SPRINGER BRIEFS IN MATHEMATICAL PHYSICS 4

Naofumi Honda
Takahiro Kawai
Yoshitsugu Takei

Virtual Turning Points

Springer

SpringerBriefs in Mathematical Physics

Volume 4

More information about this series at http://www.springer.com/series/11953

Naofumi Honda · Takahiro Kawai
Yoshitsugu Takei

Virtual Turning Points

Springer

Naofumi Honda
Associate Professor
Department of Mathematics
Hokkaido University
Sapporo
Japan

Yoshitsugu Takei
Associate Professor
Research Institute for Mathematical Sciences
Kyoto University
Kyoto
Japan

Takahiro Kawai
Professor Emeritus
Research Institute for Mathematical Sciences
Kyoto University
Kyoto
Japan

ISSN 2197-1757 ISSN 2197-1765 (electronic)
SpringerBriefs in Mathematical Physics
ISBN 978-4-431-55701-2 ISBN 978-4-431-55702-9 (eBook)
DOI 10.1007/978-4-431-55702-9

Library of Congress Control Number: 2015942818

Springer Tokyo Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer Japan KK is part of Springer Science+Business Media (www.springer.com)

Dedicated to Professor H. Komatsu on his Sanju—80 years of age in the traditional Japanese counting

Preface

As M.V. Fedoryuk, a renowned expert in asymptotic analysis in the 1980s, once lamented ([F]), global asymptotic analysis of higher order differential equations was thought to be impossible to construct in his days. At such a time H.L. Berk, W.M. Nevins, and K.V. Roberts published a remarkable paper in J. Math. Phys., **23** (1982), which shows that the traditional Stokes geometry cannot globally describe the Stokes phenomena of WKB solutions of higher order differential equations; a "new" Stokes curve is necessary for the complete description. Later T. Aoki, T. Kawai, and Y. Takei discovered the notion of a virtual turning point by applying microlocal analysis to Borel transformed WKB solutions; a "new" Stokes curve is a Stokes curve emanating from a virtual turning point. An important point is that a virtual turning point is intrinsically defined in the sense that it does not depend on the argument of the large parameter contained in the equation. At the same time, as the qualifier "virtual" indicates, a virtual turning point cannot be detected by a cosmetic study of ordinary WKB solutions; we need the conversion of the study to the one on a different space, the Borel plane on which the Borel transformed WKB solutions are analyzed. This is the reason why a virtual turning point was not found before the advent of the exact WKB analysis, the analysis of Borel transformed WKB solutions.

The aim of the monograph is to explain the core part of this novel and important notion so that it may be appreciated not only by mathematicians but also physicists and engineers and be practically used in concrete problems. To be more concrete, we present in Chap. 2 several concrete figures of Stokes geometry related to some higher order Painlevé equations (the Noumi-Yamada system), and we analyze in Chap. 3 the non-adiabatic transition problems for three-levels (the generalized three-level Landau-Zener model). In both subjects, the reader will be impressed by the importance of the role of virtual turning points in their analysis. We also note that the employment of graph-theoretic notions in Chap. 2 is a natural and reasonable approach in view of the practical way of locating virtual turning points (Sect. 1.6). The results reported in Chaps. 2 and 3 are still in progress; for example, much remains to be done in investigating the role of the total value integral of a tree (Sect. 2.4) in the study of the Noumi-Yamada system, and the more precise study of connection coefficients with the help of the exact steepest descent method (Appendix) has only just been instigated. We hope to come back to these problems in the future, but we feel it important to publish this short monograph now so that virtual turning points may find their proper place in the tool box of a mathematical scientist.

In ending this preface we would like to thank several people for their help without which this could not have been completed in this form. The most important person is Prof. T. Aoki, whose collaboration with two of us (T.K. and Y.T.) is the essential core of this monograph. We also thank sincerely Prof. T. Nishimoto, who kindly called the attention of (some of) Aoki, Kawai, and Takei to the paper [BNR] in a private conversation at the occasion of an RIMS conference; it was really an excellent instruction. Further, we are gratefully indebted to Dr. Shinji Sasaki for having allowed us to include in Sects. 3.2 and 3.3 his unpublished results on the effect of the virtual turning points in calculating the transition probabilities for three-levels. Therewith, we are most obliged to him for having drawn many figures together with several important comments on the draft of this monograph. The heartiest thanks of one of us, T. Kawai, also goes to Ms. K. Kohno for her excellent typing.

