} M_{d_\pi}$ where M_{d_π} is the full matrix algebra of size $d_\pi \times d_\pi$ and the matrix unit e^π_{pq} identified with ρ^π_{pq} is given by $|e^\pi_p >< e^\pi_q|$, where $\{e^\pi_1, \ldots, e^\pi_{d_\pi}\}$ is the standard orthonormal basis. Clearly we get a nondegenerate pairing between $\widehat{\mathcal{S}_0} = \oplus_{\pi \in Rep(\mathcal{S})} M_{d_\pi}$ and $\mathcal{S}_0$ given by $< q^\pi_{ij}, e^{\pi'}_{ps} >= \delta_{ip}\delta_{js}\delta_{\pi\pi'}$. In fact it can be shown that $\widehat{\mathcal{S}_0}$ is a multiplier Hopf $*$-algebra in the sense of [62]. It has a unique C^* norm and by taking the completion we get a C^* algebra. It is called the dual discrete quantum group of $\mathcal{S}$ denoted by $\widehat{\mathcal{S}}$. It has a coassociative $*$-homomorphism $\widehat{\Delta} : \widehat{\mathcal{S}} \to \mathcal{M}(\widehat{\mathcal{S}} \otimes \widehat{\mathcal{S}})$ called the dual coproduct satisfying $(id \otimes \widehat{\Delta})\widehat{\Delta} = (\widehat{\Delta} \otimes id)\widehat{\Delta}$.

Theorem 1.4.1 *Let U be a unitary corepresentation of a CQG $\mathcal{S}$ on a Hilbert space $\mathcal{H}$. Then $\Pi_U : \widehat{\mathcal{S}} \to \mathcal{B}(\mathcal{H})$ defined by $\Pi_U(\omega)(\xi) := (\mathrm{id} \otimes \omega)U(\xi)$ is a non degenerate $*$-homomorphism and hence extends as a $*$-homomorphism from $\mathcal{M}(\widehat{\mathcal{S}})$ to $\mathcal{B}(\mathcal{H})$. Conversely, any nondegenerate $*$-homomorphism from $\widehat{\mathcal{S}}$ to $\mathcal{H}$ is of the form Π_U for some unitary corepresentation U of $\mathcal{S}$ on $\mathcal{H}$.*

Proof Consider the spectral decomposition $\mathcal{H} = \oplus_{\pi \in \mathcal{I}, 1 \le i \le m_\pi} \mathcal{H}^\pi_i$, $U|_{\mathcal{H}^\pi_i}$, $i = 1, \ldots, m_\pi$ is equivalent to the irreducible corepresentation of type π. Moreover, fix an orthonormal basis e^π_{ij}, $j = 1, \ldots, d_\pi$, $i = 1, \ldots, m_\pi$ for $\mathcal{H}^\pi_i$ such that

$$U(e^\pi_{ij}) = \sum_k e^\pi_{ik} \otimes u^\pi_{kj},$$

for all $\pi \in \mathrm{Rep}(\mathcal{S})$. Now for a fixed $\pi \in Rep(\mathcal{S})$, $p, r = 1, \ldots, d_\pi$ observe that $\Pi_U(\rho^\pi_{pr})(\xi) = 0$ for all $\xi \in \mathcal{H}^{\pi'}_i$ such that $\pi \neq \pi'$. Also $\Pi_U(\rho^\pi_{pr})(e^\pi_{ij}) = \delta_{jr} e^\pi_{ip}$, i.e., $\Pi_U(\rho^\pi_{pr})|_{\mathcal{H}^\pi_i}$ is nothing but the rank one operator $|e^\pi_{ip} >< e^\pi_{ir}|$. This proves that $\Pi_U(\omega)$ is bounded for $\omega \in \widehat{\mathcal{S}_0}$, and moreover identifying $\widehat{\mathcal{S}_0}$ with the direct sum of matrix algebras $\oplus_{\pi \in Rep(\mathcal{S})} M_{d_\pi}$, we see that Π_U is nothing but the map which sends $X \in M_{d_\pi}$ to $X \otimes 1_{\mathbb{C}^{m_\pi}}$ in $\mathcal{B}(\mathcal{H})$. This proves that Π_U extends to a nondegenerate $*$-homomorphism.

Now we prove the converse direction. Given a nondegenerate $*$-homomorphism $\rho : \widehat{S} \to \mathcal{B}(\mathcal{H})$ and π in $\mathrm{Rep}(\mathcal{S})$, let P_π denote the projection in $\mathcal{H}$ given by $P_\pi := \rho(\sum_{i=1}^{d_\pi} |e^\pi_i >< e^\pi_i|)$. Clearly, P_π's are orthogonal for different π's and $\sum_{\pi \in \mathrm{Rep}(\mathcal{S})} P_\pi = I_{\mathcal{B}(\mathcal{H})}$. Now, the restriction of ρ to the direct summand $M_{d_\pi} \subseteq \widehat{\mathcal{S}}$ is a $*$-homomorphism of the full matrix algebra M_{d_π}. Hence, we must have an orthogonal decomposition of $P_\pi \mathcal{H}$ into m_π many copies of $\mathbb{C}^{d_\pi}$ for some non negative integers m_π such that $m_\pi d_\pi = \dim(\mathcal{H}_\pi)$ and $\rho|_{M_{d_\pi}}$ is unitarily equivalent to m_π copies of $\mathrm{id}_{M_{d_\pi}}$. It is now easy to define a corepresentation V of $\mathcal{H}$ as a direct sum of m_π copies of the irreducible corepresentation π, for π in $\mathrm{Rep}(\mathcal{S})$, such that $\Pi_V = \rho$. □

There is an abstract definition of discrete quantum groups due to van Daele [17] which is given below:

Definition 1.4.2 A discrete quantum group is a pair $(\mathcal{C}, \hat{\Delta})$ where

(i) $\mathcal{C}$ is a C^* algebra isomorphic with the C^*-direct sum of finite dimensional matrix algebras, say $\mathcal{C} \cong \oplus_{\alpha \in I} M_{n_\alpha}(\mathbb{C})$, where I is some index set and n_α are positive integers,

(ii) $\hat{\Delta}$ is a coassociative nondegenerate $*$-homomorphism from $\mathcal{C}$ to $\mathcal{M}(\mathcal{C} \otimes \mathcal{C})$, and

(iii) The linear maps T_1, T_2 defined by $T_1(a \otimes b) = \hat{\Delta}(a)(1 \otimes b)$, $T_2(a \otimes b) = \hat{\Delta}(a)(b \otimes 1)$, $a, b \in \mathcal{C}_0$ (where $\mathcal{C}_0$ denotes the dense $*$-subalgebra obtained by taking the algebraic direct sum of M_{n_α}, $\alpha \in I$) map $\mathcal{C}_0 \otimes_{\mathrm{alg}} \mathcal{C}_0$ bijectively onto itself.

It is not difficult to see that the duals of compact quantum groups are discrete quantum groups. The converse is also proved by van Daele, thus establishing a one-to-one correspondence between the abstract discrete quantum groups defined above and the duals of compact quantum groups. For more details on discrete quantum groups, we refer to [62, 63].

1.5 Coaction on von Neumann Algebras by Conjugation of Unitary Corepresentation

We now discuss an analogue of 'coaction' (as in [34]) in the context of von Neumann algebra implemented by a unitary corepresentation of the CQG. Let $\mathcal{Q}$ be a CQG, with the Haar state h and the canonical dense Hopf $*$-algebra $\mathcal{Q}_0$. Moreover, let $\mathcal{Q}^u$ and $\mathcal{Q}_r$ be the corresponding universal and reduced versions of $\mathcal{Q}$ as discussed in

Sect. 1.2.2. We will use the various definitions and notations of Sect. 1.2.2, especially those related with the spectral decomposition of unitary corepresentations.

Given a unitary corepresentation V of a CQG $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$, often we consider the $*$ homomorphism ad_V on $\mathcal{B}(\mathcal{H})$ or on some suitable von Neumann subalgebra $\mathcal{M}$ of it. We say ad_V leaves $\mathcal{M}$ invariant if $(id \otimes \phi)\mathrm{ad}_V(\mathcal{M}) \subset \mathcal{M}$ for every state ϕ of $\mathcal{Q}$. We define $\mathcal{M}^\pi = P_\pi(\mathcal{M})$, where $P_\pi = (id \otimes \rho^\pi) \circ \mathrm{ad}_V : \mathcal{M} \to \mathcal{M}$ is the spectral projection corresponding to the irreducible corepresentation π. Clearly each P_π is SOT continuous. We define $\mathcal{M}_0 := Sp\{\mathcal{M}^\pi; \pi \in Rep(\mathcal{Q})\}$, which is called the von Neumann algebraic spectral subalgebra. Then we have the following, in analogy with the spectral subalgebra corresponding to a C^* coaction:

Proposition 1.5.1 *$\mathcal{M}_0$ is dense in $\mathcal{M}$ in any of the natural locally convex topologies of $\mathcal{M}$, i.e., $\mathcal{M}_0{}'' = \mathcal{M}$.*

Proof This result must be quite well known and available in the literature but we could not find it written in this form, so we give a very brief sketch. The proof is basically almost verbatim adaptation of some arguments in [16, 29]. First, observe that the spectral algebra $\mathcal{M}_0$ remains unchanged if we replace $\mathcal{Q}$ by the reduced quantum group $\mathcal{Q}_r$ which has the same irreducible corepresentations and dense Hopf-$*$ algebra $\mathcal{Q}_0$. This means we can assume without loss of generality that the Haar state is faithful. The injective normal map $\beta := \mathrm{ad}_V$ restricted to $\mathcal{M}$ can be thought of as a coaction of the quantum group (in the von Neumann algebra setting as in [16, 29]) $\mathcal{Q}_r{}''$, where the double commutant is taken in the GNS space of the Haar state and it follows from the results in [29] about the implementability of locally compact quantum group coactions that there is a faithful normal state, say ϕ, on $\mathcal{M}$ such that β is $\mathcal{Q}_r{}''$-invariant, i.e. $(\phi \otimes \mathrm{id})(\beta(a)) = \phi(a)1\ \forall a \in \mathcal{M}$. We can replace $\mathcal{M}$, originally embedded in $\mathcal{B}(\mathcal{H})$, by its isomorphic image (to be denoted by $\mathcal{M}$ again) in $\mathcal{B}(L^2(\phi))$ and as the ultraweak topology is intrinsic to a von Neumann algebra, it will suffice to argue the ultraweak density of $\mathcal{M}_0$ in $\mathcal{M} \subset \mathcal{B}(L^2(\phi))$. To this end, note that Vaes has shown in [29] that $\beta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{Q}_r{}''$ extends to a unitary corepresentation on $L^2(\phi)$, which implies in particular that $\mathcal{M}_0$ is dense in the Hilbert space $L^2(\phi)$. From this the ultraweak density follows by standard arguments very similar to those used in the proof of Proposition 1.5 of [16], applying Takesaki's theorem about existence of conditional expectation. For the sake of completeness let us sketch it briefly. Using the notations of [29] and noting that $\delta = 1$ for a CQG, we get from Proposition 2.4 of [29] that V_ϕ commutes with the positive self adjoint operator $\nabla_\phi \otimes Q$ where ∇_ϕ denotes the modular operator, i.e., generator of the modular automorphism group σ_t^ϕ of the normal state ϕ. Clearly, this implies that $\beta := \mathrm{ad}_{V_\phi}$ satisfies the following:

$$\beta \circ \sigma_t^\phi = \sigma_t^\phi \otimes \tau_{-t},$$

where τ_t is the automorphism group generated by Q^{-1}. Next, as in Proposition 1.5 of [16], consider the ultrastrong $*$ closure $\mathcal{M}_l$ of the subspace spanned by the elements of the form $(id \otimes \omega)(\beta(x))$, $x \in M$, ω is a bounded normal functional on $\mathcal{Q}_r''$. It is enough to prove that $\mathcal{M}_l = \mathcal{M}$, as that will prove the ultrastrong $*$ density (hence also the ultraweak density) of $\mathcal{M}_l$ in $\mathcal{M}$. This is clearly a von Neumann subalgebra as β is coaasociative, and $\sigma_t^{\phi}(\mathrm{id} \otimes \omega)(\beta(x)) = (\mathrm{id} \otimes \omega \circ \tau_t)(\beta(\sigma_t^{\phi}(x)))$. Then by Takesaki's theorem ([1], 10.1) there exists a unique normal faithful conditional expectation E from $\mathcal{M}$ to $\mathcal{M}_l$ satisfying $E(x)P = Pxp$ where P is the orthogonal projection as in [16]. Clearly, the range of P contains elements of the form $(id \otimes \omega)(\beta(x))$. So in particular it contains $\mathcal{M}_0$, which is dense in $L^2(\phi)$. Thus $P = 1$ and $E(x) = x$ proving $\mathcal{M}_l = \mathcal{M}$. □

Lemma 1.5.2 *1.* $\mathrm{ad}_V|_{\mathcal{M}_0}$ *is algebraic, i.e.,* $\mathrm{ad}_V(\mathcal{M}_0) \subset \mathcal{M}_0 \otimes_{alg} \mathcal{Q}_0$.

2. $\mathcal{M}_0$ *is the maximal subspace over which* ad_V *is algebraic, i.e.,* $\mathcal{M}_0 = \{x \in \mathcal{M} | \mathrm{ad}_V(x) \in \mathcal{M} \otimes_{alg} \mathcal{Q}_0\}$.

3. If $\mathcal{M}_1 \subset \mathcal{M}$ *is SOT dense* $*$*-subalgebra such that* ad_V *leaves* $\mathcal{M}_1$ *invariant, then* $Sp\{P_\pi(\mathcal{M}_1) | \pi \in Rep(\mathcal{Q})\}$ *is SOT dense in* $\mathcal{M}_0$.

Proof Part 1. of the Lemma can be proved by verbatim adaptation of the arguments of [34, 64] and hence we omit the proof.

For part 2, we reproduce the arguments of [64] for the sake of completeness. For any $b \in \mathcal{M}$ such that $\mathrm{ad}_V(b) \subset \mathcal{M} \otimes_{alg} \mathcal{Q}_0$, we have

$$\mathrm{ad}_V(b) = \sum_{\pi \in \mathcal{S} \subset Rep(\mathcal{Q})} \sum_{i,j=1}^{d_\pi} b_{ij}^{\pi} \otimes u_{ij}^{\pi},$$

where $\mathcal{S}$ is a finite subset of $Rep(\mathcal{Q})$. Observe that for each $\pi \in \mathcal{S}$, $i, j = 1, \ldots, d_\pi$, $b_{ij}^{\pi} = (id \otimes \rho_{ij}^{\pi}) \circ \mathrm{ad}_V(b) \in \mathcal{M}_0$. As $(\mathrm{ad}_V \otimes id) \circ \mathrm{ad}_V(b) = (id \otimes \Delta) \circ \mathrm{ad}_V(b)$, we have

$$\sum_{\pi \in \mathcal{S} \subset Rep(\mathcal{Q})} \sum_{i,j=1}^{d_\pi} \mathrm{ad}_V(b_{ij}^{\pi}) \otimes u_{ij}^{\pi} = \sum_{\pi \in \mathcal{S} \subset Rep(\mathcal{Q})} \sum_{i,j,s=1}^{d_\pi} b_{ij}^{\pi} \otimes u_{is}^{\pi} \otimes q_{sj}^{\pi}.$$

Applying $(id \otimes id \otimes \rho_{kl}^{\pi})$ on both sides, we get

$$\mathrm{ad}_V(b_{kl}^{\pi}) = \sum_{i=1}^{d_\pi} b_{il}^{\pi} \otimes u_{ik}^{\pi} \tag{1.5.1}$$

Now if we take $b^{'} = \sum_{\pi \in \mathcal{S}} \sum_{i=1}^{d_\pi} b_{ii}^{\pi}$, we get by (4),

$$\mathrm{ad}_V(b^{'}) = \sum_{\pi \in \mathcal{S}} \sum_{k=1}^{d_\pi} b_{ki}^{\pi} \otimes u_{ki}^{\pi}.$$

So $\mathrm{ad}_V(b) = \mathrm{ad}_V(b^{'})$ and as ad_V is one-one, $b = b^{'} \in \mathcal{M}_0$.

Part 3 follows from part 2 and the SOT continuity of each spectral projection P_π, where π is in $\mathrm{Rep}(\mathcal{Q})$. □

Notations:
We conclude this section on quantum groups by fixing some notations which will be used throughout this book. In particular, given a compact quantum group $(\mathcal{S}, \Delta)$, the dense unital Hopf $*$-subalgebra of $\mathcal{S}$ generated by the matrix elements of the irreducible unitary corepresentations will be denoted by $\mathcal{S}_0$. Moreover, given a coaction $\gamma : \mathcal{B} \to \mathcal{B} \otimes \mathcal{S}$ of the compact quantum group $(\mathcal{S}, \Delta)$ on a unital C^*-algebra $\mathcal{B}$, the dense, unital $*$-subalgebra of $\mathcal{B}$ on which γ is a coaction by the Hopf $*$-algebra $\mathcal{S}_0$ will be denoted by $\mathcal{B}_0$. We shall use the Sweedler type convention of abbreviating $\gamma(b) \in \mathcal{B}_0 \otimes_{\text{alg}} \mathcal{S}_0$ by $b_{(0)} \otimes b_{(1)}$, for b in $\mathcal{B}_0$. However, for the coproduct Δ of a CQG, we will use the notation $\Delta(q) = q_{(1)} \otimes q_{(2)}$.

Moreover, for a linear functional f on $\mathcal{S}$ and an element c in $\mathcal{S}_0$ we recall the 'convolution' maps $f \triangleleft c := (f \otimes \text{id})\Delta(c)$ and $c \triangleright f := (\text{id} \otimes f)\Delta(c)$. We also define convolution of two functionals f and g by $(f \diamond g)(c) = (f \otimes g)(\Delta(c))$.

References

1. Takesaki, M.: Theory of Operator Algebras I. Springer, New York (Encyclopedia of Mathematical Sciences) (2002)
2. Kadison, R.V., Ringrose, J.R.: Fundamentals of the theory of operator algebras, vol. I. Elementary Theory, Graduate Studies in Mathematics, vol. 15. American Mathematical Society, Providence (1997)
3. Kadison, R.V., Ringrose, J.R.: Fundamentals of the theory of operator algebras, vol. II. Advanced Theory, Graduate Studies in Mathematics, vol. 16. American Mathematical Society, Providence (1997)
4. Pedersen, G.K.: C^*-algebras and their automorphism groups, London Mathematical Society Monographs, vol. 14. Academic Press, New York
5. Lance, E.C.: Hilbert C^* modules: a toolkit for operator algebraists. London Mathematical Society Lecture Note Series, vol. 210. Cambridge University Press, Cambridge (1995)
6. Goswami, D.: Quantum group of isometries in classical and non commutative geometry. Commun. Math. Phys. **285**(1), 141–160 (2009)
7. Rudin, W.: Functional Analysis. Tata-McGraw-Hill, New Delhi (1974)
8. Woronowicz, S.L.: Compact quantum groups, pp. 845–884. In: Connes, A. et al. (eds.): Syme'tries quantiques (Quantum Symmetries). Les Houches. Elsevier, Amsterdam (1995)
9. Woronowicz, S.L.: Compact matrix pseudogroups, Commun. Math. Phys. **111**(4), 613–665 (1987)
10. Sweedler, M.E.: Hopf Algebras. Benjamin, W.A (1969)
11. Klimyk, A., Schmudgen, K.: Quantum Groups and their Representations. Springer, New York (1998)
12. Kassel, C.: Quantum groups. Graduate Texts in Mathematics, vol. 155. Springer, New York (1995)
13. Chari, V., Pressley, A.: A Guide to Quantum Groups. Cambridge University Press, Cambridge (1994)
14. Masuda, T., Nakagami, Y., Woronowicz, S.L.: A C^*-algebraic framework for quantum groups. Int. J. Math. **14**(9), 903–1001 (2003)
15. Reshetikhin, N.Y., Takhtajan, L.A., Faddeev, L.D.: Quantization of Lie groups and Lie algebras, Algebra i analiz 1, **178** (1989) (Russian) (English translation in Leningrad Math. J. **1**(1), 193–225 (1990))
16. Kustermans, J., Vaes, S.: Locally compact quantum groups. Ann. Sci. Ecole Norm. Sup. **33**(4, 6), 837–934 (2000)

17. Van Daele, A.: Discrete quantum groups. J. Algebra **180**, 431–444 (1996)
18. Majid, S.: Foundations of Quantum Group Theory. Cambridge University Press, Cambridge (1995)
19. Mahanta, S., Mathai, V.: Operator algebra quantum homogeneous spaces of universal gauge groups. Lett. Math. Phys. **97**(3), 263–277 (2011)
20. Korogodski, L.I., Soibelman, Ya.S.: Algebras of functions on quantum groups I, Math. Surv. Monographs AMS **56** (1998)
21. Woronowicz, S.L.: Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups. Inventiones Mathematicae **93**(1), 35–76 (1988)
22. Timmermann, T.: An invitation to quantum groups and duality. From Hopf algebras to multiplicative unitaries and beyond. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zurich (2008)
23. Neshveyev, S., Tuset, L.: Compact Quantum Groups and their Representation Categories, Cours Spe'cialise's [Specialized Courses], 20. Socie'te' Mathe'matique de France, Paris (2013)
24. Van Daele, A.: The Haar measure on a compact quantum group. Proc. Am. Math. Soc. **123**, 3125–3128 (1995)
25. Wang, S.: Free products of compact quantum groups. Commun. Math. Phys. **167**(3), 671–692 (1995)
26. Maes, A., Van Daele, A.: Notes on compact quantum groups, Nieuw Arch. Wisk. **16**(4) (1998) ((1–2), 73–112)
27. Rieffel, M., Van Daele, A.: A bounded operator approach to Tomita-Takesaki Theory. Pacific J. Math. **69**(1), 187–221 (1977)
28. Kustermans, J.: Locally compact quantum groups in the universal setting. Int. J. Math. **12**(3), 289–338 (2001)
29. Vaes, S.: The unitary implementation of a locally compact quantum group action. J. Funct. Anal. **180**(2), 426–480 (2001)
30. Enock, M., Schwartz, J.-M.: Kac algebras and Duality of Locally Compact Groups. Springer, Berlin (1991)
31. Wang, S.: Tensor products and crossed products of compact quantum groups. Proc. London Math. Soc. **71**(3), 695–720 (1995)
32. Bhowmick, J., Goswami, D., Skalski, A.: Quantum isometry groups of 0-dimensional manifolds. Trans. Am. Math. Soc. **363**, 901–921 (2011)
33. Schmudgen, K., Wagner, E.: Dirac operators and a twisted cyclic cocycle on the standard Podles' quantum sphere. arXiv:0305051v2
34. Podles', P.: Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups. Commun. Math. Phys. **170**, 1–20 (1995)
35. Wang, S.: Quantum symmetry groups of finite spaces. Commun. Math. Phys. **195**, 195–211 (1998)
36. Boca, F.: Ergodic actions of compact matrix pseudogroups on C^*-algebras, recent advances in operator algebras (Orlans, 1992). Astrisque No. **232**, 93–109 (1995)
37. Banica, T., Bichon, J., Collins, B.: Quantum permutation groups: a survey. Noncommutative harmonic analysis with applications to probability. Banach Center Publ. (Polish Acad. Sci. Inst. Math., Warsaw) **78**, 13–34 (2007)
38. Banica, T., Moroianu, S.: On the structure of quantum permutation groups. Proc. Amer. Math. Soc. **135**(1), 21–29 (2007)
39. Van Daele, A., Wang, S.: Universal quantum groups. Int. J. Math. **7**(2), 255–263 (1996)
40. Wang, S.: Structure and isomorphism classification of compact quantum groups $A_u(Q)$ and $B_u(Q)$. J. Oper. Theory **48**, 573–583 (2002)
41. Bhowmick, J., D'Andrea, F., Das, B., Dabrowski, L.: Quantum gauge symmetries in noncommutative geometry. J. Noncommutat. Geoemtry **8**(2), 433–471 (2014)
42. Wang, S.: Ergodic actions of universal quantum groups on operator algebras. Commun. Math. Phys. **203**(2), 481–498 (1999)
43. Banica, T.: Theorie des representations du groupe quantique compact libre O(n), C.R. Acad. Sci. Paris Ser. I Math. **322**(3), 241–244 (1996)

44. Banica, T.: Le groupe quantique compact libre $U(n)$, Commun. Math. Phys. **190**, 143–172 (1997)
45. Banica, T., Speicher, R.: Liberation of orthogonal Lie groups. Adv. Math. **222**(4), 1461–1501 (2009)
46. Banica, T., Vergnioux, R.: Invariants of the half-liberated orthogonal group. Ann. Inst. Fourier (Grenoble) **60**(6), 2137–2164 (2010)
47. Bhowmick, J., D'Andrea, F., Dabrowski, L.: Quantum Isometries of the finite noncommutative geometry of the Standard Model. Commun. Math. Phys. **307**, 101–131 (2011)
48. Bichon, J., Dubois-Violette, M.: Half-commutative orthogonal Hopf algebras. Pacific J. Math. **263**(1), 13–28 (2013)
49. Banica, T., Curran, S., Speicher, R.: Classification results for easy quantum groups. Pacific J. Math. **247**(1), 1–26 (2010)
50. Banica, T., Curran, S., Speicher, R.: De Finetti theorems for easy quantum groups. Ann. Probab. **40**(1), 401–435 (2012)
51. Banica, T., Curran, S., Speicher, R.: Stochastic aspects of easy quantum groups. Probab. Theory Related Fields **149**(3–4), 435–462 (2011)
52. Banica, T., Bichon, J., Curran, S.: Quantum automorphisms of twisted group algebras and free hypergeometric laws. Proc. Am. Math. Soc. **139**(11), 3961–3971 (2011)
53. Banica, T., Bichon, J., Collins, B., Curran, S.: A maximality result for orthogonal quantum groups. Commun. Algebra **41**(2), 656–665 (2013)
54. Weber, M.: On the classification of easy quantum groups. Adv. Math. **245**, 500–533 (2013)
55. Raum, S.: Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications. Proc. Am. Math. Soc. **140**(9), 3207–3218 (2012)
56. Raum, S., Weber, M.: A Connection between easy quantum groups, varieties of groups and reflection groups. arXiv:1212.4742
57. Raum, S., Weber, M.: The combinatorics of an algebraic class of easy quantum groups, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17**(3), 1450016, 17 pp (2014)
58. Tarrago, P., Weber, M.: Unitary easy quantum groups: the free case and the group case. arXiv:1512.00195
59. Podles', P.: Quantum spheres. Lett. Math. Phys. **14**, 193–202 (1987)
60. Dabrowski, L., D'Andrea, F., Landi, G., Wagner, E.: Dirac operators on all Podles' quantum spheres. J. Noncommun. Geom. **1**, 213–239 (2007)
61. Bhowmick, J.: Quantum isometry groups, Ph.D. thesis (Indian Statistical Institute) (2010). arxiv:0907.0618
62. Van Daele, A.: Multiplier Hopf algebras. Trans. Am. Math. soc. **342**(2) (1994)
63. Effros, E.G., Ruan, Z.: Discrete quantum groups. I. The Haar measure. Int. J. Math. **5**(5), 681–723 (1994)
64. Soltan, P.: On actions of compact quantum groups. Illinois J. Math. **55**(3), 953–962 (2011)

Chapter 2
Classical and Noncommutative Geometry

Abstract We discuss classical Riemannian geometry and its noncommutative geometric counterparts. At first the definition and properties of the Hodge Laplacian and the Dirac operator are given. We also derive the characterizations of isometries (resp. orientation preserving isometries) in terms of the Laplacian (resp. Dirac operator). This is followed by discussion on noncommutative manifolds given by spectral triples, including the definitions of noncommutative space of forms and the Laplacian in this set up. The last section of this chapter deals with the quantum group equivariance in noncommutative geometry where we discuss some natural examples of equivariant spectral triples on the Podles' spheres.

2.1 Classical Riemannian Geometry

In this section, we recall some classical facts regarding classical differential geometry manifolds that will be useful for us.

2.1.1 Forms and Connections

Let M be an n-dimensional compact Riemannian manifold. Let $\chi(M)$ denote the $C^\infty(M)$-module of smooth vector fields on the manifold M. A linear or affine connection ∇ on M is given by an assignment $\chi(M) \ni X \mapsto \nabla_X$, where ∇_X is an $\mathbb{R}$-linear map from $\chi(M)$ to $\chi(M)$ such that $\chi(M) \ni X \mapsto \nabla_X$ is $C^\infty(M)$-linear and $\nabla_X(fY) = f\nabla_X(Y) + X(f)Y$, for all $Y \in \chi(M)$, $f \in C^\infty(M)$. Given a local chart in M and coordinates x_i, the Christoffel symbols of the connection ∇ are the functions Γ^k_{ij} defined by: $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}$. A linear connection is called symmetric or torsionless if $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$ for all $X, Y \in \chi(M)$. It is said to be compatible with the Riemannian metric if $\langle \nabla_X(Y), Z\rangle + \langle Y, \nabla_X(Z)\rangle = X\langle Y, Z\rangle$ for all $X, Y, Z \in \chi(M)$, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product on the tangent bundle. There is a unique linear connection on M [1], which is torsionless and compatible with the metric, called the Levi-Civita connection on M.

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_2

Let $\Omega^k(M)$ $(k = 0, 1, 2, ...n)$ be the space of smooth k-forms. Set $\Omega^k(M) = \{0\}$ for $k > n$. The de-Rham differential d maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$. Let $\Omega \equiv \Omega(M) = \oplus_k \Omega^k(M)$. We will denote the Riemannian volume element by dvol. We recall that the Hilbert space $L^2(M)$ is obtained by completing the space of compactly supported smooth functions on M with respect to the pre-inner product given by $\langle f_1, f_2 \rangle = \int_M \overline{f_1} f_2 dvol$.

In an analogous way, one can construct a canonical Hilbert space of forms. The Riemannian metric $\langle . , . \rangle_m$ (for m in M) on $T_m M$ induces an inner product on the vector space $T_m^* M$ and hence also $\Lambda^k T_m^* M$, which will be again denoted by $\langle . , . \rangle_m$. This gives a natural pre-inner product on the space of compactly supported k-forms by integrating the compactly supported smooth function $m \mapsto \langle \omega(m), \eta(m) \rangle_m$ over M. We will denote the completion of this space by $\mathcal{H}^k(M)$. Let $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$.

Then, one can view $d : \Omega \to \Omega$ as an unbounded, densely defined operator (again denoted by d) on the Hilbert space $\mathcal{H}$ with the domain Ω. It can be verified that it is closable.

2.1.2 The Hodge Laplacian of a Riemannian Manifold

We recall that the Laplacian $\mathcal{L}$ on M is an unbounded densely defined self-adjoint operator $-d^* d$ on the space of zero forms $\mathcal{H}^0(D) = L^2(M, \text{dvol})$ which has the local expression

$$\mathcal{L}(f) = \frac{1}{\sqrt{det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (g^{ij} \sqrt{\det(g)} \frac{\partial}{\partial x_i} f)$$

for f in $C^\infty(M)$ and where $g = ((g_{ij}))$ is the Riemannian metric and $g^{-1} = ((g^{ij}))$.

We begin with a well-known characterization of the isometry group of a (classical) compact Riemannian manifold. Let (M, g) be a compact Riemannian manifold and let $\Omega^1 = \Omega^1(M)$ be the space of smooth one forms, which has a right Hilbert-$C^\infty(M)$-module structure given by the $C^\infty(M)$-valued inner product $<< \cdot, \cdot >>$ defined by

$$\langle \omega, \eta \rangle (m) = \langle \omega(m), \eta(m) \rangle |_m,$$

where $< \cdot, \cdot > |_m$ is the Riemannian metric on the cotangent space $T_m^* M$ at the point $m \in M$. The Riemannian volume form allows us to make Ω^1 a pre-Hilbert space, and we denote its completion by $\mathcal{H}_1$. Let $\mathcal{H}_0 = L^2(M, \text{dvol})$ and consider the de-Rham differential d as an unbounded linear map from $\mathcal{H}_0$ to $\mathcal{H}_1$, with the natural domain $C^\infty(M) \subset \mathcal{H}_0$, and also denote its closure by d. Let $\mathcal{L} := -d^* d$. The following identity can be verified by direct and easy computation using the local coordinates:

$$(\partial \mathcal{L})(\phi, \psi) \equiv \mathcal{L}(\bar{\phi}\psi) - \mathcal{L}(\bar{\phi})\psi - \bar{\phi}\mathcal{L}(\psi) = 2 << d\phi, d\psi >> \quad \text{for } \phi, \psi \in C^\infty(M). \tag{2.1.1}$$

Let us recall a few well-known facts about the Laplacian $\mathcal{L}$, viewed as a negative self-adjoint operator on the Hilbert space $L^2(M, \text{dvol})$. It is known (see [2] and references therein) that $\mathcal{L}$ has compact resolvents and all its eigenvectors belong to $C^\infty(M)$. Moreover, it follows from the Sobolev Embedding Theorem that

$$\bigcap_{n\geq 1} \mathrm{Dom}(\mathcal{L}^n) = C^\infty(M).$$

Let $\{e_{ij}, j = 1, ..., d_i; i = 0, 1, 2, ...\}$ be the set of (normalized) eigenvectors of $\mathcal{L}$, where $e_{ij} \in C^\infty(M)$ is an eigenvector corresponding to the eigenvalue $\lambda_i, 0 = |\lambda_0| < |\lambda_1| < |\lambda_2| < ....$ We have the following:

Lemma 2.1.1 *The complex linear span of* $\{e_{ij}\}$ *is norm-dense in* $C(M)$.

Proof This is a consequence of the asymptotic estimates of eigenvalues λ_i, as well as the uniform bound of the eigenfunctions e_{ij}. For example, it is known ([3], Theorem 1.2) that there exist constants C, C' such that $\|e_{ij}\|_\infty \leq C|\lambda_i|^{\frac{n-1}{4}}$, $d_i \leq C'|\lambda_i|^{\frac{n-1}{2}}$, where n is the dimension of the manifold M. Now, for $f \in C^\infty(M) \subseteq \bigcap_{k\geq 1} \mathrm{Dom}(\mathcal{L}^k)$, we write f as an a priori L^2-convergent series $\sum_{ij} f_{ij} e_{ij}$ ($f_{ij} \in \mathbb{C}$), and observe that $\sum |f_{ij}|^2 |\lambda_i|^{2k} < \infty$ for every $k \geq 1$. Choose and fix sufficiently large k such that $\sum_{i\geq 0} |\lambda_i|^{n-1-2k} < \infty$, which is possible due to the well-known Weyl asymptotics of eigenvalues of $\mathcal{L}$. Now, by the Cauchy–Schwarz inequality and the estimate for d_i, we have

$$\sum_{ij} |f_{ij}| \|e_{ij}\|_\infty \leq C(C')^{\frac{1}{2}} \left(\sum_{ij} |f_{ij}|^2 |\lambda_i|^{2k} \right)^{\frac{1}{2}} \left(\sum_{i\geq 0} |\lambda_i|^{n-1-2k} \right)^{\frac{1}{2}} < \infty.$$

Thus, the series $\sum_{ij} f_{ij} e_{ij}$ converges to f in sup-norm, so $\mathrm{Sp}\{e_{ij}, j = 1, 2, ..., d_i; i = 0, 1, 2, ...\}$ is dense in sup-norm in $C^\infty(M)$, hence in $C(M)$ as well. $\square$

2.1.3 Spin Groups and Spin Manifolds

We begin with the Clifford algebras. Let Q be a quadratic form on an n-dimensional vector space V. Then $Cl(V, Q)$ will denote the universal associative algebra $\mathcal{C}$ equipped with a linear map $i : V \to \mathcal{C}$, such that $i(V)$ generates $\mathcal{C}$ as a unital algebra satisfying $i(V)^2 = Q(V).1$

Let $\beta : V \to Cl(V, Q)$ be defined by $\beta(x) = -i(x)$. Then, $Cl(V, Q) = Cl^0(V, Q) \oplus Cl^1(V, Q)$ where $Cl^0(V, Q) = \{x \in Cl(V, Q) : \beta(x) = x\}$, $Cl^1(V, Q) = \{x \in Cl(V, Q) : \beta(x) = -x\}$.

We will denote by $\mathcal{C}_n$ and $\mathcal{C}_n^{\mathbb{C}}$ the Clifford algebras $Cl(\mathbb{R}^n, -x_1^2 - ... - x_n^2)$ and $Cl(\mathbb{C}^n, z_1^2 + ... + z_n^2)$, respectively.

We will denote the vector space $\mathbb{C}^{2^{[\frac{n}{2}]}}$ by the symbol Δ_n. It follows that $\mathcal{C}_n^{\mathbb{C}} = \mathrm{End}(\Delta_n)$ if n is even and equals $\mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ is n is odd. There is a representation $\mathcal{C}_n^{\mathbb{C}} \to \mathrm{End}(\Delta_n)$ that is the isomorphism with $\mathrm{End}(\Delta_n)$ when n is even and in the odd case, it is the isomorphism with $\mathrm{End}(\Delta_n) \oplus \mathrm{End}(\Delta_n)$ followed by the projection onto the first component. This representation restricts to $\mathcal{C}_n$, to be denoted by κ_n and called the spin representation. This representation is irreducible when n is odd and for even n, it decomposes into two irreducible representations, which decomposes Δ_n into a direct sum of two vector spaces Δ_n^+ and Δ_n^-.

$\mathrm{Pin}(n)$ is defined to be the subgroup of $\mathcal{C}_n$ generated by elements of the form $\{x : \|x\| = 1, x \in \mathbb{R}^n\}$. Spin (n) is the group given by $\mathrm{Pin}(n) \cap \mathcal{C}_n^0$. There exists a continuous group homomorphism from Pin(n) to $O(n)$, which restricts to a 2-covering map $\lambda : \mathrm{Spin(n)} \to SO(n)$.

Let M be an n-dimensional orientable Riemannian manifold. Then we have the oriented orthonormal bundle of frames over M (which is a principal $SO(n)$ bundle) which we will denote by F.

Such a manifold M is said to be a **spin manifold** if there exists a pair (P, Λ) (called a spin structure) where

(1) P is a Spin(n) principal bundle over M.

(2) Λ is a map from P to F such that it is a 2-covering as well as a bundle map over M.

(3) $\Lambda(p.\widehat{g}) = \Lambda(p).g$ where $\lambda(\widehat{g}) = g,\ \widehat{g} \in \mathrm{Spin}(n)$.

Given such a spin structure, we consider the associated bundle $S = P \times_{\mathrm{Spin}(n)} \Delta_n$ called the **'bundle of spinors'**.

2.1.4 Dirac Operators

We follow the notations of the previous subsection. On the space of smooth sections of the bundle of spinors S on a compact Riemannian spin manifold M, one can define an inner product by

$$\langle s_1, s_2 \rangle_S = \int_M \langle s_1(x), s_2(x) \rangle \, dvol(x).$$

The Hilbert space obtained by completing the space of smooth sections with respect to this inner product is denoted by $L^2(S)$ and its members are called the square integrable spinors. The Levi-Civita connection on M induces a canonical connection on S which we will denote by ∇^S.

Definition 2.1.2 The **Dirac operator** on M is the self-adjoint extension of the following operator D defined on the space of smooth sections of S:

$$(Ds)(m) = \sum_{i=1}^{n} \kappa_n(X_i(m))(\nabla^S_{X_i} s)(m),$$

where $(X_1, ...X_n)$ are local orthonormal (with respect to the Riemannian metric) vector fields defined in a neighborhood of m. In this definition, we have viewed

$X_i(m)$ belonging to $T_m(M)$ as an element of the Clifford algebra $Cl_{\mathbb{C}}(T_m M)$, hence $\kappa_n(X_i(m))$ is a map on the fiber of S at m, which is isomorphic with Δ_n. The self-adjoint extension of D is again denoted by the same symbol.

We recall three important facts about the Dirac operator:

Proposition 2.1.3 *(1) $C^\infty(M)$ acts on S by multiplication and this action extends to a representation, say π, of the C^* algebra $C(M)$ on the Hilbert space $L^2(S)$.*

(2) For f in $C^\infty(M)$, $[D, \pi(f)]$ has a bounded extension.

(3) Furthermore, the Dirac operator on a compact manifold has compact resolvents.

As the action of an element f in $C^\infty(M)$ on $L^2(S)$ is by multiplication operator, we will use the symbol M_f in place of $\pi(f)$.

The Dirac operator carries a lot of geometric and topological information. We give two examples.

(a) The Riemannian metric of the manifold is recovered by

$$d(p,q) = \sup_{\phi \in C^\infty(M),\ \|[D, M_\phi]\| \leq 1} |\phi(P) - \phi(q)| . \tag{2.1.2}$$

(b) For a compact manifold, the operator e^{-tD^2} is trace class for all $t > 0$. Then the volume form of the manifold can be recovered by the formula

$$\int_M f dvol = c(n) \lim_{t \to 0} \frac{\mathrm{Tr}(M_f e^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$$

where $\dim M = n,\ c(n)$ *is a constant depending on the dimension.*

2.1.5 Isometry Groups of Classical Manifolds

Let M be a Riemannian manifold of dimension n. Then the collection of all isometries of M has a natural group structure and is denoted by ISO(M). The aim of this subsection is to prepare the necessary background for defining the notion of "quantum isometry"of a noncommutative manifold. Therefore, for a classical Riemannian (resp, spin) manifold, we give characterizations of an isometry (resp, orientation preserving isometry) in terms of the Hodge Laplacian (resp, Dirac operator). moreover, motivated by the work of Woronowicz and Soltan on "quantum families", we give characterizations of classical families of isometries (resp, orientation preserving isometries). We should mention that Proposition 2.1.4 and Theorem 2.1.12 are well known [4, 5], but for the sake of completeness, we give detailed proofs.

The topology on ISO(M) is defined in the following way. Let C and U be, respectively, a compact and open subset of M and let $W(C, U) = \{h \in \mathrm{ISO}(M) : h.C \subseteq U\}$. The compact open topology on ISO(M) is the smallest topology on ISO(M) for which the sets $W(C, U)$ are open. It follows (see [4]) that under this topology, ISO(M) is a closed locally compact topological group. Moreover, if M is compact, ISO(M) is also compact.

Characterization of ISO(M) **for a Riemannian manifold**

We start with the characterization of a single isometry.

Proposition 2.1.4 *A smooth map $\gamma : M \to M$ is a Riemannian isometry if and only if γ commutes with $\mathcal{L}$ in the sense that $\mathcal{L}(f \circ \gamma) = (\mathcal{L}(f)) \circ \gamma$ for all $f \in C^\infty(M)$.*

Proof If γ commutes with $\mathcal{L}$ then from the identity (2.1.1), we get for $m \in M$ and $\phi, \psi \in C^\infty(M)$:

$$
\begin{aligned}
&< d\phi|_{\gamma(m)}, d\psi|_{\gamma(m)} > |_{\gamma(m)} \\
&=<< d\phi, d\psi >> (\gamma(m)) \\
&= \frac{1}{2}(\partial\mathcal{L}(\phi, \psi) \circ \gamma)(m) \\
&= \frac{1}{2}\partial\mathcal{L}(\phi \circ \gamma, \psi \circ \gamma)(m) \\
&=<< d(\phi \circ \gamma), d(\psi \circ \gamma) >> (m) \\
&=< d(\phi \circ \gamma)|_m, d(\psi \circ \gamma)|_m > |_m \\
&=< (d\gamma|_m)^*(d\phi|_{\gamma(m)}), (d\gamma|_m)^*(d\psi|_{\gamma(m)}) > |_m,
\end{aligned}
$$

which proves that $(d\gamma|_m)^* : T^*_{\gamma(m)}M \to T^*_m M$ is an isometry. Thus, γ is a Riemannian isometry.

Conversely, if γ is an isometry, both the maps induced by γ on $\mathcal{H}_0$ and $\mathcal{H}_1$, i.e., $U^0_\gamma : \mathcal{H}_0 \to \mathcal{H}_0$ given by $U^0_\gamma(f) = f \circ \gamma$ and $U^1_\gamma : \mathcal{H}^1 \to \mathcal{H}^1$ given by $U^1_\gamma(f d\phi) = (f \circ \gamma)d(\phi \circ \gamma)$ are unitaries. Moreover, $d \circ U^0_\gamma = U^1_\gamma \circ d$ on $C^\infty(M) \subset \mathcal{H}_0$. From this, it follows that $\mathcal{L} = -d^*d$ commutes with U^0_γ. □

Next, we move on to the characterization of a family of isometries, which will need the following lemma.

Lemma 2.1.5 *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and for $i = 1, 2$, let $\mathcal{L}_i$ be (possibly unbounded) self-adjoint operator on $\mathcal{H}_i$ with compact resolvents, and let $\mathcal{V}_i$ be the linear span of eigenvectors of $\mathcal{L}_i$. Moreover, assume that there is an eigenvalue of $\mathcal{L}_i$ for which the eigenspace is one-dimensional, say spanned by a unit vector ξ_i. Let Ψ be a linear map from $\mathcal{V}_1$ to $\mathcal{V}_2$ such that $\mathcal{L}_2\Psi = \Psi\mathcal{L}_1$ and $\Psi(\xi_1) = \xi_2$. Then we have*

$$\langle \xi_2, \Psi(x)\rangle = \langle \xi_1, x\rangle \ \ \forall x \in \mathcal{V}_1. \tag{2.1.3}$$

Proof By hypothesis on Ψ, it is clear that there is a common eigenvalue, say λ_0, of $\mathcal{L}_1$ and $\mathcal{L}_2$, with the eigenvectors ξ_1 and ξ_2, respectively. Let us write the set of eigenvalues of $\mathcal{L}_i$ as a disjoint union $\{\lambda_0\} \bigcup \Lambda_i$ $(i = 1, 2)$, and let the corresponding orthogonal decomposition of $\mathcal{V}_i$ be given by $\mathcal{V}_i = \mathbb{C}\xi_i \bigoplus_{\lambda \in \Lambda_i} \mathcal{V}_i^\lambda \equiv \mathbb{C}\xi_i \oplus \mathcal{V}_i'$, say, where $\mathcal{V}_i^\lambda$ denotes the eigenspace of $\mathcal{L}_i$ corresponding to the eigenvalue λ. By assumption, Ψ maps $\mathcal{V}_1^\lambda$ to $\mathcal{V}_2^\lambda$ whenever λ is an eigenvalue of $\mathcal{L}_2$, i.e., $\mathcal{V}_2^\lambda \neq \{0\}$, and otherwise it maps $\mathcal{V}_1^\lambda$ into $\{0\}$. Thus, $\Psi(\mathcal{V}_1') \subseteq \mathcal{V}_2'$. Now, (2.1.3) is obviously satisfied for $x = \xi_1$, so it is enough to prove (2.1.3) for all $x \in \mathcal{V}_1'$. But we have $\langle \xi, x \rangle = 0$ for $x \in \mathcal{V}_1'$, and since $\Psi(x) \in \mathcal{V}_2' = \mathcal{V}_2 \bigcap \{\xi_2\}^\perp$, it follows that $\langle \xi_2, \Psi(x) \rangle = 0 = \langle \xi_1, x \rangle$. $\square$

Now let us consider a compact metrizable (i.e., second countable) space Y with a continuous map $\theta : M \times Y \to M$. We abbreviate $\theta(m, y)$ as my and denote by ξ_y the map $M \ni m \mapsto my$. Let $\alpha : C(M) \to C(M) \otimes C(Y) \cong C(M \times Y)$ be the map given by $\alpha(f)(m, y) := f(my)$ for $y \in Y$, $m \in M$ and $f \in C(M)$. For a state ϕ on $C(Y)$, denote by α_ϕ the map $(\mathrm{id} \otimes \phi) \circ \alpha : C(M) \to C(M)$. We shall also denote by $\mathcal{C}$ the subspace of $C(M) \otimes C(Y)$ generated by elements of the form $\alpha(f)(1 \otimes \psi)$, $f \in C(M)$, $\psi \in C(Y)$. Since $C(M)$ and $C(Y)$ are commutative algebras, it is easy to see that $\mathcal{C}$ is a $*$-subalgebra of $C(M) \otimes C(Y)$. Then we have the following

Theorem 2.1.6 *(i) $\mathcal{C}$ is norm-dense in $C(M) \otimes C(Y)$ if and only if for every $y \in Y$, ξ_y is one-to-one.*
(ii) The map ξ_y is C^∞ for every $y \in Y$ if and only if $\alpha_\phi(C^\infty(M)) \subseteq C^\infty(M)$ for all ϕ.
(iii) Under the hypothesis of (ii), each ξ_y is also an isometry if and only if α_ϕ commutes with $(\mathcal{L} - \lambda)^{-1}$ for all state ϕ and all λ in the resolvent of $\mathcal{L}$ (equivalently, α_ϕ commutes with the Laplacian $\mathcal{L}$ on $C^\infty(M)$).

Proof (i) First, assume that ξ_y is one-to-one for all y. By Stone-Weierstrass Theorem, it is enough to show that $\mathcal{C}$ separates points. Take $(m_1, y_1) \neq (m_2, y_2)$ in $M \times Y$. If $y_1 \neq y_2$, we can choose $\psi \in C(Y)$ that separates y_1 and y_2, hence $(1 \otimes \psi) \in \mathcal{C}$ separates (m_1, y_1) and (m_2, y_2). So, we can consider the case when $y_1 = y_2 = y$ (say), but $m_1 \neq m_2$. By injectivity of ξ_y, we have $m_1 y \neq m_2 y$, so there exists $f \in C(M)$ such that $f(m_1 y) \neq f(m_2 y)$, i.e., $\alpha(f)(m_1, y) \neq \alpha(f)(m_2, y)$. This proves the density of $\mathcal{C}$.

For the converse, we argue as in the proof of Proposition 3.3 of [6]. Assume that $\mathcal{C}$ is dense in $C(M) \otimes C(Y)$, and let $y \in Y$, $m_1, m_2 \in M$ such that $m_1 y = m_2 y$. That is, $\alpha(f)(1 \otimes \psi)(m_1, y) = \alpha(f)(1 \otimes \psi)(m_2, y)$ for all $f \in C(M)$, $\psi \in C(Y)$. By the density of $\mathcal{C}$, we get $\chi(m_1, y) = \chi(m_2, y)$ for all $\chi \in C(M \times Y)$, so $(m_1, y) = (m_2, y)$, i.e., $m_1 = m_2$.

(ii) The 'if part' of (ii) follows by considering the states corresponding to point evaluation, i.e., $C(Y) \ni \psi \mapsto \psi(y)$, $y \in Y$. For the converse, we note that an arbitrary state ϕ corresponds to a regular Borel measure μ on Y so that $\phi(h) = \int h d\mu$, and thus, $\alpha_\phi(f)(m) = \int f(my) d\mu(y)$ for $f \in C(M)$. From this, by interchanging differentiation and integration (which is allowed by the Dominated Convergence Theorem, since μ is a finite measure), we can prove that $\alpha_\phi(f)$ is C^∞ whenever f is so.

The assertion (iii) follows from Proposition 2.1.4 in a straightforward way. $\square$

Lemma 2.1.7 *Let Y and α be as in Theorem 2.1.6 and let $\mathcal{A}_0^\infty$ denote the complex linear span of the eigenvectors of $\mathcal{L}$, where $\mathcal{A}^\infty = C^\infty(M)$. Then the following are equivalent.*
(a) For every $y \in Y$, ξ_y is smooth isometric.
(b) For every state ϕ on $C(Y)$, we have $\alpha_\phi(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty$, and $\alpha_\phi \mathcal{L} = \mathcal{L}\alpha_\phi$ on $\mathcal{A}_0^\infty$.

Proof We prove only the nontrivial implication $(b) \Rightarrow (a)$. Assume that α_ϕ leaves $\mathcal{A}_0^\infty$ invariant and commutes with $\mathcal{L}$ on it, for every state ϕ. To prove that α is smooth and isometric, it is enough (see the proof of Theorem 2.1.6) to prove that $\alpha_y(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$ for all $y \in Y$, where $\alpha_y(f) := (\mathrm{id} \otimes \mathrm{ev}_y)(f) = f \circ \xi_y$, ev_y being the evaluation at the point y. Let $M_1, ..., M_k$ be the connected components of the compact manifold M. Thus, the Hilbert space $L^2(M, \mathrm{dvol})$ admits an orthogonal decomposition $\oplus_{i=1}^k L^2(M_i, \mathrm{dvol})$, and the Laplacian $\mathcal{L}$ is of the form $\oplus_i \mathcal{L}_i$, where $\mathcal{L}_i$ denotes the Laplacian on M_i. Since each M_i is connected, we have $\mathrm{Ker}(\mathcal{L}_i) = \mathbb{C}\chi_i$, where χ_i is the constant function on M_i equal to 1. Now, we note that for fixed y and i, the image of M_i under the continuous function ξ_y must be mapped into a component, say M_j. Thus, by applying Lemma 2.1.5 with $\mathcal{H}_1 = L^2(M_i), \mathcal{H}_2 = L^2(M_j)$, $\Psi = \xi_y$ and the L^2-continuity of the map $f \mapsto \alpha_y(f) = f \circ \xi_y$, we have

$$\int_{M_j} \alpha_y(f)(x)\mathrm{dvol}(x) = \int_{M_i} f(x)\mathrm{dvol}(x)$$

for all f in the linear span of eigenvectors of $\mathcal{L}_i$, hence (by density) for all f in $L^2(M_i)$. It follows that $\int_M \alpha_y(f)\mathrm{dvol} = \int_M f\mathrm{dvol}$ for all $f \in L^2(M)$, in particular for all $f \in C(M)$. Since α_y is a $*$-homomorphism on $C(M)$, we have

$$\langle \alpha_y(f), \alpha_y(g) \rangle = \int_M \alpha_y(\overline{f}g)\mathrm{dvol} = \int_M \overline{f}g\mathrm{dvol} = \langle f, g \rangle,$$

for all $f, g \in C(M)$. Thus, α_y extends to an isometry on $L^2(M)$, to be denoted by the same notation, which by our assumption commutes with the self-adjoint operator $\mathcal{L}$ on the core $\mathcal{A}_0^\infty$, and hence α_y commutes with $\mathcal{L}^n$ for all n. In particular, it leaves invariant the domains of each $\mathcal{L}^n$, which implies $\alpha_y(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$. □

Consider the category with objects being the pairs (G, α), where G is a compact metrizable group acting on M by the smooth and isometric action α. If (G_1, α) and (G_2, β) are two objects in this category, $\mathrm{Mor}((G_1, \alpha), (G_2, \beta))$ consists of group homomorphisms π from G_1 to G_2 such that $\beta \circ \pi = \alpha$. Then the isometry group of M is the universal object in this category.

More generally, the isometry group of a classical compact Riemannian manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object class consists of subsets (not generally subgroups) of the set of smooth isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. Thus, motivated by the ideas of Woronowicz and Soltan [7, 8], one can consider a bigger category with objects as the pair (S, f) where S is a compact metrizable space and

$f : S \times M \to M$ such that the map from M to itself defined by $m \mapsto f(s, m)$ is a smooth isometry for all s in S. The morphism set is defined as above (replacing group homomorphisms by continuous set maps). Thus, summarizing the above discussion and recalling that the span of eigenvectors of the Laplacian is norm-dense in $C(M)$, we have the following result.

Theorem 2.1.8 *Let M be a smooth Riemannian compact manifold and $(C^\infty(M))_0$ denote the span of eigenvectors of the Laplacian. Then* $\mathrm{ISO}(M)$ *is the universal object of the category with objects as pairs $(C(Y), \alpha)$ where Y is a compact metrizable space and α is a unital C^*-homomorphism from $C(M)$ to $C(M) \otimes C(Y)$ satisfying the following:*

a. $\overline{\mathrm{Sp}}(\alpha(C(M))(1 \otimes C(Y)) = C(M) \otimes C(Y)$,

b. $\alpha_\phi = (\mathrm{id} \otimes \phi)\alpha$ *maps* $(C^\infty(M))_0$ *into itself and commutes with* $\mathcal{L}$ *on* $(C^\infty(M))_0$, *for every state ϕ on $C(Y)$.*

Example 2.1.9 1. The isometry group of the n-sphere S^n is $O(n+1)$ where the action is given by the usual action of $O(n+1)$ on $\mathbb{R}^{n+1}$. The subgroup of $O(n+1)$ consisting of all orientation preserving isometries on S^n is $SO(n+1)$.

2. The isometry group of the circle S^1 is $S^1 \rtimes Z_2$. Here the $\mathbf{Z}_2(=\{0, 1\})$ action on S^1 is given by $1.z = \overline{z}$, where z is in S^1 while the action of S^1 is its action on itself.

3. $\mathrm{ISO}(\mathbb{T}^n) \cong \mathbb{T}^n \rtimes (\mathbf{Z}_2^n \rtimes S_n)$ where S_n is the permutation group on n symbols. Here an element of S_n acts on an element $(z_1, z_2, ..., z_n) \in \mathbb{T}^n$ by permutation. If the generator of i-th copy of $\mathbf{Z}_2^n$ is denoted by 1_i, then the action of 1_i is given by $1_i(z_1, z_2, ..., z_n) = (z_1, ..., z_{i-1}, \overline{z_i}, z_{i+1}, ..., z_n)$ where $(z_1, z_2, ..., z_n) \in \mathbb{T}^n$. Lastly, the action of $\mathbb{T}^n$ on itself is its usual action.

Characterization of orientation preserving isometries of a spin manifold

This characterization is in the terms of the Dirac operator [9]. For the characterization of isometries of a Riemannian manifold in terms of the Hodge Dirac operator, we refer to [10].

We begin with a few basic facts about topologizing the space $C^\infty(M, N)$ where M, N are smooth manifolds. Let Ω be an open set of $\mathbb{R}^n$. We endow $C^\infty(\Omega)$ with the usual Fre'chet topology coming from uniform convergence (over compact subsets) of partial derivatives of all orders. The space $C^\infty(\Omega)$ is complete with respect to this topology, so is a Polish space in particular. Moreover, by the Sobolev imbedding Theorem (Corollary 1.21, [2]), $\cap_{k \geq 0} H_k(\Omega) = C^\infty(\Omega)$ as a set, where $H_k(\Omega)$ denotes the k-th Sobolev space. Thus, $C^\infty(\Omega)$ has also the Hilbertian seminorms coming from the Sobolev spaces, hence the corresponding Frechet topology. We claim that these two topologies on $C^\infty(\Omega)$ coincide. Indeed, the inclusion map from $C^\infty(\Omega)$ into $\cap_k H_k(\Omega)$, is continuous and surjective, so by the open mapping theorem for Frechet space, the inverse is also continuous, proving our claim.

Given two second countable smooth manifolds M, N, we shall equip $C^\infty(M, N)$ with the weakest locally convex topology making $C^\infty(M, N) \ni \phi \mapsto f \circ \phi \in C^\infty(M)$ Frechet continuous for every f in $C^\infty(N)$.

For topological or smooth fiber or principal bundles E, F over a second countable smooth manifold M, we shall denote by $\mathrm{Hom}(E, F)$ the set of bundle morphisms from E to F. We remark that the total space of a locally trivial topological bundle such that the base and the fiber spaces are locally compact Hausdorff second countable must itself be so, hence in particular Polish (that is, a complete separable metric space).

In particular, if E, F are locally trivial principal G-bundles over a common base, such that the (common) base as well as the structure group G are locally compact Hausdorff and second countable, then $\mathrm{Hom}(E, F)$ is a Polish space.

We need a standard fact, stated below as Lemma 2.1.11, about the measurable lift of Polish space valued functions.

Before that, we introduce some notions.

A multifunction $G : X \to Y$ is a map with domain X and whose values are non-empty subsets of Y. For $A \subseteq Y$, we put $G^{-1}(A) = \{x \in X : G(x) \cap A \neq \phi\}$.

A selection of a multifunction $G : X \to Y$ is a point map $s : X \to Y$ such that $s(x)$ belongs to $G(x)$ for all x in X. Now let Y be a Polish space and σ_X a σ-algebra on X. A multifunction $G : X \to Y$ is called σ_X measurable if $G^{-1}(U)$ belongs to σ_X for every open set U in Y.

The following well-known selection theorem is Theorem 5.2.1 of [11] and was proved by Kuratowski and Ryll-Nardzewski.

Proposition 2.1.10 *Let σ_X be a σ algebra on X and Y a Polish space. Then, every σ_X measurable, closed valued multifunction $F : X \to Y$ admits a σ_X measurable selection.*

A trivial consequence of this result is the following:

Lemma 2.1.11 *Let M be a compact metrizable space, $B, \tilde{B}$ Polish spaces such that there is an n-covering map $\Lambda : \tilde{B} \to B$. Then any continuous map $\xi : M \to B$ admits a lifting $\tilde{\xi} : M \to \tilde{B}$, which is Borel measurable and $\Lambda \circ \tilde{\xi} = \xi$. In particular, if $\tilde{B}$ and B are topological bundles over M, with Λ being a bundle map, any continuous section of B admits a lifting which is a measurable section of $\tilde{B}$.*

We shall now give an operator-theoretic characterization of the classical group of orientation preserving Riemannian isometries, which will be the motivation of our definition of its quantum counterpart. Let M be a compact Riemannian n-dimensional spin manifold, with a fixed choice of orientation. We recall the notations as in Sect. 2.1.3. In particular, the spinor bundle S is the associated bundle of a principal $Spin(n)$-bundle P on M which has a canonical 2-covering bundle-map Λ from P to the frame-bundle F (which is an $SO(n)$-principal bundle), such that Λ is locally of the form $(\mathrm{id}_M \otimes \lambda)$ where λ is the two covering map from $Spin(n)$ to $SO(n)$. Moreover, the spinor space will be denoted by Δ_n. Let f be a smooth orientation preserving Riemannian isometry of M, and consider the bundles $E = \mathrm{Hom}(F, f^*(F))$

and $\tilde{E} = \mathrm{Hom}(P, f^*(P))$ (where Hom denotes the set of bundle maps). We view df as a section of the bundle E in the natural way. By the Lemma 2.1.11 we obtain a measurable lift $\tilde{df} : M \to \tilde{E}$, which is a measurable section of $\tilde{E}$. Using this, we define a map on the space of measurable section of $S = P \times_{Spin(n)} \Delta_n$ as follows: given a (measurable) section ξ of S, say of the form $\xi(m) = [p(m), v]$, with $p(m)$ in P_m, v in Δ_n, we define $U\xi$ by $(U\xi)(m) = [\tilde{df}(f^{-1}(m))(p(f^{-1}(m))), v]$. Note that sections of the above form constitute a total subset in $L^2(S)$, and the map $\xi \mapsto U\xi$ is clearly a densely defined linear map on $L^2(S)$, whose fiber-wise action is unitary since the $Spin(n)$ action is so on Δ_n. Thus it extends to a unitary U on $\mathcal{H} = L^2(S)$. Any such U, induced by the map f, will be denoted by U_f. It is not unique since the choice of the lifting used in its construction is not unique.

Theorem 2.1.12 *Let M be a compact Riemannian spin manifold (hence orientable, and fix a choice of orientation) with the usual Dirac operator D acting as an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$ of the square integrable spinors, and let S denote the spinor bundle, with $\Gamma(S)$ being the $C^\infty(M)$ module of smooth sections of S. Let $f : M \to M$ be a smooth one-to-one map which is a Riemannian orientation preserving isometry. Then the unitary U_f on $\mathcal{H}$ commutes with D and $U_f M_\phi U_f^* = M_{\phi \circ f}$, for any ϕ in $C(M)$, where M_ϕ denotes the operator of multiplication by ϕ on $L^2(S)$. Moreover, when the dimension of M is even, U_f commutes with the canonical grading γ on $L^2(S)$.*

Conversely, suppose that U is a unitary on $\mathcal{H}$ such that $UD = DU$ and the map $\alpha_U(X) = UXU^{-1}$ for X in $\mathcal{B}(\mathcal{H})$ maps $\mathcal{A} = C(M)$ into $L^\infty(M) = \mathcal{A}''$. If the dimension of M is even, assume furthermore that U commutes with the grading operator γ. Then there is a smooth one-to-one orientation preserving Riemannian isometry f on M such that $U = U_f$.

Proof From the construction of U_f, it is clear that $U_f M_\phi U_f^{-1} = M_{\phi \circ f}$. Moreover, since the Dirac operator D commutes with the $Spin(n)$-action on S, we have $U_f D = DU_f$ on each fiber, hence on $L^2(S)$. In the even dimensional case, it is easy to see that the $Spin(n)$ action commutes with γ (the grading operator), hence U_f does so.

For the converse, first note that α_U is a unital $*$-homomorphism on $L^\infty(M, dvol)$ and thus must be of the form $\psi \mapsto \psi \circ f$ for some measurable f. We claim that f must be smooth. Fix any smooth g on M and consider $\phi = g \circ f$. We have to argue that ϕ is smooth. Let δ_D denote the generator of the strongly continuous one-parameter group of automorphism $\beta_t(X) = e^{itD} X e^{-itD}$ on $\mathcal{B}(\mathcal{H})$ (with respect to the weak operator topology, say). From the assumption that D and U commute it is clear that α_U maps $\mathcal{D} := \bigcap_{n \geq 1} \mathrm{Dom}(\delta_D^n)$ into itself and since $C^\infty(M) \subset \mathcal{D}$, we conclude that $\alpha_U(M_\phi) = M_{\phi \circ g}$ belongs to $\mathcal{D}$. We claim that this implies the smoothness of ϕ. Let m be a point of M and choose a local chart (V, ψ) at m, with the coordinates $(x_1, ..., x_n)$, such that $\Omega = \psi(V) \subseteq \mathbb{R}^n$ has compact closure, $S|_V$ is trivial and D has the local expression $D = i \sum_{j=1}^n \mu(e_j) \nabla_j$, where $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$ denotes the covariant derivative (with respect to the canonical Levi-Civita connection) operator along the vector field $\frac{\partial}{\partial x_j}$ on $L^2(\Omega)$ and $\mu(v)$ denotes the Clifford multiplication by a vector v. Now, $\phi \circ \psi^{-1} \in L^\infty(\Omega) \subseteq L^2(\Omega)$ and it is easy to observe from the above local structure of D that $[D, M_\phi]$ has the local expression $\sum_j iM_{\frac{\partial}{\partial x_j}\phi} \otimes \mu(e_j)$. Thus, the fact $M_\phi \in \bigcap_{n\geq 1} \mathrm{Dom}(\delta_D^n)$ implies $\phi \circ \psi^{-1}$ is in $\mathrm{Dom}(d_{j_1}...d_{j_k})$ for every integer tuple $(j_1, ..., j_k)$, $j_i \in \{1, ..., n\}$, where $d_j := \frac{\partial}{\partial x_j}$. In other words, $\phi \circ \psi^{-1}$ is in $H^k(\Omega)$ for all $k \geq 1$, where $H^k(\Omega)$ denotes the k-th Sobolev space on Ω (see [2]). By Sobolev's theorem (see, for example. [2], Corollary 1.21, page 24) it follows that $\phi \circ \psi^{-1}$ is in $C^\infty(\Omega)$.

We note that f is one-to-one as $\phi \to \phi \circ f$ is an automorphism of L^∞. Now, we shall show that f is an isometry of the metric space (M, d), where d is the metric coming from the Riemannian structure, and we have the explicit formula (2.1.2)

$$d(p, q) = \sup_{\phi \in C^\infty(M), \|[D, M_\phi]\| \leq 1} |\phi(p) - \phi(q)|.$$

Since U commutes with D, we have $\|[D, M_{\phi\circ f}]\| = \|[D, UM_\phi U^*]\| = \|U[D, M_\phi]U^*\| = \|[D, M_\phi]\|$ for every ϕ, from which it follows that $d(f(p), f(q)) = d(p, q)$. Finally, f is orientation preserving if and only if the volume form (say ω), which defines the choice of orientation, is preserved by the natural action of df on the space of n-forms. This will follow from the explicit description of ω in terms of D, given by (see [12] page 26, also see [13])

$$\omega(\phi_0 d\phi_1 ... d\phi_n) = \tau(\epsilon M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]),$$

where $\phi_0, ..., \phi_n$ belong to $C^\infty(M)$, $\epsilon = 1$ in the odd case and $\epsilon = \gamma$ (the grading operator) in the even case and τ denotes the volume integral. In fact, $\tau(X) = \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(e^{-tD^2}X)}{\mathrm{Tr}(e^{-tD^2})}$ (where Lim is as in Sect. 2.2.2), which implies $\tau(UXU^*) = \tau(X)$ for all X in $\mathcal{B}(\mathcal{H})$ (using the fact that D and U commute). Thus,

$$\omega(\phi_0 \circ f \; d(\phi_1 \circ f) \dots d(\phi_n \circ f))$$

$$= \tau(\epsilon UM_{\phi_0}U^*U[D, M_{\phi_1}]U^*...U[D, M_{\phi_n}]U^*)$$

$$= \tau(U\epsilon M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}]U^*)$$

$$= \tau(\epsilon M_{\phi_0}[D, M_{\phi_1}]...[D, M_{\phi_n}])$$

$$= \omega(\phi_0 d\phi_1 ... d\phi_n).$$

□

Now we turn to the case of a family of maps. We are ready to state and prove the operator-theoretic characterization of a 'family of orientation preserving isometries'.

Theorem 2.1.13 *Let X be a compact metrizable space and $\psi : X \times M \to M$ is a map such that ψ_x defined by $\psi_x(m) = \psi(x, m)$ is a smooth orientation preserving Riemannian isometry and $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous with respect to the locally convex topology of $C^\infty(M, M)$ mentioned before.*

Then there exists a ($C(X)$-linear) unitary U_ψ on the Hilbert $C(X)$-module $\mathcal{H} \otimes C(X)$ (where $\mathcal{H} = L^2(S)$ as in Theorem 2.1.12) such that for all x belonging to X, $U_x := (id \otimes ev_x)U_\psi$ is a unitary of the form U_{ψ_x} on the Hilbert space $\mathcal{H}$ commuting with D and $U_x M_\phi U_x^{-1} = M_{\phi \circ \psi_x^{-1}}$. If in addition, the manifold is even dimensional, then U_{ψ_x} commutes with the grading operator γ.

Conversely, if there exists a $C(X)$-linear unitary U on $\mathcal{H} \otimes C(X)$ such that $U_x := (\mathrm{id} \otimes \mathrm{ev}_x)(U)$ is a unitary commuting with D for all x, (and U_x commutes with the grading operator γ if the manifold is even dimensional) and $(\mathrm{id} \otimes \mathrm{ev}_x)\alpha_U(L^\infty(M)) \subseteq L^\infty(M)$ for all x in X, then there exists a map $\psi : X \times M \to M$ satisfying the conditions mentioned above such that $U = U_\psi$.

Proof Consider the bundles $\hat{F} = X \times F$ and $\hat{P} = X \times P$ over $X \times M$, with fibers at (x, m) isomorphic with F_m and P_m, respectively, and where F and P are, respectively, the bundles of orthonormal frames and the Spin(n) bundle discussed before. Moreover, denote by Ψ the map from $X \times M$ to itself given by $(x, m) \mapsto (x, \psi(x, m))$. Let $\pi_X : \mathrm{Hom}(\hat{F}, \Psi^*(\hat{F})) \to X$ be the obvious map obtained by composing the projection map of the $X \times M$ bundle with the projection from $X \times M$ to X and let us denote by B the closed subset of the Polish space $C(X, \mathrm{Hom}(\hat{F}, \Psi^*(\hat{F})))$ consisting of those f such that for all x, $\pi_X(f(x)) = x$. Define $\tilde{B}$ in a similar way replacing $\hat{F}$ by $\hat{P}$. The covering map from P to F induces a covering map from $\tilde{B}$ to B as well. Let $d_\psi^{'} : M \to B$ be the map given by $d'_\psi(m)(x) \equiv d'_\psi(x, m) = d\psi_x|_m$. Then by Lemma 2.1.11 there exists a measurable lift of $d_\psi^{'}$, say $\widetilde{d_\psi^{'}}$ from M into $\tilde{B}$. Since $d'_\psi(x, m) \in \mathrm{Hom}(F_m, F_{\psi(x,m)})$, it is clear that the lift $\widetilde{d'_\psi}(x, m)$ will be an element of $\mathrm{Hom}(P_m, P_{\psi(x,m)})$.

We can identify $\mathcal{H} \otimes C(X)$ with $C(X \to \mathcal{H})$, and since $\mathcal{H}$ has a total set $\mathcal{F}$ (say) consisting of sections of the form $[p(\cdot), v]$, where $p : M \to P$ is a measurable section of P and v belongs to Δ_n, we have a total set $\tilde{\mathcal{F}}$ of $\mathcal{H} \otimes C(X)$ consisting of $\mathcal{F}$ valued continuous functions from X. Any such function can be written as $[\Xi, v]$ with $\Xi : X \times M \to P$, $v \in \Delta_n$, and $\Xi(x, m) \in P_m$, and we define U on $\tilde{\mathcal{F}}$ by $U[\Xi, v] = [\Theta, v]$, where

$$\Theta(x, m) = \widetilde{d_\psi^{'}}(x, \psi_x^{-1}(m))(\Xi(x, \psi_x^{-1}(m))).$$

It is clear from the construction of the lift that U is indeed a $C(X)$-linear isometry that maps the total set $\tilde{\mathcal{F}}$ onto itself, so extends to a unitary on the whole of $\mathcal{H} \otimes C(X)$ with the desired properties.

Conversely, given U as in the statement of the converse part of the theorem, we observe that for each x in X, by Theorem 2.1.12, $(\mathrm{id} \otimes ev_x)U = U_{\psi_x}$ for some ψ_x such that ψ_x is a smooth orientation preserving Riemannian isometry. This defines the map ψ by setting $\psi(x, m) = \psi_x(m)$. The proof will be complete if we can show that $x \mapsto \psi_x \in C^\infty(M, M)$ is continuous, which is equivalent to showing that whenever $x_n \to x$ in the topology of X, we must have $\phi \circ \psi_{x_n} \to \phi \circ \psi_x$ in the Fre'chet topology of $C^\infty(M)$, for any $\phi \in C^\infty(M)$. However, by Lemma 1.1.10, we have $(\mathrm{id} \otimes ev_{x_n})\alpha_U([D,\ M_\phi]) \to (\mathrm{id} \otimes ev_x)\alpha_U([D,\ M_\phi])$ in the strong operator topology where $\alpha_U(X) = UXU^{-1}$. Since U commutes with D, this implies

$$(\mathrm{id} \otimes ev_{x_n})[D \otimes \mathrm{id},\ \alpha_U(M_\phi)] \to (\mathrm{id} \otimes ev_x)[D \otimes \mathrm{id},\ \alpha_U(M_\phi)],$$

that is, for all ξ in $L^2(S)$,

$$[D,\, M_{\phi\circ\psi_{x_n}}]\xi \xrightarrow{L^2} [D,\ M_{\phi\circ\psi_x}]\xi.$$

By choosing ϕ with support in a local trivializing coordinate neighborhood for S, and then using the local expression of D used in the proof of Theorem 2.1.12, we conclude that $d_k(\phi \circ \psi_{x_n}) \xrightarrow{L^2} d_k(\phi \circ \psi_x)$ (where d_k is as in the proof of Theorem 2.1.12). Similarly, by taking repeated commutators with D, we can show the L^2 convergence with d_k replaced by $d_{k_1}...d_{k_m}$ for any finite tuple $(k_1, ..., k_m)$. In other words, $\phi \circ \psi_{x_n} \to \phi \circ \psi_x$ in the topology of $C^\infty(M)$ described before. □

2.2 Noncommutative Geometry

In this section, we recall those basic concepts of noncommutative geometry, which we are going to need. We refer to [14–19] for more details.

2.2.1 *Spectral Triples: Definition and Examples*

Motivated by the facts in Proposition 2.1.3, Alain Connes defined a noncommutative manifold based on the idea of a spectral triple:

Definition 2.2.1 A **spectral triple** or **spectral data** is a triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{H}$ is a separable Hilbert space, $\mathcal{A}^\infty$ is a $*$ subalgebra of $\mathcal{B}(\mathcal{H})$, (not necessarily norm closed) and D is a self-adjoint (typically unbounded) operator such that for all a in $\mathcal{A}^\infty$, the operator $[D, a]$ has a bounded extension. Such a spectral triple is also called an odd spectral triple. If in addition, we have γ in $\mathcal{B}(\mathcal{H})$ satisfying $\gamma = \gamma^* = \gamma^{-1},\ D\gamma = -\gamma D$ and $[a, \gamma] = 0$ for all a in $\mathcal{A}^\infty$, then we say that the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ is an even spectral triple. The operator D is called the Dirac operator corresponding to the spectral triple.

Furthermore, given an abstract $*$-algebra $\mathcal{B}$, an odd (even) spectral triple on $\mathcal{B}$ is an odd (even) spectral triple $(\pi(\mathcal{B}), \mathcal{H}, D)$ (respectively, $(\pi(\mathcal{B}), \mathcal{H}, D, \gamma)$) where $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism.

Since in the classical case, the Dirac operator has compact resolvent if the manifold is compact, we say that the spectral triple is of **compact type** if $\mathcal{A}^\infty$ is unital and D has compact resolvent. For nonunital C^* algebras, interesting spectral triples are not of compact type. Examples of such spectral triples include semifinite spectral triples for which we refer to [20, 21], and the references therein. Since, our final goal is to study quantum isometry groups of spectral triples of compact type, all the spectral triples under discussion will be assumed to be of compact type.

Definition 2.2.2 We say that two spectral triples $(\pi_1(\mathcal{A}), \mathcal{H}_1, D_1)$ and $(\pi_2(\mathcal{A}), \mathcal{H}_2, D_2)$ are said to be unitarily equivalent if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $D_2 = UD_1U^*$ and $\pi_2(.) = U\pi_1(.)U^*$ where π_j, $j = 1, 2$ are the representations of $\mathcal{A}$ in $\mathcal{H}_j$,, respectively.

Real structure on a spectral triple

We now give a definition of the real structure along the lines of [22, 23], which is a suitable modification of Connes' original definition (see [14, 24]) to accommodate the examples coming from quantum groups and quantum homogeneous spaces.

Definition 2.2.3 An odd spectral triple with a real structure is given by a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ along with a (possibly unbounded, invertible) closed antilinear operator $\widetilde{J}$ on $\mathcal{H}$ such that $\mathcal{D} := \mathrm{Dom}(D) \subseteq \mathrm{Dom}(\widetilde{J})$, $\widetilde{J}\mathcal{D} \subseteq \mathcal{D}$, $\widetilde{J}$ commutes with D on $\mathcal{D}$, and the antilinear isometry J obtained from the polar decomposition of $\widetilde{J}$ satisfies the usual conditions for a real structure in the sense of [23], for a suitable sign-convention given by $(\epsilon, \epsilon') \in \{\pm 1\} \times \{\pm 1\}$ as described in [12], page 30, i.e., $J^2 = \epsilon I$, $JD = \epsilon' DJ$, and for all $x, y \in \mathcal{A}^\infty$, the commutators $[x, JyJ^{-1}]$ and $[JxJ^{-1}, [D, y]]$ are compact operators.

If the spectral triple is even, a real structure with the sign-convention given by a triplet $(\epsilon, \epsilon', \epsilon'')$ as in [12], page 30, is similar to a real structure in the odd case (with the sign-convention (ϵ, ϵ')), but with the additional requirement that $J\gamma = \epsilon''\gamma J$.

Next, we give a few examples of spectral triples in classical and noncommutative geometry. We will give more examples in the later chapters of the book.

Example 2.2.4 Let M be a smooth spin manifold. Then from Proposition 2.1.3, we see that $(C^\infty(M), \mathcal{H}, D)$ is a spectral triple over $C^\infty(M)$ and it is of compact type if M is compact.

We recall that when the dimension of the manifold is even, $\Delta_n = \Delta_n^+ \oplus \Delta_n^-$. An L^2 section s has a decomposition $s = s_1 + s_2$ where $s_1(m), s_2(m)$ belongs to $\Delta_n^+(m)$ and $\Delta_n^-(m)$ (for all m), respectively, where $\Delta_n^\pm(m)$ denotes the subspace of the fiber over m. This decomposition of $L^2(S)$ induces a grading operator γ on $L^2(S)$. It can be seen that D anticommutes with γ.

Example 2.2.5 This example comes from the classical Hilbert space of forms discussed in Sect. 2.2.2. One considers the self-adjoint extension of the operator $d + d^*$ on $\mathcal{H} = \oplus_k \mathcal{H}^k(M)$, which is again denoted by $d + d^*$. $C^\infty(M)$ has a representation on each $\mathcal{H}^k(M)$ which gives a representation, say π on $\mathcal{H}$. Then it can be seen that $(C^\infty(M), \mathcal{H}, d + d^*)$ is a spectral triple and $d + d^*$ is called the Hodge Dirac operator. When M is compact, this spectral triple is of compact type.

Remark 2.2.6 Let us make it clear that by a 'classical spectral triple' we always mean the spectral triple obtained by the Dirac operator on the spinors (so, in particular, manifolds are assumed to be Riemannian spin manifolds), and not just any spectral triple on the commutative algebra $C^\infty(M)$.

Example 2.2.7 **The Noncommutative torus**

We recall from Sect. 1.1.1 that the noncommutative 2-torus $\mathcal{A}_\theta$ is the universal C^* algebra generated by two unitaries U and V satisfying $UV = e^{2\pi i\theta} VU$, where θ is a number in [0, 1].

There are two derivations d_1 and d_2 on $\mathcal{A}_\theta$ obtained by extending linearly the rule:

$$d_1(U) = U,\ d_1(V) = 0,$$

$$d_2(U) = 0,\ d_2(V) = V.$$

Then d_1 and d_2 are well defined on the following dense $*$-subalgebra of $\mathcal{A}_\theta$:

$$\mathcal{A}_\theta^\infty = \{ \sum_{m,n\in\mathbf{Z}} a_{mn} U^m V^n : \sup_{m,n} \left| m^k n^l a_{mn} \right| < \infty \ \text{ for all } k, l \text{ in } I\!N\}.$$

There is a faithful trace on $\mathcal{A}_\theta$ defined as follows:

$$\tau(\sum a_{mn} U^m V^n) = a_{00}.$$

Let $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ where $L^2(\tau)$ denotes the GNS Hilbert space of $\mathcal{A}_\theta$ with respect to the state τ. We note that $\mathcal{A}_\theta^\infty$ is embedded as a subalgebra of $\mathcal{B}(\mathcal{H})$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Now, we define $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$.

Then, $(\mathcal{A}_\theta^\infty, \mathcal{H}, D)$ is a spectral triple of compact type. In particular, for $\theta = 0$, this coincides with the classical spectral triple on $C(\mathbb{T}^2)$.

Example 2.2.8 **Spectral triples on** $SU_\mu(2)$

In this example, we discuss the spectral triple on $SU_\mu(2)$ constructed by Chakraborty and Pal in [25]. We recall from Sect. 1.2.4 that by the symbols $t^n_{i,j}$, we will denote the (i, j)-th matrix element of the $(2n + 1)$ dimensional corepresentation of $SU_\mu(2)$. Moreover, e^n_{ij}'s will denote the normalized (with respect to the Haar state h) t^n_{ij}'s.

Then the spectral triple is given by $(\mathcal{O}(SU_\mu(2)), L^2(SU_\mu(2), h), D^{SU_\mu(2)})$, where $D^{SU_\mu(2)}$ is defined by

$$\begin{aligned} &D^{SU_\mu(2)}(e^n_{ij}) \\ &= (2n+1)e^n_{ij},\ n \neq i \\ &= -(2n+1)e^n_{ij},\ n = i. \end{aligned}$$

Example 2.2.9 **A class of spectral triples on the Podles' spheres**
We discuss the spectral triples on $S^2_{\mu c}$ discussed in [26].

Let $s = -c^{-\frac{1}{2}}\lambda_-, \ \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{\frac{1}{2}}$.
For all j belonging to $\frac{1}{2}I\!N$,
$u_j = (\alpha^* - s\gamma^*)(\alpha^* - \mu^{-1}s\gamma^*)......(\alpha^* - \mu^{-2j+1}s\gamma^*)$,
$w_j = (\alpha - \mu s\gamma)(\alpha - \mu^2 s\gamma)........(\alpha - \mu^{2j}s\gamma)$,
$u_{-j} = E^{2j} \triangleright w_j$,
$u_0 = w_0 = 1$,
$y_1 = (1 + \mu^{-2})^{\frac{1}{2}}(c^{\frac{1}{2}}\mu^2\gamma^{*2} - \mu\gamma^*\alpha^* - \mu c^{\frac{1}{2}}\alpha^{*2})$,
$N^l_{kj} = \left\| F^{l-k} \triangleright ({y_1}^{l-|j|}u_j) \right\|^{-1}$.
Define

$$v^l_{k,j} = N^l_{k,j} F^{l-k} \triangleright (y_1^{l-|j|}u_j), \ l \in \frac{1}{2}I\!N_0, \ \ j, k = -l, -l+1, ...l. \tag{2.2.1}$$

Let $\mathcal{M}_N$ be the Hilbert subspace of $L^2(SU_\mu(2))$ with the orthonormal basis $\{v^l_{m,N} : l = |N|, \ |N|+1, \ ...., \ m = -l, ...l\}$.

Set

$$\mathcal{H} = \mathcal{M}_{-\frac{1}{2}} \oplus \mathcal{M}_{\frac{1}{2}}.$$

Then it is easy to check that x_i keeps $\mathcal{H}$ for all $i \in \{-1, 0, 1\}$. In particular,

$$x_i.v^l_{m,N} = \alpha^-_i(l, m; N)v^{l-1}_{m+i,N} + \alpha^0_i(l, m; N)v^l_{m+i,N} + \alpha^+_i(l, m; N)v^{l+1}_{m+i,N}, \tag{2.2.2}$$

where $\alpha^-_i, \ \alpha^0_i, \ \alpha^+_i$ are some constants.

Thus, (2.2.2) defines a representation π of $S^2_{\mu,c}$ on $\mathcal{H}$.

We will often identify $\pi(S^2_{\mu,c})$ with $S^2_{\mu,c}$.

Finally by Proposition 7.2 of [26], the following Dirac operator D gives a spectral triple $(\mathcal{O}(S^2_{\mu,c}), \mathcal{H}, D)$ which we are going to work with :

$$D(v^l_{m,\pm\frac{1}{2}}) = (c_1 l + c_2)v^l_{m,\mp\frac{1}{2}}, \tag{2.2.3}$$

where c_1, c_2 are elements of $\mathbb{R}$, $c_1 \neq 0$.

2.2.2 *The Noncommutative Space of Forms*

We start this subsection by recalling the universal space of one forms corresponding to an algebra.

Proposition 2.2.10 *Given an algebra $\mathcal{B}$, there is a (unique upto isomorphism) $\mathcal{B} - \mathcal{B}$ bimodule $\Omega^1(\mathcal{B})$ and a derivation $\delta : \mathcal{B} \to \Omega^1(\mathcal{B})$ (that is, $\delta(ab) = \delta(a)b + a\delta(b)$ for all a, b in $\mathcal{B}$), satisfying the following properties:*

(i) $\Omega^1(\mathcal{B})$ is spanned as a vector space by elements of the form $a\delta(b)$ with a, b belonging to $\mathcal{B}$; and

(ii) for any $\mathcal{B} - \mathcal{B}$ bimodule E and a derivation $d : \mathcal{B} \to E$, there is an unique $\mathcal{B} - \mathcal{B}$ linear map $\eta : \Omega^1(\mathcal{B}) \to E$ such that $d = \eta \circ \delta$.

The bimodule $\Omega^1(\mathcal{B})$ is called the space of universal 1-forms an $\mathcal{B}$ and δ is called the universal derivation.

We can also introduce universal space of higher forms on $\mathcal{B}$, $\Omega^k(\mathcal{B})$, say, for $k = 2, 3, ...$, by defining them recursively as follows: $\Omega^{k+1}(\mathcal{B}) = \Omega^k(\mathcal{B}) \otimes_{\mathcal{B}} \Omega^1(\mathcal{B})$ and also set $\Omega^0(\mathcal{B}) = \mathcal{B}$.

Next, we briefly discuss the notion of the noncommutative Hilbert space of forms for a spectral triple of compact type. We refer to [27] (page 124 -127) and the references therein for more details.

Definition 2.2.11 A spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type is said to be Θ-summable if e^{-tD^2} is of trace class for all $t > 0$. A Θ-summable spectral triple is called finitely summable when there is some $p > 0$ such that $t^{\frac{p}{2}}\,\mathrm{Tr}(e^{-tD^2})$ is bounded on $(0, \delta]$ for some $\delta > 0$. The infimum of all such p, say p', is called the dimension of the spectral triple and the spectral triple is called p'-summable.

Remark 2.2.12 We remark that the definition of Θ-summability to be used in this book is stronger than the one in [14] (page 390, Definition 1.) in which a spectral triple is called Θ-summable if $\mathrm{Tr}(e^{-D^2}) < \infty$.

For a Θ-summable spectral triple, let $\sigma_\lambda(T) = \frac{\mathrm{Tr}(Te^{-\frac{1}{\lambda}D^2})}{\mathrm{Tr}(e^{-\frac{1}{\lambda}D^2})}$ for $\lambda > 0$. We note that $\lambda \mapsto \sigma_\lambda(T)$ is bounded.

Let

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_a^\lambda \sigma_u(T)\frac{du}{u} \text{for } \lambda \geq a \geq e.$$

Now consider the quotient C^* algebra $\mathcal{B}_\infty = C_b([a, \infty))/C_0([a, \infty))$. Let for T in $\mathcal{B}(\mathcal{H})$, $\tau(T)$ in $\mathcal{B}_\infty$ be the class of $\lambda \to \tau_\lambda(T)$.

For any state ω on the C^* algebra B_∞, $Tr_\omega(T) = \omega(\tau(T))$ for all T in $\mathcal{B}(\mathcal{H})$ defines a functional on $\mathcal{B}(\mathcal{H})$. As we are not going to need the choice of ω in this book, we will suppress the suffix ω and simply write $\mathrm{Lim}_{t\to 0^+}\frac{\mathrm{Tr}(Te^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ for $Tr_\omega(T)$. This is a kind of Banach limit because if $\lim_{t\to 0^+}\frac{\mathrm{Tr}(Te^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ exists, then it agrees with the functional $\mathrm{Lim}_{t\to 0^+}$. Moreover, $\mathrm{Tr}_\omega(T)$ coincides (upto a constant) with the Dixmier trace (see Chapter IV, [14]) of the operator $T|D|^{-p}$ when the spectral triple has a finite dimension $p > 0$, where $|D|^{-p}$ is to be interpreted as the inverse of the restriction of $|D|^p$ on the closure of its range. In particular, this functional gives back the volume form for the classical spectral triple on a compact Riemannian manifold.

Let $\Omega^k(\mathcal{A}^\infty)$ be the space of universal k-forms on the algebra $\mathcal{A}^\infty$ which is spanned by $a_0\delta(a_1)\cdots\delta(a_k)$, a_i belonging to $\mathcal{A}^\infty$, where δ is as in Proposition 2.2.10. There is a natural graded algebra structure on $\Omega \equiv \bigoplus_{k\geq 0}\Omega^k(\mathcal{A}^\infty)$, which also has a natural involution given by $(\delta(a))^* = -\delta(a^*)$, and using the spectral triple, we get a $*$-representation $\Pi : \Omega \to \mathcal{B}(\mathcal{H})$ which sends $a_0\delta(a_1)\cdots\delta(a_k)$ to $a_0 d_D(a_1)\cdots d_D(a_k)$, where $d_D(a) = [D, a]$. Consider the state τ on $\mathcal{B}(\mathcal{H})$ given by, $\tau(X) = \mathrm{Lim}_{t\to 0^+} \frac{\mathrm{Tr}(Xe^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$, where Lim is as above. Using τ, we define a positive semi definite sesquilinear form on $\Omega^k(\mathcal{A}^\infty)$ by setting $\langle w, \eta\rangle = \tau(\Pi(w)^*\Pi(\eta))$. Let $K^k = \{w \in \Omega^k(\mathcal{A}^\infty) : \langle w,\ w\rangle = 0\}$, for $k \geq 0$, and $K^{-1} := (0)$. Let $\overline{\Omega^k_D}$ be the Hilbert space obtained by completing the quotient $\Omega^k(\mathcal{A}^\infty)/K^k$ with respect to the inner product mentioned above, and we define $\mathcal{H}^k_D := P_k^\perp\overline{\Omega^k_D}$, where P_k denotes the projection onto the closed subspace generated by $\delta(K^{k-1})$. The map $D' := d + d^* \equiv d_D + d_D^*$ on $\mathcal{H}_{d+d^*} := \bigoplus_{k\geq 0}\mathcal{H}^k_D$ has a self-adjoint extension (which is again denoted by $d + d^*$). Clearly, $\mathcal{H}^k_D$ has a total set consisting of elements of the form $[a_0\delta(a_1)\cdots\delta(a_k)]$, with a_i in $\mathcal{A}^\infty$ and where $[\omega]$ denotes the equivalence class $P_k^\perp(w + K^k)$ for ω belonging to $\Omega^k(\mathcal{A}^\infty)$. There is a $*$-representation $\pi_{d+d^*} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{d+d^*})$, given by $\pi_{d+d^*}(a)([a_0\delta(a_1)\cdots\delta(a_k)]) = [aa_0\delta(a_1)\cdots\delta(a_k)]$. Then it is easy to see that

Proposition 2.2.13 *$(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$ is a spectral triple.*

Let us mention that for the classical spectral triple $(C^\infty(M), L^2(S), D)$ on a compact Riemannian spin manifold M, the above construction does give the usual Hilbert space of forms discussed in Sect. 2.1.1. Moreover, the volume form on $C^\infty(M)$ using $D' = d + d^*$ in place of D agrees with the usual volume form. It is enough to explain this for smooth functions supported in a small coordinate neighborhood on which the restriction of the spinor bundle S is trivial. Combining the local expressions (5.45), (5.48) and (5.49) in [15], one can easily see that D^2 has the following local expression:

$$D^2 = \Delta \otimes I_{\mathbb{C}^k} + A,$$

where A is a first order differential operator, $\Delta = -\sum_{i,j} g^{ij}\frac{\delta}{\delta x_i}\frac{\delta}{\delta x_j}$ is the Laplacian on the manifold, k is the dimension of the fiber of S, $\{x_1, x_2, ..., x_n\}$ are local coordinates, $((g_{ij}))$ is the Riemannian metric and $((g^{ij})) = ((g_{ij}))^{-1}$.

On the other hand, we can obtain the following local expression for $(D')^2$ on a suitable trivializing neighborhood for the bundle of forms:

$$(D')^2 = \mathcal{L} \otimes I_{\mathbb{C}^m},$$

where $\mathcal{L}$ is the Hodge Laplacian on M as in Sect. 2.1.2 and m is the dimension of the fiber of $\Lambda^* M$.

A direct calculation shows that $\mathcal{L} - \Delta$ is a first order differential operator. As $\mathcal{L}^{-\frac{n}{2}}$ and $\Delta^{-\frac{n}{2}}$ are of Dixmier trace class, it follows from the discussion in page 307 of [14] and the references cited there that

$$\mathrm{Tr}_\omega(M_f\mathcal{L}^{-\frac{n}{2}}) = \mathrm{Tr}_\omega(M_f\Delta^{-\frac{n}{2}}),$$

where M_f denotes the operator of multiplication by a smooth function f supported in a small enough coordinate neighborhood on which both S and $\Lambda^* M$ are trivial. Hence we have

$$\frac{\mathrm{Tr}_\omega(M_f(D')^{-\frac{n}{2}})}{\mathrm{Tr}_\omega((D')^{-\frac{n}{2}})} = \frac{\mathrm{Tr}_\omega(M_f(D^2)^{-\frac{n}{2}})}{\mathrm{Tr}_\omega((D^2)^{-\frac{n}{2}})}.$$

2.2.3 Laplacian in Noncommutative Geometry

Now we want to formulate and study an analog of the Hodge Laplacian in noncommutative geometry. We recall that in the classical case of a compact Riemannian manifold, $\mathcal{L} = -d_D^* d_D$ coincides with the Hodge Laplacian $-d^* d$ (restricted on the space of smooth functions), where d denotes the de-Rham differential. We need some mild technical assumptions on the spectral triple to define the associated Laplacian.

Definition 2.2.14 Let $(\mathcal{A}^\infty, \mathcal{B}(\mathcal{H}), D)$ be a Θ-summable spectral triple of compact type. Assume furthermore that it satisfies the following conditions:

(1) It is QC^∞, that is, $\mathcal{A}^\infty$ and $\{[D, a],\ a \in \mathcal{A}^\infty\}$ are contained in the domains of all powers of the derivation $[|D|, \cdot]$.

(2) Under condition (1), τ defined by $\tau(X) = \mathrm{Lim}_{t\to 0} \frac{\mathrm{Tr}(Xe^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ is a positive trace on the C^*-subalgebra generated by $\mathcal{A}^\infty$ and $\{[D, a] :\ a \in \mathcal{A}^\infty\}$. We assume that τ is also faithful on this subalgebra.

(3) The unbounded densely defined map d_D from $\mathcal{H}_D^0$ to $\mathcal{H}_D^1$ given by $d_D(a) = [D, a]$ for a in $\mathcal{A}^\infty$, is closable and let d_D also denote the closure.

(4) $\mathcal{L} := -d_D^* d_D$ has $\mathcal{A}^\infty$ in its domain.

Then, we call $\mathcal{L}$ the noncommutative Laplacian and $T_t = e^{t\mathcal{L}}$ the noncommutative heat semigroup. Moreover, the $*$-subalgebra of $\mathcal{A}^\infty$ generated by $\mathcal{A}_0^\infty$ will be denoted by $\mathcal{A}_0$.

Let us record the following observation.

Lemma 2.2.15 *Under the conditions of the Definition 2.2.14, then for $x \in \mathcal{A}^\infty$, we have $\mathcal{L}(x^*) = (\mathcal{L}(x))^*$.*

Proof It follows by simple calculation using the facts that τ is a trace and $d_D(x^*) = -(d_D(x))^*$ that

$$\begin{aligned}
&\tau(\mathcal{L}(x^*)^* y)\\
&= \tau(d_D(x)d_D(y)) = \tau(d_D(y)d_D(x)) = -\tau((d_D(y^*))^* d_D(x))\\
&= < y^*, \mathcal{L}(x) > = \tau(y\mathcal{L}(x)) = \tau(\mathcal{L}(x)y),
\end{aligned}$$

for all $y \in \mathcal{A}^\infty$. By density of $\mathcal{A}^\infty$ in $\mathcal{H}_D^0$ (a) follows. □

It is well known that for compact Riemannian spin manifolds, the conditions (1) and (2) of Definition 2.2.14 are satisfied. On the other hand, we know from Sect. 2.1.2, (for example, Lemma 2.1.1) that the Hodge Laplacian on a compact Riemannian manifold satisfies the properties (3) and (4).

In the noncommutative case, the conditions (1) and (2) hold for many spectral triples including those coming from Rieffel deformations. The content of the next lemma is about the other conditions.

Lemma 2.2.16 *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple of compact type and of finite dimension, say p. Suppose that for every element $a \in \mathcal{A}^\infty$, the map $\mathbb{R} \ni t \mapsto \alpha_t(X) := \exp(itD)X\exp(-itD)$ is differentiable at $t = 0$ in the norm-topology of $\mathcal{B}(\mathcal{H})$, where $X = a$ or $[D, a]$. Then the conditions (3) and (4) of Definition 2.2.14 are satisfied. Moreover, we have:*

(a) $\mathcal{L}$ maps $\mathcal{A}^\infty$ into the weak closure of $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H}_D^0)$.

(b) If $T_t = \exp(t\mathcal{L})$ maps $\mathcal{H}_D^0$ into $\mathcal{A}^\infty$ for all $t > 0$, then any eigenvector of $\mathcal{L}$ belongs to $\mathcal{A}^\infty$.

Proof We first observe that $\tau(\alpha_t(A)) = \tau(A)$ for all t and for all $A \in \mathcal{B}(\mathcal{H})$, since $\exp(itD)$ commutes with $|D|^{-p}$. If moreover, A belongs to the domain of norm-differentiability (at $t = 0$) of α_t, i.e., $\frac{\alpha_t(A)-A}{t} \to i[D, A]$ in operator-norm, then it follows from the property of the Dixmier trace that $\tau([D, A]) = \frac{1}{i} \lim_{t\to 0} \frac{\tau(\alpha_t(A))-\tau(A)}{t} = 0$. Now, since by assumption we have the norm- differentiability at $t = 0$ of $\alpha_t(A)$ for A belonging to the $*$-subalgebra (say $\mathcal{B}$) generated by $\mathcal{A}^\infty$ and $[D, \mathcal{A}^\infty]$, it follows that $\tau([D, A]) = 0\ \forall A \in \mathcal{B}$. Let us now fix $a, b, c \in \mathcal{A}^\infty$ and observe that

$$\begin{aligned}
&< a\, d_D(b), d_D(c) > \\
&= \tau((a\, d_D(b))^* d_D(c) > \\
&= -\tau([D, [D, b^*]a^*c]) + \tau([D, [D, b^*]a^*]c) \\
&= \tau([D, [D, b^*]a^*]c),
\end{aligned}$$

using the fact that $\tau([D, [D, b^*]a^*c]) = 0$. This implies

$$| < a\, d_D(b), d_D(c) > | \le \|[D, [D, b^*]a^*]\|\tau(c^*c)^{\frac{1}{2}} = \|[D, [D, b^*]a^*]\|\,\|c\|_2,$$

where $\|c\|_2 = \tau(c^*c)^{\frac{1}{2}}$ denotes the L^2-norm of $c \in \mathcal{H}_D^0$. This proves that $a\, d_D(b)$ belongs to the domain of d_D^* for all $a, b \in \mathcal{A}^\infty$, so in particular d_D^* is dense, i.e., d_D is closable. Moreover, taking $a = 1$, we see that $d_D(\mathcal{A}^\infty) \subseteq \mathrm{Dom}(d_D^*)$, or in other words, $\mathcal{A}^\infty \subseteq \mathrm{Dom}(d_D^* d_D)$. This proves (3) and (4). The statement (a) can be proved along the line of Theorem 2.9, page 129, [27]. To prove (b), we note that if $x \in \mathcal{H}_D^0$ is an eigenvector of $\mathcal{L}$, say $\mathcal{L}(x) = \lambda x$ ($\lambda \in \mathbb{C}$), then we have $T_t(x) = e^{\lambda t}x$, hence $x = e^{-\lambda t}T_t(x) \in \mathcal{A}^\infty$. □

2.3 Quantum Group Equivariance in Noncommutative Geometry

We have already seen (Theorem 2.1.12) that the classical Dirac operator is equivariant with respect to the natural action of the group of orientation preserving Riemannian isometries. It is natural to explore similar equivariance of a spectral triple with respect to quantum group coactions. Let us begin by giving a precise definition of quantum group equivariance.

Definition 2.3.1 Consider a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ along with a coaction α of a CQG $\mathcal{Q}$ on the C^*-algebra A obtained by taking the norm closure of $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H})$. We say that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a $\mathcal{Q}$-equivariant spectral triple if there is a unitary corepresentation U of $\mathcal{Q}$ on $\mathcal{H}$ such that
(i) $\mathrm{ad}_U(.) = \alpha(.)$,
(ii) $\widetilde{U}(D \otimes I) = (D \otimes I)\widetilde{U}$.

It was not very easy to get examples of spectral triples, which are equivariant with respect to "a genuine (i.e., noncommutative as a C^* algebra) quantum group". In [25] (i.e., Example 2.2.8), the first example of an $SU_\mu(2)$-equivariant spectral triple was constructed. It was followed by the work of a number of mathematicians, see [26, 28–30] and the references therein. In the next two subsections, we show that the spectral triples of Examples 2.2.8 and 2.2.9 are indeed equivariant.

2.3.1 *The Example of $SU_\mu(2)$*

We deal with Example 2.2.8 here. Let U be the regular corepresentation of $SU_\mu(2)$ on $L^2(SU_\mu(2), h)$. Then $\mathrm{ad}_U(x) = \Delta(x)$ for all x in $SU_\mu(2)$. We recall from Example 2.2.8 the normalized vectors e^n_{ij}'s. Then $U(e^n_{ij}) = \sum_k \frac{1}{\|t^n_{ij}\| \|t^n_{ik}\|} e^n_{ik} \otimes t^n_{kj}$ from which it easily follows

Proposition 2.3.2 *([25]) The spectral triple* $(\mathcal{O}(SU_\mu(2), L^2(SU_\mu(2), h), D^{SU_\mu(2)})$ *of Example 2.2.8 is $SU_\mu(2)$-equivariant.*

2.3.2 *The Example of the Podles' Spheres*

Here, we consider the spectral triple constructed in [26] and explained in Example 2.2.9. We will use the notations of Example 2.2.9. From [26], we see that the vector spaces $\nu^l_{\pm\frac{1}{2}} = \mathrm{Span}\{v^l_{m,\pm\frac{1}{2}} : m = -l, ...l\}$ are $(2l + 1)$ dimensional Hilbert spaces on which the $SU_\mu(2)$ corepresentation is unitarily equivalent to the standard l-th unitary irreducible corepresentation of $SU_\mu(2)$, that is, if the corepresentation is denoted by U_0, then $U_0(v^l_{i,\pm\frac{1}{2}}) = \sum v^l_{j,\pm\frac{1}{2}} \otimes t^l_{j,i}$ where $t^l_{i,j}$ denotes the matrix elements in the l-th unitary irreducible corepresentation of $SU_\mu(2)$.

We now recall Theorem 3.5 of [31].

Proposition 2.3.3 *Let R_0 be an operator on $\mathcal{H}$ defined by $R_0(v^n_{i,\pm\frac{1}{2}}) = \mu^{-2i\mp 1} v^n_{i,\pm\frac{1}{2}}$. Then* $\mathrm{Tr}(R_0 e^{-tD^2}) < \infty$ (for all $t > 0$) *and one has*

$$(\tau_{R_0} \otimes \mathrm{id})(\widetilde{U_0}(x \otimes 1)\widetilde{U_0}^*) = \tau_{R_0}(x).1,$$

for all x in $\mathcal{B}(\mathcal{H})$, where $\tau_{R_0}(x) = \mathrm{Tr}(xR_0 e^{-tD^2})$.

We define a positive, unbounded operator R on $\mathcal{H}$ by $R(v^n_{i,\pm\frac{1}{2}}) = \mu^{-2i} v^n_{i,\pm\frac{1}{2}}$.

Proposition 2.3.4 ad_{U_0} *preserves the R-twisted volume. In particular, for x in $\pi(S^2_{\mu,c})$ and $t > 0$, we have $h(x) = \frac{\tau_R(x)}{\tau_R(1)}$, where $\tau_R(x) := \mathrm{Tr}(xRe^{-tD^2})$, and h denotes the restriction of the Haar state of $SU_\mu(2)$ to the subalgebra $S^2_{\mu,c}$, which is the unique $SU_\mu(2)$-invariant state on $S^2_{\mu,c}$.*

Proof It is enough to prove that τ_R is α_{U_0}-invariant. Let us denote by $P_{\frac{1}{2}}$, $P_{-\frac{1}{2}}$ the projections onto the closed subspaces generated by $\{v^l_{i,\frac{1}{2}}\}$ and $\{v^l_{i,-\frac{1}{2}}\}$, respectively. Moreover, let $\tau_\pm$ be the functionals defined by $\tau_\pm(x) = \mathrm{Tr}(xR_0 P_{\pm\frac{1}{2}} e^{-tD^2})$. Now observing that R_0, e^{-tD^2} and U_0 commute with $P_{\pm\frac{1}{2}}$ and using Proposition 2.3.3, we have, for x belonging to $\mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
&(\tau_\pm \otimes \mathrm{id})(\alpha_{U_0}(x)) \\
&= (\mathrm{Tr} \otimes \mathrm{id})(\widetilde{U_0}(x \otimes 1)\widetilde{U_0}^*(R_0 P_{\pm\frac{1}{2}} e^{-tD^2} \otimes \mathrm{id})) \\
&= (\mathrm{Tr} \otimes \mathrm{id})(\widetilde{U_0}(xP_{\pm\frac{1}{2}} \otimes 1)\widetilde{U_0}^*(R_0 e^{-tD^2} \otimes \mathrm{id})) \\
&= (\tau_{R_0} \otimes \mathrm{id})(\alpha_{U_0}(xP_{\frac{1}{2}})) \\
&= \tau_{R_0}(xP_{\pm\frac{1}{2}}) \\
&= \tau_\pm(x).1,
\end{aligned}$$

that is, $\tau_\pm$ are α_{U_0}-invariant.

Thus, $x \mapsto \mathrm{Tr}(xR_0 P_{\pm\frac{1}{2}} e^{-tD^2})$ is invariant under α_{U_0}. Moreover, since we have $RP_{\pm\frac{1}{2}} = \mu^{\pm} R_0 P_{\pm\frac{1}{2}}$, the functional τ_R coincides with $\mu^{-1}\tau_+ + \mu\tau_-$, hence is α_{U_0}-invariant. □

Theorem 2.3.5 *The spectral triple described on the Podles' sphere $S^2_{\mu,c}$ as described in Example 2.2.9 is $SU_\mu(2)$ equivariant. If $\alpha : S^2_{\mu,c} \to S^2_{\mu,c} \otimes SO_\mu(3) \subseteq SU_\mu(2) \otimes SO_\mu(3)$ denotes the canonical coaction of $SO_\mu(3)$ on $S^2_{\mu,c}$ (Sect. 1.3.3) and U_0 is as above, then $\mathrm{ad}_{U_0}(\pi(x)) = (\pi \otimes \mathrm{id})\alpha(x)$ Moreover, ad_{U_0} preserves τ_R.*

Proof

$$\begin{aligned}(D \otimes \mathrm{id})U_0(v^l_{i,\pm\frac{1}{2}}) &= (D \otimes \mathrm{id})(\sum v^l_{j,\pm\frac{1}{2}} \otimes t^l_{j,i}) \\ &= (c_1 l + c_2) \sum v^l_{j,\mp\frac{1}{2}} \otimes t^l_{j,i} \\ &= (c_1 l + c_2) U_0(v^l_{i,\mp\frac{1}{2}}) \\ &= U_0 D(v^l_{i,\pm\frac{1}{2}}).\end{aligned}$$

Thus, the above spectral triple is equivariant w.r.t. the corepresentation U_0.

For the second statement, let U denote the right regular corepresentation of $SU_\mu(2)$ on $L^2(SU_\mu(2), h)$, so that $U_0 = U|_{\mathcal{H}}$. We already noted that the coaction α of $SU_\mu(2)$ is the restriction of the coproduct, that is, $\alpha(x) = U(x \otimes 1)U^*$ for $x \in S^2_{\mu,c} \subseteq \mathcal{B}(L^2(S^2_{\mu,c}))$. Now, $\pi(x) = x|_{\mathcal{H}}$, and we also observed that both x and U (hence U^*) leaves $\mathcal{H}$ invariant. Thus,

$\mathrm{ad}_{U_0}(\pi(x)) = U_0(\pi(x) \otimes \mathrm{id})U_0^* = (U(x \otimes \mathrm{id})U^*)|_{\mathcal{H}\otimes SO_\mu(3)} = \alpha(x)|_{\mathcal{H}\otimes SO_\mu(3)}$ $= (\pi \otimes \mathrm{id})(\alpha(x))$.

Finally, ad_{U_0} preserves τ_R by Proposition 2.3.4. □

2.3.3 Constructions from Coactions by Quantum Isometries

In this subsection, we shall briefly discuss the relevance of quantum isometry group to the problem of constructing quantum group equivariant spectral triples, which is important to understand the role of quantum groups in the framework of noncommutative geometry. There has been a lot of activity in this direction recently, see, for example, the articles by Chakraborty and Pal [25], Connes [32], Landi et al. [28], and the references therein. In the classical situation, there exists a natural unitary representation of the isometry group $G = \mathrm{ISO}(M)$ of a manifold M on the Hilbert space of forms, so that the operator $d + d^*$ (where d is the de-Rham differential operator) commutes with the representation. Indeed, $d + d^*$ is also a Dirac operator for the spectral triple given by the natural representation of $C^\infty(M)$ on the Hilbert space of forms, so we have a canonical construction of G-equivariant spectral triple. Our aim in this subsection is to generalize this to the noncommutative framework, by proving that $d_D + d_D^*$ is equivariant with respect to a canonical unitary corepresentation on the Hilbert space of 'noncommutative forms'.

Consider an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and moreover, make the assumption of Lemma 2.2.16, i.e., assume that $t \mapsto e^{itD}xe^{-itD}$ is norm-differentiable at $t = 0$ for all x in the $*$-algebra $\mathcal{B}$ generated by $\mathcal{A}^\infty$ and $[D, \mathcal{A}^\infty]$.

Lemma 2.3.6 *In the notation of Lemma 2.2.16, we have the following (where $b, c \in \mathcal{A}^\infty$):*

$$d_D^*(d_D(b)c) = -\frac{1}{2}\left(b\mathcal{L}(c) - \mathcal{L}(b)c - \mathcal{L}(bc)\right). \qquad (2.3.1)$$

Proof Denote by $\chi(b, c)$ the right hand side of Eq. (2.3.1) and fix any $a \in \mathcal{A}^\infty$. Using the facts the the functional τ is a faithful trace on the $*$-algebra $\mathcal{B}$, $\mathcal{L} = -d_D^* d_D$ and that $\tau([D, X]) = 0$ for any X in $\mathcal{B}$, we have,

$$\begin{aligned}
&\tau(a^* \chi(b, c)) \\
&= -\frac{1}{2}\{\tau(a^* b\mathcal{L}(c)) - \tau(ca^*\mathcal{L}(b)) - \tau(a^*\mathcal{L}(bc))\} \\
&= \frac{1}{2}\{\tau([D, a^*b][D, c]) - \tau([D, ca^*][D, b]) - \tau([D, a^*][D, bc])\} \\
&= \frac{1}{2}\{\tau(a^*[D, b][D, c]) - \tau([D, c]a^*[D, b]) - \tau(c[D, a^*][D, b]) - \tau([D, a^*][D, b]c)\} \\
&= -\tau([D, a^*][D, b]c) \\
&= \tau([D, a]^*[D, b]c) \\
&= \langle d_D(a), d_D(b)c\rangle \\
&= \tau(a^*(d_D^*(d_D(b)c))).
\end{aligned}$$

From this, we get the following by a simple computation:

$$\langle ad_D(b), a' d_D(b')\rangle = -\frac{1}{2}\tau(b^*\Psi(a^*a', b')), \tag{2.3.2}$$

for $a, b, a', b' \in \mathcal{A}^\infty$, and where $\Psi(x, y) := \mathcal{L}(x)y - x\mathcal{L}(y)$. Now, let us denote the quantum isometry group of the given spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ by $(\mathcal{G}, \Delta, \alpha)$. Let $\mathcal{A}_0$ denote the $*$-algebra generated by $\mathcal{A}_0^\infty$ and $\mathcal{G}_0$ denote the $*$-algebra of $\mathcal{G}$ generated by matrix elements of irreducible corepresentations. Clearly, $\alpha : \mathcal{A}_0 \to \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ is a Hopf-algebraic coaction of $\mathcal{G}_0$ on $\mathcal{A}_0$. Define a $\mathbb{C}$-bilinear map $\tilde{\Psi} : (\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0) \times (\mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0) \to \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ by setting

$$\tilde{\Psi}((x \otimes q), (x' \otimes q')) := \Psi(x, x') \otimes (qq').$$

It follows from the relation $(\mathcal{L} \otimes \text{id}) \circ \alpha = \alpha \circ \mathcal{L}$ on $\mathcal{A}_0$ that

$$\tilde{\Psi}(\alpha(x), \alpha(y)) = \alpha(\Psi(x, y)). \tag{2.3.3}$$

We now define a linear map $\alpha^{(1)}$ from the linear span of $\{ad_D(b) : \ a, b \in \mathcal{A}_0\}$ to $\mathcal{H}_D^1 \otimes \mathcal{G}$ by setting

$$\alpha^{(1)}(ad_D(b)) := \sum_{i,j} a_i^{(1)} d_D(b_j^{(1)}) \otimes a_i^{(2)} b_j^{(2)},$$

where for any $x \in \mathcal{A}_0$ we write $\alpha(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in \mathcal{A}_0 \otimes_{\text{alg}} \mathcal{G}_0$ (summation over finitely many terms). We shall sometimes use the Sweedler convention of writing the above simply as $\alpha(x) = x^{(1)} \otimes x^{(2)}$. It then follows from the identities (2.3.2) and (2.3.3), and also the fact that $(\tau \otimes \text{id})(\alpha(a)) = \tau(a)1$ for all $a \in \mathcal{A}_0$ that

$$\begin{aligned}
&\langle \alpha^{(1)}(a\ d_D(b)), \alpha^{(1)}(a'\ d_D(b'))\rangle_{\mathcal{G}} \\
&= -\frac{1}{2}(\tau \otimes \mathrm{id})(\alpha(b^*)\tilde{\Psi}(\alpha(a^*a'), \alpha(b'))) \\
&= -\frac{1}{2}(\tau \otimes \mathrm{id})(\alpha(b^*)\alpha(\Psi(a^*a', b'))) \\
&= -\frac{1}{2}(\tau \otimes \mathrm{id})(\alpha(b^*\Psi(a^*a', b'))) \\
&= -\frac{1}{2}\tau(b^*\Psi(a^*a', b'))1_{\mathcal{G}} \\
&= \langle a d_D(b), a' d_D(b')\rangle 1_{\mathcal{G}}.
\end{aligned}$$

This proves that $\alpha^{(1)}$ is indeed well-defined and extends to a $\mathcal{G}$-linear isometry on $\mathcal{H}_D^1 \otimes \mathcal{G}$, to be denoted by $U^{(1)}$, which sends $(ad_D(b)) \otimes q$ to $\alpha^{(1)}(ad_D(b))(1 \otimes q)$, $a, b \in \mathcal{A}_0$, $q \in \mathcal{G}$. Moreover, since the linear span of $\alpha(\mathcal{A}_0^\infty)(1 \otimes \mathcal{G})$ is dense in $\mathcal{H}_D^0 \otimes \mathcal{G}$, it is easily seen that the range of the isometry $U^{(1)}$ is the whole of $\mathcal{H}_D^1 \otimes \mathcal{G}$, i.e., $U^{(1)}$ is a unitary. In fact, from its definition it can also be shown that $U^{(1)}$ is a unitary corepresentation of the compact quantum group $\mathcal{G}$ on $\mathcal{H}_D^1$.

In a similar way, we can construct unitary corepresentation $U^{(n)}$ of $\mathcal{G}$ on the Hilbert space of n-forms for any $n \geq 1$, by defining

$$\begin{aligned}
&U^{(n)}((a_0 d_D(a_1) d_D(a_2) ... d_D(a_n)) \otimes q) \\
&= a_0^{(1)} d_D(a_1^{(1)}) ... d_D(a_n^{(1)}) \otimes (a_0^{(2)} a_1^{(2)} ... a_n^{(2)} q), \\
&(\text{where } a_i \in \mathcal{A}_0^\infty, q \in \mathcal{G}, \ \ \text{and Sweedler convention is used}),
\end{aligned}$$

and verifying that it extends to a unitary. We also denote by $U^{(0)}$ the unitary corepresentation $\tilde{\alpha}$ on $\mathcal{H}_D^0$ discussed before. Finally, we have a unitary corepresentation $U = \bigoplus_{n \geq 0} U^{(n)}$ of $\mathcal{G}$ on $\tilde{\mathcal{H}} := \bigoplus_n \mathcal{H}_D^n$, and also extend d_D as a closed densely defined operator on $\tilde{\mathcal{H}}$ in the obvious way, by defining $d_D(a_0 d_D(a_1) ... d_D(a_n)) = d_D(a_0) ... d_D(a_n)$. It is now straightforward to see the following:

Theorem 2.3.7 *The operator $D' := d_D + d_D^*$ is equivariant in the sense that $U(D' \otimes 1) = (D' \otimes 1)U$.*

We point out that there is a natural corepresentation π of $\overline{\mathcal{A}}$ on $\tilde{\mathcal{H}}$ given by $\pi(a)(a_0 d_D(a_1) ... d_D(a_n)) = a a_0 d_D(a_1) ... d_D(a_n)$, and $(\pi(\mathcal{A}^\infty), \tilde{\mathcal{H}}, D')$ is indeed a spectral triple, which is $\mathcal{G}$-equivariant.

Although the relation between spectral properties of D and D' is not clear in general, in many cases of interest (e.g., when there is an underlying type $(1, 1)$ spectral data in the sense of [27]) these two Dirac operators are closely related. As an illustration, consider the canonical spectral on the noncommutative 2-torus $\mathcal{A}_\theta$, which is discussed in some details in the next section. In this case, the Dirac operator D acts on $L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^2$, and it can easily be shown (see [27]) that the Hilbert space of forms is isomorphic with $L^2(\mathcal{A}_\theta, \tau) \otimes \mathbb{C}^4 \cong L^2(\mathcal{A}_\theta) \otimes \mathbb{C}^2$; thus D' is essentially same as D in this case.

2.3.4 R-twisted Volume Form Coming from the Modularity of a Quantum Group

Let $(\mathcal{S}, \Delta)$ be a compact quantum group and $(\mathcal{A}^\infty, \mathcal{H}, D)$ be an $\mathcal{S}$-equivariant spectral triple with a unitary corepresentation V on $\mathcal{H}$ commuting with D. In this subsection, our aim is to show the existence of a densely defined positive functional on $\mathcal{B}(\mathcal{H})$, to be interpreted as a generalization of "volume form", which is kept invariant under ad_V.

The Hilbert space $\mathcal{H}$, on which D acts decomposes into finite dimensional eigenspaces $\mathcal{H}_k$ $(k \geq 1)$ of the operator D, i.e., $\mathcal{H} = \oplus_k \mathcal{H}_k$. Since D commutes with V, V preserves each of the $\mathcal{H}_k$'s and on each $\mathcal{H}_k$, V is a unitary corepresentation of the compact quantum group $\mathcal{Q}$. Then we have the decomposition of each $\mathcal{H}_k$ into the irreducibles, say

$$\mathcal{H}_k = \oplus_{\pi \in \mathcal{I}_k} \mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}},$$

where $m_{\pi,k}$ is the multiplicity of the irreducible corepresentation of type π on $\mathcal{H}_k$ and $\mathcal{I}_k$ is some finite subset of $\mathrm{Rep}(\mathcal{Q})$. Since R commutes with V, R preserves direct summands of $\mathcal{H}_k$. Let $\{e_i^\pi : i = 1, 2, \cdots d_\pi\}$ be an orthonormal basis of $\mathbb{C}^{d_\pi}$ such that $V(e_i^\pi) = \sum_j e_j^\pi \otimes u_{ji}^\pi$.

Let $\mathcal{E}_D$ denote the WOT-dense $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by rank one operators of the form $|\xi >< \eta|$, where ξ, η are eigenvectors of D. We note that since V maps $\mathcal{H}_k$ into $\mathcal{H}_k \otimes_{\mathrm{alg}} \mathcal{S}_0$ for all k, ad_V will map $\mathcal{E}_D$ into $\mathcal{E}_D \otimes_{\mathrm{alg}} \mathcal{S}_0$.

With the above set up and notations, we give the following definition.

Definition 2.3.8 An R-twisted spectral data (of compact type) is given by a quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, R)$, where

1. $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a spectral triple of compact type.
2. R a positive (possibly unbounded) invertible operator such that R commutes with D.

We shall also sometimes refer to $(\mathcal{A}^\infty, \mathcal{H}, D)$ as an R-twisted spectral triple.

Remark 2.3.9 We remark that in the above definition, we do not need the full strength of Definition 2.2 in [33].

Definition 2.3.10 The functional τ_R defined below on the weakly dense $*$-subalgebra $\mathcal{E}_D$ of $\mathcal{B}(\mathcal{H})$ will be called the R-twisted volume form:

$$\tau_R(x) = Tr(Rx), \quad x \in \mathcal{E}_D.$$

We now characterize those R for which ad_V preserves the functional τ_R.

Theorem 2.3.11 *Let $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ be an R-twisted spectral data of compact type which is equivariant with respect to a corepresentation V of a CQG $\mathcal{S}$ on $\mathcal{H}$. Then ad_V preserves the R-twisted volume form if and only if R is of the following form:*

$$R|_{\mathcal{H}_k} = \oplus_{\pi \in \mathcal{I}_k} F^\pi \otimes T_{\pi,k}, \tag{2.3.4}$$

for some $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$, where F^π's are as in Sect. 1.2.2.

Proof Let $\{f_j^{\pi,k}\}_{j=1}^{m_{\pi,k}}$ be an orthonormal basis for $\mathbb{C}^{m_{\pi,k}}$. Then $\{e_i^\pi \otimes f_j^{\pi,k} : i = 1, 2, \cdots d_\pi,\ j = 1, \ldots m_{\pi,k}\}$ is an orthonormal basis for $\mathcal{H}_k$. As R commutes with D, it leaves $\mathcal{H}_k$ invariant. Let us write

$$R(e_i^\pi \otimes f_j^{\pi,k}) = \sum_{s,t} R^{\pi,k}(s,t,i,j) e_s \otimes f_t.$$

Let h denote the extension of the Haar state of $\mathcal{S}$ to a vector state on $\mathcal{B}(L^2(\mathcal{S}, h))$ given by $h(x) =< 1, x1 >$.

For a fixed π, k, denoting $e_i^\pi,\ f_j^{\pi,k},\ R^{\pi,k}(s,t,i,j)$ by $e_i,\ f_j,\ R(s,t,i,j)$, respectively, and for a in $\mathcal{E}_D$, we have the following:

$$\begin{aligned}
(\tau_R \otimes h)\mathrm{ad}_V(a) &= \sum_{i,j} < V^*(e_i \otimes f_j \otimes 1_{\mathcal{Q}}), (a \otimes 1)V^* R(e_i \otimes f_j) > \\
&= \sum_{i,j,k,s,t,u} < e_k \otimes f_j \otimes (u_{ik}^\pi)^*, R(s,t,i,j) a(e_u \otimes f_t) \otimes (u_{su}^\pi)^* > \\
&= \sum_{i,j,k,s,t,u} \frac{R(s,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > \delta_{is} F^\pi(k,u) \\
&= \sum_{i,j,k,t,u} \frac{R(i,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > F^\pi(k,u).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\tau_R(a) = &\ Tr(a.R) \\
&= \sum_{i,j} < e_i \otimes f_j, aR(e_i \otimes f_j) > \\
&= \sum_{k,j,u,t} R(u,t,k,j) < e_k \otimes f_j, a(e_u \otimes f_t) > .
\end{aligned}$$

Now observe that if R is of the form given in the theorem, then $R(s,t,i,j) = F^\pi(i,s) T_{\pi,k}(j,t)$. Plugging this in the expressions for $(\tau_R \otimes h)\mathrm{ad}_V(a)$ and $\tau_R(a)$ obtained above, and using the fact that $M_\pi = \sum_i F^\pi(i,i)$, we get $(\tau_R \otimes h)\mathrm{ad}_V(a) = \tau_R(a)$. It follows easily that ad_V preserves τ_R.

We now prove the necessity part of the theorem. We note that $(\tau_R \otimes h)\mathrm{ad}_V(a) = \tau_R(a)$ implies:

$$\sum_{i,j,k,t,u} \frac{R(i,t,i,j)}{M_\pi} < e_k \otimes f_j, a(e_u \otimes f_t) > F^\pi(k,u)$$
$$= \sum_{k,j,u,t} R(u,t,k,j) < e_k \otimes f_j, a(e_u \otimes f_t) > . \tag{2.3.5}$$

Now fix u_0, t_0 and consider $a \in \mathcal{B}(\mathcal{H})$ such that $a(e_{u_0} \otimes f_{t_0}) = e_p \otimes f_q$ and zero on the other basis elements. Then from (2.3.5), we get

$$\sum_{i,j,k} \frac{R(i,t_0,i,j)}{M_\pi} < e_k \otimes f_j, e_p \otimes f_q > F^\pi(k,u_0)$$
$$= \sum_{k,j} R(u_0,t_0,k,j) < e_k \otimes f_j, e_p \otimes f_q >,$$

which gives $\sum_i \frac{R(i,t_0,i,q)}{M_\pi} F^\pi(p,u_0) = R(u_0,t_0,p,q)$.

This proves that $R|_{\mathcal{H}_k} = \oplus_{\pi \in \mathcal{I}_k} F^\pi \otimes T_{\pi,k}$ with some $T_{\pi,k} \in \mathcal{B}(\mathbb{C}^{m_{\pi,k}})$ given by $T_{\pi,k}(t_0,q) = \sum_i \frac{R(i,t_0,i,q)}{M_\pi}$. □

As an immediate corollary, we get the following:

Proposition 2.3.12 *Let $R = \Pi_V(\phi_1) \in \mathcal{B}(\mathcal{H})$, where ϕ_1 is the functional defined in Proposition 1.2.19 and Π_V is as in Theorem 1.4.1. Suppose also that $L \in \mathcal{B}(\mathcal{H})$ is $(\mathcal{S}, V)$ equivariant. Then we have:*

a. R is a (possibly unbounded) positive operator with $\mathrm{Dom}(R)$ *containing the subspaces $\mathcal{H}_k,\ k \geq 1$.*

b. $RD = DR$.

c. ad_V *preserves the functional τ_R.*

Thus, given a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ (of compact type) which is $\mathcal{S}$-equivariant with respect to a corepresentation V of a CQG $\mathcal{S}$ on $\mathcal{H}$, we can always construct a positive (possibly unbounded) invertible operator R on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is a twisted spectral data and ad_V *preserves the functional τ_R.*

Proof This follows from Theorem 2.3.11 as R is of the form (2.3.4) with $T_{\pi,k} = I$ for all π, k. □

Remark 2.3.13 If L in Proposition 2.3.12 is such that RL is trace class, then the functional χ is defined and bounded on $\mathcal{B}(\mathcal{H})$ and the conclusion of the proposition holds as well.

Remark 2.3.14 (a)When the spectral triple in question has a real structure as in Definition 2.2.3, there is a canonical choice of R (see Remark 3.3.3).

(b) When the Haar state of $\mathcal{S}$ is tracial, then it follows from the definition of R and Theorem 1.5 part 1. of [34] that R can be chosen to be I.

We record the following lemma for future use.

Lemma 2.3.15 *The* ad_V*-invariance of the functional* τ_R *on* $\mathcal{E}_D$ *is equivalent to the* ad_V*-invariance of the functional* $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ *on* $\mathcal{E}_D$ *for each* $t > 0$. *If, furthermore, the* R*-twisted spectral triple is* Θ*-summable in the sense that* Re^{-tD^2} *is trace class for every* $t > 0$, *then* ad_V *preserves the functional* $\mathcal{B}(\mathcal{H}) \ni x \mapsto \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(xRe^{-tD^2})}{\mathrm{Tr}(Re^{-tD^2})}$, *where* Lim *is as defined in Sect.* 2.2.2.

Proof If W_λ denotes the eigenspace of D corresponding to the eigenvalue, say λ, it is clear that $\tau_R(X) = e^{t\lambda^2}\mathrm{Tr}(Re^{-tD^2}X)$ for all $X = |\xi >< \eta|$ with ξ, η belonging to W_λ and for any $t > 0$. Thus, the ad_V-invariance of the functional τ_R on $\mathcal{E}_D$ is equivalent to the ad_V-invariance of the functional $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ on $\mathcal{E}_D$ for each $t > 0$. This can be argued as follows. Let ad_V be τ_R invariant on $\mathcal{E}_D$, that is, for all $|\xi >< \eta|$ with ξ, η belonging to W_λ, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_V(|\xi >< \eta|) = \tau_R(|\xi >< \eta|).1$ Therefore, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_V(|\xi >< \eta|) = \tau_R(|\xi >< \eta|).1 = e^{t\lambda^2}Tr(Re^{-tD^2}|\xi >< \eta|)$. On the other hand, $(\tau_R \otimes \mathrm{id})\mathrm{ad}_V(|\xi >< \eta|) = e^{t\lambda^2}(Tr(Re^{tD^2}.) \otimes \mathrm{id})\mathrm{ad}_V(|\xi >< \eta|)$. If the R-twisted spectral triple is Θ-summable, the above is also equivalent to the ad_V-invariance of the bounded normal functional $X \mapsto \mathrm{Tr}(XRe^{-tD^2})$ on the whole of $\mathcal{B}(\mathcal{H})$. In particular, this implies that ad_V preserves the functional $\mathcal{B}(\mathcal{H}) \ni x \mapsto \mathrm{Lim}_{t\to 0+} \frac{\mathrm{Tr}(xRe^{-tD^2})}{\mathrm{Tr}(Re^{-tD^2})}$.
□

References

1. Lee, J.M.: Riemannian Manifolds, An Introduction to Curvature, Graduate Texts in Mathematics, vol. 176. Springer (1997)
2. Rosenberg, S.: The Laplacian on a Riemannian Manifold. Cambridge University Press, Cambridge (1997)
3. Donnelly, H.: Eigenfunctions of laplacians on compact riemannian manifolds. Assian J. Math. **10**(1), 115–126 (2006)
4. Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York (1978)
5. Connes, A., Dubois-Violette, M.: Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. Comm. Math. Phys. **230**(3), 539–579 (2002)
6. Maes, A., Van Daele, A.: Notes on compact quantum groups. Nieuw Arch. Wisk. **4, 16**(1–2), 73–112 (1998)
7. Soltan, P.M.: Quantum families of maps and quantum semigroups on finite quantum spaces, preprint. J. Geom. Phys. **59**(3), 354–368 (2009)
8. Woronowicz, S.L.: Pseudogroups, pseudospaces and Pontryagin duality. In: Proceedings of the International Conference on Mathematical Physics. Lecture Notes in Physics, vol. 116, pp. 407–412. Lausanne (1979)
9. Friedrich, T.: Dirac Operators in Riemannian Geometry. Graduate Studies in Mathematics 25. American Mathematical Society, Providence (2000)
10. Park, E.: Isometries of noncommutative metric spaces. Proc. Am. Math. Soc. **123**(1), 97–105 (1995)
11. Srivastava, S.M.: A Course on Borel Sets. Graduate Texts in Mathematics, vol. 180. Springer, New York (1998)
12. Varilly, J.C.: An Introduction to Noncommutative Geometry. EMS Series of Lectures in Mathematics (2006)

13. Connes, A.: On the spectral characterization of manifolds. J. Noncommut. Geom. **7**(1), 1–82 (2013)
14. Connes, A.: Noncommutative Geometry. Academic Press, London (1994)
15. Landi, G.: An Introduction to Noncommutative Spaces and their Geometries. Lecture notes in Physics Monographs, vol. 51. Springer, Berlin (1997)
16. Connes, A., Marcolli, M.: Noncommutative Geometry, Quantum Fields and Motives. American Mathematical Society Colloquium Publications, vol. 55. American Mathematical Society, Providence, Hindustan Book Agency, New Delhi (2008)
17. Blackadar, B.: K-theory for Operator Algebras. Mathematical Sciences Research Institute Publications, vol. 5, 2nd edn. Cambridge University Press, Cambridge (1998)
18. Khalkhali, M.: Basic Noncommutative Geometry. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich (2009)
19. Higson, N., Roe, J.: Analytic K-homology. Oxford Science Publications, Oxford Mathematical Monographs. Oxford University Press, Oxford (2000)
20. Carey, A.L., Phillips, J., Rennie, A.: Semifinite spectral triples associated with graph C^*-algebras. Traces in number theory, geometry and quantum fields, pp. 35–56, Aspects Math., E38, Friedr. Vieweg, Wiesbaden (2008)
21. Carey, A.L., Phillips, J., Rennie, A., Sukochev, F.A.: The Hochschild class of the Chern character for semifinite spectral triples. J. Funct. Anal. **213**(1), 111–153 (2004)
22. Dabrowski, L.: Geometry of quantum spheres. J. Geom. Phys. **56**(1), 86–107 (2006)
23. Dabrowski, L., Landi, G., Paschke, M., Sitarz, A.: The spectral geometry of the equatorial Podles' sphere. C.R. Math. Acad. Sci. Paris **340**(11), 819–822 (2005)
24. Connes, A.: Noncommutative geometry and reality. J. Math. Phys. **36**, 6194–6231 (1995)
25. Chakraborty, P.S., Pal, A.: Equivariant spectral triples on the quantum $SU(2)$ group. K. Theory **28**, 107–126 (2003)
26. Dabrowski, L., D'Andrea, F., Landi, G., Wagner, E.: Dirac operators on all Podles' quantum spheres. J. Noncomm. Geom. **1**, 213–239 (2007)
27. Fröhlich, J., Grandjean, O., Recknagel, A.: Supersymmetric quantum theory and noncommutative geometry. Comm. Math. Phys. **203**(1), 119–184 (1999)
28. Dabrowski, L., Landi, G., Sitarz, A., van Suijlekom, W., Varilly, J.C.: The Dirac operator on $\mathrm{SU}_q(2)$. Commun. Math. Phys. **259**(3), 729–759 (2005)
29. D'Andrea, F., Dabrowski, L., Landi, G.: The noncommutative geometry of the quantum projective plane. Rev. Math. Phys. **20**(8), 979–100 (2008)
30. Neshveyev, S., Tuset, L.: The Dirac operator on compact quantum groups. J. Reine Angew. Math. **641**, 1–20 (2010)
31. Goswami, D.: Some Noncommutative Geometric Aspects of $SU_q(2)$. arXiv:0108003v4
32. Connes, A.: Cyclic cohomology, quantum group symmetries and the local index formula for $\mathrm{SU}_q(2)$. J. Inst. Math. Jussieu **3**(1), 17–68 (2004)
33. Goswami, D.: Twisted entire cyclic cohomology, JLO cocycles and equivariant spectral triples. Rev. Math. Phys. **16**(5), 583–602 (2004)
34. Woronowicz, S.L.: Compact quantum groups, pp. 845–884. In: Connes, A., et al. (eds.): Syme'tries quantiques (Quantum Symmetries) (Les Houches). Elsevier, Amsterdam (1995)

Chapter 3
Definition and Existence of Quantum Isometry Groups

Abstract Under some reasonable assumptions on a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ (which we call admissibility), we prove the existence of a universal object in the category of compact quantum groups admitting coaction on the closure of $\mathcal{A}^\infty$ which commutes with the (noncommutative) Laplacian. This universal object is called the quantum isometry group w.r.t. the Laplacian. Moreover, we discuss analogous formulations of the quantum group of orientation (and volume or a given real structure) preserving isometries. Sufficient conditions under which the action of the quantum isometry group keeps the C^* algebra invariant and is a C^* action are given. We also mention some sufficient conditions for the existence of the quantum group of orientation preserving isometries without fixing a choice of the 'volume-form'.

In this chapter, we describe in details the definition and existence of quantum isometry groups in different set-ups. We begin with the approach based on the Laplacian of the spectral triple and then discuss the concept of quantum group of orientation preserving isometries. In the first case, our construction makes sense only when the Laplacian obtained from the spectral triple is sufficiently nice. In the case of quantum group of orientation preserving isometries, we only need the data coming from the spectral triple. In a private communication, L. Dabrowski suggested an alternative terminology, namely, 'Laplace type'/ Dirac type quantum isometry group to describe the quantum isometry groups coming from these two approaches mentioned above. Towards the end of the chapter, we also describe the natural analogue of a quantum isometry group of a real spectral triple.

3.1 The Approach Based on Laplacian

The motivation for our definition comes from Theorem 2.1.8. Thus, for defining the quantum isometry group, we are led to consider a category of CQG's which coact on a noncommutative manifold such that it preserves the Riemannian structure in a suitable sense. Then we prove that under some assumptions, a universal object in this category exists.

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_3

3.1.1 The Definition and Existence of the Quantum Isometry Group

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a Θ-summable spectral triple of compact type satisfying the conditions of Definition 2.2.14. We recall from Sect. 2.2 the Hilbert spaces of k-forms $\mathcal{H}^k_D$, $k = 0, 1, 2, \ldots$ and also the Laplacian $\mathcal{L} = -d^*_D d_D$ as in Definition 2.2.14.

To define the quantum isometry group, we need the following additional assumptions:

Assumptions

1. $\mathcal{L}$ has compact resolvents. Thus, it has a discrete spectrum and $\mathcal{H}^0_D$ has a complete orthonormal basis consisting of eigenvectors of $\mathcal{L}$.

2. $\mathcal{L}$ keeps the subspace $\mathcal{A}^\infty$ invariant.

3. Every eigenvector of $\mathcal{L}$ belongs to $\mathcal{A}^\infty$.

4. Let $\mathcal{A}^\infty_0 = \mathrm{Span}\{a : a \text{ is an eigenvector of } \mathcal{L}\}$. We assume that $\mathcal{A}^\infty_0$, is norm dense in $\mathcal{A}^\infty$.

5. The kernel of $\mathcal{L}$ is one dimensional, spanned by the identity 1 of $\mathcal{A}^\infty$, viewed as a unit vector in $\mathcal{H}^0_D$. This assumption will be referred to as the "connectedness assumption".

Definition 3.1.1 We say that a spectral triple satisfying the assumptions **1.–4.** admissible.

Remark 3.1.2 Our terminology in this book differs slightly from that of [1], where the definition of admissibility included the connectedness assumption **5**. We know from Sect. 2.1.2, (for example, Lemma 2.1.1) that the Hodge Laplacian on a compact Riemannian manifold satisfies the properties **1.–4.** as above. Moreover, if the manifold is connected, then the condition **5.** is satisfied. In the noncommutative world, some of these assumptions follow from some additional conditions on the spectral triple (see for example, Lemma 2.2.16). The necessity of having the admissibility conditions as well as the connectedness assumption **5.** is discussed in details in Sect. 3.1.2.

In view of the characterization of smooth isometric action on a classical compact manifold (Theorem 2.1.8 in Chap. 2), the following definition was given in [1].

Definition 3.1.3 Let $\overline{\mathcal{A}}$ be the C^* algebra obtained by completing $\mathcal{A}^\infty$ in the norm of $\mathcal{B}(\mathcal{H}^0_D)$. A quantum family of smooth isometries of $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a pair $(\mathcal{S}, \alpha)$, where $\mathcal{S}$ is a separable unital C^* algebra and $\alpha : \overline{\mathcal{A}} \to \overline{\mathcal{A}} \otimes \mathcal{S}$ is a unital C^* homomorphism, satisfying the following:

a. $\overline{\mathrm{Span}}(\alpha(\overline{\mathcal{A}})(1 \otimes \mathcal{S})) = \overline{\mathcal{A}} \otimes \mathcal{S}$,

b. for every state ϕ on $\mathcal{S}$, $\alpha_\phi = (\mathrm{id} \otimes \phi)\alpha(\mathcal{A}^\infty_0) \subseteq \mathcal{A}^\infty_0$ and α_ϕ commutes with $\mathcal{L}$ on $\mathcal{A}^\infty_0$.

The quantum family of isometries $(\mathcal{S}, \alpha)$ is called volume-preserving if for all $a \in \mathcal{A}^\infty$, $(\tau \otimes \mathrm{id})(\alpha(a)) = \tau(a)1_{\mathcal{S}}$.

In case the C^* algebra has a coproduct Δ such that $(\mathcal{S}, \Delta)$ is a compact quantum group and α is a coaction of $(\mathcal{S}, \Delta)$ on $\overline{\mathcal{A}}$, we say that $(\mathcal{S}, \Delta)$ coacts smoothly and isometrically on the noncommutative manifold.

We point out that the volume preserving condition for a quantum family of smooth isometries is automatic if the connectedness condition **5** is satisfied. This is the content of Lemma 3.1.5. To prove it, we need the following observation.

Lemma 3.1.4 *Suppose* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *is admissible satisfying the connectedness assumption* **5**. *Let* $\Psi : \mathcal{A}_0^\infty \to \mathcal{A}_0^\infty$ *be a norm bounded linear map, such that* $\Psi(1) = 1$, *and* $\Psi \circ \mathcal{L} = \mathcal{L} \circ \Psi$ *on the subspace* $\mathcal{A}_0^\infty$. *Then* $\tau(\Psi(x)) = \tau(x)$ *for all* $x \in \mathcal{A}^\infty$.

Proof By Lemma 2.1.5 with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_D^0$, $\xi_1 = \xi_2 = 1$, we have $\tau(\Psi(x)) = \tau(x)$ for all $x \in \mathcal{A}_0^\infty$. By the norm-continuity of Ψ and τ it extends to the whole of $\mathcal{A}^\infty$. □

Before proceeding to the proof of the existence of the quantum isometry group, we fix our notations and some basic facts.

Notations and Some Basic Facts

1. The category with the object class consisting of all quantum families of isometries $(\mathcal{S}, \alpha)$ of $(\mathcal{A}^\infty, \mathcal{H}, D)$ and morphisms $\mathrm{Mor}((\mathcal{S}, \alpha), (\mathcal{S}', \alpha'))$ being the set of unital C^* homomorphisms $\phi : \mathcal{S} \to \mathcal{S}'$ satisfying $(\mathrm{id} \otimes \phi)\alpha = \alpha'$ will be denoted by $\mathbf{Q}^{\mathcal{L}}$.

2. We will denote by $\mathbf{Q}'_{\mathcal{L}}$ the category whose objects are $(\mathcal{S}, \Delta, \alpha)$ where $(\mathcal{S}, \Delta)$ is a CQG coacting smoothly and isometrically on $(\mathcal{A}^\infty, \mathcal{H}, D)$ and α is the coaction. The morphisms of $\mathbf{Q}'_{\mathcal{L}}$ are the morphisms of compact quantum groups which are also morphisms of the underlying quantum families.

3. The forgetful functor $F : \mathbf{Q}'_{\mathcal{L}} \to \mathbf{Q}^{\mathcal{L}}$ is faithful so that we can view $F(\mathbf{Q}'_{\mathcal{L}})$ as a subcategory of $\mathbf{Q}^{\mathcal{L}}$. Let ${\mathbf{Q}^{\mathcal{L}}}_0$ and $(\mathbf{Q}'_{\mathcal{L}})_0$ denote the full subcategories of $\mathbf{Q}^{\mathcal{L}}$ and $\mathbf{Q}'_{\mathcal{L}}$ respectively obtained by restricting the object-classes to the volume-preserving quantum families.

4. Given such an admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, the Laplacian $\mathcal{L}$ has a countable set of eigenvalues each with finite multiplicity, let us denote them by $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots,$ where $V_0, V_1, \ldots$ are the corresponding finite dimensional eigenspaces. We have $V_i \subseteq \mathcal{A}^\infty$ for each i. As $\mathcal{L}(x^*) = (\mathcal{L}(x))^*$, V_i is closed under $*$. Clearly, $\mathcal{H}_D^0 = \oplus_i V_i$.

5. Let $\{e_{ij}, j = 1, \ldots, d_i\}$ be an orthonormal basis of V_i. Then, $\mathcal{A}_0^\infty = \mathrm{Span}\{e_{ij}, j = 1, \ldots, d_i;\ i \geq 0\}$ and $\{e_{ij}^*, j = 1, \ldots, d_i\}$ is another orthonormal basis for V_i as $\tau(x^*y) = \tau(yx^*)$ for $x, y \in \mathcal{A}^\infty$. $\mathcal{A}_0^\infty$ is norm dense in $\overline{\mathcal{A}^\infty}$. If the spectral triple also satisfies the connectedness assumption **5.**, we have $V_0 = \mathbb{C}1$.

6. We shall denote by $\mathcal{U}_i$ the quantum group $A_{u,d_i}(I)$, (as defined in Chap. 1, Sect. 1.3.2) where d_i is the dimension of the subspace V_i. We fix a corepresentation $\beta_i : V_i \to V_i \otimes_{\mathrm{alg}} \mathcal{U}_i$ of $\mathcal{U}_i$ on the Hilbert space V_i, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{(i)}$, for $j = 1, \ldots, d_i$, where $u_i \equiv ((u_{kj}^{(i)}))$ are the generators of $\mathcal{U}_i$ satisfying

$$u_i u_i^* = I_{d_i} = u_i^* u_i, \;\; u_i' \overline{u_i} = I_{d_i} = \overline{u_i} u_i',$$

with $u_k' = ((u_{ji}^{(k)}))$ and $\overline{u_k} = ((u_{ij}^{(k)^*}))$. Thus, both u_i and $\bar{u_i}$ are unitaries in $M_{d_i}(A_{u,d_i}(I))$.

7. We recall from Sect. 1.2.2 that the corepresentations β_i canonically induce a corepresentation $\beta = *_i \beta_i$ of $\mathcal{U} := *_i \mathcal{U}_i$ on $\mathcal{H}_D^0$, such that the restriction of β on V_i coincides with β_i for all i.

Lemma 3.1.5 *For an admissible spectral triple* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *which also satisfies the connectedness assumption* **5***, any quantum family of smooth isometries is automatically volume-preserving; hence* $(\mathbf{Q}^{\mathcal{L}})_0 = \mathbf{Q}^{\mathcal{L}}$ *and* $(\mathbf{Q}'_{\mathcal{L}})_0 = \mathbf{Q}'_{\mathcal{L}}$ *as categories.*

Proof Let $(\mathcal{S}, \alpha)$ be any quantum family of smooth isometries, and ω be an arbitrary state on $\mathcal{S}$. By Lemma 3.1.4, it follows that $\tau(\alpha_\omega(x)) = \tau(x)\omega(1)$ for all $x \in \overline{\mathcal{A}}$. Since ω is arbitrary, we have $(\tau \otimes id)(\alpha(x)) = \tau(x)1_{\mathcal{S}}$ for all $x \in \overline{\mathcal{A}}$. □

Lemma 3.1.6 *Let* $(\mathcal{S}, \alpha)$ *be a quantum family of volume-preserving smooth isometries of an admissible spectral triple* $(\mathcal{A}^\infty, \mathcal{H}, D)$*. Moreover, assume that* α *is faithful, i.e., there is no proper* C^**-subalgebra* $\mathcal{S}_1$ *of* $\mathcal{S}$ *such that* $\alpha(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty \otimes \mathcal{S}_1$.

Then $\widetilde{\alpha} : \mathcal{A}^\infty \otimes \mathcal{S} \to \mathcal{A}^\infty \otimes \mathcal{S}$ *defined by*

$$\widetilde{\alpha}(a \otimes b) := \alpha(a)(1 \otimes b)$$

extends to an $\mathcal{S}$*-linear unitary on the Hilbert* $\mathcal{S}$*-module* $\mathcal{H}_D^0 \otimes \mathcal{S}$*, denoted again by* $\widetilde{\alpha}$*. Moreover, there exists a* C^**-ideal* $\mathcal{I}$ *of* $\mathcal{U}$ *and a* C^**-isomorphism* $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ *such that* $\alpha = (\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$ *on* $\mathcal{A}^\infty \subseteq \mathcal{H}_D^0$*, where* $\Pi_{\mathcal{I}}$ *is the canonical map from* $\mathcal{U}$ *to* $\mathcal{U}/\mathcal{I}$*.*

In case there is a CQG structure on $\mathcal{S}$ *given by a coproduct* Δ *such that* $(\mathcal{S}, \Delta, \alpha)$ *is an object in* $(\mathbf{Q}'_{\mathcal{L}})_0$*, the map* $\alpha : \mathcal{A}^\infty \to \mathcal{A}^\infty \otimes \mathcal{S}$ *extends to a unitary corepresentation (denoted again by* α*) of the CQG* $(\mathcal{S}, \Delta)$ *on* $\mathcal{H}_D^0$*. In this case,* $\mathcal{I}$ *is a Woronowicz* C^**-ideal and the* C^**-isomorphism* $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ *is a morphism of compact quantum groups.*

Proof We have $(\tau \otimes \mathrm{id})(\alpha(a)) = \tau(a)1_{\mathcal{S}}$ for all $a \in \mathcal{A}^\infty$ and therefore for all $a \in \overline{\mathcal{A}}$ by continuity. Moreover, since $\alpha(\overline{\mathcal{A}})(1 \otimes \mathcal{S})$ is dense in $\overline{\mathcal{A}} \otimes \mathcal{S}$, we see that $\alpha(\mathcal{A}^\infty)(1 \otimes \mathcal{S})$ is norm-total in $\bar{\mathcal{A}} \otimes \mathcal{S}$. Therefore, the $\mathcal{S}$-linear span of the range of $\alpha(\mathcal{A}^\infty)$ is dense in the Hilbert module $\mathcal{H}_D^0 \otimes \mathcal{S}$. Thus, we are in a position to apply Lemma 1.3.4 to deduce that $\widetilde{\alpha}$ is a $\mathcal{S}$-linear unitary on the Hilbert $\mathcal{S}$-module $\mathcal{H}_D^0 \otimes \mathcal{S}$.

Next, since α_ω leaves each V_i invariant for every state ω on $\mathcal{S}$, it is clear that for all i, $\alpha(V_i) \subseteq V_i \otimes_{\mathrm{alg}} \mathcal{S}$. Let us write $\alpha(e_{ij}) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$ for some $v_{kj}^{(i)}$ $(j, k = 1, \ldots, d_i)$ in $\mathcal{S}$. Then $v_i := ((v_{kj}^{(i)}))$ is a unitary in $M_{d_i}(\mathbb{C}) \otimes \mathcal{S}$ and the $*$-subalgebra generated by $\{v_{kj}^{(i)}, i \geq 0, , j, k \geq 1\}$ is dense in $\mathcal{S}$ since α is faithful.

Since α is a C^*-coaction on $\overline{\mathcal{A}}$, we have $\alpha(e_{ij}^*) = (\alpha(e_{ij}))^* = \sum_k e_{ik}^* \otimes v_{kj}^{(i)^*}$, and since $\{e_{ij}^*\}$ is also an orthonormal basis of V_i, the matrix $((v_{kj}^{(i)^*}))$ is unitary. By universality of $\mathcal{U}_i$, there is a C^*-homomorphism from $\mathcal{U}_i$ to $\mathcal{S}$ sending $u_{kj}^{(i)}$ to $v_{kj}^{(i)}$, and by definition of the free product, this induces a C^*-homomorphism, say Π, from $\mathcal{U}$ onto $\mathcal{S}$, so that $\mathcal{U}/\mathcal{I} \cong \mathcal{S}$, where $\mathcal{I} := \mathrm{Ker}(\Pi)$.

In case $\mathcal{S}$ has a coproduct so that $(\mathcal{S}, \Delta)$ is a CQG and α is a quantum group coaction, it is easy to see that the subalgebra of $\mathcal{S}$ generated by $\{v_{kj}^{(i)},\ i \geq 0,\ j, k = 1, \ldots, d_i\}$ is a Hopf algebra and $\Delta(v_{kj}^{(i)}) = \sum_l v_{kl}^{(i)} \otimes v_{lj}^{(i)}$. It follows that Π is Hopf-algebra morphism and therefore, $\mathcal{I}$ is a Woronowicz C^*-ideal. □

Theorem 3.1.7 *For any admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category $(\mathbf{Q}^{\mathcal{L}})_0$ has a universal (initial) object, say $(\mathcal{G}, \alpha_0)$. Moreover, $\mathcal{G}$ has a coproduct Δ_0 such that $(\mathcal{G}, \Delta_0)$ is a CQG and $(\mathcal{G}, \Delta_0, \alpha_0)$ is a universal object in the category $(\mathbf{Q}'_{\mathcal{L}})_0$. The coaction α_0 is faithful.*

Proof We will use the C^*-algebra $\mathcal{U}$ and the map $\beta : \mathcal{H}_D^0 \to \mathcal{H}_D^0 \otimes \mathcal{U}$. By the definition of β, $\beta(\mathcal{A}_0^\infty) \subseteq \mathcal{A}_0^\infty \otimes_{\mathrm{alg}} \mathcal{U}$. Note that β is only a linear map (unitary) but need not be a $*$-homomorphism.

We construct the universal object as a suitable quotient of $\mathcal{U}$. We will denote by $\Pi_{\mathcal{I}}$ the quotient map from $\mathcal{U}$ onto $\mathcal{U}/\mathcal{I}$. Let $\mathcal{F}$ be the collection of all those C^*-ideals $\mathcal{I}$ of $\mathcal{U}$ such that

$$\Gamma_{\mathcal{I}} := (\mathrm{id} \otimes \Pi_{\mathcal{I}}) \circ \beta : \mathcal{A}_0^\infty \to \mathcal{A}_0^\infty \otimes_{\mathrm{alg}} (\mathcal{U}/\mathcal{I})$$

extends to a C^*-homomorphism from $\bar{\mathcal{A}}$ to $\bar{\mathcal{A}} \otimes (\mathcal{U}/\mathcal{I})$, and

$$(\tau \otimes \mathrm{id})(\Gamma_{\mathcal{I}}(a)) = \tau(a) 1_{\mathcal{U}/\mathcal{I}}\ \forall\, a \in \mathcal{A}_0^\infty,\ \text{ i.e. } (\tau \otimes \mathrm{id})(\beta(a)) - \tau(a) 1_{\mathcal{U}} \in \mathcal{I}.$$

The set $\mathcal{F}$ is nonempty because the C^*-algebra $\mathbb{C}$ is an object in $(\mathbf{Q}^{\mathcal{L}})_0$ and by Lemma 3.1.6 we have an element of $\mathcal{F}$.

Now, let $\mathcal{I}_0$ be the intersection of all ideals in $\mathcal{F}$. We claim that $\mathcal{I}_0$ is again a member of $\mathcal{F}$. In order to prove this claim, we first observe that $\mathrm{Ker}(\mathrm{id} \otimes \Pi_{\mathcal{I}}) = \{X \in \overline{\mathcal{A}} \otimes \mathcal{U} :\ \forall \omega \in \mathcal{A}^* : (\omega \otimes \mathrm{id})(X) \in \mathcal{I}\} = \overline{\mathcal{A}} \otimes \mathcal{I}$. Then we use Lemma 1.1.9 to conclude that $\cap_{\mathcal{I} \in \mathcal{F}} \mathrm{Ker}(\mathrm{id} \otimes \Pi_{\mathcal{I}}) = \mathrm{Ker}(\mathrm{id} \otimes \Pi_{\mathcal{I}_0})$. Therefore, for $a \in \mathcal{A}_0^\infty$ and $\mathcal{I} \in \mathcal{F}$, we have

$||\Gamma_{\mathcal{I}_0}(a)|| = ||\beta(a) + \overline{\mathcal{A}} \otimes \mathcal{I}_0|| = ||\beta(a) + \mathrm{Ker}(\mathrm{id} \otimes \Pi_{\mathcal{I}_0})|| = ||\beta(a) + \cap_{\mathcal{I} \in \mathcal{F}} \mathrm{Ker}(\mathrm{id} \otimes \Pi_{\mathcal{I}})||$

$= \sup_{\mathcal{I} \in \mathcal{F}} ||\beta(a) + \mathrm{Ker}(\mathrm{id} \otimes \Pi_{\mathcal{I}})|| = \sup_{\mathcal{I} \in \mathcal{F}} ||\Gamma_{\mathcal{I}}(\mathrm{a})|| \leq \sup_{\mathcal{I} \in \mathcal{F}} ||\mathrm{a}|| = ||\mathrm{a}||$, where we have used Lemma 1.1.1 and the contractivity of the C^*-homomorphism $\Gamma_{\mathcal{I}}$.

Since $\mathcal{A}_0^\infty$ is dense in $\bar{\mathcal{A}}$, $\Gamma_{\mathcal{I}_0}$ extends to a norm-contractive map on $\bar{\mathcal{A}}$. Moreover, for $a, b \in \bar{\mathcal{A}}$ and for $\mathcal{I} \in \mathcal{F}$, we have $\Gamma_{\mathcal{I}}(ab) = \Gamma_{\mathcal{I}}(a)\Gamma_{\mathcal{I}}(b)$. Since $\Pi_{\mathcal{I}} = \Pi_{\mathcal{I}} \circ \Pi_{\mathcal{I}_0}$, we can rewrite the homomorphic property of $\Gamma_{\mathcal{I}}$ as

$$\Gamma_{\mathcal{I}_0}(ab) - \Gamma_{\mathcal{I}_0}(a)\Gamma_{\mathcal{I}_0}(b) \in \bar{\mathcal{A}} \otimes (\mathcal{I}/\mathcal{I}_0).$$

This holds for all $\mathcal{I} \in \mathcal{F}$ and hence we get $\Gamma_{\mathcal{I}_0}(ab) - \Gamma_{\mathcal{I}_0}(a)\Gamma_{\mathcal{I}_0}(b) \in \bigcap_{\mathcal{I} \in \mathcal{F}} \bar{\mathcal{A}} \otimes (\mathcal{I}/\mathcal{I}_0) = (0)$, i.e. $\Gamma_{\mathcal{I}_0}$ is a homomorphism. Similarly, it can be shown that it is a $*$-homomorphism. Since each β_i is a unitary corepresentation of the compact quantum group $\mathcal{U}_i$ on the finite dimensional space V_i, it follows that $\beta_i(V_i)(1 \otimes \mathcal{U}_i)$ is total in $V_i \otimes \mathcal{U}_i$. Thus, $\forall\, i$ and $\forall\, w \in V_i$, $w \otimes 1_{\mathcal{U}_i} \equiv w \otimes 1_{\mathcal{U}}$ belongs to the linear span of $\beta_i(V_i)(1 \otimes \mathcal{U}_i) \subset \beta(V_i)(1 \otimes \mathcal{U})$. Therefore, $\mathcal{A}_0^\infty \otimes 1_{\mathcal{U}}$ is a subset of the span of $\beta(\mathcal{A}_0^\infty)(1 \otimes \mathcal{U})$ and hence $\mathcal{A}_0^\infty \otimes 1_{\frac{\mathcal{U}}{\mathcal{I}_0}}$ is spanned by $\Gamma_{\mathcal{I}_0}(\mathcal{A}_0^\infty)(1 \otimes \mathcal{U}/\mathcal{I}_0)$. Since $\mathcal{A}_0^\infty$ is norm-dense in $\overline{\mathcal{A}}$ and the quotient map is contractive, $\overline{\mathcal{A}} \otimes \mathcal{U}/\mathcal{I}_0$ is the closure of the span of $\Gamma_{\mathcal{I}_0}(\mathcal{A}_0^\infty)(1 \otimes \mathcal{U}/\mathcal{I}_0)$. Thus, we have proved that $(\mathcal{U}/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})$ is an object of the category $\mathbf{Q}$. We also observe that $(\mathcal{U}/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})$ is an object of the category $(\mathbf{Q}^{\mathcal{L}})_0$. This follows because for $a \in \mathcal{A}_0^\infty$, we have $(\tau \otimes \mathrm{id})(\beta(a)) - \tau(a)1_{\mathcal{U}} \in \mathcal{I}\ \forall\, \mathcal{I} \in \mathcal{F}$ and thus $(\tau \otimes \mathrm{id})(\beta(a)) - \tau(a)1_{\mathcal{U}} \in \mathcal{I}_0$.

Now, we want to show that $\mathcal{G} := \mathcal{U}/\mathcal{I}_0$ is a universal object in $(\mathbf{Q}^{\mathcal{L}})_0$. To this end, let $(\mathcal{S}, \alpha)$ be any object of the category $(\mathbf{Q}^{\mathcal{L}})_0$. We note that we can assume the coaction to be faithful. Indeed, if the coaction is not faithful, we can always replace $\mathcal{S}$ by the C^*-subalgebra generated by the elements $\{v_{kj}^{(i)}\}$ which appears in the proof of Lemma 3.1.6. However, by virtue of Lemma 3.1.6 we can also assume that there exists an ideal $\mathcal{I}$ in $\mathcal{F}$ such that $\mathcal{S}$ is isomorphic with $\mathcal{U}/\mathcal{I}$. Since $\mathcal{I}_0$ is contained in $\mathcal{I}$, we have a C^*-homomorphism from $\mathcal{U}/\mathcal{I}_0$ onto $\mathcal{U}/\mathcal{I}$ which sends $x + \mathcal{I}_0$ to $x + \mathcal{I}$. This C^*-homomorphism is a morphism in $(\mathbf{Q}^{\mathcal{L}})_0$ and it is indeed the unique such morphism as it is uniquely determined on the dense subalgebra generated by $\{u_{kj}^{(i)} + \mathcal{I}_0,\ i \geq 0,\ j, k \geq 1\}$ of $\mathcal{G}$.

We now construct the coproduct on $\mathcal{G} = \mathcal{U}/\mathcal{I}_0$. Let $\alpha^{(2)}$ denote the map $(\Gamma_{\mathcal{I}_0} \otimes \mathrm{id}) \circ \Gamma_{\mathcal{I}_0} : \overline{\mathcal{A}} \to \overline{\mathcal{A}} \otimes \mathcal{G} \otimes \mathcal{G}$. Clearly, $(\mathcal{G} \otimes \mathcal{G}, \alpha^{(2)})$ is an object in the category $(\mathbf{Q}^{\mathcal{L}})_0$. By the universality of $(\mathcal{G}, \Gamma_{\mathcal{I}_0})$, we have a unique unital C^*-homomorphism $\Delta_0 : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ which satisfies

$$(\mathrm{id} \otimes \Delta_0) \circ \Gamma_{\mathcal{I}_0}(x) = \alpha^{(2)}(x)\ \forall x \in \overline{\mathcal{A}}.$$

Putting $x = e_{ij}$, we get

$$\sum_l e_{il} \otimes (\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0})\left(\sum_k u_{lk}^{(i)} \otimes u_{kj}^{(i)}\right) = \sum_l e_{il} \otimes \Delta_0(\pi_{\mathcal{I}_0}(u_{lj}^{(i)})).$$

Let $\widetilde{\Delta}$ denote the coproduct on $\mathcal{U}$. Since $\widetilde{\Delta}(u_{lj}^{(i)}) = \sum_k u_{lk}^{(i)} \otimes u_{kj}^{(i)}$, by comparing coefficients of e_{il}, we deduce that

$$(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) \circ \widetilde{\Delta} = \Delta_0 \circ \pi_{\mathcal{I}_0} \tag{3.1.1}$$

on the linear span of $\{u_{jk}^{(i)}, i \geq 0,\ j, k \geq 1\}$, and thus on the whole of $\mathcal{U}$. Therefore, $\Delta_0(\mathcal{I}_0) \subseteq \mathrm{Ker}(\pi_{\mathcal{I}_0} \otimes \pi_{\mathcal{I}_0}) = \mathrm{Span}(\mathcal{I}_0 \otimes 1 + 1 \otimes \mathcal{I}_0) \subset \mathcal{U} \otimes \mathcal{U}$. Hence, $\mathcal{I}_0$ is a Woronowicz C^*-ideal so that $\mathcal{G} = \mathcal{U}/\mathcal{I}_0$ has the canonical compact quantum group structure as a quantum subgroup of $\mathcal{U}$. It is clear from the relation (3.1.1) that Δ_0 coincides with the canonical coproduct of the quantum subgroup $\mathcal{U}/\mathcal{I}_0$ inherited from that of $\mathcal{U}$. Since Lemma 3.1.6 implies that any CQG $(\mathcal{G}, \Phi)$ coacting smoothly and isometrically on the given spectral triple is isomorphic with a quantum subgroup

$\mathcal{U}/\mathcal{I}$, for some Woronowicz C^*-ideal $\mathcal{I}$ of $\mathcal{U}$, it is easy to check that the object $(\mathcal{G}, \Delta_0, \Gamma_{\mathcal{I}_0})$ is universal in the category $(\mathbf{Q}'_{\mathcal{L}})_0$.

We sketch the argument for the faithfulness of the coaction α_0. Let $\mathcal{G}_1 \subset \mathcal{G}$ be a $*$-subalgebra of $\mathcal{G}$ such that $\alpha_0(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}} \otimes \mathcal{G}_1$. Then it can be easily checked that $(\mathcal{G}_1, \Delta_0, \alpha_0)$ is also a universal object. Moreover, by the universality of $\mathcal{G}$, there is a unique morphism, say j, from $\mathcal{G}$ to $\mathcal{G}_1$. If $i : \mathcal{G}_1 \to \mathcal{G}$ denotes the inclusion, the map $i \circ j$ is a morphism from $\mathcal{G}$ to itself. By applying universality again, we deduce that $i \circ j = \mathrm{id}_{\mathcal{G}}$, so in particular, i is onto, i.e. $\mathcal{G}_1 = \mathcal{G}$. □

Definition 3.1.8 We shall call the universal object $(\mathcal{G}, \Delta_0)$ obtained in Theorem 3.1.7 the quantum isometry group of the admissible spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and denote it by $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}^\infty, \mathcal{H}, D)$, or just $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}^\infty)$ (or sometimes $\mathrm{QISO}^{\mathcal{L}}(\bar{\mathcal{A}})$) if the spectral triple is understood from the context.

Remark 3.1.9 It follows from Lemma 3.1.5 that in case the spectral triple satisfies both the admissibility and the connectedness assumption, $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}^\infty, \mathcal{H}, D)$ is the universal object in the categories $\mathbf{Q}^{\mathcal{L}}$ and $\mathbf{Q}'_{\mathcal{L}}$.

Remark 3.1.10 $\mathrm{QISO}^{\mathcal{L}}$ is a quantum subgroup of the CQG $\mathcal{U} = *_i A_{u,d_i}(I)$. As $A_{u,d_i}(I)$ satisfies $\kappa^2 = \mathrm{id}$, (by Remark 1.3.11) the same is satisfied by $QISO^{\mathcal{L}}$ so that by Remark 1.2.16, $\mathrm{QISO}^{\mathcal{L}}$ has tracial Haar state.

3.1.2 Discussions on the Admissibility Conditions

In this subsection, we discuss the assumptions of admissibility as well as the connectedness condition **5.** which were used to prove the existence of the quantum isometry group. Indeed, some of these assumptions can be avoided and we point out alternative conditions to ensure the existence of the universal object.

In our definition of quantum family of smooth isometries, we assumed that the coaction α keeps $\mathcal{A}_0^\infty$ invariant. As seen in Chap. 2, in the classical case, this condition is equivalent to the condition that α keeps the possibly bigger subspace $\mathcal{A}^\infty = C^\infty(M)$ invariant. Therefore, in the noncommutative situation, we are led to look for sufficient conditions for invariance of $\mathcal{A}^\infty$ under α. We state and prove below one such result.

Lemma 3.1.11 *If an admissible spectral triple* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *satisfies the condition* $\bigcap \mathrm{Dom}(\mathcal{L}^n) = \mathcal{A}^\infty$, *and if* $\alpha : \bar{\mathcal{A}} \to \bar{\mathcal{A}} \otimes \mathcal{S}$ *is a smooth isometric coaction on* $\mathcal{A}^\infty$ *by a CQG* $\mathcal{S}$, *then for every state* ϕ *on* $\mathcal{S}$, $\alpha_\phi (= (\mathrm{id} \otimes \phi)\alpha)$ *keeps* $\mathcal{A}^\infty$ *invariant.*

Proof By Lemma 1.3.4, the map $\widetilde{\alpha}$ from $\overline{\mathcal{A}} \otimes \mathcal{S}$ to itself extends to an $\mathcal{S}$-linear unitary on the Hilbert $\mathcal{S}$-module $\mathcal{H}_D^0 \otimes \mathcal{S}$ so that $\widetilde{\alpha}$ is a unitary in $\mathcal{M}(\mathcal{K}(\mathcal{H}_D^0) \otimes \mathcal{S})$. For any state ϕ on $\mathcal{S}$, $\alpha_\phi = (\mathrm{id} \otimes \phi)(\widetilde{\alpha}) \in \mathcal{B}(\mathcal{H}_D^0)$. Since α is a smooth and isometric coaction, the bounded operator α_ϕ commutes with the self-adjoint operator $\mathcal{L}$ on $\mathcal{A}_0^\infty$. Since $\mathcal{A}_0^\infty$ is a core for $\mathcal{L}$, α_ϕ commutes with $\mathcal{L}^n$ for all n so that it keeps $\mathcal{A}^\infty = \bigcap_n \mathrm{Dom}(\mathcal{L}^n)$ invariant. □

Next, we come to the connectedness assumption **5.** in Lemma 3.1.5. We remark that the condition **5.** is not actually necessary there. We do have admissible spectral triples not satisfying **5.** for which the coaction automatically preserves the volume form. For example, let X be a set consisting of n points and view $\mathcal{A} = C(X)$ on X as diagonal matrices acting on $\mathcal{H} = \mathbb{C}^n$. If we take the Dirac operator D to be the identity operator, it follows that the volume form τ is given by $\tau(\delta_i) = 1$ for $i = 1, \ldots, n$, where δ_i denotes the characteristic function of the singleton set $\{i\}$. Now if $(\mathcal{S}, \Delta, \alpha)$ is a CQG acting faithfully on $\mathcal{A}$, with $\alpha(e_i) = \sum_{i=1}^{n} e_j \otimes x_{ji}$, then the elements $\{x_{ji}\}_{i,j}$ satisfy the relations of the quantum permutation group $A_s(n)$ as in Chap. 1. Therefore, $\sum_j x_{ji} = 1_{\mathcal{S}}$ for each i and hence α is automatically τ-invariant. Therefore, $(\mathbf{Q}'_{\mathcal{L}})_0 = \mathbf{Q}'_{\mathcal{L}}$ even if $(\mathcal{A}, \mathcal{H}, D)$ is not admissible whenever $n \geq 2$ as $\mathcal{L} = 0$ has n-dimensional kernel.

On the other hand, with $\mathcal{A} = M_n(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^n$, $D = I$, we get an example of admissible spectral triple not satisfying **5** for which volume-preserving property is not automatic. Indeed, it suffices to observe that τ is equal to the usual trace of $M_n(\mathbb{C})$, and there are coactions of compact quantum groups on $M_n(\mathbb{C})$ which do not preserve the trace (see [2]). Observe that any quantum group coaction is trivially smooth and isometric in this case since $\mathcal{L} = 0$.

In [3], Connes and Moscovici introduced the notion of twisted or type-III spectral triples. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a σ-twisted, Lipschitz-regular spectral triple of finite-dimension n. Thus, σ is a unital algebra automorphism of $\mathcal{A}^\infty$ such that $\forall a \in \mathcal{A}^\infty, \sigma(a)^* = \sigma^{-1}(a^*)$ and $Da - \sigma(a)D \in \mathcal{B}(\mathcal{H})$. Using the faithful positive linear functional τ given by $\tau(X) = \frac{Tr_\omega(X|D|^{-n})}{Tr_\omega(|D|^{-n})}$, we can construct analogues of the Hilbert spaces $\mathcal{H}^0_D$ and $\mathcal{H}^1_D$, by replacing the map d_D by d^σ_D which is defined by $d^\sigma_D(a) = Da - \sigma(a)D$. We will say that a σ-twisted spectral triple is admissible if the conditions **1.–4.** hold with d_D replaced by d^σ_D. We briefly discuss how to generalize our formulation of quantum isometry groups in this framework. We shall use the notations V_i, $\{\lambda_i\}$ and $\{e_{ij}\}$ as before and define quantum families of smooth isometries, smooth isometric coactions by compact quantum groups and volume preserving coactions as in the case of usual spectral triples. We observe that in this case τ is not necessarily a trace and hence $\{e^*_{ij}\}$ need not be orthogonal. However, they are still linearly independent. Therefore, $Q_i = ((\langle e^*_{ij}, e^*_{ik}\rangle))^{d_i}_{j,k=1} = ((\tau(e_{ij}e^*_{ik})))^{d_i}_{j,k=1}$ is a nonsingular $d_i \times d_i$ matrix.

Let $\mathcal{U}^\sigma_i = A_u(Q_i)$ be the compact quantum groups discussed in Sect. 1.3.2, so that it is the universal C^*-algebra generated by $\{u^{Q_i}_{kj}, k, j = 1, \ldots, d_i\}$ such that $u := ((u^{Q_i}_{kj}))$ satisfies

$$uu^* = I_n = u^*u, \quad u'Q_i\overline{u}Q_i^{-1} = I_n = Q_i\overline{u}Q_i^{-1}u'. \tag{3.1.2}$$

We set $\beta^\sigma_i : V_i \to V_i \otimes_{\text{alg}} \mathcal{U}^\sigma_i$ by

$$\beta^\sigma_i(e_{ij}) = \sum_k e_{ik} \otimes u^{Q_i}_{kj},$$

and then define a unitary corepresentation β^σ of the free product $\mathcal{U}^\sigma = *_i \mathcal{U}_i^\sigma$ on $\mathcal{H}$ by taking $\beta^\sigma = *_i \beta_i^\sigma$ as before.

By replacing $\mathcal{U}$ with $\mathcal{U}^\sigma$, it is easy to derive an analogue of Lemma 3.1.6. We continue to use the notations used in the proof of Lemma 3.1.6. We observe that the map $\widetilde{\alpha}$ is an $\mathcal{S}$-linear unitary on the Hilbert module $\mathcal{H}_D^0 \otimes \mathcal{S}$, and it keeps the subspace $V_i' \otimes \mathcal{S} \equiv \mathrm{Sp}\{e_{ij}^*, j = 1, \ldots, d_i\} \otimes \mathcal{S}$ invariant so that

$$\alpha(e_{ij}^*) = \sum_k e_{ik}^* \otimes v_{kj}^{(i)*}. \tag{3.1.3}$$

Therefore, $\mathcal{S}$-valued matrix $\overline{v_i} \equiv ((v_{kj}^{(i)^*}))$ is invertible in $M_{d_i}(\mathbb{C}) \otimes \mathcal{S}$. Now we take the $\mathcal{S}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ on the Eq. (3.1.3) and using the fact that $\langle \alpha(x), \alpha(y) \rangle_{\mathcal{S}} = \tau(x^* y) 1_{\mathcal{S}}$, we obtain $Q_i = v_i' Q_i \overline{v_i}$. Therefore, $Q_i^{-1} v_i' Q_i$ is the inverse of $\overline{v_i}$, from which we see that the relations (1.3.2) are satisfied with u replaced by v_i. This induces natural C^*-homomorphism from $\mathcal{U}_i^\sigma$ into $\mathcal{S}$. By replacing $\mathcal{U}$ and β by $\mathcal{U}^\sigma$ and β^σ respectively, the rest of the proof of Lemma 3.1.6 as well as the proof of Theorem 3.1.7 go through verbatim. This allows us to define quantum isometry group of an admissible σ-twisted spectral triple.

3.2 Definition and Existence of the Quantum Group of Orientation Preserving Isometries

3.2.1 Motivation

The formulation of quantum isometry groups in terms of the Laplacian as in the previous section has a drawback from the viewpoint of noncommutative geometry since it needs a 'good' Laplacian to exist. In noncommutative geometry it may not be always easy to verify such an assumption about the Laplacian. Therefore, it is more natural to have a formulation of the quantum isometry group in terms of the Dirac operator directly. This is what we aim to achieve in this section.

We have already seen in Theorem 2.1.12 that a group action on a Riemannian spin manifold lifts to a unitary representation on the Hilbert space $\mathcal{H}$ of spinors which commutes with the Dirac operator D if and only if the action on the manifold is an orientation preserving and isometric. This motivates us to define the quantum analogue of the group of orientation-preserving Riemannian isometries of a possibly noncommutative manifold given by a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ by considering a category $\mathbf{Q}'$ of compact quantum groups having unitary corepresentation, say U, on $\mathcal{H}$, which commutes with D, and the coaction on $\mathcal{B}(\mathcal{H})$ obtained by conjugation maps $\mathcal{A}^\infty$ into its weak closure. A universal object in this category, if it exists, should be called the 'quantum group of orientation preserving Riemannian isometries' of the underlying spectral triple. The subtle point to note here is that unlike the classical group actions on $\mathcal{B}(\mathcal{H})$ which always preserve the usual trace, a quantum group coaction may not do so. In fact, in the view of Theorem 2.3.12, it makes it quite natural to work in the setting of twisted spectral data.

As in the previous section, we actually consider a bigger category called the category of 'quantum families of smooth orientation preserving Riemannian isometries' motivated by the ideas of Woronowicz and Soltan [4, 5], We present several explicit computations of such universal quantum groups in later chapters of the book. It may be relevant to point out here that it was not clear whether one could accommodate the spectral triples on $SU_\mu(2)$ and the Podles' spheres $S^2_{\mu,c}$ constructed in [6, 7] respectively in the framework of $\mathrm{QISO}^{\mathcal{L}}$, since it is very difficult to get a workable description of the space of 'noncommutative' forms and the Laplacian for these examples. However, formulation presented in this section makes it possible to accommodate these examples. More precisely, we have identified $U_\mu(2)$ and $SO_\mu(3)$ as the universal quantum group of orientation preserving (and suitable twisted volume preserving for $S^2_{\mu,c}$) isometries for the spectral triples on $SU_\mu(2)$ and $S^2_{\mu,c}$ respectively. The computations for $S^2_{\mu,c}$ have been presented in Chap. 4.