Last but not least, we sincerely thank Mr. M. Nakamura of Springer Japan for having invited us to write this monograph.

Kyoto

Naofumi Honda
Takahiro Kawai
Yoshitsugu Takei

Acknowledgments

N. Honda is supported by JSPS KAKENHI Grant Number 15K04887, T. Kawai is supported by JSPS KAKENHI Grant Number 24340026, and Y. Takei is supported by JSPS KAKENHI Grant Number 24340026 and 26287015.

Contents

Chapter 1
Definition and Basic Properties of Virtual Turning Points

As we mentioned in Preface, global asymptotic analysis of higher order differential equations was thought impossible to construct in 1980s among specialists in asymptotic analysis. In hindsight we find it reasonable, because neither new Stokes curves nor virtual turning points were not in their toolboxes. At that time, physicists H.L. Berk et al. observed in [BNR] that the totality of ordinary Stokes curves was insufficient to describe the Stokes phenomena for WKB solutions of higher order equations and that "new" Stokes curves were needed. Unfortunately it seems that the importance of their observation was not properly appreciated by mathematical specialists in asymptotic analysis as the publication date of the survey article [F] indicates. We believe the reason was that they were not familiar with the Borel transformation depending on parameters, whereas such notion is crucially important in formulating the notion of virtual turning points. Thus we begin our discussion by recalling the core part of "WKB analysis based upon the Borel transformation with parameters" of the Schrödinger equation, a typical second order differential equation. Such analysis is usually referred to as the "exact WKB analysis"; here the adjective "exact" is used in contrast to "asymptotic".

1.1 A Brief Survey of the Exact WKB Analysis of the Schrödinger Equation

In this section we show the core part of the exact WKB analysis of the following Eq. (1.1.1), the one-dimensional stationary Schrödinger equation:

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi(x,\eta) = 0, \tag{1.1.1}$$

where the potential $Q(x)$ is a polynomial and η is a large parameter. See [KT2] for the details. To fix the situation we assume $\eta > 0$ at the beginning, but eventually

N. Honda et al., *Virtual Turning Points*, SpringerBriefs in Mathematical Physics,
DOI 10.1007/978-4-431-55702-9_1

η may be allowed to be a complex number. We use a large parameter, not a small parameter like the Planck constant, as it naturally fits in with the framework of the Borel transformation with parameters. WKB analysis of (1.1.1) begins by giving the definition of a WKB solution ψ of (1.1.1); it is a formal solution that has the form

$$\psi(x,\eta) = \exp\left(\int^x S(x,\eta)dx\right), \tag{1.1.2}$$

where $S(x,\eta)$ has the following form

$$S(x,\eta) = \eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \cdots . \tag{1.1.3}$$

It is clear that $S(x,\eta)$ is a solution of the following Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2 Q(x), \tag{1.1.4}$$

and an important point is that $S_j(x)$ $(j \geq 0)$ is uniquely determined in a recursive manner once we fix

$$S_{-1}(x) = \pm\sqrt{Q(x)}. \tag{1.1.5}$$

Furthermore, if we set

$$S_{\mathrm{odd}} = \sum_{l \geq 0} S_{2l-1}(x)\eta^{-2l+1} \tag{1.1.6}$$

and

$$S_{\mathrm{even}} = \sum_{k \geq 0} S_{2k}(x)\eta^{-2k}, \tag{1.1.7}$$

the comparison of the odd degree (in η^{-1}) part of (1.1.4) entails

$$S_{\mathrm{even}} = -\frac{1}{2}\frac{(\partial S_{\mathrm{odd}}/\partial x)}{S_{\mathrm{odd}}} = -\frac{1}{2}\frac{\partial}{\partial x}\log S_{\mathrm{odd}}. \tag{1.1.8}$$

Hence the right-hand side of (1.1.2) may be rewritten as

$$\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\int^x S_{\mathrm{odd}}\,dx\right). \tag{1.1.9}$$