We conclude this subsection with an important remark about the use of the phrase 'orientation-preserving' in our terminology. We recall from Remark 2.2.6 that by a 'classical spectral triple' we always mean the spectral triple obtained by the Dirac operator on the spinors. This is important since the Hodge Dirac operator $d + d^*$ on the L^2-space of differential forms also gives a spectral triple of compact type on any compact Riemannian (not necessarily with a spin structure) manifold M, and the action of the full isometry group $\mathrm{ISO}(M)$ (and not just the subgroup of orientation preserving isometries $\mathrm{ISO}^+(M)$, even when M is orientable) lifts to a unitary representation on this space commuting with $d + d^*$. In fact, the category of groups acting smoothly on M such that the action is implemented by a unitary representation commuting with $d + d^*$, has $\mathrm{ISO}(M)$, (and not $\mathrm{ISO}^+(M)$) as its universal object. So, we need to stick to the Dirac operator on spinors to obtain the group of orientation preserving isometries in the usual geometric sense. This also has a natural quantum generalization, as we shall see in Sect. 3.2.4.

3.2.2 *Quantum Group of Orientation-Preserving Isometries of an R-twisted Spectral Triple*

In view of the characterization of orientation-preserving isometric action on a classical manifold (Theorem 2.1.13), we give the following definitions.

Definition 3.2.1 A quantum family of orientation preserving isometries for the (odd) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is given by a pair $(\mathcal{S}, U)$ where $\mathcal{S}$ is a separable unital C^*-algebra and U is a linear map from $\mathcal{H}$ to $\mathcal{H} \bar{\otimes} \mathcal{S}$ such that $\widetilde{U}$ given by $\widetilde{U}(\xi \otimes b) = U(\xi)b$ $(\xi \in \mathcal{H},\ b \in \mathcal{S})$ extends to a unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S})$ satisfying the following:

(i) for every state ϕ on $\mathcal{S}$ we have $U_\phi D = DU_\phi$, where $U_\phi := (\mathrm{id} \otimes \phi)(\widetilde{U})$;

(ii) $(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_U(a) \in (\mathcal{A}^\infty)''$ for all a in $\mathcal{A}^\infty$ and state ϕ on $\mathcal{S}$, where $\mathrm{ad}_U(x) := \widetilde{U}(x \otimes 1)\widetilde{U}^*$ for x belonging to $\mathcal{B}(\mathcal{H})$.

In case the C^*-algebra $\mathcal{S}$ has a coproduct Δ such that $(\mathcal{S}, \Delta)$ is a compact quantum group and U is a unitary corepresentation of $(\mathcal{S}, \Delta)$ on $\mathcal{H}$, we say that $(\mathcal{S}, \Delta)$ coacts by orientation-preserving isometries on the spectral triple.

In case the spectral triple is even with the grading operator γ, a quantum family of orientation preserving isometries $(\mathcal{A}^\infty, \mathcal{H}, D, \gamma)$ will be defined exactly as above, with the only extra condition being that $\widetilde{U}$ commutes with $\gamma \otimes 1$.

From now on, we will mostly consider odd spectral triples. However let us remark that in the even case, all the definitions and results obtained by us will go through with some obvious modifications. We also remark that all our spectral triples are of compact type.

Consider the category $\mathbf{Q} \equiv \mathbf{Q}(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}(D)$ with the object-class consisting of all quantum families of orientation preserving isometries $(\mathcal{S}, U)$ of the given spectral triple, and the set of morphisms $\mathrm{Mor}((\mathcal{S}, U), (\mathcal{S}', U'))$ being the set of unital C^*-homomorphisms $\Phi : \mathcal{S} \to \mathcal{S}'$ satisfying $(\mathrm{id} \otimes \Phi)(U) = U'$. We also consider another category $\mathbf{Q}' \equiv \mathbf{Q}'(\mathcal{A}^\infty, \mathcal{H}, D) \equiv \mathbf{Q}'(D)$ whose objects are triplets $(\mathcal{S}, \Delta, U)$, where $(\mathcal{S}, \Delta)$ is a compact quantum group acting by orientation preserving isometries on the given spectral triple, where U is the corresponding unitary corepresentation. The morphisms are the homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families of orientation preserving isometries. The forgetful functor $F : \mathbf{Q}' \to \mathbf{Q}$ is clearly faithful, and we can view $F(\mathbf{Q}')$ as a subcategory of $\mathbf{Q}$.

Unfortunately, in general $\mathbf{Q}'$ or $\mathbf{Q}$ will not have a universal object. It is easily seen by taking the standard example $\mathcal{A}^\infty = M_n(\mathbb{C})$, $\mathcal{H} = \mathbb{C}^n$, $D = I$. Any CQG having a unitary corepresentation on $\mathbb{C}^n$ is an object of $\mathbf{Q}'(M_n(\mathbb{C}), \mathbb{C}^n, I)$. But by Proposition 1.3.9, there is no universal object in this category. However, the fact that comes to our rescue is that a universal object exists in each of the subcategories which correspond to the CQG coactions preserving a given faithful functional on M_n. On the other hand, given any equivariant spectral triple, we recall from Theorem 2.3.12 of Chap. 2 that there is a (not necessarily unique) faithful functional which is preserved by the CQG coaction. This motivates the following definition.

Before that, let us recall the definition of an R-twisted spectral data from Sect. 2.3.4. Given an R-twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D, R)$, we note that $(\mathcal{A}^\infty, \mathcal{H}, D, R^{-1})$ is also an R^{-1} twisted spectral data. Let us denote by $\tau_{R^{-1}}$ the canonical functional attached to the spectral data.

Definition 3.2.2 Given an R-twisted spectral data $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ of compact type, a quantum family of orientation preserving isometries $(\mathcal{S}, U)$ of $(\mathcal{A}^\infty, \mathcal{H}, D)$ is said to preserve the R and R^{-1}-twisted volume form, (simply said to be volume-preserving if R is understood) if one has $(\tau_R \otimes \mathrm{id})(\mathrm{ad}_U(x)) = \tau_R(x).1_{\mathcal{S}}$ and $(\tau_{R^{-1}} \otimes \mathrm{id})(\mathrm{ad}_{U^{-1}}(x)) = \tau_{R^{-1}}(x).1_{\mathcal{S}}$ for all x in $\mathcal{E}_D$, where $\mathcal{E}_D$ and τ_R are as in Proposition 2.3.12. We shall also call $(\mathcal{S}, U)$ a quantum family of orientation-preserving isometries of the R-twisted spectral triple.

If, furthermore, the C^*-algebra $\mathcal{S}$ has a coproduct Δ such that $(\mathcal{S}, \Delta)$ is a CQG and U is a unitary corepresentation of $(\mathcal{S}, \Delta)$ on $\mathcal{H}$, we say that $(\mathcal{S}, \Delta)$ coacts by volume and orientation-preserving isometries on the R-twisted spectral triple.

We shall consider the categories $\mathbf{Q}_R \equiv \mathbf{Q}_R(D)$ and $\mathbf{Q}'_R \equiv \mathbf{Q}'_R(D)$ which are the full subcategories of $\mathbf{Q}$ and $\mathbf{Q}'$ respectively, obtained by restricting the object-classes to the volume-preserving quantum families.

Remark 3.2.3 We have already observed in Chap. 1 that for any object $(\mathcal{S}, \Delta, U)$ of $\mathbf{Q}'$, preservation of one of the two functionals τ_R and $\tau_{R^{-1}}$ by ad_U implies the preservation of both. Thus, in the definition of $\mathbf{Q}'_R$, it is enough to have the condition of τ_R preservation.

Let us mention here that in our paper [8] where the notion of quantum group of orientation preserving isometries was originally formulated, we made a mistake by not adding the condition of $\tau_{R^{-1}}$ invariance in the definition of quantum families of orientation and volume preserving isometries. The definition given above rectifies this mistake.

Let us now fix a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ which is of compact type. The C^*-algebra generated by $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{A}$. Let $\lambda_0 = 0, \lambda_1, \lambda_2, \cdots$ be the eigenvalues of D with V_i denoting the (d_i-dimensional, $0 \leq d_i < \infty$) eigenspace for λ_i. Let $\{e_{ij}, j = 1, \ldots, d_i\}$ be an orthonormal basis of V_i. We also assume that there is a positive invertible R on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is an R-twisted spectral data. The operator R must have the form $R|_{V_i} = R_i$, say, with R_i positive invertible $d_i \times d_i$ matrix. Let us denote the CQG $A_{u,d_i}(R_i^T)$ by $\mathcal{U}_i$, with its canonical unitary corepresentation β_i on $V_i \cong \mathbb{C}^{d_i}$, given by $\beta_i(e_{ij}) = \sum_k e_{ik} \otimes u_{kj}^{R_i^T}$. Let $\mathcal{U}$ be the free product of $\mathcal{U}_i, i = 1, 2, \ldots$ and $\beta = *_i \beta_i$ be the corresponding free product corepresentation of $\mathcal{U}$ on $\mathcal{H}$. We shall also consider the corresponding unitary element $\widetilde{\beta}$ in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U})$.

The next two results are analogues of Lemma 3.1.6 and Theorem 3.1.7 and so we sketch their proofs and refer to [8] for the details.

Lemma 3.2.4 *Consider the R-twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ and let $(\mathcal{S}, U)$ be a quantum family of volume and orientation preserving isometries of the given spectral triple. Moreover, assume that the map U is faithful in the sense that there is no proper C^*-subalgebra $\mathcal{S}_1$ of $\mathcal{S}$ such that $\widetilde{U}$ belongs to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{S}_1)$. Then we can find a C^*-isomorphism $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ between $\mathcal{S}$ and a quotient of $\mathcal{U}$ by a C^*-ideal $\mathcal{I}$ of $\mathcal{U}$, such that $U = (\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$, where $\Pi_{\mathcal{I}}$ denotes the quotient map from $\mathcal{U}$ to $\mathcal{U}/\mathcal{I}$.*

If, furthermore, there is a compact quantum group structure on $\mathcal{S}$ given by a coproduct Δ such that $(\mathcal{S}, \Delta, U)$ is an object in $\mathbf{Q}'_R$, the ideal $\mathcal{I}$ is a Woronowicz C^*-ideal and the C^*-isomorphism $\phi : \mathcal{U}/\mathcal{I} \to \mathcal{S}$ is a morphism of compact quantum groups.

Proof It is clear that U maps V_i into $V_i \otimes_{\mathrm{alg}} \mathcal{S}$ for each i. Thus, if $v_{kj}^{(i)}$ $(j, k = 1, \ldots, d_i)$ are the elements of $\mathcal{S}$ such that $U(e_{ij}) = \sum_k e_{ik} \otimes v_{kj}^{(i)}$, then as in Lemma 3.1.6,

$v_i := ((v_{kj}^{(i)}))$ is a unitary in $M_{d_i}(\mathbb{C}) \otimes_{\text{alg}} \mathcal{S}$ and the $*$-subalgebra generated by all $\{v_{kj}^{(i)}, i \geq 0, , j, k \geq 1\}$ is dense in $\mathcal{S}$.

Consider the $*$-homomorphism α_i from the finite dimensional C^* algebra $\mathcal{A}_i \cong M_{d_i}(\mathbb{C})$ generated by the rank one operators $\{|e_{ij} >< e_{ik}|, j, k = 1, \dots, d_i\}$ to $\mathcal{A}_i \otimes_{\text{alg}} \mathcal{S}$ given by $\alpha_i(y) = \widetilde{U}(y \otimes 1)\widetilde{U}^*|_{V_i}$. Clearly, the restrictions of the functionals τ_R and $\tau_{R^{-1}}$ on $\mathcal{A}_i$ are nothing but the functionals given by $\text{Tr}(R_i \cdot)$ and $\text{Tr}(R_i^{-1} \cdot)$ respectively, where Tr denotes the usual trace of matrices. Since α_i preserves these functionals by assumption, we get, by the universality of $\mathcal{U}_i$ given in Lemma 1.3.12, a C^*-homomorphism from $\mathcal{U}_i$ to $\mathcal{S}$ sending $u_{kj}^{(i)} \equiv u_{kj}^{R_i^T}$ to $v_{kj}^{(i)}$, and by definition of the free product, this induces a C^*-homomorphism, say Π, from $\mathcal{U}$ onto $\mathcal{S}$, so that $\mathcal{U}/\mathcal{I} \cong \mathcal{S}$, where $\mathcal{I} := \text{Ker}(\Pi)$. The rest of the proof is exactly as in Lemma 3.1.6 and hence omitted. □

Theorem 3.2.5 *For any R-twisted spectral triple of compact type $(\mathcal{A}^\infty, \mathcal{H}, D)$, the category $\mathbf{Q}_R$ of quantum families of volume and orientation preserving isometries has a universal (initial) object, say $(\widetilde{\mathcal{G}}, U_0)$. Moreover, $\widetilde{\mathcal{G}}$ has a coproduct Δ_0 such that $(\widetilde{\mathcal{G}}, \Delta_0)$ is a compact quantum group and $(\widetilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category $\mathbf{Q}'_R$. The corepresentation U_0 is faithful.*

Proof For any C^*-ideal $\mathcal{I}$ of $\mathcal{U}$, we shall denote by $\Pi_{\mathcal{I}}$ the canonical quotient map from $\mathcal{U}$ onto $\mathcal{U}/\mathcal{I}$, and let $\Gamma_{\mathcal{I}} = (\text{id} \otimes \Pi_{\mathcal{I}}) \circ \beta$. Clearly, $\widetilde{\Gamma_{\mathcal{I}}} = (\text{id} \otimes \pi_{\mathcal{I}}) \circ \widetilde{\beta}$ is a unitary element of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U}/\mathcal{I})$. Let $\mathcal{F}$ be the collection of all those C^*-ideals $\mathcal{I}$ of $\mathcal{U}$ such that $(\text{id} \otimes \omega) \circ \text{ad}_{\widetilde{\Gamma_{\mathcal{I}}}}$ maps $\mathcal{A}^\infty$ into $\mathcal{A}''$ for every state ω (equivalently, every bounded linear functional) on $\mathcal{U}/\mathcal{I}$. This collection is nonempty, since the trivial one-dimensional C^*-algebra $\mathbb{C}$ gives an object in $\mathbf{Q}_R$ and by Lemma 3.2.4 we do get a member of $\mathcal{F}$. Now, let $\mathcal{I}_0$ be the intersection of all the ideals in $\mathcal{F}$. We claim that $\mathcal{I}_0$ is again a member of $\mathcal{F}$. Indeed, in the notation of Lemma 1.1.2, it is clear that for a in $\mathcal{A}^\infty$, $(\text{id} \otimes \phi) \circ \widetilde{\Gamma}_{\mathcal{I}_0}(a)$ belongs to $\mathcal{A}''$ for all ϕ in the convex hull of $\Omega \bigcup (-\Omega)$. Now, for any state ω on $\mathcal{U}/\mathcal{I}_0$, we can find, by Lemma 1.1.2, a net ω_j in the above convex hull (so in particular $\|\omega_j\| \leq 1$ for all j) such that $\omega(x + \mathcal{I}_0) = \lim_j \omega_j(x + \mathcal{I}_0)$ for all x in $\mathcal{U}/\mathcal{I}_0$.

It follows from Lemma 1.1.10 that $(\text{id} \otimes \omega_j)(X) \to (\text{id} \otimes \omega)(X)$ in the strong operator topology for all X belonging to $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{U}/\mathcal{I}_0)$. Thus, for a in $\mathcal{A}^\infty$, $(\text{id} \otimes \omega) \circ \text{ad}_{\widetilde{\Gamma}_{\mathcal{I}_0}}(a)$ is the limit of $(\text{id} \otimes \omega_i) \circ \text{ad}_{\widetilde{\Gamma}_{\mathcal{I}_0}}(a)$ in the strong operator topology, hence belongs to $\mathcal{A}''$.

We are left to show that $(\widetilde{\mathcal{G}} := \mathcal{U}/\mathcal{I}_0, \Gamma_{\mathcal{I}_0})$ is a universal object in $\mathbf{Q}_R$ and that $(\widetilde{\mathcal{G}}, \Delta_0, \Gamma_{\mathcal{I}_0})$ is universal in the category $\mathbf{Q}'_R$. These facts follow exactly as in the proof of Theorem 3.1.7 and hence omitted. □

Consider the $*$-homomorphism $\text{ad}_0 := \text{ad}_{U_0}$, where $(\widetilde{\mathcal{G}}, U_0)$ is the universal object obtained in the previous theorem. For every state ϕ on $\widetilde{\mathcal{G}}$, $(\text{id} \otimes \phi) \circ \text{ad}_0$ maps $\mathcal{A}$ into $\mathcal{A}''$. However, in general ad_0 may not be faithful even if U_0 is so, and let $\mathcal{G}$ denote the C^*-subalgebra of $\widetilde{\mathcal{G}}$ generated by the elements $\{(f \otimes \text{id}) \circ \text{ad}_0(a) : f \in \mathcal{A}^*, a \in \mathcal{A}\}$.

Definition 3.2.6 We shall call $\mathcal{G}$ the quantum group of orientation-preserving isometries of R-twisted spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ and denote it by QISO_R^+

$(\mathcal{A}^\infty, \mathcal{H}, D, R)$ or even simply as $\mathrm{QISO}_R^+(D)$. The quantum group $\widetilde{\mathcal{G}}$ is denoted by $\widetilde{\mathrm{QISO}_R^+}(D)$.

In a private communication to us, Shuzhou Wang kindly pointed out that an alternative approach to the formulation of quantum isometry group may involve the category of compact quantum groups which has a C^*-coaction on the underlying C^* algebra and a unitary corepresentation with respect to which the Dirac operator is equivariant. However, Corollary 4.5.16 of Chap. 4 shows that the category proposed by Wang in general does not admit a universal object in general.

At this point, let us remark that in case of quantum group of orientation preserving Riemannian isometry, $\widetilde{\mathrm{QISO}_R^+}(\mathcal{D})$ depends on the von Neumann algebra $(\mathcal{A}^\infty)''$ and not on the algebra $\mathcal{A}^\infty$ itself. More precisely, we have the following proposition:

Proposition 3.2.7 *Let* $(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D}, R)$ *be as in [8]. If we have a SOT dense subalgebra* $\mathcal{A}_0$ *of* $(\mathcal{A}^\infty)'' \subset \mathcal{B}(\mathcal{H})$ *such that* $[\mathcal{D}, a] \in \mathcal{B}(\mathcal{H})$ *for all* $a \in \mathcal{A}_0$*, then* $(\mathcal{A}_0, \mathcal{H}, \mathcal{D})$ *is again a spectral triple and*

$$\widetilde{\mathrm{QISO}_R^+}(\mathcal{A}_0, \mathcal{H}, \mathcal{D}) \cong \widetilde{\mathrm{QISO}_R^+}(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D}),$$

and hence $\mathrm{QISO}_R^+(\mathcal{A}_0, \mathcal{H}, \mathcal{D}) \cong \mathrm{QISO}_R^+(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$.

The Proposition follows from the definition of $\widetilde{\mathrm{QISO}_R^+}(\mathcal{D})$ and hence we omit its proof.

We end this subsection with the following observation for which we will need to recall the notion of universal version of a compact quantum group as mentioned in Sect. 1.2.2.

Theorem 3.2.8 *The quantum groups* $\widetilde{\mathrm{QISO}_R^+}(D)$, $\mathrm{QISO}_R^+(D)$ *and* $\mathrm{QISO}^{\mathcal{L}}(D)$ *are universal compact quantum groups.*

Proof Let us prove the statement for $\widetilde{\mathrm{QISO}_R^+}(D)$ only, since the other cases follow by very similar arguments. Let $\mathcal{Q} := \widetilde{\mathrm{QISO}_R^+}(D)$ and $\mathcal{Q}_0$ and $\mathcal{Q}^u$ be the canonical Hopf $*$-algebra of $\mathcal{Q}$ and the universal CQG corresponding to $\mathcal{Q}_0$ respectively. By definition, we have a surjective CQG morphism $\pi : \mathcal{Q}^u \to \mathcal{Q}$ which is the identity map on $\mathcal{Q}_0$. To prove that π is an isomorphism, it suffices to get a surjective morphism from $\mathcal{Q}$ to $\mathcal{Q}^u$ which is the identity map on $\mathcal{Q}_0$.

Let V be the corepresentation of $\mathcal{Q}$ on $\mathcal{H}$ and let V_u be the corresponding corepresentation of $\mathcal{Q}^u$ (see Chap. 1). Clearly, $(\mathrm{id} \otimes \pi)V_u = U$ and moreover, $V_u = U$ on the dense subspace $\mathcal{N}$ spanned by the finite dimensional eigenspaces of D. As V commutes with D, so does V_u. We also have $\mathrm{ad}_{V_u} = \mathrm{ad}_V$ on the weakly dense subalgebra $\mathcal{E}_D$ of Proposition 2.3.12. Thus, by Lemma 2.3.15, τ_R is preserved by ad_{V_u}. Finally, let $\mathcal{A}_0$ be the dense subalgebra of $(\mathcal{A}^\infty)''$ as in Lemma 1.5.2 so that $\mathrm{ad}_V(\mathcal{A}_0) \subseteq \mathcal{A}_0 \otimes_{\mathrm{alg}} \mathcal{Q}_0$. So $(\mathrm{id} \otimes \pi)(\mathrm{ad}_{V_u}(\mathcal{A}_0)) \subseteq \mathcal{A}_0 \otimes_{\mathrm{alg}} \mathcal{Q}_0$. But since $\pi = \mathrm{id}$ on $\mathcal{Q}_0$, we have $\mathrm{ad}_{V_u}(\mathcal{A}_0) \subseteq \mathcal{A}_0 \otimes_{\mathrm{alg}} \mathcal{Q}_0$. This implies that ad_{V_u} leaves $\mathcal{A}_0'' = (\mathcal{A}^\infty)''$ invariant, i.e., $(\mathcal{Q}^u, V_u)$ is an object of $\mathbf{Q}'_{\mathbf{R}}(D)$, giving the required surjective morphism from $\mathcal{Q}$ to $\mathcal{Q}^u$.

3.2.3 Stability and C^* Coaction

In this subsection, we are going to use the notations as in Sect. 3.2.2, in particular $\widetilde{\mathcal{G}}, \mathcal{G}, U_0, \mathrm{ad}_0$. It is not clear whether the coaction ad_0 of $\mathrm{QISO}_R^+(D)$ preserves the C^* algebra $\mathcal{A}$ generated by $\mathcal{A}^\infty$ in the sense that $(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_0$ maps $\mathcal{A}$ into $\mathcal{A}$ for every ϕ. Moreover, even if $\mathcal{A}$ is stable under ad_0, the question remains whether ad_0 (which is always a von Neumann algebraic coaction) is a C^*-coaction of the compact quantum group $\mathrm{QISO}_R^+(D)$. In Chap. 4, Sect. 4.5.2, we have answered this question in the negative, by proving the following:

There is an R-twisted spectral triple on the Podles' sphere $S^2_{\mu,c}$ for which ad_0 *does not keep $S^2_{\mu,c}$ invariant.*

In this section, we will show that the CQG $\mathrm{QISO}_R^+(D)$ has a C^*-coaction for any spectral triple for which there is a 'reasonable' Laplacian. This includes all classical spectral triples as well as their cocycle deformations (with $R = I$).

We begin with a sufficient condition for stability of $\mathcal{A}^\infty$ under ad_0. Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple such that the condition (1) of Definition 2.2.14 is satisfied, that is: $\mathcal{A}^\infty$ and $\{[D, a],\ a \in \mathcal{A}^\infty\}$ are contained in the domains of all powers of the derivations $[D, \cdot]$ and $[|D|, \cdot]$.

Let us denote by $\widetilde{T}_t$ the one parameter group of $*$-automorphisms on $\mathcal{B}(\mathcal{H})$ given by $\widetilde{T}_t(S) = e^{itD} S e^{-itD}$ for all S in $\mathcal{B}(\mathcal{H})$. This semigroup which is continuous in the strong operator topology. Moreover, let us denote the generator of this group by δ. For X such that $[D, X]$ is bounded, we have $\delta(X) = i[D, X]$ and thus

$$\left\| \widetilde{T}_t(X) - X \right\| = \left\| \int_0^t \widetilde{T}_s([D, X]) ds \right\| \leq t \|[D, X]\|.$$

Definition 3.2.9 We say that a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ satisfies the *Sobolev regularity condition* if

$$\mathcal{A}^\infty = \mathcal{A}'' \bigcap_{n \geq 1} \mathrm{Dom}(\delta^n).$$

The following result is a natural generalization of the classical fact that under some conditions, a measurable isometric action automatically becomes smooth.

Theorem 3.2.10 *(i) For every state ϕ on $\mathcal{G}$,*

$$(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_0(\mathcal{A}^\infty) \textit{ belongs to } \mathcal{A}'' \bigcap_{n \geq 1} \mathrm{Dom}(\delta^n).$$

(ii) If the spectral triple satisfies the Sobolev regularity condition then $\mathcal{A}^\infty$ (and hence $\mathcal{A}$) is stable under ad_0.

Proof Since U_0 commutes with $D \otimes I$, it is clear that $\widetilde{T}_t$ commutes with $\mathrm{ad}_0^\phi \equiv (\mathrm{id} \otimes \phi) \circ \mathrm{ad}_0$. Therefore, by the continuity of ad_0 in the strong operator topology, for a in $\mathrm{Dom}(\delta)$, we have

$$\begin{aligned}
&\lim_{t\to 0+}\frac{\widetilde{T}_t(\mathrm{ad}_0^{\phi}(a))-\mathrm{ad}_0^{\phi}(a)}{t}\\
&=\lim_{t\to 0+}\mathrm{ad}_0^{\phi}(\frac{\widetilde{T}_t(a)-a}{t})\\
&=\mathrm{ad}_0^{\phi}(\delta(a)).
\end{aligned}$$

Thus, ad_0^{ϕ} keeps $\mathrm{Dom}(\delta)$ invariant and commutes with δ. In a similar way, (i) can be proved. The assertion (ii) follows by applying (i) and the Sobolev regularity condition. □

Let us now add a strengthened version of assumption (2), Definition 2.2.14, that is:

Assumption (2′)

Let $\mathcal{S}^{\infty}$ denote the $*$-algebra generated by $\{\widetilde{T}_s(\mathcal{A}^{\infty}), \widetilde{T}_s(\mathcal{A}^{\infty})([D, a]) : s \geq 0,\ a \in \mathcal{A}^{\infty}\}$ and Lim be as in Sect. 2.2.2. We assume that the spectral triple is Θ-summable, that is, for every $t > 0$, e^{-tD^2} is trace-class and the functional $\tau(X) = \mathrm{Lim}_{t\to 0}\frac{\mathrm{Tr}(Xe^{-tD^2})}{\mathrm{Tr}(e^{-tD^2})}$ is a positive faithful trace on $\mathcal{S}^{\infty}$.

The functional τ is to be interpreted as the volume form (we refer to [1, 9] for the details). The completion of $\mathcal{S}^{\infty}$ in the norm of $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{S}$, and we shall denote by $\|a\|_2$ and $\|\cdot\|_{\infty}$ the L^2-norm $\tau(a^*a)^{\frac{1}{2}}$ and the operator norm of $\mathcal{B}(\mathcal{H})$ respectively.

By using the definition of τ, it is easy to see that $\widetilde{T}_t$ preserves τ. Thus, $\widetilde{T}_t$ extends to a group of unitaries on $\mathcal{N} := L^2(\mathcal{S}^{\infty}, \tau)$. Moreover, for any X such that $[D, X]$ is in $\mathcal{B}(\mathcal{H})$, so in particular for X in $\mathcal{S}^{\infty}$, we have

$$\begin{aligned}
&\left\|\widetilde{T}_s(X)-X\right\|_2^2\\
&=\tau(\widetilde{T}_s(X)^*(\widetilde{T}_s(X)-X))+\tau(X^*(X-\widetilde{T}_s(X)))\\
&\leq 2\left\|X-\widetilde{T}_s(X)\right\|_{\infty}\|X\|_2\\
&\leq 2s\|[D, X]\|_{\infty}\|X\|_2.
\end{aligned}$$

Thus, for X in $\mathcal{S}^{\infty}$, the map $s \mapsto \widetilde{T}_s(X)$ is L^2-continuous. The unitarity of $\widetilde{T}_s$ implies that the L^2-continuity is true for any X in belonging to $\mathcal{N}$, so that it is a strongly continuous one-parameter group of unitaries. Let us denote its generator by $\widetilde{\delta}$. Then $\widetilde{\delta}$ is a skew adjoint map, that is, $i\widetilde{\delta}$ is self adjoint, and $\widetilde{T}_t = \exp(t\widetilde{\delta})$. Clearly, $\widetilde{\delta} = \delta = [D, \cdot]$ on $\mathcal{S}^{\infty}$.

We will denote $L^2(\mathcal{A}^{\infty}, \tau) \subset \mathcal{N}$ by $\mathcal{H}_D^0$ and the restriction of $\widetilde{\delta}$ to $\mathcal{H}_D^0$ (which is a closable map from $\mathcal{H}_D^0$ to $\mathcal{N}$) by d_D. Thus, d_D is closable too.

Throughout the rest of this subsection, we will assume that the spectral triple is admissible in the sense of Definition 3.1.1.

Clearly, $\mathcal{L}(\mathcal{A}_0^{\infty}) \subseteq \mathcal{A}_0^{\infty}$. The $*$-subalgebra of $\mathcal{A}^{\infty}$ generated by $\mathcal{A}_0^{\infty}$ is denoted by $\mathcal{A}_0$. We also note that $\mathcal{L} = P_0\widetilde{\mathcal{L}}P_0$, where $\widetilde{\mathcal{L}} := (i\widetilde{\delta})^2$ (which is a self adjoint operator on $\mathcal{N}$) and P_0 denotes the orthogonal projection in $\mathcal{N}$ whose range is the subspace $\mathcal{H}_D^0$.

Theorem 3.2.11 *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be an admissible spectral triple satisfying the condition (1) of Definition 2.2.14 and assumption (2′). In addition, assume that at least one of conditions (a) and (b) mentioned below is satisfied:*

(a) $\mathcal{A}'' \subseteq \mathcal{H}_D^0$.

(b) For any state ϕ on $\mathcal{G} = \mathrm{QISO}_I^+(D)$, $\mathrm{ad}_0^\phi(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty$.

Then ad_0 is a C^*-coaction of $\mathrm{QISO}_I^+(D)$ on $\mathcal{A}$.

Proof Let ϕ be a fixed state on $\mathcal{G}$. Under either of the conditions (a) and (b), the map $\mathrm{ad}_0^\phi(\mathcal{A}^\infty) \subseteq \mathcal{H}_D^0 \subseteq \mathcal{N}$. Since ad_0^ϕ also commutes with $[D, \cdot]$ on $\mathcal{A}^\infty$, $\mathrm{ad}_0^\phi(\mathcal{S}^\infty) \subseteq \mathcal{N}$. In fact, using the complete positivity of ad_0^ϕ and the fact that ad_0 preserves τ, we have

$$\tau(\mathrm{ad}_0^\phi(a)^* \mathrm{ad}_0^\phi(a)) \leq \tau(\mathrm{ad}_0^\phi(a^*a)) = (\mathrm{id} \otimes \phi)((\tau \otimes \mathrm{id})\mathrm{ad}_0(a^*a)) = \tau(a^*a).1.$$

Therefore, ad_0^ϕ extends to a bounded operator from $\mathcal{N}$ to itself. Since U_0 commutes with D, it is clear that ad_0^ϕ (viewed as a bounded operator on $\mathcal{N}$) will commute with the group of unitaries $\widetilde{T}_t$. Hence ad_0^ϕ commutes with the generator $\widetilde{\delta}$ of $\widetilde{T}_t$ and also with the self adjoint operator $\widetilde{\mathcal{L}} = (i\widetilde{\delta})^2$.

On the other hand, the definition of $\mathcal{G} = \mathrm{QISO}_I^+(D)$ implies that $(\tau \otimes \mathrm{id})$ $(\mathrm{ad}_0(X)) = \tau(X)1_\mathcal{G} \forall\ X$ in $\mathcal{B}(\mathcal{H})$ and so in particular for all $X \in \mathcal{S}^\infty$. Therefore, the map $\mathcal{S}^\infty \otimes_{\mathrm{alg}} \mathcal{G} \ni (a \otimes q) \mapsto \mathrm{ad}_0(a)(1 \otimes q)$ extends to a $\mathcal{G}$-linear unitary on the Hilbert $\mathcal{G}$-module $\mathcal{N} \otimes \mathcal{G}$. Let us denote this unitary by W. We remark that we have used the fact that $(\mathrm{id} \otimes \phi)(W)(\mathcal{S}^\infty \otimes_{\mathrm{alg}} \mathcal{G}) \subseteq \mathcal{N}$ for any ϕ, as $\mathrm{ad}_0^\phi(\mathcal{S}^\infty) \subseteq \mathcal{N}$. Since for all ϕ, ad_0^ϕ commutes with $\widetilde{T}_t$, W and $\widetilde{T}_t \otimes \mathrm{id}_\mathcal{G}$ commute on $\mathcal{N} \otimes \mathcal{G}$. Moreover, $\mathrm{ad}_0^\phi(\mathcal{H}_D^0) \subseteq \mathcal{H}_D^0$ so that $W(\mathcal{H}_D^0 \otimes \mathcal{G}) \subseteq \mathcal{H}_D^0 \otimes \mathcal{G}$. Therefore, by the unitarity of W, it commutes with the projection $P_0 \otimes 1$. It follows that ad_0^ϕ commutes with P_0, and since it also commutes with $\widetilde{\mathcal{L}}$, ad_0^ϕ commutes with $\mathcal{L} = P_0\widetilde{\mathcal{L}}P_0$ as well.

Thus, all the (finite dimensional) eigenspaces of the Laplacian $\mathcal{L}$ are preserved by ad_0^ϕ. Therefore, ad_0 is a Hopf algebraic coaction on the subalgebra $\mathcal{A}_0$ spanned algebraically by these eigenvectors. Moreover, the $\mathcal{G}$-linear unitary W clearly restricts to a unitary corepresentation on each of the above eigenspaces. Let $((q_{ij}))_{(i,j)}$ denote the $\mathcal{G}$-valued unitary matrix corresponding to one such particular eigenspace. Then, by Proposition 1.2.19, q_{ij} must belong to $\mathcal{G}_0$ and we must have $\epsilon(q_{ij}) = \delta_{ij}$. Thus, $(\mathrm{id} \otimes \epsilon) \circ \mathrm{ad}_0 = \mathrm{id}$ on each of the eigenspaces and hence on the norm-dense subalgebra $\mathcal{A}_0$ of $\mathcal{A}$, completing the proof of the fact that ad_0 extends to a C^* coaction on $\mathcal{A}$. □

A combination of the above theorem with Theorem 3.2.10 yields the following immediate corollary.

Corollary 3.2.12 *If the spectral triple satisfies the Sobolev regularity condition of Definition 3.2.9, in addition to the assumptions of Theorem 3.2.11, then* $\mathrm{QISO}_I^+(D)$

has a C^**-coaction. In particular, for a classical spectral triple,* $\mathrm{QISO}_I^+(D)$ *has* C^**-coaction.*

Remark 3.2.13 Let us remark here that in case the restriction of τ on $\mathcal{A}^\infty$ is normal, $\mathcal{A}'' \subseteq \mathcal{H}_D^0$, and hence by the double commutant theorem, the condition (a) of Theorem 3.2.11 is satisfied and hence its conclusion holds.

3.2.4 Comparison with the Approach Based on Laplacian

Throughout this section, we shall assume the set-up of Sect. 3.2.3 for the existence of a 'Laplacian' $\mathcal{L} \equiv \mathcal{L}_D$ in the sense of Sect. 3.1.1 so that we work with admissible spectral triples (Definition 3.1.1) satisfying the condition (1) of Definition 2.2.14 and assumption (2′) of Sect. 3.2.3. Let us also use the notation of the previous subsections. In particular, $\mathcal{A}_0^\infty$ will denote the complex linear span of the eigenvectors of $\mathcal{L}$. We will denote the category of all compact quantum groups coacting smoothly and isometrically on $\mathcal{A}$ by the symbol $\mathbf{Q}'_{\mathcal{L}_\mathbf{D}}$. Since our spectral triples are assumed to be admissible, there exists a universal object in $\mathbf{Q}'_{\mathcal{L}_\mathbf{D}}$ to be denoted by $\mathrm{QISO}^{\mathcal{L}} \equiv \mathrm{QISO}^{\mathcal{L}_D}$.

The following result now follows immediately from Theorem 3.2.11.

Corollary 3.2.14 *If* $(\mathcal{A}^\infty, \mathcal{H}, D)$ *is a spectral triple satisfying any of the two conditions (a) or (b) of Theorem 3.2.11, then* $\mathrm{QISO}_I^+(D)$ *is a sub-object of* $\mathrm{QISO}^{\mathcal{L}_D}$ *in the category* $\mathbf{Q}'_{\mathcal{L}_\mathbf{D}}$.

Proof The Corollary follows from the fact that $\mathrm{QISO}_I^+(D)$ has the C^*-coaction ad_0 on $\mathcal{A}$, and the observation already made in the proof of the Theorem 3.2.11 that this coaction commutes with the Laplacian $\mathcal{L}_D$. □

Now, we will need the Hilbert space of forms $\mathcal{H}_{d+d^*}$, $d \equiv d_D$ corresponding to a Θ-summable spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ as discussed in Sect. 2.2.2. We recall that one obtains an associated spectral triple $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d + d^*)$. We assume that this spectral triple is of compact type, that is, $d + d^*$ has compact resolvents.

We will denote the inner product on the space of k forms coming from the spectral triples $(\mathcal{A}^\infty, \mathcal{H}, D)$ and $(\mathcal{A}^\infty, \mathcal{H}_{d+d^*}, d+d^*)$ by $\langle\ ,\ \rangle_{\mathcal{H}_D^k}$ and $\langle\ ,\ \rangle_{\mathcal{H}_{d+d^*}^k}$ respectively, $k = 0, 1$.

We will denote by π_D, π_{d+d^*} the corepresentations of $\mathcal{A}^\infty$ in $\mathcal{H}$ and $\mathcal{H}_{d+d^*}$ respectively.

Let U_{d+d^*} be the canonical unitary corepresentation of $\mathrm{QISO}_I^+(d+d^*)$ on $\mathcal{H}_{d+d^*}$. Then the Hilbert space $\mathcal{H}_{d+d^*}$ decomposes into finite dimensional orthogonal subspaces corresponding to the distinct eigenvalues of $(d + d^*)^2 = d^*d + dd^*$. It can be easily seen that each of the subspaces $\mathcal{H}_D^i$ are kept invariant by $(d + d^*)^2$. Let us denote by $V_{\lambda,i}$ the subspace of $\mathcal{H}^i_{d+d^*}$ spanned by eigenvectors of $(d+d^*)^2$ corresponding to the eigenvalue λ and $\{e_{j\lambda i}\}_j$ be an orthonormal basis of $V_{\lambda,i}$. We observe that $\mathcal{L}_D$ is the restriction of $(d+d^*)^2$ to $\mathcal{H}^0_D$.

At this point, we recall Theorem 2.3.7 which says that $\mathrm{QISO}^{\mathcal{L}_D}$ has a unitary corepresentation $U \equiv U_{\mathcal{L}}$ on $\mathcal{H}^{d+d^*}$ such that U commutes with $d+d^*$. Thus, $(\mathcal{A}^\infty,\ \mathcal{H}_{d+d^*},\ d+d^*)$ is a $\mathrm{QISO}^{\mathcal{L}_D}$ equivariant spectral triple. Moreover, by Remark 3.1.10, the Haar state of $\mathrm{QISO}^{\mathcal{L}_D}$ is a trace so that by Proposition 2.3.12 and Remark 2.3.14, we have that ad_U keeps the functional τ_I invariant. Summarizing, we have the following result:

Proposition 3.2.15 *The quantum isometry group* $(\mathrm{QISO}^{\mathcal{L}_D},\ U_{\mathcal{L}})$ *is a sub-object of* $(\mathrm{QISO}^+_I(d+d^*),\ U_{d+d^*})$ *in the category* $\mathbf{Q}_I(d+d^*)$*, so in particular,* $\mathrm{QISO}^{\mathcal{L}_D}$ *is isomorphic to a quotient of* $\mathrm{QISO}^+_I(d+d^*)$ *by a Woronowicz* C^* *ideal.*

Under mild conditions, we shall give a concrete description of the above Woronowicz ideal.

Let $\mathcal{I}$ be the C^* ideal of $\mathrm{QISO}^+_I(d+d^*)$ generated by

$$\cup_{\lambda\in\sigma((d+d^*)^2)}\{\big\langle(P_0^\perp\otimes \mathrm{id})U_{d+d^*}(e_{j\lambda 0}),\, e_{j\lambda i'}\otimes 1\big\rangle : j, i' \geq 1\},$$

where P_0 is the projection onto $\mathcal{H}^0_D$, $\langle .\ ,\ .\rangle$ denotes the $\mathrm{QISO}^+_I(d+d^*)$ valued inner product and $\sigma((d+d^*)^2)$ denotes the spectrum of $(d+d^*)^2$.

Since U_{d+d^*} keeps the eigenspaces of $(d+d^*)^2$ invariant, we can write

$$U_{d+d^*}(e_{j\lambda 0}) = \sum_k e_{k\lambda 0}\otimes\ q_{kj\lambda 0} + \sum_{i'\neq 0,k'} e_{k'\lambda i'}\otimes\ q_{k'j\lambda i'},$$

for some $q_{kj\lambda 0},\ q_{k'j\lambda i'}$ in $\mathrm{QISO}^+_I(d+d^*)$.

We note that $q_{k'j\lambda i'}$ is in $\mathcal{I}$ if $i'\neq 0$.

Lemma 3.2.16 $\mathcal{I}$ *is a Woronowicz* C^**-ideal of* $\mathrm{QISO}^+_I(d+d^*)$.

Proof It suffices to verify that $\Delta(X)\in \mathrm{Span}(\mathcal{I}\otimes \mathrm{QISO}^+_I(d+d^*)+\mathrm{QISO}^+_I(d+d^*)\otimes\mathcal{I})$ for the elements X in $\mathcal{I}$ of the form $\big\langle(P_0^\perp\otimes\mathrm{id})U_{d+d^*}(e_{j\lambda 0}), e_{j\lambda i_0}\otimes 1\big\rangle$. We have:

$$\Delta(\big\langle(P_0^\perp\otimes \mathrm{id})U_{d+d^*}(e_{m\lambda 0}), e_{j\lambda i_0}\otimes 1\big\rangle)$$

$$= \big\langle(P_0^\perp\otimes\mathrm{id})(\mathrm{id}\otimes\Delta)U_{d+d^*}(e_{m\lambda 0}), e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle)$$

$$= \big\langle(P_0^\perp\otimes\mathrm{id})U_{(12)}U_{(13)}(e_{m\lambda 0}), e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$= \left\langle (P_0^\perp\otimes\mathrm{id})U_{(12)}(\sum_k e_{k\lambda 0}\otimes 1\otimes q_{km\lambda 0})\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\right\rangle$$

$$+\sum_{i'\neq 0,l}\big\langle (P_0^{\perp}\otimes \mathrm{id})U_{(12)}(e_{l\lambda i'}\otimes 1\otimes q_{lm\lambda i'})\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$=\sum_{k,k'}\big\langle (P_0^{\perp}\otimes \mathrm{id})(e_{k'\lambda 0}\otimes q_{k'k\lambda 0}\otimes q_{km\lambda 0})\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$+\sum_{i'\neq 0,\ k,\ k''}\big\langle (P_0^{\perp}\otimes \mathrm{id})(e_{k''k\lambda i'}\otimes q_{k'',k,\lambda,i'}\otimes q_{km\lambda 0})\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$+\sum_{i'\neq 0,\ l,\ l'}\big\langle (P_0^{\perp}\otimes \mathrm{id})(e_{l'\lambda i'}\otimes q_{l'l\lambda i'}\otimes q_{lm\lambda i'})\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$+\sum_{i'\ \neq 0,\ i''\neq i',\ l,\ l''}\big\langle (P_0^{\perp}\otimes \mathrm{id})(e_{l''\lambda i''}\otimes q_{l''l\lambda i''}\otimes q_{lm\lambda i'})\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$=\sum_{i'\neq 0,\ k',\ k''}\big\langle e_{k''\lambda i'}\otimes q_{k''k\lambda i'}\otimes q_{km\lambda 0}\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$+\sum_{i'\neq 0,\ l,\ l'}\big\langle e_{l'\lambda i'}\otimes q_{l'l\lambda i'}\otimes q_{lm\lambda i'}\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle$$

$$+\sum_{i'\neq 0,\ i''\neq i',\ i''\neq 0,\ l,\ l''}\big\langle e_{l''\lambda i''}\otimes q_{l''l\lambda i''}\otimes q_{lm\lambda i'}\ \ ,\ \ e_{j\lambda i_0}\otimes 1\otimes 1\big\rangle,$$

which is clearly an element of $\mathcal{I}\otimes \mathrm{QISO}_I^+(d+d^*)+\mathrm{QISO}_I^+(d+d^*)\otimes\mathcal{I}$, since $q_{kj\lambda i'}$ is an element of $\mathcal{I}$ for $i'\neq 0$. □

Theorem 3.2.17 *If* $\mathrm{ad}_{U_{d+d^*}}$ *is a* C^* *coaction on* $\mathcal{A}$, *then we have* $\mathrm{QISO}^{\mathcal{L}_D}\cong\mathrm{QISO}_I^+(d+d^*)/\mathcal{I}$.

Proof By Theorem 2.3.7, we conclude that there exists a surjective CQG morphism $\pi:\mathrm{QISO}_I^+(d+d^*)\to\mathrm{QISO}^{\mathcal{L}_D}$. By construction, the unitary corepresentation $U_{\mathcal{L}}$ of $\mathrm{QISO}^{\mathcal{L}_D}$ preserves each of the $\mathcal{H}_D^i$, in particular $\mathcal{H}_D^0$. The definition of $\mathcal{I}$ implies that π induces a surjective CQG morphism $\pi':\mathrm{QISO}_I^+(d+d^*)/\mathcal{I}\to\mathrm{QISO}^{\mathcal{L}_D}$, which is also a morphism of $\mathbf{Q}'_I(d+d^*)$.

Conversely, if $V=(\mathrm{id}\otimes\rho_{\mathcal{I}})\circ U_{d+d^*}$ is the corepresentation of $\mathrm{QISO}_I^+(d+d^*)/\mathcal{I}$ on $\mathcal{H}_{d+d^*}$ induced by U_{d+d^*}, where $\rho_{\mathcal{I}}:\mathrm{QISO}_I^+(d+d^*)\to\mathrm{QISO}_I^+(d+d^*)/\mathcal{I}$ denotes the quotient map, then by the definition of $\mathcal{I}$, $\mathcal{H}_D^0$ is preserved by V preserves $\mathcal{H}_D^0$. Thus, V commutes with P_0. Since V also commutes with $(d+d^*)^2$, it follows that V must commute with $(d+d^*)^2P_0=\mathcal{L}$, that is,

$$\widetilde{V}(d^*d\ P_0\otimes 1)=(d^*dP_0\otimes 1)\widetilde{V}.$$

Recall that ad_V is a C^* coaction on $\mathcal{A}$ since $\mathrm{ad}_{U_{d+d^*}}$ is so by assumption. It is easy to show from the above that ad_V is a smooth isometric coaction of $\mathrm{QISO}_I^+(d+d^*)/\mathcal{I}$

with respect to the Laplacian $\mathcal{L}_D$. Thus, $\mathrm{QISO}_I^+(d+d^*)/\mathcal{I}$ is a sub-object of $\mathrm{QISO}^{\mathcal{L}_D}$ in the category $\mathbf{Q}'_{\mathcal{L}_\mathbf{D}}$ which completes the proof. □

Now we prove that under some further assumptions, one even has the isomorphism $\mathrm{QISO}^{\mathcal{L}_D} \cong \mathrm{QISO}_I^+(d+d^*)$. These assumptions are valid for classical manifolds as well as their cocycle deformations.

The assumptions are as follows:

(**A**) Both the spectral triples $(\mathcal{A}^\infty,\ \mathcal{H},\ D)$ and $(\mathcal{A}^\infty,\ \mathcal{H}_{d+d^*},\ d+d^*)$ are admissible satisfying the condition (1) of Definition 2.2.14 and assumption (2′). Thus, if $D' = d + d^*$, both $\mathrm{QISO}^{\mathcal{L}_D}$ and $\mathrm{QISO}^{\mathcal{L}_{D'}}$ exist.

(**B**) We have

$$\forall\, a,\ b\ \in \mathcal{A}^\infty,\ \langle a,\ b\rangle_{\mathcal{H}^0_D} = \langle a,\ b\rangle_{\mathcal{H}^0_{D'}}\ ,\ \langle d_D a,\ d_D b\rangle_{\mathcal{H}^1_D} = \langle d_{D'} a,\ d_{D'} b\rangle_{\mathcal{H}^1_{D'}}.$$

Remark 3.2.18 The above assumptions do hold for classical spin manifolds. It follows from the discussion at the end of Sect. 2.2.2 where we saw that the volume forms corresponding to D^2 and $(D')^2$ coincide, which is a consequence of the fact that $D^2 - (D')^2$ is a first order differential operator.

By assumption (**B**), we observe that the identity map on $\mathcal{A}^\infty$ extends to a unitary, say Σ, from $\mathcal{H}^0_D$ to $\mathcal{H}^0_{D'}$. Moreover, we have

$$\mathcal{L}_D = \Sigma^* \mathcal{L}_{D'} \Sigma,$$

from which we get the following Proposition:

Proposition 3.2.19 *Under the above assumptions,* $\mathrm{QISO}^{\mathcal{L}_D} \cong \mathrm{QISO}^{\mathcal{L}_{D'}}$.

The next result identifies the quantum isometry group $\mathrm{QISO}^{\mathcal{L}_D}$ of Sect. 3.1 as the QISO_I^+ of a spectral triple, and thus, in some sense, accommodates the construction of [1] in the framework of the present section.

Theorem 3.2.20 *If in addition to the assumptions already made, the spectral triple* $(\mathcal{A}^\infty, \mathcal{H}_{D'}, D')$ *also satisfies the conditions of Theorem 3.2.11, so that* $\mathrm{QISO}_I^+(D')$ *has a C^*-coaction, then we have the following isomorphism of CQG s:*

$$\mathrm{QISO}^{\mathcal{L}_D} \cong \mathrm{QISO}_I^+(D') \cong \mathrm{QISO}^{\mathcal{L}_{D'}}.$$

Proof By applying Proposition 3.2.15 we see that $\mathrm{QISO}^{\mathcal{L}_D}$ is a sub-object of $\mathrm{QISO}_I^+(D')$ in the category $\mathbf{Q}'_I(D')$. On the other hand, by Corollary 3.2.14, $\mathrm{QISO}_I^+(D')$ as a sub-object of $\mathrm{QISO}^{\mathcal{L}_{D'}}$ in the category $\mathbf{Q}'_{\mathcal{L}_{\mathbf{D}'}}$. Combining these facts with Proposition 3.2.19, we get the required isomorphism. □

3.3 The Case of $\widetilde{J}$ Preserving Quantum Isometries

In this section, we are going to work with spectral triples $(\mathcal{A}^\infty, \mathcal{H}, D)$ equipped a conjugate linear invertible map $\widetilde{J}$ on $\mathcal{H}$ such that the linear span $\mathcal{D}_0$ of eigenvectors of $\mathcal{D}$ is contained in the domain of $\widetilde{J}$, $\widetilde{J}\mathcal{D}_0 \subseteq \mathcal{D}_0$ and $\widetilde{J}$ commutes with D on Dom(D). Thus, the real spectral triples defined in Sect. 2.2.1 are a particular class of examples. We shall work with odd spectral triples. However, in the even case, all the arguments will go through, with some obvious and minor changes at places.

Given a C^*-algebra $\mathcal{S}$ we shall denote by $\widetilde{J}_{\mathcal{S}}$ the antilinear map $a \mapsto a^*$. If $\mathcal{S}$ is separable, a faithful state always exists. We shall denote the antilinear map $a \mapsto a^*$ by $\widetilde{J}_{\mathcal{S}}$. We can view this map as a closable unbounded antilinear map on the G.N.S. space of the state. By a slight abuse of notation, we will continue to denote the closed extension of this map by $\widetilde{J}$.

Definition 3.3.1 Suppose that the (odd) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is equipped with a conjugate linear invertible map $\widetilde{J}$ as above. We say that a quantum family of orientation preserving isometries $(\mathcal{S}, U)$ preserves $\widetilde{J}$ if the following equation holds on $\mathcal{D}_0$:

$$(\widetilde{J} \otimes \widetilde{J}_{\mathcal{S}}) \circ U = U \circ \widetilde{J}. \tag{3.3.1}$$

If the C^*-algebra $\mathcal{S}$ has a coproduct Δ such that $(\mathcal{S}, \Delta)$ is a CQG and U is a unitary corepresentation of $(\mathcal{S}, \Delta)$ on $\mathcal{H}$, we will say that $(\mathcal{S}, \Delta)$ coacts by orientation and $\widetilde{J}$ preserving isometries on the spectral triple.

The even case can be similarly dealt with by assuming in addition that U commutes with γ.

As in Sect. 3.2, we consider the category $\mathbf{Q}_{\widetilde{J}} \equiv \mathbf{Q}_{\widetilde{J}}(D)$ with the object-class consisting of all quantum families of orientation and $\widetilde{J}$ preserving isometries $(\mathcal{S}, U)$ of the given spectral triple, and the set of morphisms as before. We also consider another category $\mathbf{Q}'_{\widetilde{J}} \equiv \mathbf{Q}'_{\widetilde{J}}(D)$ whose objects are triplets $(\mathcal{S}, \Delta, U)$, where $(\mathcal{S}, \Delta)$ is a compact quantum group coacting by orientation and $\widetilde{J}$ preserving isometries on the given spectral triple, with U being the corresponding unitary corepresentation. We fix the following set of notations.

The C^*-algebra generated by $\mathcal{A}^\infty$ in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{A}$. As before, let $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots$ be the eigenvalues of D with V_i denoting the (d_i-dimensional, $d_i < \infty$) eigenspace for λ_i. Let $\{e_{ij}, j = 1, \ldots, d_i\}$ be an orthonormal basis of V_i. Clearly, $\{\widetilde{J}(e_{ij}), i \geq 0, 1 \leq j \leq d_i\}$ is a linearly independent (but not necessarily orthogonal) set, and let T_i denote the positive nonsingular matrix $\left(\langle \widetilde{J}(e_{ij}), \widetilde{J}(e_{ik})\rangle\right)_{j,k=1}^{d_i}$.

Let $\mathcal{U}_i = A_u(T_i)$ and β be its canonical unitary corepresentation on $V_i \cong \mathbb{C}^{d_i}$ and $\beta = *_i \beta_i$ as in earlier subsections.

Lemma 3.3.2 *Let $(\mathcal{A}^\infty, \mathcal{H}, D, \widetilde{J})$ be as above and let $(\mathcal{S}, U)$ be a quantum family of orientation and $\widetilde{J}$ preserving isometries of the given spectral triple. If U is faithful,* *then there exists a C^*-ideal $\mathcal{I}$ of $\mathcal{U}$ and a $*$-isomorphism $\phi : \mathcal{U}/I \to \mathcal{S}$ such that $U = (\mathrm{id} \otimes \phi) \circ (\mathrm{id} \otimes \Pi_I) \circ \beta$, where Π_I denotes the quotient map from $\mathcal{U}$ to $\mathcal{U}/I$.*

If, furthermore, there is a CQG structure on $\mathcal{S}$ given by a coproduct Δ such that $(\mathcal{S}, \Delta, U)$ is an object in $\mathbf{Q}'_{\widetilde{J}}(D)$, I is a Woronowicz C^*-ideal and ϕ is a morphism of CQG's.

Proof We follow the line of arguments of a similar result in Sect. 3.2. We observe that for all i, $U(V_i) \subseteq V_i \otimes_{\mathrm{alg}} \mathcal{S}$. Let for $j, k = 1, \ldots, d_i$, $v^{(i)}_{kj}$ be the elements of $\mathcal{S}$ such that $U(e_{ij}) = \sum_k e_{ik} \otimes v^{(i)}_{kj}$. Then $v_i := ((v^{(i)}_{kj}))$ is a unitary in $M_{d_i}(\mathbb{C}) \otimes_{\mathrm{alg}} \mathcal{S}$ and since U is faithful, the $*$-subalgebra generated by all $\{v^{(i)}_{kj}, i \geq 0, j, k = 1, \ldots, d_i\}$ must be dense in $\mathcal{S}$.

Now, we shall use (3.3.1). Fix any i and let $\Lambda_i = ((\tau_{lm}))$ be the matrix defined by $\widetilde{J}(e_{ij}) = \sum_l \tau_{lj} e_{il}$. By assumption, Λ_i is invertible and $\Lambda_i^* \Lambda_i = T_i$. The equation

$$U(\widetilde{J} e_{ij}) = \sum_k \widetilde{J} e_{ik} \otimes (v^{(i)}_{kj})^* \tag{3.3.2}$$

yields

$$\sum_m e_{im} \otimes \left(\sum_l \tau_{lj} v^{(i)}_{ml} \right) = \sum_m e_{im} \otimes \left(\sum_l \tau_{ml} (v^{(i)}_{lj})^* \right). \tag{3.3.3}$$

By comparing coefficients of e_{im} in both sides of (3.3.3), we get $\sum_l \tau_{lj} v^{(i)}_{ml} = \sum_l \tau_{ml} (v^{(i)}_{lj})^*$, so that $v_i \Lambda_i = \Lambda_i \overline{v_i}$. Therefore, $\overline{v_i} = \Lambda_i^{-1} v_i \Lambda_i$ and thus $\overline{v_i}$ is invertible, since v_i is so. Moreover, taking the $\mathcal{S}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ on both sides of the Eq. (3.3.2), we get $T_i = v'_i T_i \overline{v_i}$. Thus, $T_i^{-1} v'_i T_i$ is the inverse of $\overline{v_i}$, from which we see that the relations (1.3.2) are satisfied with u replaced by v_i.

The rest of the proof is very similar to that of Lemma 3.2.4 and hence is omitted. □

Remark 3.3.3 An examination of the proof of Lemma 3.3.2 reveals that if $(\mathcal{S}, U)$ is an object of $\mathbf{Q}'_{\widetilde{\mathbf{J}}}(D)$, then ad_U preserves the R-twisted volume form, where R is defined by $R|_{V_i} = T'_i$. This connects the approach of the present section to that of Sect. 3.2.

The rest of the arguments in Sect. 3.2 go through more or less verbatim and we have the following result.

Theorem 3.3.4 *For any (odd or even) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ equipped with a conjugate invertible linear map $\widetilde{J}$ as above, the category $\mathbf{Q}_{\widetilde{J}}(D)$ of quantum families of orientation and $\widetilde{J}$ preserving isometries has a universal (initial) object, say $(\widetilde{\mathcal{G}}, U_0)$. Moreover, there exists a coproduct Δ_0 on $\widetilde{\mathcal{G}}$ such that $(\widetilde{\mathcal{G}}, \Delta_0)$ is a CQG and $(\widetilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category $\mathbf{Q}'_{\widetilde{J}}(D)$. The corepresentation U_0 is faithful.*

Definition 3.3.5 Let $< \cdot, \cdot >$ denotes the $\widetilde{\mathcal{G}}$-valued inner product of the Hilbert module $\mathcal{H} \otimes \widetilde{\mathcal{G}}$. Then let $\mathcal{G}$ be defined by the Woronowicz subalgebra of $\widetilde{\mathcal{G}}$ generated by $\{< \xi \otimes 1, \mathrm{ad}_{U_0}(a)(\eta \otimes 1) >, \ \xi, \eta \in \mathcal{H}, \ a \in \mathcal{A}^\infty\}$. We shall call $\mathcal{G}$ the quantum group of orientation and $\widetilde{J}$ preserving isometries of the given spectral triple, and denote it by $\mathrm{QISO}^+(\mathcal{A}^\infty, \mathcal{H}, D, \widetilde{J})$ (or $\mathrm{QISO}^+_J(\mathcal{A}^\infty, \mathcal{H}, D)$ in short). The quantum group $\widetilde{\mathcal{G}}$ is denoted by $\widetilde{\mathrm{QISO}^+}(\mathcal{A}^\infty, \mathcal{H}, D, \widetilde{J})$ (or $\widetilde{\mathrm{QISO}^+_J}$ in short).

Given a compact quantum group $\mathcal{Q}$ acting on $\mathcal{A}$, such that the coaction is implemented by a unitary corepresentation U of the quantum group on $\mathcal{H}$, it is easy to see that the definition of equivariance of the spectral triple with the real structure as proposed in [10] is equivalent to saying that $(\mathcal{Q}, U)$ is an object of $\mathbf{Q}'_{\widetilde{\mathbf{J}}}$. We refer to [10] for related discussions and examples of such equivariant real spectral triples.

3.4 A Sufficient Condition for Existence of Quantum Isometry Groups Without Fixing the Volume Form

In this subsection, we make further investigations on the conditions on a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ which can ensure the existence of a universal object in the category $\mathbf{Q}$ or $\mathbf{Q}'$. In case such an universal object, say, $(\mathcal{S}, U)$ exists, we will denote it by $\widetilde{\mathrm{QISO}^+}(D)$. Moreover, the largest Woronowicz subalgebra of $\widetilde{\mathrm{QISO}^+}(D)$ for which the coaction ad_U is faithful on $\mathcal{A}^\infty$ will be denoted by $\mathrm{QISO}^+(D)$.

Remark 3.4.1 If $\widetilde{\mathrm{QISO}^+}(D)$ exists, then by Proposition 2.3.12, there exists an R such that $\widetilde{\mathrm{QISO}^+}(D)$ is an object in the category $\mathbf{Q}'_R(D)$. Since the universal object in this category, that is, $\widetilde{\mathrm{QISO}^+_R}(D)$, is clearly a sub-object of $\widetilde{\mathrm{QISO}^+}(D)$, we have $\widetilde{\mathrm{QISO}^+}(D) \cong \widetilde{\mathrm{QISO}^+_R}(D)$ for this choice of R.

The following theorem gives some sufficient conditions for the existence of $\widetilde{\mathrm{QISO}^+}(D)$.

Theorem 3.4.2 *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple and assume that D has an one-dimensional eigenspace spanned by a unit vector ξ, which is cyclic and separating for the algebra $\mathcal{A}^\infty$. Moreover, assume that each eigenvector of D belongs to the dense subspace $\mathcal{A}^\infty\xi$ of $\mathcal{H}$. Then there is a universal object $(\widetilde{\mathcal{G}}, U_0)$ in the category* $\mathbf{Q}$. *Moreover, $\widetilde{\mathcal{G}}$ admits a coproduct Δ_0 such that $(\widetilde{\mathcal{G}}, \Delta_0)$ is a compact quantum group and $(\widetilde{\mathcal{G}}, \Delta_0, U_0)$ is a universal object in the category* $\mathbf{Q}'$.

If $\langle \cdot\, , \ \cdot \rangle_{\widetilde{\mathcal{G}}}$ denotes the $\widetilde{\mathcal{G}}$ valued inner product of $\mathcal{H} \otimes \widetilde{\mathcal{G}}$, let us define $\mathcal{G}$ as the Woronowicz C^* subalgebra of $\widetilde{\mathcal{G}}$ generated by the set: $\{\big\langle \mathrm{ad}_{U_0}(a)(\eta \otimes 1), \ \eta' \otimes 1 \big\rangle_{\widetilde{\mathcal{G}}} : \eta, \ \eta' \in \mathcal{H}, \ a \in \mathcal{A}^\infty\}$.

Then $\widetilde{\mathcal{G}} \cong \mathcal{G} * C(\mathbb{T})$.

Proof Let V_i, $\{e_{ij}\}$ be as before, and by assumption, for each i, j, there exists a unique x_{ij} in $\mathcal{A}^\infty$ such that $e_{ij} = x_{ij}\xi$. Since ξ is separating for $\mathcal{A}^\infty$, the vectors $\{\overline{e_{ij}} = x^*_{ij}\xi, \ j = 1, \ldots, d_i\}$ are linearly independent. If $(\mathcal{S}, U)$ denotes an object in $\mathbf{Q}$, such that U is faithful, then there exists an element q in $\mathcal{S}$ such that $\widetilde{U}(\xi \otimes 1) = \xi \otimes q$. By the unitarity of $\widetilde{U}$ it follows that q is a unitary.

Let $W : \mathcal{H} \to \mathcal{H} \otimes \mathcal{S}$ be defined by $W(v) = U(v)q^{-1}$, so that $\widetilde{W} = \widetilde{U}(1 \otimes q^{-1})$. Then, $\mathrm{ad}_W = \mathrm{ad}_U$, $W\xi = \xi \otimes 1$ and $\widetilde{W}$ commutes with $D \otimes 1$. Therefore, $(\mathcal{S}, W)$ is also an object of $\mathbf{Q}$. Let $\mathcal{D}_0 = \mathrm{Span}\{\mathrm{e_{ij}} : \mathrm{j} = 1, 2, \ldots, \mathrm{d_i}, \mathrm{i} \geq 0\}$ and define a conjugate linear invertible operator $\widetilde{J} : \mathcal{D}_0 \to \mathcal{D}_0$ by $\widetilde{J}(x\xi) = x^*\xi$. We see that $\widetilde{J}(\mathcal{D}_0) \subseteq \mathcal{D}_0$. Moreover, as $\widetilde{U}$ and hence $\widetilde{W}$ leaves $\mathcal{D}_0$ invariant and $\widetilde{W}^*(\xi \otimes 1) = \xi \otimes 1$, we have $\mathrm{ad}_W(x_{ij})\xi \in \mathcal{D}_0$ for all i, j. Therefore, we have

$W\widetilde{J}(x\xi) = W(x^*\xi) = \mathrm{ad}_W(x^*)\widetilde{W}(\xi \otimes 1) = (\mathrm{ad}_W(x))^*(\xi \otimes 1)$
$= (\widetilde{J} \otimes \widetilde{J_{\mathcal{S}}})(\mathrm{ad}_W(x)(\xi \otimes 1)) = (\widetilde{J} \otimes \widetilde{J_{\mathcal{S}}})(W(x\xi))$.

Thus, $(\mathcal{S}, W)$ is an object of $\mathbf{Q}_{\widetilde{J}}(D)$ and clearly, W is faithful because $W(v) = U(v)q^{-1}$ and q is a unitary. By Theorem 3.3.4, we have a surjective C^*-homomorphism, say $\pi : \widetilde{\mathrm{QISO}^+}(\mathcal{A}^\infty, \mathcal{H}, D, \widetilde{J}) \to \mathcal{S}$. Hence, we can identify $\mathcal{S}$ with the quotient of $\widetilde{\mathrm{QISO}^+}(\mathcal{A}^\infty, \mathcal{H}, D, \widetilde{J})$ by a closed both sided ideal. Since the rest of the arguments for showing the existence of $\widetilde{\mathcal{G}}$ are very similar to those in the proof of Theorem 3.2.5, we omit the proof.

Now we come to the proof of the last part of the theorem. If U_0 denotes the unitary corepresentation of $\widetilde{\mathcal{G}}_0$ with $U_0(\xi) = \xi \otimes q_0$, then for a in $\mathcal{A}^\infty$, we have $\widetilde{U_0}(a\xi \otimes 1) = \mathrm{ad}_{U_0}(a)\widetilde{U_0}(\xi \otimes 1) = \mathrm{ad}_{U_0}(a)(\xi \otimes q_0)$. Since $\mathrm{Span}\{a\xi : a \in \mathcal{A}^\infty\}$ is dense in $\mathcal{H}$, it follows that $\widetilde{\mathcal{G}}$ is generated as a C^*-algebra by $\mathcal{G}$ and q_0. Taking the free product of the inclusion map $i : \mathcal{G} \to \widetilde{\mathcal{G}}$ and the canonical map from $C(\mathbb{T})$ to $C^*(q_0)$ sending the co-ordinate function z of $C(\mathbb{T})$ to q_0, we get a surjective $*$-homomorphism from $\mathcal{G} * C(\mathbb{T})$ to $\widetilde{\mathcal{G}}$. Finally, $C(\mathbb{T})$ has a unitary corepresentation given by $\xi \to \xi \otimes z$, $e_{ij} \to e_{ij} \otimes 1$ for all $i, j \geq 1$ and taking the free product of this corepresentation with the restriction of W_0 (where $W_0(v) = U_0(v)q_0^{-1}$) to the subspace $\xi^\perp$, we obtain a unitary corepresentation of $\mathcal{G} * C(\mathbb{T})$ which identifies $\mathcal{G} * C(\mathbb{T})$ as a subobject of the category $\mathbf{Q}_{\widetilde{J}}$ and completes the proof. $\square$

Remark 3.4.3 The proof of Theorem 4.18 of [8] regarding the quantum isometry group of $C(\mathbb{T}^2)$ shows that $\widetilde{\mathrm{QISO}}^+(D)$ can exist without the assumption of the existence of a single cyclic separating eigenvector as above.

Next, we prove a corollary to this theorem which helps in computing $\mathrm{QISO}^+(D)$ for a number of spectral triples satisfying the conditions of the above theorem. The examples include the Chakraborty-Pal spectral triple on $SU_q(2)$ (see [8]) as well as Connes' spectral triples on group algebras coming from length functions, the latter being the content of Chap. 8.

Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple which satisfies the conditions of Theorem 3.4.2. Let τ denote the faithful state $\langle \cdot \xi, \xi \rangle$ and let $\mathcal{A}_{00} = \mathrm{Span}\{a \in \mathcal{A}^\infty : a\xi$ is an eigenvector of $D\}$.

Moreover, assume that $\mathcal{A}_{00}$ is norm dense in $\mathcal{A}^\infty$.
We define $\widehat{D} : \mathcal{A}_{00} \to \mathcal{A}_{00}$ by:
$\widehat{D}(a)\xi = D(a\xi)$.
Since ξ is cyclic and separating for $\mathcal{A}^\infty$, $\widehat{D}$ is well defined.

Definition 3.4.4 Let $\mathcal{A}$ be the C^*-closure of $\mathcal{A}^\infty$ as above.

Let $\widehat{\mathbf{C}}$ be the category with objects $(\mathcal{Q}, \alpha)$ such that $\mathcal{Q}$ is a CQG with a C^* coaction α on $\mathcal{A}$ such that the following hold:

1. α is τ preserving (where τ is as above), that is, $(\tau \otimes \mathrm{id})(\alpha(a)) = \tau(a).1$.
2. $\alpha(\mathcal{A}_{00}) \subseteq \mathcal{A}_{00} \otimes_{\mathrm{alg}} \mathcal{Q}$.
3. $\alpha\widehat{D} = (\widehat{D} \otimes I)\alpha$.

Corollary 3.4.5 *There exists a universal object $\widehat{\mathcal{Q}}$ of the category $\widehat{\mathcal{C}}$ and it is isomorphic to the CQG $\mathcal{G} = \mathrm{QISO}^+(D)$ introduced in Theorem 3.4.2.*

Proof Since D hence $\widehat{D}$ has compact resolvent, the proof of the existence of the universal object follows exactly as in the proof of Theorem 3.1.7 by replacing $\mathcal{L}$ with $\widehat{D}$. We will denote the coaction of $\widehat{\mathcal{Q}}$ on $\mathcal{A}$ by $\widehat{\alpha}$.

Now, we prove that $\widehat{\mathcal{Q}}$ is isomorphic to $\mathcal{G}$.

Each eigenvector of $\widehat{D}$ is in $\mathcal{A}^\infty$ by assumption. From the proof of Theorem 3.4.2, it easily follows that ad_{U_0} maps the norm-dense $*$-subalgebra $\mathcal{A}_{00}$ into $\mathcal{A}_{00} \otimes_{alg} \mathcal{G}_0$, and $(\mathrm{id} \otimes \epsilon) \circ \mathrm{ad}_{U_0} = \mathrm{id}$. Therefore, ad_{U_0} is indeed a C^* coaction of the compact quantum group $\mathcal{G}$. Moreover, it also follows that τ preserves ad_{U_0} and that ad_{U_0} commutes with $\widehat{D}$. Hence, $(\mathcal{G}, \mathrm{ad}_{U_0})$ is an element of $\mathrm{Obj}(\widehat{\mathbf{C}})$ so that by the universality of $\widehat{\mathcal{Q}}$, $\mathcal{G}$ is a quantum subgroup of $\widehat{\mathcal{Q}}$.

For the opposite containment, we claim that $\widehat{\alpha}$ induces a unitary corepresentation W of $\widehat{\mathcal{Q}} * C(\mathbb{T})$ on $\mathcal{H}$ which commutes with D and ad_W coincides with $\widehat{\alpha}$.

Define $W(a\xi) = \widehat{\alpha}(a)(\xi)q^*$ for all a in $\mathcal{A}_0^\infty$ where q is the standard generator of $C(\mathbb{T})$.

Since α preserves the functional τ, $\widetilde{W}$ is a ($\widehat{\mathcal{Q}} * C(\mathbb{T})$-linear) isometry on the dense subspace $\mathcal{A}_{00}\xi \otimes_{\mathrm{alg}} \widehat{\mathcal{Q}}$ and thus extends to $\mathcal{H} \otimes (\widehat{\mathcal{Q}} * C(\mathbb{T}))$ as an isometry. Moreover, since $\widehat{\alpha}$ is a CQG coaction, $\mathrm{Span}\{\widehat{\alpha}(\mathcal{A})(1 \otimes \widehat{\mathcal{Q}})\}$ is norm dense in $\mathcal{A} \otimes \widehat{\mathcal{Q}}$ so that the range of $\widetilde{W}$ is dense. Therefore, $\widetilde{W}$ is indeed a unitary. It can be easily checked that it is actually a unitary corepresentation of $\widehat{\mathcal{Q}} * C(\mathbb{T})$.

We also have,

$$\begin{aligned}
&WD(a\xi)\\
&\quad = W(\widehat{D}(a)\xi) = \widehat{\alpha}(\widehat{D}(a))(\xi)q^*\\
&\quad = (D \otimes I)(\widehat{\alpha}(a)\xi)q^* = (D \otimes I)W(a\xi),
\end{aligned}$$

and hence W commutes with D.

Moreover, it is easy to observe that $\mathrm{ad}_W = \widehat{\alpha}$. This gives a surjective CQG morphism from $\widetilde{\mathcal{G}} = \mathcal{G} * C(\mathbb{T})$ to $\widehat{\mathcal{Q}} * C(\mathbb{T})$, sending $\mathcal{G}$ onto $\widehat{\mathcal{Q}}$, which completes the proof. $\square$

References

1. Goswami, D.: Quantum group of isometries in classical and non commutative geometry. Comm. Math. Phys. **285**(1), 141–160 (2009)
2. Wang, S.: Quantum symmetry groups of finite spaces. Comm. Math. Phys. **195**, 195–211 (1998)
3. Connes, A.; Moscovici, H.: Type III and spectral triples, traces in number theory, geometry and quantum fields. Aspects Mathematics, vol. E38, pp. 57–71. Friedr. Vieweg, Wiesbaden (2008)
4. Woronowicz, S.L.: Pseudogroups, pseudospaces and Pontryagin duality. In: Proceedings of the International Conference on Mathematical Physics, Lecture Notes in Physics, vol. 116, pp. 407–412. Lausane (1979)
5. Soltan, P.M.: Quantum families of maps and quantum semigroups on finite quantum spaces, preprint. J. Geom. Phys. **59**(3), 354–368 (2009)
6. Chakraborty, P.S., Pal, A.: Equivariant spectral triples on the quantum $SU(2)$ group. K. Theory **28**, 107–126 (2003)
7. Dabrowski, L., D'Andrea, F., Landi, G., Wagner, E.: Dirac operators on all Podles' quantum spheres. J. Noncomm. Geom. **1**, 213–239 (2007)
8. Bhowmick, J., Goswami, D.: Quantum group of orientation preserving Riemannian isometries. J. Funct. Anal. **257**(8), 2530–2572 (2009)
9. Fröhlich, J., Grandjean, O., Recknagel, A.: Supersymmetric quantum theory and non-commutative geometry. Comm. Math. Phys. **203**(1), 119–184 (1999)
10. Dabrowski, L.: Geometry of quantum spheres. J. Geom. Phys. **56**(1), 86–107 (2006)

Chapter 4
Quantum Isometry Groups of Classical and Quantum Spheres

Abstract We begin by explicitly computing the quantum isometry groups for the classical spheres and show that these coincide with the commutative C^* algebra of continuous functions on their classical isometry groups. The rest of the chapter is devoted to the computation of the quantum group of orientation preserving isometries for two different families of spectral triples on the Podles' spheres, one constructed by Dabrowski et al and the other by Chakraborty and Pal.

In this as well as the following chapters, we explicitly compute the quantum isometry groups of the classical sphere and the quantum Podles' spheres. These calculations may appear slightly boring and tedious at places. In fact, the example of the classical sphere can be derived from the general theory developed in Chap. 6. However, we decide to present some of the computations which do illustrate quite a few general techniques and principles, to motivate and prepare the reader for the materials of Chaps. 6 and 7. For quantum isometry groups of free spheres, we refer to Chap. 10.

4.1 Classical Spheres: No Quantum Isometries

Let $\mathcal{S}$ be the quantum isometry group of $S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1\}$ and let α be the coaction of $\mathcal{S}$ on $C(S^{n-1})$. Let $\mathcal{L}$ be the standard Laplacian on S^{n-1} given by

$$\mathcal{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

where $x_1, x_2, \ldots, x_n$ are the Cartesian coordinates.

The eigenspaces of $\mathcal{L}$ on S^{n-1} are of the form

$$E_k = \text{Span}\{(\sum_{i=1}^{n} c_i x_i)^k \ : \ c_i \in \mathbb{C}, i = 1, 2, \ldots, n, \ \sum_{j=1}^{n} c_i^2 = 0\},$$

where $k \geq 1$ (see [1], pp. 29–30).

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_4

Lemma 4.1.1 *The coaction α satisfies $\alpha(x_i) = \sum_{j=1}^{n} x_j \otimes Q_{ij}$, where $Q_{ij} \in \mathcal{S}$, $i = 1, 2, \ldots, n$.*

Proof Since α is an isometric coaction, $\alpha(E_k) \subseteq E_k \otimes_{\mathrm{alg}} \mathcal{S}$. The lemma follows by considering the containment $\alpha(E_1) \subseteq E_1 \otimes_{\mathrm{alg}} \mathcal{S}$. Now, $E_1 = \mathrm{Span}\{(\sum_{i=1}^{n} c_i x_i) : c_i \in \mathbb{C}, i = 1, 2, \ldots, n, \sum_{i=1}^{n} c_i^2 = 0\}$.

For $j \neq l$, both $x_j + i x_l, x_j - i x_l \in E_1$ and hence, $x_j, x_l \in E_1$. Therefore, $E_1 = \mathrm{span}\{x_1, x_2, \ldots, x_n\}$, which completes the proof. $\square$

We note that the functions $x_1, \ldots, x_n$ do not satisfy any nontrivial homogeneous quadratic relations, that is, $\{x_i x_j : 1 \leq i < j \leq n\}$ is linearly independent. This will imply that $\mathcal{S}$ is commutative as a C^* algebra. We deduce it from the theorem stated and proved below, which is going to play a crucial role later in Chap. 6 where we discuss CQG coactions on smooth Riemannian manifolds.

Theorem 4.1.2 *Let $\mathcal{A}$ be a unital commutative C^* algebra and $x_1, \ldots, x_n$ be self-adjoint elements of $\mathcal{A}$ such that the $*$-algebra generated by $\{x_1, \ldots, x_n\}$ is norm dense in $\mathcal{A}$. Moreover, we assume that $\mathcal{V} = \mathrm{Span}\{x_1, \ldots, x_n\}$ is quadratically independent, that is, the set $\{x_i x_j,\ 1 \leq i \leq j \leq n\}$ is linearly independent. If $\mathcal{S}$ is a compact quantum group acting faithfully on $\mathcal{A}$, such that the coaction is injective and it maps $\mathrm{Span}\{x_1, \ldots, x_n\}$ into itself, then $\mathcal{S}$ must be commutative as a C^* algebra, i.e., $\mathcal{S} \cong C(G)$ for some compact group G.*

Proof Let us denote the coaction of $\mathcal{S}$ by α. Since α is injective and the finite-dimensional subspace $\mathcal{V} = \mathrm{Span}\{x_1, \ldots, x_n\}$ is left invariant by α, the restriction of α to $\mathcal{V}$ must be a nondegenerate corepresentation of $\mathcal{S}$, hence $\alpha(\mathcal{V}) \subseteq \mathcal{V} \otimes_{\mathrm{alg}} \mathcal{S}_0$. Let $\mathcal{A} \cong C(X)$ for some compact Hausdorff space X, and fix any faithful (i.e., with full support) Borel probability measure μ on X, and denote by ϕ_μ the corresponding state on $\mathcal{A}$. Consider the convolved state $\overline{\phi_\mu} = (\phi_\mu \otimes h) \circ \alpha$ on $\mathcal{A}$. This is a faithful $\mathcal{S}$-invariant state, where faithfulness follows from the injectivity of α and faithfulness of h on $\mathcal{S}_0$. Let $\overline{\mu}$ and $\mathcal{H}$ be the corresponding Borel measure and the L^2-space, respectively, with the inner product $< \cdot, \cdot >_{\overline{\phi_\mu}}$. As $\mathcal{A}$ is commutative and x_i are self-adjoint, so are $x_i x_j\ \forall\, i, j$, and hence the inner product $< x_i, x_j >_{\overline{\phi_\mu}}$'s are real numbers. Thus, Gram–Schmidt orthogonalization on $\{x_1, \ldots x_n\}$ will give an orthonormal (w.r.t. $< \cdot, \cdot >_{\overline{\phi_\mu}}$) set $\{y_1, \ldots, y_n\}$ consisting of self-adjoint elements, with the same span as $\mathrm{Span}\{x_1, \ldots, x_n\}$. Replacing x_i's by y_i's, let us assume for the rest of the proof that $\{x_1, \ldots, x_n\}$ is an orthonormal set, there are $Q_{ij} \in \mathcal{S}$, $i, j = 1, \ldots, n$ such that $\mathcal{S} = C^*(Q_{ij}, i, j = 1, \ldots, n)$ and

$$\alpha(x_i) = \sum_{j=1}^{n} x_j \otimes Q_{ij}, \quad \forall i = 1, \ldots, n.$$

Since $x_i^* = x_i$, $\alpha(x_i^*) = \alpha(x_i)^*$ implies that $Q_{ij}^* = Q_{ij}\ \forall i, j = 1, 2, \ldots, n$.
Next, the relation $\alpha(x_i x_j) = \alpha(x_j x_i)$ gives

$$Q_{ij} Q_{kj} = Q_{kj} Q_{ij} \forall i, j, k, \tag{4.1.1}$$

$$Q_{ik}Q_{jl} + Q_{il}Q_{jk} = Q_{jk}Q_{il} + Q_{jl}Q_{ik}. \tag{4.1.2}$$

Now, it follows from Lemma 1.3.4 that α extends to a unitary corepresentation of $\mathcal{S}$ on $L^2(X, \overline{\mu})$. But α keeps $V = \mathrm{Span}\{x_1, x_2, \ldots, x_n\}$ invariant. Thus α is a unitary corepresentation of $\mathcal{S}$ on V, i.e., $Q = ((Q_{ij})) \in M_n(\mathcal{S})$ is a unitary, hence $Q^{-1} = Q^* = Q^T := ((Q_{ji}))$, as $Q_{ij}^* = Q_{ij}\ \forall\, i, j$.

Since the matrix Q defines an n-dimensional unitary corepresentation of $\mathcal{S}$, by Proposition 1.2.19, we obtain

$$\kappa(Q_{ij}) = Q_{ij}^{-1} = Q_{ij}^{T} = Q_{ji}. \tag{4.1.3}$$

Now from (4.1.1), we have $Q_{ij}Q_{kj} = Q_{kj}Q_{ij}$. Applying the antipode on this equation and using (4.1.3), we have $Q_{jk}Q_{ji} = Q_{ji}Q_{jk}$ Similarly, an application of κ on (4.1.2) yields

$$Q_{lj}Q_{ki} + Q_{kj}Q_{li} = Q_{li}Q_{kj} + Q_{ki}Q_{lj}\ \forall i, j, k, l.$$

Interchanging between k and i and also between l, j gives

$$Q_{jl}Q_{ik} + Q_{il}Q_{jk} = Q_{jk}Q_{il} + Q_{ik}Q_{jl}\ \forall i, j, k, l. \tag{4.1.4}$$

By subtracting (4.1.4) from (4.1.2), we have

$$[Q_{ik}, Q_{jl}] = [Q_{jl}, Q_{ik}].$$

Therefore,

$$[Q_{ik}, Q_{jl}] = 0.$$

Since α is a faithful coaction, the C^* subalgebra generated by $\{q_{ij} : i, j = 1, 2, \ldots n\}$ equals $\mathcal{S}$ and so $\mathcal{S}$ is commutative. $\square$

As S^{n-1} is connected, it follows from the previous chapter that the coaction of $\mathcal{S} = \mathrm{QISO}^{\mathcal{L}}(C(S^{n-1}))$ preserves the faithful state coming from the usual Riemannian volume form coming from S^{n-1}. Hence, the coaction of $\mathcal{S}$ is injective and we conclude that $\mathcal{S}$ coincides with $C(\mathrm{ISO}(\mathrm{S}^{n-1})) = \mathrm{C}(\mathrm{O}(\mathrm{n}))$.

Remark 4.1.3 We refer the reader to [2] for a more general and elegant no-go result for Hopf algebra coactions on commutative algebras.

4.2 Quantum Isometry Group of a Spectral Triple on Podles' Sphere

In this section, we turn to a noncommutative example, namely the quantum group of orientation preserving isometries for the spectral triples on the Podles' spheres $S^2_{\mu,c}$ already discussed in Chap. 2 and identify it with $SO_\mu(3)$.

4.3 Descriptions of the Podles' Spheres

We had introduced the Podles' spheres in Sect. 1.3.3 which is the universal C^* algebra generated by elements A and B satisfying the relations:

$$A^* = A, \ AB = \mu^{-2} BA,$$

$$B^*B = A - A^2 + cI, \ BB^* = \mu^2 A - \mu^4 A^2 + cI.$$

These are the relations which we are going to exploit for getting homomorphism conditions while computing the quantum group of orientation preserving isometries.

Notation: We will denote the coordinate $*$ algebra of $S^2_{\mu,c}$ by $\mathcal{O}(S^2_{\mu,c})$.

We will start with yet another equivalent description of the Podles' sphere given in [3].

4.3.1 The Description as in [3]

We need the quantum group $\mathcal{U}_\mu(su(2))$ for this description.

Let

$$X_c = \mu^{\frac{1}{2}} (\mu^{-1} - \mu)^{-1} c^{-\frac{1}{2}} (1 - K^2) + EK + \mu FK \quad \text{for all } c \in (0, \ \infty),$$

$$X_0 = 1 - K^2.$$

One has $\Delta(X_c) = 1 \otimes X_c + X_c \otimes K^2$.

Then (as in page 9, [3]) we have the following:

$$\mathcal{O}(S^2_{\mu,c}) = \{x \in \mathcal{O}(SU_\mu(2)) : x \triangleleft X_c = 0\},$$

where $\triangleleft$ is as in Sect. 1.2.5. The following is a vector space basis of $\mathcal{O}(S^2_{\mu,c})$:

$$\{A^k, A^k B^l, A^k B^{*m}, \ k \geq 0, \ l, m > 0\}.$$

Let ψ be the densely defined linear map on $L^2(SU_\mu(2))$ defined by $\psi(x) = x \triangleleft X_c$.

From [4], page 5, we recall that $v^l_{j,k} \triangleleft E = \mu \alpha^l_{k-1} v^l_{j,k-1}$, $v^l_{j,k} \triangleleft F = \frac{\alpha^l_k}{\mu} v^l_{j,k+1}$, $v^l_{j,k} \triangleleft K = \mu^k v^l_{j,k}$ for some constants α^l_j.

Lemma 4.3.1 *Let $\overline{S^2_{\mu,c}}$ be the closure of the subspace of $L^2(SU_\mu(2))$ generated by $S^2_{\mu,c}$. Then the map ψ is closable and we have $\overline{S^2_{\mu,c}} \subseteq \mathrm{Ker}(\overline{\psi})$, where $\overline{\psi}$ is the closed extension of ψ. Moreover, $\mathcal{O}(S^2_{\mu,c}) = \mathcal{O}(SU_\mu(2)) \bigcap \mathrm{Ker}(\overline{\psi}) = \mathcal{O}(SU_\mu(2)) \bigcap \mathrm{Ker}(\psi)$.*

Proof We observe that $v^l_{j,k} \triangleleft (1-K^2) = (1-\mu^{2k})v^l_{j,k}$, $v^l_{j,k} \triangleleft EK = \mu^k \alpha^l_{k-1} v^l_{j,k-1}$, $v^l_{j,k} \triangleleft FK = \mu^k \alpha^l_k v^l_{j,k+1}$. Hence, $v^l_{j,k} \in \mathrm{Dom}(\psi)$ with $(\psi^*)(v^l_{j,k}) = \mu^{\frac{1}{2}} (\mu^{-1}-\mu)^{-1} c^{-\frac{1}{2}}$.

$(1-\mu^{2k})v^l_{j,k} + \mu^k \alpha^l_{k-1} v^l_{j,k+1} + \mu^{k+1}\alpha^l_k v^l_{j,k-1}$. Hence, $\mathcal{O}(SU_\mu(2)) \subseteq \mathrm{Dom}(\psi^*)$ which shows that ψ is closable and thus $\mathrm{Ker}(\overline{\psi})$ is closed. The proof is finished by observing that $\mathcal{O}(S^2_{\mu,c}) = \mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\overline{\psi})$. □

4.3.2 'Volume Form' on the Podles' Spheres

We now derive a few useful facts and formulas concerning the canonical faithful, $SO_\mu(3)$-invariant state on $S^2_{\mu,c}$ obtained by restricting the Haar state of $SU_\mu(2)$ to it.

We recall from [3], page 33 that for every bounded complex Borel function f on $\sigma(A)$,

$$h(f(A)) = \gamma_+ \sum_{n=0}^{\infty} f(\lambda_+\mu^{2n})\mu^{2n} + \gamma_- \sum_{n=0}^{\infty} f(\lambda_-\mu^{2n})\mu^{2n}, \tag{4.3.1}$$

where $\lambda_+ = \frac{1}{2} + (c+\frac{1}{4})^{\frac{1}{2}}$, $\lambda_- = \frac{1}{2} - (c+\frac{1}{4})^{\frac{1}{2}}$, $\gamma_+ = (1-\mu^2)\lambda_+(\lambda_+-\lambda_-)^{-1}$, $\gamma_- = (1-\mu^2)\lambda_-(\lambda_- - \lambda_+)^{-1}$.

Lemma 4.3.2 $\{x_{-1}, x_0, x_1\}$ *is a set of orthogonal vectors.*

Proof From (1.3.11) and (1.3.12), we note that $x^*_{-1}x_0 \in \mathrm{Span}\{\alpha^{*2}\gamma^*\alpha, \alpha^{*2}, \alpha^{*2}\gamma^*\gamma, \alpha^{*2}\gamma\alpha^*, \gamma^*\alpha^*\gamma^*\alpha, \gamma^*\alpha^*, \gamma^*\alpha^*\gamma^*\gamma, \gamma^*\alpha^*\gamma\alpha^*, \gamma^{*3}\alpha, \gamma^{*2}, \gamma^{*3}\gamma, \gamma^{*2}\gamma\alpha^*\}$.

Further, using (1.2.11)–(1.2.15), we note that this coincides with $\mathrm{Span}\{\alpha^*\gamma^*, \alpha^*\gamma^{*2}\gamma, \alpha^{*2}, \alpha^{*2}\gamma^*\gamma, \alpha^{*3}\gamma, \gamma^{*2}, \gamma^{*3}\gamma, \alpha^*\gamma^*, \alpha\gamma^{*3}, \gamma^{*3}\}$.

Then $h(x^*_{-1}x_0) = 0$ follows using (1.2.16). Similarly, one can prove $h(x^*_0 x_1) = h(x^*_1 x_{-1}) = 0$. □

Lemma 4.3.3

$$h(A) = \frac{1}{1+\mu^2}, \quad h(A^2) = \frac{(1-\mu^2)(\lambda^3_+ - \lambda^3_-)}{(\lambda_+ - \lambda_-)(1-\mu^6)}.$$

Proof Recalling (4.3.1), we have

$$\begin{aligned} h(A) &= \gamma_+ \sum_{n=0}^{\infty} \lambda_+\mu^{4n} + \gamma_- \sum_{n=0}^{\infty} \lambda_-\mu^{4n} \\ &= \frac{(1-\mu^2)(\lambda^2_+ - \lambda^2_-)}{(\lambda_+ - \lambda_-)(1-\mu^4)} \\ &= \frac{\lambda_+ + \lambda_-}{1+\mu^2} = \frac{1}{1+\mu^2}. \end{aligned}$$

Similarly, putting $f(x) = x^2$ in (4.3.1), we have

$$h(A^2) = \gamma_+ \sum_{n=0}^{\infty} \lambda_+^2 \mu^{6n} + \gamma_- \sum_{n=0}^{\infty} \lambda_-^2 \mu^{6n} = \frac{(1-\mu^2)(\lambda_+^3 - \lambda_-^3)}{(\lambda_+ - \lambda_-)(1-\mu^6)}.$$

□

Lemma 4.3.4 $h(x_{-1}^* x_{-1}) = h(x_0^* x_0) = h(x_1^* x_1) = t^2(1-\mu^2)(1-\mu^6)^{-1}[\mu^2 + t^{-1}(\mu^4 + 2\mu^2 + 1) + t(-\mu^4 - 2\mu^2 - 1)]$.

Proof From (1.3.10) we have $x_{-1}^* x_{-1} = \frac{t^2(1+\mu^2)}{\mu^2} B^* B$ and hence, using Lemma 4.3.3, we obtain

$$\begin{aligned} &h(x_{-1}^* x_{-1}) \\ &= \frac{t^2(1+\mu^2)}{\mu^2}[h(A) - h(A^2) + (t^{-1} - t).1] \end{aligned}$$

$$= \frac{1 - \mu^6 - (1-\mu^4)(t^{-1} - t + 1) + (t^{-1} - t)(1+\mu^2)(1-\mu^6)}{(1+\mu^2)(1-\mu^6)},$$

from which we get the desired result.

Similarly, the second equality is derived from $h(x_0^* x_0) = t^2 - 2t^2(1+\mu^2)h(A) + (1+\mu^2)^2 t^2 h(A^2)$.

From (1.3.9), $x_1 = -\mu x_{-1}^*$ and hence, $x_1^* x_1 = t^2(1+\mu^2)(\mu^2 A - \mu^4 A^2 + c.I)$, from which $h(x_1^* x_1)$ is obtained and can be shown to be equal to the same value as $h(x_{-1}^* x_{-1}) = h(x_0^* x_0)$. □

4.4 Computation of the Quantum Isometry Groups

We recall the spectral triple on $(S^2_{\mu,c}, \mathcal{H}, D)$ discussed in Example 2.2.9 (taken from [5]) and let R as in Theorem 2.3.5. Note that a restatement of Theorem 2.3.5 yields the following:

Theorem 4.4.1 *$(SU_\mu(2),\ U_0)$ is an object in $\mathbf{Q}'_R(D)$ and ad_{U_0} is a faithful coaction of $SO_\mu(3)$.*

Here we will compute $\widetilde{\mathrm{QISO}}^+_R(S^2_{\mu,c})$ of the above-mentioned spectral triple and show that it is isomorphic with $SO_\mu(3)$.

Let $(\widetilde{\mathcal{S}}, U)$ be an object in $\mathbf{Q}'_R(D)$ and $< \cdot, \cdot >_{\widetilde{\mathcal{S}}}$ be the $\widetilde{\mathcal{S}}$ valued inner product of $\mathcal{H} \otimes \widetilde{\mathcal{S}}$ and $\mathcal{S}$ be the Woronowicz C^* subalgebra of $\widetilde{\mathcal{S}}$ generated by $\{< (\xi \otimes 1), \mathrm{ad}_U(a)(\eta \otimes 1) >_{\widetilde{\mathcal{S}}} : \xi, \eta \in \mathcal{H},\ a \in S^2_{\mu,c}\}$.

Notation: We shall denote ad_U by ϕ from now on.

The computation has the following steps:

Step 1. We prove that ϕ keeps the span of Span$\{1, A, B, B^*\}$ invariant. We will refer to this property of ϕ as the affineness of the coaction.

Step 2. We use the facts that ϕ is a $*$-homomorphism and that it preserves the canonical volume form on $S^2_{\mu,c}$ which is the restriction of the Haar state of $SU_\mu(2)$.

Step 3. We will compute the antipode of $\mathcal{S}$ and apply it to get some more relations.

Step 4. We will use the above steps to identify $\mathcal{S}$ as a subobject of $SO_\mu(3)$ which will finish the proof.

We shall be brief in our exposition. For the full details, we refer to Chap. 5 of [6] and [7].

Remark 4.4.2 **Step 1**. Does not use that ϕ preserves the canonical volume form on $S^2_{\mu,c}$. Thus the conclusion of **Step 1**. remains valid for any object in $\mathbf{Q}'(D)$.

Lemma 4.4.3 *We have the following:*

1. The eigenspaces of D for the eigenvalues $(c_1 l + c_2)$ *and* $-(c_1 l + c_2)$ *are* $\mathrm{Span}\{\mathrm{v}^{\mathrm{l}}_{\mathrm{m},\frac{1}{2}} + \mathrm{v}^{\mathrm{l}}_{\mathrm{m},-\frac{1}{2}} : -\mathrm{l} \le \mathrm{m} \le \mathrm{l}\}$ *and* $\mathrm{Span}\{\mathrm{v}^{\mathrm{l}}_{\mathrm{m},\frac{1}{2}} - \mathrm{v}^{\mathrm{l}}_{\mathrm{m},-\frac{1}{2}} : -\mathrm{l} \le \mathrm{m} \le \mathrm{l}\}$, *respectively.*

2. The eigenspace of $|D|$ for the eigenvalue $(c_1.\frac{1}{2} + c_2)$ *is* Span$\{\alpha,\ \gamma,\ \alpha^*,\ \gamma^*\}$.

Proof 1. Follows from (2.2.3). To prove 2., we note that by 1, it is sufficient to identify $\mathrm{Span}\{\mathrm{v}^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}},\ \mathrm{v}^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}},\ \mathrm{v}^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}},\ \mathrm{v}^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}\}$.

Using (2.2.1) and (1.2.21), we have

$$\begin{aligned}
&v^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}\\
&= N^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} F \triangleright (y^0_1 u_{\frac{1}{2}}) \;=\; N^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} F \triangleright (\alpha^* - s\gamma^*)\\
&= N^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} (\gamma + \mu^{-1} s\alpha) \;=\; N^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} \gamma + \mu^{-1} s N^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}} \alpha.
\end{aligned}$$

$$\begin{aligned}
&v^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}\\
&= N^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} F^0 \triangleright (y^0_1 u_{\frac{1}{2}}) \;=\; N^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} (\alpha^* - s\gamma^*)\\
&= N^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} \alpha^* - s N^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} \gamma^*.
\end{aligned}$$

$$\begin{aligned} & v^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} \\ &= N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} F \triangleright (y^0_1 u_{-\frac{1}{2}}) \;=\; N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} F \triangleright (E \triangleright w_{\frac{1}{2}}) \\ &= N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} F \triangleright (E \triangleright (\alpha - \mu s \gamma)) \;=\; N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} F \triangleright (-\mu\gamma^* - \mu s \alpha^*) \\ &= N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} (\alpha - \mu s \gamma) \;=\; N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} \alpha - \mu s N^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}} \gamma. \end{aligned}$$

$$\begin{aligned} & v^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} \\ &= N^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} F^0 \triangleright (y^0_1 u_{-\frac{1}{2}}) \;=\; N^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} (E \triangleright (\alpha - \mu s \gamma)) \\ &= N^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} (-\mu\gamma^* - \mu s \alpha^*) \;=\; -\mu N^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} \gamma^* - \mu s N^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}} \alpha^*. \end{aligned}$$

Combining these, we have the result. □

Lemma 4.4.4 1. $\pi(A) v^l_{m,N} \in \mathrm{Span}\{v^{l-1}_{m,N}, v^l_{m,N}, v^{l+1}_{m,N}\}$,
$\pi(B) v^l_{m,N} \in \mathrm{Span}\{v^{l-1}_{m-1,N}, v^l_{m-1,N}, v^{l+1}_{m-1,N}\}$,
$\pi(B^*) v^l_{m,N} \in \mathrm{Span}\{v^{l-1}_{m+1,N}, v^l_{m+1,N}, v^{l+1}_{m+1,N}\}$.

2. $\pi(A^k)(v^l_{m,N}) \in \mathrm{Span}\{v^{l-k}_{m,N}, v^{l-k+1}_{m,N}, \ldots, v^{l+k}_{m,N}\}$.

3. $\pi(A^{m'} B^{n'})(v^l_{m,N}) \in \mathrm{Span}\{v^{l-m'-n'}_{m-n',N}, v^{l-(n'+m'-1)}_{m-n',N}, \ldots, v^{l+n'+m'}_{m-n',N}\}$.

4. $\pi(A^r B^{*s})(v^l_{m,N}) \in \mathrm{Span}\{v^{l-s-r}_{m+s,N}, v^{l-s-r+1}_{m+s,N}, \ldots v^{l+s+r}_{m+s,N}\}$.

Proof Using (1.3.10) and (2.2.2), we have

$$\begin{aligned} & \pi(A) v^l_{m,N} \\ &= \frac{1}{1+\mu^2} v^l_{m,N} - \frac{t^{-1}}{1+\mu^2} [\alpha^-_0(l,m;N) v^{l-1}_{m,N} + \alpha^0_0(l,m;N) v^l_{m,N} + \alpha^+_0(l,m;N) v^{l+1}_{m,N}]. \end{aligned}$$

Thus, $\pi(A) v^l_{m,N} \in \mathrm{Span}\{v^l_{m,N}, v^{l-1}_{m,N}, v^{l+1}_{m,N}\}$.

Similarly, using the expressions for B and B^* from (1.3.10) and then using (2.2.2) just as above, we get the required statements for $\pi(B)$ and $\pi(B^*)$. This proves 1.

Repeated use of 1. now yields 2., 3., and 4. □

4.4.1 *Affineness of the Coaction*

Let v be a vector in $\mathcal{H}$. Let $T_v : \mathcal{B}(\mathcal{H}) \to L^2(SU_\mu(2))$ be defined as $T_v(x) = xv$, where we view $xv \in \mathcal{H}$ as an element of $L^2(SU_\mu(2))$. The map T_v is a continuous map w.r.t. the strong operator topology on $\mathcal{B}(\mathcal{H})$ and the Hilbert space topology of $L^2(SU_\mu(2))$.

Now if $a \in SU_\mu(2)$, R_a will denote the bounded linear map on $L^2(SU_\mu(2))$ defined by right multiplication by a. Then the map $R_a T_v$ is a bounded linear map from $\mathcal{B}(\mathcal{H})$ (with Strong Operator Topology) to the Hilbert space $L^2(SU_\mu(2))$. We now define

$$T = R_{\alpha^*} T_\alpha + \mu^2 R_\gamma T_{\gamma^*}.$$

Lemma 4.4.5 *Let ω be a state on $\widetilde{\mathcal{S}}$ and $x \in S^2_{\mu,c}$. Then $T(\phi_\omega(x)) = \phi_\omega(x) \equiv R_1(\phi_\omega(x)) \in \overline{S^2_{\mu,c}} \subseteq L^2(SU_\mu(2))$, where $\phi_\omega(x) = (\mathrm{id} \otimes \omega)(\phi(x))$.*

Proof Let $x \in S^2_{\mu,c} \subseteq \mathcal{B}(\mathcal{H})$. Using the relation $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$, it follows that $T(x) = x \equiv R_1(x)$, where we view x in the right-hand side of the equation as an element of $L^2(SU_\mu(2))$. Since $\phi_\omega(x) \in (S^2_{\mu,c})''$, which is the Strong Operator Topology closure of $S^2_{\mu,c}$ and T is continuous in the Strong Operator Topology, the conclusion of the lemma follows. □

Let

$$\mathcal{V}^l = \mathrm{Span}\{v^{l'}_{i,\pm\frac{1}{2}}, -l' \leq i \leq l',\ l' \leq l\}.$$

The eigenspace of $|D|$ for the eigenvalue $c_1 l + c_2$, is $\mathrm{Span}\{v^l_{i,\pm\frac{1}{2}}, -l \leq i \leq l\}$. Therefore, $\forall\, l$, the subspace $\mathcal{V}^l$ is kept invariant by $\widetilde{U}$ and $\widetilde{U}^*$.

Lemma 4.4.6 *There exists a subspace $\mathcal{V}$ of $\mathcal{O}(SU_\mu(2))$ which is finite dimensional such that $R_{\alpha^*}(\phi_\omega(A)v^{\frac{1}{2}}_{j,\pm\frac{1}{2}})$, $R_\gamma(\phi_\omega(A)v^{\frac{1}{2}}_{j,\pm\frac{1}{2}})$ belong to $\mathcal{V}$ for all states ω on $\widetilde{\mathcal{S}}$.*

The same holds when A is replaced by B or B^*.

Proof We have $\phi(A)(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1) = \widetilde{U}(\pi(A) \otimes 1)\widetilde{U}^*(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1)$.

Now, $\widetilde{U}^*(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1) \in \mathcal{V}^{\frac{1}{2}} \otimes_{\mathrm{alg}} \widetilde{\mathcal{S}}$, and then using the definition of π as well as the Lemma 4.4.4, we get $(\pi(A) \otimes 1)\widetilde{U}^*(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}} \otimes 1) \in \mathrm{Span}\{v^{l'}_{j,\pm\frac{1}{2}} : -l' \leq j \leq l', l' \leq \frac{3}{2}\} \otimes_{\mathrm{alg}} \widetilde{\mathcal{S}} = \mathcal{V}^{\frac{3}{2}} \otimes_{\mathrm{alg}} \widetilde{\mathcal{S}}$. Again, $\widetilde{U}$ keeps $\mathcal{V}^{\frac{3}{2}} \otimes_{\mathrm{alg}} \widetilde{\mathcal{S}}$ invariant, so $R_{\alpha^*}(\phi_\omega(A)(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}}))$ is in $\mathrm{Span}\{v\alpha^*, v \in \mathcal{V}^{\frac{3}{2}}\}$. Similarly, $R_\gamma(\phi_\omega(A)(v^{\frac{1}{2}}_{j,\pm\frac{1}{2}})) \in \mathrm{Span}\{v\gamma : v \in \mathcal{V}^{\frac{3}{2}}\}$. So, the statement for A follows by taking $\mathcal{V} = \mathrm{Span}\{v\alpha^*, v\gamma :\ v \in \mathcal{V}^{\frac{3}{2}}\} \subset \mathcal{O}(SU_\mu(2))$. A similar argument proves the lemma for B and B^*. □

Corollary 4.4.7 *There exists a subspace (finite dimensional) $\mathcal{V}$ of $\mathcal{O}(SU_\mu(2))$ so that for all states (hence for all elements $\omega \in \widetilde{\mathcal{S}}^*$), we have $T(\phi_\omega(X)) \in \mathcal{V}$, where X is A, B or B^*.*

Proof This follows since $\alpha, \gamma^* \in \mathrm{Span}\{v^{\frac{1}{2}}_{j,\pm\frac{1}{2}}\}$. □

Proposition 4.4.8 *$\phi(A)$, $\phi(B)$, $\phi(B^*)$ belong to $\mathcal{O}(S^2_{\mu,c}) \otimes_{\mathrm{alg}} \mathcal{S}$.*

Proof By applying Corollary 4.4.7, Lemmas 4.4.5 and 4.3.1, it follows that for all $\omega \in \widetilde{\mathcal{S}}^*$, we have $T(\phi_\omega(A)) \in \mathcal{V} \bigcap \overline{S^2_{\mu,c}} \subset \mathcal{O}(SU_\mu(2)) \bigcap \mathrm{Ker}(\psi)$ and hence $\mathcal{V} \bigcap \overline{S^2_{\mu,c}} = \mathcal{V} \bigcap \mathcal{O}(S^2_{\mu,c})$, where $\mathcal{V}$ is as in Corollary 4.4.7. Since $\mathcal{V} \bigcap \mathcal{O}(S^2_{\mu,c})$ is finite dimensional, there exists a positive integer m such that for all ω, $T(\phi_\omega(A)) \in \mathcal{W} := \mathrm{Span}\{A^k, \ A^k B^l, \ A^k B^{*l} : 0 \leq k, l \leq m\}$. Since for all states (and hence for all bounded linear functional) ω on $\widetilde{\mathcal{S}}$, we have $T(\phi_\omega(A)) = R_1(\phi_\omega(A)) \equiv \phi_\omega(A).1$, it is clear that $\phi_\omega(A)$ is in $\mathcal{W}$ for every ω in $\widetilde{\mathcal{S}}^*$.

We recall from Lemma 3.2.4 that $\widetilde{S}$ is a quotient of the separable C^* algebra $\mathcal{U}$ and is thus separable. Let us choose and fix a faithful state ω on $\widetilde{\mathcal{S}}$ and let $\mathcal{K} = L^2(\mathcal{S}, \omega)$. Thus, we can view $\widetilde{\mathcal{S}}$ and $\mathcal{L}(\mathcal{H} \otimes \widetilde{\mathcal{S}})$ as embedded in $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, respectively. Then $\phi(A)$ can be viewed as an element of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. If $\{q_1, q_2, \ldots\}$ be elements of $\widetilde{\mathcal{S}}$ such that it is a countable orthonormal basis of $\mathcal{K}$, then $\phi(A)$ can be written as $\sum_{i,j=1}^{\infty} \phi^{ij}(A) \otimes |q_i >< q_j|$, where the series is weakly convergent. Let $\omega_{ij}(\cdot) = \omega(q_i^* \cdot q_j)$ so that $\phi^{ij}(A) = (\mathrm{id} \otimes \omega_{ij})(\phi(A))$. Therefore, $\forall\, i, j$, $\phi^{ij}(A)$ belong to $\mathcal{W}$. As W is finite dimensional and the series $\sum_{i,j=1}^{n} \phi^{ij}(A) \otimes |q_i >< q_j| \in \mathcal{W} \otimes \mathcal{B}(\mathcal{K})$ is weakly convergent, we conclude that $\phi(A) \in \mathcal{W} \otimes \mathcal{B}(\mathcal{K})$. Hence, if $A_1, \ldots, A_k$ is a basis of $\mathcal{W}$, there exists $B_1, B_2, \ldots B_k$ such that $\phi(A) = \sum_{i=1}^{k} A_i \otimes B_i$.

At this point, we claim that $B_1, B_2, \ldots B_k \in \widetilde{\mathcal{S}}$. To this end, let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $\mathcal{H}$ so that any trace class positive operator ρ in $\mathcal{H}$ is of the form $\rho = \sum_j \lambda_j |e_j >< e_j|$, where λ_j are positive scalars such that $\sum_j \lambda_j < \infty$. Moreover, let $\psi_\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be the normal functional defined by $\psi_\rho(x) = \mathrm{Tr}(\rho x)$. Let $\widetilde{\psi_\rho} : \mathcal{L}(\mathcal{H} \otimes \mathcal{S}) \to \widetilde{\mathcal{S}}$ be the bounded linear map defined by $\widetilde{\psi_\rho}(X) = \sum_j \lambda_j < e_j \otimes 1, X(e_j \otimes 1) >_{\widetilde{\mathcal{S}}}$, where $X \in \mathcal{L}(\mathcal{H} \otimes \widetilde{\mathcal{S}})$ and $< \cdot, \cdot, >_{\widetilde{\mathcal{S}}}$ denotes that $\widetilde{\mathcal{S}}$ valued inner product of $\mathcal{H} \otimes \widetilde{\mathcal{S}}$.

By the linear independence of $\{A_1, \ldots, A_k\}$, there exists trace class operators $\rho_1, \ldots, \rho_k$ such that $\psi_{\rho_i}(A_i) = 1$ and $\psi_{\rho_i}(A_j) = 0$ for $j \neq i$. Thus $B_i = \widetilde{\psi}_{\rho_i}(\phi(A)) \in \widetilde{\mathcal{S}}$. Since $\mathcal{S}$ is the Woronowicz subalgebra of $\widetilde{\mathcal{S}}$ generated by $\{< \xi \otimes 1, \phi(x)(\eta \otimes 1) >_{\widetilde{\mathcal{S}}} : \xi, \eta \in \mathcal{H}\}$, it follows that $B_i \in \mathcal{S}\ \forall\, i = 1, 2, \ldots, k$. This finishes the proof for $\phi(A)$.

The proofs for $\phi(B)$ and $\phi(B^*)$ being similar, are omitted. □

Proposition 4.4.9 *ϕ keeps the span of $1, A, B, B^*$ invariant.*

Proof We prove the Proposition for $\phi(A)$. The proofs for B and B^* are similar.

By Proposition 4.4.8, $\phi(A)$ can be written as a finite sum of the form $\sum_{k \geq 0} A^k \otimes Q_k + \sum_{m',n',n' \neq 0} A^{m'} B^{n'} \otimes R_{m',n'} + \sum_{r,s,s \neq 0} A^r B^{*s} \otimes R'_{r,s}$.

Let $\xi = v^l_{m_0,N_0}$. We have $U(\xi) \in \mathrm{Span}\{v^l_{m,N}, m = -l, \ldots l, \ N = \pm\frac{1}{2}\}$. Let

$$\widetilde{U}(\xi \otimes 1) = \sum_{m=-l,\ldots l, N=\pm\frac{1}{2}} v^l_{m,N} \otimes q^l_{(m,N),(m_0,N_0)},$$

where $q^l_{(m,N),(m_0,N_0)}$ belong to $\mathcal{S}$. Since ϕ preserves the R-twisted volume, we have

$$\sum_{m',N'} q^l_{(m,\ N),\ (m',\ N')} q^{l*}_{(m,N),\ (m',\ N')} = 1. \tag{4.4.1}$$

It also follows that $U(A\xi) \in \mathrm{Span}\{v^{l'}_{m,\ N}, m = -l', \ldots l', l' = l-1, l, l+1,\ N = \pm\frac{1}{2}\}$.

By Lemma 4.4.4, we have $\phi(A)\widetilde{U}(\xi \otimes 1) = \sum_{k,\ m=-l,\ldots l, N=\pm\frac{1}{2}} A^k v^l_{m,\ N} \otimes Q_k q^l_{(m,\ N),\ (m_0,N_0)} + \sum_{m',\ n',n'\neq 0,\ m=-l,\ldots l, N=\pm\frac{1}{2}} A^{m'} B^{n'} v^l_{m,N} \otimes R_{m',\ n'}\ q^l_{(m,N),\ (m_0,N_0)} + \sum_{r,s,\ s\neq 0,\ m=-l,\ldots l,\ N=\pm\frac{1}{2}} A^r B^{*s} v^l_{m,N} \otimes R'_{r,s} q^l_{(m,N),\ (m_0,N_0)}$.

Let m'_0 denote the largest integer m' such that there is a nonzero coefficient of $A^{m'}B^{n'}$, $n' \geq 1$ in the expression of $\phi(A)$. We claim that the coefficient of $v^{l-m'_0-n'}_{m-n',N}$ in $\phi(A)\widetilde{U}(\xi \otimes 1)$ is $R_{m'_0,n'} q^l_{(m,N),\ (m_0,N_0)}$.

Indeed, the term $v^{l-m'_0-n'}_{m-n',N}$ can arise in the following ways: it can either come from a term of the form $A^{m''}B^{n''}v^l_{m,N}$ or $A^k v^l_{m,N}$ or $A^r B^{*s} v^l_{m.N}$ for some m'', n'', k, r, s.

In the first case, by Lemma 4.4.4, we must have $l - m'_0 - n' = l - m'' - n'' + t,\ 0 \leq t \leq 2m''$ and $m - n' = m - n''$ so that $m'' = m'_0 + t$, and since m'_0 is the largest integer such that $A^{m'_0}B^{n'}$ appears in $\phi(A)$, we can conclude that $t = 0$, that is $v^{l-m'_0-n'}_{m-n',N}$ appears only in $A^{m'_0}B^{n'}$.

In the second case, we have $m - n' = m$ implying $n' = 0$, which is a contradiction. In the last case, we have $m - n' = m + s$ so that $-n' = s$ which is possible only when $n' = s = 0$ which is again a contradiction.

Now, coefficient of $v^{l-m'_0-n}_{m-n',N}$ in $\widetilde{U}(A\xi \otimes 1)$ is zero if $m'_0 \geq 1$ (as $n' \neq 0$). It now follows from the above claim, using Lemma 4.4.4 and comparing coefficients in the equality $\widetilde{U}(A\xi \otimes 1) = \phi(A)\widetilde{U}(\xi \otimes 1)$, that $R_{m'_0,n'} q^l_{(m,N),\ (m_0,N_0)} = 0\ \forall\ n' \geq 1,\ \forall\ m, N$ when $m'_0 \geq 1$. Varying $(m_0,\ N_0)$, we conclude that the above holds $\forall\ (m_0,\ N_0)$. Using (4.4.1), we get $R_{m'_0,n'} \sum_{m',N'} q^l_{(m,N),\ (m',N')} q^{l*}_{(m,\ N),(m',\ N')} = 0\ \forall\ n' \geq 1$,

that is, $R_{m'_0,\ n'} = 0\ \forall\ n' \geq 1$ if $m'_0 \geq 1$. Proceeding by induction on m'_0, we deduce $R_{m',\ n'} = 0\ \forall\ m' \geq 1,\ n' \geq 1$.

Similarly, we have $Q_k = 0\ \forall\ k \geq 2$ and $R'_{r,\ s} = 0\ \forall\ r \geq 1,\ s \geq 1$.

Thus, $\phi(A) \in \mathrm{Span}\{1, A, B, B^*, B^2, \ldots B^n, B^{*2}, \ldots B^{*m}\}$. But the coefficient of $v^{l-n'}_{m-n',\ N}$ in $\phi(A)\widetilde{U}(\xi \otimes 1)$ is $R_{0,n'}$. Arguing as before, we arrive at the conclusion that $R_{0,\ n'} = 0\ \forall\ n' \geq 2$. In a similar way, we can prove $R'_{0,\ n'} = 0\ \forall\ n' \geq 2$. □

Thus, we can write

$$\phi(A) = 1 \otimes T_1 + A \otimes T_2 + B \otimes T_3 + B^* \otimes T_4, \tag{4.4.2}$$

$$\phi(B) = 1 \otimes S_1 + A \otimes S_2 + B \otimes S_3 + B^* \otimes S_4, \tag{4.4.3}$$

for some T_i, S_i in $\mathcal{S}$.

4.4.2 Homomorphism Conditions

We start with Eqs. (4.4.2) and (4.4.3). We shall make use of the relations among T_i, S_i coming from the facts that ϕ is a unital $*$-homomorphism and that it preserves the R-twisted volume.

Lemma 4.4.10

$$T_1 = \frac{1 - T_2}{1 + \mu^2},$$

$$S_1 = \frac{-S_2}{1 + \mu^2}.$$

Proof We use the formulas of A and B in terms of the $SU_\mu(2)$ elements from Eqs. (1.3.12), (1.3.12), and (1.3.13). It follows that $h(A) = (1 + \mu^2)^{-1}$ and $h(B) = 0$. Using Proposition 2.3.4 and the equations $(h \otimes \mathrm{id})\phi(A) = h(A).1$ and $(h \otimes \mathrm{id})\phi(B) = h(B).1$, we arrive at the conclusion. □

Lemma 4.4.11 $T_1^* = T_1,\ T_2^* = T_2,\ T_4^* = T_3.$

Proof It follows by comparing the coefficients of 1, A, and B, respectively, in the equation $\phi(A^*) = \phi(A)$. □

We shall now assume that $\mu \neq 1$.

Lemma 4.4.12

$$S_2^* S_2 + c(1 + \mu^2)^2 S_3^* S_3 + c(1 + \mu^2)^2 S_4^* S_4$$
$$= (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)^2 T_3 T_3^* - c(1 + \mu^2)^2 T_3^* T_3 + c(1 + \mu^2)^2.1, \tag{4.4.4}$$

$$-2S_2^* S_2 + (1 + \mu^2) S_3^* S_3 + \mu^2 (1 + \mu^2) S_4^* S_4 = (\mu^2 + 2T_2 - 1) T_2 - \mu^2 (1 + \mu^2) T_3 T_3^* - (1 + \mu^2) T_3^* T_3, \tag{4.4.5}$$

$$S_2^* S_2 - S_3^* S_3 - \mu^4 S_4^* S_4 = -T_2^2 + \mu^4 T_3 T_3^* + T_3^* T_3, \tag{4.4.6}$$

$$S_2^* S_4 + S_3^* S_2 = -(\mu^2 + T_2) T_3^* + T_3^* (1 - T_2), \tag{4.4.7}$$

$$S_2^* S_3 + \mu^2 S_4^* S_2 = -T_2 T_3 - \mu^2 T_3 T_2, \tag{4.4.8}$$

$$S_4^* S_3 = -T_3^2. \tag{4.4.9}$$

Proof The Lemma is proved by comparing the coefficients of $1, A, A^2, B^*, AB$, and B^2 in the equation $\phi(B^*B) = \phi(A) - \phi(A^2) + cI$ and then substituting $S_1,\ T_1,\ T_2^*$, T_4 by $\frac{-S_2}{1+\mu^2},\ \frac{1-T_2}{1+\mu^2},\ T_2,\ T_3^*$, respectively, using the relations in Lemmas 4.4.10 and 4.4.11. □

Lemma 4.4.13

$$S_2S_2^* + c(1+\mu^2)^2S_3S_3^* + c(1+\mu^2)^2S_4S_4^*$$

$$= \mu^2(1-T_2)(1+\mu^2T_2) + c(1+\mu^2)^2T_3T_3^* + c(1+\mu^2)^2T_3^*T_3 + c(1+\mu^2)^2.1, \tag{4.4.10}$$

$$-2S_2S_2^* + \mu^2(1+\mu^2)S_3S_3^* + (1+\mu^2)S_4S_4^*$$

$$= \mu^2(1+\mu^2)T_2 - 2\mu^4(1-T_2)T_2 - \mu^6(1+\mu^2)T_3T_3^* - \mu^4(1+\mu^2)T_3^*T_3, \tag{4.4.11}$$

$$-\ S_2S_4^* - S_3S_2^* = \mu^2(1+\mu^2)T_3 - \mu^4(1-T_2)T_3 - \mu^4T_3(1-T_2), \tag{4.4.12}$$

$$S_2S_2^* - \mu^4S_3S_3^* - S_4S_4^* = -\mu^4T_2^2 + \mu^8T_3T_3^* + \mu^4T_3^*T_3, \tag{4.4.13}$$

$$S_2S_4^* + \mu^2S_3S_2^* = -\mu^4T_2T_3 - \mu^6T_3T_2, \tag{4.4.14}$$

$$S_3S_4^* = -\mu^4T_3^2. \tag{4.4.15}$$

Proof This follows by equating the coefficients of $1,\ A,\ B,\ A^2,\ AB,\ B^2$ in the equation $\phi(BB^*) = \mu^2\phi(A) - \mu^4\phi(A^2) + c.1$ and then using Lemmas 4.4.10 and 4.4.11. □

Similarly, we can derive some more relations from the equation $\phi(BA) = \mu^2\phi(AB)$.

4.4.3 Relations Coming from the Antipode

Now, we compute the antipode, say κ of $\widetilde{\mathcal{S}}$.

To begin with, we recall from Lemmas 4.3.2 and 4.3.4 that $\{x_{-1}, x_0, x_1\}$ is a set of orthogonal vectors with same norm.

Lemma 4.4.14 *If $x'_{-1},\ x'_0,\ x'_1$ is the normalized basis corresponding to $\{x_{-1},\ x_0,\ x_1\}$, then from (4.4.2) and (4.4.3) we obtain*

$$\phi(x'_{-1}) = x'_{-1} \otimes S_3 + x'_0 \otimes -\mu^{-1}(1+\mu^2)^{-\frac{1}{2}}S_2 + x'_1 \otimes -\mu^{-1}S_4,$$
$$\phi(x'_0) = x'_{-1} \otimes -\mu(1+\mu^2)^{\frac{1}{2}}T_3 + x'_0 \otimes T_2 + x'_1 \otimes (1+\mu^2)^{\frac{1}{2}}T_4,$$
$$\phi(x'_1) = x'_{-1} \otimes -\mu S_4^* + x'_0 \otimes (1+\mu^2)^{-\frac{1}{2}}S_2^* + x'_1 \otimes S_3^*.$$

Proof As $x_{-1},\ x_0,\ x_1$ have the same norm, it follows that $x_i' = Kx_i,\quad$ where $K = \|x_i\|^{-1},\ i = \{-1, 0, 1\}$.

Now, using (1.3.10) and (4.4.3), we have

$$\begin{aligned}
&\phi(x_{-1}') \\
&= \frac{Kt(1+\mu^2)^{\frac{1}{2}}}{\mu}\phi(B) \\
&= \frac{Kt(1+\mu^2)^{\frac{1}{2}}}{\mu}[1\otimes S_1 + \frac{1-t^{-1}x_0}{1+\mu^2}\otimes S_2 + \frac{\mu x_{-1}}{t(1+\mu^2)^{\frac{1}{2}}}\otimes S_3 + \frac{\mu(-\mu^{-1}x_1)}{t(1+\mu^2)^{\frac{1}{2}}}\otimes S_4 \\
&= Kx_{-1}\otimes S_3 + Kx_0\otimes -\frac{S_2}{\mu(1+\mu^2)^{\frac{1}{2}}} + Kx_1\otimes -\frac{S_4}{\mu}
\end{aligned}$$

(by Lemma 4.4.10)

$$= x_{-1}'\otimes S_3 + x_0'\otimes -\mu^{-1}(1+\mu^2)^{-\frac{1}{2}}S_2 + x_1'\otimes -\mu^{-1}S_4.$$

By similar calculations, we get the second and the third equations. □

Hence, ϕ keeps the span of the orthonormal set $\{x_{-1}',\ x_0',\ x_1'\}$ invariant. Moreover, the Haar state h of $SU_\mu(2)$ is preserved by ϕ. Therefore, we have a unitary corepresentation of the CQG $\widetilde{\mathcal{S}}$ on $\mathrm{Span}\{\mathrm{x}_{-1}',\ \mathrm{x}_0',\ \mathrm{x}_1'\}$.

Using $T_4 = T_3^*$ from Lemma 4.4.11, the unitary matrix, say Z corresponding to ϕ and the ordered basis $\{x_{-1}',\ x_0',\ x_1'\}$ is given by

$$Z = \begin{pmatrix} S_3 & -\mu\sqrt{1+\mu^2}T_3 & -\mu S_4^* \\ \frac{-S_2}{\mu\sqrt{1+\mu^2}} & T_2 & \frac{S_2^*}{\sqrt{1+\mu^2}} \\ -\mu^{-1}S_4 & \sqrt{1+\mu^2}T_3^* & S_3^* \end{pmatrix}. \tag{4.4.16}$$

By Proposition 1.2.19, we have

$$\kappa(T_2) = T_2,\ \kappa(T_3) = \frac{S_2^*}{\mu^2(1+\mu^2)},\ \kappa(S_2) = \mu^2(1+\mu^2)T_3^*,$$

$$\kappa(S_3) = S_3^*,\ \kappa(S_4) = \mu^2 S_4,\ \kappa(T_3^*) = \frac{S_2}{1+\mu^2},$$

$$\kappa(S_2^*) = (1+\mu^2)T_3,\ \kappa(S_3^*) = S_3,\ \kappa(S_4^*) = \mu^{-2}S_4^*.$$

Now we derive some more equations by applying κ on the equations obtained by homomorphism condition. For the sake of simplicity, we give one example.

Lemma 4.4.15

$$\mu^4(1+\mu^2)^2 T_3T_3^* + c\mu^2(1+\mu^2)^2 S_3S_3^* + c\mu^2(1+\mu^2)^2 S_4^*S_4$$
$$= \mu^4(1-T_2)(1+\mu^2T_2) + cS_2S_2^* + cS_2^*S_2 + c\mu^2(1+\mu^2)^2.1, \quad (4.4.17)$$

$$S_3S_4 = -\frac{\mu^2 S_2^2}{(1+\mu^2)^2}, \quad (4.4.18)$$

$$-\mu^2(1+\mu^2)^2 S_4^*T_3^* - \mu^2(1+\mu^2)^2 T_3S_3^* = \mu^2(1+\mu^2)S_2^* - \mu^4 S_2^*(1-T_2) - \mu^4(1-T_2)S_2^*, \quad (4.4.19)$$

$$(1+\mu^2)^2 S_4^*T_3^* + \mu^2(1+\mu^2)^2 T_3S_3^* = -\mu^2 S_2^*T_2 - \mu^4 T_2S_2^*. \quad (4.4.20)$$

Proof The lemma is proved by applying the antipode on (4.4.10), (4.4.15), (4.4.12), and (4.4.14), respectively. □

4.4.4 Identification of $SO_\mu(3)$ as the Quantum Isometry Group

Now we want to prove that there is a $*$-homomorphism from $SO_\mu(3)$ to $\mathcal{S}$ which sends M, L, G, N, C to $-(1+\mu^2)^{-1}S_2,\ S_3,\ -\mu^{-1}S_4,\ (1+\mu^2)^{-1}(1-T_2),\ \mu T_3$, respectively.

To prove this, it is enough to show that all the relations of $SO_\mu(3)$ (as in (1.2.23)) when translated to relations of $\mathrm{QISO}_R^+(\mathcal{O}(S^2_{\mu,c}),\ \mathcal{H},\ D)$ via the above map are satisfied. We spell out some of the relations for which we also provide the proofs and remark that the verification of the remaining relations are quite similar and straightforward.

$G^*G = GG^* = N^2$ gives $-\frac{S_4^*}{\mu}(-\frac{S_4}{\mu}) = (-\frac{S_4}{\mu})(-\frac{S_4^*}{\mu}) = \frac{(1-T_2)^2}{(1+\mu^2)^2}$ implying

$$S_4^*S_4 = S_4S_4^* = \frac{\mu^2(1-T_2)^2}{(1+\mu^2)^2}.$$

$M^*M = N - N^2$ gives

$$\frac{S_2^*S_2}{(1+\mu^2)^2} = \frac{1-T_2}{1+\mu^2} - \frac{(1-T_2)^2}{(1+\mu^2)^2},$$

which means $\frac{S_2^*S_2}{(1+\mu^2)^2} = \frac{(1+\mu^2)(1-T_2)-(1-T_2)^2}{(1+\mu^2)^2}$ implying

$$S_2^*S_2 = (1-T_2)(\mu^2 + T_2).$$

$M^*L = \mu^{-1}(I - N)C$ gives $-\frac{S_2^*}{1+\mu^2} S_3 = \mu^{-1}(1 - \frac{1-T_2}{1+\mu^2})\mu T_3$, that is,

$$-S_2^* S_3 = (\mu^2 + T_2)T_3.$$

Lemma 4.4.16 $S_2^* S_2 = (1 - T_2)(\mu^2 + T_2)$.

Proof Subtracting the equation obtained by multiplying $c(1 + \mu^2)$ with (4.4.5) from (4.4.4), we have

$$(1 + 2c(1 + \mu^2))S_2^* S_2 + c(1 + \mu^2)^2(1 - \mu^2)S_4^* S_4$$

$$= (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)(\mu^2 + 2T_2 - 1)T_2 + c(1 + \mu^2)^2(\mu^2 - 1)T_3 T_3^* + c(1 + \mu^2)^2.1. \tag{4.4.21}$$

Again, by adding (4.4.4) with $c(1 + \mu^2)^2$ times (4.4.6) gives

$$(1 + c(1 + \mu^2)^2)S_2^* S_2 + c(1 - \mu^4)(1 + \mu^2)^2 S_4^* S_4.$$

$$= (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)^2 T_2^2 + c(1 + \mu^2)^2(\mu^4 - 1)T_3 T_3^* + c(1 + \mu^2)^2.1. \tag{4.4.22}$$

Subtracting the equation obtained by multiplying $(\mu^2 + 1)$ with (4.4.21) from (4.4.22) we obtain

$$-(\mu^2 + c(1 + \mu^2)^2)S_2^* S_2 = (1 - T_2)(\mu^2 + T_2) - c(1 + \mu^2)^2 T_2^2$$

$$-(1 + \mu^2)(1 - T_2)(\mu^2 + T_2) - c\mu^2(1 + \mu^2)^2.1 + c(1 + \mu^2)^2(\mu^2 + 2T_2 - 1)T_2.$$

The right-hand side is equal to $-(\mu^2 + c(1 + \mu^2)^2)(1 - T_2)(\mu^2 + T_2)$. Thus, $S_2^* S_2 = (1 - T_2)(\mu^2 + T_2)$. □

Lemma 4.4.17

$$\mu^2(1 + \mu^2)^2 T_3^* T_3 = (1 - T_2)(\mu^2 + T_2), \tag{4.4.23}$$

$$(1 + \mu^2)^2 T_3 T_3^* = (1 - T_2)(1 + \mu^2 T_2), \tag{4.4.24}$$

$$S_2 S_2^* = \mu^2(1 - T_2)(1 + \mu^2 T_2). \tag{4.4.25}$$

Proof Equation (4.4.23) is obtained by applying the antipode on Lemma 4.4.16.

Since Z is a unitary matrix, the (2, 2) position of the matrix Z^*Z equals 1. Therefore, $\mu^2(1+\mu^2)T_3^*T_3 + T_2^2 + (1+\mu^2)T_3T_3^* = 1$.

Using (4.4.23) we deduce $-(1+\mu^2)^2T_3T_3^* = (T_2-1)(1+\mu^2T_2)$. Thus we obtain (4.4.24).

Applying κ on (4.4.24), we deduce (4.4.25). □

Lemma 4.4.18 $S_4^*S_4 = S_4S_4^* = (1+\mu^2)^{-2}\mu^2(1-T_2)^2$.

Proof Adding the equations obtained by applying the antipode on (4.4.5) and (4.4.6), we have $-\mu^4(1+\mu^2)^3(1-\mu^2)T_3^*T_3 + \mu^4(1+\mu^2)^2\ (1-\mu^2)S_4S_4^* = -\mu^2(1+\mu^2)(1-\mu^2)T_2(1-T_2) - \mu^2\ (1-\mu^2)S_2S_2^*$.

Using $\mu^2 \neq 1$, we obtain,

$$-\mu^4(1+\mu^2)^3T_3^*T_3 + \mu^4(1+\mu^2)^2S_4S_4^* = -\mu^2(1+\mu^2)T_2(1-T_2) - \mu^2S_2S_2^*.$$

Now using (4.4.23) and (4.4.25), we reduce the above equation to

$$\begin{aligned}
&\mu^4(1+\mu^2)^2S_4S_4^* \\
&= -\mu^2(1-T_2)(T_2+\mu^2T_2+\mu^2+\mu^4T_2) + \mu^2(1+\mu^2)(1-T_2)(\mu^2+T_2) \\
&= \mu^6(1-T_2)^2.
\end{aligned}$$

Thus,

$$S_4S_4^* = \frac{\mu^6}{\mu^4(1+\mu^2)^2}(1-T_2)^2 = \frac{\mu^2}{(1+\mu^2)^2}(1-T_2)^2.$$

Applying κ, we have $S_4^*S_4 = \frac{\mu^2}{(1+\mu^2)^2}(1-T_2)^2$.

Thus, $S_4^*S_4 = S_4S_4^* = \frac{\mu^2}{(1+\mu^2)^2}(1-T_2)^2$. □

Lemma 4.4.19 $-S_2^*S_3 = (\mu^2+T_2)T_3$.

Proof By applying adjoint and then multiplying by μ^2 on (4.4.7) we have $\mu^2S_2^*S_3 + \mu^2S_4^*S_2 = -\mu^2T_3(\mu^2+T_2) + \mu^2(1-T_2)T_3$. Subtracting this from (4.4.8) we have $(1-\mu^2)S_2^*S_3 = -T_2T_3 - \mu^2T_3T_2 + \mu^2T_3(\mu^2+T_2) - \mu^2(1-T_2)T_3$ which implies $-S_2^*S_3 = (\mu^2+T_2)T_3$ as $\mu^2 \neq 1$. □

We can verify the other relations among S_i's and T_i's to prove the following.

Theorem 4.4.20 *Assume* $\mu \neq 1$. *The map* $SO_\mu(3) \to \mathcal{S}$ *sending* M, L, G, N, C *to* $-(1+\mu^2)^{-1}S_2,\ S_3,\ -\mu^{-1}S_4,\ (1+\mu^2)^{-1}(1-T_2),\ \mu T_3$*, respectively, is a* $*$ *homomorphism.*

We now prove the main result of this subsection.

Theorem 4.4.21 *For* $\mu \neq 1$, $\mathrm{QISO}_R^+(\mathcal{O}(S^2_{\mu,c}),\ \mathcal{H},\ D) \cong SO_\mu(3)$.

Proof We have seen in Theorem 4.4.1 that $SU_\mu(2)$ is an object in $\mathbf{Q}'_{\mathbf{R}}(D)$ and $SO_\mu(3)$ is the corresponding maximal Woronowicz subalgebra for which the coaction is faithful. Thus, $SO_\mu(3)$ is a quantum subgroup of $\mathrm{QISO}_R^+(D)$. Now, Theorem 4.4.20 implies that $\mathrm{QISO}_R^+(D)$ is a quantum subgroup of $SO_\mu(3)$, thereby completing the proof. □

Remark 4.4.22 We observe that in the proof of Theorem 4.4.21, the only place where the structure of D was used was in Proposition 4.4.9 and there we used the fact that the unitary commutes with $|D|$. Thus, if we replace this spectral triple by $(\mathcal{O}(S^2_{\mu,c}), \mathcal{H}, |D|)$, everything remains same and we deduce that

$$\mathrm{QISO}_R^+(\mathcal{O}(S^2_{\mu,c}), \mathcal{H}, |D|) \cong \mathrm{QISO}_R^+(\mathcal{O}(S^2_{\mu,c}), \mathcal{H}, D) \cong SO_\mu(3).$$

Remark 4.4.23 With a bit of extra book keeping, the case $\mu = 1$ can be dealt similarly. Alternatively, it also follows from Subsection 5.3.5 of [6].

There are two ways to view the classical group $SO(3)$: as the group of (orientation preserving) isometries of S^2 and as the automorphism group of $M_2(\mathbb{C})$. In the paper [8], P.M. Soltan has proved that the universal object in the category of compact quantum group acting on $M_2(\mathbb{C})$ such that the coaction preserves a certain functional ω_μ defined in [8] is the quantum group $SO_\mu(3)$. The characterization of $SO_\mu(3)$ as the quantum isometry group of the quantum sphere $S^2_{\mu,c}$ can be seen as the noncommutative analogue of the characterization of $SO(3)$.

4.5 Another Spectral Triple on the Podles' Sphere: A Counterexample

In this section, we present a concrete example in support of the claim made in Chap. 3 that the homomorphism α_U may not be a C^*-coaction of the quantum isometry group of a spectral triple. The same example also illustrates the nonexistence of the universal object in the category (suggested at the end of Sect. 3.2.2) of CQG' s which have a C^*-coaction on the underlying C^* algebra. This is done by computing the quantum group of orientation preserving isometries for spectral triples on the Podles' sphere $S^2_{\mu,c}$. constructed by Chakraborty and Pal in [9] for $c > 0$. The quantum isometry group turns out to be $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$, where $\mathbf{Z}^\infty$ denotes countably infinite copies of the group of integers.

For the purpose of computation in this section, we are going to use the description of the Podles' spheres as the universal C^* algebra generated by elements A and B satisfying the relations:

$$A^* = A,\ AB = \mu^{-2}BA,$$

$$B^*B = A - A^2 + cI,\ BB^* = \mu^2 A - \mu^4 A^2 + cI.$$

In this section, we will work with $c > 0$.

4.5.1 The Spectral Triple

Let us describe the spectral triple on $S^2_{\mu,c}$ introduced and studied in [9].

Let $\mathcal{H}_+ = \mathcal{H}_- = l^2(I\!N \bigcup \{0\})$, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Let $\{e_n, n \geq 0\}$ be the canonical orthonormal basis of $\mathcal{H}_+ = \mathcal{H}_-$ and N be the operator defined on it by $N(e_n) = ne_n$.

We recall the irreducible representations π_+ and $\pi_- : \mathcal{H}_\pm \to \mathcal{H}_\pm$ as in [9].

$$\pi_\pm(A)e_n = \lambda_\pm \mu^{2n} e_n, \tag{4.5.1}$$

$$\pi_\pm(B)e_n = c_\pm(n)^{\frac{1}{2}} e_{n-1}, \tag{4.5.2}$$

where

$$e_{-1} = 0,\ \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{\frac{1}{2}},\ c_\pm(n) = \lambda_\pm \mu^{2n} - (\lambda_\pm \mu^{2n})^2 + c. \tag{4.5.3}$$

Let $\pi = \pi_+ \oplus \pi_-$ and $D = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$.

Then $(S^2_{\mu,c}, \pi, \mathcal{H}, D)$ is a spectral triple.

We observe that the eigenvalues of D are $\{n : n \in \mathbf{Z}\}$ and the eigenspace for the eigenvalue n is spanned by $\begin{pmatrix} e_n \\ e_n \end{pmatrix}$ when n is nonnegative and by $\begin{pmatrix} e_n \\ -e_n \end{pmatrix}$ when n is negative.

Lemma 4.5.1

$$\pi_+(B^*)(e_n) = c_+(n+1)^{\frac{1}{2}} e_{n+1},$$

$$\pi_-(B^*)(e_n) = c_-(n+1)^{\frac{1}{2}} e_{n+1}.$$

Proof $\left\langle \pi(B)(\sum_n c_n \begin{pmatrix} e_n \\ 0 \end{pmatrix}), \begin{pmatrix} e_{n'} \\ 0 \end{pmatrix} \right\rangle = \sum_n c_n c_+(n)^{\frac{1}{2}} \langle e_{n-1},\ e_{n'} \rangle = c_{n'+1}$
$c_+(n'+1)^{\frac{1}{2}} = \sum_n c_n c_+(n'+1)^{\frac{1}{2}} \langle e_n,\ e_{n'+1} \rangle = \sum_n c_n$
$\left\langle e_n,\ \overline{c_+(n'+1)^{\frac{1}{2}}} e_{n'+1} \right\rangle = \left\langle \sum_n c_n \begin{pmatrix} e_n \\ 0 \end{pmatrix},\ \begin{pmatrix} c_+(n'+1)^{\frac{1}{2}} e_{n'+1} \\ 0 \end{pmatrix} \right\rangle.$

Hence, $\pi_+(B^*)(e_n) = c_+(n+1)^{\frac{1}{2}} e_{n+1}$.
Similarly, $\pi_-(B^*)(e_n) = c_-(n+1)^{\frac{1}{2}} e_{n+1}$. □

Lemma 4.5.2 *If P_n, Q_n denote the projections onto the subspace generated by $\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_n \end{pmatrix}$, respectively, then P_n, Q_n belong to $\pi(S^2_{\mu,c})$.*

Proof We claim that $\forall\, n \neq 0,\ c_+(n)$ and $c_-(n)$ are distinct.

Let $c_+(n) = c_-(n)$. Therefore, $\lambda_+\mu^{2n} - (\lambda_+\mu^{2n})^2 + c = \lambda_-\mu^{2n} - (\lambda_-\mu^{2n})^2 + c$. This implies $(\lambda_+ + \lambda_-)\mu^{2n} = 1$. Thus, $\mu^{2n} = 1$ and so n has to be 0.

Now, $\forall\, n \geq 1,\ \pi(B^*B)\begin{pmatrix} e_n \\ 0 \end{pmatrix} = c_+(n)^{\frac{1}{2}}\pi(B^*)\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} = c_+(n)\begin{pmatrix} e_n \\ 0 \end{pmatrix}$.

Similarly, $\forall\, n \geq 1,\ \pi(B^*B)\begin{pmatrix} 0 \\ e_n \end{pmatrix} = c_-(n)\begin{pmatrix} 0 \\ e_n \end{pmatrix}$.

Hence, $\forall\, n \geq 1,\ c_+(n)$ and $c_-(n)$ are eigenvalues of B^*B with the corresponding eigenspaces spanned by $\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_n \end{pmatrix}$, respectively. It follows that the eigen-projections corresponding to these eigenvalues belong to $C^*(B^*B) \subseteq \pi(S^2_{\mu,c})$, and hence $P_n,\ Q_n$ belong to $\pi(S^2_{\mu,c})\ \forall\, n \geq 1$.

Moreover, $\pi(A)\begin{pmatrix} e_0 \\ 0 \end{pmatrix} = \lambda_+\begin{pmatrix} e_0 \\ 0 \end{pmatrix}$ and $\pi(A)\begin{pmatrix} 0 \\ e_0 \end{pmatrix} = \lambda_-\begin{pmatrix} 0 \\ e_0 \end{pmatrix}$.

Thus, by the same arguments as above, $P_0,\ Q_0$ belong to $\pi(S^2_{\mu,c})$. □

Lemma 4.5.3 $\pi(S^2_{\mu,c})'' = \{\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) : X_{12} = X_{21} = 0\}$.

Proof It is sufficient to prove that the commutant $\pi(S^2_{\mu,c})'$ is the von Neumann algebra of operators of the form $\begin{pmatrix} c_1 I & 0 \\ 0 & c_2 I \end{pmatrix}$ for some c_1, c_2 in $\mathbb{C}$. To this end, we use that π_+ and π_- are irreducible representations.

Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \pi(S^2_{\mu,c})'$. Since X commutes with $\pi(A),\ \pi(B),\ \pi(B^*)$, we have $X_{11} \in \pi_+(S^2_{\mu,c})' \cong \mathbb{C}$ and $X_{22} \in \pi_-(S^2_{\mu,c})' \cong \mathbb{C}$, so that $X_{11} = c_1 I,\ X_{22} = c_2 I$ for some c_1, c_2.

Moreover,

$$X_{12}\pi_-(A) = \pi_+(A)X_{12}, \tag{4.5.4}$$

$$X_{12}\pi_-(B) = \pi_+(B)X_{12}, \tag{4.5.5}$$

$$X_{12}\pi_-(B^*) = \pi_+(B^*)X_{12}. \tag{4.5.6}$$

Now, (4.5.5) implies that $X_{12}e_0 \in \mathrm{Ker}(\pi_+(B)) = \mathbb{C}e_0$. Let $X_{12}e_0 = p_0 e_0$.

We have $\pi_+(B)(X_{12}e_1) = c_-^{\frac{1}{2}}(1)X_{12}e_0 = p_0 c_-^{\frac{1}{2}}(1)e_0$, that is, $\pi_+(B)(X_{12}e_1) \in \mathbb{C}e_0$.

Since it follows from the definition of $\pi_+(B)$ that $\pi_+(B)$ maps $\overline{\text{Span}}\ \{e_i : i \geq 2\}$ to $(\mathbb{C}e_0)^\perp = \overline{\text{Span}}\{e_i : i \geq 1\}$, $X_{12}e_1$ must belong to $\text{Span}\{\text{e}_0,\ \text{e}_1\}$.

By an inductive argument, we see that $\forall\, n,\ X_{12}(e_n) \in \text{Span}\{e_0, e_1, \ldots, e_n\}$.

Using the definition of $\pi_\pm(B^*)e_n$ along with (4.5.6), we have $c_-^{\frac{1}{2}}(1)X_{12}e_1 = p_0 c_+^{\frac{1}{2}}(1)e_1$, that is, $X_{12}e_1 = p_0 \frac{c_+^{\frac{1}{2}}(1)}{c_-^{\frac{1}{2}}(1)} e_1$.

We argue in a similar way by induction that $X_{12}e_n = c'_n e_n$ for some constants c'_n.

Now we apply (4.5.6) and (4.5.5) on the vectors e_n and e_{n+1} to get $c'_{n+1} = \frac{c'_n c_+^{\frac{1}{2}}(n+1)}{c_-^{\frac{1}{2}}(n+1)}$ and $c'_{n+1} = \frac{c'_n c_-^{\frac{1}{2}}(n+1)}{c_+^{\frac{1}{2}}(n+1)}$. Since $c_+(n+1) \neq c_-(n+1)$ for $n \geq 0$, we have $c'_n = 0$. Thus, $c'_n = 0\ \forall\, n$ so that $X_{12} = 0$. It follows similarly that $X_{21} = 0$. $\square$

4.5.2 Computation of the Quantum Isometry Group

Let $(\widetilde{\mathcal{S}}, \Delta, U)$ be an object in $\mathbf{Q}'(D)$, with $\alpha = \text{ad}_U$ and the corresponding Woronowicz C^* subalgebra of $\widetilde{\mathcal{S}}$ generated by $\{< (\xi \otimes 1), \alpha(x)(\eta \otimes 1) >_{\widetilde{\mathcal{S}}}\colon\ \xi, \eta \in \mathcal{H},\ x \in S^2_{\mu,c}\}$ is denoted by $\mathcal{S}$. Assume, without loss of generality, that the corepresentation U is faithful.

As U commutes with D, it preserves the one-dimensional subspaces spanned by each of the eigenvectors $\begin{pmatrix} e_n \\ e_n \end{pmatrix}$ and $\begin{pmatrix} e_n \\ -e_n \end{pmatrix}$.

Let $U \begin{pmatrix} e_n \\ e_n \end{pmatrix} = \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^+$,

$U \begin{pmatrix} e_n \\ -e_n \end{pmatrix} = \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes q_n^-$,

for some q_n^+, q_n^- in $\widetilde{\mathcal{S}}$.

Lemma 4.5.4 *We have*

$$q_n^+ q_n^{-*} = q_n^- q_n^{+*}\ \forall n, \tag{4.5.7}$$

$$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} - q_{n-1}^- q_n^{-*}) + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{-*} - q_{n-1}^- q_n^{+*}) = 0$$
$$\forall n \geq 1, \tag{4.5.8}$$

$$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} - q_{n-1}^- q_n^{-*}) + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^- q_n^{+*} - q_{n-1}^+ q_n^{-*}) = 0$$
$$\forall n \geq 1, \tag{4.5.9}$$

$$(c_+(n+1)^{\frac{1}{2}} + c_-(n+1)^{\frac{1}{2}})(q^+_{n+1}q^{+*}_n - q^-_{n+1}q^{-*}_n) = (c_+(n+1)^{\frac{1}{2}} - c_-(n+1)^{\frac{1}{2}}) \times (q^-_{n+1}q^{+*}_n - q^+_{n+1}q^{-*}_n)\ \forall n. \tag{4.5.10}$$

Proof Since α maps $\pi(S^2_{\mu,c})$ into its double commutant, the description of $\pi(S^2_{\mu,c})''$ as in Lemma 4.5.3 implies that the coefficient of $\begin{pmatrix} 0 \\ e_n \end{pmatrix}$ in $\alpha(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ must be 0.

To compute this coefficient, we start computing $\alpha(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ as follows:

$$\begin{aligned}
&\alpha(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix} \\
&= \widetilde{U}(A \otimes 1)\widetilde{U}^*(\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes 1) \\
&= \tfrac{1}{2}\widetilde{U}(\pi(A) \otimes 1)\widetilde{U}^*(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes 1 + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes 1) \\
&= \tfrac{1}{2}\widetilde{U}(\pi(A) \otimes 1)[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q^{+*}_n + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes q^{-*}_n] \\
&= \tfrac{1}{2}\widetilde{U}[\begin{pmatrix} \pi_+(A)e_n \\ \pi_-(A)e_n \end{pmatrix} \otimes q^{+*}_n + \begin{pmatrix} \pi_+(A)e_n \\ -\pi_-(A)e_n \end{pmatrix} \otimes q^{-*}_n].
\end{aligned}$$

Using (4.5.1), we see that it is equal to

$$\begin{aligned}
&= \tfrac{1}{2}\widetilde{U}[\begin{pmatrix} \lambda_+\mu^{2n}e_n \\ \lambda_-\mu^{2n}e_n \end{pmatrix} \otimes q^{+*}_n + \begin{pmatrix} \lambda_+\mu^{2n}e_n \\ -\lambda_-\mu^{2n}e_n \end{pmatrix} \otimes q^{-*}_n] \\
&= \tfrac{1}{2}\widetilde{U}[\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \lambda_+\mu^{2n}(q^{+*}_n + q^{-*}_n) + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \lambda_-\mu^{2n}(q^{+*}_n - q^{-*}_n)] \\
&= \tfrac{1}{4}\widetilde{U}[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \{\lambda_+\mu^{2n}(q^{+*}_n + q^{-*}_n) + \lambda_-\mu^{2n}(q^{+*}_n - q^{-*}_n)\} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes \\
&\{\lambda_+\mu^{2n}\ (q^{+*}_n + q^{-*}_n) - \lambda_-\mu^{2n}(q^{+*}_n - q^{-*}_n)\}] \\
&= \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \tfrac{1}{4}q^+_n\{\lambda_+\mu^{2n}(q^{+*}_n + q^{-*}_n) + \lambda_-\mu^{2n}(q^{+*}_n - q^{-*}_n)\} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes \\
&\tfrac{1}{4}q^-_n\{\lambda_+\ \mu^{2n}(q^{+*}_n + q^{-*}_n) - \lambda_-\mu^{2n}(q^{+*}_n - q^{-*}_n)\} \\
&= \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \tfrac{1}{4}\{\lambda_+\mu^{2n}(1 + q^+_nq^{-*}_n) + \lambda_-\mu^{2n}(1 - q^+_nq^{-*}_n) + \lambda_+\mu^{2n}(1 + q^-_nq^{+*}_n) \\
&- \lambda_-\mu^{2n}(q^-_nq^{+*}_n - 1)\} + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes \tfrac{1}{4}\{\lambda_+\mu^{2n}(1 + q^+_nq^{-*}_n) + \lambda_-\mu^{2n}(1 - q^+_nq^{-*}_n) \\
&- \lambda_+\mu^{2n}\ (1 + q^-_nq^{+*}_n) + \lambda_-\mu^{2n}(q^-_nq^{+*}_n - 1)\}.
\end{aligned}$$

Thus, we conclude

$\lambda_+[1 + q^+_nq^{-*}_n - (1 + q^-_nq^{+*}_n)] + \lambda_-[1 - q^+_nq^{-*}_n + q^-_nq^{+*}_n - 1] = 0$ and hence $(\lambda_+ - \lambda_-)(q^+_nq^{-*}_n - q^-_nq^{+*}_n) = 0$. Therefore, $(q^+_nq^{-*}_n - q^-_nq^{+*}_n) = 0$.

Proceeding in a similar way, (4.5.8), (4.5.9), (4.5.10) follow by equating the coefficients of $\begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix}$, $\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_{n+1} \end{pmatrix}$ in $\alpha(B)\begin{pmatrix} e_n \\ 0 \end{pmatrix}$, $\alpha(B)\begin{pmatrix} 0 \\ e_n \end{pmatrix}$, and $\alpha(B^*)\begin{pmatrix} e_n \\ 0 \end{pmatrix}$ (respectively). □

We are now in a position to write formulas for $\alpha(A)$ and $\alpha(B)$ in terms of the projections P_n and Q_ns.

Corollary 4.5.5 *We have*

$$\alpha(A) = \sum_{n=0}^{\infty} AP_n \otimes \frac{1}{2\lambda_+}\{\lambda_+(1 + q_n^+ q_n^{-*}) + \lambda_-(1 - q_n^+ q_n^{-*})\}$$

$$+ \sum_{n=0}^{\infty} AQ_n \otimes \frac{1}{2\lambda_-}\{\lambda_+(1 - q_n^+ q_n^{-*}) + \lambda_-(1 + q_n^+ q_n^{-*})\}.$$

$$\alpha(B) = \sum_{n=1}^{\infty} BP_n \otimes \frac{1}{4c_+(n)}[(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*})$$

$$+ (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*})] + \sum_{n=1}^{\infty} BQ_n \otimes \frac{1}{4c_-(n)}[(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}}).$$

$$(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{-*} + q_{n-1}^- q_n^{+*})].$$

Proof We note that $\pi(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_+(A) & 0 \\ 0 & \pi_-(A) \end{pmatrix}\begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_+(A)e_n \\ 0 \end{pmatrix} = \lambda_+\mu^{2n}\begin{pmatrix} e_n \\ 0 \end{pmatrix}$.

Thus, $\begin{pmatrix} e_n \\ 0 \end{pmatrix} = \frac{\pi(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix}}{\lambda_+\mu^{2n}}$.

Similarly, $\begin{pmatrix} 0 \\ e_n \end{pmatrix} = \frac{\pi(A)\begin{pmatrix} 0 \\ e_n \end{pmatrix}}{\lambda_-\mu^{2n}}$.

Now, using (4.5.7),

$$\alpha(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix} = \begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{2}\{\lambda_+\mu^{2n}(1 + q_n^+ q_n^{-*}) + \lambda_-\mu^{2n}(1 - q_n^+ q_n^{-*})\}$$

$$= \pi(A)\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes \frac{1}{2\lambda_+}\{\lambda_+(1 + q_n^+ q_n^{-*}) + \lambda_-(1 - q_n^+ q_n^{-*})\}.$$

Similarly,

$$\alpha(A)\begin{pmatrix}0\\e_n\end{pmatrix} = \pi(A)\begin{pmatrix}0\\e_n\end{pmatrix} \otimes \frac{1}{2\lambda_-}\{\lambda_+(1-q_n^+q_n^{-*}) + \lambda_-(1+q_n^+q_n^{-*})\}.$$

Thus, $\alpha(A) = \sum_{n=0}^{\infty} AP_n \otimes \frac{1}{2\lambda_+}\{\lambda_+(1+q_n^+q_n^{-*}) + \lambda_-(1-q_n^+q_n^{-*})\} + + \sum_{n=0}^{\infty} AQ_n \otimes \frac{1}{2\lambda_-}\{\lambda_+(1-q_n^+q_n^{-*}) + \lambda_-(1+q_n^+q_n^{-*})\}$.

Considering $\alpha(B)\begin{pmatrix}e_n\\0\end{pmatrix}$ and $\alpha(B)\begin{pmatrix}0\\e_n\end{pmatrix}$, the expression for $\alpha(B)$ is obtained. □

Lemma 4.5.6 *Let $\widetilde{P}_n = P_n + Q_n$. Then for each vector v in $\mathcal{H}$, $\alpha(\widetilde{P}_n)v = \widetilde{P}_n v \otimes 1$.*

Proof To start with, we recall that P_n and Q_n belong to $\pi(S^2_{\mu,c})$ (Lemma 4.5.2). Hence, $\widetilde{P}_n \in \pi(S^2_{\mu,c})$.

We compute

$\alpha(\widetilde{P}_n)\begin{pmatrix}e_n\\e_n\end{pmatrix} = \widetilde{U}(\widetilde{P}_n \otimes 1)\widetilde{U}^*\begin{pmatrix}e_n\\e_n\end{pmatrix} = \widetilde{U}(\begin{pmatrix}e_n\\e_n\end{pmatrix} \otimes q_n^{+*})$

$= \begin{pmatrix}e_n\\e_n\end{pmatrix} \otimes 1 = (\widetilde{P}_n \otimes 1)(\begin{pmatrix}e_n\\e_n\end{pmatrix} \otimes 1)$.

Next, for $k \neq n$,

$\widetilde{U}(\widetilde{P}_k \otimes 1)\widetilde{U}^*\begin{pmatrix}e_n\\e_n\end{pmatrix} = \widetilde{U}(\widetilde{P}_k \otimes 1)(\begin{pmatrix}e_n\\e_n\end{pmatrix} \otimes q_n^{+*}) = 0 = (\widetilde{P}_k \otimes 1)(\begin{pmatrix}e_n\\e_n\end{pmatrix} \otimes 1)$.

Similarly, $\alpha(\widetilde{P}_n)\begin{pmatrix}e_n\\-e_n\end{pmatrix} = (\widetilde{P}_n \otimes 1)(\begin{pmatrix}e_n\\-e_n\end{pmatrix} \otimes 1)$,

and for $k \neq n$, $\alpha(\widetilde{P}_k)\begin{pmatrix}e_n\\-e_n\end{pmatrix} = 0 = (\widetilde{P}_k \otimes 1)(\begin{pmatrix}e_n\\-e_n\end{pmatrix} \otimes 1)$.

Combining all these, we get the required result. □

Theorem 4.5.7 $\widetilde{\mathcal{S}} = C^*\{q_n^+, y_0 = q_0^{-*}q_0^+ : \quad n \geq 0\}$ *Moreover,* $\mathcal{S} = C^*\{z_n = q_{n-1}^+q_n^{+*}, w' : n \geq 1\}$, *where w' is a self-adjoint unitary.*

Proof Replacing $n+1$ by n in (4.5.10) we have

$$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_n^+q_{n-1}^{+*} - q_n^-q_{n-1}^{-*}) - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_n^-q_{n-1}^{+*} - q_n^+q_{n-1}^{-*}) = 0 \quad \forall n \geq 1. \tag{4.5.11}$$

Subtracting (4.5.11) from the equation obtained by applying $*$ on (4.5.8), we have $2(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})(q_n^-q_{n-1}^{+*} - q_n^+q_{n-1}^{-*}) = 0 \ \forall \ n \geq 1$. Now, from the proof of Lemma 4.5.2, $(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}}) \neq 0 \ \forall \ n \geq 1$. This implies

$$q_n^-q_{n-1}^{+*} = q_n^+q_{n-1}^{-*} \ \text{ for all } n \geq 1. \tag{4.5.12}$$

Using (4.5.12) in (4.5.11), we have

$$q_n^+ q_{n-1}^{+*} = q_n^- q_{n-1}^{-*} \text{ for all } n \geq 1. \tag{4.5.13}$$

Let $y_n = q_n^{-*} q_n^+$. Clearly, y_n is a unitary because it is a product of unitaries. Moreover, using (4.5.7), we have $q_n^{-*} q_n^+ = q_n^{+*} q_n^-$, that is, $y_n = y_n^*$. Now, from (4.5.12), we have $q_n^- = q_n^+ y_{n-1}^* \ \forall\, n \geq 1$, so that

$$q_n^- = q_n^+ y_{n-1} \text{ for all } n \geq 1. \tag{4.5.14}$$

Next, from (4.5.13), we obtain $q_n^{-*} q_n^+ = q_{n-1}^{-*} q_{n-1}^+ \ \forall\, n \geq 1$, implying

$$y_n = y_{n-1} \text{ for all } n \geq 1. \tag{4.5.15}$$

By Eqs. (4.5.14), (4.5.15) and the faithfulness of the corepresentation U, we conclude that $\widetilde{\mathcal{S}}$ is generated by $\{q_n^+\}_{n \geq 0}$ and y_0.

Now we prove the second part of the theorem.

Using Lemma 4.5.6, we observe that for all v in $\mathcal{H}$, $\alpha(A\widetilde{P}_k)v = \alpha(A)(\widetilde{P}_k v \otimes 1) = AP_k v \otimes \frac{1}{2\lambda_+}\{\lambda_+(1 + q_k^+ q_k^{-*}) + \lambda_-(1 - q_k^+ q_k^{-*})\} + AQ_k v \otimes \frac{1}{2\lambda_-}\{\lambda_+(1 - q_k^+ q_k^{-*}) + \lambda_-(1 + q_k^+ q_k^{-*})\}$. Therefore, $\alpha(A\widetilde{P}_k) = AP_k \otimes \frac{1}{2\lambda_+}\{\lambda_+(1 + q_k^+ q_k^{-*}) + \lambda_-(1 - q_k^+ q_k^{-*})\} + AQ_k \otimes \frac{1}{2\lambda_-}\{\lambda_+(1 - q_k^+ q_k^{-*}) + \lambda_-(1 + q_k^+ q_k^{-*})\}$.

Now, AP_k and AQ_k being distinct elements, there exist linear functional ϕ such that $\phi(AP_k) = 1,\ \phi(AQ_k) = 0$. Hence, $(\phi \otimes \mathrm{id})\alpha(A\widetilde{P}_k) = \lambda_+(1 + q_m^+ q_m^{-*}) + \lambda_-(1 - q_m^+ q_m^{-*}) \in \mathcal{S}$. Similarly, $\lambda_+(1 - q_m^+ q_m^{-*}) + \lambda_-(1 + q_m^+ q_m^{-*}) \in \mathcal{S}\ \forall\, m$.

Subtracting, we deduce that $q_m^+ q_m^{-*} \in \mathcal{S}$.

Using the formula of $\alpha(B)$ in a similar way, we prove that

$(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) + (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}}) \quad (q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*}) \in \mathcal{S}\ \forall\ n \geq 1$ and $(c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}})(q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}) - (c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})\ (q_{n-1}^+ q_n^{-*} + q_{n-1}^- q_n^{+*}) \in \mathcal{S}\ \forall\, n \geq 1$.

Adding and subtracting, we have

$$q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*} \in \mathcal{S}\ \forall n \geq 1, \tag{4.5.16}$$

$$q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*} \in \mathcal{S}\ \forall n \geq 1. \tag{4.5.17}$$

Recalling (4.5.15), we have $q_n^- = q_n^+ y_{n-1} = q_n^+ y_0$. Using this in (4.5.16), we obtain $q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*} = q_{n-1}^+ q_n^{+*} + q_{n-1}^+ y_0 y_0^* q_n^{+*} = q_{n-1}^+ q_n^{+*} + q_{n-1}^+ q_n^{+*} = 2q_{n-1}^+ q_n^{+*}$.

Similarly, using $q_n^- = q_n^+ y_0$ in (4.5.17), one has $q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*} = q_{n-1}^+ y_0 q_n^{+*} + q_{n-1}^+ y_0 q_n^{+*} = 2q_{n-1}^+ y_0 q_n^{+*}$.

This implies that $q_{n-1}^+ q_n^{+*}$ and $q_{n-1}^+ y_0 q_n^{+*}$ are in $\mathcal{S}\ \forall\, n \geq 1$.

Let $\forall\, n \geq 1,\ z_n = q_{n-1}^+ q_n^{+*},\ w_n = q_{n-1}^+ y_0 q_n^{+*}$, and we observe that

$$z_n^* w_n = q_n^+ y_0 q_n^{+*} = q_n^+ q_n^{-*}.$$

Moreover,

$$q_0^+ q_0^{-*} = q_0^+ (q_0^+ y_0^*)^* = q_0^+ y_0 q_0^{+*} = q_0^+ y_0 q_1^{+*} q_1^+ q_0^{+*} = w_1 z_1^*.$$

Hence $\forall\, n \geq 0,\ q_n^+ q_n^{-*} \in C^*(\{z_n, w_n\}_{n\geq 1})$.

For all $n \geq 2,\ q_{n-1}^- q_n^{-*} = q_{n-1}^+ y_{n-2}^*\ (q_n^+ y_{n-1}^*)^* = q_{n-1}^+\ y_0^* y_0 q_n^{+*} = q_{n-1}^+\ q_n^{+*} = q_{n-1}^+ q_n^{+*} = z_n$,

$$q_0^- q_1^{-*} = q_0^+ y_0 (q_1^+ y_0)^* = q_0^+ y_0 y_0^* q_1^{+*} = q_0^+ q_1^{+*} = z_1.$$

Finally,

$$q_{n-1}^- q_n^{+*} = q_{n-1}^+ y_0^* q_n^{+*} = w_n$$

and

$$q_0^- q_1^{+*} = q_0^+ y_0 q_1^{+*} = w_1.$$

From the formulas of $\alpha(A)$ and $\alpha(B)$, it follows that $\mathcal{S}$ is generated by $q_n^+ q_n^{-*}$, $q_{n-1}^+ q_n^{+*} + q_{n-1}^- q_n^{-*}$, $q_{n-1}^- q_n^{+*} + q_{n-1}^+ q_n^{-*}$. By the above observations, these belong to $C^*(\{z_n, w_n\}_{n\geq 1})$ which implies that $\mathcal{S}$ is a C^* subalgebra of C^* $(\{z_n, w_n\}_{n\geq 1})$. Moreover, from the definitions of $z_n,\ w_n$ it is clear that $C^*(\{z_n, w_n\}_{n\geq 1})$ is a C^* subalgebra of $\mathcal{S}$.

Therefore, $\mathcal{S} \cong C^*(\{z_n, w_n\}_{n\geq 1})$, by the assumed faithfulness.

In fact, there is a more simple description which can be derived by observing that $z_n w_{n+1} = q_{n-1}^+ q_n^{+*} q_n^+ y_0 q_{n+1}^{+*} = q_{n-1}^+ y_0 q_{n+1}^{+*} = q_{n-1}^+ y_0 q_n^{+*} q_n^+ q_{n+1}^{+*} = w_n z_{n+1}$ and so, $w_{n+1} = z_n{}^* w_n z_{n+1}$ so that $\{w_n\}_{n\geq 1}$ is a subset of $C^*(\{z_n\}_{n\geq 1},\ w_1)$.

Let us define $w' = w_1^* z_1$. Then we note that $z_1 = q_0^+ y_0 q_1^{+*} q_1^+ y_0^* q_1^{+*} = w_1 (z_1^* w_1)$, hence $w_1^* z_1 = z_1^* w_1$. Thus, w' is self-adjoint. It is a unitary as it is a product of unitaries.

Thus $\mathcal{S} \cong C^*\{\{z_n\}_{n\geq 1},\ w'\}$. □

Lemma 4.5.8 $\Delta(q_n^\pm) = q_n^\pm \otimes q_n^\pm,\ \Delta(y_1) = y_1 \otimes y_1$.

Proof We use the fact that U is a unitary corepresentation.

$(\mathrm{id} \otimes \Delta) U \begin{pmatrix} e_n \\ e_n \end{pmatrix} = (\mathrm{id} \otimes \Delta)(\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^+) = (\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes \Delta(q_n^+)$.

$U_{(12)} U_{(13)} \begin{pmatrix} e_n \\ e_n \end{pmatrix} = \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes q_n^+ \otimes q_n^+$.

Hence, $\Delta(q_n^+) = q_n^+ \otimes q_n^+$.

Similarly, $\Delta(q_n^-) = q_n^- \otimes q_n^-$.

Moreover, $\Delta(y_1) = \Delta(q_n^- q_n^{+*}) = (q_n^- \otimes q_n^-)(q_n^{+*} \otimes q_n^{+*}) = q_n^- q_n^{+*} \otimes q_n^- q_n^{+*} = y_1 \otimes y_1$. □

At this point, we consider the quantum group $\widetilde{\mathcal{S}} \cong C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$, where $\mathbf{Z}^\infty = \mathbf{Z} * \mathbf{Z} * \cdots$ denotes the free product of countably infinitely many copies of $\mathbf{Z}$. By Remarks 1.1.7, 1.1.4, and 1.1.3, $\widetilde{\mathcal{S}} \cong C(\mathbf{Z}_2) * C(\mathbb{T}) * C(\mathbb{T}) * \cdots$.

Let r_n^+ denotes the generator of the n-th copy of $C(\mathbb{T})$ and by y the generator of $C(\mathbf{Z}_2)$.

The coproduct Δ_0 on $\widetilde{\mathcal{S}}$ is given by $\Delta_0(r_n^+) = r_n^+ \otimes r_n^+,\ \Delta_0(y) = y \otimes y$.

Define

$$V\begin{pmatrix} e_n \\ e_n \end{pmatrix} = \begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes r_n^+.$$

$$V\begin{pmatrix} e_n \\ -e_n \end{pmatrix} = \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes r_n^+ y.$$

Lemma 4.5.9 $(\widetilde{\mathcal{S}}, \Delta_0, V)$ *is an object in* $\mathbf{Q}'(D)$.

Proof As the eigenspaces corresponding to distinct eigenvalues of D are spanned by $\begin{pmatrix} e_n \\ e_n \end{pmatrix}$ and $\begin{pmatrix} e_n \\ -e_n \end{pmatrix}$, V commutes with D.

The co-associativity of V follows from the proof of Lemma 4.5.8. As r_n^+ and y are unitaries, it is easy to see that V is a unitary corepresentation.

Finally, putting $q_n^+ = r_n^+$ and $q_n^- = r_n^+ y$, it can be checked that the conditions in Lemma 4.5.4 are satisfied, implying that $\mathrm{ad}_{\widetilde{V}}$ leaves $\pi(S^2_{\mu,c})''$ invariant. □

By defining $r_n^- = r_n^+ y$, we note that r_n^- is a unitary satisfying the following equations:

$$r_n^- r_{n-1}^{+*} = r_n^+ r_{n-1}^{-*} \quad \text{for all } n \geq 1. \tag{4.5.18}$$

$$r_n^+ r_{n-1}^{+*} = r_n^- r_{n-1}^{-*} \quad \text{for all } n \geq 1. \tag{4.5.19}$$

Using $r_n^- = r_n^+ r_{n-1}^{-*} r_{n-1}^+$ (from 4.5.18) in (4.5.19) we have $r_{n-1}^+ = r_{n-1}^- r_{n-1}^{+*} r_{n-1}^-$.

This implies

$$r_n^+ r_n^{-*} = r_n^- r_n^{+*} \quad \text{for all } n. \tag{4.5.20}$$

Moreover, taking adjoint on (4.5.18) and (4.5.19), respectively, we get

$$r_{n-1}^+ r_n^{-*} - r_{n-1}^- r_n^{+*} = 0 \quad \text{for all } n \geq 1. \tag{4.5.21}$$

$$r_{n-1}^+ r_n^{+*} - r_{n-1}^- r_n^{-*} = 0 \quad \text{for all } n \geq 1. \tag{4.5.22}$$

Thus, Eqs. (4.5.7)–(4.5.10) in Lemma 4.5.4 are satisfied with $q_n^\pm$'s replaced by $r_n^\pm$'s and therefore, it follows that there is a C^*-homomorphism from $\widetilde{\mathcal{S}}$ to $\widetilde{\mathcal{S}}$ sending y, r_n^+ to y_0 and q_n^+, respectively, which is surjective by Theorem 4.5.7 and is a CQG morphism by Lemma 4.5.8. In other words, $(\widetilde{\mathcal{S}}, \Delta_0, V)$ is indeed a universal object in $\mathbf{Q}'(D)$. Clearly, the maximal Woronowicz subalgebra of $\widetilde{\mathcal{S}}$ for which the coaction is faithful, that is $\mathrm{QISO}^+(D)$, is generated by $r^+_{n-1}r^{+^*}_n$, $n \geq 1$ and $r^+_0 y r^{+^*}_1$, so again isomorphic with $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$.

Thus we have the following theorem:

Theorem 4.5.10 *The universal object in the category $\mathbf{Q}'(D)$, that is $\widetilde{\mathrm{QISO}^+}(D)$, exists and is isomorphic with $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$. Moreover, $\mathrm{QISO}^+(D)$ is again isomorphic with $C^*(\mathbf{Z}_2 * \mathbf{Z}^\infty)$.*

Remark 4.5.11 For a classical compact Riemannian manifold, the isometry group is always a compact Lie group, hence has an embedding into the group of orthogonal matrices of some finite dimension. Theorem 4.5.10 shows that QISO^+ need not be a compact matrix quantum group, that is, it may not have a finite-dimensional fundamental unitary corepresentation.

We end this chapter by noting that α gives an example where the quantum group of orientation preserving isometries does not have a C^* coaction. Before that, we recall some useful properties of the so-called Toeplitz algebra from [10].

Proposition 4.5.12 *Let τ_1 be the unilateral shift operator on $l^2(\mathbb{N})$ defined by $\tau_1(e_n) = e_{n-1},\ n \geq 1,\ \tau(e_0) = 0$. Then the C^* algebra $C^*(\tau_1)$ generated by τ_1, called the Toeplitz algebra, contains all compact operators. Moreover, the commutator of any two elements of $C^*(\tau_1)$ is compact.*

Let τ be the operator on $\mathcal{H}$ defined by $\tau = \tau_1 \otimes \mathrm{id}$.

Lemma 4.5.13 $B = \tau|B|$.

Proof We note that

$$
\begin{aligned}
|B|\left(\begin{pmatrix} e_n \\ 0 \end{pmatrix}\right) &= (A - A^2 + cI)^{\frac{1}{2}}\left(\begin{pmatrix} e_n \\ 0 \end{pmatrix}\right) \\
&= \sqrt{\lambda_+\mu^{2n} - \lambda_+^2\mu^{4n} + c}\left(\begin{pmatrix} e_n \\ 0 \end{pmatrix}\right) = c_+(n)^{\frac{1}{2}}\begin{pmatrix} e_n \\ 0 \end{pmatrix}
\end{aligned}
$$

and hence $\tau|B|\left(\begin{pmatrix} e_n \\ 0 \end{pmatrix}\right) = c_+(n)^{\frac{1}{2}}\left(\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix}\right) = B\left(\begin{pmatrix} e_n \\ 0 \end{pmatrix}\right)$.

Similarly, $\tau|B|\begin{pmatrix} 0 \\ e_n \end{pmatrix} = B\begin{pmatrix} 0 \\ e_n \end{pmatrix}$. This proves the lemma. □

Lemma 4.5.14

$$\alpha(\tau) = \sum_{n \geq 1} \tau(P_n + Q_n) \otimes r^+_{n-1} r^{+^*}_n,$$

where $r^\pm_n$ are the elements of $\widetilde{\mathrm{QISO}^+}(D)$ introduced before.

Proof We have $\forall\, n \geq 1$,

$$\begin{aligned}
\alpha(\tau)\begin{pmatrix} e_n \\ 0 \end{pmatrix} &= \widetilde{U}(\tau \otimes \mathrm{id})\widetilde{U}^*\begin{pmatrix} e_n \\ 0 \end{pmatrix}\\
&= \frac{1}{2}\widetilde{U}(\tau \otimes \mathrm{id})\widetilde{U}^*[\begin{pmatrix} e_n \\ e_n \end{pmatrix} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix}]\\
&= \frac{1}{2}\widetilde{U}(\tau \otimes \mathrm{id})[\begin{pmatrix} e_n \\ e_n \end{pmatrix} \otimes r_n^{+*} + \begin{pmatrix} e_n \\ -e_n \end{pmatrix} \otimes r_n^{-*}]\\
&= \frac{1}{2}\widetilde{U}(\tau \otimes \mathrm{id})[\begin{pmatrix} e_n \\ 0 \end{pmatrix} \otimes (r_n^{+*} + r_n^{-*}) + \begin{pmatrix} 0 \\ e_n \end{pmatrix} \otimes (r_n^{+*} - r_n^{-*})]\\
&= \frac{1}{2}\widetilde{U}[\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes (r_n^{+*} + r_n^{-*}) + \begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix} \otimes (r_n^{+*} - r_n^{-*})]\\
&= \frac{1}{4}\widetilde{U}[\begin{pmatrix} e_{n-1} \\ e_{n-1} \end{pmatrix} \otimes (2r_n^{+*}) + \begin{pmatrix} e_{n-1} \\ -e_{n-1} \end{pmatrix} \otimes 2r_n^{-*}]\\
&= \frac{1}{2}[\begin{pmatrix} e_{n-1} \\ e_{n-1} \end{pmatrix} \otimes r_{n-1}^{+} r_n^{+*} + \begin{pmatrix} e_{n-1} \\ -e_{n-1} \end{pmatrix} \otimes r_{n-1}^{-} r_n^{-*}] = \begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes r_{n-1}^{+} r_n^{+*}.
\end{aligned}$$

Similarly, $\alpha(\tau)\begin{pmatrix} 0 \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ e_{n-1} \end{pmatrix} \otimes r_{n-1}^{+} r_n^{+*}\ \forall\, n \geq 1$.

Moreover, $\alpha(\tau)\begin{pmatrix} e_0 \\ 0 \end{pmatrix} = \alpha(\tau)\begin{pmatrix} 0 \\ e_0 \end{pmatrix} = 0$.