It is also evident by the way of constructing S_j recursively that S_{2l-1} is a sum of terms of the form

$$a_p(x)\big(\sqrt{Q(x)}\big)^{2p-1}, \tag{1.1.10}$$

and hence the sign in (1.1.5) is shared also by S_{2l-1} ($l \geq 1$). Therefore

$$\psi_{\pm}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\pm \int_{x_*}^{x} S_{\mathrm{odd}}\, dx\Big) \tag{1.1.11}$$

are linearly independent solutions of (1.1.1), where x_* is an appropriately fixed reference point. It is also clear $\psi_{\pm}(x,\eta)$ may be rewritten as

$$\exp\left(\pm\eta \int_{x_*}^{x} \sqrt{Q(x)}\, dx\right)\left(\sum_{j=0}^{\infty} \psi_{\pm,j}(x)\eta^{-(j+1/2)}\right) \tag{1.1.12}$$

for some $\psi_{\pm,j}(x)$. WKB solutions expressed in this form are convenient to analyze and often used in our subsequent discussion.

It seems that WKB solutions have not been very popular among mathematicians despite their clean definition, and we imagine that one reason is that WKB solutions are divergent in general. But, as several pioneering works ([BW, V, S] and references cited therein) of physicists and chemists have shown, the divergence of WKB solutions naturally leads to the algebraic analysis of WKB solutions via the Borel transformation; Borel transformed WKB solutions define analytic functions depending on the "parameter" x. To concretize this statement, let us first recall the definition of the Borel transformation and the Borel resummation of WKB solutions given in the form (1.1.12). First, the Borel transform $\psi_{\pm,B}(x,y)$ is, by definition,

$$\sum_{j\geq 0} \frac{\psi_{\pm,j}(x)}{\Gamma(j+1/2)} (y + y_{\pm}(x))^{j-1/2}, \tag{1.1.13}$$

where

$$y_{\pm}(x) = \pm \int_{x_*}^{x} \sqrt{Q(x)}\, dx. \tag{1.1.14}$$

It is known (e.g. [AKT1]) that $\psi_{+,B}(x,y)$ (resp., $\psi_{-,B}(x,y)$) is convergent near $y = -y_{+}(x)$ (resp., $-y_{-}(x)$) when $Q(x) \neq 0$ and defines an analytic function there. Furthermore it has recently been proved [KoS] that $\psi_{+,B}(x,y)$ can be analytically continued to a neighborhood of

$$\big\{y \in \mathbb{C};\ \mathrm{Im}(y + y_{+}(x)) = 0 \quad \text{and} \quad \mathrm{Re}(y + y_{+}(x)) > 0\big\} \tag{1.1.15}$$

and that

$$\int_{-y_+(x)}^{\infty} \exp(-y\eta)\psi_{+,B}(x, y)dy \tag{1.1.16}$$

is well-defined for $\eta \gg 1$, unless $\psi_{+,B}(x, y)$ has singularities on (1.1.15), and a similar result also holds for $\psi_{-,B}$. When the integral (1.1.16) is well-defined, we say it is the Borel sum of $\psi_+(x, \eta)$.

Remark 1.1.1 (i) The integration path of the integral (1.1.16) is a half line parallel to the real axis; this is a counterpart of the fact $\arg \eta = 0$.
(ii) We usually abbreviate the expression "Borel resummation" to "Borel summation".
(iii) Borel resummation is a classical notion in mathematics, but one important novel feature of the Borel sum of a WKB solution is that it depends on the "parameter" x. At the same time one can readily confirm that $\psi_{\pm,B}(x, y)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2} - Q(x)\frac{\partial^2}{\partial y^2}\right)\psi_{\pm,B}(x, y) = 0 \tag{1.1.17}$$

outside $\{Q(x) = 0\}$, and the core of our analysis in this book is the study of the Eq. (1.1.17) and its counterpart for higher order equations, where the roles of variables x and y are symmetric. Parenthetically we note that mathematicians usually use (ξ, η) as the dual variable of (x, y), and this is the reason we use the symbol η to denote the large parameter in (1.1.1).

Let us now recall some terminologies in WKB analysis.