Thus, $\alpha(\tau) = \sum_{n\geq 1} \tau P_n \otimes r_{n-1}^{+} r_n^{+*} + \sum_{n \geq 1} \tau Q_n \otimes r_{n-1}^{+} r_n^{+*} = \sum_{n\geq 1} \tau(P_n + Q_n) \otimes r_{n-1}^{+} r_n^{+*}$ □

Theorem 4.5.15 *The $*$-homomorphism α is not a C^* coaction.*

Proof We begin with the observation that each of the C^* algebras $\pi_{\pm}(S^2_{\mu,c})$ is nothing but the Toeplitz algebra. For example, consider $\mathcal{C} := \pi_+(S^2_{\mu,c})$. Clearly, $T = \pi_+(B)$ in an invertible operator with the polar decomposition given by $T = \tau_1|T|$, hence $\tau_1 \in \mathcal{C}$. Thus, $\mathcal{C}$ contains the Toeplitz algebra $C^*(\tau_1)$. However, by Proposition 4.5.12, $C^*(\tau_1)$ contains all compact operators. In particular, $C^*(\tau_1)$ must contain $\pi_+(A)$ as well as all the eigenprojections P_n of $|\pi_+(B)|$, so it must contain the whole of $\mathcal{C}$. Hence $\mathcal{C} = C^*(\tau_1)$. Similar arguments will work for $\pi_-(S^2_{\mu,c})$.

Thus, $\tau = \tau_1 \oplus \tau_1 = \pi(B)|\pi(B)|^{-1} \in \pi(S^2_{\mu,c})$. If α is a C^* coaction, then for any state ϕ on $\mathrm{QISO}^+(D)$ we have $\alpha_\phi(\tau) \equiv (\mathrm{id} \otimes \phi) \circ \alpha(\tau) \in \pi(S^2_{\mu,c})$, hence $\alpha_\phi(\tau)P_+$ must belong to $\mathcal{C} = \pi_+(S^2_{\mu,c})$, where P_+ denotes the projection onto $\mathcal{H}_+$. By Proposition 4.5.12, this implies that $[\alpha_\phi(\tau)P_+, \tau_1]$ is a compact operator. We show that for suitably chosen ϕ, this compactness condition is violated, which will complete the proof.

To this end, fix an irrational number θ and consider the sequence $\lambda_n = e^{2\pi i n\theta}$. We note that the linear functionals which send the generator of $C(Z_2)$ (which is y) to 1 and the generator of the n-th copy of $C(\mathbb{T})$ (which is $r_{n-1}^{+} r_n^{+*}$ by Theorem 4.5.7) to λ_n are evaluation maps and hence homomorphisms. Using Remark 1.1.6, we have a unital $*$-homomorphism $\phi : \mathrm{QISO}^+(D) = C(Z_2) * C(\mathbb{T})^{*\infty} \to \mathbb{C}$ which extends the above-mentioned homomorphisms. Therefore, $\alpha_\phi(\tau) = \sum_n \lambda_n \tau(P_n + Q_n)$. Moreover,

$$\begin{aligned}
&[\alpha_\phi(\tau)P_+, \tau_1]\begin{pmatrix} e_n \\ 0 \end{pmatrix} \\
&= (\mathrm{id} \otimes \phi)\alpha(\tau)\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} - \tau(\mathrm{id} \otimes \phi)[\begin{pmatrix} e_{n-1} \\ 0 \end{pmatrix} \otimes r^+_{n-1} r^{+*}_n] \\
&= (\lambda_{n-1} - \lambda_n)\begin{pmatrix} e_{n-2} \\ 0 \end{pmatrix}, \quad n \geq 2.
\end{aligned}$$

Similarly, $[\alpha_\phi(\tau)P_+, \tau_1]\begin{pmatrix} 0 \\ e_n \end{pmatrix} = (\lambda_{n-1} - \lambda_n)\begin{pmatrix} 0 \\ e_{n-2} \end{pmatrix}$. Hence, the above commutator cannot be compact since $\lambda_n - \lambda_{n-1}$ does not go to 0 as $n \to \infty$. □

Corollary 4.5.16 *The subcategory of* $\mathbf{Q}'(D)$ *consisting of objects* $(\widetilde{\mathcal{S}}, U)$ *where* ad_U *is a* C^* *coaction does not have a universal object.*

Proof By Theorem 4.5.15, it suffices to prove that if a universal object exists for the subcategory (say $\mathbf{Q}'_1$) mentioned above, then it must be isomorphic with $\widetilde{\mathrm{QISO}^+}(D)$. To this end, we consider the quantum subgroups $\widetilde{\mathcal{S}}_N$, $N = 1, 2, \ldots$, of $\widetilde{\mathrm{QISO}^+}(D)$ generated by r^+_n, $n = 1, \ldots, N$ and y. Let $\pi_N : \widetilde{\mathrm{QISO}^+}(D) \to \widetilde{\mathcal{S}}_N$ be the CQG morphism given by $\pi_N(y) = y$, $\pi_N(r^+_n) = r^+_n$ for $n \leq N$ and $\pi_N(r^+_n) = 1$ for $n > N$.

Let V denotes the unitary corepresentation of $\widetilde{\mathrm{QISO}^+}(D)$ on $\mathcal{H}$. We claim that $(\widetilde{\mathcal{S}}_N, U_N := (\mathrm{id} \otimes \pi_N) \circ V)$ is an object in $\mathbf{Q}'_1$. To see this, we observe that for all N, $(\mathrm{id} \otimes \pi_N)\alpha(A) = \sum_{n=0}^N AP_n \otimes \frac{1}{2\lambda_+}\{\lambda_+(1 + r^+_n y r^{+*}_n) + \lambda_-(1 - r^+_n y r^{+*}_n)\} + \sum_{n=0}^N AQ_n \otimes \frac{1}{2\lambda_-}\{\lambda_+(1 - r^+_n y r^{+*}_n) + \lambda_-(1 + r^+_n y r^{+*}_n)\} + \sum_{n=N+1}^\infty AP_n \otimes \frac{1}{2\lambda_+}$ $\{\lambda_+(1 + y) + \lambda_-(1 - y)\} + \sum_{n=N+1}^\infty AQ_n \otimes \frac{1}{2\lambda_-}\{\lambda_+(1 - y) + \lambda_-(1 + y)\}$.

Among the four summands, the first two clearly belong to $\mathcal{A} \otimes \widetilde{\mathcal{S}}_N$. Moreover, the sum of the third and the fourth summand equals $A(1 - \sum_{n=1}^N P_n) \otimes \frac{1}{2\lambda_+}\{\lambda_+(1 + y) + \lambda_-(1 - y)\} + A(1 - \sum_{n=1}^N Q_n) \otimes \frac{1}{2\lambda_-}\{\lambda_+(1 - y) + \lambda_-(1 + y)\}$ which is an element of $\mathcal{A} \otimes \widetilde{\mathcal{S}}_N$.

We proceed similarly in the case of B, to observe that it suffices to show that for all N,

$$\sum_{n=N+2}^\infty BP_n \otimes \left(\frac{c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}}}{2c_+(n)}\right)1 + \sum_{n=N+2}^\infty BP_n \otimes \frac{(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})y}{2c_+(n)}$$

$$+ \sum_{n=N+2}^\infty BQ_n \otimes \left(\frac{c_+(n)^{\frac{1}{2}} + c_-(n)^{\frac{1}{2}}}{2c_-(n)}\right)1 - \sum_{n=N+2}^\infty BQ_n \otimes \frac{(c_+(n)^{\frac{1}{2}} - c_-(n)^{\frac{1}{2}})y}{2c_-(n)}$$

belongs to $\mathcal{A} \otimes \widetilde{\mathcal{S}_N}$. The norm of the second and the fourth term are finite.

The first term equals $\frac{1}{2}B(1 - \sum_{n=1}^{N+1} P_n)[(A - A^2 + cI)^{-\frac{1}{2}} + (A - A^2 + cI)^{-1}$ $\{\frac{\lambda_-}{\lambda_+}A - (\frac{\lambda_-}{\lambda_+}A)^2 + cI\}^{\frac{1}{2}}] \otimes 1$ and therefore belongs to $\mathcal{A} \otimes \widetilde{\mathcal{S}_N}$. The other terms can be treated similarly.

Let $\widetilde{\mathcal{G}}$ be the universal object of $\mathbf{Q}'_1$. Then, there is surjective CQG morphism ψ_N from $\widetilde{\mathcal{G}}$ to $\widetilde{\mathcal{S}_N}$. Clearly, $(\widetilde{\mathcal{S}_N})_{N\geq 1}$ form an inductive system of objects in $\mathbf{Q}'(D)$, with the inductive limit being $\widetilde{\mathrm{QISO}^+(D)}$, and the morphisms ψ_N induce a surjective CQG morphism (say ψ) from $\widetilde{\mathcal{G}}$ to $\widetilde{\mathrm{QISO}^+(D)}$. But $\widetilde{\mathcal{G}}$ is an object in $\mathbf{Q}'(D)$, so must be a quantum subgroup of $\widetilde{\mathrm{QISO}^+(D)}$. This gives the CQG morphism from $\widetilde{\mathrm{QISO}^+(D)}$ onto $\widetilde{\mathcal{G}}$, which is the inverse of ψ, and hence we get the desired isomorphism between $\widetilde{\mathcal{G}}$ and $\widetilde{\mathrm{QISO}^+(D)}$. □

References

1. Helgason, S.: Topics in Harmonic Analysis on Homogeneous Spaces. Birkhauser, Boston, Basel, Stuttgart (1981)
2. Etingof, P., Goswami, D., Mandal, A., Walton, C.: Hopf coactions on commutative algebras generated by a quadratically independent comodule. Commun. Algebra (To appear). arXiv:1507.08486
3. Schmudgen, K., Wagner, E.: Representation of cross product algebras of Podles' quantum spheres. J. Lie Theory **17**(4), 751–790 (2007)
4. Schmudgen, K., Wagner, E.: Dirac operators and a twisted cyclic cocycle on the standard Podles' quantum sphere. arXiv:0305051v2
5. Dabrowski, L., D'Andrea, F., Landi, G., Wagner, E.: Dirac operators on all Podles' quantum spheres. J. Noncommut. Geom. **1**, 213–239 (2007)
6. Bhowmick, J.: Quantum isometry groups. Ph.D thesis, Indian Statistical Institute (2010). arxiv:0907.0618
7. Bhowmick, J., Goswami, D.: Quantum isometry groups of the Podles' Spheres. J. Funct. Anal. **258**, 2937–2960 (2010)
8. Soltan, P.M.: Quantum $SO(3)$ groups and quantum group actions on M_2. J. Noncommut. Geom. **4**(1), 1–28 (2010)
9. Chakraborty, P.S., Pal, A.: Spectral triples and associated Connes-de Rham complex for the quantum SU(2) and the quantum sphere. Commun. Math. Phys. **240**(3), 447–456 (2003)
10. Davidson, K.R.: C^* Algebras by Examples. Hindustan Book Agency (1996)

Chapter 5
Quantum Isometry Groups of Discrete Quantum Spaces

Abstract We show that the definitions of quantum symmetry groups for finite metric spaces and graphs given by Banica and Bichon can be viewed as quantum isometry groups in our sense. Next, we prove that the quantum group of orientation preserving isometries of a spectral triple on some approximately finite dimensional C^* algebras arise as the inductive limit of the quantum group of orientation preserving isometries of suitable spectral triples on the constituent finite dimensional algebras.

In his seminal paper of 1998 [1], Shuzhou Wang defined and studied the quantum automorphism groups of finite-dimensional C^* algebras. The next breakthrough came in [2, 3] where the notion of quantum automorphism groups of finite graphs and finite metric spaces were respectively defined by Bichon and Banica. These quantum symmetry groups and their variants were further studied by Banica, Bichon and their collaborators in a series of papers ([4–10] etc.). It is a natural problem to see whether the quantum isometry groups for suitable spectral triples on finite metric spaces and graphs have any relations with the quantum symmetry groups of Banica and Bichon. The first part of this chapter (Sect. 5.1) provides a positive answer to this question. First, the metric data on a finite metric space or graph is shown to correspond to natural noncommutative geometric structures on the function algebra of the metric space or graph so that the quantum isometry groups of these objects make sense. Then, it is shown that the quantum symmetry groups of finite graphs or metric spaces coincide with the quantum isometry groups of the corresponding classical objects equipped with natural Laplacians.

The second question dealt in this chapter is the question of existence of quantum isometry groups for more general zero dimensional manifolds. In [11, 12] Christensen and Ivan constructed natural spectral triples on approximately finite-dimensional (AF) C^*-algebras. AF algebras provide a natural "connecting bridge" between the finite and infinite-dimensional noncommutative spaces and thus can be thought of as 0-dimensional manifolds. This is reinforced by the fact that Christensen and Ivan showed that on each AF algebra, one can construct spectral triples with arbitrarily good summability properties. We show that the "quantum group of orientation" preserving isometries' of a Christensen-Ivan type triple arises as an inductive limit of quantum isometry groups of certain finite-dimensional triples

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_5

(Theorem 5.2.1). In the case when the AF algebra in question is commutative, the resulting quantum isometry groups of relevant finite-dimensional objects fit into the framework described in Sect. 5.1 and we show that they coincide with the quantum symmetry groups of finite graphs obtained by suitable truncations of the Bratteli diagrams. This observation implies that the construction we consider can be thought of as giving a definition of a quantum symmetry group of an arbitrary Bratteli diagram. It also enables us to compute explicitly the quantum isometry group of a spectral triple associated with the middle-third Cantor set introduced first by Connes and later studied by Christensen and Ivan. It is perhaps worth mentioning that unlike the classical case a quantum isometry of the product set preserving the first factor in a suitable sense need not be a product isometry. For more general quantum symmetry groups of inductive limits, we refer to [13].

Let us fix some notations and conventions at this point. For a finite metric space (X, d), we will write $X = \{1, ..., n\}$ for simplicity. For $i, j \in X$, define $d_{ij} = d(i, j)$. Then the entire information about (X, d) is encoded in the matrix $d := ((d_{ij}))$. As before, we will denote by δ_i the indicator function of the point $i \in X$ and by $D_{X\times X}$ the diagonal in $X \times X$.

We recall the notation $\alpha^{(2)}$ introduced in Lemma 1.3.6. For a finite set X and coaction α of a CQG $\mathcal{S}$ on $C(X)$, we have a unitary $\widetilde{\alpha} \in \mathcal{B}(l^2(X)) \otimes \mathcal{S}$ given by $\widetilde{\alpha}(f \otimes q) = \alpha(f)(1 \otimes q)$. Moreover, we define $\alpha^{(2)} : C(X) \otimes C(X) \to C(X) \otimes C(X) \otimes \mathcal{S}$ by $\alpha^{(2)} = (\mathrm{id}_2 \otimes m_{\mathcal{S}})\sigma_{23}(\alpha \otimes \alpha)$, where $m_{\mathcal{S}}$ denotes the multiplication map from $\mathcal{S} \otimes \mathcal{S}$ to $\mathcal{S}$, and id_2 denotes the identity map on $C(X) \otimes C(X)$, Throughout this chapter, we will frequently use Lemma 1.3.6 and Corollary 1.3.7, sometimes without mentioning explicitly.

5.1 Quantum Isometry Groups of Finite Metric Spaces and Finite Graphs

As mentioned above, the aim of this section is to see how one can accommodate the rich and well developed theory of quantum permutation and quantum automorphism groups of "finite" structures in the more general set-up of quantum isometry groups. We will identify the quantum group of automorphisms of a finite metric space or a finite graph in the sense of Banica and Bichon with the quantum group of orientation (and suitable "volume-form") preserving isometries of a natural spectral triple, thus successfully unifying the approaches of [2, 3] with that of Sects. 3.1 and 3.2.

5.1.1 The Works of Banica [3] and Bichon [2]

Let us recall the work of Banica in [3]. For a finite metric space (X, d), let $\mathbf{C}^{\mathbf{Ban}}_{\mathbf{X},\mathbf{d}}$ denote the category with objects $(\mathcal{S}, \alpha)$, where α defined by $\alpha(\delta_i) = \sum_j \delta_j \otimes v_{ji}$ is the coaction of a compact quantum group $\mathcal{S}$ on $C(X)$ such that the matrices $d := ((d_{ij}))$

and $v := ((v_{ij}))$ commute. The morphisms of the category are compact quantum group morphisms intertwining the coactions. It is proved by Banica that there exists a universal object in $\mathbf{C}^{\mathbf{Ban}}_{\mathbf{X,d}}$, to be called the quantum symmetry group of (X, d) and denoted by $\mathrm{QISO}^{\mathrm{Ban}}$ in this chapter.

We now recall the definition of the quantum automorphism group of a finite directed graph given in [2]. Let (V, E) be a graph with V denoting the set of vertices and E the set of edges. Let $s : E \to V$ (respectively $t : E \to V$) be the source map (respectively the target map). The target and source maps induce *-homomorphisms $s^*, t^* : C(V) \to C(E)$. Let $m : C(E) \otimes C(E) \to C(E)$ be the pointwise multiplication map on E and given a quantum group coaction α on $C(V)$ let $\alpha^{(2)}$ be defined as in Lemma 1.3.6.

Definition 5.1.1 [2] A coaction of a CQG $\mathcal{S}$ on a finite graph $G = (V, E)$ consists of a coaction α of $\mathcal{S}$ on the set of vertices, $\alpha : C(V) \to C(V) \otimes \mathcal{S}$ and a coaction β of $\mathcal{S}$ on the set of edges, $\beta : C(E) \to C(E) \otimes \mathcal{S}$, such that

$$((m(s_* \otimes t_*)) \otimes \mathrm{id}_{\mathcal{S}}) \circ \alpha^{(2)} = \beta \circ (m(s^* \otimes t^*)).$$

It is also called a quantum symmetry of the graph (V, E).

The *quantum automorphism group of the finite graph* is the universal object in the category of compact quantum groups with coactions as above. We refer to [2] for the details.

5.1.2 Noncommutative Geometry on Finite Metric Spaces

Let X be a finite metric space. We recall the natural Dirac operator (see [12, 14]) on a particular representation space of $C(X)$, for which the Rieffel-type metric [14] on the state space of $C(X)$ restricts to the original metric on X. It can be seen, as pointed out earlier in [3], that this framework can be related to the one of quantum symmetry groups of finite graphs.

The metric structure on X allows the construction of a natural spectral triple on $C(X)$. Let $Y = X \times X - D_{X\times X}$, define the Hilbert space $\mathcal{H} = \oplus_{(x_0,x_1),x_0\neq x_1}\mathcal{H}_{(x_0,x_1)}$, where $\mathcal{H}_{(x_0,x_1)} = \mathbb{C}^2$, and let the Dirac operator be given by

$$D = \oplus_{(x_0,x_1),x_0\neq x_1} d^{-1}(x_0, x_1) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Viewing $\mathcal{H}$ as $\mathbb{C}^2 \otimes l^2(Y)$, we have

$$D = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes M_{d^{-1}},$$

where $M_{d^{-1}}$ denotes multiplication by the function d^{-1} on Y.

Let $s, t : Y \to X$ be given by the formulas $s(x_0, x_1) = x_0$, $t(x_0, x_1) = x_1$. Then for each $f \in C(X)$, there is $s^*(f) = (f \otimes 1)\chi_Y$, $t^*(f) = (1 \otimes f)\chi_Y$, where χ_Y denotes the characteristic function of Y. Let for $f \in C(X)$

$$\pi_1(f) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes s^*(f), \quad \pi_2(f) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes t^*(f),$$

define

$$\pi(f) = \pi_1(f) + \pi_2(f)$$

and consider the resulting spectral triple $(\pi(C(X)), \mathcal{H}, D)$.

As X is finite, it is clear that

$$\mathrm{Lim}_{t \to 0+} t^d \mathrm{Tr}(Te^{-tD^2}) = \mathrm{Tr(T)} \text{ for } d = 0 \text{ and } 0 \text{ otherwise.}$$

This means that the spectral triple is zero dimensional in the sense of Chap. 2 and the inner product on the space of one forms is given by

$$\langle d_D f, d_D g \rangle = \mathrm{Tr}([D, \pi(f)]^* [D, \pi(g)]) \text{ for all } f, g \in C(X).$$

It is easy to describe the "Laplacian" in the sense of Sect. 3.1 for this spectral triple.

Lemma 5.1.2 *The Laplacian $\mathcal{L}$ on $C(X)$ associated with the spectral triple $(\pi(C(X)), \mathcal{H}, D)$ constructed above via the prescription in Sect. 3.1 is given by the formula*

$$\mathcal{L}(\delta_i) = \frac{4}{2n-1} \sum_{j \in X} c(i,j)\delta_j, \ i \in X \tag{5.1.1}$$

where δ_i is the function on X taking value 1 at the point i and zero elsewhere, and for $i, j \in X$ we have $c(i,j) = d^{-2}(i,j)$ if $i \neq j$ and $c(i,i) = -\sum_{j \neq i} \frac{1}{d^2(i,j)}$.

Proof Let $\tau_0(f) := \sum_{i=1}^n f(i)$. Denote by $\mathcal{H}_0$ the Hilbert space obtained from $\pi(C(X))$ with respect to the norm coming from the functional $Tr(\pi(f)) = (2n-1)\tau_0(f)$.

From the definition of the inner product it follows that for all $f, g \in C(X)$

$$\langle \mathcal{L}(f), g \rangle = \sum_{i \in X} g(i)(2n-1)\overline{\mathcal{L}(f)(i)}.$$

On the other hand, $\langle \mathcal{L}(f), g \rangle = -\left\langle d_D^* d_D f, g \right\rangle = -\langle d_D f, d_D g \rangle$ where $d_D(\cdot) = [D, \cdot]$.

A straightforward computation gives $[D, \pi(f)] = i \oplus_{x,y \in X, x \neq y} \frac{f(y)-f(x)}{d(x,y)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus

$- \langle d_D f, d_D g \rangle$
$= -\mathrm{Tr}([D, \pi(f)]^* [D, \pi(g)]) = -2 \sum_{i \neq j} \frac{(\overline{f(j)} - \overline{f(i)})(g(j)-g(i))}{d^2(i,j)}$
$= 4 \sum_{i \neq j} \frac{g(i)(\overline{f(j)} - \overline{f(i)})}{d^2(i,j)}$.
Thus for each $i \in X$

$$\mathcal{L}(f)(i) = \frac{4}{2n-1} \sum_{j \neq i} \frac{f(j) - f(i)}{d^2(i,j)}.$$

Comparison of the above with formula (5.1.1) ends the proof. □

It is now easy to verify that $\mathcal{L}$ is admissible in the sense of Sect. 3.1, so that the corresponding quantum isometry group $\mathrm{QISO}^{\mathcal{L}}(X) := \mathrm{QISO}^{\mathcal{L}}(\pi(C(X)), \mathcal{H}, D)$ exists. We recall that $\mathrm{QISO}^{\mathcal{L}}(X)$ is the universal object in the category $\mathbf{C}^{\mathcal{L}}_{\mathbf{X},\mathbf{d}}$, with the objects being pairs $(\mathcal{S}, \alpha)$, where $\mathcal{S}$ is a compact quantum group and α is the coaction of $\mathcal{S}$ on $C(X)$ satisfying $(\mathcal{L} \otimes \mathrm{id}) \circ \alpha = \alpha \circ \mathcal{L}$.

5.1.3 Quantum Symmetry Groups of Banica and Bichon as Quantum Isometry Groups

In this subsection, we show that given a finite metric space X the quantum symmetry group of X defined in [3] coincides with the quantum isometry group of the algebra of functions on X equipped with the natural Laplacian. We also discuss the connections with the spectral triple defined above. We begin by observing an alternative characterization of isometric coactions of compact quantum groups on finite metric spaces considered in [3].

Lemma 5.1.3 *Given a coaction α of a compact quantum group $\mathcal{S}$ on a finite metric space (X, d) (i.e. a coaction of $\mathcal{S}$ on $C(X)$), the following are equivalent:*
(i) $(\mathcal{S}, \alpha)$ is a quantum isometry in the sense of Banica;
(ii) $\alpha^{(2)}(d) = d \otimes 1$;
(iii) $\alpha^{(2)}(c) = c \otimes 1$, where $c \in C(X \times X)$ is as in Lemma 5.1.2;
(iv) $(\mathcal{S}, \alpha)$ is an object in $\mathbf{C}^{\mathcal{L}}_{\mathbf{X},\mathbf{d}}$.

Proof (i) $\Leftrightarrow$ (ii)
Write $d = \sum_{i,j \in X} d_{ij} \delta_i \otimes \delta_j$, where d_{ij} are defined as in the beginning of this section. Let us write the coaction α as

$$\alpha(\delta_i) = \sum_{j \in X} \delta_j \otimes q_{ij}, \tag{5.1.2}$$

where $q_{ij} \in \mathcal{S}$. Then it follows by using the relations of the quantum permutation group that the relation $\alpha^{(2)}(d) = d \otimes 1$ is equivalent to the following equation being satisfied for all $k, l \in X$:

$$d_{kl} 1 = \sum_{i,j \in X} d_{ij} q_{ki} q_{lj}. \tag{5.1.3}$$

Thus, to prove the lemma, it is enough to show equivalence of (5.1.3) with Banica's definition of quantum isometry.

Begin by noting that Banica's definition implies that for all $k, l \in X$

$$\sum_{i \in X} d_{il} q_{ki} = \sum_{i \in X} d_{ki} q_{il}.$$

From this, it follows that for all $k, m \in X$,

$$\sum_{l,i \in X} d_{il} q_{ki} q_{ml} = \sum_{l,i \in X} d_{ki} q_{il} q_{ml} = \sum_{l \in X} d_{km} q_{ml} = d_{km} 1,$$

which is exactly (5.1.3). Note that we have used the relations of the quantum permutation group, namely, $q_{il} q_{ml} = \delta_{im} q_{ml}$ for all i, m, l, where δ_{im} denotes the Kronecker delta function.

For the converse direction rewrite (5.1.3) as

$$\begin{aligned}
\sum_{i \in X} d_{ki} q_{il} &= \sum_{i \in X} \Big(\sum_{j,m \in X} d_{jm} q_{kj} q_{im} \Big) q_{il} = \sum_{i,j,m \in X} d_{jm} q_{kj} q_{im} q_{il} \\
&= \sum_{i,j,l \in X} q_{kj} \delta_{m,l} d_{jm} q_{im} q_{il} \text{ (where } \delta_{m,l} \text{ denotes the Kronecker delta)} \\
&= \sum_{i,j \in X} q_{kj} d_{jl} q_{il} = \sum_{j \in X} d_{jl} q_{kj}.
\end{aligned}$$

Thus Banica's condition is derived.

The proof of (iii) $\Leftrightarrow$ (iv) is very similar to the above proof of equivalence of (i) and (ii), hence omitted.

Finally, the equivalence of (ii) and (iii) follows from the relation between c and d, i.e.,

$$d = \chi_Y c^{-\frac{1}{2}}, \quad c = \chi_Y d^{-2} - (1 - \chi_Y)((\tau_0 \otimes \mathrm{id})(\chi_Y d^{-2}) \otimes 1).$$

As

$$\alpha^{(2)}(\chi_Y) = \chi_Y \otimes 1, \tag{5.1.4}$$

(iii) implies (ii). On the other hand, the implication of (iii) from (ii) follows from (5.1.4) and the fact that τ_0 is α-invariant. $\square$

Next, we consider the spectral triple $(\pi_1(C(X)), \mathcal{H}, D)$. Denote by $\mathbf{C_{X,d}}$ the category of compact quantum groups acting by volume (corresponding to $R = I$) and orientation preserving isometries on the spectral triple $(\pi_1(C(X), \mathcal{H}, D)$. We want to show that $\mathrm{QISO}^{\mathrm{Ban}}$, that is, the universal object in $\mathbf{C}^{\mathbf{Ban}}_{\mathbf{X},\mathbf{d}}$, is isomorphic to the quantum group $\mathrm{QISO}^+_I(\pi_1(C(X), \mathcal{H}, D)$. The proof of this fact is contained in the two following lemmas.

Lemma 5.1.4 *Let $(\mathcal{S}, \alpha)$ be a quantum isometry of (X, d) in the sense of Banica, i.e. an object in $\mathbf{C}^{\mathbf{Ban}}_{\mathbf{X},\mathbf{d}}$. Then there is a unitary corepresentation U of $\mathcal{S}$ on $\mathcal{H}$ such that $(\mathcal{S}, U) \in \mathrm{Obj}(\mathbf{C}_{\mathbf{X},\mathbf{d}})$, with $\alpha_U = \alpha$ on $C(X)$.*

Proof Define $\widetilde{U} = I_{\mathbb{C}^2} \otimes \widetilde{\alpha^{(2)}}$ on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{S}$. Then $\widetilde{U}$ gives a unitary corepresentation since $\widetilde{\alpha^{(2)}}$ is one.

Moreover, it is easy to see that $Tr(\pi_1(f)) = (n-1)\tau_0(f)$. As α preserves τ_0, the coaction α_U already preserves the volume form corresponding to $R = I$.

We now claim that U commutes with D. By using (1.3.1), we note that the condition $\alpha^{(2)}(c) = c \otimes 1$ implies $\widetilde{\alpha}_{(13)}\widetilde{\alpha}_{(23)}(c \otimes 1) = c \otimes 1$.

By the discussion in Sect. 1.3.1 and Lemma 1.3.8, we obtain

$$\widetilde{\alpha}_{(23)}(c \otimes 1) = (\mathrm{id} \otimes \alpha)(M_c)\mathbf{1} = \widetilde{\alpha}_{(23)}(M_c \otimes 1)\widetilde{\alpha}^{-1}_{(23)}(\mathbf{1}) \tag{5.1.5}$$

and

$$\widetilde{\alpha}^{-1}_{(13)}(c \otimes 1) = \Sigma_{23}((\mathrm{id} \otimes \kappa)\alpha)(M_c)(\mathbf{1}) = \widetilde{\alpha}^{-1}_{(13)}(M_c \otimes 1)\widetilde{\alpha}_{(13)}(\mathbf{1}). \tag{5.1.6}$$

Here, Σ_{23} denotes the flip in the second and the third tensor copy and $\mathbf{1}$ denotes the vector $1_{l^2(X \times X)} \otimes 1_{\mathcal{S}}$, which is clearly separating for the algebra $C(X \times X) \otimes S$. As $\widetilde{\alpha}_{(23)}(M_c \otimes 1)\widetilde{\alpha}^{-1}_{(23)}$ and $\widetilde{\alpha}^{-1}_{(13)}(M_c \otimes 1)\widetilde{\alpha}_{(13)}$ belong to $C(X \times X) \otimes \mathcal{S}$ and from (5.1.5), (5.1.6), they agree on $\mathbf{1}$, we conclude that $\widetilde{\alpha}_{(23)}(M_c \otimes 1)\widetilde{\alpha}^{-1}_{(23)} = \widetilde{\alpha}^{-1}_{(13)}(M_c \otimes 1)\widetilde{\alpha}_{(13)}$, i.e., $\widetilde{\alpha}_{(13)}\widetilde{\alpha}_{(23)}(M_c \otimes 1)\widetilde{\alpha}^{-1}_{(23)}\widetilde{\alpha}^{-1}_{(13)} = M_c \otimes 1$.

Thus, $\widetilde{\alpha^{(2)}}(M_c \otimes 1)\widetilde{\alpha^{(2)}}^{-1} = M_c \otimes 1$.

It follows that $\widetilde{U} = I_{\mathbb{C}^2} \otimes \widetilde{\alpha^{(2)}}$ commutes with $D = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes M_c$ on $\mathcal{H}$.

Further, it is easy to see that for any $f \in C(X)$

$$\begin{aligned}
&\widetilde{U}(\pi_1(f) \otimes 1)\widetilde{U}^{-1} \\
&\qquad = \widetilde{U}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes s^*(f) \otimes 1\right)\widetilde{U}^{-1} \\
&\qquad = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes M_{\widetilde{\alpha^{(2)}(s^* f)}} \\
&\qquad = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes s^*(f_{(0)}) \otimes f_{(1)} \\
&\qquad = (\pi_1 \otimes \mathrm{id})\alpha(f) \subseteq \pi_1(C(X)) \otimes \mathcal{S},
\end{aligned}$$

where we have used the Sweedler notation $\alpha^{(2)}(f) = f_{(0)} \otimes f_{(1)}$. This implies that $(\mathcal{S}, U) \in \mathrm{Obj}(\mathbf{C_{X,d}})$. It is obvious from the construction that $\alpha_U = \alpha$. □

Lemma 5.1.5 *Let $(\widetilde{\mathcal{S}}, U) \in \mathrm{Obj}(\mathbf{C_{X,d}})$, and let $\mathcal{S}$ be the largest Woronowicz C^*-subalgebra of $\widetilde{\mathcal{S}}$ such that the coaction α_U maps $C(X)$ into $C(X) \otimes \mathcal{S}$. Then $(\mathcal{S}, \alpha_U)$ is an object of $\mathbf{C^{Ban}_{X,d}}$.*

Proof The fact that U commutes with D implies that U commutes with $D^2 = I_{\mathbb{C}^2} \otimes M_c$ on $\mathbb{C}^2 \otimes l^2(Y)$. Since $U = I \otimes \alpha^{(2)}$, it follows that $\alpha^{(2)}(c) = c \otimes 1$, hence (by Lemma 5.1.3) $(\mathcal{S}, \alpha)$ is a quantum isometry in the sense of Banica. □

Lemmas 5.1.3, 5.1.4 and 5.1.5 put together imply immediately the following:

Theorem 5.1.6 *We have the following isomorphisms of compact quantum groups:*

$$\mathrm{QISO}^{\mathrm{Ban}}(X) \cong \mathrm{QISO}^+_I(\pi_1(C(X)), \mathcal{H}, D) \cong \mathrm{QISO}^{\mathcal{L}}(X).$$

Remark 5.1.7 We can accommodate graphs in the framework of the above theorem if we view a finite non-directed graph (V, E) as a metric space (V, d_E) where

$$d_E(v, w) = 1 \text{ if } (v, w) \in E, \quad d_E(v, w) = \infty \text{ if } (v, w) \notin E$$

$(v, w \in V, v \neq w)$. A similar observation was made in [3]. Here the Theorem 5.1.6 shows that quantum symmetry groups of finite graphs of [2] can be viewed as quantum isometry groups associated to the natural Laplacians on such graphs.

In Theorem 5.2.7, we will see that the quantum symmetry group of a particular type of a finite graph (as defined in [2]) coincides with the quantum isometry group of a certain spectral triple.

5.2 Quantum Isometry Groups for Inductive Limits

The aim of this section is to show that the quantum isometry groups of spectral triples associated with approximately finite-dimensional C^*-algebras arise as inductive limits of quantum symmetry groups of corresponding truncated Bratteli diagrams. We use this result to determine explicitly the quantum isometry group of the natural spectral triple on the algebra of continuous functions on the middle-third Cantor set. In the more generalized setting of quantum symmetry groups of orthogonal filtrations, the inductive limits coming from the algebra of continuous functions on the group of p-adic integers and the limits of quantum isometry groups of symmetric groups were considered in [13].

We will be using Lemma 1.2.20 and we note that its proof remains valid for any other indexing set for the net, not necessarily $\mathbb{N}$. The next theorem connects the inductive construction in Lemma 1.2.20 with some specific quantum isometry groups.

Theorem 5.2.1 *Suppose that A is a C^*-algebra acting on a Hilbert space $\mathcal{H}$ and that D is a (densely defined) self-adjoint operator on $\mathcal{H}$ with compact resolvent, such that D has a one-dimensional eigenspace spanned by a vector ξ which is cyclic and separating for A. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be an increasing net of a unital *-subalgebras of A and put $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. Suppose that $\mathcal{A}$ is dense in A and that for each $a \in \mathcal{A}$ the commutator $[D, a]$ is densely defined and bounded. Additionally put $\mathcal{H}_n = \overline{\mathcal{A}_n\xi}$, let P_n denote the orthogonal projection on $\mathcal{H}_n$ and assume that $P_n\mathcal{H}$ is a direct sum of eigenspaces of D. Then each $(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$ is a spectral triple satisfying the conditions of Theorem 3.4.2, there exist natural compatible CQG morphisms $\pi_{m,n} : \mathrm{QISO}^+(\mathcal{A}_m, \mathcal{H}_m, D|_{\mathcal{H}_m}) \to \mathrm{QISO}^+(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$ $(n, m \in \mathbb{N}, m \le n)$ and*

$$\mathrm{QISO}^+(\mathcal{A}, \mathcal{H}, D) = \lim_{n\in\mathbb{N}} \mathrm{QISO}^+(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n}).$$

Similar conclusions hold if we replace $\widetilde{\mathrm{QISO}^+}$ by QISO^+ everywhere above.

Proof We prove the assertion corresponding to $\widetilde{\mathrm{QISO}^+}$ only, since the proof for QISO^+ follows by very similar arguments. Let us denote $\widetilde{\mathrm{QISO}^+}(\mathcal{A}_n, \mathcal{H}_n, D_n)$ by $\mathcal{S}_n$ and the corresponding unitary corepresentation (in $\mathcal{H}_n$) by U_n. Let us denote the category of compact quantum groups acting by orientation preserving isometries on $(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$ and $(\mathcal{A}, \mathcal{H}, D)$, respectively, by $\mathbf{C_n}$ and $\mathbf{C}$.

Since U_n is a unitary which commutes with $D_n \equiv D|_{\mathcal{H}_n}$ and hence preserves the eigenspaces of D_n, it restricts to a unitary corepresentation of S_n on each H_m for $m \le n$. In other words, $(\mathcal{S}_n, U_n|_{\mathcal{H}_m}) \in \mathrm{Obj}(\mathbf{C_m})$, and by the universality of $\mathcal{S}_m$ there exists a compact quantum group morphism, say, $\pi_{m,n} : \mathcal{S}_m \to \mathcal{S}_n$ such that $(\mathrm{id} \otimes \pi_{m,n})U_m|_{\mathcal{H}_m} = U_n|_{\mathcal{H}_m}$.

Let $p \le m \le n$. Then we have $(\mathrm{id} \otimes \pi_{m,n}\pi_{p,m})U_p|_{\mathcal{H}_p} = U_n|_{\mathcal{H}_p}$. It follows by the uniqueness of the map $\pi_{p,n}$ that $\pi_{p,n} = \pi_{m,n}\pi_{p,m}$, i.e., $(\mathcal{S}_n)_{n\in\mathbb{N}}$ forms an inductive system of compact quantum groups satisfying the assumptions of Lemma 1.2.20. Denote by $\mathcal{S}_\infty$ the inductive limit CQG obtained in that lemma, with $\pi_{n,\infty} : \mathcal{S}_n \to \mathcal{S}$ denoting the corresponding CQG morphisms. The family of formulas $U|_{\mathcal{H}_n} := (\mathrm{id} \otimes \pi_{n,\infty}) \circ U_n$ combine to define a unitary corepresentation U of $\mathcal{S}_\infty$ on $\mathcal{H}$. It is also easy to see from the construction that U commutes with D. This means that $(\mathcal{S}_\infty, U) \in \mathrm{Obj}(\mathbf{C})$, hence there exists a unique surjective CQG morphism from $\mathcal{S} := \widetilde{\mathrm{QISO}^+}(\mathcal{A}, \mathcal{H}, D)$ to $\mathcal{S}_\infty$ identifying $\mathcal{S}_\infty$ as a quantum subgroup of $\mathcal{S}$.

The proof will now be complete if we can show that there is a surjective CQG morphism in the reverse direction, identifying $\mathcal{S}$ as a quantum subgroup of $\mathcal{S}_\infty$. This can be deduced from Lemma 1.2.20 by using the universality property of the inductive limit. Indeed, for each $n \in \mathbb{N}$ the unitary corepresentation V_n (say) of $\widetilde{\mathrm{QISO}^+}(\mathcal{A}, \mathcal{H}, D)$ restricts to $\mathcal{H}_n$ and commutes with D on that subspace, thus inducing a CQG morphism ρ_n from $\mathcal{S}_n = \widetilde{\mathrm{QISO}^+}(\mathcal{A}_n, \mathcal{H}_n, D_n)$ into $\mathcal{S}$. The family of morphisms $(\rho_n)_{n\in\mathbb{N}}$ satisfies the compatibility conditions required in Lemma 1.2.20. It remains to show that the induced CQG morphism ρ_∞ from $\mathcal{S}_\infty$ into $\mathcal{S}$ is surjective. By the faithfulness of the corepresentation V of $\widetilde{\mathrm{QISO}^+}(\mathcal{A}, \mathcal{H}, D)$, we know that the span of matrix elements corresponding to all V_n forms a norm-dense subset of $\mathcal{S}$. As the range of ρ_n contains the matrix elements corresponding to $V_n = V|_{\mathcal{H}_n}$, the surjectivity of ρ_∞ follows. □

Remark 5.2.2 Theorem 5.2.1 was first observed in [15]. However, as noted in [13], the statement of Theorem 1.2 as stated in [15] is not correct. The assumption that the projections P_n commute with D as was made in [15] is not enough to have the desired conclusion. We actually need to assume that $P_n\mathcal{H}$ is a direct sum of eigenspaces of D. However, in all the applications of [15], the assumptions of the correct formulation are satisfied. A generalization of Theorem 5.2.1 was proved in [13] in terms of the quantum symmetry groups of orthogonal filtrations developed in [16].

5.2.1 Examples Coming from AF Algebras

In this subsection, we show how our theory applies to AF algebras. In particular, we will show the relationship between the quantum isometry groups for a particular class of spectral triples on AF algebras and the quantum symmetry groups of finite graphs as discussed before. In the next subsection, we will deal with the particular example of the AF algebra constructed out of the Cantor set.

A C^* algebra A is called AF (approximately finite-dimensional) if it is the closure of an increasing union of finite-dimensional C^* algebras A_n. We will denote $\cup_{n\geq 0} A_n$ by $\mathcal{A}$. We will only be dealing with unital AF algebras so that we take $A_0 = \mathbb{C}1_A$ and assume that the unit in each A_n is the unit of $\mathcal{A}$. Moreover, the embedding of A_n inside A_{n+1} will be assumed to be unital. We refer to [17] and the references therein for the details on AF algebras. In particular, we will need the following theorem:

Theorem 5.2.3 *Let ϕ be a unital $*$-homomorphism from $B_1 = \oplus_{i=1}^k M_{n_i}$ to $B_2 = \oplus_{i=1}^l M_{m_i}$. Then upto unitary equivalence in B_2, the map ϕ is determined by an $l \times k$ matrix $((a_{ij}))$ where a_{ij} are such that $\sum_{j=1}^k a_{ij} n_j = m_i$ for all i.*

Thus, the embedding of B_1 into B_2 can be described by the following data: a k-tuple $\{(1,1) = n_1, \cdots (1,k) = n_k\}$, an l-tuple $\{(2,1) = m_1, \cdots (2,l) = m_l\}$, a_{ij} arrows from $(1,j)$ to $(2,i)$.

Thus, one is led to consider the graph with vertices parametrized by the set $\cup_{p\in N\cup\{0\}}(p,i)$ and $a_{ji}^{(p)}$ number of edges from (p,i) to $(p+1,j)$, where $a_{ji}^{(p)}$ corresponds to the numbers appearing in the embedding of A_p inside A_{p+1}.

This diagram is known as the Bratteli diagram of the AF algebra.

Now we describe a natural family of spectral triples on AF algebras, constructed in [11], for which the situation is exactly as in Theorem 5.2.1. We derive basic properties of the compact quantum groups appearing in the related inductive system and relate them for commutative AF algebras with the quantum symmetry groups of truncated Bratteli diagrams.

We first recall the construction of natural spectral triples on AF algebras due to Christensen and Ivan [11]. Let A be a unital AF C^*-algebra, the norm closure of an increasing sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of finite-dimensional C^*-algebras. We always put $\mathcal{A}_0 = \mathbb{C}1_A$, $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ and assume that the unit in each $\mathcal{A}_n$ is the unit of $\mathcal{A}$. Suppose that A is acting on a Hilbert space $\mathcal{H}$ and that $\xi \in \mathcal{H}$ is a separating and cyclic unit vector for A. Let P_n denote the orthogonal projection onto the subspace $\mathcal{H}_n := \mathcal{A}_n\xi$ of $\mathcal{H}$ and write $Q_0 = P_0 = P_{\mathbb{C}\xi}$, $Q_n = P_n - P_{n-1}$ for $n \in \mathbb{N}$. There exists a (strictly increasing) sequence of real numbers $(\alpha_n)_{n=1}^{\infty}$ such that the selfadjoint operator $D = \sum_{n\in\mathbb{N}} \alpha_n Q_n$ yields a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and the topology on the state space of A induced by the Rieffel metric [14] coincides with the weak*-topology. Due to the existence of a cyclic and separating vector the orientation preserving quantum isometry group exists by Theorem 3.4.2.

In [11] it was additionally observed that if A is infinite-dimensional and $p > 0$ then one can choose $(\alpha_n)_{n=1}^{\infty}$ in such a way that the resulting Fredholm module is p-summable [18]. This reflects the fact that AF algebras should be thought of as 0-dimensional noncommutative spaces.

Note that for each $n \in \mathbb{N}$ by restricting we obtain a (finite-dimensional) spectral triple $(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$. As we are precisely in the framework of Theorem 5.2.1, to compute $\mathrm{QISO}^+(\mathcal{A}, \mathcal{H}, D)$ we need to understand the quantum isometry groups $\mathrm{QISO}^+(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$ and embeddings relating them. To simplify the notation we will write $\mathcal{S}_n := \mathrm{QISO}^+(\mathcal{A}_n, \mathcal{H}_n, D|_{\mathcal{H}_n})$.

We begin with some general observations. We recall from Sect. 1.3 the compact quantum group $A_{\mathrm{aut}}(\mathcal{A}_n, \omega_\xi)$ which is the universal compact quantum group acting on $\mathcal{A}_n$ and preserving the (faithful) state on $\mathcal{A}_n$ given by the vector ξ (see [1]).

Lemma 5.2.4 *There exists a CQG morphism from $A_{\mathrm{aut}}(\mathcal{A}_n, \omega_\xi)$ to $\mathcal{S}_n$.*

Proof Consider the spectral triple given by $(\mathcal{A}_n, \mathcal{H}_n, D'_n)$, where $D'_n = P_n - P_0$. It is then easy to see that $\mathrm{QISO}^+(\mathcal{A}_n, \mathcal{H}_n, D'_n)$ is isomorphic to the universal compact quantum group acting on $\mathcal{A}_n$ and preserving ω_ξ. On the other hand, universality assures the existence of the CQG morphism from $\mathrm{QISO}^+(\mathcal{A}_n, \mathcal{H}_n, D'_n)$ to $\mathcal{S}_n$. □

Lemma 5.2.5 *Assume that each $\mathcal{A}_n$ is commutative, i.e., $\mathcal{A}_n = \mathbb{C}^{k_n}$ for some positive integer k_n, $n \in \mathbb{N}$. There exists a CQG morphism from $A_s(k_n)$ to $\mathcal{S}_{k_n}$, where $A_s(k_n)$ denotes the quantum permutation group as in Sect. 1.3.1.*

Proof The proof is identical to the one above—we only need to observe additionally that for any measure μ on the set $\{1, \ldots, k_n\}$ which has full support there is a natural CQG morphism from $A_s(k_n)$ to $A_{\mathrm{aut}}(\mathbb{C}^{k_n}, \mu)$. In case when μ is uniformly distributed, we simply have $A_{\mathrm{aut}}(\mathbb{C}^{k_n}, \mu) = A_s(k_n)$, which follows from the first part of Lemma 1.3.6. □

Let $\alpha_n : \mathcal{A}_n \to \mathcal{A}_n \otimes \mathcal{S}_n$ denote the universal coaction at the n-th level. Then we have the following important property, being the direct consequence of Theorem 5.2.1. We have

$$\alpha_{n+1}(\mathcal{A}_n) \subseteq \mathcal{A}_n \otimes \mathcal{S}_{n+1}, \tag{5.2.1}$$

identifying $\mathcal{A}_n$ with a subalgebra of $\mathcal{A}_{n+1}$ and $\mathcal{S}_n$ is generated exactly by these coefficients of $\mathcal{S}_{n+1}$ which appear in the image of $\mathcal{A}_n$ under α_{n+1}. This, in conjunction with the previous lemma, suggests the strategy for computing relevant quantum isometry groups inductively. Suppose that we have determined the generators of $\mathcal{S}_n$. Then $\mathcal{S}_{n+1}$ is generated by generators of $\mathcal{S}_n$ and these of the $A_{\text{aut}}(\mathcal{A}_n, \omega_\xi)$, with the only additional relations provided by the Eq. (5.2.1).

This will be used below to determine the concrete form of relations determining $\mathcal{S}_n$ for commutative AF algebras.

Lemma 5.2.6 *Let A be a commutative AF algebra. Suppose that $\mathcal{A}_n$ is isomorphic to $\mathbb{C}^m$ and the embedding of $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ is given by a sequence $(l_i)_{i=1}^m$. Let $m' = \sum_{i=1}^m l_i$. Suppose that the 'copy' of $A_s(m)$ in $\mathcal{S}_n$ is given by the family of projections $a_{i,j}$ $(i,j \in \{1,\dots m\})$ and that the "copy" of $A_s(m')$ in $\mathcal{S}_{n+1}$ is given by the family of projections $a_{(i,r_i),(j,s_j)}$ $(i,j \in \{1,\dots,m\},\ r_i \in \{1,\dots,l_i\},\ s_j \in \{1,\dots,l_j\})$. Then the formula (5.2.1) is equivalent to the following system of equalities:*

$$a_{i,j} = \sum_{r_i=1}^{l_i} a_{(j,s_j),(i,r_i)} \tag{5.2.2}$$

for each $i,j \in \{1,\dots,m\}$, $s_j \in \{1,\dots,l_j\}$.

Proof For the universal coaction $\alpha : \mathcal{A}_n \to \mathcal{A}_n \otimes \mathcal{S}_n$, we have

$$\alpha(\widetilde{e_i}) = \sum_{j=1}^m \widetilde{e_j} \otimes a_{j,i}$$

where by $\widetilde{e_i}$ we denote the image of the basis vector $e_i \in \mathcal{A}_n$ in $\mathcal{A}_{n+1}$. As $\widetilde{e_j} = \sum_{r_i=1}^{l_i} e_{(i,r_i)}$,

$$\alpha(\widetilde{e_i}) = \sum_{r_i=1}^{l_i} \alpha(e_{i,r_i}) = \sum_{r_i=1}^{l_i}\sum_{j=1}^{m}\sum_{s_j=1}^{l_j} e_{(j,s_j)} \otimes a_{(j,s_j),(i,r_i)}.$$

On the other hand, we have

$$\alpha(\widetilde{e_i}) = \sum_{j=1}^{m}\sum_{s_j=1}^{l_j} e_{(j,s_j)} \otimes a_{j,i},$$

and the comparison of the formulas above yields precisely (5.2.2). □

As each AF algebra can be described via its Bratteli diagram, it is natural to ask whether the construction in this paper can be compared to the one in [2].

Let us now restrict our attention to a truncation of a Bratteli diagram (up to n-th level, say), of a commutative *AF* algebra. The set of vertices V is a disjoint union of sets V_1, ..., V_n with V_1 being a singleton, and there exist surjective maps $\pi_j : V_j \to V_{j-1}$ $(j \geq 2)$ determining the graph structure. Denote by π the map from

V to V defined by the formulas $\pi|_{V_j} = \pi_j$ for $j \geq 2$, and $\pi = \text{id}$ on V_1. Then $\pi^* : C(V) \to C(V)$ is a C^*-homomorphism, with $\pi^*|_{C(V_j)}$ injective for each $j \leq n-1$. The corresponding graph is obtained by joining $\pi_{i+1}(v) \in V_i$ with $v \in V_{i+1}$ for each $i = 1, \ldots, n$, $v \in V_i$.

Denote by $D(V)$ the diagonal subalgebra of $C(V) \otimes C(V)$, i.e. the span of $\{\delta_v \otimes \delta_v : v \in V\}$. Since the map $m(s_* \otimes t_*)$ is onto, $C(E) \cong (C(V) \otimes C(V))/\text{Ker}(m(s_* \otimes t_*))$. Indeed, for the graph corresponding to the commutative AF algebra described above, $C(E)$ is isomorphic to the subalgebra $(\text{id} \otimes \pi^*)(D(V))$ of $C(V) \otimes C(V)$, and the condition on β in the above definition of a quantum symmetry of the graph (V, E) amounts to saying that $\alpha^{(2)}$ leaves the algebra $\mathcal{C} := (\text{id} \otimes \pi^*)(D(V))$ invariant, in which case β is, up to the obvious identification, nothing but the restriction of $\alpha^{(2)}$ on $\mathcal{C}$.

In other words, an equivalent description of the objects in the category of the quantum symmetry of such a finite commutative (i.e. all the matrix algebras in the vertices are one-dimensional) Bratteli diagram is obtained from the observation that they correspond precisely to these CQG coactions α on $C(V)$ for which $\mathcal{C}$ is left invariant by $\alpha^{(2)}$. This leads to the following result:

Theorem 5.2.7 *Let A be a commutative AF algebra. Then the quantum isometry group $\mathcal{S}_n$ described in the beginning of this section coincides with the quantum symmetry group of the graph given by the restriction of the Bratteli diagram of A to the n-th level.*

Proof Suppose first that we are given a quantum isometry of the canonical spectral triple on the respective 'finite' part of the AF algebra in question, so that we have a CQG coaction $(\mathcal{S}, \alpha)$ on $C(V)$ such that each $\alpha_j \equiv \alpha|_{C(V_j)}$ leaves $C(V_j)$ invariant ($j = 1, \ldots, n$) and that α commutes with the embeddings π_j, that is

$$\alpha_{j+1}{\pi_{j+1}}^* = ({\pi_{j+1}}^* \otimes \text{id})\alpha_j.$$

We deduce that

$$\begin{aligned}\alpha^{(2)}_{j+1}(\text{id} \otimes {\pi_{j+1}}^*) &= (\text{id}_2 \otimes m_{\mathcal{S}})\sigma_{23}(\alpha_{j+1} \otimes \alpha_{j+1}{\pi_{j+1}}^*) \\ &= (id_2 \otimes m_{\mathcal{S}})\sigma_{23}(\alpha_{j+1} \otimes ({\pi_{j+1}}^* \otimes \text{id})\alpha_j) = ({\pi_{j+1}}^* \otimes \text{id})\alpha^{(2)}_j.\end{aligned}$$

Using the above expression and the fact that $\alpha^{(2)}$ leaves $D(V)$ invariant (by the second part of Lemma 1.3.6), we see that $\alpha^{(2)}$ leaves $\mathcal{C}$ invariant.

Conversely, we need to show that a coaction of a CQG on the Bratteli diagram induces a quantum symmetry of the corresponding part of a spectral triple on the AF algebra. Let $(\mathcal{S}, \alpha)$ be a coaction on $C(V)$ such that $\alpha^{(2)}$ leaves $\mathcal{C}$ invariant. It follows from the discussion before the lemma that we have the corresponding coaction β on $C(E)$. Therefore we can start with a coaction α on the Bratteli diagram such that $\alpha^{(2)}|_{C(V_n)}$ preserves $(\text{id} \otimes {\pi_n}^*)(D(V_{n-1}))$. We first show by induction that α leaves each $C(V_j)$ invariant. Consider $j = n$ first. Observe that $C(V_n)$ is nothing but $\text{Ker}(\Psi)$, where $\Psi : C(V) \to C(E)$ is the map $f \mapsto (m \circ (s^* \otimes t^*))(f \otimes 1)$. It is clear from the definition of a quantum symmetry of a graph that α will leave this subalgebra invariant. As α is a unitary w.r.t. the counting measure on $C(V)$, it follows that it will leave $C(V_1 \cup ... \cup V_{n-1})$ invariant as well, and thus restricts to a quantum symmetry of the reduced graph obtained by deleting V_n and the corresponding edges. Then the inductive arguments complete the proof that each $C(V_j)$ is left invariant.

The proof will now be complete if we can show that

$$\alpha_{m+1}\pi^*_{m+1} = (\pi^*_{m+1} \otimes \mathrm{id})\alpha_m \tag{5.2.3}$$

for each $m = 1, \ldots, n-1$. Let $V_m = \{v^m_1, \ldots, v^m_{t_m}\}$, and let $q_{m,lj}$ be elements of $\mathcal{S}$ such that

$$\alpha(\delta_{v^m_l}) = \sum_j \delta_{v^m_j} \otimes q_{m,lj}.$$

We set $\Lambda_j = \{m : \pi(m) = j\}$. Then we have

$$\begin{aligned}\alpha^{(2)}(\mathrm{id} \otimes \pi^*_{m+1})(\delta_{v^m_i} \otimes \delta_{v^m_i}) &= \alpha^{(2)}(\delta_{v^m_i} \otimes \sum_{k\in\Lambda_i} \delta_{v^{m+1}_k}) \\ &= \sum_{j,p}\sum_{k\in\Lambda_i} \delta_{v^m_j} \otimes \delta_{v^{m+1}_p} \otimes q_{m,ij}q_{m+1,kp}.\end{aligned}$$

Furthermore, for all $p \notin \Lambda_j$

$$\sum_{k\in\Lambda_i} q_{m,ij}q_{m+1,kp} = 0.$$

Multiplying by $q_{m+1,k'p}$, where $k' \in \Lambda_i$, we obtain

$$\forall_{p\notin\Lambda_j, k\in\Lambda_i} \quad q_{m,ij}q_{m+1,kp} = 0. \tag{5.2.4}$$

As stated after Lemma 1.3.6, $\alpha^{(2)}$ leaves the ortho-complement of the diagonal algebra in $l^2(V \times V)$ invariant. This means that if $k \notin \Lambda_i$ then $\alpha^{(2)}(\delta_{v^m_i} \otimes \delta_{v^{m+1}_k})$ belongs to $C(D^c) \otimes \mathcal{S}$. On the other hand

$$\alpha^{(2)}(\delta_{v^m_i} \otimes \delta_{v^{m+1}_k}) = \sum_{j,p} \delta_{v^m_j} \otimes \delta_{v^{m+1}_p} \otimes q_{m,ij}q_{m+1,kp},$$

hence

$$\forall_{k\notin\Lambda_i, p\in\Lambda_j} \quad q_{m,ij}q_{m+1,kp} = 0. \tag{5.2.5}$$

Further

$$\alpha_{m+1}\pi^*_{m+1}(\delta_{v^m_i}) = \sum_j \delta_{v^{m+1}_j} \otimes \sum_{k\in\Lambda_i} q_{m+1,kj}.$$

We also have

$$(\pi_{m+1}^* \otimes \mathrm{id})\alpha_m(\delta_{v_i^m}) = \sum_j \delta_{v_j^{m+1}} \otimes \sum_{r:\, j\in\Lambda_r} q_{m,ir} = \sum_j \delta_{v_j^{m+1}} \otimes q_{m,i\pi(j)}.$$

Finally we get

$$\begin{aligned}\sum_{k\in\Lambda_i} q_{m+1,kp} &= \sum_{k\in\Lambda_i}(\sum_j q_{m,ij})q_{m+1,kp} = \sum_{k\in\Lambda_i}(\sum_j q_{m,ij}q_{m+1,kp})\\ &= \sum_{k\in\Lambda_i} q_{m,i\pi(p)}q_{m+1,kp} = \sum_k q_{m,i\pi(p)}q_{m+1,kp} - \sum_{k\notin\Lambda_i} q_{m,i\pi(p)}q_{m+1,kp}\\ &= q_{m,i\pi(p)},\end{aligned}$$

where in the third equality we have used (5.2.4) and in the final equality we have used (5.2.5) as well as the relation $\sum_k q_{m+1,kp} = 1$. This shows that (5.2.3) holds and the proof is complete. □

The above result justifies the statement that the quantum isometry groups of Christensen-Ivan triples on *AF* algebras provide natural notions of quantum symmetry groups of the corresponding Bratteli diagrams. The theorem could be proved directly by comparing the commutation relations of [2] with these listed in Lemma 5.2.6, but the method we gave has the advantage of being more functorial and transparent.

5.2.2 *The Example of the Middle-Third Cantor Set*

Now we use the results proved in the previous subsections to compute the quantum isometry group for Connes's spectral triple related to the Cantor set.

In the special case when A is the commutative AF algebra of continuous functions on the middle-third Cantor set we can use the observations of the last section to provide explicit description of the quantum isometry groups $\mathcal{S}_n$, and therefore also of $\mathrm{QISO}^+(\mathcal{A}, \mathcal{H}, D)$. Note that several variants of the spectral triple we consider here were studied in [11] and in [12], where its construction is attributed to the unpublished work of Connes.

Theorem 5.2.8 *Let A be the AF algebra arising as a limit of the unital embeddings*

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \cdots .$$

Suppose that the state ω_ξ is the canonical trace on $\mathcal{A}$. Then $\mathcal{S}_1 = C(\mathbb{Z}_2)$ and for $n \in \mathbb{N}$

$$\mathcal{S}_{n+1} = (\mathcal{S}_n \star \mathcal{S}_n) \oplus (\mathcal{S}_n \star \mathcal{S}_n).$$

Proof Begin by noticing that for each $n \in \mathbb{N}$ we have $\mathcal{A}_n = \mathbb{C}^{2^n}$ and therefore we can use a natural multi-index notation for the indexing sets discussed in the paragraph after Lemma 5.2.6. To be precise, for each $n \in \mathbb{N}$ we denote by τ_n the set $\{i_1 i_2 \cdots i_n : i_j \in \{1, 2\} \text{ for } j = 1, \ldots, n\}$. Multi-indices in $\tau := \bigcup_{n\in\mathbb{N}} \tau_n$ will be denoted by capital letters $I, J, \ldots$ and let the basis of the algebra $\mathcal{A}_n$ be indexed by elements of τ_n. Hence, the basis vectors of $\mathcal{A}_n$ will be denoted by e_I, where I belongs to τ_n. Moreover, as in Lemma 5.2.6, for I in τ_n, $\widetilde{e_I}$ will denote the image of e_I in $\mathcal{A}_{n+1}$.

Then the natural embeddings of $\mathcal{A}_n$ in $\mathcal{A}_{n+1}$ can be conveniently described by the formula

$$e_I \longrightarrow e_{I1} + e_{I2}, \quad I \in \tau_n,$$

where we use the standard concatenation of multi-indices.

To understand the quantum isometry group, we start with $\mathcal{A}_1$. As $\mathcal{A}_0 = \mathbb{C}$, the invariance condition on the embedding means simply that the coaction is unital and we see that $\mathcal{S}_1$ is simply the universal quantum group acting on 2 points, i.e., $C(\mathbb{Z}_2)$. If we denote a_{11} by p, then the unitary matrix corresponding to $\{a_{I,J} : I, J \in \tau_1\}$ looks as follows:

$$\begin{pmatrix} p & p^{\perp} \\ p^{\perp} & p \end{pmatrix}.$$

Now we look at the coaction on $\mathcal{A}_n$. For I in τ_n, we write $\alpha(e_I) = \sum_{J\in\tau_n} e_J \otimes a_{JI}$. Thus, for the coaction on the $(n+1)$-th level, we write for each I in τ_n and $j = 1, 2$,

$$\alpha(e_{Ij}) = \sum_{K\in\tau_n, l=1,2} e_{Kl} \otimes a_{Kl,Ij}.$$

Thus, $\alpha(\widetilde{e_I}) = \alpha(\sum_{j=1}^{2} e_{Ij}) = \sum_{j=1}^{2} \sum_{K\in\tau_n,l=1,2} e_{Kl} \otimes a_{Kl,Ij} = \sum_{K\in\tau_n,l=1,2} e_{Kl} \otimes \sum_{j=1}^{2} a_{Kl,Ij}$.

Since the coaction on the $(n+1)$-th level has to preserve $\mathrm{Span}\{\widetilde{e_I} : I \in \tau_n\} = \mathrm{Span}\{\sum_{i=1}^{2} e_{Ii} : I \in \tau_n\}$, we deduce that for all $l, l' \in \{1, 2\}$ and for all $K, I \in \tau_n$, we have

$$\sum_{j=1}^{2} a_{Kl,Ij} = \sum_{j=1}^{2} a_{Kl',Ij}.$$

Hence

$$\alpha(\widetilde{e_I}) = \sum_{K\in\tau_n} \widetilde{e_K} \otimes \sum_{j=1}^{2} a_{K1,Ij}. \tag{5.2.6}$$

But

$$\alpha(\widetilde{e_I}) = \sum_{K} e_K \otimes a_{KI}. \tag{5.2.7}$$

Thus (5.2.6) and (5.2.7) together imply that

$$a_{KI} = \sum_{j=1}^{2} a_{K1,Ij}. \tag{5.2.8}$$

Applying the antipode κ on (5.2.8) and applying part 5 of Proposition 1.2.19, we deduce

$$a_{K1,I1} + a_{K2,I1} = a_{KI}. \tag{5.2.9}$$

Therefore, from (5.2.8) and (5.2.9), we have $a_{K1,I2} = a_{K2,I1}$.

Thus

$$\mathcal{S}_{n+1} = C^*\{a_{K1,I1}, a_{K1,I2}, a_{K2,I2} : I, K\tau_n\}. \tag{5.2.10}$$

Next, we prove that $\mathcal{S}_{n+1}$ is a quantum subgroup of $(\mathcal{S}_n * \mathcal{S}_n) \oplus (\mathcal{S}_n * \mathcal{S}_n)$. To start with, we claim that if $I, J \in \tau_n$ such that $I \neq J$, we have

$$a_{KI}a_{KJ} = 0 = a_{IK}a_{JK}. \tag{5.2.11}$$

This follows by induction on n. Let us suppose that we have proved the claim for all $k \leq n$. Since the coaction of $\mathcal{S}_{k+1}$ on the finite-dimensional commutative C^* algebra $\mathcal{A}_{k+1}$ is by quantum permutations, for all K in τ_k and $j = 1, 2$, we have

$$\sum_{I \in \tau_k, j=1,2} a_{Kl,Ij} = 1$$

by virtue of the relations of the quantum permutation groups as in Sect 1.3.1.

Moreover, by the relations of the quantum permutation group, each $a_{Kl,Ij}$ is a projection. Thus, we have a finite set of projections summing up to 1 and therefore $a_{Kl,Ij}a_{Kl,I'j'} = 0$ if $(Ij) \neq (I'j')$. This proves one of the equations of (5.2.11). The other equation follows similarly by using the other relations of the quantum permutation group.

The matrix associated with the coaction is given by

$$\begin{pmatrix} A_{K1,I1} & A_{K2,I1} \\ A_{K1,I2} & A_{K2,I2} \end{pmatrix}, \tag{5.2.12}$$

where $A_{Ki,Ij}$ denotes the matrix $((a_{Ki,Ij}))_{K,I\in\tau_n}$.

By an inductive argument on n, it is easy to see that for all I, K in τ_n and $i, j = 1, 2$, the matrix $A_{Ki,Ij}$ defines an isometric coaction on $\mathcal{A}_n$. Thus, $C^*\{a_{Ki,Ij} : I, K \in \tau_n\}$ is a quantum subgroup of $\mathcal{S}_n$. This observation taken together with (5.2.10) and (5.2.11) imply that $\mathcal{S}_{n+1}$ is a quantum subgroup of $(\mathcal{S}_n * \mathcal{S}_n) \oplus (\mathcal{S}_n * \mathcal{S}_n)$.

On the other hand, it is easy to see that the matrix in (5.2.12) defines an isometric coaction of $(\mathcal{S}_n * \mathcal{S}_n) \oplus (\mathcal{S}_n * \mathcal{S}_n)$ on $\mathcal{A}_{n+1}$. This completes the proof. □

It is clear from the proof of the Theorem 5.2.8 and also from the discussion before Lemma 5.2.6 that the quantum group coactions we consider are coactions on the tensor product of algebras preserving in some sense one of the factors. In the classical world such actions have to have a product form, as is confirmed by the following result which we state without proof (see [15] for the proof).

Proposition 5.2.9 *Suppose that* $(X, d_X), (Y, d_Y)$ *are compact metric spaces and* $T : X \times Y \to X \times Y$ *is an isometry satisfying the following condition:* $\alpha_T(C(X) \otimes 1_Y) \subset C(X) \otimes 1_Y$, *where* $\alpha_T : C(X \times Y) \to C(X \times Y)$ *is given simply by the composition with* T. *Then* T *must be a product isometry.*

Theorem 5.2.8 shows that Proposition 5.2.9 has no counterpart for quantum group coactions, even on classical spaces. We could have thought of elements of $\mathcal{S}_2$ as quantum isometries acting on the Cartesian product of 2 two-point set, "preserving" the first coordinate in the sense analogous to the one in the proposition above. If this forced elements of $\mathcal{S}_2$ to be product isometries, we would necessarily have $\mathcal{S}_2 = \mathcal{S}_1 \otimes \mathcal{S}_1$; in particular $\mathcal{S}_2$ would have to be commutative.

References

1. Wang, S.: Quantum symmetry groups of finite spaces. Commun. Math. Phys. **195**, 195–211 (1998)
2. Bichon, J.: Quantum automorphism groups of finite graphs. Proc. Am. Math. Soc. **131**(3), 665–673 (2003)
3. Banica, T.: Quantum automorphism groups of small metric spaces. Pac. J. Math. **219**(1), 27–51 (2005)
4. Banica, T.: Quantum automorphism groups of homogeneous graphs. J. Funct. Anal. **224**(2), 243–280 (2005)
5. Banica, T., Bichon, J.: Quantum automorphism groups of vertex-transitive graphs of order ≤ 11. J. Algebraic Comb. **26**, 83–105 (2007)
6. Banica, T., Bichon, J., Chenevier, G.: Graphs having no quantum symmetry. Ann. Inst. Fourier **57**, 955–971 (2007)
7. Banica, T., Bichon, J.: Quantum groups acting on 4 points. J. Reine Angew. Math. **626**, 75–114 (2009)
8. Banica, T., Bichon, J., Collins, B.: Quantum permutation groups: a survey. noncommutative harmonic analysis with applications to probability. Polish Acad. Sci. Inst. Math. **78**, 13–34, Banach Center Publications, Warsaw (2007)
9. Banica, T., Moroianu, S.: On the structure of quantum permutation groups. Proc. Am. Math. Soc. **135**(1), 21–29 (2007) (electronic)
10. Bichon, J.: Algebraic quantum permutation groups. Asian-Eur. J. Math. **1**(1), 1–13 (2008)
11. Christensen, E., Ivan, C.: Spectral triples for AF C^*-algebras and metrics on the Cantor set. J. Oper. Theory **56**(1), 17–46 (2006)
12. Christensen, E., Ivan, C.: Sums of two dimensional spectral triples. Math. Scand. **100**(1), 35–60 (2007)

13. Skalski, A., Soltan, P.M.: Projective limits of quantum symmetry groups and the doubling construction for Hopf algebras. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17**(2), 1450012 (27 pages) (2014)
14. Rieffel, M.A.: Metrics on state spaces. Doc. Math. **4**, 559–600 (1999)
15. Bhowmick, J., Goswami, D., Skalski, A.: Quantum isometry groups of 0-dimensional manifolds. Trans. Am. Math. Soc. **363**, 901–921 (2011)
16. Banica, T., Skalski, A.: Quantum symmetry groups Of C^*-algebras equipped with orthogonal filtrations. Proc. Lond. Math. Soc. **3**(5), 980–1004, 106 (2013)
17. Davidson, K.R.: C^* Algebras by Examples. Hindustan Book Agency (1996)
18. Connes, A.: Compact metric spaces, Fredholm modules, and hyperfiniteness. Ergodic Theory Dynam. Syst. **9**(2), 207–220 (1989)

Chapter 6
Nonexistence of Genuine Smooth CQG Coactions on Classical Connected Manifolds

Abstract The question of existence of coactions of Hopf algebras and compact quantum groups on connected Riemannian manifolds is investigated. We give a very brief outline of the proof of some important cases of the conjecture as well as a number of examples supporting the conjecture.

The goal of this chapter is to look into the quantum isometric coactions, and more generally, smooth coactions (in a sense to be precisely stated later) of compact quantum groups on the algebra of smooth functions $C^\infty(M)$ for arbitrary compact smooth manifolds M. As our computations presented in the previous chapters already indicate, when M is connected, we do not expect to have genuine quantum isometries. We can even go a step further and conjecture that there cannot be any faithful coaction of a genuine compact quantum group on $C^\infty(M)$ where M is a compact connected manifold and the coaction is smooth in a natural sense. At the moment the conjecture is still open. However, it has been proved in several important cases: for example, for isometric coactions as well as for smooth coactions of finite-dimensional compact quantum groups. We briefly discuss these results in this chapter. We do not want to present the proofs which involve too many technical details but sketch only the main ideas. For the details the reader is referred to the archived article [1] and [2]. In this chapter, we will also discuss a few results of Etingof and Walton on the same question in an algebraic context. For yet another example in the algebraic context, we refer to the work of [3] where the classical nodal cubic is shown to be a quantum homogeneous space of a noncommutative Hopf algebra.

6.1 Smooth Coaction of a Compact Quantum Group and the No-Go Conjecture

6.1.1 Definition of Smooth Coaction

Let us fix a compact smooth manifold M of dimension n and consider the natural Fréchet topology on $C^\infty(M)$. This is given by the family of seminorms $\{\| \cdot \|_{U,K,\underline{i}}\}$, where U is a local coordinate neighborhood with the corresponding local coordinates

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_6

$x_1, \ldots, x_n$ (say), $K \subset U$ compact, $\underline{i} = (i_1, \ldots, i_k)$ some multi-index with $1 \le i_j \le n$ for all j and

$$\|f\|_{U,K,\underline{i}} := \sup_{x \in K} |\frac{\partial}{\partial x_{i_1}} \ldots \frac{\partial}{\partial x_{i_k}}(f)|. \tag{6.1.1}$$

An action of a compact group G on M is called smooth if for every $g \in G$, the map $M \ni M \mapsto g.m$ is smooth. This is equivalent to saying that the corresponding $*$-automorphism of $C(M)$, which sends f to $\alpha_g(f)(m) := f(g.m)$, satisfies $\alpha_g(C^\infty(M)) \subseteq C^\infty(M)$ for all g. This gives a natural way to generalize the notion of smooth action for CQG coacting on $C(M)$. To do this, we need to consider the subalgebra $C^\infty(M, \mathcal{A})$ of the C^* algebra $C(M, \mathcal{A}) \cong C(M) \otimes \mathcal{A}$, where $\mathcal{A}$ is any unital C^* algebra. Here, $C^\infty(M, \mathcal{A})$ denotes the set of all $\mathcal{A}$-valued functions on M which are C^∞ with respect to the Fréchet norm of $\mathcal{A}$. This is a Fréchet space, having a $*$-algebra structure inherited from that of $C(M, \mathcal{A})$ and the $*$-algebra operations are easily seen to be continuous in the natural Fréchet topology of $C^\infty(M, \mathcal{A})$, given by the seminorms analogous to the ones appearing in the Eq. (6.1.1) but the absolute value on the right hand side replaced by the norm of $\mathcal{A}$. Indeed, it is known that $C(M)$ is *nuclear* as a C^* algebra, which means in particular that the C^* tensor product $C(M) \otimes \mathcal{A}$ is unique. A similar property is shared by $C^\infty(M)$ as a locally convex space, making $C^\infty(M, \mathcal{A})$ the unique tensor product of the locally convex space with $\mathcal{A}$, but we do not want to go into this issue which will lead to unnecessary technicalities.

Now we can introduce the notion of a smooth CQG coaction.

Definition 6.1.1 A coaction α of a CQG $\mathcal{Q}$ on $C(M)$ is called a smooth coaction on M if $\alpha(C^\infty(M)) \subseteq C^\infty(M, \mathcal{Q})$ and the linear span of elements of the form $\alpha(f)(1 \otimes q)$, $f \in C^\infty(M)$, $q \in \mathcal{Q}$, is dense in the Fréchet topology of $C^\infty(M, \mathcal{Q})$.

One can actually prove the following analogue of the Podles' condition [4] for coactions on C^* algebras

Proposition 6.1.2 ([1]) *Suppose that we are given a C^* coaction α of $\mathcal{Q}$ on $C(M)$. Then the following are equivalent:*

(1) α is smooth.
(2) $(\mathrm{id} \otimes \phi)\alpha(C^\infty(M)) \subset C^\infty(M)$ for every state ϕ on $\mathcal{Q}$, and there is a Fréchet dense subalgebra $\mathcal{A}$ of $C^\infty(M)$ over which α is algebraic.

The proof is very similar to the proof by Podles in [5] for C^*-coactions. All one has to do is to replace the norm topology by the Fréchet topology at every step and observe that the arguments go through.

6.1.2 *Statement of the Conjecture and Some Positive Evidence*

For a smooth manifold with at least four connected components, it is easy to see that the quantum permutation group of four objects, which is an infinite-dimensional genuine CQG, has a natural (faithful) coaction on the manifold. We have used the term "genuine' to mean that the underlying C^* algebra of the quantum group is noncommutative, i.e., it is not of the form $C(G)$ for some compact group G. However, for a connected manifold such a construction is not possible. But if one forgets smoothness, it is indeed possible to produce examples of genuine CQG coactions (faithful) on $C(X)$ where X is compact and connected. One can even have examples of such coactions, where X is a smooth manifold except at just one point. Let us give a couple of such examples before we state the conjecture about nonexistence of smooth faithful CQG coactions on smooth compact connected manifolds.

Our first example is adapted from P. Etingof and C. Walton (see Remark 4.3 of [6]). Let

$$X = \{(u, v) : \; -1 \leq u, v \leq 1, \; u = 0 \text{ or } v = 0\} \subseteq \mathbb{R}^2.$$

This is basically a compact subset of the algebraic variety given by the equation $xy = 0$. This is not irreducible and has a singularity at $(0, 0)$. The C^* algebra $C(X)$ is generated by the two coordinate functions, viewed as self-adjoint operators x, y satisfying $x^2 \leq 1$, $y^2 \leq 1$, $xy = 0$. Consider a faithful coaction α of $\mathcal{Q} := C^*(S_3)$, where S_3 is the group of permutations of three objects, given by

$$\alpha(x) = x \otimes \delta_a, \;\; \alpha(y) = y \otimes \delta_b.$$

Here, $a = (12)$, $b = (13)$ (transpositions) and δ_a, δ_b denote the corresponding characteristic functions viewed as elements of $\mathcal{Q}$.

The second example is due to Huichi Huang ([7] Example 3.10,(1)), where the space is again taken as a topological join. Let $Y = [0, 1]$, $X_N = \{1, 2, \ldots, N\}$ and $Y_1 = \{0\}$ in the notation of [7] and S be the topological join or wedge sum of N unit intervals identifying $(x_i, 0)$ for all $i = 1, \ldots, N$. We have a coaction of the quantum permutation group $A_s(N)$ given by

$$\alpha(f) = \sum_{ij} f \otimes e_i \otimes a_{ij} \text{ for } f \in C(Y),$$

where a_{ij} are the canonical generators of $A_s(N)$ as in Sect. 1.3.1. For $N \geq 4$, this is a faithful coaction of a genuine (i.e., noncommutative as a C^* algebra) CQG on a compact connected topological space. However, the space S is not a smooth manifold.

We can now state the following:

Conjecture I If a CQG $\mathcal{Q}$ faithfully coacts smoothly on a compact, connected, smooth manifold M then $\mathcal{Q}$ must be isomorphic with $C(G)$ for some compact group G acting smoothly on M.

Let us mention that so far, we have been able to verify this conjecture at least in the following two cases:

(i) When the CQG is finite dimensional.
(ii) When the coaction is isometric for some Riemannian structure on the manifold. On the other hand, there is a purely algebraic result obtained by Etingof and Walton in [6] which also gives some indication in favor of the truth of the conjecture.

This says the following:

Proposition 6.1.3 ([6]) *Let $\mathcal{Q}$ be a finite dimensional, semisimple Hopf algebra over an algebraically closed field of characteristic zero which has a faithful coaction α on a commutative domain $\mathcal{C}$. Then $\mathcal{Q}$ must be a commutative algebra.*

Note that in the work of Etingof-Walton, the term "inner faithful" was used, but as $\mathcal{Q}$ is finite dimensional, this coincides with our notion of faithfulness.

Let us conclude this subsection with some examples of Hopf algebras (of noncompact type) faithfully coacting on commutative domains, indicating that the conjecture made above is not expected to hold for locally compact quantum groups.

Example 6.1.4 Let $\mathcal{A} \equiv \mathbb{C}[x]$, and let $\mathcal{Q}_0$ be the universal algebra generated by a, a^{-1}, b subject to the following relations

$$aa^{-1} = a^{-1}a = 1, ab = q^2 ba,$$

where q is a parameter described in [8]. This has a Hopf algebra structure which corresponds to the quantum $ax + b$ group. There are at least two different (non-isomorphic) constructions of the analytic versions of this quantum group, i.e., as locally compact quantum groups in the sense of [9], one by Woronowicz [8] the other by Baaj-Skandalis [10] and Vaes-Vainermann [11]. The coproduct is given by (see [8] for details)

$$\Delta(a) = a \otimes a, \Delta(b) = a \otimes b + b \otimes 1.$$

We have a coaction $\alpha : \mathbb{C}[x] \to \mathbb{C}[x] \otimes \mathcal{Q}_0$ given by

$$\alpha(x) = x \otimes a + 1 \otimes b.$$

It is clearly an inner faithful coaction in the sense of Etingof and Walton.

Example 6.1.5 This is taken from Example 2.20. of [12]. Let k be a field, q be a nonzero element of k and let $\mathcal{G}_q$ be the universal unital algebra generated by a, b, c, d, D, D^{-1} subject to the following relations:
$ab = ba$, $cd = dc$, $db = q^{-1}bd$, $ac = qca$, $ad - bc = da - cb = da - q^{-1}bc = ad - qbc = D$. This has a Hopf algebra structure for which $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a fundamental corepresentation, i.e., the coproduct Δ is given by $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = c \otimes a + d \otimes c$, $\Delta(d) = c \otimes b + d \otimes d$, $\Delta(D) = D \otimes D$, $\Delta(D^{-1}) = D^{-1} \otimes D^{-1}$. The counit ϵ and antipode κ are given by

$$\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon(D) = \epsilon(D^{-1}) = 1,$$

$$\begin{pmatrix} \kappa(a) & \kappa(b) \\ \kappa(c) & \kappa(d) \end{pmatrix} = \begin{pmatrix} dD^{-1} & -bD^{-1} \\ -cD^{-1} & aD^{-1} \end{pmatrix}, \quad \kappa(D) = D^{-1}, \ \kappa(D^{-1}) = D.$$

For a generic value of the parameter $q \neq 1$, this is known to be a noncommutative and non-cocommutative, infinite-dimensional Hopf algebra. It has a (inner faithful) coaction α on the commutative domain $k[x, y]$, i.e., the classical 2-plane, given by

$$\alpha(x) = x \otimes a + y \otimes c, \quad \alpha(y) = x \otimes b + y \otimes d.$$

Remark 6.1.6 Note that the quotient of $\mathcal{G}_q$ by the ideal generated by D, D^{-1} is not the same as $SL_q(2)$, even if there seems to be some similarity of certain defining relations. For example, the commutativity of a, b, and c, d is quite different from the relations among the natural generators of $SL_q(2)$.

6.1.3 Defining the 'Differential' of the Coaction

If a compact group G acts on a compact smooth manifold M by diffeomorphism, one has a natural G-action on the bundle of one (and higher) forms obtained by the differential dg, where $g \in G$ also denotes the diffeomorphism on M. To put it in an algebraic language, we consider the Fréchet space of the form $\Omega^1(M, \mathcal{C})$ for a C^* algebra $\mathcal{C}$, to be called the space of $\mathcal{C}$-valued smooth one forms, consisting of smooth, $C^\infty(M)$-linear maps from the set $\chi(M)$ of smooth vector fields on M to $C^\infty(M, \mathcal{C})$. Here, the $C^\infty(M)$-module structure on $C^\infty(M, \mathcal{C})$ is given by $(f \cdot F)(m) = f(m)F(m)$, for $f \in C^\infty(M)$, $F \in C^\infty(M, \mathcal{C})$. There is a natural Fréchet topology on $\Omega^1(M, \mathcal{C})$ determined by the following: a sequence Ω_k converges to an element Ω in this topology if and only if for every compactly supported smooth vector field X, $\Omega_k(X)$ converges to $\Omega(X)$ in the Fréchet topology of $C^\infty(M, \mathcal{C})$. Moreover, any $\Omega \in \Omega^1(M, \mathcal{C})$ is determined by $\Omega(m) \in T_m^*(M) \otimes_{\mathrm{alg}} \mathcal{C}$, $m \in M$, which is defined as follows. Given a tangent vector τ at m, choose any smooth vector field X such that $X(m) = \tau$ and then define $\Omega(m) = \Omega(X)(m) \in \mathcal{C}$. By standard arguments using $C^\infty(M)$-linearity of Ω, one can verify that this definition is independent of the choice of X satisfying $X(m) = \tau$. Thus, $\Omega^1(M, \mathcal{C})$ can be viewed as the space of smooth sections of a bundle over M with the fiber at m given by $T_m^*(M) \otimes_{\mathrm{alg}} \mathcal{C}$.

For a smooth vector field X on M, we have a unique, Fréchet-continuous map $\tilde{X}$ on $C^\infty(M, \mathcal{C})$ satisfying $\tilde{X}(f \otimes a) = X(f) \otimes a$, for $f \in C^\infty(M)$, $a \in \mathcal{C}$. Indeed, it can be easily defined in terms of integral curves of X. For $m \in M$, let $\gamma(t)$ be the integral curve for X such that $\gamma(0) = m$, for all t in some interval containing 0. Then, for any $F \in C^\infty(M, \mathcal{C})$,

$$\tilde{X}(F)(m) := \frac{d}{dt}|_{t=0} F(\gamma(t)).$$

Using this, we have a map $\tilde{d} : C^\infty(M, \mathcal{C}) \to \Omega^1(M, \mathcal{C})$ given by

$$\tilde{d}(F)(X) := \tilde{X}(F).$$

It is easily seen that $\tilde{d}$ satisfies the following Leibniz rule

$$\tilde{d}(FG)(m) = \tilde{d}(F)(m)G(m) + F(m)\tilde{d}(G)(m).$$

The smooth G-action on M can be encoded by a map Γ from $\Omega^1(M)$ to $\Omega^1(M, C(G))$. Before we introduce Γ, let us note that $\Omega^1(M, C(G))$ can be identified with $\Omega^1(M)$-valued continuous functions on G. Indeed, for $\Omega \in \Omega^1(M, C(G))$, we have $\Omega(m) \in T_m^* M \otimes_{\rm alg} C(G)$, so evaluating it at g gives a cotangent vector in $T_m^* M$. Let us denote this $\Omega^1(M)$-valued function on G by Ω_g, i.e., $\Omega_g(m) = \Omega(m)(g) \in T_m^* M$. Now, we define $\Gamma(\omega)_g := dg(\omega)$, where dg is the G-action on $\Omega^1(M)$ given by the differential of the diffeomorphism g on M. Clearly, $\Gamma(\phi d\psi)(g) = \alpha_g(\phi) d(\alpha_g(\psi))$, where $\psi, \phi \in C^\infty(M)$ and $\alpha_g(f)(m) = f(gm)$. It is natural to use the notation $d\alpha$ for the map Γ. The space $\Omega^1(M, C(G)) \equiv C(G, \Omega^1(M))$ has a $C^\infty(M)$-bimodule structure given by $(\Phi \cdot f)(g) = (f \cdot \Phi)(g) = \alpha_g(f)\Phi(g)$. Clearly, $\Gamma \equiv d\alpha$ is a bimodule-morphism from $\Omega^1(M)$ to $C(G, \Omega^1(M))$.

We can now ask a similar question for a smooth coaction, say α, of a CQG $\mathcal{Q}$ on a smooth manifold M. That is, we consider $\Gamma(df) := \tilde{d}(\alpha(f))$ and want to extend it to a well-defined $C^\infty(M)$-bimodule morphism from $\Omega^1(M)$ to $\Omega^1(M, \mathcal{Q})$, where the module structure on $\Omega^1(M, \mathcal{Q})$ is given by $f \cdot \Phi = \alpha(f)\Phi$, $\Phi \cdot f = \Phi\alpha(f)$, for $f \in C^\infty(M)$, $\Phi \in \Omega^1(M, \mathcal{Q})$. However, we have $(df)\phi = \phi df$ in $\Omega^1(M)$, so a necessary condition for Γ to be well defined is

$$\tilde{d}(\alpha(f))\alpha(h) = \alpha(h)\tilde{d}(\alpha(f)) \ \ \forall f, h \in C^\infty(M). \tag{6.1.2}$$

Now, we do have the commutativity of $\alpha(f)(m)$ and $\alpha(h)(m)$ for all m, as $(\mathrm{ev}_m \otimes \mathrm{Id}) \circ \alpha$ is a homomorphism from a commutative algebra to $\mathcal{A}$. However, the condition (6.1.2) requires also the commutativity of the partial derivatives of $\alpha(f)$ at the point m with the value of $\alpha(h)$ at m. As the partial derivatives of $\alpha(f)$ at m possibly depend on values of $\alpha(f)(x)$ for x other than m, in a small enough neighborhood, it is not so clear why they would commute with $\alpha(h)(m)$. To illustrate it further, consider one parameter smooth curve $\gamma(t)$ in M with $\gamma(0) = m$ and consider the Taylor expansion of the $\mathcal{Q}$-valued function $t \mapsto F(t) := [\alpha(f)(\gamma(t)), \alpha(h)(m)]$, where $[A, B] = AB - BA$. We have $F(0) = 0$ but the condition (6.1.2) would imply $F(t)$ has no first-order term, i.e., $F'(0) = 0$.

In fact, (6.1.2) turns out to be also sufficient, as stated in the theorem below, for the well-defined extension of Γ as a bimodule morphism.

Theorem 6.1.7 *The following are equivalent:*
(i) There is a well-defined Fréchet continuous map $\beta : \Omega^1(M) \to \Omega^1(M, \mathcal{Q})$ *which is a* $C^\infty(M)$ *bimodule morphism, i.e.,* $\beta(f\omega) = \alpha(f)\beta(\omega)$ *and* $\beta(\omega f) = \beta(\omega)\alpha(f)$ *for all* $\omega \in \Omega^1(M)$ *and* $f \in C^\infty(M)$ *and also* $\beta(df) = (d \otimes \mathrm{id})(\alpha(f))$ *for all* $f \in C^\infty(M)$.
(ii)

$$(\nu \otimes \mathrm{id})(\alpha(f))\alpha(g) = \alpha(g)(\nu \otimes \mathrm{id})(\alpha(f)) \tag{6.1.3}$$

for all $f, g \in C^\infty(M)$ *and all smooth vector fields* ν *on* M.