Definition 1.1.1 (i) A zero of $Q(x)$ is called a turning point of the Eq. (1.1.1).
(ii) A Stokes curve emanating from a turning point a is, by definition, the curve defined by

$$\operatorname{Im} \int_a^x \sqrt{Q(x)}\, dx = 0. \tag{1.1.18}$$

In what follows we shed a new light upon these traditional notions from the viewpoint of the exact WKB analysis, and our discussion naturally leads to the notion of virtual turning points for higher order equations. To begin with, we consider the case where the potential $Q(x)$ is x. In this case Eq. (1.1.1) is normally referred to as the **Airy equation**. For the Airy equation we can readily confirm S_j in (1.1.3) has the form

$$S_j = c_j x^{-1-3j/2} \quad (c_j : \text{a constant}), \tag{1.1.19}$$

and, by making use of this expression, we obtain

$$\psi_{+,B}(x,y) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}s^{-1/2}F\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};s\right), \tag{1.1.20}$$

$$\psi_{-,B}(x,y) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{1}{x}(s-1)^{-1/2}F\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};1-s\right), \tag{1.1.21}$$

where

$$s = \frac{3y}{4x^{3/2}} + \frac{1}{2} \tag{1.1.22}$$

and $F(\alpha,\beta,\gamma;s)$ denotes Gauss' hypergeometric function. (See [KT2, Sect. 2.2] for the detailed computation. We note that the origin is chosen as the endpoint in the integral (1.1.11), which is legitimate as a complex contour integral. We will explain the details later in a general context after Theorem 1.1.2.) Thanks to this explicit expression of $\psi_{\pm,B}(x,y)$ we observe the following:

Fact A. $\psi_{\pm,B}(x,y)$ can be analytically continued to all over the y-plane with singularities described in Fact B.

Fact B. $\psi_{+,B}(x,y)$ (resp., $\psi_{-,B}(x,y)$) has its singularity at $s=1$, that is, $y=-y_-(x)$ $(=y_+(x))$ besides its obvious singularity at $s=0$, that is, $y=-y_+(x)$ (resp., at $s=0$ besides its obvious singularity at $s=1$).

Fact C. The growth order of $\psi_{\pm,B}(x,y)$ near $y=\infty$ is tame.

Fact B implies that the Borel summation of the series $\psi_+(x,\eta)$ in η^{-1}, which is given by the integral (1.1.16), may break down when the "parameter" x satisfies

$$\mathrm{Im}\,\frac{2}{3}x^{3/2} = 0. \tag{1.1.23}$$

We note that, as we are studying the case where $Q=x$, (1.1.23) is nothing but

$$\mathrm{Im}\int_0^x \sqrt{Q}\,dx = 0, \tag{1.1.24}$$

that is, the defining equation of a Stokes curve emanating from a turning point $\{x=0\}$. This is our interpretation of a Stokes curve; a curve along which the Borel summation may break down.

Next we let the "parameter" x move so that we may see how the Borel sum $\psi_+(x,\eta)$ changes when x crosses the Stokes curve. Thus we encounter the configurations of Fig. 1.1 that describes the relative location of the path of integration in (1.1.16) and the singularities of its integrand. In Fig. 1.1 we set $x = 1+i\varepsilon$ ($|\varepsilon| \ll 1$); as we change ε from a negative value to a positive one, the singular point $\{y = 2x^{3/2}/3\}$ of the integrand hits the path of integration at $\varepsilon=0$. Hence to

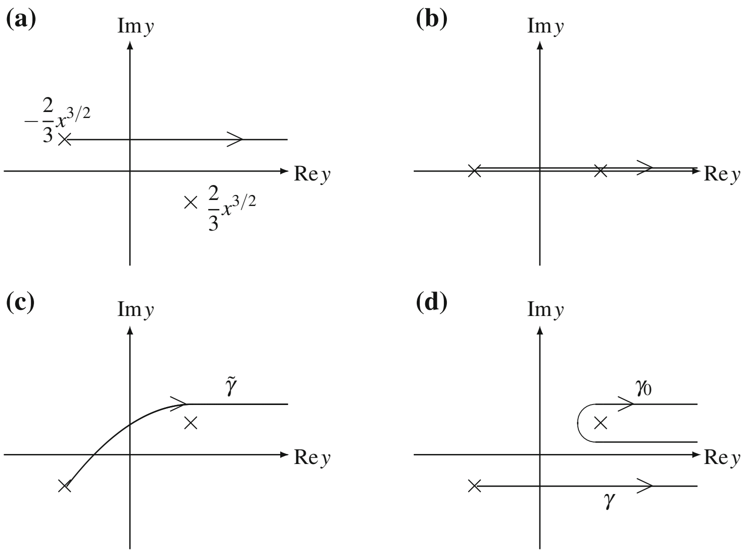


Fig. 1.1 Relative locations of singularities of the integrand and the path of integration of (1.1.16)

perform the analytic continuation of the Borel sum $\psi_+(x, \eta)$ from the region $\{\varepsilon < 0\}$ to $\{\varepsilon \geq 0\}$ we have to deform the path of integration to find $\tilde{\gamma}$ in Fig. 1.1c, which is allowed by Cauchy's theorem. We further decompose it as $\gamma + \gamma_0$ in Fig. 1.1d, again by using Cauchy's theorem, where γ_0 is a path that encircles the half line l_0 given by

$$\{(x, y) \in \mathbb{C}^2; \operatorname{Im} y = \operatorname{Im}(2x^{3/2}/3),\ \operatorname{Re} y \geq \operatorname{Re}(2x^{3/2}/3)\}. \tag{1.1.25}$$

Since the integral along the path γ gives the Borel sum $\psi_+(x, \eta)$ for $\varepsilon > 0$, we find that the Borel sum $\psi_+(x, \eta)$ changes by

$$\int_{\gamma_0} \exp(-y\eta)\psi_{+,B}(x, y)dy \tag{1.1.26}$$

as x crosses the Stokes curve. Thus our next task is to describe the discontinuity $(\Delta_{l_0}\psi_{+,B})(x, y)$, i.e., the difference of $\psi_{+,B}(x, y)$ evaluated above l_0 and that below l_0. Fortunately, by using Gauss' connection formula for hypergeometric functions, we find the following discontinuity formula for the Borel transformed WKB solutions:

$$(\Delta_{l_0}\psi_{+,B})(x, y) = i\psi_{-,B}(x, y). \tag{1.1.27}$$

(See [KT2, Sect. 2.2] for the detailed computation.) Since

$$\int_{l_0} \exp(-y\eta)\psi_{-,B}(x, y)dy \tag{1.1.28}$$

is the Borel sum of $\psi_-(x,\eta)$, we thus find the Borel sum $\psi_+(x,\eta)$ acquires $i\psi_-(x,\eta)$ as x crosses the Stokes curve from $\{\varepsilon < 0\}$ to $\{\varepsilon > 0\}$. We note that in the current configuration the point $\{y = -2x^{3/2}/3\}$ is located far away from the path of integration l_0 that appears in the definition of the Borel sum $\psi_-(x,\eta)$, it is kept intact even when x crosses the Stokes curve in question. We also note that it is exponentially small compared with $\psi_+(x,\eta)$ as l_0 starts from $\{y = 2x^{3/2}/3\}$ with $x = 1 + i\varepsilon$ ($|\varepsilon| \ll 1$). These phenomena are what we call "Stokes phenomena of WKB solutions" in the exact WKB analysis. We emphasize that all relations we have used are *exact*, versus asymptotic. At the same time, the reader might imagine that such exact relations would be consequences of the simple form of the Schrödinger equation in question, i.e.,

$$\left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi(x,\eta) = 0. \tag{1.1.29}$$

Fortunately we can find a counterpart of the exact relation (1.1.27) for a general Schrödinger equation

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 \tilde{Q}(\tilde{x})\right)\tilde{\psi}(\tilde{x},\eta) = 0 \tag{1.1.30}$$

in a neighborhood of its simple turning point $\tilde{x} = a$; here a simple turning point is, by definition, a simple zero of $\tilde{Q}(\tilde{x})$, that is, a satisfies

$$\tilde{Q}(a) = 0, \quad \left.\frac{d\tilde{Q}}{d\tilde{x}}\right|_{\tilde{x}=a} \neq 0. \tag{1.1.31}$$

An important step in finding such an exact relation is to show Theorem 1.1.1; with the notation in it, we call the transformation

$$\tilde{x} \longmapsto x(\tilde{x},\eta) \tag{1.1.32}$$

a formal coordinate transformation. As Silverstone first observed in [S], the chemistry of the formal coordinate transformation and the Borel summation is excellent. An important point is that a formal coordinate transformation determines an integro-differential operator via the Borel transformation, as was emphasized in [AKT5].