We do not give the proof here, which is somewhat long and technical, and refer the reader to [1].

Let us say that α is *liftable* if the conditions of the above theorem hold. We make the following

Conjecture II Any smooth coaction of a CQG on a compact smooth manifold is liftable.

Let us remark that the above conjecture does not hold even for algebraic coactions of Hopf algebras of noncompact type on smooth manifolds. To see this, consider the algebra $\mathcal{A} = \mathbb{C}[x]$ of Example 6.1.4 which is the algebro–geometric analogue of $C^\infty(\mathbb{R})$ and there is a canonical derivation $\delta : \mathcal{A} \to \mathcal{A}$ corresponding to the vector field $\frac{d}{dt}$ of $\mathbb{R}$

$$\delta(p) = p',$$

where p' denotes the usual derivative of the polynomial p. However, one has

$$(\delta \otimes \mathrm{id})\alpha(x) = 1 \otimes a,$$

which does not commute with $\alpha(x)$ as $ab \neq ba$.

6.2 Brief Sketch of Proof of Nonexistence of Genuine Quantum Isometries

We give here a very brief outline of the following result:

Theorem 6.2.1 *Let* $\mathcal{Q}$ *be a CQG which has a faithful isometric coaction on a compact connected Riemannian manifold* M. *Then* $\mathcal{Q} \cong C(G)$, *where* G *is a subgroup of the group of (classical) isometries of* M.

We refer to [1] for the details of the arguments. Before we go into the steps of the proof, let us recall a few geometric facts. We say that a manifold is parallelizable if it admits a set $\{X_1, \ldots, X_n\}$ of globally defined smooth vector fields such that at each point $m \in M$, $\{X_1(m), \ldots, X_n(m)\}$ is a basis of $T_m(M)$. It is known that the bundle of orthogonal frames of a Riemannian manifold is always parallelizable. Moreover, any parallelizable Riemannian manifold admits an isometric embedding in some Euclidean space such that the corresponding normal bundle is trivial. We refer the reader to [13] for a proof of this fact and related discussion.

The key idea of proof of this theorem consists of a chain of arguments to lift the given CQG coaction to a subset of $\mathbb{R}^n$ (for large enough n), which has nonempty interior and where the lifted coaction is affine in the Euclidean coordinates. To explain the motivation of these steps, let us first think of a similar line of arguments in the classical setup, i.e., a proof of the fact that any smooth action by a compact group G on a classical compact connected manifold M can be made affine with respect to Euclidean coordinates of some $\mathbb{R}^n$ where M is embedded. This can be done as follows:

(a) Using an averaging trick we reduce the problem to the case of isometric coaction.
(b) We can assume without loss of generality that M is parallelizable and hence has an isometric embedding in some $\mathbb{R}^n$ with a trivial normal bundle $N(M)$, by passing to the parallelizable total space $T(O(M))$ of the orthonormal frame bundle $O(M)$ where the G-action on M can be lifted to the natural isometric action by taking the differential dg of the map $g \in G$.
(c) We can lift the action to an isometric action on the total space $T(N)$ of the trivial normal bundle $N(M)$.
(d) Finally, noting that $T(N)$ is locally isometric to the flat Euclidean space $\mathbb{R}^n$, we can argue that any isometric action on it must be affine in the corresponding coordinates.

The proof in the quantum case is a natural (but rather nontrivial due to noncommutativity at every stage) adaptation of the above lines of arguments in the framework of isometric CQG coaction.

As an immediate corollary of Theorem 4.1.2, we get the following

Lemma 6.2.2 *Let W be a subset with nonempty interior of some Euclidean space $\mathbb{R}^N$. If a faithful coaction α of a CQG $\mathcal{Q}$ on X is affine in the sense that α leaves the linear span of the coordinate functions and the constant function* 1 *invariant, then $\mathcal{Q} \cong C(G)$ for some group G.*

Now, let us describe briefly the main steps of the proof of Theorem 6.2.1.

Fix a faithful, isometric coaction α of a CQG $\mathcal{Q}$ on a compact-connected Riemannian manifold M. Just like $\Omega^1(M, \mathcal{Q})$ described before, we can consider the spaces of $\mathcal{Q}$-valued smooth k-forms, denoted by $\Omega^k(M, \mathcal{Q})$. Clearly, these are $C^\infty(M, \mathcal{Q})$-bimodules in a natural way. Moreover, there is a natural Hilbert-$C^\infty(M)$ structure on $\Omega^k(M)$ coming from the Riemannian structure. Consider the $C^\infty(M, \mathcal{Q})$-module structure on $\Omega^k(M, \mathcal{Q})$ given by the $C^\infty(M, \mathcal{Q})$-valued inner product $\langle \cdot, \cdot \rangle_{C^\infty(M,\mathcal{Q})}$ defined as follows:

$$\langle \theta, \eta \rangle_{C^\infty(M,\mathcal{Q})}(m) := \langle \theta(m), \eta(m) \rangle_{\mathcal{Q}},$$

where $\theta, \eta \in \Omega^k(M, \mathcal{Q})$ viewed as smooth sections of a bundle with the fiber at m being $\Lambda^k(T_m^* M) \otimes \mathcal{Q}$ and $\langle \cdot, \cdot \rangle_{\mathcal{Q}}$ denotes the natural $\mathcal{Q}$-valued inner product on $T_m^* M \otimes \mathcal{Q}$.

Let $\chi(M)$ denote the space of smooth vector fields on M and ∇ denote the Levi-Civita connection for the given Riemannian metric.

Step 1: From the isometry condition, it is easy to verify that the coaction is liftable. Hence we get the commutativity of the algebra $\mathcal{Q}_m$ for every fixed m, where $\mathcal{Q}_m$ is generated by $(X \otimes \mathrm{id})(\alpha(f))(m), \alpha(g)(m)$ for $f, g \in C^\infty(M)$, $X \in \chi(M)$. It is also easily seen that the lift $\Gamma = d\alpha$ is equivariant in the sense that

$$\langle \Gamma(\omega), \Gamma(\eta) \rangle_{C^\infty(M,\mathcal{Q})} = \alpha(\langle \omega, \eta \rangle_{C^\infty(M)}) \ \ \forall\, \omega, \eta \in \Omega^1(M).$$

Step 2: Using the lift $\Gamma = d\alpha$, we can further construct equivariant $\Gamma^k := d\alpha^{(k)}$ on $\Omega^k(M)$ which is a bimodule morphism in the sense discussed before. Moreover, for each k, Γ^k is equivariant for the natural $C^\infty(M)$-valued inner product of $\Omega^k(M)$. The Riemannian metric gives a conjugate linear $C^\infty(M)$-module isomorphism between the set $\Omega^1(M)$ of smooth one forms and the set $\chi(M)$ of smooth vector fields, so we can view Γ as a corepresentation on $\chi(M)$ as well.

Step 3: Using isometry, we show that the Levi-Civita connection is “preserved" in the following sense: $\Gamma(\nabla_X(Y)) = \tilde{\nabla}_{\Gamma(X)}(\Gamma(Y))$. Here, ∇ is the covariant derivative operator corresponding to the Levi-Civita connection and $\tilde{\nabla}$ is the extension of ∇ on (a suitable topological tensor product) $\chi(M)\hat{\otimes}\mathcal{Q}$ satisfying $\tilde{\nabla}_{X_1 \otimes q_1}(X_2 \otimes q_2) := \nabla_{X_1}(X_2) \otimes q_1 q_2$. This implies $\alpha(\phi \Gamma_{ij}^k) \in \mathcal{Q}_m$, where ϕ is any smooth function supported in the domain of some coordinate chart for which the Christoffel symbols are denoted by Γ_{ij}^k.

Step 4: With little more calculation, this further implies a “second order commutativity”: for each $m \in M$ and local coordinates $(x_1, \ldots, x_n)$ around m, the algebra generated by $\{\alpha(f)(m), \frac{\partial}{\partial x_i}(\alpha(g))(m), \frac{\partial^2}{\partial x_i \partial x_j}(\alpha(h))(m) : f, g, h \in C^\infty(M)\}$ is commutative.

Step 5: Commutativity of $\mathcal{Q}_m$ further allows us to prove that Γ^k naturally induces a $*$-homomorphic coaction $\tilde{\alpha}$ (say) on $C^\infty(T(O_M))$, where $T(O_M)$ is the total space of the orthogonal frame bundle on M, identifying $C^\infty(T(O_M))$ with a suitable completion of the symmetric algebra of the $C^\infty(M)$-module $\Omega^1(M)$. The second-order commutativity of α implies first order commutativity for the lift $\tilde{\alpha}$, hence using an averaging trick and other arguments, we get a Riemannian structure on E for which $\tilde{\alpha}$ is isometric.

Step 6: As E is parallelizable, it has an embedding in some $\mathbb{R}^m$ with trivial normal bundle (with respect to the Riemannian metric chosen above) $N(E)$ (say). We can lift $\tilde{\alpha}$ as an isometric coaction Φ on some suitable $\epsilon(> 0)$-neighborhood W of E in the total space $N \equiv \mathbb{R}^m$ of $N(E)$.

Step 7: It now suffices to prove that Φ is affine, so that Lemma 6.2.2 will apply. We will actually use connectedness of M for the first time here. We give some details of this step which is interesting in its own right. A different argument was given in [1] for proving this step. Let $y_1, \ldots y_m$ be the usual coordinates of $W \subseteq \mathbb{R}^m$. As Φ

commutes with the Laplacian $\mathcal{L}_{\mathbb{R}^N} := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ and $\mathcal{L}_{\mathbb{R}^N} \frac{\partial}{\partial y_j} = \frac{\partial}{\partial y_j} \mathcal{L}_{\mathbb{R}^N}$, $\mathcal{L}_{\mathbb{R}^N} y_j = 0$ for all j, we get

$$(\mathcal{L}_{\mathbb{R}^N} \otimes id)(\frac{\partial}{\partial y_j} \otimes id)\Phi(y_i) = (\frac{\partial}{\partial y_j} \otimes id)\Phi(\mathcal{L}_{\mathbb{R}^N} y_i) = 0.$$

Let $D_{ij}(y) = ((\frac{\partial}{\partial y_i} \otimes id)\Phi(y_j))(y)$. Note that as $d\Phi$ is a Φ-equivariant unitary corepresentation, so $D(y) := ((D_{ij}(y)))_{i,j=1,\ldots,N}$ is unitary for all $y \in W$. Pick y_0 in the interior of W(which is non empty). Then the new $\mathcal{Q}$ valued matrix $G(y) := ((G_{ij}(y))) = D(y)(D(y_0))^{-1}$ is unitary since $D(y)$ is so. As $G(y)$ is unitary for all y, we have $|\psi(G_{ij}(y))| \leq 1$ and $|\psi(G_{ii}(y_0))| = 1$, where ψ is a state on $\mathcal{Q}$. Since M is connected, $\mathrm{Int}(W)$ is connected. Thus $\psi(G_{ii}(y))$ is a harmonic function on the open connected set $\mathrm{Int}(W)$ which attains the supremum of its absolute value at an interior point. Hence by Corollary 1.9 of [14] we conclude that $\psi(G_{ii}(y)) = \psi(G_{ii}(y_0)) = 1$. As this is true for every state ψ on $\mathcal{Q}$, we have $G_{ii}(y) = 1$. But $G(y)$ is a unitary, so $G_{ii}(y) = 1$ implies $G_{ij}(y) = \delta_{ij} 1$ for all i, j. Since the matrix $((G_{ij}(y)))$ is unitary for all y, we get $G_{ij} = \delta_{ij}.1_{\mathcal{Q}}$. Then $D(y)(D(y_0))^{-1} = 1_{M_N(\mathcal{Q})}$, i.e., $D(y) = D(y_0)$ for all $y \in W$. Hence Φ is affine with $q_{ij} = D_{ij}(y_0)$. This completes the proof of Theorem 6.2.1.

6.3 An Example of No-Go Result Without Quadratic Independence

We have taken this example from the unpublished dissertation thesis by Promit Ghosal with his kind permission, who did this work under the supervision of the second author. Here, we give an example of a non-smooth, algebraic variety in $\mathbb{R}^3$ generated by a set of quadratically dependent coordinates, for which an analogue of the no-go result, i.e., Theorem 6.2.1 holds. For this, we need yet another approach [15] to quantum isometries of classical spaces, formulated in the setup of a compact metric space which will be briefly discussed in Chap. 10. In particular, the existence of a quantum group analogue of the isometry group of a compact metric space (X, d) is proved for a large class of (X, d) including those which are isometrically embedded in $\mathbb{R}^n$ for some n. This quantum group is denoted by $\mathrm{QISO}^{\mathrm{metric}}(X, d)$.

We consider the compact conical surface $\mathcal{M} = \{(x, y, z) : x^2 + y^2 = z^2, |z| \leq 1\} \subseteq \mathbb{R}^3$. Let X_1, X_2, X_3 be the restriction of the Euclidean coordinates of $\mathbb{R}^3$ to M. Let α be the coaction of $\mathcal{Q} = \mathrm{QISO}^{\mathrm{metric}}(X, d)$ where d denotes the restriction of the Euclidean metric on $\mathbb{R}^3$. It is known (see Chap. 10 and [15]) that α must be of the form given by (10.5.2) of Chap. 10, i.e., $\alpha(X_i) = \sum_j X_j \otimes q_{ji} + 1 \otimes r_i$, where q_{ij}, r_i are self-adjoint elements such that the matrices $((q_{ij}))_{i,j}$ and $((q_{ji}))_{i,j}$ are unitary elements in $M_n(\mathcal{Q})$ and the C^* algebra generated by $\{q_{ij}, r_i : i, j\}$ is $\mathcal{Q}$.

Before proceeding further, let us make the convention of denoting the symmetrized product $a_1 a_2 + a_2 a_1$ by the symbol $\overline{a_1 a_2}$. Using the relation

$$X_1^2 + X_2^2 = X_3^2, \tag{6.3.1}$$

we can obtain the following.

Lemma 6.3.1 *We have*

$$\begin{aligned} q_{11}^2 + q_{12}^2 - q_{13}^2 &= q_{12}^2 + q_{22}^2 - q_{23}^2 \\ &= -q_{31}^2 - q_{32}^2 + q_{33}^2, \end{aligned} \tag{6.3.2}$$

$$\begin{aligned} \overline{q_{11}r_1} + \overline{q_{12}r_2} &= \overline{q_{13}r_3}, \\ \overline{q_{21}r_1} + \overline{q_{22}r_2} &= \overline{q_{23}r_3}, \\ \overline{q_{31}r_1} + \overline{q_{32}r_2} &= \overline{q_{33}r_3}, \end{aligned} \tag{6.3.3}$$

$$\begin{aligned} \overline{q_{11}q_{21}} + \overline{q_{12}q_{22}} &= \overline{q_{13}q_{23}}, \\ \overline{q_{21}q_{31}} + \overline{q_{22}q_{32}} &= \overline{q_{23}q_{23}}, \\ \overline{q_{31}q_{11}} + \overline{q_{32}q_{12}} &= \overline{q_{33}q_{13}}. \end{aligned} \tag{6.3.4}$$

Proof It can be easily checked that $\{1, X_1, X_2, X_3, X_1X_2, X_2X_3, X_3X_1\}$ are linearly independent. Equating the coefficients of X_1^2, X_2^2 and X_3^2 on both sides of the homogeneous relation $X_1^2 + X_2^2 = X_3^3$, we get (6.3.2). Similarly equating coefficients of X_i's and X_iX_j for $i \neq j$, we can obtain (6.3.3) and (6.3.4), respectively. □

Lemma 6.3.2 *We have the following:*

$$q_{ki}q_{lj} + q_{li}q_{kj} = q_{kj}q_{li} + q_{lj}q_{kj} \quad \textit{for } k \neq l, i \neq j \tag{6.3.5}$$

$$q_{ki}r_j + r_iq_{kj} = q_{kj}r_i + r_jq_{ki} \quad \textit{for } i \neq j \tag{6.3.6}$$

$$q_{ki}q_{kj} + q_{3i}q_{3j} = q_{kj}q_{ki} + q_{3j}q_{3i} \quad \textit{for } k = 1, 2, i \neq j \tag{6.3.7}$$

Proof Equating the coefficients of $1, X_1, X_2, X_3, X_1X_2, X_2X_3, X_3X_1$ on both sides of $\alpha(X_iX_j) = \alpha(X_jX_i)$ along with the use of (6.3.1) on the relation $\alpha(X_iX_j) = \alpha(X_jX_i)$ would give the result stated above. □

Lemma 6.3.3 *For all i, $r_i = 0$, i.e., α is linear.*

Proof Let μ be a probability measure on $\mathcal{M}$ such that the corresponding state on $C(M)$ is α-invariant. Then

$$r_i.1_{\mathcal{Q}} = (\int X_i d\mu)1_{\mathcal{Q}} - \sum_j (\int X_j d\mu)q_{ji}.$$

We claim that for all $j = 1, 2, 3$, $\int X_j d\mu = 0$ which proves the lemma. To this end, we note that we have a natural $\mathbf{Z}_2$ action on $\mathcal{M}$ which sends X_i to $-X_i$. Clearly, this is an isometry. Therefore, $C(\mathbf{Z}_2)$ is identified with a quantum subgroup of $\mathcal{Q}$. Then, the α invariant state for $\mathcal{Q}$ must be preserved by this $\mathbf{Z}_2$ action too. This implies that $\int X_j d\mu = -\int X_j d\mu$, i.e., $\int X_j d\mu = 0$ for all j. □

We now prove the main result of this section.

Theorem 6.3.4 *The quantum isometry group $\mathcal{Q}$ is commutative as C^*-algebra.*

Proof As $\mathcal{Q}$ is a quantum subgroup of $A_o(3)$ via the surjective morphism sending the canonical generators of $A_o(3)$ to q_{ij}'s, we have $\kappa(q_{ij}) = q_{ji}$, where κ is the antipode of $\mathcal{Q}$. Applying this to (6.3.4), we get the following new relations:

$$\overline{q_{1i}q_{1j}} + \overline{q_{2i}q_{2j}} = \overline{q_{3i}q_{3j}} \quad \text{for } i \neq j. \tag{6.3.8}$$

On the other hand, rewriting (6.3.7) for $k = 1$, we obtain

$$q_{1i}q_{1j} - q_{1j}q_{1i} = q_{3j}q_{3i} - q_{3i}q_{3j}. \tag{6.3.9}$$

By subtracting (6.3.9) from (6.3.8), we will get

$$q_{1i}q_{1j} + \frac{1}{2}\overline{q_{2i}q_{2j}} = q_{3j}q_{3i}. \tag{6.3.10}$$

Interchanging the role of i and j, one can get the following relation:

$$q_{1j}q_{1i} + \frac{1}{2}\overline{q_{2i}q_{2j}} = q_{3i}q_{3j}. \tag{6.3.11}$$

Let us recall the fact that

$$q_{1i}q_{1j} + q_{2i}q_{2j} + q_{3i}q_{3j} = 0. \tag{6.3.12}$$

Plugging the relations obtained from (6.3.11) into (6.3.12), we have

$$\overline{q_{1i}q_{1j}} + \frac{1}{2}\overline{q_{2i}q_{2j}} = -q_{2i}q_{2j}. \tag{6.3.13}$$

But it can observed that the left-hand side of (6.3.13) is symmetric with respect to i, j. Hence, q_{2i} and q_{2j} will commute with each other. But utilizing (6.3.7), we can further write down

$$q_{1i}q_{1j} - q_{1j}q_{1i} = q_{2i}q_{2j} - q_{2j}q_{2i} = q_{3j}q_{3i} - q_{3i}q_{3j}.$$

Hence we have the following relations:

$$q_{ki}q_{kj} = q_{kj}q_{ki}, \tag{6.3.14}$$

whereas after applying κ on both sides that we can further conclude that

$$q_{ik}q_{jk} = q_{jk}q_{ik}. \tag{6.3.15}$$

On applying κ on (6.3.5), we have the following:

$$q_{jl}q_{ik} + q_{jk}q_{il} = q_{il}q_{jk} + q_{ik}q_{jl} \quad \text{for } k \neq l, i \neq j. \tag{6.3.16}$$

Interchanging the role of k and i and also of l, j in (6.3.5) gives

$$q_{ik}q_{jl} + q_{jk}q_{il} = q_{il}q_{jk} + q_{jl}q_{ik} \quad \text{for } k \neq l, i \neq j. \tag{6.3.17}$$

Now from (6.3.16)–(6.3.17), one can have

$$[q_{jl}, q_{ik}] = [q_{ik}, q_{jl}] \quad \text{for } k \neq l, i \neq j,$$

which implies

$$q_{jl}q_{ik} = q_{ik}q_{jl} \quad \text{for } k \neq l, i \neq j.$$

Hence $\mathcal{Q}$ is commutative as a C^*- algebra. □

Remark 6.3.5 From the proof of the theorem above, it is clear that there is nothing special about $n = 3$, and the result holds for a cone in $\mathbb{R}^{n+1}$ defined by the equation $\beta_1 X_1^2 + \cdots + \beta_{n-1} X_{n-1}^2 = X_n^2$, where $\beta_1, \beta_2, \ldots \beta_{n_1}$ are positive constants.

Remark 6.3.6 The cones given by $\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 = x_{n+1}^2$ are irreducible algebraic sets, i.e., the corresponding coordinate algebra is a commutative domain. However, we cannot apply Proposition 6.1.3 because the quantum group is not finite dimensional. Thus, in a sense, the above computations give an indication that the results of Etingof and Walton may hold beyond finite-dimensional Hopf algebras.

References

1. Goswami, D., Joardar, S.: Rigidity of action of compact quantum groups on compact connected manifolds. arxiv:1309.1294v4
2. Goswami, D., Joardar, S.: An averaging trick for smooth actions of compact quantum groups on manifolds. Indian J. Pure Appl. Math. **46**(4), 477–488 (2015)
3. Krahmer, U., Tabiri, A.: The nodal cubic is a quantum homogeneous space. arXiv: 1604.00199
4. Podles', P.: Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups. Commun. Math. Phys. **170**, 1–20 (1995)
5. Podles', P.: Quantum spheres. Lett. Math. Phys. **14**, 193–202 (1987)
6. Etingof, P., Walton, C.: Semisimple Hopf actions on commutative domains. Adv. Math. **251**, 47–61 (2014)
7. Huang, H.: Faithful compact quantum group actions on connected compact metrizable spaces. J. Geom. Phys. **70**, 232–236 (2013)

8. Woronowicz, S.L., Zakrzewski, S.: Quantum $ax + b$ group. Rev. Math. Phys. **14**, Nos 7 & 8, 797–828 (2002)
9. Kustermans, J., Vaes, S.: Locally compact quantum groups. Ann. Sci. Ecole Norm. Sup. (4) **33**(6), 837–934 (2000)
10. Baaj, S., Skandalis, G.: Duality for locally compact quantum groups, Mathematisches Forschungsinstitut Oberwolfach, Tagungsbericht 46/1991, C^*-algebren (1991)
11. Vaes, S., Vainerman, L.: Extensions of locally compact quantum groups and the bicrossed product construction. Adv. Math. **175**(1), 1–101 (2003)
12. Walton, C., Wang, X.: On quantum groups associated to non-Noetherian regular algebras of dimension 2. arXiv:1503.0918
13. Kolsinsky, A.A.: Differential Manifolds. Academic Press (1993)
14. Axler, S., Bourdon, P., Ramey, W.: Harmonic Function Theory, Graduate Texts in Mathematics, vol. 137, 2nd edn. Springer, New York (2001)
15. Goswami, D.: Existence and examples of quantum isometry groups for a class of compact metric spaces. Adv. Math. **280**, 340–359 (2015)

Chapter 7
Deformation of Spectral Triples and Their Quantum Isometry Groups

Abstract This chapter deals with $\mathrm{QISO}^{\mathcal{L}}$ and QISO^{+}_{R} of a cocycle twisted noncommutative manifold. We first discuss the cocycle deformation of compact quantum groups, von Neumann algebras and spectral triples, followed by the proof of the fact that QISO^{+}_{R} and $\mathrm{QISO}^{\mathcal{L}}$ of a cocycle twist of a (noncommutative) manifold is a cocycle twist of the QISO^{+}_{R} and $\mathrm{QISO}^{\mathcal{L}}$ (respectively) of the original (undeformed) manifold.

Most of the examples of noncommutative manifolds are obtained by deforming classical spectral triples. In particular, the spectral triples on the noncommutative torus discussed in Sect. 2.2.1 fall in this class. It is a natural question whether the quantum isometry group of the deformed noncommutative manifold is related to the quantum isometry group of the original manifold. In the context of quantum automorphism groups, such a question was addressed in [1]. The aim of this chapter is to show that the quantum isometry group of a cocycle-deformed noncommutative manifold can be obtained by a similar deformation of the quantum isometry group of the original (undeformed) noncommutative manifold. Combining this with the fact that the quantum isometry group of a classical compact, connected, Riemannian manifold is the same as the classical isometry group of such manifold, it follows that the quantum isometry group of non-commutative manifolds obtained from classical manifolds using dual unitary 2-cocycles are now known. We should also mention that Sadeleer has extended the results of [2] to the setup of monoidal deformations in the paper [3].

7.1 Cocycle Twisting

We formulate a notion of deformation by dual unitary 2-cocycles in the framework of von Neumann algebras. In fact, this can be thought of as a special (and simpler) case of the C^*-algebraic deformation theory developed by Neshveyev and Tuset in [4] for coactions of locally compact quantum groups. Basically, the von Neumann algebra obtained by deformation in our sense will be the closure in the weak operator topology of the corresponding C^*-algebraic deformation obtained by the theory of

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_7

Neshveyev and Tuset. However, we must mention that our theory differs from that of [4] in two aspects: (i) the quantum groups considered by us are universal CQG's whereas those in [4] are in the reduced form and (ii) coactions in our setup are not necessarily C^*-coactions.

7.1.1 Cocycle Twist of a Compact Quantum Group

Let $\mathcal{Q}$ be a compact quantum group with the dual discrete quantum group $\widehat{\mathcal{Q}}$ and the canonical dense Hopf $*$-algebra $\mathcal{Q}_0$. Recall from Sect. 1.2.2 the dense Hopf $*$-algebra $\mathcal{Q}_0$ spanned by the matrix coefficients of its inequivalent irreducible corepresentations. We now define a dual unitary 2-cocycle.

Definition 7.1.1 A dual unitary 2-cocycle on a CQG $\mathcal{Q}$ is a unitary element $\sigma \in \mathcal{M}(\widehat{\mathcal{Q}} \otimes \widehat{\mathcal{Q}})$ satisfying

$$(1 \otimes \sigma)(\mathrm{id} \otimes \widehat{\Delta})(\sigma) = (\sigma \otimes 1)(\widehat{\Delta} \otimes \mathrm{id})(\sigma).$$

The cocycle σ is said to be normalized if $(1 \otimes p_\epsilon)\sigma = 1 \otimes p_\epsilon$ and $(p_\epsilon \otimes 1)\sigma = p_\epsilon \otimes 1$, where p_π's are the minimal projections of $\oplus_{\pi \in \mathrm{Rep}(\mathcal{Q})} M_{d_\pi}$. It is possible to assume without loss of generality that a cocycle is normalized. We will do so in the rest of this chapter.

If we view σ as a linear functional on $\mathcal{Q}_0 \otimes_{alg} \mathcal{Q}_0$ it will satisfy the following cocycle condition (see p. 64 of [5]):

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c),$$

for $a, b, c \in \mathcal{Q}_0$. We can deform $\mathcal{Q}_0$ using σ to obtain a new Hopf $*$-algebra $\mathcal{Q}_0^\sigma$. The algebra multiplication, involution, and antipode κ are given by the following formulas, whereas the coproduct remain unchanged. For $a, b \in \mathcal{Q}_0$,

$$\begin{aligned} a \cdot_\sigma b &:= \sigma^{-1}(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma(a_{(3)}, b_{(3)}), \\ a^{*_\sigma} &:= (v_\sigma^{-1} \otimes \mathrm{id} \otimes v_\sigma)(\Delta \otimes \mathrm{id})\Delta(a^*) \\ \kappa_\sigma(a) &:= W_\sigma(a_{(1)})\kappa(a_{(2)})W_\sigma^{-1}(a_{(3)}). \end{aligned} \tag{7.1.1}$$

where

$$W_\sigma(a) = \sigma(a_{(1)}, \kappa(a_{(2)})), \quad v_\sigma(a) = W_\sigma(\kappa^{-1}(a)). \tag{7.1.2}$$

The reader is referred to the p. 65 of [5] and [3] for a proof of the fact that, $\mathcal{Q}_0^\sigma$ with the operations defined above indeed becomes a Hopf-$*$ algebra.

We observe that the unit of the deformed Hopf $*$-algebra is the same as before. It follows from the following:

$$(a \cdot_\sigma 1) = \sigma^{-1}(a_{(1)(1)}, 1)a_{(1)(2)}\sigma(a_{(2)}, 1).$$

As the cocycle is normalized, we have

$$\sigma(a, 1) = \sigma^{-1}(a, 1) = \epsilon(a)$$

for all $a \in \mathcal{Q}_0$, implying $a._\sigma 1 = \epsilon(a_{(1)(1)})a_{(1)(2)}\epsilon(a_{(2)}) = a$. The identity $1._\sigma a = a$ for all $\mathcal{Q}_0$ can be shown in a similar way.

Remark 7.1.2 As pointed out by Sadeleer in [3], the formula for the involution on the deformed algebra presented in [2] was incorrect. The formula we have given here takes care of the correction noted in [3].

We recall from Sect. 1.4 the discrete quantum group $\widehat{\mathcal{Q}_0}$ with the coproduct $\widehat{\Delta}$ defined by $\widehat{\Delta}(\omega)(a \otimes b) := \omega(ab)$, $w \in \widehat{\mathcal{Q}_0}$ *and* $a, b \in \mathcal{Q}_0$. This can also be deformed using $\sigma \in M(\widehat{\mathcal{Q}} \otimes \widehat{\mathcal{Q}})$ by keeping the $*$ algebraic structure unchanged while changing the coproduct by the one given below

$$\widehat{\Delta}_\sigma(.) = \sigma\widehat{\Delta}(.)\sigma^{-1}.$$

The coassociativity of $\widehat{\Delta}_\sigma$ follows from the cocycle condition of σ. Indeed, $((\widehat{\mathcal{Q}_0})^\sigma, \widehat{\Delta}_\sigma)$ turns out to be a discrete quantum group and hence we get a dual CQG by the Proposition 3.12 and discussions preceeding Proposition 4.5 of [6]. It is also clear that the natural pairing between $\mathcal{Q}_0^\sigma$ and $\widehat{\mathcal{Q}_0^\sigma}$ is non-degenerate. In fact, any CQG having $(\widehat{\mathcal{Q}_0})^\sigma$ as the dual discrete quantum group will contain $\mathcal{Q}_0^\sigma$ as a dense Hopf $*$-algebra. Hence it makes sense to consider the universal CQG corresponding to $\mathcal{Q}_0^\sigma$.

Definition 7.1.3 Call the universal CQG containing $\mathcal{Q}_0^\sigma$ as a dense Hopf $*$-algebra to be the cocycle twist of the CQG by a dual unitary 2-cocycle σ and denote it by $(\mathcal{Q}^\sigma, \Delta)$.

Next we explain how to get dual unitary 2-cocycles on a CQG from similar cocycles on quantum subgroups of it. Given two CQG's $\mathcal{Q}_1$, $\mathcal{Q}_2$ and a surjective CQG morphism $\pi : \mathcal{Q}_1 \to \mathcal{Q}_2$ which identifies $\mathcal{Q}_2$ as a quantum subgroup of $\mathcal{Q}_1$, one can prove that $\pi((\mathcal{Q}_1)_0) = (\mathcal{Q}_2)_0$. Dualizing, we get a map $\widehat{\pi} : (\mathcal{Q}_2)'_0 \to (\mathcal{Q}_1)'_0$. It is not difficult to see that the image of the dense multiplier Hopf $*$-algebra $(\mathcal{Q}_2)'_0 \subset \widehat{\mathcal{Q}_2}$ under $\widehat{\pi}$ coincides with $(\mathcal{Q}_1)'_0$. In fact, we get an extension of $\widehat{\pi}$, to be denoted by the same symbol again, as a non-degenerate $*$-homomorphism from $\mathcal{M}(\widehat{\mathcal{Q}_2})$ to $\mathcal{M}(\widehat{\mathcal{Q}_1})$. For a dual unitary 2-cocycle σ on $\mathcal{Q}_2$, we define $\sigma' := (\widehat{\pi} \otimes \widehat{\pi})(\sigma) \in \mathcal{M}(\widehat{\mathcal{Q}_1} \otimes \widehat{\mathcal{Q}_1})$ and verify easily that σ' is a dual unitary 2-cocycle on $\mathcal{Q}_1$. By a slight notational abuse, we will often denote σ' by σ.

Lemma 7.1.4 *$\mathcal{Q}_2^\sigma$ is a quantum subgroup of $\mathcal{Q}_1^{\sigma'}$.*

Proof We begin with the observation that $\pi : (\mathcal{Q}_1)_0^{\sigma'} \to (\mathcal{Q}_2)_0^\sigma$ is a surjective Hopf $*$-algebra morphism. Since the coproducts do not change, it suffices to verify that π is a $*$-algebra homomorphism. To this end, observe that for $a, b \in (\mathcal{Q}_1)_0^{\sigma'}$,

$$\begin{aligned}\pi(a._{\sigma'}b) &= \pi[\sigma'(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}(\sigma')^{-1}(a_{(3)}, b_{(3)})] \\ &= \sigma'(a_{(1)}, b_{(1)})\pi(a_{(2)}, b_{(2)})(\sigma')^{-1}(a_{(3)}, b_{(3)}) \\ &= \sigma(\pi(a_{(1)}), \pi(b_{(1)}))\pi(a_{(2)})\pi(b_{(2)})\sigma^{-1}(\pi(a_{(3)}), \pi(b_{(3)})) \\ &= \pi(a)._{\sigma}\pi(b).\end{aligned}$$

It can be shown in a similar way that $\pi(a^{*_{\sigma'}}) = (\pi(a))^{*_\sigma}$. Hence $\mathcal{Q}_1^{\sigma'}$ contains $(\mathcal{Q}_2)_0^\sigma$ as a Hopf $*$-algebra and by the fact that $\mathcal{Q}_1^{\sigma'}$ is a universal CQG, we conclude that there is a surjective CQG morphism from $\mathcal{Q}_1^{\sigma'}$ to $\mathcal{Q}_2^\sigma$. □

Lemma 7.1.5 *Let σ be a dual unitary 2-cocycle on a universal CQG $\mathcal{Q}$. Then we have $(\mathcal{Q}^\sigma)^{\sigma^{-1}} \cong \mathcal{Q}$.*

Proof As $\mathcal{Q}$ is universal, it suffices to show that there exists a Hopf-$*$ algebra isomorphism from $(\mathcal{Q}_0^\sigma)^{\sigma^{-1}}$ to $\mathcal{Q}_0$. Indeed, for $a, b \in \mathcal{Q}_0$.

$$\begin{aligned}a(._{\sigma})_{\sigma^{-1}}b &= \sigma^{-1}(a_{(1)}, b_{(1)})a_{(2)}._{\sigma}b_{(2)}\sigma(a_{(3)}, b_{(3)}) \\ &= \sigma^{-1}(a_{(1)(1)}, b_{(1)(1)})a_{(1)(2)}._{\sigma}b_{(1)(2)}\sigma(a_{(2)}, b_{(2)}) \\ &= \sigma^{-1}(a_{(1)(1)}, b_{(1)(1)})\sigma(a_{(1)(2)(1)(1)}, b_{(1)(2)(1)(1)})a_{(1)(2)(1)(2)}b_{(1)(2)(1)(2)} \\ &\quad \sigma^{-1}(a_{(1)(2)(2)}, b_{(1)(2)(2)})\sigma(a_{(2)}, b_{(2)}) \\ &= \sigma^{-1}(a_{(1)(1)}, b_{(1)(1)})\sigma(a_{(1)(2)}, b_{(1)(2)})a_{(2)(1)}b_{(2)(1)} \\ &\quad \sigma^{-1}(a_{(2)(2)(1)}, b_{(2)(2)(1)})\sigma(a_{(2)(2)(2)}, b_{(2)(2)(2)}) \\ &= \epsilon(a_{(1)})\epsilon(b_{(1)})a_{(2)(1)}b_{(2)(1)}\epsilon(a_{(2)(2)})\epsilon(b_{(2)(2)}) \\ &= \epsilon(a_{(1)})\epsilon(b_{(1)})a_{(2)}b_{(2)} \\ &= ab.\end{aligned}$$

The fact that the $(*_\sigma)^{\sigma^{-1}} = *$ can be argued in a similar way. □

7.1.2 Unitary Corepresentations of a Twisted Compact Quantum Group

Consider a universal CQG $\mathcal{Q}$ and a dual unitary 2-cocycle σ on it. Our goal in this subsection is to discuss (proofs briefly sketched) some standard facts about the corepresentations and the Haar state of $\mathcal{Q}^\sigma$.

Proposition 7.1.6 *(i) For* $\pi \in \mathrm{Rep}(\mathcal{Q})$*, we get an irreducible unitary corepresentation π_σ of $\mathcal{Q}^\sigma$, which coincides with π as a linear map.*
(ii) $\mathrm{Rep}(\mathcal{Q}^\sigma) = \{\pi_\sigma | \pi \in \mathrm{Rep}(\mathcal{Q})\}$.

Proof (i) This is a consequence of the fact that the C^* tensor categories of unitary corepresentations of $\mathcal{Q}$ and $\mathcal{Q}^\sigma$ are isomorphic (see, e.g. [7]). However, let us prove it using the isomorphism of $\widehat{\mathcal{Q}_0}$ and $(\widehat{\mathcal{Q}_0})^\sigma$ as $*$-algebras with $\oplus_{\pi \in \mathrm{Rep}(\mathcal{Q})} M_{d_\pi}$. Recall the functionals m_{pq}^π. We have

$$
\begin{aligned}
& < \kappa_\sigma(u^\pi_{ij})^{*_\sigma}, m^\pi_{pq} > \\
&= \overline{< u^\pi_{ij}, m^{\pi *}_{pq} >} \\
&= \overline{< u^\pi_{ij}, m^\pi_{qp} >} (since\ the\ *\ structure\ does\ not\ change) \\
&= \delta_{iq}\delta_{jp}.
\end{aligned}
$$

But we know $< u^\pi_{ji}, m^\pi_{pq} > = \delta_{iqjp}$. Hence by non-degeneracy of the pairing, we have $\kappa_\sigma(u_{ij}) = u^{*_\sigma}_{ji}$, i.e. π_σ is again a unitary corepresentation of $\mathcal{Q}^\sigma$. □

Let U be a (possibly infinite dimensional) unitary corepresentation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$, which is decomposed into the spectra subspaces, say $\mathcal{H} = \oplus_{\pi \in \mathrm{Rep}(\mathcal{Q}), i} \mathcal{H}^\pi_i$. Define U_σ by $U_\sigma|_{\mathcal{H}^\pi_i} = \pi_\sigma$ for all i. Clearly, U_σ is a unitary corepresentation of $\mathcal{Q}^\sigma$ and $U \mapsto U_\sigma$ gives a one-to-one onto correspondence.

Proposition 7.1.7 *Let h and h_σ be the Haar states of $\mathcal{Q}$ and $\mathcal{Q}^\sigma$ respectively. Then we have $h = h_\sigma$ on the common dense algebra $\mathcal{Q}_0$.*

Proof This follows from the fact that the coalgebra structures of $\mathcal{Q}$ and $\mathcal{Q}^\sigma$ are isomorphic. From Proposition 7.1.6 we note that the matrix coefficients of π_σ ($\pi \in \mathrm{Rep}(\mathcal{Q})$) are $\{u^\pi_{ij} : i, j = 1, \ldots, d_\pi\}$, hence $h_\sigma(u^\pi_{ij}) = h(u^\pi_{ij}) = 0$ if π is nontrivial and 1 for the trivial corepresentation. □

7.1.3 Deformation of a von Neumann Algebra by Dual Unitary 2-Cocycles

Given a dual unitary 2-cocycle σ on a CQG $\mathcal{Q}$ and a unitary corepresentation V of $\mathcal{Q}$ on some Hilbert space $\mathcal{H}$, along with a subspace $\mathcal{N}$ of $\mathcal{H}$ which is dense and $V(\mathcal{N}) \subseteq \mathcal{N} \otimes_{\mathrm{alg}} \mathcal{Q}_0$, we define a new $*$-homomorphism on the spectral subalgebra $\mathcal{M}_0$ (which is SOT-dense in $\mathcal{M}$ and $\mathrm{ad}_V(\mathcal{M}_0) \subseteq \mathcal{M}_0 \otimes_{\mathrm{alg}} \mathcal{Q}_0$ by Proposition 1.5.1) on $\mathcal{N}$ given by the following:

$$\rho_\sigma(b)(\xi) := b_{(0)}\xi_{(0)}\sigma^{-1}(b_{(1)}, \xi_{(1)}),\ \ for\ \xi \in \mathcal{N},$$

where $\mathrm{ad}_V(b) = b_{(0)} \otimes b_{(1)}$ and $V(\xi) = \xi_{(0)} \otimes \xi_{(1)}$ in Sweedler type notation.

Lemma 7.1.8 *(1) For all $b \in \mathcal{M}_0$, the linear map $\rho_\sigma(b)$ extends to a bounded operator on $\mathcal{H}$.*
(2) $\rho_\sigma P_\pi$ is SOT continuous, where P_π denotes the spectral projection corresponding to $\pi \in \mathrm{Rep}(\mathcal{Q})$.

Proof Let $\mathrm{ad}_V(b) = \sum_{i=1}^k b^i_{(0)} \otimes b^i_{(1)}$ and $V(\xi) = \sum_{j=1}^l \xi^j_{(0)} \otimes \xi^j_{(1)}$, where $k, l \geq 1$ are integers. Define $\sigma_i \in \mathcal{M}(\widehat{\mathcal{Q}})$ by $\sigma_i(q_0) := \sigma^{-1}(b^i_{(1)}, q_0)$ for all $i = 1, \ldots, k$. By Theorem 1.4.1, we have $\Pi_V(\sigma_i) \in \mathcal{B}(\mathcal{H})$ for all $i = 1, \ldots, k$. By definition of ρ_σ on $\mathcal{N}$,

$$\rho_\sigma(b)\xi = \sum_{i=1}^{k} b^i_{(0)} \Pi_V(\sigma_i)(\xi).$$

Using the facts that $\Pi_V(\sigma_i)$ and $b^i_{(0)}$'s are in $\mathcal{B}(\mathcal{H})$ we observe that $\rho_\sigma(b) \in \mathcal{B}(\mathcal{H})$ for all $b \in \mathcal{M}_0$, which proves (1).
We omit the proof of (2) which is very similar. □

Definition 7.1.9 We call $\rho_\sigma(\mathcal{M}_0)$ and $(\rho_\sigma(\mathcal{M}_0))''$ the deformations of $\mathcal{M}_0$ and $\mathcal{M}$ respectively and denote them by $\mathcal{M}_0^\sigma$ and $\mathcal{M}^\sigma$ respectively.

Remark 7.1.10 It is clear from (2) of Lemma 7.1.8 that $\rho_\sigma(\mathcal{B})'' = \rho_\sigma(\mathcal{M}_0)''$ if $\mathcal{B} \subseteq \mathcal{M}_0$ is a SOT dense $*$-subalgebra of $\mathcal{M}_0$.

7.2 Deformation of Spectral Triples by Unitary Dual Cocycles

Fix a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type, a positive, possibly unbounded operator R on $\mathcal{H}$ such that $RD = DR$, an object $(\mathcal{Q}, V)$ of $\widetilde{\mathbf{Q}'_{\mathbf{R}}}(D)$ and a dual unitary 2-cocycle σ on $\mathcal{Q}$. As $\mathcal{Q}$ is a quantum subgroup of $\widetilde{\mathrm{QISO}_R^+}(D)$, we have a dual unitary 2-cocycle, say σ', on $\widetilde{\mathrm{QISO}_R^+}(D)$ induced by σ as discussed before. Let U be the unitary corepresentation of $\widetilde{\mathrm{QISO}_R^+}(D)$ on $\mathcal{H}$. Then $U(D \otimes I) = (D \otimes I)U$. Henceforth we will denote the von Neumann subalgebra $(\mathcal{A}^\infty)'' \subseteq \mathcal{B}(\mathcal{H})$ by $\mathcal{M}$. As in Sect. 3.2.2, we consider the decomposition of the Hilbert space $\mathcal{H}$ into the eigenspaces $\mathcal{H}_k$, $k \geq 1$ of D and the further decomposition of each $\mathcal{H}_k$ into spectral subspaces:

$$\mathcal{H}_k = \oplus_{\pi \subseteq \mathcal{I}_k} \mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}}, \tag{7.2.1}$$

where $\mathcal{I}_k \subseteq \mathrm{Rep}(\mathcal{Q})$ and $m_{\pi,k}$ is the multiplicity of the d_π-dimensional irreducible corepresentation π in $\mathcal{I}_k$. As $(\mathrm{id} \otimes \phi)\mathrm{ad}_U(\mathcal{M}) \subseteq \mathcal{M}$ for every bounded linear functional ϕ on $\widetilde{\mathrm{QISO}_R^+}(D)$, the corresponding spectral subalgebra is SOT dense. Moreover, if $\mathcal{A}_0 \subseteq \mathcal{A}$ is any subalgebra on which ad_U is algebraic, one may consider its deformation by a dual unitary 2-cocycle. It is easy to see that ad_V is algebraic over $\mathcal{A}_0$ too and $\mathcal{A}_0^{\sigma'} = \mathcal{A}_0^\sigma$. From now on, let us use the same symbol σ to denote both σ and σ'.

As $\mathcal{Q}$ is an object in $\mathbf{Q}'_R(D)$, R must be of the form: $R = \oplus_{\pi,k} F^\pi \otimes T_{\pi,k}$ by Theorem 2.3.11.

Theorem 7.2.1 *There exists a SOT dense $*$-subalgebra $\mathcal{A}_0$ in $\mathcal{M}$ satisfying*
(1) ad_U *is algebraic over* $\mathcal{A}_0$,
(2) $[D, a] \in \mathcal{B}(\mathcal{H})$ *for all* $a \in \mathcal{A}_0$,
(3) $(\mathcal{A}_0^\sigma)'' = \mathcal{M}^\sigma$,
(4) $(\mathcal{A}_0^\sigma, \mathcal{H}, D)$ *is again a spectral triple.*

Proof Let $\mathcal{G} = \{a \in \mathcal{M} : [D, a] \in \mathcal{B}(\mathcal{H})\}$, which is a SOT-dense unital $*$-subalgebra of $\mathcal{M}$. For $b \in \mathcal{G}$ and a state ϕ of $\widetilde{\mathrm{QISO}_R^+}(D)$, one has $(\mathrm{id} \otimes \phi)\mathrm{ad}_U(b) \in \mathcal{M}$, and also

$$[D, ((\mathrm{id} \otimes \phi)\mathrm{ad}_U(b))] = (\mathrm{id} \otimes \phi)\mathrm{ad}_U([D, b]),$$

using the commutativity of D and U. Hence $(\mathrm{id} \otimes \phi)\mathrm{ad}_U(b) \in \mathcal{G}$ for all bounded linear functionals ϕ of $\widetilde{\mathrm{QISO}_R^+}(D)$ i.e., $\mathcal{G}$ is ad_U invariant SOT-dense $*$ subalgebra of $\mathcal{M}$.

The proof of (1) and (2) is now a straightforward consequence of part (3) of Lemma 1.5.2, with $\mathcal{A}_0$ being the subspace spanned by $\{P_\pi(\mathcal{G}), \pi \in \mathrm{Rep}(\mathcal{Q})\}$. Moreover, (3) follows from Remark 7.1.10. For the proof of (4), we note that $\forall a \in \mathcal{A}_0$, $\rho_\sigma(a) \in \mathcal{B}(\mathcal{H})$ as observed in the proof of (1) of the Lemma 7.1.8. Therefore, it is sufficient to show that $[D, \rho_\sigma(a)] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}_0$. With the notations used before, consider $a \in \mathcal{A}_0$ and ξ in the linear span of $\mathcal{H}_k$'s, writing $\mathrm{ad}_U(a) = \sum_{i=1}^k a^i_{(0)} \otimes a^i_{(1)}$, we have

$$\begin{aligned} &[D, \rho_\sigma(a)](\xi) \\ &= D\rho_\sigma(a)(\xi) - \rho_\sigma(a)D(\xi) \\ &= \sum_{i=1}^k (Da^i_{(0)}\Pi_U(\sigma_i)(\xi) - a^i_{(0)}\Pi_U(\sigma_i)D)(\xi) \\ &= \sum_{i=1}^k [D, a^i_{(0)}]\Pi_U(\sigma_i)(\xi) \ (as\ U(D \otimes I) = (D \otimes I)U\ ). \end{aligned}$$

As $[D, a^i_{(0)}]$ is bounded for all $i = 1, \ldots, k$, it follows that $[D, \rho_\sigma(a)]$ is bounded for all $a \in \mathcal{A}_0$. □

7.3 Quantum Isometry Groups of Deformed Spectral Triples

We continue to work in the setup of the previous section. Let U, V be the unitary corepresentations of $\widetilde{\mathrm{QISO}_R^+}(D)$ and $\mathcal{Q}$ on $\mathcal{H}$ respectively. Let $\mathcal{A}_0$ be any SOT dense $*$-subalgebra of $(\mathcal{A}^\infty)'' = \mathcal{M}$ satisfying (1) of Theorem 7.2.1 and σ, σ' be the dual unitary 2-cocycles on $\mathcal{Q}$ and $\widetilde{\mathrm{QISO}_R^+}(D)$ respectively. The category $\mathbf{Q}'_{R^\sigma}(\mathcal{A}_0^\sigma, \mathcal{H}, D)$ for some unbounded operator R^σ does not depend on the choice of the SOT dense subalgebra $\mathcal{A}_0$. Let us abbreviate it as $\mathbf{Q}'_{R^\sigma}(D_\sigma)$. We have the following:

Lemma 7.3.1 *$(\mathcal{Q}^\sigma, V_\sigma)$ is an object in the category $\mathbf{Q}'_{R^\sigma}(D_\sigma)$, where $R^\sigma = \oplus_{\pi \in \mathcal{I}_k, k \geq 1} F^{\pi_\sigma} \otimes T_{\pi,k}$.*

Proof Recall the decomposition (7.2.1)

$$\mathcal{H} = \oplus_{k \geq 1, \pi \in \mathcal{I}_k \subseteq \mathrm{Rep}(\mathcal{Q})} \mathbb{C}^{d_\pi} \otimes \mathbb{C}^{m_{\pi,k}}.$$

By Proposition 7.1.6, V_σ is again a unitary corepresentation of $\mathcal{Q}^\sigma$ on the Hilbert space $\mathcal{H}$. Also note that ad_V is algebraic over $\mathcal{A}_0$. Let $a \in \mathcal{A}_0$ and $\xi \in \mathcal{N}$, where $\mathcal{N}$ is the subspace of $\mathcal{H}$ spanned by the $\mathcal{H}_k$'s. Then we have

$$\begin{aligned} V_\sigma(\rho_\sigma(a)(\xi)) &= V_\sigma(a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)})) \\ &= a_{(0)(0)}\xi_{(0)(0)}\sigma^{-1}(a_{(1)}, \xi_{(1)}) \otimes a_{(0)(1)}\xi_{(0)(1)} \\ &= a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \otimes a_{(1)(1)}\xi_{(1)(1)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & (\rho_\sigma \otimes \mathrm{id})\mathrm{ad}_V(V_\sigma(\xi)) \\ &= (\rho_\sigma(a_{(0)})\xi_{(0)}) \otimes a_{(1)} \cdot_\sigma \xi_{(1)} \\ &= a_{(0)(0)}\xi_{(0)(0)}\sigma^{-1}(a_{(0)(1)}, \xi_{(0)(1)}) \otimes \sigma(a_{(1)(1)(1)}, \xi_{(1)(1)(1)})a_{(1)(1)(2)}\xi_{(1)(1)(2)} \\ &\quad \sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \\ &= a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)(1)(1)}, \xi_{(1)(1)(1)})\sigma(a_{(1)(1)(2)}, \xi_{(1)(1)(2)}) \otimes a_{(1)(2)(1)}\xi_{(1)(2)(1)} \\ &\quad \sigma^{-1}(a_{(1)(2)(2)}, \xi_{(1)(2)(2)}) \\ &= a_{(0)}\xi_{(0)}\epsilon(a_{(1)(1)})\epsilon(\xi_{(1)(1)})a_{(1)(2)(1)}\xi_{(1)(2)(1)}\sigma^{-1}(a_{(1)(2)(2)}, \xi_{(1)(2)(2)}) \\ &= a_{(0)}\xi_{(0)}\epsilon(a_{(1)(1)(1)})\epsilon(\xi_{(1)(1)(1)}) \otimes a_{(1)(1)(2)}\xi_{(1)(1)(2)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \\ &= a_{(0)}\xi_{(0)}\sigma^{-1}(a_{(1)(2)}, \xi_{(1)(2)}) \otimes a_{(1)(1)}\xi_{(1)(1)}. \end{aligned}$$

So $\mathrm{ad}_{V_\sigma}(\rho_\sigma(a))(V_\sigma\xi) = V_\sigma(\rho_\sigma(a)\xi) = (\rho_\sigma \otimes \mathrm{id})\mathrm{ad}_V(a)(V_\sigma(\xi))$. By the density of $\mathcal{N}$ in $\mathcal{H}$ we conclude that

$$\mathrm{ad}_{V_\sigma}(\rho_\sigma(a)) = (\rho_\sigma \otimes \mathrm{id})\mathrm{ad}_V(a), \tag{7.3.1}$$

for all $a \in \mathcal{A}_0$. Also for $\phi \in (\mathcal{Q}^\sigma)^*$,

$$(\mathrm{id} \otimes \phi)\mathrm{ad}_{V_\sigma}(\rho_\sigma(a)) = \rho_\sigma(a_{(0)})\phi(a_{(1)}) \in \mathcal{A}_0^\sigma.$$

So in particular $(\mathrm{id} \otimes \phi)\mathrm{ad}_{V_\sigma}(\mathcal{A}_0) \subseteq (\mathcal{A}_0)''$.

V_σ commutes with D since as a linear map V_σ is the same as V. As R^σ is of the form $\oplus_{\pi \in \mathcal{I}_k, k \geq 1} F^{\pi_\sigma} \otimes T_{\pi,k}$, by the "if part" of Theorem 2.3.11, we conclude that ad_{V_σ} preserves the R^σ-twisted volume. □

Remark 7.3.2 One can conclude the following from the proof above: if $(\mathcal{Q}, V)$ is an object in the category $\mathbf{Q}'(D)$, then $(\mathcal{Q}^\sigma, V_\sigma)$ is an object in the category $\mathbf{Q}'(D_\sigma)$.

Now replacing $\mathcal{Q}$ with $\widetilde{\mathrm{QISO}_R^+}(D)$ and using the dual unitary 2-cocycle on $\widetilde{\mathrm{QISO}_R^+}(D)$ induced from σ on its quantum subgroup, we get

Corollary 7.3.3 $(\widetilde{\mathrm{QISO}_R^+}(D))^\sigma$ *is a sub-object of* $\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma)$ *in the category* $\mathbf{Q}'(D_\sigma)$.

Thus we have the dual unitary 2-cocycle σ^{-1} on $\mathcal{Q}^\sigma$ which is a quantum subgroup of $\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma)$. Thus, it makes sense to deform the Dirac operator D_σ by σ^{-1}. We have the following:

Lemma 7.3.4 $(\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma))^{\sigma^{-1}}$ *is a sub-object of* $\widetilde{\mathrm{QISO}_R^+}(D)$ *in the category* $\mathbf{Q}'_R(D)$.

Proof For convenience, we choose $\mathcal{A}_0 = \mathcal{M}_0$, which is the maximal subalgebra of $\mathcal{M}$ on which ad_V is algebraic. Consider $\mathcal{M}_1 = \mathrm{Span}\{P_{\pi_\sigma}((\mathcal{M}_0)'') : \pi \in \mathrm{Rep}(\mathcal{Q})\}$, where $P_{\pi_\sigma} = (\mathrm{id} \otimes \rho^{\pi_\sigma})\mathrm{ad}_{V_\sigma}$. Clearly, $\rho_{\sigma^{-1}}$ is defined on $\mathcal{M}_1$ and $\mathcal{M}_1$ the maximal subspace on which ad_{V_σ} is algebraic. As ad_{U_σ} is again a von Neumann algebraic coaction of $\mathcal{Q}^\sigma$ on $(\mathcal{A}_0)'' = \rho_\sigma(\mathcal{M}_0)''$, we have a SOT dense subalgebra $\mathcal{C}_0$ of $\rho_\sigma(\mathcal{M}_0)''$ over which ad_{U_σ} is algebraic. Then ad_{V_σ} is also algebraic over $\mathcal{C}_0$. Hence, by the maximality of $\mathcal{M}_1$ for ad_{V_σ}, we have $\mathcal{C}_0 \subseteq \mathcal{M}_1$. Again by the SOT continuity of $\rho_{\sigma^{-1}}$ on the image of P_{π_σ}, we have $\rho_{\sigma^{-1}}(\mathcal{C}_0)'' = \rho_{\sigma^{-1}}(\mathcal{M}_1)''$. On the other hand, it follows from (7.3.1) that ad_{V_σ} is algebraic over $\rho_\sigma(\mathcal{M}_0)$, so that we have $\rho_\sigma(\mathcal{M}_0) \subseteq \mathcal{M}_1$, implying $\rho_{\sigma^{-1}}(\rho_\sigma(\mathcal{M}_0)) \subseteq \rho_{\sigma^{-1}}(\mathcal{M}_1)$. Thus, $\mathcal{M}_0 \subseteq \rho_{\sigma^{-1}}(\mathcal{M}_1)$. Substituting V by V_σ and σ by σ^{-1} in (7.3.1), we get

$$\mathrm{ad}_V(\rho_{\sigma^{-1}}(\mathcal{M}_1)) \subseteq \rho_{\sigma^{-1}}(\mathcal{M}_1) \otimes_{\mathrm{alg}} \mathcal{Q}_0,$$

so by maximality of $\mathcal{M}_0$ for ad_V, we conclude that $\rho_{\sigma^{-1}}(\mathcal{M}_1) \subseteq \mathcal{M}_0$, i.e., $\mathcal{M}_0 = \rho_{\sigma^{-1}}(\mathcal{M}_1)$. Hence,

$$\rho_{\sigma^{-1}}(\mathcal{C}_0)'' = \mathcal{M}_0'' = (\mathcal{A}^\infty)'',$$

which implies that $\mathbf{Q}'_{(R^\sigma)^{\sigma^{-1}}}((D_\sigma)_{\sigma^{-1}}) = \mathbf{Q}'_R(D)$. Also observe that $\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma)^{\sigma^{-1}}$ preserves volume τ_R and

$$\mathrm{ad}_U(a) = \mathrm{ad}_{U_\sigma}(\rho_{\sigma^{-1}}(a)),$$

for all $a \in \mathcal{C}_0$.

But by definition $(\mathrm{id} \otimes \phi)\mathrm{ad}_{U_\sigma}(\rho_{\sigma^{-1}}(a)) \subseteq \rho_{\sigma^{-1}}(\mathcal{C}_0)'' = (\mathcal{A}^\infty)''$.

Finally, $(R^\sigma)^{\sigma^{-1}} = \oplus_{\pi,k} F^\pi \otimes T_{\pi,k} = R$, as $(\pi_\sigma)_\sigma = \pi$, hence $\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma)$ is an object in $\mathbf{Q}'_R(D)$. □

Combining the above results, we are in a position to state and prove the main result of this chapter.

Theorem 7.3.5 $\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma) \cong (\widetilde{\mathrm{QISO}_R^+}(D))^\sigma$ *and hence* $\mathrm{QISO}_{R^\sigma}^+(D_\sigma) \cong (\mathrm{QISO}_R^+(D))^\sigma$.

Proof From Lemma 7.3.4, we have $(\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma))^{\sigma^{-1}}$ is a sub-object of $\widetilde{\mathrm{QISO}_R^+}(D)$ in the category $\mathbf{Q}'_R(D_\sigma)$. Then by Lemma 7.1.4, we have $((\widetilde{\mathrm{QISO}_{R^\sigma}^+}(D_\sigma))^{\sigma^{-1}})^\sigma$ is a sub-object of $(\widetilde{\mathrm{QISO}}^+_R(D))^\sigma$ in the category $\mathbf{Q}'_R(D_\sigma)$. Since quantum isometry groups are universal, we can conclude by Lemma 7.1.5 that $\widetilde{\mathrm{QISO}}^+_{R^\sigma}(D_\sigma)$ is a sub-object of $(\widetilde{\mathrm{QISO}}^+_R(D))^\sigma$ in the category $\mathbf{Q}'_R(D_\sigma)$. Combining this with Corollary 7.3.3, we complete the proof of the theorem. □

We recall that in general, the category $\mathbf{Q}'(D)$ does not admit a universal object. However, by looking at the proofs in this section and with the notations here, we have the following:

Corollary 7.3.6 *If both of the quantum groups* $\mathrm{QISO}^+(\mathcal{A}^\infty, \mathcal{H}, D)$ *and* $\mathrm{QISO}^+(\mathcal{A}_0^\sigma, \mathcal{H}, D)$ *exist, then* $\mathrm{QISO}^+(\mathcal{A}^\infty, \mathcal{H}, D)^\sigma \cong \mathrm{QISO}^+(\mathcal{A}_0^\sigma, \mathcal{H}, D)$.

Remark 7.3.7 We can also obtain similar result for the approach based on the Laplacian, i.e., in the category $\mathbf{Q}'_{\mathcal{L}}$, whenever it makes sense. The particular case of Rieffel deformations was proved in Theorem 3.13 of [8].

We remark that our results do not include the examples constructed by Neshveyev-Tuset in [9]. In that paper, they took up the problem of deforming a spectral triple by actions of classical compact Lie groups using a Drinfeld twist. It should be noted that although any 2-cocycle is an example of a Drinfeld twist, there are Drinfeld twists which are not 2-cocycles and our machineries do not apply to such twists. The main problem lies in transporting a unitary Drinfeld twist to a compact quantum group from its quantum subgroup. We will briefly mention some partial results and open questions about this matter in the last chapter of this book.

Remark 7.3.8 Sadeleer [3] has extended the results of [2] to a bigger class of examples called monoidal deformation by them. However, it still does not include the examples coming from general Drinfeld twists.

7.4 Examples and Computations

The Rieffel deformation [10] of a unital C^* algebra $\mathcal{A}$ by the action of the compact group $\mathbb{T}^n$ and the resulting isospectral deformation of the spectral triple [11, 12] on $\mathcal{A}$ are particular examples of cocycle deformations explained in this chapter. The Rieffel deformation of compact quantum groups were defined and studied by Wang in [13, 14] and by Rieffel in [15]. For the identification of Rieffel deformation of algebras, compact quantum groups and spectral triples as particular cases of cocycle deformations, we refer to [4, 16–18] and the references therein. However, let us spell out some of the details for the example of the noncommutative tori. For simplicity, we discuss only the case of noncommutative 2-torus.

Fix an irrational number θ and let $\lambda = \exp(2\pi i\theta)$. The dual group of $\mathbb{T}^2$ is $\mathbf{Z}^2$ and so a dual unitary 2-cocycle on $\mathbb{T}^2$ is a unitary element in the multiplier of $C_0(\mathbf{Z}^2) \otimes C_0(\mathbf{Z}^2)$, which can be identified with $C_b(\mathbf{Z}^2 \times \mathbf{Z}^2)$. Let σ be the element of $C_b(\mathbf{Z}^2 \times \mathbf{Z}^2)$ given by

$$\sigma((m,n),(k,l)) = \lambda^{-nk}.$$

It can be easily checked that σ defines a unitary dual cocycle which is also normalized. As done before, we identify this with a functional, still denoted by σ (by an abuse of notation), on $\mathcal{Q}_0 \otimes_{\mathrm{alg}} \mathcal{Q}_0$, where $\mathcal{Q}_0$ is the canonical dense Hopf algebra of $C(\mathbb{T}^2)$, given by $\sigma(z_1^m z_2^n \otimes z_1^k z_2^l) = \lambda^{-nk}$. Here, we have denoted the canonical coordinate functions of the 2-torus by z_1 and z_2 respectively.

We can use the above cocycle to deform $C(\mathbb{T}^2)$ both as a (compact quantum) group or as a C^* algebra. However, the deformation of $C(\mathbb{T}^2)$ as a CQG does not produce anything new. Indeed, in the notation introduced before, $\widehat{\Delta}_\sigma = \sigma\widehat{\Delta}\sigma^{-1} = \widehat{\Delta}$, as $C_b(\mathbb{Z}^2 \times \mathbb{Z}^2)$ is commutative. Thus, we have

Proposition 7.4.1 *The compact quantum group $(C(\mathbb{T}^2))^\sigma$ is isomorphic to $C(\mathbb{T}^2)$ itself.*

Nevertheless, the deformation of $C(\mathbb{T}^2)$ as a C^* algebra, i.e., $\rho_\sigma(C(\mathbb{T}^2))$ in our notation, is indeed noncommutative and isomorphic with $\mathcal{A}_\theta$. To see this, consider the standard spectral triple on $C(\mathbb{T}^2)$ which is nothing but the one considered in Example 2.2.7 of Chap. 2, with $\theta = 0$. In order to understand $\rho_\sigma(C(\mathbb{T}^2))$, it suffices to compute $\rho_\sigma(z_1^m z_2^n)$ on the vectors of the Hilbert space $L^2(\mathbb{T}^2 \otimes \mathbb{C}^2)$ However, the action of $\mathbb{T}^2$ is trivial on the tensor component $\mathbb{C}^2$ and the (undeformed) representation of $C(\mathbb{T}^2)$ is given by $M_f \otimes I_{\mathbb{C}^2}$, where M_f denotes the operator of multiplication by $f \in C(\mathbb{T}^2)$. Moreover, we note that $\Delta(z_1^m z_2^n) = z_1^m z_2^n \otimes z_1^m z_2^n$. Hence, using the defining formula of ρ_σ in Sect. 7.1.3, we have

$$\rho_\sigma(z_1^m z_2^n)(e_{kl} \otimes w) = \lambda^{nk} e_{m+k,n+l} \otimes w,$$

where e_{kl} denotes $z_1^k z_2^l$ viewed as a unit vector in $L^2(\mathbb{T}^2)$ and $w \in \mathbb{C}^2$. From this, it follows easily that the unitaries $U := \rho_\sigma(z_1)$ and $V := \rho_\sigma(z_2)$ satisfies the commutation relation of the noncommutative torus, i.e., $UV = \lambda VU$. As every nonzero $*$-representation of $\mathcal{A}_\theta$ is an isomorphism, which follows from the fact that $\mathcal{A}_\theta$ is a simple C^* algebra, we get $\rho_\sigma(C(\mathbb{T}^2)) \cong \mathcal{A}_\theta$. Hence, we have the following proposition.

Proposition 7.4.2 *The deformation of $C(\mathbb{T}^2)$ as a C^* algebra is isomorphic to $\mathcal{A}_\theta$. Moreover, the cocycle deformation of the spinorial spectral triple on $C^\infty(\mathbb{T}^2)$ is isomorphic to the spectral triple on the noncommutative torus as explained in Example 2.2.7.*

Next, we compute the quantum isometry group of the noncommutative 2-torus using the machinery developed in this chapter. We recall that as mentioned in Remark 7.3.7, the quantum group $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$ is again a cocycle deformation of $\mathrm{QISO}^{\mathcal{L}}(C^\infty(\mathbb{T}^2))$. Thus, we will start with the quantum group of orientation preserving isometries $\mathrm{QISO}^+(\mathcal{A}_\theta)$ and then pass on to $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$.

Theorem 7.4.3 $\widetilde{\mathrm{QISO}}^{+}(\mathcal{A}_\theta^\infty, \mathcal{H}, D) = \widetilde{\mathrm{QISO}}^{+}(C^\infty(\mathbb{T}^2)) = C(\mathbb{T}^2) * C(\mathbb{T})$, *and* $\mathrm{QISO}^+(\mathcal{A}_\theta^\infty) = \mathrm{QISO}^+(C^\infty(\mathbb{T}^2)) = C(\mathbb{T}^2)$.

Now, we pass on to the computation of $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$. We work with the spectral triple of Example 2.2.7 as before so that the corresponding Laplacian $\mathcal{L}$ is given by $\mathcal{L}(U^{m_1}V^{m_2}) = -(m_1^2 + m_2^2)U^{m_1}V^{m_2}$, and it is also easy to see that all the assumptions in Sect. 3.1 required for defining $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$ are satisfied.

Now, we recall that $\mathcal{A}_\theta$ is obtained as a cocycle deformation of $\mathbb{T}^2$ using the dual unitary cocycle σ on $\mathbf{Z} \times \mathbf{Z}$ defined by $\sigma((m, n), (k, l)) = \lambda^{-nk}$. On the other hand, $C(\mathbb{T}^2)$ sits as a quantum subgroup of $QISO^{\mathcal{L}}(C(\mathbb{T}^2)) \cong C(\mathbb{T}^2 > \!\!\lhd (\mathbf{Z}_2^2 > \!\!\lhd S_2))$. Then, as in Sect. 7.1.1, we can induce a dual cocycle $\widehat{\sigma}$ on $C(\mathbb{T}^2 > \!\!\lhd (\mathbf{Z}_2^2 > \!\!\lhd S_2))$. Thus, by Remark 7.3.7, we can conclude that $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$ is again a $\widehat{\sigma}$ deformation of $C(\mathbb{T}^2 > \!\!\lhd (\mathbf{Z}_2^2 > \!\!\lhd S_2))$.

Let us use the notation Γ for the discrete group $\mathbf{Z}_2^2 > \!\!\lhd \mathbf{Z}_2$. Moreover, let $\gamma_1, \gamma_2, \gamma_3$ denote the nontrivial elements of the first, second, and the third copy of $\mathbf{Z}_2$ in Γ respectively, while $\underline{z} = (z_1, z_2)$ will denote an arbitrary element of $\mathbb{T}^2$. We will need an explicit description of the two semi-direct products in Γ. The action of $\mathbf{Z}_2$ on $\mathbf{Z}_2^2$ is given by $\gamma_3(x, y) = (y, x)$ $((x, y) \in \mathbf{Z}_2^2)$ and the action of $\mathbf{Z}_2^2 > \!\!\lhd \mathbf{Z}_2$ on $\mathbb{T}^2$ (denoted by $\circ$) is given by

$$\gamma_1 \circ (z_1, z_2) = (\overline{z_1}, z_2),\ \gamma_2 \circ (z_1, z_2) = (z_1, \overline{z_2}),\ \gamma_3 \circ (z_1, z_2) = (z_2, z_1).$$

By an expression of the form $(\gamma_1\gamma_2\gamma_3)(z_1, z_2)$, we will mean $\gamma_1(\gamma_2(\gamma_3(z_1, z_2)))$.

For C in $C(\mathbb{T}^2)$ and $\underline{\gamma}$ in Γ, the symbol $C_{\underline{\gamma}}$ will denote the element of $C(\mathbb{T}^2 > \!\!\lhd \Gamma)$ defined by

$$C_{\underline{\gamma}}(\underline{z}, \underline{\gamma'}) = C(\underline{z})\delta_{\gamma\gamma'},$$

where $\delta_{\gamma\gamma'}$ is the Kronecker delta. In particular, we will denote the generators of $\mathrm{QISO}^{\mathcal{L}}(\mathbb{T}^2)$ by $\{A_{\underline{\gamma}},\ B_{\underline{\gamma}} : \underline{\gamma} \in \mathbf{Z}_2^3\}$, i.e.,

$$A_{\underline{\gamma}}(z_1, z_2, \underline{\gamma'}) = z_1\delta_{\gamma,\gamma'},\ \ B_{\underline{\gamma}}(z_1, z_2, \underline{\gamma'}) = z_2\delta_{\gamma,\gamma'}.$$

For elements $C, D \in C(\mathbb{T}^2)$, we observe that

$$\widehat{\sigma}(C_{\underline{\gamma}}, D_{\underline{\gamma'}}) = \delta_{\underline{\gamma}\underline{0}}\delta_{\underline{\gamma'}\underline{0}}\sigma(C, D). \tag{7.4.1}$$

Moreover, we have the following expression of the coproduct Δ of $\mathrm{QISO}^{\mathcal{L}}(\mathbb{T}^2)$ for any group-like element C (i.e., $\Delta(C) = C \otimes C$) in $C(\mathbb{T}^2)$, e.g., $C = z_1^m z_2^n$:

$$\Delta(C_{\underline{\gamma}}) = \sum_{\gamma' \in \Gamma} C_{\underline{\gamma'}} \otimes (\gamma' C)_{\underline{\gamma'^{-1}\gamma}} \text{ where } (\underline{\gamma'}.C)(\underline{z}) = C(\underline{\gamma'}\underline{z}).$$

Indeed, we have

$$\begin{aligned}
&\Delta(C_{\underline{\gamma}}) \\
&\qquad = C_{\underline{\gamma}}(\underline{z}.(\underline{\alpha}.\underline{w}), \underline{\alpha}.\underline{\beta}) \\
&\qquad = \delta_{\underline{\gamma},\underline{\alpha\beta}} C(\underline{z}) C(\underline{\alpha w}) \\
&\qquad = (\sum_{\gamma' \in \Gamma} C_{\underline{\gamma'}} \otimes (\gamma' C)_{\underline{\gamma'^{-1}}\underline{\gamma}})((\underline{z}, \underline{\alpha}), (\underline{w}, \beta)).
\end{aligned}$$

This gives

$$\begin{aligned}
&(\Delta \otimes \mathrm{id})\Delta(C_{\underline{\gamma}}) \\
&\qquad = \sum_{\gamma', \gamma'' \in \Gamma} C_{\gamma''} \otimes (\underline{\gamma''}.C)_{\underline{\gamma''^{-1}}\underline{\gamma'}} \otimes (\underline{\gamma'} C)_{\underline{\gamma'^{-1}}\underline{\gamma}} \\
&\qquad = \sum_{\underline{\gamma_1}, \underline{\gamma_2}, \underline{\gamma_3} : \underline{\gamma_1 \gamma_2 \gamma_3} = \underline{\gamma}} C_{\underline{\gamma_1}} \otimes (\underline{\gamma_1}.C)_{\underline{\gamma_2}} \otimes (\underline{\gamma_1 \gamma_2}.C)_{\underline{\gamma_3}}.
\end{aligned}$$

Now we obtain from the definition of the twisted product as well as (7.4.1) the following:

$$C_{\underline{\gamma}} \times_{\widehat{\sigma}} D_{\underline{\gamma'}} = \sigma^{-1}(C, D) C_{\underline{\gamma}} D_{\underline{\gamma'}} \sigma(\underline{\gamma}.C, \underline{\gamma'} D) = \delta_{\underline{\gamma\gamma'}} (CD)_{\underline{\gamma}} \sigma^{-1}(C, D) \sigma(\underline{\gamma}.C, \underline{\gamma}.D). \tag{7.4.2}$$

This equation in turn implies that

$$A_{\underline{\gamma}} \times_{\widehat{\sigma}} B_{\underline{\gamma}} = \lambda^{-1} \sigma(\underline{\gamma} A, \underline{\gamma} B) \sigma^{-1}(\underline{\gamma} B, \underline{\gamma} A) B_{\underline{\gamma}} \times_{\widehat{\sigma}} A_{\underline{\gamma}}. \tag{7.4.3}$$

Using the above formulas, it is easy to observe the following result:

Proposition 7.4.4 *$\{A_{\underline{\gamma}}, B_{\underline{\gamma}}\}$ are unitaries for the multiplication $\times_{\widehat{\sigma}}$ and for $\underline{\gamma}$ belonging to Γ. Moreover, $A^*_{\underline{\gamma}} \times_{\widehat{\sigma}} A_{\underline{\gamma}}(u_1, u_2, \underline{\gamma'}) = 0$ if $\underline{\gamma} \neq \underline{\gamma'}$. We have a similar result for $B_{\underline{\gamma}}$.*

Next, using (7.4.3) and the formula for σ, we get the following commutation relations among $A_{\underline{\gamma}}$ and $B_{\underline{\gamma}}$:

Proposition 7.4.5 *If $\underline{0}$ denotes the element $(0, 0, 0)$ of Γ, then*

$$A_{\underline{0}} \times_{\widehat{\sigma}} B_{\underline{0}} = B_{\underline{0}} \times_{\widehat{\sigma}} A_{\underline{0}}, \;\; A_{\gamma_3} \times_{\widehat{\sigma}} B_{\gamma_3} = \lambda^{-2} B_{\gamma_3} \times_{\widehat{\sigma}} A_{\gamma_3},$$

$$A_{\gamma_2\gamma_3} \times_{\widehat{\sigma}} B_{\gamma_2\gamma_3} = B_{\gamma_2\gamma_3} \times_{\widehat{\sigma}} A_{\gamma_2\gamma_3}, \;\; A_{\gamma_1\gamma_2} \times_{\widehat{\sigma}} B_{\gamma_1\gamma_2} = \lambda^{-2} B_{\gamma_1\gamma_2} \times_{\widehat{\sigma}} A_{\gamma_1\gamma_2},$$

$$A_{\gamma_1\gamma_3} \times_{\widehat{\sigma}} B_{\gamma_1\gamma_3} = B_{\gamma_1\gamma_3} \times_{\widehat{\sigma}} A_{\gamma_1\gamma_3}, \;\; A_{\gamma_1} \times_{\widehat{\sigma}} B_{\gamma_1} = \lambda^{-2} B_{\gamma_1} \times_{\widehat{\sigma}} A_{\gamma_1},$$

$$A_{\gamma_1\gamma_2\gamma_3} \times_{\widehat{\sigma}} B_{\gamma_1\gamma_2\gamma_3} = \lambda^{-2} B_{\gamma_1\gamma_2\gamma_3} \times_{\widehat{\sigma}} A_{\gamma_1\gamma_2\gamma_3}, \;\; A_{\gamma_2} \times_{\widehat{\sigma}} B_{\gamma_2} = \lambda^{-2} B_{\gamma_2} \times_{\widehat{\sigma}} A_{\gamma_2}.$$

Proof We will only prove the cases when $\underline{\gamma} = \gamma_3$ and $\underline{\gamma} = \gamma_1\gamma_3$, the proofs of the others being similar.

We have $(\gamma_3 A)(z_1, z_2) = A(z_2, z_1) = B(z_1, z_2)$. Thus, $\gamma_3 A = B$. Similarly, $\gamma_3 B = A$.

Thus, $\lambda^{-1}\sigma(\gamma_3 A, \gamma_3 B)\sigma^{-1}(\gamma_3 B, \gamma_3 A) = \lambda^{-1}\sigma(B, A)\sigma^{-1}(A, B) = \lambda^{-1}$ $\lambda^{-1}.1 = \lambda^{-2}$.

This proves the second equation of the proposition.

Next, $\gamma_1\gamma_3 A(z_1, z_2) = \gamma_1 A(z_2, z_1) = A(\overline{z_2}, z_1) = \overline{z_2} = B^{-1}(z_1, z_2)$, while $\gamma_1\gamma_3 B(z_1, z_2) = \gamma_1 B(z_2, z_1) = B(\overline{z_2}, z_1) = A(z_1, z_2)$.

Thus, $\gamma_1\gamma_3 A = B^{-1}$ and $\gamma_1\gamma_3 B = A$. Therefore, we have $\lambda^{-1}\sigma(\gamma_1\gamma_3 A, \gamma_1\gamma_3 B)\sigma^{-1}(\gamma_1\gamma_3 B, \gamma_1\gamma_3 A) = \lambda^{-1}\sigma(B^{-1}, A)\sigma^{-1}(A, B^{-1}) = \lambda^{-1}$ $\lambda = 1$.

This proves the fifth equation. □

Let $\mathcal{B}$ be the C^*-algebra defined below, having 8 direct summands with four of them commutative and the others isomorphic with $\mathcal{A}_{2\theta}$.

Now we are in a position to describe $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$ explicitly.

Theorem 7.4.6 $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$ *is isomorphic with* $\mathcal{B} = C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta} \oplus C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta} \oplus C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta} \oplus C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta}$.

Proof This is a direct application of Propositions 7.4.4 and 7.4.5. □

Remark 7.4.7 In particular, for $\theta = 1/2$, we get a commutative compact quantum group as the quantum isometry group of a noncommutative C^* algebra.

Let us state the following result without proof, which gives an identification of the "quantum double torus" discovered and studied by Hajac and Masuda [19] with an interesting quantum subgroup of $\mathrm{QISO}^{\mathcal{L}}(\mathcal{A}_\theta)$. Let us call a smooth isometric coaction γ on $\mathcal{A}_\theta$ "holomorphic" if γ leaves the subalgebra generated by $\{U^m V^n,\ m, n \geq 0\}$ invariant. Then we have the following:

Theorem 7.4.8 ([8]) *Let* A_0, D_0 *and* B_0, C_0 *be the canonical generators of* $C(\mathbb{T}^2)$ *and* $\mathcal{A}_{2\theta}$ *respectively. Consider the* C^* *algebra* $\mathcal{Q}^{\mathrm{hol}} = C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta}$, *with the following coproduct:*

$$\Delta_h(A_0) = A_0 \otimes A_0 + C_0 \otimes B_0, \quad \Delta_h(B_0) = B_0 \otimes A_0 + D_0 \otimes B_0,$$

$$\Delta_h(C_0) = A_0 \otimes C_0 + C_0 \otimes D_0, \quad \Delta_h(D_0) = B_0 \otimes C_0 + D_0 \otimes D_0.$$

Then $(\mathcal{Q}^{\mathrm{hol}}, \Delta_h)$ *is a compact quantum group isomorphic with the quantum double torus of [19]. It has a coaction* β_0 *on* $\mathcal{A}_\theta$ *given by*

$$\beta_0(U) = U \otimes A_0 + V \otimes B_0, \quad \beta_0(V) = U \otimes C_0 + V \otimes D_0.$$

Moreover, $\mathcal{Q}^{\mathrm{hol}} = C(\mathbb{T}^2) \oplus \mathcal{A}_{2\theta}$ *is universal among the compact quantum groups coacting "holomorphically" on* $\mathcal{A}_\theta$ *in the sense discussed above.*

Remark 7.4.9 For the isospectral deformation of the classical spectral triple on $C^\infty(S^{n-1})$, we can also identify $\mathrm{QISO}^{\mathcal{L}}(S^{n-1}_\theta)$ as the compact quantum group $O_\theta(n)$. This follows from the results of [12] where it has been shown that S^{n-1}_θ and $O_\theta(n)$ are Rieffel deformation of $C(S^{n-1})$ (as a C^* algebra) and $C(O(n))$ (as a compact quantum group) respectively and by observing that the quantum isometry group of the classical sphere S^{n-1} is the commutative C^* algebra $C(O(n))$.

References

1. Banica, T., Bichon, J., Curran, S.: Quantum automorphisms of twisted group algebras and free hypergeometric laws. Proc. Am. Math. Soc. **139**(11), 3961–3971 (2011)
2. Goswami, D., Joardar S.: Quantum isometry groups of non commutative manifolds obtained by deformation using unitary 2-cocycle. SIGMA **10**, 076, 18 (2014)
3. De Sadeleer, L.: Deformations of spectral triples and their quantum isometry groups via monoidal equivalences. arXiv:1603.02931
4. Neshveyev, S., Tuset, L.: Deformation of C*-algebras by cocycles on locally compact quantum groups. Adv. Math. **254**, 454–496 (2014)
5. Majid, S.: Foundations of Quantum Group Theory. Cambridge University Press (1995)
6. Bichon, J., De Rijdt, A., Vaes, S.: Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups. Commun. Math. Phys. **262**(3), 703–728 (2006)
7. Bichon, J.: Quantum automorphism groups of finite graphs. Proc. Am. Math. Soc. **131**(3), 665–673 (2003)
8. Bhowmick, J., Goswami, D.: Quantum isometry groups: examples and computations. Comm. Math. Phys. **285**(2), 421–444 (2009)
9. Neshveyev, S., Tuset, L.: The Dirac operator on compact quantum groups. J. Reine Angew. Math. **641**, 1–20 (2010)
10. Rieffel, M.A.: Deformation quantization for actions of R^d. Mem. Am. Math. Soc. **106**(506) (1993)
11. Connes, A., Landi, G.: Noncommutative manifolds, the instanton algebra and isospectral deformations. Commun. Math. Phys. **221**(1), 141–159 (2001)
12. Connes, A., Dubois-Violette, M.: Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. Commun. Math. Phys. **230**(3), 539–579 (2002)
13. Wang, S.: Deformation of compact quantum groups via Rieffel's quantization. Commun. Math. Phys. **178**, 747–764 (1996)
14. Wang, S.: Rieffel type discrete deformation of finite quantum groups. Commun. Math. Phys. **202**, 291–307 (1999)
15. Rieffel, M.A.: Compact quantum groups associated with toral subgroups. Contemp. Math. **145**, 465–491 (1992)
16. Yamashita, M.: Connes-Landi deformation of spectral triples. **94**(3), 263–291 (2010)
17. Yamashita, M.: Deformation of algebras associated to group cocycles. arXiv:1107.2512
18. Bhowmick, J., Neshveyev, S., Sangha, A.S.: Deformation of operator algebras by Borel cocycles. J. Funct. Anal. **265**, 983–1001 (2013)
19. Hajac, P., Masuda, T.: Quantum double-torus. Comptes Rendus Acad. Sci. Paris Ser. I Math. **327**(6), 553–558 (1998)

Chapter 8
Spectral Triples and Quantum Isometry Groups on Group C^*-Algebras

Abstract We discuss the quantum isometry group of the reduced C^* algebra of a finitely generated discrete group. The relevant spectral triple are the ones defined by Connes which arise from length functions. We prove the existence of quantum isometry groups for such spectral triples using results of Sect. 3.4 of Chap. 3 and then present detailed computation for a number of interesting examples.

One of the earliest examples of spectral triples on noncommutative spaces was those by Connes for group algebras [1]. For a finitely generated discrete group Γ, the spectral triple is given by the left regular representation of the group ring $\mathbb{C}[\Gamma]$ on $l^2(\Gamma)$ and an operator built out of a length function on Γ. When Γ is finitely generated, properties of the natural word length function on Γ and its associated spectral triple reflect combinatorial and geometric aspects of Γ. For example, the spectral triple is finitely summable if and only if the group Γ has polynomial growth [1]. Gromov hyperbolicity or the Rapid Decay condition on the group implies that the spectral triple gives rise to compact quantum metric spaces (a la Rieffel cite-metricrieffelold,rieffelmetriconstatespace,rieffelexpository on the reduced group C^* algebra [5–7]. In [8], isometric actions of groups on the spectral triples mentioned above were studied. This chapter deals with the quantum isometry groups for these spectral triples carried out by a number of authors.

In the first section, we describe Connes' construction of these spectral triples. Then, we relate the existence and computations of quantum isometry groups of these spectral triples with Theorem 3.4.2 and Corollary 3.4.5 of Chap. 3. In the following sections, we present several results on explicit computations of the quantum isometry groups.

Throughout this chapter, $\mathbb{Z}_\infty$ will stand for the infinite group $\mathbb{Z}$. However, whenever we write $\mathbb{Z}_n$, n will be assumed to be finite unless otherwise mentioned.

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_8

8.1 Connes' Spectral Triple on Group C^*-Algebras and Their Quantum Isometry Groups

We fix the following notations to begin with. If g belongs to a discrete group Γ, we will denote by δ_g the function in $l_2(\Gamma)$ which takes the value 1 at the point g and 0 at all other points. The natural generators of the algebra $\mathbb{C}[\Gamma]$ and their images in the left regular representation will be denoted by λ_g. The functional τ will denote the canonical tracial state on $C_r^*(\Gamma)$ defined by $\tau(\lambda_\gamma) = \delta_{\gamma,e}$ where e denotes the identity element of the group Γ.

Let Γ be a discrete group equipped with a length function $l : \Gamma \to \mathbb{R}_+$. Recall that l has to satisfy the following conditions:

$$l(g) = 0 \text{ iff } g = e, \quad l(g^{-1}) = l(g), \quad l(gh) \le l(g) + l(h), \quad g, h \in \Gamma.$$

We will assume that the length takes only integer values and that for each $n \in \mathbb{N}$ the set $W_n := \{g \in \Gamma : l(g) = n\}$ is finite.

The most important example is given by the word length induced by a fixed finite symmetric set of generators in a finitely generated group Γ. The word length is defined as follows: we fix a set S of generators of Γ which is inverse closed, that is, S is of the form $\{s_1, \ldots, s_n\}$ such that S generates Γ, s_i belongs to S if and only if s_i^{-1} belongs to S. Then one defines the word length by $l(g) = \min\{n : g = g_1 g_2 \ldots g_m : g_i \in S\}$.

For our purpose, it will be enough to restrict our attention to word length functions for finitely generated groups. Thus, from now on, we fix a generating set S of a finitely generated discrete group Γ as above and l will denote the word length function corresponding to S.

Define the operator D_Γ on $l^2(\Gamma)$ by

$$\mathrm{Dom}(D_\Gamma) = \{\xi \in l^2(\Gamma) : \sum_{g \in \Gamma} l(g)^2 |\xi(g)|^2 < \infty\},$$

$$(D_\Gamma(\xi))(g) = l(g)\xi(g), \quad \xi \in \mathrm{Dom}(D_\Gamma), g \in \Gamma.$$

Consider the left regular representation of Γ on $l_2(\Gamma)$ and extend it to the defining representation of $C_r^*(\Gamma)$. Thus, we can view $\mathbb{C}[\Gamma]$ as a subalgebra of $\mathcal{B}(l^2(\Gamma))$. It is easy to check that $(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$ is a spectral triple. In particular, the finite generation of Γ implies that the sets $\{g \in \Gamma : l(g) = n\} \le (\mathrm{card}(S))^n < \infty$ for all n. Hence, all the eigenspaces of D are finite-dimensional implying that D has compact resolvent.