Theorem 1.1.1 ([KT2, Theorem 2.15]) *Let $\tilde{x} = 0$ be a simple turning point of the Schrödinger equation* (1.1.30). *Then we can find a formal series*

$$x(\tilde{x},\eta) = x_0(\tilde{x}) + x_1(\tilde{x})\eta^{-1} + x_2(\tilde{x})\eta^{-2} + \cdots, \tag{1.1.33}$$

which satisfies the conditions (1.1.34)–(1.1.38) *below:*

There exists a neighborhood $\tilde{U}$ *of* $\tilde{x} = 0$ *and* $x_j(\tilde{x})$ *is holomorphic on* $\tilde{U}$. (1.1.34)

$$\tilde{Q}(\tilde{x}) = \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 x(\tilde{x}, \eta) - \frac{\eta^{-2}}{2}\{x; \tilde{x}\} \tag{1.1.35}$$

holds, where $\{x; \tilde{x}\}$ *stands for the Schwarzian derivative, i.e.,*

$$\frac{\partial^3 x/\partial \tilde{x}^3}{\partial x/\partial \tilde{x}} - \frac{3}{2}\left(\frac{\partial^2 x/\partial \tilde{x}^2}{\partial x/\partial \tilde{x}}\right)^2.$$

$$\frac{dx_0}{d\tilde{x}} \neq 0 \quad \text{on } U. \tag{1.1.36}$$

$$x_{2p+1}(\tilde{x}) \ (p \geq 0) \text{ identically vanishes.} \tag{1.1.37}$$

For any compact set K *in* $\tilde{U}$ *there exist some constants* A_K *and* C_K *for which*

$$\sup_{\tilde{x} \in K} |x_j(\tilde{x})| \leq A_K C_K^j j! \tag{1.1.38}$$

holds.

Referring the reader to [KT2] for the proof, we concentrate our attention on explaining how this result is related to WKB analysis. First of all, we note the following

Theorem 1.1.2 *In the situation assumed in the precedent theorem, let us consider the following two Riccati equations:*

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial \tilde{x}} = \eta^2 \tilde{Q}(\tilde{x}), \tag{1.1.39}$$

$$S^2 + \frac{\partial S}{\partial x} = \eta^2 x. \tag{1.1.40}$$

Assume that the branches of $\tilde{S}_{-1} = \sqrt{\tilde{Q}(\tilde{x})}$ *and* $S_{-1} = \sqrt{x}$ *are chosen so that*

$$\tilde{S}_{-1}(\tilde{x}) = S_{-1}(x_0(\tilde{x}))\,\frac{dx_0}{d\tilde{x}}. \tag{1.1.41}$$

Then we find

$$\tilde{S}(\tilde{x}, \eta) = \left(\frac{\partial x}{\partial \tilde{x}}\right) S(x(\tilde{x}, \eta), \eta) - \frac{1}{2}\frac{\partial^2 x/\partial \tilde{x}^2}{\partial x/\partial \tilde{x}}. \tag{1.1.42}$$

Remark 1.1.2 Comparing the degree 0 part in (1.1.35) we find

$$\tilde{Q}(\tilde{x}) = \left(\frac{dx_0}{d\tilde{x}}\right)^2 x_0(\tilde{x}). \tag{1.1.43}$$

Hence the condition (1.1.41) concerns only with the signs.

We note that (1.1.42) combined with (1.1.37) entails the following

Corollary 1.1.1 *We find*

$$\tilde{S}_{\text{odd}}(\tilde{x}, \eta) = \left(\frac{\partial x}{\partial \tilde{x}}\right) S_{\text{odd}}(x(\tilde{x}, \eta), \eta). \tag{1.1.44}$$

Proof of Theorem 1.1.2. In what follows we often abbreviate $\partial x/\partial \tilde{x}$ etc. to x' etc. It follows from (1.1.40) that we obtain

$$\begin{aligned}
&\left(x' S(x(\tilde{x}, \eta), \eta) - \frac{1}{2}\frac{x''}{x'}\right)^2 + \left(x' S(x(\tilde{x}, \eta), \eta) - \frac{1}{2}\frac{x''}{x'}\right)' \\
&= x'^2 S(x(\tilde{x}, \eta), \eta)^2 - x'' S(x(\tilde{x}, \eta), \eta) + \frac{1}{4}\frac{x''^2}{x'^2} \\
&\quad + x'' S(x(\tilde{x}, \eta), \eta) + x'^2 \left(\frac{\partial S}{\partial x}\right)(x(\tilde{x}, \eta), \eta) + \frac{1}{2}\frac{x''^2}{x'^2} - \frac{1}{2}\frac{x'''}{x'} \\
&= x'^2 \eta^2 x(\tilde{x}, \eta) - \frac{1}{2}\{x; \tilde{x}\}.
\end{aligned} \tag{1.1.45}$$

Hence (1.1.35) implies

$$\left(x'S - \frac{1}{2}\frac{x''}{x'}\right)^2 + \left(x'S - \frac{1}{2}\frac{x''}{x'}\right) = \eta^2 \tilde{Q}(\tilde{x}), \tag{1.1.46}$$

that is, we find

$$T \underset{\text{def}}{=} x' S(x(\tilde{x}, \eta), \eta) - \frac{1}{2}\frac{x''}{x'} \tag{1.1.47}$$

satisfies (1.1.39). On the other hand, (1.1.41) guarantees the degree 1 (in η) part T_{-1} of T coincides with $\tilde{S}_{-1}$. Hence, by the uniqueness of solutions of the form (1.1.3) of the Riccati equation, we find

$$T = \tilde{S}. \tag{1.1.48}$$

This completes the proof of Theorem 1.1.2.

It is clear from the above proof that (1.1.35) follows from (1.1.42). In this sense, the existence of formal series (1.1.33) is, in essence, equivalent to the existence of

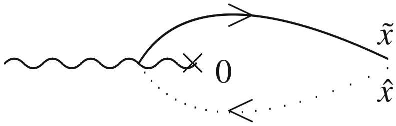


Fig. 1.2 The path of integration used to define the integral (1.1.50); the wiggly line designates a cut that fixed the branch of $\tilde{x}^{p-1/2}$

an appropriate transformation of solutions of the Riccati equation associated with the Schrödinger equation. Hence it is natural to expect that some transformation of WKB solutions may be determined by the series (1.1.33). However, some ambiguity of multiplicative constant is anticipated in such relations because of the freedom in choosing the endpoint x_* of the integral in (1.1.11). Fortunately, thanks to (1.1.10), we find that $\tilde{S}_{2l-1}(\tilde{x})$ has the form

$$\sum_p a_p(\tilde{x})\tilde{x}^{(2p-1)/2} \tag{1.1.49}$$

with holomorphic functions $a_p(\tilde{x})$ near the origin, which is, by the assumption, a simple turning point, i.e., a simple zero of the potential $\tilde{Q}(\tilde{x})$. This fact enables us to choose the origin as the endpoint x_* of the integral in (1.1.11); we define

$$\int_0^{\tilde{x}} \tilde{S}_{2l-1}(\tilde{x})d\tilde{x} = \frac{1}{2}\int_{\hat{x}}^{\tilde{x}} \tilde{S}_{2l-1}(\tilde{x})d\tilde{x} \tag{1.1.50}$$

by using the contour integral on the Riemann surface of $\sqrt{\tilde{Q}(\tilde{x})}$ near the origin, as is shown in Fig. 1.2. Here the integral in the right-hand side of (1.1.50) is given by the integration along a path from $\hat{x}$ to $\tilde{x}$, where $\hat{x}$ designates the point corresponding to $\tilde{x}$ on a sheet of the Riemann surface which is different from the sheet on which $\tilde{x}$ lies. Then, by choosing the origin, i.e., the simple turning point in question, as the endpoint of the integral in (1.1.11) both in $\psi_\pm(x,\eta)$ and in $\tilde{\psi}_\pm(\tilde{x},\eta)$, we obtain the following

Theorem 1.1.3 *Let us consider the problem in the same situation as in Theorem 1.1.1, and assume that WKB solutions $\psi_\pm(x,\eta)$ and $\tilde{\psi}_\pm(\tilde{x},\eta)$ are normalized by choosing the origin as the endpoint of the integral in* (1.1.11)*. Then we find*

$$\tilde{\psi}_\pm(\tilde{x},\eta) = \left(\frac{\partial x}{\partial \tilde{x}}\right)^{-1/2} \psi_\pm(x(\tilde{x},\eta),\eta). \tag{1.1.51}$$