Moreover, the relation $\mathrm{card}\{g \in \Gamma : l(g) = n\} \le \mathrm{card}(S)^n$ implies that the triple $(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$ is automatically *θ-summable* - for each $t > 0$ the (bounded) operator $\exp(-tD^2)$ is trace class. This follows from the following computation:

$$\mathrm{Tr}(\exp(-tD^2)) = \sum_{\gamma \in \Gamma} \mathrm{e}^{-tl(\gamma)^2} = \sum_{n=0}^{\infty} \mathrm{card}\{g \in \Gamma : l(g) = n\} e^{-tn^2} < \infty.$$

For more information on such spectral triples, extension to weighted Dirac operators and related topics we refer to the paper [9] and the references therein. We end this section with the remark that the above-mentioned spectral triples admit real structures as well as "admissible" Laplacians so that $QISO^{\mathcal{L}}$ (as in Sect. 3.1) as well as the $\widetilde{J}$ preserving quantum isometry group (Sect. 3.3) make sense. We refer to Sects. 6 and 7 of [10] for the related discussions and computations.

8.1.1 Quantum Isometry Groups of $(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$

We note that the state τ is the vector state associated to the cyclic and separating vector $\delta_e \in l^2(\Gamma)$. It is easy to see that the eigenvectors of D_Γ belong to the dense subspace $\mathbb{C}[\Gamma]\delta_e$. We also have $\mathbb{C}[\Gamma] = \mathrm{Span}\{a \in \mathbb{C}[\Gamma] : a_e \text{ is an eigenvector of } D_\Gamma\}$. Hence, by Theorem 3.4.2, $\widetilde{QISO^+}(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$ exists and is isomorphic to $QISO^+(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma) * C(\mathbb{T})$. For the purpose of computation, we will apply Corollary 3.4.5. Since the quantum isometry group depends on the choice of the word length function which in turn depends on the choice of the generating set S, we denote $QISO^+(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$ by $QISO^+(\Gamma, S)$. We will see in the next section that the quantum group structure of the quantum isometry group actually depends on the choice of S.

Remark 8.1.1 The corepresentation U_0 (say) of the quantum group $\widetilde{QISO^+}(\Gamma, S)$ on $l^2(\Gamma)$ commutes with D_Γ, hence we have

$$U_0(\delta_e) = \delta_e \otimes q \tag{8.1.1}$$

for some unitary element q in $\widetilde{QISO^+}(\Gamma, S)$. It can be seen from the proof of Theorem 3.4.2 that $C^*(q) \cong C(\mathbb{T})$ and this $C(\mathbb{T})$ appears in the expression for $\widetilde{QISO^+}(\Gamma, S)$ as given in Theorem 3.4.2.

The following proposition and the next theorem indicates the computational scheme as well as the notations that we are going to follow.

Proposition 8.1.2 *1. The coaction* α *of* $QISO^+(\Gamma, S)$ *on* $C^*(\Gamma)$ *satisfies*

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in \Gamma : l(\gamma) = l(\gamma')} \lambda_{\gamma'} \otimes q_{\gamma',\gamma}, \quad \gamma \in \Gamma. \tag{8.1.2}$$

for some elements $\{q_{\gamma',\gamma} \in QISO^+(\Gamma, S) : \gamma, \gamma' \in \Gamma\}$.

2.

$$q_{\gamma'^{-1},\gamma} = q^*_{\gamma',\gamma^{-1}} \quad \forall\, \gamma, \gamma' \in \Gamma, \text{ with } l(\gamma) = l(\gamma'). \tag{8.1.3}$$

3. The matrices $((q_{t,s}))_{t,s \in S}$ *and* $((q^*_{t,s}))_{t,s \in S}$ *are unitaries in* $M_{\mathrm{card}(S)}(QISO^+(\Gamma, S))$.

4. $\widetilde{\mathrm{QISO}^+}(\Gamma, S)$ *is the free product of* $\mathrm{QISO}^+(\Gamma, S)$ *with* $C(\mathbb{T})$.

Proof The first assertion follows from Corollary 3.4.5. Next, the condition $(\alpha(\lambda_\gamma))^* = \alpha(\lambda_\gamma{}^*) = \alpha(\lambda_{\gamma^{-1}})$ implies that $\sum_{\gamma' \in \Gamma : l(\gamma') = l(\gamma)} \lambda_{\gamma'^{-1}} \otimes q^*_{\gamma',\gamma} = \sum_{\gamma' \in S : l(\gamma') = l(\gamma^{-1})} \lambda_{\gamma'} \otimes q_{\gamma',\gamma^{-1}}$. This leads to 2. The third claim follows from the fact that α preserves the trace τ. Indeed, we observe that for $s, t \in S$, and writing $Q := \mathrm{QISO}^+(\Gamma, S)$:

$\delta_{s,t} \otimes 1_Q = \delta_{s^{-1}t,e} \otimes 1_Q$
$= \tau(\lambda_{s^{-1}t}) \otimes 1_Q = (\tau \otimes \mathrm{id}_Q)(\alpha(\lambda_{s^{-1}t}))$
$= (\tau \otimes \mathrm{id}_Q)(\alpha(\lambda_s)^* \alpha(\lambda_t)) = (\tau \otimes \mathrm{id})\big(\big(\sum_{s' \in S} \lambda_{s'} \otimes q_{s',s}\big)^* \big(\sum_{t' \in S} \lambda_{t'} \otimes q_{t',t}\big)\big)$
$= \sum_{s',t' \in S} \tau(\lambda_{s'^{-1}t'}) q^*_{s',s} q_{t',t} = \sum_{s' \in S} q^*_{s',s} q_{s',t}$.

3. follows by combining this observation with an analogous computation starting from $\tau(\lambda_{st^{-1}})$.

We note that the matrix $((q_{t,s}))_{t,s \in S}$ is a fundamental unitary corepresentation of $\mathrm{QISO}^+(\Gamma, S)$. If q is as in (8.1.1), the unitary $\widetilde{U} \in M(K(\mathcal{H}) \otimes \widetilde{\mathrm{QISO}^+}(\Gamma, S))$ implementing the coaction of $\mathrm{QISO}^+(\Gamma, S)$ is determined by the conditions

$$\widetilde{U}(\delta_e \otimes 1_{\widetilde{\mathrm{QISO}^+}(\Gamma,S)}) = \delta_e \otimes q, \tag{8.1.4}$$

$$\widetilde{U}(\delta_\gamma \otimes 1_{\widetilde{\mathrm{QISO}^+}(\Gamma,S)}) = \sum_{\gamma' \in \Gamma,\, l(\gamma') = l(\gamma)} \delta_{\gamma'} \otimes q_{\gamma',\gamma} q, \quad \gamma \in \Gamma. \tag{8.1.5}$$

Thus, 4. follows from Remark 8.1.1 combined with Corollary 3.4.5. □

Theorem 8.1.3 *Let* Γ, S *and* $\{q_{t,s} : s, t \in S\}$ *be as above and let* $\mathcal{O}_{\Gamma,\mathbf{S}}$ *be the category with objects* $(\mathcal{S}, ((b_{ts}))_{s,t \in S})$ *where* $\mathcal{S}$ *is a unital* C^* *algebra generated by the elements of the matrix* $((b_{st}))_{s,t \in S}$ *satisfying the following conditions:*

a. *The matrices* $((b_{t,s}))_{t,s \in S}$ *and* $((b^*_{t,s}))_{t,s \in S}$ *are unitaries in* $M_{\mathrm{card}(S)}(\mathcal{S})$.
b. *There exists a* $*$*-homomorphism* $\alpha_{\mathcal{S}} : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \otimes \mathcal{S}$ *such that for all* s *in* S, $\alpha_{\mathcal{S}}(\lambda_s) = \sum_{t \in S} \lambda_t \otimes b_{ts}$.

Then $(\mathrm{QISO}^+(\Gamma, S), ((q_{ts}))_{s,t \in S})$ *is the universal object in the category* $\mathcal{O}_{\Gamma,\mathbf{S}}$.

Proof Let us denote the cardinality of the set S by k.

Clearly, there is a universal C^* algebra, say $\mathcal{S}_0$, with generators $\{q^0_{ts} : s, t \in S\}$ satisfying the conditions a. and b., which is in fact a quotient of the quantum group $A_{u,k}(I)$ by a closed two-sided ideal. By Proposition 8.1.2, $(\mathrm{QISO}^+(\Gamma, S), ((q_{ts}))_{s,t \in S})$ is an object in the category $\mathcal{O}_{\Gamma,\mathbf{S}}$ and hence we get a surjective C^* homomorphism from $\mathcal{S}_0$ to $\mathrm{QISO}^+(\Gamma, S)$ which sends q^0_{ts} to q_{ts}.

We want a C^* homomorphism in the converse direction. Following the line of arguments in the proof of Theorem 3.1.7 and using the universality of $\mathcal{S}_0$, we get a coproduct, say $\Delta_{\mathcal{S}_0}$ on $\mathcal{S}_0$ given by

$$\Delta_{\mathcal{S}_0}(q^0_{ts}) = \sum_{w \in S} q^0_{tw} \otimes q^0_{ws},$$

and it can also be shown that $(\mathcal{S}_0, \Delta_{\mathcal{S}_0})$ is a compact quantum group. In fact, it is a quantum subgroup of $A_{u,k}(I)$ in the obvious way, and there is a norm bounded counit ϵ on $\mathcal{S}_0$ satisfying $\epsilon(q^0_{s,t}) = \delta_{s,t}$. Moreover, it is easy to see that $(\mathrm{id} \otimes \epsilon)\alpha_{\mathcal{S}_0}(a) = a$ for all a in $\mathbb{C}[\Gamma]$.

Now we observe that $(C^*_r(\Gamma), \Delta_\Gamma)$ is an object in $\mathcal{O}_{\Gamma,\mathbf{S}}$ and so there is a surjective unital C^* homomorphism π from $\mathcal{S}_0$ to $C^*_r(\Gamma)$ which maps $q^0_{t,s}$ to $\lambda_s \delta_{s,t}$. It can be easily seen that this map is a CQG morphism and therefore $(C^*_r(\Gamma), \Delta_\Gamma)$ is naturally identified with a quantum subgroup of $(\mathcal{S}_0, \Delta_{\mathcal{S}_0})$.

Let us consider the positive linear normalized functional ϕ on $\mathbb{C}[\Gamma]$ defined by $\phi = (\tau \otimes h)\alpha_{\mathcal{S}_0}$. Then it is easy to see that ϕ is preserved by $\alpha_{\mathcal{S}_0}$. Thus, $\Delta_\Gamma = (\mathrm{id} \otimes \pi)\alpha_{\mathcal{S}_0}$ also preserves the functional ϕ. In other words, ϕ is a Γ-invariant normalized linear functional on $\mathbb{C}[\Gamma]$ and thus coincides with τ. Thus, $\alpha_{\mathcal{S}_0}$ preserves τ and hence extends to a τ preserving homomorphism from $C^*_r(\Gamma)$ to $C^*_r(\Gamma) \otimes \mathcal{S}_0$. From the above discussion, it becomes clear that α is a coaction on $C^*_r(\Gamma)$ by the quantum group $\mathcal{S}_0$. Moreover, by Lemma 1.3.4, we have a unitary corepresentation $\widetilde{\alpha_{\mathcal{S}_0}}$ of $\mathcal{S}_0$ on $l^2(\Gamma)$.

Lastly, we claim that the coaction $\alpha_{\mathcal{S}_0}$ commutes with the operator $\widehat{D_\Gamma}$ as introduced in Sect. 3.4. To this end, we define $V_0 = \mathbb{C}1_{C^*_r(\Gamma)}$, $V_n = \mathrm{Span}\{\lambda_\gamma : l(\gamma) = n\}$ and $W_n = \mathrm{Span}\{V_k : 0 \le k \le n\}$. It can be easily seen that we have an orthogonal decomposition (w.r.t. τ) $W_{n+1} = W_n \oplus V_{n+1}$. Clearly, $\alpha_{\mathcal{S}_0}$ keeps V_0 and V_1 invariant, and consequently, by the homomorphic property of $\alpha_{\mathcal{S}_0}$ and the inequality $l(\gamma\gamma') \le l(\gamma) + l(\gamma')$, $\alpha_{\mathcal{S}_0}$ keeps each of the subspaces W_n invariant. Thus, $\widetilde{\alpha_{\mathcal{S}_0}}(W_n) \subseteq W_n \otimes \mathcal{S}_0$ for all n and since $\widetilde{\alpha_{\mathcal{S}_0}}$ is a unitary corepresentation, we deduce that $\widetilde{\alpha_{\mathcal{S}_0}}$ keeps $W_{n+1} \cap W_n^\perp = V_n$ invariant. This proves the claim.

Combining the arguments above, we conclude that $(\mathcal{S}_0, \alpha_{\mathcal{S}_0})$ is an object of the category $\widehat{C}$ of Corollary 3.4.5. This gives us the required morphism from $\mathrm{QISO}^+(\Gamma, S)$ to $\mathcal{S}_0$ which will be the inverse of the morphism from $\mathcal{S}_0$ to $\mathrm{QISO}^+(\Gamma, S)$ mentioned before. This finishes the proof. □

8.2 The Case of Finitely Generated Abelian Groups

We start this section with the cases of the groups $\mathbb{Z}$ and $\mathbb{Z}_n := \mathbb{Z}/_{n\mathbb{Z}}$. All these quantum isometry groups turn out to be classical with the only exception of $\mathbb{Z}/_{4\mathbb{Z}}$. We recall that a similar phenomenon was earlier observed for quantum symmetry groups of n-gons in [11]. The section ends with the discussion on the general case of finitely generated abelian groups.

8.2.1 *Computation for the Groups $\mathbb{Z}_n$ and $\mathbb{Z}$*

To begin with, a straightforward calculation based on the approach described in the last section shows that $\mathrm{QISO}^+(\mathbb{Z}_2, \{1\})$ is isomorphic (as a compact quantum group) to $C^*(\mathbb{Z}_2)$ (so to $\mathbb{C}^2$ as a C^*-algebra). The next simplest possible cases fitting in the framework described in the previous section are those of $\Gamma = \mathbb{Z}_n$, $n \geq 3$ and $\Gamma = \mathbb{Z}$.

Theorem 8.2.1 *Let $n \in \mathbb{N} \setminus \{1, 2, 4\}$ and consider $\Gamma = \mathbb{Z}_n$ with the standard symmetric generating set $S = \{1, n-1\}$ ($n \geq 3$). Then $\mathrm{QISO}^+(\mathbb{Z}_n, S)$ is isomorphic to $C^*(\mathbb{Z}_n) \oplus C^*(\mathbb{Z}_n)$ as a C^*-algebra. Its coaction on $C^*(\mathbb{Z}_n)$ is given by the formula*

$$\alpha(\lambda_1) = \lambda_1 \otimes A + \lambda_{n-1} \otimes B,$$

where A and B are identified respectively with $\lambda_1 \oplus 0, 0 \oplus \lambda_1 \in C^*(\mathbb{Z}_n) \oplus C^*(\mathbb{Z}_n)$. The coproduct of $\mathrm{QISO}^+(\mathbb{Z}_n, S)$ is determined by the condition that $\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$ is the fundamental corepresentation.

Proof We begin by observing that $C^*(\mathbf{Z}_n) \oplus C^*(\mathbf{Z}_n)$ with the coaction given above is an object of the category $\widehat{\mathcal{C}}$ of Corollary 3.4.5. Thus, $C^*(\mathbf{Z}_n) \oplus C^*(\mathbf{Z}_n)$ is a subobject of $\mathrm{QISO}^+(\mathbf{Z}_n, S)$ in that category.

Now we prove the other direction. The coaction of $\mathrm{QISO}^+(\mathbb{Z}_n, S)$ on $C^*(\mathbb{Z}_n)$ is determined by the formula

$$\alpha(\lambda_1) = \lambda_1 \otimes A + \lambda_{n-1} \otimes B, \tag{8.2.1}$$

where A, B are some elements in $\mathrm{QISO}^+(\mathbb{Z}_n, S)$. Due to the fact that $\lambda_{n-1} = \lambda_1^*$ and α is *-preserving, we must have

$$\alpha(\lambda_{n-1}) = \lambda_1 \otimes B^* + \lambda_{n-1} \otimes A^*. \tag{8.2.2}$$

It follows from the discussion in the previous section that the matrix $\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$ is the fundamental corepresentation of $\mathrm{QISO}^+(\mathbb{Z}_n, S)$ and so in particular it is a unitary in $M_2(\mathrm{QISO}^+(\mathbb{Z}_n, S))$. Thus $AB + BA = 0$ and both A and B are normal. We have then

$$\alpha(\lambda_2) = \lambda_2 \otimes A^2 + \lambda_{n-2} \otimes B^2. \tag{8.2.3}$$

Let $n = 3$. Then the comparison of (8.2.2) and (8.2.3) yields

$$A^2 = A^*, B^2 = B^*,$$

so that the unitarity of the fundamental corepresentation yields $A^3 + B^3 = 1$. Further

$$\alpha(\lambda_0) = \alpha(\lambda_1)\alpha(\lambda_2) = \lambda_0 \otimes (AA^* + BB^*) + \lambda_1 \otimes BA^* + \lambda_2 \otimes AB^*.$$

As α commutes with D_Γ, this implies $AB^* = BA^* = 0$. We claim that A^3 is a projection. Indeed,

$$(A^3)^2 = A^3(1 - B^3) = A^3 - A^3 B^* B = A^3.$$

Moreover, $(A^3)^* = (A^2)^3 = A^3$. Similarly we show that B^3 is a projection. Finally, $AB = A(B^2)^* = 0$. Hence, $BA = -AB = 0$, so that A and B generate 'orthogonal' copies of $C^*(\mathbb{Z}_3)$.

Let then $n > 4$. Then by formulae (8.2.1) and (8.2.3) we have

$$\alpha(\lambda_3) = \alpha(\lambda_2)\alpha(\lambda_1) = \lambda_3 \otimes A^3 + \lambda_{n+1} \otimes A^2 B + \lambda_{n-1} \otimes B^2 A + \lambda_{n-3} \otimes B^3.$$

This means that $A^2 B = B^2 A = 0$. As a consequence, for all $k = 1, 2, \ldots, n-1$ we have

$$\alpha(\lambda_k) = \lambda_k \otimes A^k + \lambda_{n-k} \otimes B^k. \tag{8.2.4}$$

Indeed, the cases $k = 1, 2$ are covered by formulas (8.2.1) and (8.2.3) and the inductive reasoning for $k \geq 2$ gives

$$\begin{aligned} \alpha(\lambda_{k+1}) &= \alpha(\lambda_k)\alpha(\lambda_1) = (\lambda_k \otimes A^k + \lambda_{n-k} \otimes B^k)(\lambda_1 \otimes A + \lambda_{n-1} \otimes B) \\ &= \lambda_{k+1} \otimes A^{k+1} + \lambda_{n-k-1} \otimes B^{k+1}, \end{aligned}$$

so that (8.2.4) follows. Combining it (in the case of $k = n-1$) with (8.2.2) we see that $A^{n-1} = A^*$, $B^{n-1} = B^*$ and further $A^* B = A^{n-1} B = A^{n-3} A^2 B = 0$. This together with the equation $A^* B + BA^* = 0$ which follows from the unitarity condition for the fundamental corepresentation implies that $BA^* = 0$. Hence, A^* commutes with B and as A and B are normal, A^* commutes with B^* too, so that and $AB + BA = 0$ implies $AB = BA = 0$. Thus again A and B are "partial unitaries" satisfying the conditions $A^{n-1} = A^*$, $B^{n-1} = B^*$ with orthogonal ranges summing up to 1, which completes the proof. $\square$

Theorem 8.2.1 specifically excluded $n = 4$. Curiously, the quantum group of orientation preserving isometries $\mathrm{QISO}^+(\mathbb{Z}_4, S)$ (for $S = \{1, 3\}$) is a noncommutative C^*-algebra.

Theorem 8.2.2 ([10, 12, 13]) *As a C^* algebra, the quantum isometry group* $\mathrm{QISO}^+(\mathbb{Z}_4, S)$ *is isomorphic with the universal C^*-algebra generated by two normal elements A, B satisfying the following relations:*

$$AB + BA = AB^* + BA^* = A^* B + BA^* = 0, \;\; A^2 + B^2 = (A^*)^2 + (B^*)^2,$$

$$A^2 B + B^3 = B^*, \;\; B^2 A + A^3 = A^*, A^4 + B^4 + 2A^2 B^2 = 1, AA^* + BB^* = 1.$$

The coaction of $\mathrm{QISO}^+(\mathbb{Z}_4, S)$ *on $C^*(\mathbb{Z}_4)$ is given by the formula*

$$\alpha(\lambda_1) = \lambda_1 \otimes A + \lambda_3 \otimes B.$$

The coproduct of $\mathrm{QISO}^+(\mathbb{Z}_4, S)$ *can be read out from the condition that* $\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$ *is the fundamental corepresentation. The* C^**-algebra* $\mathrm{QISO}^+(\mathbb{Z}_4, S)$ *is not commutative and is isomorphic to* $C^*(D_\infty \times \mathbb{Z}_2)$ *as a* C^* *algebra and to* $\mathbb{Z}_2 \wr_* \mathbb{Z}_2$ *as a quantum group. Here,* D_∞ *denotes the infinite dihedral group.*

Proof The description of the algebra in terms of generators and relations follow exactly as in the previous computations. The identifications with $C^*(D_\infty \times \mathbb{Z}_2)$ and $\mathbb{Z}_2 \wr_* \mathbb{Z}_2$ were made in [12, 13], respectively. □

Remark 8.2.3 Already in this simple case we can see that if one chooses a non-minimal generating set in Γ, the resulting quantum isometry group will be different. In particular if we put $S' = \{1, 2, 3\}$, the quantum group $\mathrm{QISO}^+(\mathbb{Z}_4, S')$ is not isomorphic to the one obtained in the theorem above. This can be shown by analyzing the quotients of the $\mathrm{QISO}^+(\mathbb{Z}_4, S)$ and $\mathrm{QISO}^+(\mathbb{Z}_4, S')$ by respective commutator ideals and checking that in the first case one obtains the algebra $C(S_2 \times S_2)$ and in the second the algebra $C(S_4)$.

Consider now $\Gamma = \mathbb{Z}$ with the standard symmetric generating set $S = \{1, -1\}$. Theorem 8.2.5 describes the quantum isometry group $\mathrm{QISO}^+(\mathbb{Z}, S)$, the proof of which needs the following Lemma.

Lemma 8.2.4 *Let* α *be a faithful, smooth coaction of a compact quantum group* $(\mathcal{Q}, \Delta)$ *on* $C(S^1)$ *defined by* $\alpha(z) = z \otimes A + \overline{z} \otimes B$. *Then* $\mathcal{Q}$ *is a commutative* C^* *algebra.*

Proof By the assumption of faithfulness, it is clear that $\mathcal{Q}$ is generated (as a unital C^* algebra) by A and B. Moreover, recall that smoothness in particular means that A and B must belong to the algebra $\mathcal{Q}_0$ spanned by matrix elements of irreducible corepresentations of $\mathcal{Q}$. Since $z\overline{z} = \overline{z}z = 1$ and α is a $*$-homomorphism, we have $\alpha(z)\alpha(\overline{z}) = \alpha(\overline{z})\alpha(z) = 1 \otimes 1$.

Comparing coefficients of z^2, $\overline{z}^2$ and 1 in both hand sides of the relation $\alpha(z)\alpha(\overline{z}) = 1 \otimes 1$, we get

$$AB^* = BA^* = 0, \quad AA^* + BB^* = 1. \tag{8.2.5}$$

Similarly, $\alpha(\overline{z})\alpha(z) = 1 \otimes 1$ gives

$$B^*A = A^*B = 0, \quad A^*A + B^*B = 1. \tag{8.2.6}$$

Let $U = A + B$, $P = A^*A$, $Q = AA^*$. Then it follows from (8.2.5) and (8.2.6) that U is a unitary and P is a projection since P is self-adjoint and

$$P^2 = A^*AA^*A = A^*A(1 - B^*B) = A^*A - A^*AB^*B = A^*A = P.$$

Moreover,

$$\begin{aligned} &UP \\ &= (A+B)A^*A \quad = AA^*A + BA^*A \quad = AA^*A \\ &\qquad \text{(since } BA^* = 0 \text{ from (8.2.5))} \\ &= A(1-B^*B) \quad = A - AB^*B \quad = A. \end{aligned}$$

Thus, $A = UP$, $B = U - UP = U(1-P) \equiv UP^{\perp}$, so $\mathcal{Q} = C^*(A, B) = C^*(U, P)$.

We can rewrite the coaction α as follows:

$$\alpha(z) = z \otimes UP + \overline{z} \otimes UP^{\perp}.$$

The coproduct Δ can easily be calculated from the requirement $(\mathrm{id} \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id})\alpha$, and it is given by:

$$\Delta(UP) = UP \otimes UP + P^{\perp}U^{-1} \otimes UP^{\perp}, \tag{8.2.7}$$

$$\Delta(UP^{\perp}) = UP^{\perp} \otimes UP + PU^{-1} \otimes UP^{\perp}. \tag{8.2.8}$$

From this, we get

$$\Delta(U) = U \otimes UP + U^{-1} \otimes UP^{\perp}, \tag{8.2.9}$$

$$\Delta(P) = \Delta(U^{-1})\Delta(UP) = P \otimes P + UP^{\perp}U^{-1} \otimes P^{\perp}. \tag{8.2.10}$$

It can be checked that Δ given by the above expression is coassociative.

Let h denote the right-invariant Haar state on $\mathcal{Q}$. By the general theory of compact quantum groups, h must be faithful on $\mathcal{Q}_0$. We have (by right-invariance of h):

$$(\mathrm{id} \otimes h)(P \otimes P + UP^{\perp}U^{-1} \otimes P^{\perp}) = h(P)1.$$

That is, we have

$$h(P^{\perp})UP^{\perp}U^{-1} = h(P)P^{\perp}. \tag{8.2.11}$$

Since P is a positive element in $\mathcal{Q}_0$ and h is faithful on $\mathcal{Q}_0$, $h(P) = 0$ if and only if $P = 0$. Similarly, $h(P^{\perp}) = 0$, that is $h(P) = 1$, if and only if $P = 1$. However, if P is either 0 or 1, clearly $\mathcal{Q} = C^*(U, P) = C^*(U)$, which is commutative. On the other hand, if we assume that P is not a trivial projection, then $h(P)$ is strictly between 0 and 1, and we have from (8.2.11)

$$UP^{\perp}U^{-1} = \frac{h(P)}{1-h(P)}P^{\perp}.$$

Since both $UP^{\perp}U^{-1}$ and $P^{\perp}$ are nontrivial projections, they can be scalar multiples of each other if and only if they are equal, so we conclude that $UP^{\perp}U^{-1} = P^{\perp}$, that is U commutes with $P^{\perp}$, hence with P, and $\mathcal{Q}$ is commutative. $\square$

Theorem 8.2.5 *The quantum group* $\mathrm{QISO}^{+}(\mathbb{Z}, S)$ *is isomorphic to the (commutative) compact quantum group* $C(\mathbb{T} \rtimes \mathbb{Z}_2)$*. Its coaction on* $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ *is given by the standard (isometric) action of the group* $\mathbb{T} \rtimes \mathbb{Z}_2$ *on* $\mathbb{T}$*.*

Proof Let the coaction of $\mathrm{QISO}^{+}(\mathbb{Z}, S)$ on $C^*(\mathbb{Z})$ be determined by the formula

$$\alpha(\lambda_1) = \lambda_1 \otimes A + \lambda_{-1} \otimes B,$$

where $A, B \in \mathrm{QISO}^{+}(\mathbb{Z}, S)$. Via the canonical identification of $C^*(\mathbb{Z})$ with $C(\mathbb{T})$ the coaction can be written as

$$\alpha(z) = z \otimes A + \overline{z} \otimes B. \tag{8.2.12}$$

Moreover, it is easy to see that the conditions for the coaction on $C^*(\mathbb{Z})$ to commute with the operator $\hat{D_{\Gamma}}$ constructed from the length function are exactly the same as the conditions for the coaction on $C(\mathbb{T})$ to commute with the standard Laplacian on $C(\mathbb{T})$. Thus $\mathrm{QISO}^{+}(\mathbb{Z}, S) \cong \mathrm{QISO}^{\mathcal{L}}(C(\mathbb{T}))$ where $\mathrm{QISO}^{\mathcal{L}}$ is as in Sect. 3.1.

But the isometric coaction of $\mathrm{QISO}^{\mathcal{L}}(\mathbb{T})$ satisfies the conditions of Lemma 8.2.4 and hence $\mathrm{QISO}^{\mathcal{L}}(\mathbb{T}) \cong C(\mathrm{ISO}(\mathbb{T})) \cong C(\mathbb{T} \rtimes \mathbb{Z}_2)$. $\square$

Remark 8.2.6 Since the quantum group of orientation preserving isometries of $C(\mathbb{T})$(w.r.t the classical Dirac operator) is a quantum subgroup of $\mathrm{QISO}^{\mathcal{L}}(C(\mathbb{T}))$ by Theorem 3.2.11, it follows from the above discussion that the quantum group of orientation preserving isometries is commutative as a C^* algebra and hence coincides with $C(\mathbb{T})$.

It may seem surprising that the $\mathrm{QISO}^{+}(\mathbb{Z}, S)$ is strictly bigger than the quantum group of orientation preserving isometries of $C(\mathbb{T})$. This is due to the fact that the Dirac operator coming from the length function on $\mathbb{Z}$ has (in the $L^2(\mathbb{T})$-picture) a spectral decomposition of the form $\sum_{n\in\mathbb{Z}} |n|z^n$ and its eigenspaces coincide with those of the classical Laplacian, and not the usual Dirac operator on $L^2(\mathbb{T})$.

8.2.2 Results for the General Case

In this subsection, we identify (without proof) the quantum isometry groups of almost all finitely generated discrete groups. For a finitely generated discrete group Γ and a generating set S as before, we denote the universal object in the category of all compact quantum groups of the form $C(G)$, where G is a (classical) compact group coacting isometrically on the spectral triple $(\mathbb{C}[\Gamma], l^2(\Gamma), D_\Gamma)$ by the symbol $C(\mathrm{ISO}(\Gamma))$. Thus, the quantum isometry group of Γ is commutative as a C^* algebra if and only if it is isomorphic with $C(\mathrm{ISO}(\Gamma))$.

If $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$, with Γ_is to be either $\mathbb{Z}_n$ or $\mathbb{Z}$, the generating set S of Γ with which we will work is given by $S = \cup_i S_i'$ where $S_i' = (0, 0, \ldots S_i \ldots)$ and S_i equals $\{1, n_1\}$ for $\mathbb{Z}_n$ and $\{1, -1\}$ for $\mathbb{Z}$. Having fixed this generating set S, we will write $\mathrm{QISO}^+(\Gamma)$ for the quantum isometry group of Γ. Then we have the following result.

Proposition 8.2.7 *For* $\Gamma, \Gamma_i S, S_i$ *as above,* $\mathrm{QISO}^+(\Gamma_1) \otimes^{\max} \cdots \otimes^{\max} \mathrm{QISO}^+(\Gamma_n)$ *is a quantum subgroup of* $\mathrm{QISO}^+(\Gamma)$.

The following result computes $C(\mathrm{ISO}(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdot\cdot \mathbb{Z}_n))$.

Proposition 8.2.8 ([14]) *Let* $\Gamma = \underbrace{(\mathbb{Z}_n \times \mathbb{Z}_n \times \cdot\cdot \mathbb{Z}_n)}_{k \text{ copies}}$ *and* S_k *be the group of permutation of k elements. We have:*

1. *If* $n = 2$ *then we have* $C(\mathrm{ISO}(\Gamma)) \cong C(\underbrace{(\widehat{\mathbb{Z}_2} \times \widehat{\mathbb{Z}_2} \times \cdot\cdot \widehat{\mathbb{Z}_2})}_{k \text{ copies}} \rtimes S_k)$.
2. *If* $n \neq 2, 4$ *then* $C(\mathrm{ISO}(\Gamma)) \cong C(\underbrace{(\widehat{\mathbb{Z}_n} \times \widehat{\mathbb{Z}_n} \times \cdot\cdot \widehat{\mathbb{Z}_n})}_{k \text{ copies}} \rtimes (\mathbb{Z}_2^k \rtimes S_k))$.

Next, we state the following result, the $n = 2$ case of which was proved in [13] and the rest of the cases were dealt in [14].

Theorem 8.2.9 *If* $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n \cdots \times \mathbb{Z}_n$*, where* $n = \infty$ *is allowed but* $n \neq 2, 4$*, then* $\mathrm{QISO}^+(\Gamma) \cong C(\mathrm{ISO}(\Gamma))$*. If* $n = 2$*, then* $\mathrm{QISO}^+(\Gamma) \cong O_n^{-1}$.

Here, O_n^{-1} (see [15] and references therein) is the quantum subgroup of $A_o(n)$ defined by the ideal generated by the following relations:

1. $u_{ij}u_{ik} = -u_{ik}u_{ij},\ u_{ji}u_{ki} = -u_{ki}u_{ji}$ for $i \neq j$,
2. $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $i \neq k,\ j \neq l$.

Remark 8.2.10 It easily follows from Theorem 8.2.2 and Proposition 8.2.7 that $\mathrm{QISO}^+(\mathbb{Z}_4 \times \mathbb{Z}_4 \cdots \mathbb{Z}_4)$ is noncommutative as a C^* algebra. To the best of our knowledge, apart from the case of $\mathrm{QISO}^+(\mathbb{Z}_4)$, the structures of these quantum groups have not been understood.

More generally, we have the following result.

Theorem 8.2.11 ([14])

Let $\Gamma = \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_l$ *where* $\Gamma_i = \underbrace{(\mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i} \cdot\cdot \times \mathbb{Z}_{n_i})}_{k_i \text{ copies}}$ *and* $n_i = \infty$ *is allowed. Also assume that* $n_1 \neq n_2 \neq \cdot\cdot \neq n_l$ *and at most one* n_i *is 2 or 4.*

If none of the n_i s equal 4, then $\mathrm{QISO}^+(\Gamma) \cong C(\mathrm{ISO}\,(\underbrace{\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdot\cdot \times \mathbb{Z}_{n_1}}_{k_1\ copies}))$

$\otimes C(\mathrm{ISO}\,(\underbrace{\mathbb{Z}_{n_2} \times \mathbb{Z}_{n_2} \cdot\cdot \times \mathbb{Z}_{n_2}}_{k_2\ copies})) \otimes \cdot\cdot \otimes C(\mathrm{ISO}\,(\underbrace{\mathbb{Z}_{n_l} \times \mathbb{Z}_{n_l} \cdot\cdot \times \mathbb{Z}_{n_l}}_{k_l\ copies}))$.

If one of the n_i equals 4, then $\mathrm{QISO}^+(\Gamma)$ *is noncommutative and isomorphic to* $\mathrm{QISO}^+(\Gamma_1) \otimes^{\max} \mathrm{QISO}^+(\Gamma_2) \otimes^{\max} \cdots \otimes^{\max} \mathrm{QISO}^+(\Gamma_l)$.

As a corollary to the above results, the following necessary and sufficient conditions for the quantum isometry group to be commutative (as a C^* algebra) follows.

Corollary 8.2.12 ([14]) *Let Γ be a finitely generated abelian group. Then* $\mathrm{QISO}^+(\Gamma)$ *is commutative if and only if Γ is of the form*

$$\underbrace{(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \cdot\cdot \times \mathbb{Z}_{n_1})}_{k_1\ copies} \times \underbrace{(\mathbb{Z}_{n_2} \times \mathbb{Z}_{n_2} \cdot\cdot \times \mathbb{Z}_{n_2})}_{k_2\ copies} \times \cdots \times \underbrace{(\mathbb{Z}_{n_l} \times \mathbb{Z}_{n_l} \cdot\cdot \times \mathbb{Z}_{n_l})}_{k_l\ copies},$$

where $n_i \neq 4$ for all i and if $n_j = 2$ for some j, then k_j must be 1.

8.3 The Case of Free Products of Groups

If $\Gamma = \Gamma_1 * \cdots * \Gamma_n$, with Γ_i to be either $\mathbb{Z}_n$ or $\mathbb{Z}$, the generating set S of Γ with which we will work is given by $S = \cup_i S_i'$ where S_i' is as in Sect. 8.2.2. As before, we will write $\mathrm{QISO}^+(\Gamma)$ for the quantum isometry group of Γ without mentioning the generating set S explicitly. Moreover, we will denote the free group on n-generators by the symbol $\mathbb{F}_n$.

8.3.1 Some Quantum Groups

Before going to the main results, we need to introduce the free wreath product by the quantum permutation group $A_s(n)$ as well as some other quantum groups appearing in [12, 13, 16].

In [17], Bichon defined the notion of the free wreath product.

Definition 8.3.1 ([17]) Let $\mathcal{S}$ be a compact quantum group and $n > 1$. The free wreath product of $\mathcal{S}$ by the quantum permutation group $A_s(n)$ is the quotient of $\mathcal{S}^{*n} * A_s(n)$ by the closed two-sided ideal generated by the elements

$$\nu_k(a)t_{ki} - t_{ki}\nu_k(a),\ 1 \le i, k \le N,\ a \in \mathcal{S},$$

where $((t_{ij}))$ is the matrix coefficients of the quantum permutation group $A_s(n)$ and $\nu_k(a)$ denotes the natural image of $a \in \mathcal{S}$ in the k-th factor of $\mathcal{S}^{*n}$. This is denoted by $\mathcal{S} \wr_* A_s(n)$.

The comultiplication satisfies the following rules:

$$\Delta(\nu_i(a)) = \sum_{k=1}^{n} \nu_i(a_{(1)}) t_{ik} \otimes \nu_k(a_{(2)}),$$

where we have used the Sweedler convention of writing $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Next, we define the two parameter family $H^+(p, q)$ discovered in [16]. There is a natural fundamental $2p + q$ dimensional corepresentation of $H^+(p, q)$. Thus, in [16], Banica and Skalski are led to make the following notations and conventions: for a $(2p + q) \times (2p + q)$ matrix, they denote the indices for the p-part by pairs $i\alpha$, where $i \in \{0, 1\}$ and $\alpha \in \{1, 2, \dots, p\}$ and to the q-part by letters running from 1 to q. They write

$$\tau_p = \{i\alpha : i \in \{0, 1\}, \alpha \in \{1, 2, \cdots p\}\}, \tau_{p,q} = \tau_p \cup \{1, 2, \cdots q\}.$$

They also use the notation $\bar{\cdot} : \overline{0} = 1, \overline{1} = 0$. Moreover, $\overline{z} = \overline{i}\alpha$ if $z = i\alpha \in \tau_p$, $\overline{z} = M$ if $z = M \in \{1, 2, \cdots q\}$.

Then the quantum group $H^+(p, q)$ is defined as follows:

Definition 8.3.2 ([16]) $H^+(p, q)$ is the quantum subgroup of $A_u(2p + q)$ with a fundamental corepresentation U with entries $U_{z,y} : z, y \in \tau_{p,q}$ such that $U_{z,y}$ are all partial isometries and for all $i\alpha, j\beta \in \tau_p$, $M, N \in \{1, 2, \cdots q\}$, we have

$$U^*_{i\alpha, j\beta} = U_{\overline{i}\alpha, \overline{j}\beta}, U^*_{i\alpha, N} = U_{\overline{i}\alpha, N},$$

$$U^*_{M, j\beta} = U_{M, \overline{j}\beta}, \ U^*_{M,N} = U_{M,N}.$$

Associated to $H^+(p, q)$ are the following quantum groups:

Definition 8.3.3 ([13, 16])

K_n^+ is the quantum subgroup of $H^+(n, 0)$ defined by the additional condition that each U_{ij} is a normal, partial isometry. $H_s^+(n, 0)$ is the quantum subgroup of K_n^+ satisfying the extra condition $U_{ij}^* = U_{ij}^{s-1}$.

Definition 8.3.4 ([12]) Let $u = ((u_{ij}))$ denote the defining corepresentation of the quantum group $A_o(n)$. Then H_n^+ is the quantum subgroup of $A_o(n)$ defined by the relations $u_{ij} u_{ik} = 0, u_{ji} u_{ki} = 0 \quad \forall i, j, k$ with $j \neq k$.

In [18], it was shown that $H_n^+ \cong C^*(\mathbb{Z}_2) \wr_* A_s(n)$. A similar description of the quantum groups $H_s^+(n, 0)$ and K_n^+ were obtained by Mandal in [19] and is the content of Theorem 8.3.9.

At this point, we want to remind the reader that we are following the convention mentioned in Remark 1.2.14. Thus, our $A_s(n)$, $H^+(p,q)$ etc., correspond to the notations $C(S_n^+)$, $C(H^+(p,q))$ in [16] and other papers.

8.3.2 Results for the Free Groups $\mathbb{F}_n$

The description of $\mathrm{QISO}^+(\mathbb{F}_2)$ in terms of generators and relations were derived in [10]. Subsequently, the following description of the quantum isometry group for the general case was derived in [16]. In fact, the free group case led to the discovery of new two parameter family of compact quantum groups $H_s^+(p,q)$ in the same paper.

Theorem 8.3.5 ([16])
$\mathrm{QISO}^+(\mathbb{F}_n) \cong H^+(n,0)$.

8.3.3 Quantum Isometry Groups of Free Product of Finite Cyclic Groups

As a follow up of [16], Banica and Skalski studied quantum isometry groups of duals of free product of cyclic groups for several cases in [12]. They showed the following:

Theorem 8.3.6 ([12])

$$\mathrm{QISO}^+(\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 \cdots * \mathbb{Z}_2}_{n\ copies}) \cong H_n^+,$$

$$\text{For } s \neq 2,4,\ \mathrm{QISO}^+(\underbrace{\mathbb{Z}_s * \mathbb{Z}_s \cdots * \mathbb{Z}_s}_{n\ copies}) \cong H_s^+(n,0).$$

It is a natural question whether one can prove an analogue of Theorem 8.2.11 for the free product case. The next result is the first step in this direction. In [12], Banica and Skalski showed that $\mathrm{QISO}^+(\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 \cdots * \mathbb{Z}_2}_{n\ copies})$ is the wreath product (Sect. 8.2, [17]) of $\mathrm{QISO}^+(\mathbb{Z}_2)$ by the quantum permutation group $A_s(n)$ introduced in Chap. 1. This was generalized by Mandal in [19].

Theorem 8.3.7 ([12, 19])

$$\mathrm{QISO}^+(\underbrace{\mathbb{Z}_s * \mathbb{Z}_s \cdots * \mathbb{Z}_s}_{n\ copies}) \cong \mathrm{QISO}^+(\mathbb{Z}_s) \wr_* A_s(n).$$

The structure of the quantum isometry groups of almost all free products of the above form is given by the following:

Theorem 8.3.8 ([14]) *Let* $\Gamma_i = \underbrace{(\mathbb{Z}_{n_i} * \mathbb{Z}_{n_i} \cdots * \mathbb{Z}_{n_i})}_{k_i \text{ copies}}$ *for* $i = 1, 2, \ldots l$, *and* $\Gamma = \Gamma_1 * \Gamma_2 * \cdots \Gamma_l$, *where* $n_1 \neq n_2 \neq \cdot\cdot \neq n_l$ *and* $n_1 \neq 2$. *Then* $\mathrm{QISO}^+(\Gamma) = \mathrm{QISO}^+(\Gamma_1) \cdots \mathrm{QISO}^+(\Gamma_l)$.

As a by-product of the proof of Theorem 8.3.7, new characterizations of the quantum groups K_n^+ and $H_s^+(n, 0)$ were obtained in [19].

Theorem 8.3.9 *For* $s > 2$, *we have*

$$H_s^+(n, 0) \cong C^*(\mathbb{Z}_s) \oplus C^*(\mathbb{Z}_s) \wr_* A_s(n), \quad K_n^+ \cong C^*(\mathbb{Z}) \oplus C^*(\mathbb{Z}) \wr_* A_s(n).$$

Here, the quantum group structures on $C^*(\mathbb{Z}_s) \oplus C^*(\mathbb{Z}_s)$ and $C^*(\mathbb{Z}) \oplus C^*(\mathbb{Z})$ are the ones obtained on the doublings of a CQG (corresponding to the involution $\lambda_a \to \lambda_{a^{-1}}$) to be defined in the next section.

Corollary 8.3.10 *Using Theorems 8.3.7, 8.3.9 and the results of [18], we can conclude that for every finite s,*

$$\mathrm{QISO}^+\underbrace{(\mathbb{Z}_s * \mathbb{Z}_s \cdots * \mathbb{Z}_s)}_{n \text{ copies}} \cong \mathrm{QISO}^+(\mathbb{Z}_s) \wr_* A_s(n).$$

Remark 8.3.11 If we consider $\Gamma = \mathbb{Z}_n * \mathbb{Z}_n$ where n is finite, then $\mathrm{QISO}^+(\Gamma)$ is a doubling of the quantum group $\mathrm{QISO}^+(\mathbb{Z}_n) \star \mathrm{QISO}^+(\mathbb{Z}_n)$. In particular for $n = 2$, $\mathrm{QISO}^+(\Gamma)$ becomes doubling of the group algebra as $\mathrm{QISO}^+(\mathbb{Z}_2) \cong (C^*(\mathbb{Z}_2), \Delta_{\mathbb{Z}_2})$ and $C^*(\mathbb{Z}_2) \star C^*(\mathbb{Z}_2) \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$.

8.4 Quantum Isometry Groups as Doublings

We discuss the doubling procedure of the group algebra following [20, 21]. Let $(\mathcal{S}, \Delta)$ be a CQG with a CQG-automorphism θ such that $\theta^2 = id$. The doubling of this CQG with respect to θ, denoted by $(\mathcal{D}_\theta(\mathcal{S}), \tilde{\Delta})$, is given by $\mathcal{D}_\theta(\mathcal{S}) := \mathcal{S} \oplus \mathcal{S}$ as a C^*-algebra, and the coproduct is defined by the following formulas:

$$\tilde{\Delta} \circ \xi = (\xi \otimes \xi + \eta \otimes [\eta \circ \theta]) \circ \Delta,$$

$$\tilde{\Delta} \circ \eta = (\xi \otimes \eta + \eta \otimes [\xi \circ \theta]) \circ \Delta,$$

where a belongs to $\mathcal{S}$ and we have denoted the injections of $\mathcal{S}$ onto the first and second component in $\mathcal{D}_\theta(\mathcal{S})$ by ξ and η, respectively, i.e., $\xi(a) = (a, 0),\ \eta(a) = (0, a)$.

Example 8.4.1 ([20]) Let us consider the group $\Gamma = S_n$ with the generating set as the nearest neighbor transpositions, i.e., $\{s_1, s_2, \cdots s_{n-1}\}$, where $s_i = (i, i+1)$. Let θ denote the automorphism of $C^*(S_n)$ defined by $\theta(\lambda_{s_i}) = \lambda_{s_{n-i}}$. Then $\mathcal{D}_\theta(C^*(S_n)) = C^*(S_n) \oplus C^*(S_n)$ as a C^* algebra with the comultiplication is given by the following formula:

$$\Delta(\sigma_i) = \sigma_i \otimes \sigma_i + \tau_i \otimes \tau_{n-i}, \; \Delta(\tau_i) = \sigma_i \otimes \tau_i + \tau_i \otimes \sigma_{n-i},$$

where $\sigma_i = \tau_i = s_i$ and we have used the symbols λ_{σ_i} and λ_{τ_i} to denote the generating sets of the first and second copy of $C^*(S_n)$ inside $C^*(S_n) \oplus C^*(S_n)$, respectively.

In the next two subsections, we present the results on computations of the quantum isometry groups of the symmetric group S_n. This was done in the case of S_3 in [10] where the computations for two different generating sets were performed. In both cases, the C^* algebra of the quantum isometry group was shown to be $C^*(S_3) \oplus C^*(S_3)$. Dalecki and Soltan [20] later showed that the two quantum isometry groups are actually non-isomorphic as compact quantum groups. Moreover, they proved that when the generators are chosen to be nearest neighbor transpositions (i.e., as in Example 8.4.1), the quantum isometry group of S_n is isomorphic to $\mathcal{D}_\theta(S_n)$ of Example 8.4.1, thus generalizing one of the results of [10] to all n.

Before going to the case of $\Gamma = S_n$, let us mention the examples of some other groups for which their quantum isometry group is a doubling of the group algebra. For the details, we refer to the respective papers. From Theorem 8.2.1, it follows that for all $n \neq 4$, $\mathrm{QISO}^+(\mathbf{Z}_n)$ is the doubling of $C^*(\mathbf{Z}_n)$ corresponding to the automorphism $\lambda_\gamma \mapsto \lambda_{\gamma^{-1}}$. However, $\mathrm{QISO}^+(\mathbf{Z}_4)$ is not a doubling since by Theorem 8.2.2, the C^* algebra of $\mathrm{QISO}^+(\mathbf{Z}_4)$ is not commutative. The dihedral groups $D_{2(2n+1)}$ and D_{2n} for two different generating sets were dealt in [22] and [14]. The case of Baumslag–Solitar groups, Coxeter groups were dealt in [14]. Moreover, if Γ is a group generated by two elements a and b so that $a^2 = b^3 = e$, then the quantum isometry group is a doubling in the following three cases: $(ab)^3 = e$, $(ab)^4 = e$, $(ab)^5 = e$. This is also done in [14]. In [19], the class of examples was enlarged to include the groups $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, $(\mathbb{Z}_2 * \mathbb{Z}_2) \times \mathbb{Z}_2$ and the Lamplighter group. In the same paper, it was also shown that the quantum isometry group for the braid group is a double of a double of the group algebra.

8.4.1 Result for a Generating Set of Transpositions

Let s_i, σ_i and τ_i be as above. If $(\Gamma, S) = (S_n, s_1, \cdots s_{n-1}\}$, then Dalecki and Soltan proved the following theorem.

Theorem 8.4.2 ([20])
$\mathrm{QISO}^+(S_n, S) \cong \mathcal{D}_\theta(S_n)$ *and the coaction* α_1 *is given by*

$$\alpha_1(\lambda_{s_i}) = \lambda_{s_i} \otimes \lambda_{\sigma_i} + \lambda_{s_{n-i}} \otimes \lambda_{\tau_{n-i}}.$$

We present the proof only for the case $n = 3$ given in [10] and refer to [20] for the general case. In the case of S_3, the generating set S is given by $\{s_1, s_2\}$. Then s_1 and s_2 satisfy the relations $s_1^2 = s_2^2 = e$ and $s_2 s_1 s_2 = s_1 s_2 s_1$, where e as usual denotes the identity element of the group.

Let α be the coaction of $\mathrm{QISO}^+(S_3, S)$ on $C^*(S_3)$. As explained before, there exist elements A, B, C, D in $\mathrm{QISO}^+(S_3, S)$ such that

$$\alpha(\lambda_{s_1}) = \lambda_{s_1} \otimes A + \lambda_{s_2} \otimes B, \quad \alpha(\lambda_{s_2}) = \lambda_{s_1} \otimes C + \lambda_{s_2} \otimes D. \tag{8.4.1}$$

Now we derive some relations among A, B, C, D which follow from the fact that α is a $*$-homomorphism and that it commutes with the operator D_Γ associated to the generating set S.

Lemma 8.4.3 *Let A, B, C, D be the elements of* $\mathrm{QISO}^+(S_3, S)$ *determined by the formula* (8.4.1) *for the coaction of* $\mathrm{QISO}^+(S_3, S)$ *on* $C^*(S_3)$. *Then the following hold:*

$$A^2 + B^2 = 1, \tag{8.4.2}$$

$$AB = 0, \tag{8.4.3}$$

$$BA = 0, \tag{8.4.4}$$

$$C^2 + D^2 = 1, \tag{8.4.5}$$

$$CD = 0, \tag{8.4.6}$$

$$DC = 0, \tag{8.4.7}$$

$$AC + BD = 0, \tag{8.4.8}$$

$$CA + DB = 0, \tag{8.4.9}$$

$$DAC = CBD = 0, \tag{8.4.10}$$

$$ADB = BCA = 0, \tag{8.4.11}$$

$$DAD + CBC = ADA + BCB, \tag{8.4.12}$$

$$A^* = A, B^* = B, C^* = C, D^* = D. \tag{8.4.13}$$

Proof We derive (8.4.2)–(8.4.4) from $s_1^2 = e$, (8.4.5)–(8.4.7) from $s_2^2 = e$, (8.4.8) and (8.4.9) by equating the coefficients of λ_e in $\alpha(\lambda_{s_1}\lambda_{s_2})$ and $\alpha(\lambda_{s_2}\lambda_{s_1})$ to zero, (8.4.10) and (8.4.11) by equating coefficients of λ_{s_1} and λ_{s_2} in $\alpha(\lambda_{s_2 s_1 s_2})$ and $\alpha(\lambda_{s_1 s_2 s_1})$ to zero, (8.4.12) from $\alpha(\lambda_{s_2 s_1 s_2}) = \alpha(\lambda_{s_1 s_2 s_1})$ and finally (8.4.13) from the facts that $\alpha(\lambda_{s_1}{}^*) = (\alpha(\lambda_{s_1}))^*$, $\alpha(\lambda_{s_2}{}^*) = (\alpha(\lambda_{s_2}))^*$. □

Theorem 8.4.4 *The quantum isometry group* $\mathrm{QISO}^+(S_3, S)$ *for the generating set built of transpositions ($S = \{s, t\}$) is isomorphic to $\mathcal{D}_\theta(S_3)$. Its coaction on $C^*(S_3)$ is given by the formula* (8.4.1)*, where A, B, C, D are identified with $\lambda_{s_1} \oplus 0, 0 \oplus \lambda_{s_2}, 0 \oplus \lambda_{s_1}, \lambda_{s_2} \oplus 0 \in C^*(S_3) \oplus C^*(S_3)$, respectively.*

Proof It is easy to see that $\mathcal{D}_\theta(S_3)$ with the coaction given in the statement of the theorem is a sub-object of $\mathrm{QISO}^+(S_3, S)$ in the category of quantum families of orientation preserving quantum isometries. Let us show the converse direction.

We use the notations of Lemma 8.4.3. The commutation relations listed in that lemma imply that $(A^2)^2 = AA^2A = A(1 - B^2)A = A^2 - ABBA = A^2$. Thus, as A^2 is also self-adjoint, it is a projection, to be denoted by P. Similarly, B^2 is a projection. As $A^2 + B^2 = 1$, we have $B^2 = P^\perp$. Proceeding in the same way, we obtain $C^2 = Q$ and $D^2 = Q^\perp$ for another projection Q.

Multiplying (8.4.8) by A on the left and C on the right, we obtain $PQ = 0$. Hence, $Q \le P^\perp$. Similarly, multiplying the same equation by B on the left and D on the right yields $P^\perp Q^\perp = 0$ which implies $Q^\perp \le P$. This implies that $P^\perp = Q$. Thus $A^2 = D^2 = P,\ B^2 = C^2 = P^\perp$.

Next, from (8.4.12) we deduce that $DAD - ADA = BCB - CBC$. As Ran $(DAD - -ADA) \subseteq \mathrm{Ran}(P)$, $\mathrm{Ran}(BCB - -CBC) \subseteq \mathrm{Ran}(P^\perp)$, this implies

$$DAD = ADA, \tag{8.4.14}$$

$$BCB = CBC. \tag{8.4.15}$$

Now, by Corollary 3.4.5 and Proposition 8.1.2, the Eq. (8.4.2)–(8.4.13) together with the unitarity of the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (which follows from Proposition 8.1.2) provide all the conditions necessary to ensure that $C^*\{A, B, C, D\}$ is an object of $\widehat{\mathbf{C}}(S_3, S)$.

The action of the antipode κ of a compact quantum group on the matrix elements of a finite-dimensional unitary corepresentation $U^\beta \equiv (u^\beta_{pq})$ is given by $\kappa(u^\beta_{pq}) = (u^\beta_{qp})^*$ (see [23]). Thus in our situation we have $\kappa(A) = A^* = A$, $\kappa(B) = C^* = C$, $\kappa(C) = B$, $\kappa(D) = D^* = D$.

Applying the antipode to the Eq. (8.4.3) and (8.4.6) and then taking adjoints, we obtain the equations $AC = 0$ and $BD = 0$. Thus $DAC = CBD = 0$ too and the conditions in (8.4.10) follow. We deduce the equalities (8.4.8), (8.4.9), and (8.4.11) in a similar way. Thus, $C^*\{A, B, C, D\}$ is the universal C^*-algebra generated by elements A, B, C, D such that the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a unitary and the relations (8.4.2)–(8.4.7) along with (8.4.13)–(8.4.15) hold. Then it is easy to see that $C^*\{A, B, C, D\} \cong C^*(S_3) \oplus C^*(S_3)$ via the map sending A, D, C, B to

$\lambda_{s_1} \oplus 0, \lambda_{s_2} \oplus 0,\ 0 \oplus \lambda_{s_1},\ 0 \oplus \lambda_{s_2}$. The rest of the statements are easy consequences of the proof above. □

8.4.2 The Case When S Has a Cycle

This time we consider a set of generators of S_3 given by a transposition and a cycle: $S' = \{s_1, s_1 s_2, s_2 s_1\}$.

Theorem 8.4.5 ([10]) $\mathrm{QISO}^+(S_3, S')$ *is isomorphic to* $C^*(S_3) \oplus C^*(S_3)$ *as a* C^* *algebra. Its coaction on* $C^*(S_3)$ *is given by the formulas*

$$\alpha_2(\lambda_{s_1}) = \lambda_{s_1} \otimes \lambda_{\sigma_1+\tau_1}, \alpha_2(\lambda_{s_2}) = \lambda_{s_2} \otimes \lambda_{\sigma_2} + \lambda_{s_1 s_2 s_1} \otimes \lambda_{\tau_2}.$$

The coproduct Δ_2 *is given by*

$$\Delta_2(\lambda_{\sigma_1}) = \lambda_{\sigma_1} \otimes \lambda_{\sigma_1} + \lambda_{\tau_1} \otimes \lambda_{\tau_1}, \Delta_2(\lambda_{\tau_1}) = \lambda_{\tau_1} \otimes \lambda_{\sigma_1} + \lambda_{\sigma_1} \otimes \lambda_{\tau_1},$$

$$\Delta_2(\lambda_{\sigma_2}) = \lambda_{\sigma_2} \otimes \lambda_{\sigma_2} + \lambda_{\tau_1\tau_2\tau_1} \otimes \lambda_{\tau_2}, \Delta_2(\lambda_{\tau_2}) = \lambda_{\tau_2} \otimes \lambda_{\sigma_2} + \lambda_{\sigma_1\sigma_2\sigma_1} \otimes \lambda_{\tau_2}.$$

In [10], it was left as an open question to decide whether $\mathcal{D}_\theta(S_3)$ of Example 8.4.1 is isomorphic to $(C^*(S_3) \oplus C^*(S_3), \Delta_2)$ as compact quantum group. In [20], Dalecki and Soltan settled this question in the negative. Thus, the quantum isometry groups of the same group algebra but corresponding to two different sets of generators can be indeed different as compact quantum groups.

Moreover, in the same paper, Dalecki and Soltan pointed out that only two in the list of all 12-dimensional semisimple Hopf algebras over algebraically closed fields of characteristic different from 2 and 3 given in [24] are noncommutative and non-cocommutative. Since the 12-dimensional noncommutative and noncocommutative Hopf algebras $\mathcal{D}_\theta(S_3)$ and $(C^*(S_3) \oplus C^*(S_3), \Delta_2)$ are non-isomorphic, they deduced that these two quantum groups are precisely the ones mentioned in [24], thus providing noncommutative geometric interpretations of these Hopf algebras.

References

1. Connes, A.: Compact metric spaces, Fredholm modules, and hyperfiniteness. Ergodic Theor. Dyn. Syst. **9**(2), 207–220 (1989)
2. Rieffel, M.A.: Metrics on states from actions of compact groups. Doc. Math. **3**, 215–229 (1998)
3. Rieffel, M.A.: Metrics on state spaces. Doc. Math. **4**, 559–600 (1999)
4. Rieffel, M.A.: Compact quantum metric spaces, operator algebras, quantization, and noncommutative geometry. Contemp. Math. **365**, 315–330; Am. Math. Soc. Provid. RI (2004)
5. Christ, M., Rieffel, M.A.: Nilpotent group C^*-algebras as compact quantum metric spaces. arXiv:1508.00980

6. Ozawa, N., Rieffel, M.A.: Hyperbolic group C^*-algebras and free-product C^*-algebras as compact quantum metric spaces. Canad. J. Math. **57**(5), 1056–1079 (2005)
7. Antonescu, C., Christensen, E.: Metrics on group C^*-algebras and a non-commutative Arzelà-Ascoli theorem. J. Funct. Anal. **214**(2), 247–259 (2004)
8. Park, E.: Isometries of unbounded Fredholm modules over reduced group C^*-algebras. Proc. Am. Math. Soc. **123**(6), 1839–1843 (1995)
9. Bertozzini, P., Conti, R., Lewkeeratiyutkul, W.: A category of spectral triples and discrete groups with length function. Osaka J. Math. **43**(2), 327–350 (2006)
10. Bhowmick, J., Skalski, A.: Quantum isometry groups of noncommutative manifolds associated to group C^*-algebras. J. Geom. Phys. **60**(10), 1474–1489 (2010)
11. Banica, T.: Quantum automorphism groups of homogeneous graphs, J. Funct. Anal. **224**(2), 243–280 (2005)
12. Banica, T., Skalski, A.: Quantum isometry groups of duals of free powers of cyclic groups. Int. Math. Res. Not. **9**, 2094–2122 (2012)
13. Banica, T., Skalski, A.: Quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations. Proc. Lond. Math. Soc. **106**(3, 5), 980–1004 (2013)
14. Goswami, D., Mandal, A.: Quantum isometry groups of dual of finitely generated discrete groups and quantum groups. arXiv: 1408.5683
15. Banica, T., Bichon, J., Collins, B.: Quantum permutation groups: a survey. In: Noncommutative Harmonic Analysis with Applications to Probability, pp. 13–34, vol. 78. Banach Center Publications. Polish Acad. Sci. Inst. Math., Warsaw (2007)
16. Banica, T., Skalski, A.: Two-parameter families of quantum symmetry groups. J. Funct. Anal. **260**(11), 3252–3282 (2011)
17. Bichon, J.: Free wreath product by the quantum permutation group. Algebr. Represent. Theor. **7**(4), 343–362 (2004)
18. Banica, T., Vergnioux, R.: Fusion rules for quantum reflection groups. J. Noncommut. Geom. **3**(3), 327–359 (2009)
19. Mandal, A.: Quantum isometry group of dual of finitely generated discrete groups–II. Annales Mathematiques Blaise Pascal (2016). arXiv: 1504.02240
20. Liszka-Dalecki, J., Sołtan, P.M.: Quantum isometry groups of symmetric groups. Internat. J. Math. **23**(7) (2012)
21. Skalski, A.; Sołtan, P.M.: Projective limits of quantum symmetry groups and the doubling construction for Hopf algebras. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17**(2), 1450012 (27 pages) (2014)
22. Tao, J., Qiu, D.: Quantum isometry groups for Dihedral group $D_{2(2n+1)}$. J. Geom. Phys. **62**(9), 1977–1983
23. Maes, A., Van Daele, A.: Notes on compact quantum groups. Nieuw Arch. Wisk. **16**(4, 1–2), 73–112 (1998)
24. Fukuda, N.: Semisimple Hopf algebras of dimension 12, Tsukuba. J. Math. **21**, 43–54 (1997)

Chapter 9
An Example of Physical Interest

Abstract This chapter is devoted to the quantum isometry group of the finite geometry of the Connes-Chamseddine picture of the Standard Model. We begin with some generalities on real C^* algebras, followed by a brief discussion in the finite noncommutative space of the Connes-Chamseddine model. Then we compute the quantum isometry group of the corresponding spectral triple and also discuss some physical significance of our results.

This chapter deals with the noncommutative geometric approach to the particle physics of the Standard Model. For the relevant physics background, we refer to [1]. The idea of using noncommutative geometric paradigm to unify diffeomorphisms and local gauge symmetries came into light as early as 1990, in the paper [2]. The interpretation of the inner automorphism group as the gauge group appeared in [3]. Subsequently, the Connes–Chamseddine spectral action principle appeared in [3–5] and further studied in [6, 7]. For the definition of spectral action, we refer to Sect. 9.6 of this chapter. The key assumption behind the Connes–Chamseddine theory is that the four-dimensional space time can be described by a noncommutative space given by the product of the algebra of smooth functions on a four-dimensional spin manifold M and a finite-dimensional C^* algebra. For the spectral standard model presented in [8], this finite-dimensional C^* algebra is actually a real C^* algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, where $\mathbb{H}$ denotes the quaternions. Connes and Chamseddine associated a spectral triple $(A_F, H_F, D_F, \gamma_F, J_F)$, where H_F is a finite-dimensional complex Hilbert space of appropriate dimension. On M, one has the canonical spectral triple $(C^\infty(M), L^2(M, S), D_M, \gamma_M, J_M)$, where $L^2(M, S)$ denotes the Hilbert space of L^2-sections on the canonical spinor bundle S on M. Then the spectral data $(C^\infty(M) \otimes A_F, L^2(M, S) \otimes H_F, D_M \otimes I + \gamma_M \otimes D_F, \gamma_M \otimes \gamma_F, J_M \otimes J_F)$ is a spectral triple on the noncommutative manifold $C^\infty(M) \otimes A_F$. In this picture, diffeomorphisms are realized as outer automorphisms of the algebra, while inner automorphisms correspond to the gauge transformations.

In [8, 9] (also see [10, 11]), a conceptual mechanism for selecting the particular algebra A_F as well as the Hilbert space H_F and the Dirac operator D_F were presented. In addition, the fermion doubling problem pointed out in [12] was taken care of. The KO dimension (Sect. 9.2 of [13]) of $(A_F, H_F, D_F, \gamma_F, J_F)$ in this setting became 6,

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_9

the need of which was already pointed out earlier in [14, 15]. Moreover, the spectral action functional introduced in [5] was modified to accommodate an additional term. The inner fluctuations of the Dirac operator ([13]) $D \mapsto D + A + \epsilon' JAJ^{-1}$ of the product spectral triple give rise to the gauge bosons and the Higgs field in the following manner: the 1-forms given by commutators with the Dirac operator of M give the gauge bosons, while the 1-forms corresponding to the Dirac operator of F give the Higgs field. Then Connes–Chamseddine and Marcolli gave the derivation of the full Lagrangian of the Standard Model from a certain asymptotic expansion of the spectral action functional along with the correct gauge group. The predictions of the Spectral Standard Model was reconciled with the experimental value of the Higgs mass in [16] by the introducing a scalar field (also see [17]).

A Lorentzian analog of the noncommutative geometry of the standard model was derived in [15]. For more details and other recent works, we refer to the books [13, 18]. The latter contains an exposition on Suijlekom and his collaborators' works on the noncommutative geometric approach to supersymmetric quantum chromodynamics ([19]), Yang–Mills fields ([20]), and electrodynamics ([21]). The book [18] also contains expositions on the very recent probes into physics beyond the Standard Model using noncommutative geometric frameworks, which were done by Suijlekom and van den Broek in [22], and then by Chamseddine, Connes, and Suijlekom in [23–25]. For supersymmetry in the spectral Standard Model, we refer to [26]. For an exposition from a physicist's perspective, we refer to the review [27].

The idea of using quantum group symmetries to understand the conceptual significance of the finite geometry F is mentioned in a final remark by Connes in [28]. Some preliminary studies on the Hopf-algebra level appeared in [29–31]. In the papers [32, 33], compact quantum group symmetries were employed to study the spectral Standard Model of Connes and Chamseddine. While [32] deals with the quantum isometries of the concerned spectral triple, [33] deals with quantum gauge symmetries. In this chapter, we present the results of [32]. In Sect. 9.2, we introduce the spectral triple on the finite-dimensional algebra A_F. In Sect. 9.3, we state the results of the computation of the quantum isometry groups for this finite geometry. In particular, in the first part of Sect. 9.3, we begin by computing the quantum group of orientation and real structure-preserving isometries (in the sense of Definition 3.3.5 and [34]) of the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$ where $B_F \subset \mathcal{B}(\mathcal{H})$ is the smallest C^*-algebra over the complex field containing $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ as a real C^*-subalgebra. It turns out that the quantum isometry group is isomorphic to the free product $C(U(1)) * A_{\text{aut}}(M_3(\mathbb{C}), \frac{1}{3}\text{Tr})$, where $A_{\text{aut}}(M_n(\mathbb{C}), \frac{1}{n}\text{Tr})$ is Wang's quantum automorphism group of $M_n(\mathbb{C})$ as in Sect. 1.3. In Sect. 9.3.2 we explain how the result changes if we work with the real C^* algebra A_F instead of B_F. Next, we extend this quantum symmetry to the spectral triples obtained by taking a product of the above spectral triple with the natural spectral triple over the Riemannian spin manifold M. This gives genuine quantum group symmetries of the full Standard Model. In Sect. 9.5, we discuss the physical significance of our results. Finally, in Sect. 9.6, we discuss how the spectral action is kept invariant by these quantum symmetries.

9.1 Notations and Preliminaries

In this section, we fix our notations and introduce some compact quantum groups which will be needed to analyze the quantum isometry group for the spectral triple of Connes–Chamseddine.

The symbol $\otimes_{\mathbb{R}}$ will denote the tensor product over the real numbers. Usually, we assume that all the algebras are unital, associative, and over $\mathbb{C}$. The quotient of an algebra A by its commutator C^*-ideal is called the abelianization of A. For $u = ((u_{ij}))$, where u_{ij} are elements of a C^* algebra A, we denote the matrix $((u_{ij}^*))$ by $\overline{u}$.

We will use the symbol $A_{\text{aut}}(M_n(\mathbb{C}))$ in place of $A_{\text{aut}}(M_n(\mathbb{C}), \frac{1}{\text{n}}\text{Tr})$ for notational simplicity. Throughout this section, we will be freely using the quantum groups introduced in Sect. 1.3.2. In particular, a matrix B with entries in a unital $*$-algebra such that both B and B^t are unitary is called a *biunitary* ([35]). We recall the free quantum unitary group $A_{u,n}(I)$ introduced in Sect. 1.3.2 which we will denote simply by $A_u(n)$. Thus, $A_u(n)$ is generated by the biunitary matrix u such that $u^t\overline{u} = \overline{u}u^t = I_n$. We will also need the free orthogonal quantum group $A_o(n)$, the half liberated quantum groups $A_o^*(n)$, $A_u^*(n)$ and their projective versions as introduced in Sect. 1.3.2.

Definition 9.1.1 Consider n copies of the CQG $A_u(n')$ and let $Q_n(n')$ be the amalgamated free product of them over $PA_u(n')$, which is a common Woronowicz C^*-subalgebra of all the copies of $A_u(n')$. This is the universal C^* algebra generated by the entries of the matrices $u_m,\ m = 1, 2, \cdots n$, where u_m is a biunitary $n' \times n'$ matrix satisfying the following relations:

$$(u_m^*)_{i,j}(u_m)_{k,l} = (u_{m'}^*)_{i,j}(u_{m'})_{k,l} \qquad \forall\, i, j, k, l = 1, \ldots, n',\ m, m' = 1, \ldots, n\,.$$

This becomes a CQG with the matrix coproduct: $\Delta((u_m)_{ij}) = \sum_{k=1}^{n'}(u_m)_{ik} \otimes (u_m)_{kj}$ for all $m = 1, \ldots, n$.

9.1.1 Generalities on Real C* Algebras

The C^* algebra A_F of the finite noncommutative space of the Standard Model is a real C^* algebra. For details on real C^* algebras, we refer to [36]. We state some basic facts about them in this subsection.

By a real $*$-algebra, we will mean a unital, associative, involutive algebra over $\mathbb{R}$. Associated to a real $*$-algebra A is its complexification $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$. Then A can be recovered from $A_{\mathbb{C}}$ as the fixed-point subalgebra of $A_{\mathbb{C}}$ with respect to an involutive conjugate-linear real $*$-algebra automorphism σ defined by

$$\sigma(a \otimes_{\mathbb{R}} z) = a \otimes_{\mathbb{R}} \overline{z} \quad \forall\, a \in \mathcal{A}, z \in \mathbb{C}, \tag{9.1.1}$$

that is

$$\mathcal{A} = \{a \in \mathcal{A}_{\mathbb{C}} : \sigma(a) = a\} .$$

It is easy to see that the automorphisms of A are precisely those automorphisms of $A_{\mathbb{C}}$ which commute with σ (Sect. 5 of [32]). This leads to the following result.

Proposition 9.1.2 ([32]) *Let A be a finite-dimensional real C^*-algebra and let G be the automorphism group of A. Then the condition $\sigma\,\phi = \phi\,\sigma \;\forall\; \phi \in \mathrm{Aut}(A_{\mathbb{C}})$ is equivalent to*

$$(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma ,$$

where, for $a \in A_{\mathbb{C}}$ and $\phi \in C(\mathrm{Aut}(A_{\mathbb{C}}))$, $\alpha : A_{\mathbb{C}} \to A_{\mathbb{C}} \otimes C(\mathrm{Aut}(A_{\mathbb{C}}))$ is defined by $\alpha(a)(\phi) = \phi(a)$.

9.1.2 Quantum Isometries

The algebra of the spectral triple of our interest is a product spectral triple on a finite-dimensional algebra and the classical triple on a compact Riemannian spin manifold. More precisely, we will have to deal with the following situation in Sect. 9.4.

Lemma 9.1.3 *Consider two real spectral triples $(\mathcal{A}_i, \mathcal{H}_i, D_i, \gamma_i, J_i)$, $i = 1, 2$, where each $\mathcal{A}_i$ is unital, $\mathcal{H}_2$ is finite dimensional, γ_2 is non-trivial (i.e., $\neq 1$) but γ_1 can be trivial (i.e., $= 1$). Consider the product triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$, which is given by*

$$\mathcal{A} := \mathcal{A}_1 \otimes_{\mathrm{alg}} \mathcal{A}_2 , \qquad \mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 , \qquad D := D_1 \otimes \gamma_2 + 1 \otimes D_2 ,$$
$$\gamma := \gamma_1 \otimes \gamma_2 , \qquad J := J_1 \otimes J_2 .$$

Then the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ admits an orientation and real structure-preserving isometric coaction by the CQG $\widetilde{\mathrm{QISO}}^+(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$. It is given by $\mathrm{ad}_{\widehat{U}}$ where $\widehat{U}$ denotes the corepresentation $1 \otimes U$ of $\widetilde{\mathrm{QISO}}^+(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ on the product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

We refer to [32] for the proof, which is quite straightforward.

Finally, we want to attract the reader's attention to a choice of notation. The notations $\widetilde{\mathrm{QISO}}_J^+$ and QISO_J^+ used in this chapter are as in Definition 3.3.5 and should not be confused with the newly defined objects $\widetilde{\mathrm{QISO}}_{\mathbb{R}}^+$ and $\mathrm{QISO}_{\mathbb{R}}^+$ of Sect. 9.3.2 in the context of quantum isometries of real C^*-algebras.

9.2 The Finite Noncommutative Space F

The aim of this section is to present the spectral triple $(A_F, H_F, D_F, \gamma_F, J_F)$ which describes the internal space F of the Standard Model. For more details, we refer to [13] and the references therein.

9.2.1 The Elementary Particles and the Hilbert Space of Fermions

We define the Hilbert space H_F using the notations as in [32]. H_F is given by a tensor product

$$H_F := \mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^n \ ,$$

where, using the notations of [13], we get

(i) the first two factors $\mathbb{C}^2 \otimes \mathbb{C}^4$ with

$$\mathbb{C}^2 = \mathbb{C}[\uparrow, \downarrow] \ , \qquad \mathbb{C}^4 = \mathbb{C}[\ell, \{q_c\}_{c=1,2,3}] \ ,$$

where $\uparrow$ and $\downarrow$ denote the weak isospin up and down, ℓ and q_c denote lepton and quark of color c, respectively. These may be combined into

$$\mathbb{C}^8 = \mathbb{C}[\nu, e, \{u_c, d_c\}_{c=1,2,3}] \ ,$$

where ν denotes "neutrino," e the "electron," u_c and d_c the quarks with weak isospin $+1/2$ and $-1/2$ respectively and of color c. The isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^4 \to \mathbb{C}^8$ is given by the following:

$$\uparrow \otimes \ell \mapsto \nu \ , \quad \downarrow \otimes \ell \mapsto e \ , \quad \uparrow \otimes q_c \mapsto u_c \ , \quad \downarrow \otimes q_c \mapsto d_c \ .$$

(ii) a factor

$$\mathbb{C}^4 = \mathbb{C}[p_L, \overline{p}_R, \overline{p}_L, p_R] \ ,$$

where the two chiralities are denoted by L and R, p denotes "particle" and $\overline{p}$ the "antiparticle";

(iii) a factor $\mathbb{C}^n$ because each particle comes in n generations. At present only 3 generations have been observed, but we choose to work with an arbitrary $n \geq 3$ for the sake of generality.

Thus, we have

$$
\begin{aligned}
\nu_{L,k} &:= e_1 \otimes e_1 \otimes e_1 \otimes e_k \ , && \text{(left-handed neutrino, generation} k\text{)}\\
\nu_{R,k} &:= e_1 \otimes e_1 \otimes e_4 \otimes e_k \ , && \text{(right-handed neutrino, generation} k\text{)}\\
e_{L,k} &:= e_2 \otimes e_1 \otimes e_1 \otimes e_k \ , && \text{(left-handed electron, generation} k\text{)}\\
e_{R,k} &:= e_2 \otimes e_1 \otimes e_4 \otimes e_k \ , && \text{(right-handed electron, generation} k\text{)}\\
u_{L,c,k} &:= e_1 \otimes e_{c+1} \otimes e_1 \otimes e_k \ , && \text{(left-handed up-quark, color } c, \text{ generation } k\text{)}\\
u_{R,c,k} &:= e_1 \otimes e_{c+1} \otimes e_4 \otimes e_k \ , && \text{(right-handed up-quark, color } c, \text{ generation } k\text{)}\\
d_{L,c,k} &:= e_2 \otimes e_{c+1} \otimes e_1 \otimes e_k \ , && \text{(left-handed down-quark, color } c, \text{ generation } k\text{)}\\
d_{R,c,k} &:= e_2 \otimes e_{c+1} \otimes e_4 \otimes e_k \ , && \text{(right-handed down-quark, color } c, \text{ generation } k\text{)}
\end{aligned}
$$

where $\{e_i \ , \ i = 1, \ldots, r\}$ is the canonical orthonormal basis of $\mathbb{C}^r$, $c = 1, 2, 3$ and $k = 1, \ldots, n$.

These particles and their corresponding antiparticles form a linear basis of the Hilbert space H_F.

9.2.2 The Spectral Triple

Consider the following linear operator:

$$
J_0 := 1 \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes 1 \ . \tag{9.2.1}
$$

The *charge conjugation* operator J_F is defined as the composition of the operators J_0 and the operator on H_F which acts by componentwise complex conjugation.

The grading is

$$
\gamma_F := 1 \otimes 1 \otimes \operatorname{diag}(1, 1, -1, -1) \otimes 1 \ .
$$

The Dirac operator is given by

$$
\begin{aligned}
D_F := e_{11} \otimes e_{11} \otimes & \begin{pmatrix} 0 & 0 & 0 & \Upsilon_\nu \\ 0 & 0 & \Upsilon_\nu^t & \Upsilon_R \\ 0 & \overline{\Upsilon}_\nu & 0 & 0 \\ \Upsilon_\nu^* & \Upsilon_R^* & 0 & 0 \end{pmatrix} + e_{11} \otimes (1 - e_{11}) \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_u \\ 0 & 0 & \Upsilon_u^t & 0 \\ 0 & \overline{\Upsilon}_u & 0 & 0 \\ \Upsilon_u^* & 0 & 0 & 0 \end{pmatrix} \\
+ e_{22} \otimes e_{11} \otimes & \begin{pmatrix} 0 & 0 & 0 & \Upsilon_e \\ 0 & 0 & \Upsilon_e^t & 0 \\ 0 & \overline{\Upsilon}_e & 0 & 0 \\ \Upsilon_e^* & 0 & 0 & 0 \end{pmatrix} + e_{22} \otimes (1 - e_{11}) \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_d \\ 0 & 0 & \Upsilon_d^t & 0 \\ 0 & \overline{\Upsilon}_d & 0 & 0 \\ \Upsilon_d^* & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{9.2.2}
$$

Here $\Upsilon \in M_n(\mathbb{C})$, $\overline{m} := (m^*)^t$, and $\mathcal{B}(H_F) = M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes \big(M_4(\mathbb{C}) \otimes M_n(\mathbb{C})\big)$ can be identified with $M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes M_{4n}(\mathbb{C})$ by writing $M_{4n}(\mathbb{C})$ as a 4×4 matrix with entries in $M_n(\mathbb{C})$; in particular $e_{ij} \otimes m \in M_4(\mathbb{C}) \otimes M_n(\mathbb{C})$ will be the matrix with the $n \times n$ block m in position (i, j).

Finally, the algebra A_F is defined as follows:

$$A_F := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \;, \tag{9.2.3}$$

where $\mathbb{H}$ is identified with the real subalgebra of $M_2(\mathbb{C})$ with elements

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{9.2.4}$$

for $\alpha, \beta \in \mathbb{C}$.

The element $a = (\lambda, q, m) \in A_F$ (with $\lambda \in \mathbb{C}$, $q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$) is represented by

$$\begin{aligned} \pi(a) = q \otimes 1 \otimes e_{11} \otimes 1 + \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \otimes 1 \otimes e_{44} \otimes 1 \\ + 1 \otimes \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & & & \\ 0 & & m & \\ 0 & & & \end{pmatrix} \otimes (e_{22} + e_{33}) \otimes 1 \;, \end{aligned} \tag{9.2.5}$$

where m is a 3×3 block and $\{e_{ij}\}_{i,j=1,\ldots,k}$ is the canonical basis of $M_k(\mathbb{C})$.

9.2.3 A Hypothesis on the Υ Matrices

Before going to the result concerning the quantum isometry groups, we discuss some known properties of the Υ matrices. Moreover, since the quantum isometry group does not change if one replaces a spectral triple with an equivalent one, one can make some reductions about the structure of the Υ matrices which helped in the computation of the quantum isometry group in [32].

We know that Υ_R is a symmetric matrix and the other Υ matrices are nonnegative. It can be seen from Sect. 17.4 of [13] that the eigenvalues of $\Upsilon_x^* \Upsilon_x$ ($x = e, u, d, \nu, R$) are m_x^2, where m_x stand for the masses of the n generations of the particle x for $x = e, u, d$, the Dirac masses of the neutrinos for $x = \nu$ and the Majorana masses of neutrinos for $x = R$.

It follows from Theorem 1.187(3) and Lemma 1.190 of [13] that it is possible to diagonalize one element of each of the pairs $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_u, Υ_d) modulo unitary equivalence. Let us diagonalize Υ_u and Υ_e.

Thus, we make the following hypotheses on the Υ matrices:

- Υ_u and Υ_e are diagonal with strictly positive entries on the diagonal.
- Υ_d and Υ_ν are nonnegative and Υ_d is strictly positive. Let C be the $SU(n)$ matrices such that $\Upsilon_d = C\delta_\downarrow C^*$, where $\delta_\downarrow$ is some nonnegative diagonal matrix. C is called the Cabibbo–Kobayashi–Maskawa matrix, responsible for the quark mixing, cf. Sect. 9.3 of [13]. In a similar way, we get the Pontecorvo–Maki–Nakagawa–Sakata matrix, responsible for the neutrino mixing, to be denoted by Υ_ν, cf. Sect. 9.6 of [13].
- Υ_R is a symmetric matrix.
- We make further assumptions for physical reasons: we assume that Υ_x and Υ_y have distinct eigenvalues $\forall x, y \in \{\nu, e, u, d\}$ with $x \neq y$; the eigenvalues of Υ_e, Υ_u, and Υ_d are assumed to be nonzero and have multiplicity one.

9.3 Quantum Isometries of *F*

So far, we have defined the quantum isometry group for spectral triples over complex $*$-algebras. Now we want to formulate a similar notion for spectral triples over real $*$-algebras. For this, the lemma presented below will be a key step.

Lemma 9.3.1 *Let $\mathcal{A}$ be a real $*$-algebra, with $\mathcal{B} \simeq \mathcal{A}_\mathbb{C}/\ker \pi_\mathbb{C}$, where $\mathcal{A}_\mathbb{C} \simeq \mathcal{A} \otimes_\mathbb{R} \mathbb{C}$ denotes the complexification and $\pi_\mathbb{C}$ is the $*$-representation from $\mathcal{A}_\mathbb{C}$ to $\mathcal{B}(\mathcal{H})$ given by*

$$\pi_\mathbb{C}(a \otimes_\mathbb{R} z) = z\pi(a)\ , \qquad (a \in \mathcal{A}\,,\ z \in \mathbb{C})\,. \tag{9.3.1}$$

Here, we have defined the conjugation on $\mathcal{A}_\mathbb{C}$ by $(a \otimes_\mathbb{R} z)^* = a^* \otimes_\mathbb{R} \overline{z}$. Then, given any real spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ over $\mathcal{A}$, there is an associated real spectral triple $(\mathcal{B}, \mathcal{H}, D, \gamma, J)$ over $\mathcal{B}$.

It should be mentioned here that it is possible to have $\ker \pi_\mathbb{C} \neq \{0\}$. This happens when $\pi_\mathbb{C}$ is not faithful. For example, in the Standard Model case, $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, $(A_F)_\mathbb{C} := \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$, where the complex $*$-algebra isomorphism $M_n(\mathbb{C}) \otimes_\mathbb{R} \mathbb{C} \to M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ is given by

$$m \otimes_\mathbb{R} z \mapsto (mz, \overline{m}z)$$

and the inverse

$$(m, m') \mapsto \tfrac{m+\overline{m}'}{2} \otimes_\mathbb{R} 1 + \tfrac{m-\overline{m}'}{2i} \otimes_\mathbb{R} i \tag{9.3.2}$$

for all $m, m' \in M_n(\mathbb{C})$, $z \in \mathbb{C}$.

Using (9.3.1), (9.3.2), and (9.2.5) we get $\pi_\mathbb{C}(\lambda, \lambda', q, m, m') = \langle \lambda, \lambda', q, m \rangle$, where

$$\langle\lambda,\lambda',q,m\rangle := q\otimes 1\otimes e_{11}\otimes 1+\begin{pmatrix}\lambda & 0\\ 0 & \lambda'\end{pmatrix}\otimes 1\otimes e_{44}\otimes 1$$
$$+1\otimes\begin{pmatrix}\lambda & 0 & 0 & 0\\ 0 & & & \\ 0 & & m & \\ 0 & & & \end{pmatrix}\otimes(e_{22}+e_{33})\otimes 1\,. \tag{9.3.3}$$

We have $B_F := (A_F)_{\mathbb{C}}/\ker\pi_{\mathbb{C}} \simeq \mathbb{C}\oplus\mathbb{C}\oplus M_2(\mathbb{C})\oplus M_3(\mathbb{C})$ with elements of the form $\langle\lambda,\lambda',q,m\rangle$. Now we will discuss quantum isometries replacing A_F by B_F.

It is easy to see that for the spectral triple of the internal part of the Standard Model, the conditions (3.3.1) and ii. of Definition 3.2.1 are equivalent to

$$(J_0\otimes 1)\overline{U}=U(J_0\otimes 1)\ ; \tag{9.3.4a}$$
$$\mathrm{Ad}_{\mathrm{U}}(B_F)\subset B_F\otimes_{\mathrm{alg}} Q\ ; \tag{9.3.4b}$$

with J_0 given by (9.2.1). The equivalence between (3.3.1) and (9.3.4a) is an immediate consequence of the definition of J_F. The equivalence between ii. of Definition 3.2.1 and (9.3.4b) follows because the weak closure of the finite-dimensional C^* algebra B_F coincides with itself.

The main proposition describing $\widetilde{\mathrm{QISO}_J^+}$ will require the following result.

Lemma 9.3.2 *Let Q be the universal C^*-algebra generated by unitary elements x_k ($k=0,\ldots,n$), the matrix entries of 3×3 biunitaries T_m ($m=1,\ldots,n$) and of an $n\times n$ biunitary V, with relations*

$$\mathrm{diag}(x_0x_1,\ldots,x_0x_n)\Upsilon_\nu=\Upsilon_\nu\mathrm{diag}(x_0x_1,\ldots,x_0x_n)=\overline{V}\Upsilon_\nu=\Upsilon_\nu\overline{V}\,,\qquad V\Upsilon_R=\Upsilon_R\overline{V}\,, \tag{9.3.5a}$$
$$\sum\nolimits_{m=1}^{n} C_{rm}\overline{C}_{sm}(T_m)_{j,k}=0\,,\qquad \forall\, r\neq s\,,\quad (r,s=1,\ldots,n;\ j,k=1,2,3) \tag{9.3.5b}$$
$$(T_m^*)_{i,j}(T_m)_{k,l}=(T_{m'}^*)_{i,j}(T_{m'})_{k,l}\,,\qquad \forall\, m,m'\,,\quad (i,j,k,l=1,2,3,\ m,m'=1,\ldots,n) \tag{9.3.5c}$$

where $C=((C_{r,s}))$ is the CKM matrix. Then Q with matrix coproduct

$$\Delta(x_k)=x_k\otimes x_k\,,\quad \Delta((T_m)_{ij})=\sum_{l=1,2,3}(T_m)_{il}\otimes(T_m)_{lj}\,,\quad \Delta(V_{ij})=\sum_{l=1,\ldots,n}V_{il}\otimes V_{lj}\,, \tag{9.3.6}$$

is a quantum subgroup of the free product

$$\underbrace{C(U(1))*C(U(1))*\ldots*C(U(1))}_{n+1} * Q_n(3)*A_u(n)\,. \tag{9.3.7}$$

The Woronowicz C^*-ideal of (9.3.7) defining Q is determined by the relations (9.3.5a) and (9.3.5b).

Proposition 9.3.3 *The universal object $\widetilde{\mathrm{QISO}_J^+}(D_F)$ of the category $\mathfrak{C}_J$ is given by the CQG in Lemma 9.3.2 with corepresentation*

$$\begin{aligned}
U \;=\; & e_{11} \otimes e_{11} \otimes e_{11} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_0 x_k + e_{22} \otimes e_{11} \otimes (e_{11} + e_{44}) \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k \\
& + e_{11} \otimes e_{11} \otimes e_{33} \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k^* x_0^* + e_{22} \otimes e_{11} \otimes (e_{22} + e_{33}) \otimes \sum_{k=1}^{n} e_{kk} \otimes x_k^* \\
& + e_{11} \otimes e_{11} \otimes e_{22} \otimes \sum_{j,k=1}^{n} e_{jk} \otimes (V)_{jk} + e_{11} \otimes e_{11} \otimes e_{44} \otimes \sum_{j,k=1}^{n} e_{jk} \otimes (\overline{V})_{jk} \\
& + e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{11} + e_{44}) \otimes \sum_{m=1}^{n} e_{mm} \otimes (T_m)_{j,k} \\
& + e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{11} + e_{44}) \otimes \sum_{m=1}^{n} e_{mm} \otimes x_0^* (T_m)_{j,k} \\
& + e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{22} + e_{33}) \otimes \sum_{m=1}^{n} e_{mm} \otimes (\overline{T}_m)_{j,k} \\
& + e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{22} + e_{33}) \otimes \sum_{m=1}^{n} e_{mm} \otimes (\overline{T}_m)_{j,k} x_0 \,. \qquad (9.3.8)
\end{aligned}$$

$\widetilde{\mathrm{QISO}_J^+}(D_F)$ has a trivial coaction on the two summands $\mathbb{C}$ of $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ and the coaction on the other summands is given by

$$\alpha(\langle 0, 0, e_{ii}, 0\rangle) = \langle 0, 0, e_{ii}, 0\rangle \otimes 1 \,, \qquad (9.3.9a)$$
$$\alpha(\langle 0, 0, e_{12}, 0\rangle) = \langle 0, 0, e_{12}, 0\rangle \otimes x_0 \,, \qquad (9.3.9b)$$
$$\alpha(\langle 0, 0, e_{21}, 0\rangle) = \langle 0, 0, e_{21}, 0\rangle \otimes x_0^* \,, \qquad (9.3.9c)$$
$$\alpha(\langle 0, 0, 0, e_{ij}\rangle) = \textstyle\sum_{k,l=1,2,3} \langle 0, 0, 0, e_{kl}\rangle \otimes (T_1^*)_{i,k} (T_1)_{l,j} \,. \qquad (9.3.9d)$$

Definition 9.3.4 Let $Q_{n,C}(3)$ be the quantum subgroup of $Q_n(3)$, cf. Definition 9.1.1, defined by the relation $\sum_{m=1}^{n} C_{rm} \overline{C}_{sm} (u_m)_{j,k} = 0$.

Remark 9.3.5 As a C^* algebra, $Q_{n,C}(3)$ is noncommutative. Moreover, $A_u(3)$ can be identified with a quantum subgroup of $Q_{n,C}(3)$. To see this, let u be a 3×3 biunitary matrix whose entries generate $A_u(3)$ as a C^* algebra. Then the map

$$(u_m)_{jk} \mapsto u_{jk} \,, \;\; \forall\, m = 1, \ldots, n \,, \;\; j, k = 1, 2, 3 \,,$$

is a morphism of C^*-algebras as (9.3.5b) and (9.3.5c) are satisfied automatically. Thus $A_u(3)$ is a quantum subgroup of $Q_{n,C}(3)$.

The next result computes the quantum isometry group of the internal space of the Standard Model.

Proposition 9.3.6 *We have*

$$\mathrm{QISO}^{+}_{J}(D_F) = C(U(1)) * A_{\mathrm{aut}}(M_3(\mathbb{C})) \; .$$

The abelianization of the above quantum group is $C(U(1) \times PU(3))$.

Proof We observe from (9.3.9) that x_0 and $(T_1^*)_{i,k}(T_1)_{l,j}$ generate $\mathrm{QISO}^{+}_{J}(D_F)$. This implies that $\mathrm{QISO}^{+}_{J}(D_F)$ can be identified with a quantum subgroup of $C(U(1)) * PA_u(3)$ and let the corresponding surjective map from $C(U(1)) * PA_u(3)$ to $\mathrm{QISO}^{+}_{J}(D_F)$ be denoted by ϕ. Moreover, from Remark. 9.3.5, we can identify $A_u(3)$ as a quantum subgroup of $Q_{n,C}(3)$. Now, consider the map from $\widetilde{\mathrm{QISO}}^{+}_{J}(D_F)$ onto $C(U(1)) * A_u(3)$ which sends x_0 to itself, x_i to 1 for $i = 1, \ldots, n$, V to $x_0 1_n$ and $(T_m)_{jk}$ to u_{jk} $\forall\, m = 1, \ldots, n$, where u_{jk} denote the canonical generators of $A_u(3)$. This is clearly a morphism in the category $\mathfrak{C}_J$; hence we get $C(U(1)) * A_u(3)$ as a sub-object of $\widetilde{\mathrm{QISO}}^{+}_{J}(D_F)$. Moreover, the restriction of this morphism to $\mathrm{QISO}^{+}_{J}(D_F)$, which identifies $C(U(1)) * PA_u(3)$ with a quantum subgroup of $\mathrm{QISO}^{+}_{J}(D_F)$, is clearly the inverse of the morphism ϕ mentioned in the beginning of the proof. Thus, $\mathrm{QISO}^{+}_{J}(D_F) \cong C(U(1)) * PA_u(3)$. As $PA_u(3) \simeq A_{\mathrm{aut}}(M_3(\mathbb{C}))$, (cf. Definition 1.3.17 and Proposition 1.3.17) the proposition is proved. □

9.3.1 $\widetilde{QISO}^{+}_{J}$ in two special cases

Note that $\widetilde{\mathrm{QISO}}^{+}_{J}(D_F)$ in general depends on the forms of Υ_ν, Υ_R, and C. In case when Υ_ν is invertible, which happens for the Dirac operator in the moduli space as in Prop. 1.192 of [13], we have the following:

Proposition 9.3.7 *If Υ_ν is invertible, $\widetilde{\mathrm{QISO}}^{+}_{J}(D_F) \cong Q_{n,C}(3) * \mathcal{E}$, where $\mathcal{E}$ is the quotient of*

$$\underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1}$$

by the relations

$$\begin{aligned} x_i^* x_0^* &= x_0 x_j \qquad \forall\, i, j \text{ with } (\Upsilon_R)_{ij} \neq 0 \; , \\ x_i &= x_j \qquad \forall\, i, j \text{ with } (\Upsilon_\nu)_{ij} \neq 0 \; . \end{aligned}$$

Proof In this case, the first equation in (9.3.5a) gives $V = \mathrm{diag}(x_1^* x_0^*, \ldots, x_n^* x_0^*)$; hence the term $A_u(n)$ in (9.3.7) vanishes. Moreover, $(\Upsilon_\nu)_{ij} x_0 (x_i - x_j) = 0$. The latter implies $x_i = x_j$ if $(\Upsilon_\nu)_{ij} \neq 0$.

The second equation in (9.3.5a) becomes $(\Upsilon_R)_{ij}(x_i^* x_0^* - x_0 x_j) = 0$, which gives $x_i^* x_0^* = x_0 x_j$ if $(\Upsilon_R)_{ij} \neq 0$. $\square$

Next we discuss the case $\Upsilon_\nu = 0$. However, experiments suggest that such a situation cannot arise in reality. Nevertheless, it is an interesting mathematical exercise. In literature, this is the so-called *minimal* Standard Model. This is described by the following proposition.

Proposition 9.3.8 *If* $\Upsilon_\nu = 0$,

$$\widetilde{\mathrm{QISO}_J^+}(D_F) \cong \underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1} * Q_{n,C}(3) * A',$$

where $A' := A_u(n)/\sim$, $A_u(n)$ *is generated by the* $n \times n$ *biunitary* V *and "*$\sim$*" is the relation* $V\Upsilon_R = \Upsilon_R \overline{V}$.

We refer to [32] for the proof of the proposition.

9.3.2 Quantum Isometries for the Real C^* Algebra A_F

After computing the quantum isometry group for the spectral triple on B_F, let us now discuss the corresponding results for the real C^* algebra A_F. We mainly state the results without proof and refer to [32] for the details.

Motivated by Proposition 9.1.2, let $\mathfrak{C}_{J,\mathbb{R}}$ be the category of compact quantum groups coacting by orientation and real structure-preserving isometries via a unitary corepresentation U on the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$ such that there is a coaction α on $(A_F)_\mathbb{C} = A_F \otimes_\mathbb{R} \mathbb{C}$ with $\alpha|_{B_F} = \mathrm{ad}_\mathrm{U}$ and

$$(\sigma \otimes *)\alpha = \alpha\sigma \,. \tag{9.3.10}$$

It is clear that $\mathfrak{C}_{J,\mathbb{R}}$ is a subcategory of $\mathfrak{C}_J$ and for $A, B \in \mathfrak{C}_{J,\mathbb{R}}$, $\mathrm{Mor}_{\mathfrak{C}_J}(A, B) = \mathrm{Mor}_{\mathfrak{C}_{J,\mathbb{R}}}(A, B)$, where $\mathrm{Mor}_\mathcal{C}(\cdot\,,\ \cdot)$ denotes the set of morphisms in a category $\mathcal{C}$. In particular, any object Q (say) of the category $\mathfrak{C}_{J,\mathbb{R}}$ will satisfy the relations of $\widetilde{\mathrm{QISO}_J^+}(D_F)$ as given in Proposition 9.3.3. In fact, we will denote the generators of Q by the same symbols as in Proposition 9.3.3 throughout the subsection.

Theorem 9.3.9 *An object* Q *of* $\mathfrak{C}_J$ *is an object in* $\mathfrak{C}_{J,\mathbb{R}}$ *if and only if the generators satisfy*

$$(T_m)_{jk}(T_m)^*_{j'k'}(T_m)_{j''k''} = (T_m)_{j''k''}(T_m)^*_{j'k'}(T_m)_{jk} \tag{9.3.11}$$

for all $m = 1, \ldots, n$ *and all* $j, j', j'', k, k', k'' \in \{1, 2, 3\}$.

Let $\widetilde{\mathrm{QISO}_\mathbb{R}^+}(D_F)$ *be the quantum subgroup of the CQG* $\widetilde{\mathrm{QISO}_J^+}(D_F)$ *in Proposition 9.3.3 defined by the relations (9.3.11). Then* $\widetilde{\mathrm{QISO}_\mathbb{R}^+}(D_F)$ *is the universal object in the category* $\mathfrak{C}_{J,\mathbb{R}}$.

The Eq. (9.3.11) shows that the quantum group $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ is related to the half liberated unitary group $A^*_u(n)$ introduced in Sect. 1.3.2. In fact, as in (9.3.7), let $Q^*_n(n')$ be the amalgamated free product of n copies of $A^*_u(n')$ over $C(PU(n'))$. Then we have

Corollary 9.3.10 $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ *can be identified with a quantum subgroup of the following CQG:*

$$\underbrace{C(U(1)) * C(U(1)) * \ldots * C(U(1))}_{n+1} * Q^*_n(3) * A_u(n).$$

The Woronowicz C^*-ideal of the above CQG defining $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ is given by (9.3.5a) *and* (9.3.5b).

As before, let $\mathrm{QISO}^+_{\mathbb{R}}(D_F)$ be the C^*-subalgebra of $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ generated by $\langle \xi \otimes 1, \mathrm{ad}_{\mathrm{U}_{\mathbb{R}}}(a)(\eta \otimes 1)\rangle$, where $a \in B_F$, $\xi, \eta \in H_F$ and $U_{\mathbb{R}}$ is the corepresentation of $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$. Combining Proposition 1.3.19 and Corollary 9.3.10, we get

Corollary 9.3.11 $\mathrm{QISO}^+_{\mathbb{R}}(D_F) = C(U(1)) * C(PU(3))$.

9.4 Quantum Isometries of $M \times F$

It is not difficult to see that $\mathrm{QISO}^+_{\mathbb{R}}(D_F)$ or $\mathrm{QISO}^+_J(D_F)$ can also give orientation-preserving quantum isometries of the full spectral triple of the Standard Model. Indeed, considering $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ in Lemma 9.1.3 as the canonical classical spectral triple on the spin manifold M and the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$, respectively, we do get orientation-preserving isometric coactions of $\mathrm{QISO}^+_J(D_F)$ on the corresponding product spectral triple.

9.5 Physical Significance of the Results

Having obtained the description of the quantum groups $\widetilde{\mathrm{QISO}}^+_J(D_F)$ and $\mathrm{QISO}^+_J(D_F)$ and their coactions, we describe the physical significance of the results.

The relevance of the following observation stems from the fact that the neutrino masses are not known. In fact, we only know that they are all distinct ([37, 38]). We refer to [32] for the details of the proof.

Proposition 9.5.1 *1.* $\widetilde{\mathrm{QISO}}^+_J(D_F)$ *depends on Υ_ν, Υ_R, and the CKM matrix C.*

2. However, the quantum group $\mathrm{QISO}^+_J(D_F)$ *does not depend on the explicit form of these two matrices.*

3. $\mathrm{QISO}^+_J(D_F)$ *is independent of the number of generations.*

Next, we explain how the elementary particles transform under the corepresentation U. A straightforward computation using (9.3.8) proves that we have the following transformation laws for the neutrinos, electrons, and quarks (cf. the dictionary in Sect. 9.2.1).

Proposition 9.5.2

$$U(\nu_{L,k}) := \nu_{L,k} \otimes x_0 x_k \ , \qquad U(\nu_{R,k}) := \sum_{j=1}^{n} \nu_{R,j} \otimes \overline{V}_{jk} \ ,$$

$$U(e_{L,k}) := e_{L,k} \otimes x_k \ , \qquad U(e_{R,k}) := e_{R,k} \otimes x_k \ ,$$

$$U(u_{L,c,k}) := \sum_{c'=1}^{3} u_{L,c',k} \otimes (T_k)_{c'c} \ , \qquad U(u_{R,c,k}) := \sum_{c'=1}^{3} u_{R,c',k} \otimes (T_k)_{c'c} \ ,$$

$$U(d_{L,c,k}) := \sum_{c'=1}^{3} d_{L,c',k} \otimes x_0^*(T_k)_{c'c} \ , \qquad U(d_{R,c,k}) := \sum_{c'=1}^{3} d_{R,c',k} \otimes x_0^*(T_k)_{c'c} \ .$$

Antiparticles transform according to the conjugate corepresentations.

As an immediate corollary, we see that

Corollary 9.5.3 *The corepresentation U of $\widetilde{\mathrm{QISO}_J^+}(D_F)$ behaves in the following way:*

1. Each of the one-dimensional subspaces corresponding to each of the k-th generation of the left-handed neutrino, left-handed electron and the right-handed electron is kept invariant.

2. The corepresentation of the quantum subgroup of $A_u(n)$ generated by V_{jk} s keeps the n-dimensional subspace of the right-handed neutrinos invariant by (quantum) permuting the n generations.

3. The three-dimensional subspace of the k-th generation of the left-handed up-quark is kept invariant by the coaction of $Q_{n,C}(3)$ by (quantum) permuting the colors. The same holds for the right-handed up quarks.

4. There is a corepresentation of a quantum subgroup of $C(U(1)) * Q_{n,C}(3)$ which keeps the subspace of the k-th generation of the left-handed (as well as the right handed) down quarks invariant.

Now we show the connection between $\widetilde{\mathrm{QISO}_J^+}(D_F)$ and the gauge group of the Standard Model after symmetry breaking.

The classical gauge group of the Standard Model after symmetry breaking is $(U(1) \times U(3))/\mathbb{Z}_3$. It corresponds to the following action, say, α on H_F :

$$\alpha_{\tau,g}(\nu_{\bullet,k}) = \nu_{\bullet,k}, \quad \alpha_{\tau,g}(e_{\bullet,k}) = (\tau^*)^3 e_{\bullet,k},$$

$$\alpha_{\tau,g}(u_{\bullet,c,k}) = \sum_{c'=1}^{3} \tau^2 g_{c'c} u_{\bullet,c',k}, \quad \alpha_{\tau,g}(d_{\bullet,c,k}) = \sum_{c'=1}^{3} \tau^* g_{c'c} d_{\bullet,c',k},$$

where τ is the generator of $U(1)$ and g is an element of $U(3)$.

$A_u(3)$ is the free version of $U(3)$ and hence we can view the quantum group $C(U(1)) * A_u(3)$ as a free version of the classical group $U(1) \times U(3)$. Now, for a cube root of unity q, there is an action of $\mathbf{Z}_3$ on $C(U(1)) * A_u(3)$ given by the formula

$$z \mapsto qz, \; T_{jk} \mapsto qT_{jk}.$$

Thus, it is reasonable to view the fixed-point subalgebra for this action, denoted by $\{C(U(1)) * A_u(3)\}^{\mathbf{Z}_3}$ as a free version of the gauge group $(U(1) \times U(3))/\mathbb{Z}_3$.

The next proposition says that modulo terms which do not appear in the adjoint coaction, the quantum group $\widetilde{\mathrm{QISO}}_J^+(D_F)$ is the free version of the classical gauge group.

Proposition 9.5.4 *Let z be the canonical generator of $C(U(1))$, and $T = ((T_{jk}))$ be the generators of $A_u(3)$. The quantum group $\{C(U(1)) * A_u(3)\}^{\mathbf{Z}_3}$ has the following corepresentation on H_F making it a sub-object of $\widetilde{\mathrm{QISO}}_J^+(D_F)$ in the category $\mathfrak{C}_J$:*

$$\nu_{\bullet,k} \mapsto \nu_{\bullet,k} \otimes 1 \,, \qquad e_{\bullet,k} \mapsto e_{\bullet,k} \otimes (z^*)^3 \,, \tag{9.5.1a}$$

$$u_{\bullet,c,k} \mapsto \sum\nolimits_{c'=1}^{3} u_{\bullet,c',k} \otimes z^2 T_{c'c} \,, \qquad d_{\bullet,c,k} \mapsto \sum\nolimits_{c'=1}^{3} d_{\bullet,c',k} \otimes z^* T_{c'c} \,, \tag{9.5.1b}$$

where $\bullet$ is L or R.

More precisely, modulo the terms which do not appear in the adjoint coaction on B_F, we have $\widetilde{\mathrm{QISO}}_J^+(D_F) \sim \{C(U(1)) * A_u(3)\}^{\mathbf{Z}_3}$, i.e., the "free version" of the ordinary gauge group after symmetry breaking.

Finally, it follows from (9.5.1) that the coaction of the abelianization $C(U(1) \times U(3))^{\mathbf{Z}_3} \simeq C\big((U(1) \times U(3))/\mathbf{Z}_3\big)$ of $\{C(U(1)) * A_u(3)\}^{\mathbf{Z}_3}$ gives us precisely the usual global gauge transformations after symmetry breaking.

Proof The proof being routine, we merely point out that the surjective CQG homomorphism $\widetilde{\mathrm{QISO}}_J^+(D_F) \to \{C(U(1)) * A_u(3)\}^{\mathbf{Z}_3}$ is given by

$$x_0 \mapsto z^3, \quad x_m \mapsto (z^*)^3, \quad \forall m = 1, \ldots, n \,, \quad (T_m)_{jk} \mapsto z^2 T_{jk}, \quad \forall m = 1, \ldots, n \,, \quad V \mapsto 1_n \,.$$

The kernel of this map is the ideal generated by V_{jk}'s and by products $x_0 x_k$'s and $(T_m^* T_{m'})_{kl}$ for all $m \neq m'$ is given by elements that do not appear in the adjoint coaction on B_F. □

9.5.1 Analysis of the Result for the Minimal Standard Model

In this subsection, we briefly present some observations for the minimal standard model. For the details, we refer to Sect. 3.3 of [32]. It is well known that as a consequence of Noether's theorem, there exists a conservation law corresponding to each classical group of symmetries. So we try to investigate the symmetries coming from some classical subgroups of the quantum isometry group as in Proposition 9.3.8.

The minimal Standard Model considers only left-handed neutrinos and thus the symmetry coming from the quantum group A' which coacts only on the subspace $(e_{11} \otimes e_{11} \otimes (e_{22} + e_{44}) \otimes 1)H_F$ of right-handed neutrinos can be neglected. The symmetries coming from the $C(U(1))$ factors generated by $x_i, i = 1, \ldots, n$ give the conservation laws of the total number of leptons in each generation which is the sum of the number of electrons, muons, tau, and the remaining $n-3$ coming from the other lepton families. Finally, we consider the classical symmetry given by the subgroup of $Q_{n,C}(3)$ isomorphic with $U(1)$. This corresponds to a phase transformation on the subspace of quarks, i.e., $\mathbb{C}^2 \otimes (1-e_{11})\mathbb{C}^4 \otimes (e_{11}+e_{44})\mathbb{C}^4 \otimes \mathbb{C}^n$ of quarks and its inverse on the subspace of anti-quarks given by $\mathbb{C}^2 \otimes (1-e_{11})\mathbb{C}^4 \otimes (e_{22}+e_{33})\mathbb{C}^4 \otimes \mathbb{C}^n$. This is called the "baryon phase symmetry" in physics literature and gives the conservation law of the number of baryon, i.e., the difference between the number of quarks and that of anti-quarks.

9.6 Invariance of the Spectral Action

After our discussion on the case of neutrinos without mass, we want to discuss the full quantum symmetry given by $\widetilde{\mathrm{QISO}}_J^+(D_F)$ for the standard model with massive neutrinos. In some sense, we want to do a "Noether analysis" for quantum group symmetries to derive laws of conservation, leading to possible physical predictions. To this end, note that most of the classical (group) symmetries of the massless neutrino case do not extend to the models with massive neutrinos. However, we have a quite rich symmetry coming from quantum group coactions. To utilize this quantum symmetry in the spirit of Noether's theory, we need to see whether such quantum symmetries leave the spectral action invariant. We discuss this point in this section.

Let $(\mathcal{A}, \mathcal{H}, D, \gamma)$ be a spectral triple and we choose $\gamma = 1$ in the odd case. Consider the following "action functional" [5]:

$$S[A, \psi] := S_b[A] + S_f[A, \psi] ,$$

where A is a self-adjoint element of $\Omega^1_D \subset \mathcal{B}(\mathcal{H})$, ψ is either in $\mathcal{H}$ (for the Yang–Mills theories) or in $\mathcal{H}_+ := (1 + \gamma)\mathcal{H}$ for the Standard Model. The reduction of the Hilbert space to $\mathcal{H}_+$ is needed to solve the fermion doubling problem [12, 13]. This functional governs the dynamics of the physical model given by the spectral triple. Let f be some appropriate "cut off" function, which is a smooth approximation of the characteristic function of $[-1, 1]$. Denote by D_A either the operator $D + A$ (when there is no real structure to be considered) or $D_A := D + A + \epsilon' JAJ^{-1}$ in case there is a real structure given by J, where ϵ' is the sign in Definition 2.2.3. The bosonic part S_b and the fermionic part S_f of the action functional are defined as follows:

$$S_b[A] = \mathrm{Tr}\, f(D_A/\Lambda) \ ,$$

$$S_f[A, \psi] := \langle J\psi, D_A\psi \rangle \ , \tag{9.6.1}$$

in the presence of a real structure J (where $D_A := D + A + \epsilon' JAJ^{-1}$), or otherwise

$$S_f[A, \psi] = \langle \psi, D_A\psi \rangle \ , \tag{9.6.2}$$

where $D_A := D + A$.

Note that $f(D_A/\Lambda)$ is defined by the continuous functional calculus and it is a trace-class operator on $\mathcal{H}$. Hence, $S_b[A]$ is well defined.

Let us concentrate on the question of co-invariance of the fermionic part S_f in the presence of a real structure, i.e., D_A is given by (9.6.1). However, in the other case, the proof of the co-invariance will be very similar.

Let Q be a compact quantum group and $\hat{U}$ a unitary corepresentation of Q on $\mathcal{H}$ commuting with D and γ and assume furthermore that $\mathrm{ad}_{\hat{U}}(\mathcal{A}) \subseteq \mathcal{A} \otimes_{\mathrm{alg}} Q$. Then $\hat{U}$ leaves $\mathcal{H}_+$ invariant and for any 1-form $A = \sum_i a_i[D, b_i]$, with $a_i, b_i \in \mathcal{A}$, we have $\mathrm{ad}_{\hat{U}}(A) = \sum_i \mathrm{ad}_{\hat{U}}(a_i)[D \otimes 1, \mathrm{ad}_{\hat{U}}(b_i)] \in \Omega^1_D \otimes_{\mathrm{alg}} Q$. This allows us to define a coaction of Q on $\Omega^1_D \oplus \mathcal{H}_+$ by

$$\beta : (A, \psi) \mapsto \big(\hat{U}(A \otimes 1)\hat{U}^*, \hat{U}(\psi \otimes 1)\big) \ .$$

Next, we need to consider natural extensions of the action functional S_b and S_f on $\Omega^1_D \oplus \mathcal{H}_+$. Recall the Hilbert Q-module structure of $\mathcal{M} := \mathcal{H}_+ \otimes Q$ with the Q-valued inner product $\langle\, ,\, \rangle_Q : \mathcal{M} \otimes \mathcal{M} \to Q$ given by $\langle \psi \otimes q, \psi' \otimes q' \rangle_Q = q^* q' \langle \psi, \psi' \rangle$. For any unitary (resp. antiunitary) map L on $\mathcal{H}_+$, consider $L \otimes 1$ (resp. $L \otimes *$) on $\mathcal{M}$, which gives a Q-linear (resp. antilinear) extension of L. We now extend the spectral action by the following Q-valued functional:

$$\tilde{S}[\tilde{A}, \tilde{\psi}] := \tilde{S}_b[\tilde{A}] + \tilde{S}_f[\tilde{A}, \tilde{\psi}] \ ,$$

where

$$\tilde{S}_b[\tilde{A}] := (\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\, f(D_{\tilde{A}}/\Lambda) \ ,$$
$$\tilde{S}_f[\tilde{A}, \tilde{\psi}] := \big\langle (J \otimes *)\tilde{\psi}, D_{\tilde{A}}\tilde{\psi} \big\rangle_Q \ ,$$

and $\tilde{A}$ is a self-adjoint element of $\Omega^1_D \otimes_{\mathrm{alg}} Q$, $\tilde{\psi} \in \mathcal{H}_+ \otimes Q$, $D_{\tilde{A}} := D \otimes 1 + \tilde{A} + \epsilon'(J \otimes *)\tilde{A}(J \otimes *)^{-1}$.

We explain how to define $f(D_{\tilde{A}}/\Lambda)$. Consider the GNS Hilbert space $L^2(Q)$ corresponding to the Haar state of Q and the bounded self-adjoint operator $\tilde{A} + \epsilon'(J \otimes *)\tilde{A}(J \otimes *)^{-1}$ on $\mathcal{H} \otimes L^2(Q)$. Then $D_{\tilde{A}}$ is an unbounded self-adjoint operator on $\mathcal{H} \otimes L^2(Q)$ and we define $f(D_{\tilde{A}}/\Lambda)$ by the continuous functional calculus.

The action functional is called co-invariant if the following holds:

$$\tilde{S}[\beta(A,\psi)] = S[A,\psi]\cdot 1_Q. \tag{9.6.3}$$

As A is a self-adjoint 1-form, $\tilde{A} = \hat{U}(A\otimes 1)\hat{U}^*$ is a self-adjoint element of $\Omega^1_D \otimes_{\text{alg}} Q$ and hence $\tilde{S}[\beta(A,\psi)]$ is well defined. In the rest of the section we discuss the co-invariance of the action. We consider the fermionic and the bosonic part separately.

Proposition 9.6.1 *If $\hat{U}$ commutes with the Dirac operator and the grading and also satisfies (3.3.1), then*

$$\tilde{S}_f[\beta(A,\psi)] = S_f[A,\psi]\cdot 1_Q$$

for all $(A,\psi) \in \Omega^{1,s.a.}_D \oplus \mathcal{H}_+$.

Proof Since $\hat{U}$ commutes with D and $J\otimes *$, we have

$$D_{\hat{U}(A\otimes 1)\hat{U}^*} = D\otimes 1 + \hat{U}(A\otimes 1)\hat{U}^* + \epsilon'(J\otimes *)\hat{U}(A\otimes 1)\hat{U}^*(J\otimes *)^{-1} = \hat{U}(D_A\otimes 1)\hat{U}^*. \tag{9.6.4}$$

Thus,

$$\begin{aligned}
\tilde{S}_f[\beta(A,\psi)] &= \big\langle (J\otimes *)\hat{U}(\psi\otimes 1_Q), D_{\hat{U}(A\otimes 1)\hat{U}^*}\hat{U}(\psi\otimes 1_Q)\big\rangle_Q \\
&= \big\langle \hat{U}(J\psi\otimes 1_Q), \hat{U}(D_A\psi\otimes 1_Q)\big\rangle_Q \\
&= \langle J\psi, D_A\psi\rangle\cdot 1_Q = S_f[A,\psi]\cdot 1_Q\ ,
\end{aligned}$$

as $\hat{U}$ is a unitary. □

Now we will assume that $(\mathcal{A},\mathcal{H},D,J,\gamma)$ is the product of two real spectral triples and one of them is even and finite dimensional. Moreover, we assume that $\hat{U} := 1\otimes U$, where (Q,U) is an object in the category of orientation and real structure-preserving quantum isometries of the finite-dimensional spectral triple $(\mathcal{A}_2,\mathcal{H}_2,D_2,\gamma_2,J_2)$. The result stated and proved below gives the co-invariance of the bosonic part of the action functional.

Proposition 9.6.2 *For any* $A \in \Omega^{1,s.a.}_D$, $\tilde{S}_b[\mathrm{Ad}_{\hat{U}}(A)] = S_b[A]\cdot 1_Q$.

Proof From (9.6.4) we have

$$\begin{aligned}
\tilde{S}_b[\hat{U}(A\otimes 1)\hat{U}^*] &= (\mathrm{Tr}_{\mathcal{H}}\otimes \mathrm{id})\, f(D_{\hat{U}(A\otimes 1)\hat{U}^*}/\Lambda) \\
&= (\mathrm{Tr}_{\mathcal{H}}\otimes \mathrm{id})\, f\big(\hat{U}(D_A\otimes 1)\hat{U}^*/\Lambda\big)\ .
\end{aligned}$$

By continuous functional calculus,

$$f\big(\hat{U}(D_A \otimes 1)\hat{U}^*/\Lambda\big) = \hat{U} f\big((D_A \otimes 1)/\Lambda\big)\hat{U}^* = \hat{U}\big(f(D_A/\Lambda) \otimes 1\big)\hat{U}^*.$$

Now, it is easy to see that for any trace-class operator L on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$(\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})(\hat{U}(L \otimes 1)\hat{U}^*) = \mathrm{Tr}_{\mathcal{H}}(L)1_Q\ .$$

Applying this observation to the trace-class operator $L := f(D_A/\Lambda)$ we get

$$\begin{aligned}\tilde{S}_b[\hat{U}(A \otimes 1)\hat{U}^*] &= (\mathrm{Tr}_{\mathcal{H}} \otimes \mathrm{id})\,\hat{U}(L \otimes 1)\hat{U}^* \\ &= \mathrm{Tr}_{\mathcal{H}}(L)1_Q \equiv \mathrm{Tr}_{\mathcal{H}} f(D_A/\Lambda)1_Q \\ &= S_b[A]1_Q\ .\end{aligned}$$

This completes the proof. □

Proposition 9.6.3 *The bosonic and the fermionic part of the spectral action of the Standard Model are preserved by the compact quantum group* $\widetilde{\mathrm{QISO}}^+(B_F, H_F, D_F, \gamma_F, J_F)$.

Proof The compact quantum group $\widetilde{\mathrm{QISO}}^+(B_F, H_F, D_F, \gamma_F, J_F)$ has a corepresentation preserving $\mathcal{H}_+$ and it satisfies the hypothesis of Lemma 9.1.3 and Propositions 9.6.1 and 9.6.2, hence the result follows. □

Remark 9.6.4 Since $\widetilde{\mathrm{QISO}}^+_{\mathbb{R}}(D_F)$ is a quantum subgroup of $\widetilde{\mathrm{QISO}}^+_J(D_F)$, its coaction preserves the spectral action.

References

1. Cottingham, W.N., Greenwood, D.A.: An Introduction to the Standard Model of Particle Physics, 2nd edn. Cambridge University Press, Cambridge (2007)
2. Connes, A., Lott, J.: Particle models and noncommutative geometry, Recent advances in field theory (Annecy- le- Vieux, 1990). Nuclear Phys. B. Proc. Suppl **18B**(1990), 29–47 (1991)
3. Connes, A.: Gravity coupled with matter and the foundation of non-commutative geometry. Commun. Math. Phys **182**, 155–176 (1996)
4. Chamseddine, A.H., Connes, A.: Universal formula for noncommutative geometry actions: unification of gravity and the Standard Model. Phys. Rev. Lett. **77**, 4868 (1996)
5. Chamseddine, A.H., Connes, A.: The spectral action principle. Commun. Math. Phys. **186**, 731–750 (1997)
6. Chamseddine, A.H., Connes, A.: Scale invariance in the spectral action. J. Math. Phys. **47**, 063504, 19 (2006)
7. Chamseddine, A.H., Connes, A.: Inner fluctuations of the spectral action. J. Geom. Phys **57**(1), 1–21 (2006)
8. Chamseddine, A.H., Connes, A., Marcolli, M.: Gravity and the standard model with neutrino mixing. Adv. Theor. Math. Phys **11**(6), 991–1089 (2007)

9. Connes, A.: Noncommutative geometry and the standard model with neutrino mixing. J. High. Energ. Phys. **11**, 081 (2006)
10. Chamseddine, A.H., Connes, A.: Why the standard model. J. Geom. Phys. **58**(1), 38–47 (2008)
11. Chamseddine, A.H., Connes, A.: Noncommutative geometry as a framework for unification of all fundamental interactions including gravity. Part I. Fortschr. Phys. **58**(6), 553–600 (2010)
12. Lizzi, F., Mangano, G., Miele, G., Sparano, G.: Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories. Phys. Rev. D **55**, 6357–6366 (1997)
13. Connes, A., Marcolli, M.: Noncommutative Geometry, Quantum Fields and Motives. American Mathematical Society Colloquium Publications, 55 American Mathematical Society, Providence. RI, Hindustan Book Agency, New Delhi (2008)
14. Connes, A.: On the foundation of noncommutative geometry, In: The Unity of Mathematics, Progress in Mathematics, vol. 224, pp. 173–204. Birkhauser, Boston (2006)
15. Barrett, J.W.: A Lorentzian version of the noncommutative geometry of the standard model of particle physics. J. Math. Phys. **48**, 012303 (2007)
16. Chamseddine, A.H., Connes, A.: Resilience of the spectral standard model. JHEP **1209**, 104 (2012)
17. Devastato, A., Lizzi, F., Martinetti, P.: Grand Symmetry. Spectral Action, and the Higgs mass. JHEP **1**, 042 (2014)
18. van Suijlekom, W.D.: Noncommutative Geometry and Paricle Physics. Springer, Mathematical Physics Studies (2015)
19. van den Broek, T., van Suijlekom, W.D.: Supersymmetric QCD from noncommutative geometry. Phys. Lett. B **699**(1–2), 119–122 (2011)
20. Boeijink, J., van Suijlekom, W.D.: The noncommutative geometry of Yang Mills fields. J. Geom. Phys. **61**, 1122–1134 (2011)
21. van den Dungen, K., van Suijlekom, W.D.: Electrodynamics from noncommutative geometry. J. Noncommut. Geom. **7**, 433–456 (2013)
22. van den Broek, T., van Suijlekom, W.D.: Going beyond the standard model with noncommutative geometry, J. High Energy Phys. **1**(3), 12 (2013)
23. Chamseddine, A.H., Connes, A., van Suijlekom, W.D.: Inner fluctuations in noncommutative geometry without the first order condition. J. Geom. Phys. **73**, 222–234 (2013)
24. Chamseddine, A.H., Connes, A., van Suijlekom, W.D.: Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. J. High. Energ. Phys. 132 (2013)
25. Chamseddine, A.H., Connes, A., van Suijlekom, W.D.: Grand unification in the spectral Pati-Salam model. J. High. Energ. Phys. **11**, 011 (2015)
26. Beenakker, W., van den Broek, T., van Suijlekom, W.D.: Supersymmetry and Noncommutative Geometry. Springer Brief in Mathematical Physics, vol. 9. Springer (2016)
27. van den Dungen, K., van Suijlekom, W.D.: Particle physics from almost commutative space time. Rev. Math. Phys. **24**(9), 1230004 (2012)
28. Connes, A.: Noncommutative differential geometry and the structure of space-time. In: , Huggett, S.A., et al. (eds.) Proceedings of the Symposium on Geometry, pp. 49–80. Oxford Univ. Press, Oxford, UK (1998)
29. Kastler, D.: Regular and adjoint representation of $SL_q(2)$ at third root of unit. CPT internal report (1995)
30. Coquereaux, R.: On the finite dimensional quantum group $M_3 \oplus (M_{2|1}(\Lambda^2))_0$. Lett. Math. Phys. **42**, 309–328 (1997)
31. Dabrowski, L., Nesti, F., Siniscalco, P.: A finite quantum symmetry of $M(3, \mathbb{C})$, Int. J. Mod. Phys. A **13**, 4147–4162 (1998)
32. Bhowmick, J., D'Andrea, F., Dabrowski, L.: Quantum isometries of the finite noncommutative geometry of the standard model. Commun. Math. Phys. **307**, 101–131 (2011)
33. Bhowmick, J., D'Andrea, F., Das, B., Dabrowski, L.: Quantum gauge symmetries in noncommutative geometry. J. Noncommutative Geom. **8**(2), 433–471 (2014)
34. Goswami, D.: Quantum isometry group for spectral triples with real structure. SIGMA **6**, 007 (2010)

35. Banica, T., Vergnioux, R.: Invariants of the half-liberated orthogonal group. Ann. Inst. Fourier (Grenoble) **60**(6), 2137–2164 (2010)
36. Goodearl, K.R.: Notes on real and complex C^* algebras. Shiva Mathematical Series, Book 5, Birkhauser, Boston (1980)
37. Fukuda, Y., et al.: (Super-Kamiokande Collaboration), Evidence for oscillation of atmospheric neutrinos. Phys. Rev. Lett. **81**, 1562–1567 (1998)
38. Ahmad, Q.R., et al.: (SNO Collaboration), Direct evidence for neutrino flavor transformation from neutral-current interactions in the sudbury neutrino observatory. Phys. Rev. Lett. **89**, 011301 (2002)

Chapter 10
More Examples and Open Questions

Abstract We briefly discuss quantum isometry groups of few more interesting examples, including the free and half-liberated spheres, examples due to Raum and Weber as well as some Drinfeld-Jimbo quantum groups. We also give the outlines of other approaches to quantum isometry groups, such as the framework of orthogonal filtrations due to Banica, Skalski and de Chanvalon, affine quantum isometry groups in the sense of Banica and quantum isometry groups of compact metric spaces due to Banica, Goswami, Sabbe and Quaegebeur. We mention several open questions in this context.

10.1 Free and Twisted Quantum Spheres and Their Projective Spaces

In Chap. 4, the quantum isometry groups of classical and Podles' spheres were discussed. The Podles' spheres are quantum homogeneous spaces of the non-Kac type quantum group $SU_\mu(2)$. For the definition and examples of higher dimensional quantum spheres (obtained from $SU_\mu(n)$) and their projective spaces, we refer to [1–3] and the references therein. On the other hand, as a byproduct of the constructions by Banica and Speicher in [4], free versions of the quantum spheres realized as quantum homogeneous spaces of the free and half -liberated quantum groups and their projective spaces appeared in [5] and then in [6, 7] and very recently in [8]. The goal of this section is to give a very brief overview of some of the results of these papers which are related to quantum isometry groups and quantum group of affine isometries in the sense of Banica. As we are mainly concerned with quantum isometry groups or more generally quantum symmetry groups, we will refrain from discussing a lot of highly interesting results in these papers including the classification of quantum spheres (both real and complex) and their projective spaces by Banica and his collaborators, related category theoretical framework and relations with partitions as well as computation of spherical integrals and free probabilistic aspects by Banica, Speicher, Weber et al.

Let us collect the main definitions of these quantum groups, quantum spheres and the noncommutative projective spaces associated with them. We will extensively

D. Goswami and J. Bhowmick, *Quantum Isometry Groups*,
Infosys Science Foundation Series, DOI 10.1007/978-81-322-3667-2_10

use the free orthogonal quantum groups $A_o(n)$ (denoted by O_n^+ by Banica), its half-liberated version $A_o^*(n)$ and projective versions $PA_o(n)$ introduced in Sect. 1.3.2. We remark that unlike Banica, we will only deal with the C^* algebras in question and not deal with the dual "quantum spaces" following Remark 1.2.14.

The definitions of the following quantum groups will be of use to us.

Definition 10.1.1 If $u = ((u_{ij}))$ denotes the fundamental corepresentation of the quantum group $A_o(n)$, then we have the following twisted orthogonal quantum groups:

i. $\overline{A_o}(n)$ is the quotient of $A_o(n)$ by the relations $u_{ij}u_{kl} = -u_{kl}u_{ij}$ for $u_{ij} \neq u_{kl}$ on the same row or column of u and $u_{ij}u_{kl} = u_{kl}u_{ij}$ otherwise.

ii. $\overline{A}_o^*(n)$ is the quotient of $A_o(n)$ by the relations $u_{ij}u_{kl}u_{pq} = -u_{pq}u_{kl}u_{ij}$ if $r \leq 2, s = 3$ or $r = 3, s \leq 2$, $u_{ij}u_{kl}u_{pq} = u_{pq}u_{kl}u_{ij}$ for $r \leq 2$, $s \leq 2$ or $r = s = 3$, where, $r, s \in \{1, 2, 3\}$ are the number of rows or columns spanned by u_{ij}, u_{kl}, u_{pq} and i, j, k, l, p, q be natural numbers between 1 and n.

Associated with the quantum groups mentioned above, one has the following quantum homogeneous spaces:

Definition 10.1.2 i. The free orthogonal sphere S_+^{n-1} is the universal C^* algebra generated by elements $x_1, x_2, \ldots x_n$ satisfying the relations $x_i^* = x_i$ and $\sum_i x_i^2 = 1$.

ii. The half-liberated orthogonal sphere S_*^{n-1} is the quotient of S_+^{n-1} by the ideal generated by the relation $x_ix_jx_k = x_kx_jx_i$ for all $i, j, k = 1, 2, \ldots, n$.

iii. The twisted quantum spheres $\overline{S}^{n-1}$ ($\overline{S}_*^{n-1}$) are the quotients of S_+^{n-1} by the relations $x_ix_j = -x_jx_i \ \forall i \neq j$ (respectively $x_ix_jx_k = -x_kx_jx_i$ for distinct i, j, k, $x_ix_jx_k = x_kx_jx_i$ otherwise).

It can be easily seen that the commutative C^* algebra $C(S^{n-1})$ is a quotient of S_+^{n-1} by the relations $x_ix_j = x_jx_i$ for all i, j.

By an abuse of notation, let us denote the canonical generators of S_+^{n-1}, S_*^{n-1} and $C(S^{n-1})$ by the same symbols x_i and the generators of $A_o(n), A_o^*(n)$ and $C(O_n)$ by u_{ij}. Then the coactions of $A_o(n), A_o^*(n)$ and $C(O_n)$ on S_+^{n-1}, S_*^{n-1} and $C(S^{n-1})$ (respectively) are defined by the following:

$$\alpha(x_i) = \sum_j x_j \otimes u_{ji}.$$

Moreover, it is easy to see that in each case, this coaction restricts to a coaction on the canonical dense $*$-subalgebra A_n, A_n^+ or A_n^* (generated by x_i-s) of $C(S^{n-1}), S_+^{n-1}$ or S_*^{n-1} respectively.

For other examples of quantum spheres arising from similar or different considerations include the multi-parameter family of Banica in [9] and those by Speicher and Weber in [8]. We refer to those papers and the references therein for the details.

Definition 10.1.3 The noncommutative projective space associated to a C^* algebraic quotient S of S_+^{n-1} is the C^* subalgebra PS of S generated by $p_{ij} = x_ix_j$, $i, j = 1, 2, \ldots n$.

By the results in [5], it follows that $PS^{n-1} = C(P^{n-1})$ and $PS_*^{n-1} = C(P_{\mathbb{C}}^{n-1})$, where P^{n-1} and $P_{\mathbb{C}}^{n-1}$ are the real and complex projective space respectively. Moreover, PS_+^{n-1} is a noncommutative C^* algebra.

10.1.1 Quantum Isometry Groups of Free and Half-Liberated Quantum Spheres

The aim of this subsection is to discuss the realization of $A_o(n)$ and $A_o^*(n)$ as quantum isometry groups of a particular type of spectral triples over S_+^{n-1} and S_*^{n-1}. We refer to [5] for more details.

It is well known that $C(S^{n-1})$ has a unique $O(n)$-invariant trace given by the integration with respect to the Haar measure on $O(n)$, since the homogeneous space $C(S^{n-1})$ is isomorphic with the $*$-subalgebra of $C(O(n))$ generated by $\{u_{11}, u_{12}, \cdots u_{1n}\}$. A similar result holds for A_n^+ and A_n^*, the proofs of which use the orthogonal Weingarten formula.

Proposition 10.1.4 ([5]) *Both A_n^+ and A_n^* admit unique α-invariant tracial state.*

We will denote this trace by tr in both the cases.

Remark 10.1.5 It is not known whether tr is faithful.

Notation: From now on, we will use the symbol $A_n^\times$ to denote the image of A_n, A_n^+, and A_n^* on the GNS spaces with respect to tr simultaneously. Similarly, the symbol $O_n^\times$ will stand for any of $C(O(n)), A_o(n)$ and $A_o^*(n)$.

For us, the importance of the state tr stems from the fact that one can construct a spectral triple on $A_n^\times$ represented on the Hilbert space $L^2(A_n^\times, \mathrm{tr})$. In fact, the construction of the spectral triple makes sense for a larger class of algebras.

Definition 10.1.6 A spherical algebra is a C^*-algebra A, along with a family of generators $x_1, \ldots, x_n$ and a faithful positive trace tr, satisfying the following:

1. $x_1, \ldots, x_n$ are self-adjoint.
2. $x_1^2 + \cdots + x_n^2 = 1$.
3. $\mathrm{tr}(x_i) = 0$, for any i.

As a first observation, each $A_n^\times$ is indeed a spherical algebra in the above sense. We will view the identity 1 as a length 0 word in the generators $x_1, \ldots, x_n$.

Theorem 10.1.7 ([5]) *Associated to any spherical algebra $A = < x_1, \ldots, x_n >$ is the spectral triple $(\mathcal{A}, H, D)$, where the dense subalgebra $\mathcal{A}$ is chosen to be the linear span of all the finite words in the generators x_i, and the operator D acting on $H = L^2(A, \mathrm{tr})$ is defined as follows:*

Let $H_k = \mathrm{Span}(x_{i_1} \ldots x_{i_r} | i_1, \ldots, i_r \in \{1, \ldots, n\}, r \leq k)$ and $E_k = H_k \cap H_{k-1}^\perp$, so that $H = \oplus_{k=0}^\infty E_k$. Then we set $Dx = kx$, for any $x \in E_k$.

Proof We have to show that $[D, T_i]$ is bounded, where T_i is the left multiplication by x_i. Since $x_i \in A$ is self-adjoint, so is the corresponding operator T_i. As $T_i(H_k) \subseteq H_{k+1}$, by self-adjointness we get $T_i(H_k^{\perp}) \subset H_{k-1}^{\perp}$. Thus we have:

$$T_i(E_k) \subseteq E_{k-1} \oplus E_k \oplus E_{k+1}.$$

This gives a decomposition $T_i = T_i^{(-1)} + T_i^{(0)} + T_i^{(1)}$, where

$$T_i^{(-1)} = \oplus_k P_{E_{k-1}} T_i P_{E_k},\ T_i^{(0)} = \oplus_k P_{E_k} T_i P_{E_k},\ T_i^{(1)} = \oplus_k P_{E_{k+1}} T_i P_{E_k}$$

and P_{E_k} is the orthogonal projection onto E_k. It is routine to check that $[D, T_i^{(\alpha)}] = \alpha T_i^{(\alpha)}$ for any $\alpha \in \{-1, 0, 1\}$, and this completes the proof. □

The main result of this subsection is the following:

Theorem 10.1.8 ([5]) *Let A be a spherical algebra and consider the associated spectral triple $(\mathcal{A}, H, D)$. Then* $\mathrm{QISO}^+(\mathcal{A}, H, D)$ *exists. Moreover,* $\mathrm{QISO}^+(S_{\times}^{n-1}) = O_n^{\times}$.

Proof The first statement follows by an application of Theorem 3.4.2. So we prove the second statement only. Consider the standard coaction $\alpha : A_n^{\times} \to A_n^{\times} \otimes O_n^{\times}$. This extends to a unitary representation on the GNS space $H_n^{\times}$, which we denote by U.

We have $\alpha(H_k) \subseteq H_k \otimes C(O_n^{\times})$ so that $U(H_k) \subseteq H_k \otimes O_n^{\times}$. By the unitarity of U, we see that $H_k^{\perp}$ is also kept invariant by U. In particular, each E_k is U-invariant and thus U commutes with the Dirac operator D. Therefore, the coaction α is isometric with respect to D, and $O_n^{\times}$ must be a quantum subgroup of $\mathrm{QISO}^+(A_n^{\times})$.

Assume now that $\mathcal{Q}$ is compact quantum group with a unitary corepresentation V on H commuting with D, such that ad_V leaves $(A_n^{\times})''$ invariant. Since D has an eigenspace spanned by $x_1, \ldots, x_n$, both V and V^* must preserve this subspace, so we can find self-adjoint elements $b_{ij} \in \mathcal{Q}$ such that:

$$\mathrm{ad}_V(x_i) = \sum_j x_j \otimes b_{ij}.$$

Now, we note that $\mathrm{tr}(x_i^2) = \frac{1}{n}$ for all $i = 1, 2, \ldots n$, which follows from the observation that for each i, j, there exists an automorphism of $S_n^{\times}$ taking x_i to x_j and leaves the rest of the x_k's unchanged. Then, $\{y_i = \sqrt{n} x_i\}_i$ is an orthonormal set in $L^2(S_n^{\times}, \mathrm{tr})$. As $x_i = x_i^*$, we have $\mathrm{ad}_V(y_i) = \sum_j y_j \otimes b_{ij} = \sum_j y_j \otimes b_{ij}^*$. This implies the unitarity of both the matrices $((b_{ij}))$ and $((b_{ij}^*))$, hence of $((b_{ji}))$ too. It follows in particular that the antipode κ of $\mathcal{Q}$ must send b_{ij} to b_{ji}. Moreover, using the defining relations satisfied by the x_i's and the fact that ad_V and $(\mathrm{id} \otimes \kappa) \circ \mathrm{ad}_V$ are $*$-homomorphism, we can prove that the b_{ij}'s will satisfy the same relations as those of the generators u_{ij} of $C(O_n^{\times})$. Indeed, for the free case there is nothing to prove. The classical case follows from Theorem 4.1.2 and the same proof goes through almost verbatim for the half-liberated case too, replacing the words $x_i x_j$ of length two by the length-3 words $x_ix_jx_k$. This shows that $\mathcal{Q}$ is a quotient of $C(O_n^\times)$, so $\mathcal{Q}$ is a quantum subgroup of $O_n^\times$, and we are done. □

Remark 10.1.9 There is a generalization by Banica of Theorems 10.1.7 and 10.1.8 in the context of complex quantum spheres, where the quantum isometry group gets replaced by suitable quantum subgroups of $A_u(n)$. For the details, we refer to [6].

10.1.2 Quantum Group of Affine Isometries a la Banica

In this section, we discuss the recent series of works by Banica in [6, 7], focusing our attention on the theory of quantum group of affine isometries developed by Banica. For more details, we refer to the review article [10] and the references therein.

Let us fix the following notation: for a C^* algebraic quotient S of S_+^{n-1}, we will denote the quotient map from S_+^{n-1} to S by π_S. Similarly, for a quantum subgroup $\mathcal{Q}$ of $A_o(n)$, the quotient map from $A_o(n)$ to $\mathcal{Q}$ will be denoted by $\pi_{\mathcal{Q}}$.

Definition 10.1.10 Consider the categories $\mathbf{C}^{\mathrm{aff}}(S)$ ($\mathbf{C}^{\mathrm{pro}}(PS)$) of pairs $(\mathcal{Q}, \alpha)$ where $\mathcal{Q}$ is a quantum subgroup of $A_o(n)$ and $\alpha : S \to S \otimes \mathcal{Q}$ (respectively $\alpha : PS \to PS \otimes \mathcal{Q}$) is a unital C^*-homomorphism satisfying

$$\alpha(\pi_S(x_i)) = \sum_j \pi_S(x_j) \otimes \pi_{\mathcal{Q}}(u_{ij}) \;\; (\alpha(\pi_S(x_ix_j)) = \sum_{a,b} \pi_S(x_ax_b) \otimes \pi_{\mathcal{Q}}(u_{ia}u_{jb}) \text{ respectively}),$$

where x_i and u_{ij} are the canonical generators of S_+^{n-1} and $A_o(n)$ respectively. The morphisms of the above categories are CQG morphisms intertwining the $*$-homomorphisms.

An object $(\mathcal{Q}, \alpha)$ of $\mathbf{C}^{\mathrm{aff}}(S)$(respectively $\mathbf{C}^{\mathrm{pro}}(PS)$) will be said to coact in an affine (respectively projective) isometric manner on S (respectively PS).

Remark 10.1.11 The C^* homomorphism α of an object of $\mathbf{C}^{\mathrm{aff}}(S)$ or $\mathbf{C}^{\mathrm{pro}}(PS)$ is actually a coaction of $\mathcal{Q}$. Moreover, the largest Woronowicz C^*-subalgebra of $\mathcal{Q}$ ($\in \mathbf{C}^{\mathrm{pro}}(PS)$) which coacts faithfully on PS is the C^*-subalgebra of $\mathcal{Q}$ generated by $\{\pi_{\mathcal{Q}}(u_{ia}u_{jb}) : i, j, a, b \in \{1, 2, \ldots n\}\}$.

Theorem 10.1.12 ([11]) *If $S = S_+^{n-1}/I_S$, where I_S is a two sided C^*-ideal generated by relations which are polynomials in $x_1, x_2, \cdots x_n$, then the categories $\mathbf{C}^{\mathrm{aff}}(S)$ and $\mathbf{C}^{\mathrm{pro}}(PS)$ have universal objects.*

Moreover, when S is one of the C^*-algebras S_+^{n-1}, S_*^{n-1}, $C(S^{n-1})$, $\overline{S}^{n-1}$ and $\overline{S}_*^{n-1}$ respectively, the universal objects in $\mathbf{C}^{\mathrm{aff}}(S)$ coincide with $A_o(n)$, $A_o^*(n)$, $C(O(n))$, $\overline{A}_o(n)$ and $\overline{A}_o^*(n)$ respectively, while the universal objects of $\mathbf{C}^{\mathrm{pro}}(PS)$ are $PA_o(n)$, $C(PU(n))$, and $C(PO(n))$ (respectively). Here, $PU(n)$ and $PO(n)$ denote the projective unitary and projective orthogonal groups respectively.

It is natural to expect that the free unitary quantum group should also have interesting quantum subgroups and homogeneous spaces, the latter to be viewed as complex versions of the spheres mentioned above. Indeed, Banica developed a theory for such complex quantum spheres, their half-liberated and projective versions and identified their quantum groups of affine isometries as suitable quantum subgroups of $A_u(n)$. We refer to the papers [6, 11–13] for the details of this theory.

10.2 Easy Quantum Groups as Quantum Isometry Groups

One of the new areas of research in the theory of compact quantum groups that has attracted a lot of attention is the theory of easy quantum groups initiated by Banica and Speicher in [4]. These quantum groups are quantum subgroups of $A_o(n)$ which have the quantum permutation group $A_s(n)$ as a quantum subgroup and can be described by certain noncrossing partitions. For the classification program of orthogonal easy quantum groups, we refer to [4, 14–17], etc. After the realization of the easy quantum groups $A_o(n)$ and $A_o^*(n)$ as quantum isometry groups in [5], it was a natural question to ask whether all easy quantum groups can be viewed as quantum isometry groups. As a part of the classification program of orthogonal easy quantum groups, Raum and Weber showed in [18] that there are uncountably many mutually nonisomorphic easy quantum groups (in particular, those of simplifiable hyperoctahedral type) admitting coaction by quantum isometries on the full group C^* algebra of certain quotient groups of $\mathbf{Z}_2^{*n}$. We do not intend to go into the details of the simplifiable hyperoctahedral easy quantum groups. Nevertheless, let us describe very briefly the result of [18] regarding coactions by quantum isometries.

Let E be the subgroup of $\mathbf{Z}_2^{*\infty}$ consisting of all words of even length. There exists a certain subsemigroup of $\mathrm{End}(\mathbf{Z}_2^{*\infty})$ denoted by S_0 (Definition 3.5 of [18]) such that E is the unique maximal proper S_0-invariant subgroup of $\mathbf{Z}_2^{*\infty}$. The set of all S_0-invariant subgroups of E is isomorphic to a category of partitions, which were called simplifiable hyperoctahedral.

On the other hand, a simplifiable hyperoctahedral easy quantum group is an easy quantum group such that the entries u_{ij} of the fundamental corepresentation $u := ((u_{ij}))$ satisfies $u_{ij}^2 u_{kl} = u_{kl} u_{ij}^2$ for all $i, j, k, l = 1, 2, \ldots n$ and possibly some more relations. In particular, the quantum subgroup of $A_o(n)$ defined by the ideal generated by the relation $u_{ij}^2 u_{kl} = u_{kl} u_{ij}^2$ is a simplifiable hyperoctahedral quantum group denoted by $H_n^{[\infty]}$ and any other such quantum group $\mathcal{Q}$ is a quantum subgroup of $H_n^{[\infty]}$ defined by a suitable Woronowicz C^*-ideals $I_{\mathcal{Q}}$. It turns out that the ideals $I_{\mathcal{Q}}$ are in one-to-one correspondence with the hyperoctahedral category of partitions. Summarizing the above discussion, the set of all Woronowicz C^*-ideal $I_{\mathcal{Q}}$ defining the simplifiable hyperoctahedral quantum groups are in one-to-one correspondence with S_0 invariant subgroups of E.

The connection with quantum isometry groups is given by the following result:

Theorem 10.2.1 ([18]) *Let $\mathcal{Q}$ be a simplifiable hyperoctahedral easy quantum group and H be the associated S_0-invariant subgroup of E. Then the universal object in the category of compact quantum groups which are quantum subgroups of $H_n^{[\infty]}$ and coact isometrically on $C^*_{\max}(\mathbf{Z}_2^{*n}/(H)_n)$ is $\mathcal{Q}$.*

10.3 Quantum Symmetry Groups of Orthogonal Filtrations

From the proof of existence of the quantum isometry groups $\mathrm{QISO}^{\mathcal{L}}$ and QISO^{+} in Chap. 3, it is clear that the crucial ingredient for the proof is the existence of a family of finite dimensional mutually orthogonal subspaces of some Hilbert space $\mathcal{H}$ whose union is a dense subspace of $\mathcal{H}$. In the case of $\mathrm{QISO}^{\mathcal{L}}$, these are the eigenspaces of the Laplacian while for the quantum group of orientation-preserving isometries, these are the eigenspaces of the Dirac operator. Most importantly, each of these finite dimensional subspaces is kept invariant by the coaction (in the case of $\mathrm{QISO}^{\mathcal{L}}$) or the corepresentation (in the case of $\widetilde{\mathrm{QISO}^{+}}$) of the universal object in question. This naturally leads to the question of existence of quantum symmetry groups for infinite dimensional noncommutative manifolds in a broader framework. This was developed by Banica and Skalski in [19], the key result of which is the following:

Theorem 10.3.1 ([19]) *Let A be a unital C^*-algebra equipped with a faithful state ω. Moreover, suppose that there exists a family $\{V_i\}_{i\in I}$ of finite dimensional subspaces of A such that the index set I contains a distinguished element 0 satisfying the following conditions:*

1. $V_0 = \mathbb{C}1_A$,
2. *for all $i, j \in I,\ i \neq j$, and for all a in $V_i,\ b$ in V_j, we have $\omega(a^*b) = 0$.*
3. $\mathrm{span}(\cup_{i\in I} V_i)$ *is a dense $*$-subalgebra of A.*

Consider the category $\mathbf{C}_{(A,(V_i)_{i\in I})}$ with objects $(\mathcal{Q}, \alpha)$, where $\mathcal{Q}$ is a compact quantum group with a coaction $\alpha : A \to A \otimes \mathcal{Q}$ such that $\alpha(V_i) \subseteq V_i \otimes \mathcal{Q}_0$ for all i, where $\mathcal{Q}_0$ is the canonical dense Hopf $*$-subalgebra of $\mathcal{Q}$. The morphisms are CQG morphisms intertwining the coaction.

Then $\mathbf{C}_{(A,(V_i)_{i\in I})}$ admits a universal object called the quantum symmetry group of $(A, \omega, (V_i)_{i\in I})$ whose coaction on A is faithful.

When the index set I is countable, Banica and Skalski proved that there exists a spectral triple $(\cup_{i\in I} V_i, L^2(A, \omega), D)$ on $\cup_i V_i$. Here, $D = \sum_{n=0}^{\infty} nP_n$ where P_n denotes the projection onto the finite dimensional subspace V_n of $L^2(A, \omega)$. Then the connection with the quantum isometry group of this spectral triple is given by the following result.

Theorem 10.3.2 ([19]) *The quantum symmetry group of $(A, \omega, (V_i)_{i\in I})$ is isomorphic to the quantum isometry group* $\mathrm{QISO}^{+}(\cup_{i\in I} V_i, L^2(A, \omega), D)$.

Remark 10.3.3 Theorem 10.3.2 in particular says that QISO^{+} exists. In fact, the setup of Theorem 10.3.1 forces the coaction of any object in $\mathbf{C}_{(A,(V_i)_{i\in I})}$ to preserve the state ω on A. We refer Sect. 2 of to [19] for the details. Second, it should also be pointed out that in the setup of orthogonal filtrations of Banica-Skalski, the nuance of the quantum isometry group not having a C^*-coaction (Theorem 4.5.15 of Chap. 4) does not arise.

Examples of C^* algebras equipped with an orthogonal filtration include finite dimensional examples as well as the spectral triples of [20] considered in Chap. 5. Further examples were studied in [21], where Skalski and Soltan showed that under some assumptions, the inductive limit of C^* algebras equipped with orthogonal filtrations is the projective limit of the quantum symmetry groups of the constituent C^* algebras. As an example, they considered the sequence $\{C^*(\mathbf{Z}/p^n\mathbf{Z})\}_{n=1}^{\infty}$. Each of the C^* algebras $C^*(\mathbf{Z}/p^n\mathbf{Z})$ admits a filtration induced by word length. Then Skalski and Soltan showed that the quantum symmetry group of the resulting inductive system is $C_0(\mathbf{Z}_p \rtimes \mathbf{Z}_2)$, where $\mathbf{Z}_p$ denotes the group of p-adic integers and the action of $\mathbf{Z}_2$ on $\mathbf{Z}_p$ is given by $g \mapsto g^{-1}$.

In the same paper, Skalski and Soltan considered the inductive limit of $\{C^*(S_n)\}_{n\geq 1}$, where S_n denotes the permutation group on n distinct objects. However, this inductive limit does not satisfy the conditions of their theorem and the authors of [21] proved that the inductive limits of quantum symmetry groups indexed by even and odd integers are nonisomorphic.

de Chanvalon [22] extended the theory of Banica and Skalski to the context of Hilbert modules equipped with orthogonal filtration. She proved the existence of the universal object in a certain category of compact quantum groups coacting on such a Hilbert module in a filtration preserving way. This universal object is called the quantum symmetry group of the orthogonal filtration for the precise definition of the category mentioned above and the assumptions involved, we refer to [22]. One of the novelties of this work is the development of a setup which simultaneously generalizes the theories of [19, 23]. Some of the interesting examples where Chanvalon's theory works include a Hilbert module associated with the classical space $[0, 1] \times \{1, 2, \cdots d\}$, whose quantum symmetry group turns out to be the hyperoctahedral quantum group. Moreover, Chanvalon showed the existence of the universal object in certain interesting subcategories of the category mentioned above. These universal objects can be viewed as the quantum symmetry groups of some quotients of the space $[0, 1] \times \{1, 2, \ldots d\}$.

10.4 Equivariant Spectral Triples on the Drinfeld–Jimbo q-Deformation of Compact Lie Groups and Their Homogeneous Spaces

In this section, we briefly describe a general construction of equivariant spectral triples, due to Neshyvev and Tuset [24] on the compact quantum groups G_q and the associated (quantum) homogeneous spaces obtained by Drinfeld–Jimbo q-deformation of simple compact Lie groups. We also discuss some partial results and open problems about the corresponding quantum isometry groups. We mostly follow the notation and definitions of [24]. We refer the reader to [25, 26] for the basics of Lie algebras and Lie groups.

Let G be a simply connected simple compact Lie group with the complexified Lie algebra $\mathcal{G}$. Let $\mathcal{U}(\mathcal{G})$ denote the universal enveloping Hopf algebra of $\mathcal{G}$. Let $W^*(G)$ be the group von Neumann algebra of G generated by the right regular representation of the Lie group G in the GNS space $L^2(G)$ of its Haar state. It is in fact the von Neumann algebra generated by the dual discrete quantum group $\widehat{G}$ in $\mathcal{B}(L^2(G))$. The elements of $\mathcal{U}(\mathcal{G})$ can be viewed as unbounded operators affiliated to the von Neumann algebra $W^*(G)$. Let $\mathcal{C}(\mathcal{G})$ be the complexified Clifford algebra of $\mathcal{G}$, with the natural embedding $\gamma : \mathcal{G} \to \mathcal{C}(\mathcal{G})$. The adjoint action of G on $\mathcal{G}$ extends to an action of G on $\mathcal{C}(\mathcal{G})$, denoted by $\widetilde{\mathrm{ad}}$. This induces a $*$-homomorphism from the group von Neumann algebra $W^*(G)$ to $\mathcal{C}(\mathcal{G})$, which we continue to denote by $\widetilde{\mathrm{ad}}$. Let us also denote by the same symbol the corresponding homomorphism at the Lie algebra level as well as its lift to a homomorphism from $\mathcal{U}(\mathcal{G})$ to $\mathcal{C}(\mathcal{G})$. Let us choose a basis $\{x_1, \dots, x_n\}$ which is orthonormal w.r.t. the Killing form on $\mathcal{G}$. Consider the canonical $*$-structure on $\mathcal{C}(\mathcal{G})$ requiring $\gamma(x)$ to be self-adjoint for each $x \in \mathcal{G}_{\mathbb{R}}$. Let $\mathcal{D} = \sum_{k=1}^{n} \left(x_k \otimes \gamma(x_k) + \frac{1}{2} \otimes \gamma(x_k)\widetilde{\mathrm{ad}}(x_k)\right) \in \mathcal{U}(\mathcal{G}) \otimes_{\mathrm{alg}} \mathcal{C}(\mathcal{G})$. Fix a spinor module, i.e., an irreducible $*$-representation $s : \mathcal{C}(\mathcal{G}) \to \mathrm{End}(\mathcal{S})$ where $\mathcal{S}$ is a finite dimensional vector space. The corresponding spin Dirac operator D on $\mathcal{H} = L^2(G) \otimes \mathcal{S}$ is given by

$$D = (\partial \otimes s)(\mathcal{D}),$$

where ∂ denotes the homomorphism from $\mathcal{G}$ into the algebra of first-order differential operators on $L^2(G)$ induced by the right regular representation of G. Thus, D is a first-order differential operator on $L^2(G) \otimes \mathcal{S}$. There are two natural G-actions on $\mathcal{H}$, given by $L_g \otimes \mathrm{id}$ and $R_g \otimes v_g$, where L_g, R_g denote the left and right regular representations of G (respectively) and v_g denotes the G-action on the spinor module $\mathcal{S}$, coming from the adjoint action. The Dirac operator D is equivariant w.r.t. both these G-actions.

We now come to the quantum case. Fix $q \in (0, 1)$. Deforming the Serre relations for semisimple Lie algebras, Drinfeld and Jimbo defined the Hopf algebra $\mathcal{U}_q(\mathcal{G})$ called the Drinfeld–Jimbo quantized universal enveloping algebras. We do not explicitly describe the defining relations here, but refer the reader to [27–31]. It is shown by Rosso [32] that there is a compact quantum group G_q in the sense of Woronowicz such that the corresponding canonical dense Hopf algebra, say $(G_q)_0$, is the dual of the Hopf algebra $\mathcal{U}_q(\mathcal{G})$. Let $L^2(G_q)$ denote the GNS Hilbert space of the Haar state on G_q, with the GNS representation $\pi_q : G_q \to \mathcal{B}(L^2(G_q))$ and the cyclic vctor ξ_q. Let U be the right regular corepresentation given by $U(\pi_q(a)\xi_q) = (\pi_q \otimes \mathrm{id})(\Delta_q(a))\xi_q$, where Δ_q denotes the coproduct of G_q. As discussed in Chap. 1, the dual discrete quantum group $\widehat{G_q}$ has a faithful $*$-representation Π_U on $L^2(G_q)$ and let $W^*(G_q)$ denote the von Neumann algebra generated by its image in $\mathcal{B}(L^2(G_q))$. As noted in [24], there is a faithful representation of $\mathcal{U}_q(\mathcal{G})$ into an algebra of operators affiliated to $W^*(G_q)$. It is known that $W^*(G)$ and $W^*(G_q)$ are isomorphic for every $q \in (0, 1)$. Moreover, a special choice of $*$-isomorphism ϕ_q from $W^*(G_q)$ to $W^*(G)$ can be made, as in Theorem 1.1 of [24], along with a unitary element $\mathcal{F}_q$ (called the Drinfeld twist) of $W^*(G)\overline{\otimes}W^*(G)$, which satisfies the conditions of that theorem.

Let us fix such a Drinfeld twist. The isomorphism ϕ_q naturally lifts to an isomorphism between $\mathcal{U}_q(\mathcal{G})$ and $\mathcal{U}(\mathcal{G})$ as well, to be denoted by the same symbol again.

Let

$$\mathcal{D}_q = (\phi_q^{-1} \otimes \mathrm{id}) \left((\mathrm{id} \otimes \widetilde{\mathrm{ad}})(\mathcal{F}_q)\mathcal{D}(\mathrm{id} \otimes \widetilde{\mathrm{ad}})(\mathcal{F}_q^*)\right) \in \mathcal{U}_q(\mathcal{G}) \otimes_{\mathrm{alg}} \mathcal{C}(\mathcal{G}).$$

We have the following analogue ∂_q of the representation ∂. For $\omega \in \mathcal{U}_q(\mathcal{G})$, we define $\partial_q(\omega)$ to be the linear operator defined on the dense subspace spanned by elements of the form $\pi_q(a)\xi_q$, with a in the canonical dense Hopf algebra $(G_q)_0$ of G_q, as follows:

$$\partial_q(\omega)(\pi_q(a)\xi_q) = \pi_q(a_{(1)}\xi_q) \otimes \omega(a_{(2)}),$$

using the Sweedler notation and the duality between $\mathcal{U}_q(\mathcal{G})$ and $(G_q)_0$ mentioned before. We can define ∂_q on elements of $W^*(G_q)$ by the same formula and verify that it is a $*$-homomorphism. Let $\widetilde{\mathrm{ad}}_q := \widetilde{\mathrm{ad}} \circ \phi_q : W^*(G_q) \to \mathcal{C}(\mathcal{G})$. Then, $(\partial_q \otimes s \circ \widetilde{\mathrm{ad}}_q)\widehat{\Delta}_q$ is a $*$-homomorphism from $W^*(G_q)$ to $\mathcal{B}(L^2(\mathcal{H}_q))$, where $\widehat{\Delta_q}$ denotes the coproduct of the discrete quantum group $\widehat{G_q}$, extended naturally to $W^*(G_q)$. There is a unitary corepresentation, say U_r, of G_q corresponding to this $*$-homomorphism. We have another unitary corepresentation U_l on $\mathcal{H}_q$ given by $U_l(\pi_q(a)\xi_q \otimes \eta) = \pi_q(a_{(2)})\xi_q \otimes \eta \otimes \kappa^{-1}(a_{(1)})$, where $a \in (G_q)_0$, $\eta \in \mathcal{S}$ and κ denotes the antipode.

Now we define $D_q = (\partial_q \otimes s)(\mathcal{D}_q)$ to be the q-deformed Dirac operator. Let us denote by $\mathcal{A}$ the $*$ subalgebra generated by $\pi_q(a) \otimes 1_{\mathcal{S}}$ on $\mathcal{H}_q := L^2(G_q) \otimes \mathcal{S}$. then Neshveyev and Tuset proved that

Theorem 10.4.1 ([24]) *$(\mathcal{A}, \mathcal{H}_q, D_q)$ is a spectral triple of compact type. and it is also equivariant with respect to the copresentations U_l and U_r of G_q on $\mathcal{H}_q$.*

This motivates one to ask the following:

Question I: *Does* $\mathrm{QISO}^+(D_q)$ *exist?*

Question II: *If it exists, is it a q-deformation of the group of orientation-preserving isometries for the classical counterpart, i.e., q-deformation of* $C(\mathrm{ISO}^+(D))$?

Moreover, it is interesting to consider appropriate R-twisted volume forms on the above spectral triples, and then ask Question II for $\mathrm{QISO}_R^+(D_q)$.

The reader may recall at this point the equivariant spectral triple constructed by Chakraborty and Pal on $SU_q(2)$ [33] which was discussed in Chap. 2. However, it is different from the spectral triple on $SU_q(2)$ constructed in [34]. The construction in [34] is actually a particular case of the general prescription in [24]. Unlike the spectral triple of [34], the Dirac operator for the Chakraborty–Pal example does not admit a "classical limit" as $q \mapsto 1^-$, Nevertheless, in [35], we have computed the quantum isometry group of this spectral triple and observed the following:

Theorem 10.4.2 ([35], Theorem 4.13) *In the notation of Example 2.2.8 and Sect. 3.4, we have for all* $0 < \mu < 1$,

$$\mathrm{QISO}^{+}(D^{SU_\mu(2)}) \cong U_\mu(2),$$

where the coaction Γ *of* $U_\mu(2)$ *on* $SU_\mu(2)$ *is given by*

$$\Gamma(\alpha) = \alpha \otimes u_{11} + \gamma^* \otimes \mu u_{21}, \quad \Gamma(\gamma^*) = \alpha \otimes \mu^{-1} u_{12} + \gamma^* + u_{22}.$$

Towards the end of [24], Neshveyev and Tuset discuss construction of G_q-equivariant spectral triples on the quantum homogeneous spaces G_q/K_q, for suitable Lie subgroup K of G (see [36] and [37] for another construction). In fact, such spectral triples are constructed by restricting the Dirac operator D_q discussed before to a suitable subspace of $\mathcal{H}_q$, which is K_q-invariant in an appropriate sense. The $SU_q(2)$-equivariant spectral triples on the Podles' spheres $S^2_{\mu,0}$ considered in Chap. 2 are closely related to such construction. One can ask analogues of Questions I and II for the spectral triples on quantum homogeneous spaces as well. In fact, we have given an affirmative answer to Question II for the example of Podles sphere in Chap. 4. In an ongoing joint work with Mandal [38], the second author seems to have obtained an affirmative answer to Question II for a natural class of R and for q in a small enough neighborhood of 1.

10.5 Quantum Isometry Groups for Metric Spaces

It is a natural question to ask whether one can formulate a notion of quantum isometric coactions in a purely metric space setup. Indeed, for finite metric spaces, we already discussed Banica's approach in Chap. 5 and interpreted the corresponding universal quantum groups as the quantum isometry groups of suitable spectral triples or Laplacians. We now briefly discuss the more general case. Let (X, d) be a compact metric space and G be a compact group acting continuously on X. The G-action is isometric with respect to d if and only if $d(x, gy) = d(y, g^{-1}x)(= d(g^{-1}x, y))$ for all $x, y \in X$, which can be re-written as

$$(\mathrm{id}_{\mathcal{C}} \otimes \beta)(d) = \sigma_{23} \circ ((\mathrm{id}_{\mathcal{C}} \otimes \kappa) \circ \beta \otimes \mathrm{id}_{\mathcal{C}})(d),$$

where $\beta : C(X) \to C(X) \otimes C(G)$ is the coaction map given by $\beta(f)(x, g) = f(gx)$, σ_{23} denotes the flip of the second and third tensor copies and κ denotes the (bounded) antipode on $C(G)$. In order to generalize this to the case of CQG coactions, we need a well-defined bounded antipode map. However, it is a well-known fact (see Theorem 3.23, [39]) that any CQG $\mathcal{S}$ coacting faithfully on $C(X)$, where X is a compact Hausdorff space, must be of Kac type, i.e., the Haar state is tracial and the antipode κ admits a norm-bounded extension on $\mathcal{S}_r$ satisfying $\kappa^2 = \mathrm{id}$. This allows us to define the following notion of quantum isometry in the metric space setup.

Definition 10.5.1 *(*[40]*)* Let (X, d) be a compact metric space. Given a coaction β of a CQG $\mathcal{S}$ on $\mathcal{C} = C(X)$, we say that β is "isometric in the metric space sense" if the corresponding reduced coaction $\beta_r := (\mathrm{id} \otimes \pi_r) \circ \beta$ of $\mathcal{S}_r$ satisfies the following:

$$(\mathrm{id}_{\mathcal{C}} \otimes \beta_r)(d) = \sigma_{23} \circ ((\mathrm{id}_{\mathcal{C}} \otimes \kappa) \circ \beta_r \otimes \mathrm{id}_{\mathcal{C}})(d), \tag{10.5.1}$$

where as before, σ_{23} denotes the flip of the second and third tensor copies, $d \in C(X \times X)$ denotes the metric and κ denotes the (bounded) antipode.

Let us mention that A. Chirvasitu called the above notion "D-isometry" in [41]. Let us consider the category $\mathbf{Q}^{\mathrm{metric}}(X, d)$ of all CQG's coacting faithfully and isometrically on (X, d) in the above sense, with the obvious morphisms. Clearly, if X is a finite set, the above definition does coincide with Banica's definition discussed earlier and we get a universal object in the category $\mathbf{Q}^{\mathrm{metric}}(X, d)$. The general case is open, but in [40] an existence theorem has been obtained which covers a very large class of interesting metric spaces, including all those which are embeddable (as metric spaces) in some Euclidean space. We state this existence result below, and refer the reader to [40] for a proof and other discussions.

Theorem 10.5.2 ([40]) *Let (X, d) be a compact metric space. Suppose also that there are topological embedding $f : X \to \mathbb{R}^n$ and a homeomorphism ψ of $\mathbb{R}^+$ such that $(\psi \circ d)(x, y) = d_0(f(x), f(y))$ for all $x, y \in X$, where we have denoted the Euclidean metric of $\mathbb{R}^n$ by d_0. Then there is a universal object in $\mathbf{Q}^{\mathrm{metric}}(X, d)$.*

We denote the above universal object by $\mathrm{QISO}^{\mathrm{metric}}(X, d)$.

Let X be a compact subset of $\mathbb{R}^n$, with the Euclidean coordinates given by $X_1, \ldots, X_n$ and let α be a coaction of some CQG $\mathcal{Q}$ on $C(X)$. Then the coaction is isometric in the metric space sense if and only if

$$\sum_i (\alpha(X_i)(p) - \alpha(X_i)(q))^2 = d_0^2(p, q) 1_{\mathcal{Q}},$$

for all $p, q \in X$, where d_0 denotes the Euclidean distance. Moreover, the coaction α must be affine, i.e., of the form

$$\alpha(X_i) = \sum_j X_j \otimes q_{ji} + 1 \otimes r_i, \tag{10.5.2}$$

where, q_{ij}, r_i are self-adjoint elements such that the matrices $((q_{ij}))_{i,j}$ and $((q_{ji}))_{i,j}$ are unitary elements in $M_n(\mathcal{Q})$.

Remark 10.5.3 It follows from [42] that an arbitrary finite metric space satisfies the condition of Theorem 10.5.2 with $\psi(t) = t^c$ for some $c > 0$. This implies that our existence theorem does extend that of Banica for finite spaces. Examples of metric spaces satisfying the condition of Theorem 10.5.2 also include the spheres S^n for all $n \geq 1$.

If the metric space X in Theorem 10.5.2 has at least 4 components each of which is isometric to some given set, $\mathrm{QISO}^{\mathrm{metric}}(X, d)$ will have $\mathcal{S}_4^+$ as a quantum subgroup, hence genuine. It is more interesting to compute $\mathrm{QISO}^{\mathrm{metric}}(X, d)$ when X is connected and see whether it is a genuine CQG or not. We refer to [43] (see also [44]) for a rich source of such examples. In fact, for a compact connected $X \subseteq \mathbb{R}^n$, $\mathrm{QISO}^{\mathrm{metric}}(X, d)$ coincides with $C(\mathrm{ISO}(X, d))$ whenever X has nonempty interior in $\mathbb{R}^n$. We refer the reader to [40] for explicit computations of $\mathrm{QISO}^{\mathrm{metric}}(X, d)$ for more examples.

For a compact connected Riemannian manifold M one has a natural metric d coming from the geodesic distance and it is interesting to compare the corresponding quantum isometry group $\mathrm{QISO}^{\mathrm{metric}}(M, d)$ (whenever it exists) with the geometric quantum isometry group $\mathrm{QISO}^{\mathcal{L}}(M)$ defined in Chap. 3 in terms of the Hodge Laplacian. We make the following:

Conjecture $\mathrm{QISO}^{\mathrm{metric}}(M, d)$ *exists and equals* $\mathrm{QISO}^{\mathcal{L}}(M) = C(\mathrm{ISO}(M))$.

A.L. Chirvasitu has recently proved this fact for negatively curved M [45].

There is also an attempt by Sabbe and Quaegebeur in [46] to give such a formulation for even more general framework of compact quantum metric spaces a la Rieffel [47]. We do not go into the details of this approach here, as we did not define a compact quantum metric space (CQMS) in this book. However, for the metric spaces case, it is based on the following observation : an equivalent condition for a continuous action by a group G on a metric space (X, d) is the inequality $d(gx, gy) \leq d(x, y)$ for all $x, y \in X$ and for all $g \in G$. This follows because we can replace x, y by $g^{-1}x$, $g^{-1}y$ (respectively) to get the reverse inequality. This condition is also equivalent to $|f(gx) - f(gy)| \leq 1$ for all $g \in G$ for all $f \in C(X)$ satisfying $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$. This leads to the following definition of quantum isometry in [46]: call a coaction β of a CQG $\mathcal{S}$ on X to be isometric if $\|\beta(f)(x) - \beta(f)(y)\| \leq 1$ for all x, y whenever $|f(x) - f(y)| \leq d(x, y) \forall x, y \in X$. In [46], the authors took this as the definition of quantum isometry and proved that this is equivalent to Banica's definition for finite metric spaces. It is also easy to see that the definition of quantum isometry given in [40] implies that of [46]. However, the following questions are still open:

Question I: *Do the notions of quantum isometric coactions given by* [40, 46] *agree in general?*

Question II: *If the answer of Question I is negative, does there exist a universal quantum isometry group for the category of CQG's coacting isometrically in the sense of* [46]?

References

1. Vaksman, L.L., Soibelman, Y.S.: Algebra of functions on the quantum group $SU(n+1)$, and odd-dimensional quantum spheres, (Russian), Algebra i Analiz **2**(5), 101–120 (1990); translation in Leningrad Math. J. **2**(5), 1023–1042 (1991)
2. Dijkhuizen, M.S., Noumi, M.: A family of quantum projective spaces and related q-hypergeometric orthogonal polynomials. Trans. Am. Math. Soc. **350**(8), 3269–3296 (1998)
3. Klimyk, A., Schmudgen, K.: Quantum Groups and Their Representations. Springer, New York (1998)
4. Banica, T., Speicher, R.: Liberation of orthogonal Lie groups. Adv. Math. **222**(4), 1461–1501 (2009)
5. Banica, T., Goswami, D.: Quantum isometries and noncommutative spheres. Comm. Math. Phys. **298**, 343–356 (2010)
6. Banica, T.: Liberations and twists of real and complex spheres. J. Geom. Phys. **96**, 1–25 (2015)
7. Banica, T., Meszaros, S.: Uniqueness results for noncommutative spheres and projective spaces. Illinois J. Math. **59**(1), 219–233 (2015)
8. Speicher, R., Weber, M.: Quantum groups with partial commutation relations. arXiv:1603.09192
9. Banica, T.: Liberation theory for noncommutative homogeneous spaces. arXiv:1505.07755
10. Banica, T.: Quantum isometries, noncommutative spheres, and related integrals. arXiv:1601.02159
11. Banica, T.: Quantum isometries of noncommutative polygonal spheres. arXiv:1501.05229
12. Banica, T.: Half liberated manifolds, and their quantum isometries. arXiv:1505.00646
13. Banica, T.: A duality principle for noncommutative cubes and spheres. arXiv:1502.05315
14. Banica, T., Curran, S., Speicher, R.: Classification results for easy quantum groups. Pac. J. Math. **247**(1), 1–26 (2010)
15. Raum, S., Weber, M.: The full classification of orthogonal easy quantum groups. Comm. Math. Phys. **341**(3), 751–779 (2016)
16. Weber, M.: On the classification of easy quantum groups. Adv. Math. **245**, 500–533 (2013)
17. Raum, S.: Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications. Proc. Am. Math. Soc. **140**(9), 3207–3218 (2012)
18. Raum, S., Weber, M.: A Connection between Easy Quantum Groups, Varieties of Groups and Reflection Groups. arXiv:1212.4742
19. Banica, T., Skalski, A.: Quantum Symmetry Groups Of C^*-algebras equipped with orthogonal filtrations. Proc. Lon. Math. Soc. **3**, 106 (2013) (no. 5, 980–1004)
20. Christensen, E., Ivan, C.: Spectral triples for AF C^*-algebras and metrics on the Cantor set. J. Operator Theory **56**(1), 17–46 (2006)
21. Skalski, A., Soltan, P.M.: Projective limits of quantum symmetry groups and the doubling construction for Hopf algebras. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17** (2014), no. 2, 1450012 (27 pages)
22. de Chanvalon, M.T.: Quantum symmetry groups of Hilbert modules equipped with orthogonal filtrations. J. Funct. Anal. **266**(5), 3208–3235 (2014)
23. Goswami, D.: Quantum group of isometries in classical and non commutative geometry. Comm. Math. Phys. **285**(1), 141–160 (2009)
24. Neshveyev, S., Tuset, L.: The Dirac operator on compact quantum groups. J. Reine Angew. Math. **641**, 1–20 (2010)
25. Humphreys, J.E.: Introduction to Lie algebras and representation theory. In: Graduate Texts in Mathematics, no. 9. Springer (1972)
26. Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York (1978)
27. Chari, V., Pressley, A.: A Guide to Quantum Groups. Cambridge University Press (1994)
28. Drinfeld, V.G.: Quantum Groups, Proceedings of the International Congress of Mathematicians. Berkeley (1986)

29. Drinfeld, V.G.: Quasi-Hopf algebras. Leningrad Math. J. **1**, 1419–1457 (1990)
30. Jimbo, M.: Quantum R-matrix for the generalized Toda system. Commun. Math. Phys. **10**, 63–69 (1985)
31. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang-Baxter equation. Lett. Math. Phys. **10**(1), 63–69 (1985)
32. Rosso, M.: Algebres enveloppantes quantiqees, groupes quantiques compacts de matrices et calcul diffe'rentiel non commutatif. Duke Math. J. **61**, 11–40 (1990)
33. Chakraborty, P.S., Pal, A.: Equivariant spectral triples on the quantum $SU(2)$ group. K. Theory **28**, 107–126 (2003)
34. Dabrowski, L., Landi, G., Sitarz, A., van Suijlekom, W., Varilly, Joseph C.: The Dirac operator on $\mathrm{SU}_q(2)$. Comm. Math. Phys. **259**(3), 729–759 (2005)
35. Bhowmick, J., Goswami, D.: Quantum group of orientation preserving Riemannian isometries. J. Funct. Anal. **257**(8), 2530–2572 (2009)
36. Krahmer, U.: Dirac operators on quantum flag manifolds. Lett. Math. Phys. **67**(1), 49–59 (2004)
37. Krahmer, U., Tucker-Simmons, M.: On the Dolbeault-Dirac operator of quantized symmetric spaces. Trans. Lond. Math. Soc. **2**(1), 33–56 (2015)
38. Goswami, D., Mandal, A.: In preparation
39. Huang, H.: Invariant subsets under compact quantum group actions. J. Noncommut. Geom. **10**(2), 447–469 (2016)
40. Goswami, D.: Existence and examples of quantum isometry groups for a class of compact metric spaces. Adv. Math. **280**, 340–359 (2015)
41. Chirvasitu, A.L.: On quantum symmetries of compact metric spaces. J. Geom. Phys. **94**, 141–157 (2015)
42. Deza, M., Maehera, H.: Metric transformations and Euclidean embeddings. Trans. A.M.S. **317**(2), 261–271 (1990)
43. Huang, H.: Faithful compact quantum group actions on connected compact metrizable spaces. J. Geom. Phys. **70**, 232–236 (2013)
44. Etingof, P., Walton, C.: Semisimple Hopf actions on commutative domains. Adv. Math. **251**, 47–61 (2014)
45. Chirvasitu, A.L.: Quantum rigidity of negatively curved manifolds. Comm. Math. Phys. **344**(1), 193–221 (2016)
46. Sabbe, M., Quaegebeur, J.: Isometric coactions of compact quantum groups on compact quantum metric spaces. Proc. Indian Acad. Sci. Math. Sci. **122**(3), 351–373 (2012)
47. Rieffel, M.A.: Compact quantum metric spaces, Operator algebras, quantization, and noncommutative geometry, pp. 315–330, Contemporary Mathematics, vol. 365, American Mathematical Society, Providence, RI (2004)