Proof In view of (1.1.44) together with the choice (1.1.41) of the branches of the top degree part of S and $\tilde{S}$, we find it is sufficient to show

$$\int_0^{\tilde{x}} \tilde{S}_{\mathrm{odd}}(\tilde{x},\eta)d\tilde{x} = \left(\int_0^x S_{\mathrm{odd}}(x,\eta)dx\right)\Big|_{x=x(\tilde{x},\eta)}. \tag{1.1.52}$$

First we note that (1.1.44) entails

$$\int_0^{\tilde{x}} \tilde{S}_{\mathrm{odd}}(\tilde{x}, \eta) d\tilde{x} = \frac{1}{2} \int_{\hat{x}}^{\tilde{x}} \tilde{S}_{\mathrm{odd}}(\tilde{x}, \eta) d\tilde{x}$$
$$= \frac{1}{2} \int_{\hat{x}}^{\tilde{x}} S_{\mathrm{odd}}(x(\tilde{x}, \eta), \eta) \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} d\tilde{x}. \tag{1.1.53}$$

To rewrite the right-hand side of (1.1.53) further, let us introduce

$$z(\tilde{x}, \eta) = x(\tilde{x}, \eta) - x_0(\tilde{x}). \tag{1.1.54}$$

Then, by using the Taylor expansion, we find

$$S_{\mathrm{odd}}(x(\tilde{x}, \eta), \eta) \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} = \left(\sum_{n \geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n} (x_0(\tilde{x}), \eta) \frac{z^n}{n!} \right) \left(\frac{dx_0}{d\tilde{x}} + \frac{dz}{d\tilde{x}} \right)$$
$$= \sum_{n \geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n} (x_0(\tilde{x}), \eta) \frac{z^n}{n!} x_0'$$
$$+ \sum_{n \geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n} (x_0(\tilde{x}), \eta) \left(\frac{z^{n+1}}{(n+1)!} \right)'. \tag{1.1.55}$$

Here, and in what follows, x_0' etc. stand for $dx_0/d\tilde{x}$ etc. To apply the technique of integration by parts to the second sum in the right-hand side of (1.1.55), we note the following fact: the holomorphy of $x_0(\tilde{x})$ on a neighborhood of the origin entails that the point $x_0(\hat{x})$, which is reached by $x_0(\tilde{x})$ as $\tilde{x}$ moves along the contour (in the reverse direction) in Fig. 1.2, is the point that corresponds to $x_0(\tilde{x})$ on the sheet of the Riemann surface of $\sqrt{x}$ which is different from the sheet on which $x_0(\tilde{x})$ lies. This fact then implies

$$\frac{1}{2} \int_{\hat{x}}^{\tilde{x}} \frac{d}{d\tilde{x}} \left(\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n} (x_0(\tilde{x}), \eta) \frac{z^{n+1}}{(n+1)!} \right) d\tilde{x}$$
$$= \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n} (x_0(\tilde{x}), \eta) \frac{z^{n+1}}{(n+1)!}. \tag{1.1.56}$$

Hence it follows from (1.1.55) that we find the following:

$$\frac{1}{2} \int_{\hat{x}}^{\tilde{x}} S_{\mathrm{odd}}(x(\tilde{x}, \eta), \eta) \frac{dx(\tilde{x}, \eta)}{d\tilde{x}} d\tilde{x}$$
$$= \frac{1}{2} \int_{\hat{x}}^{\tilde{x}} \left(\sum_{n \geq 0} \frac{\partial^n S_{\mathrm{odd}}}{\partial x^n} (x_0(\tilde{x}), \eta) \frac{z^n}{n!} x_0'(\tilde{x}) \right) d\tilde{x}$$

$$
\begin{aligned}
&+\Big[\sum_{n\ge 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0(\tilde{x}),\eta)\frac{z^{n+1}}{(n+1)!}\\
&-\frac{1}{2}\int_{\hat{x}}^{\tilde{x}}\Big(\sum_{n\ge 0}\frac{\partial^{n+1} S_{\mathrm{odd}}}{\partial x^{n+1}}(x_0(\tilde{x}),\eta)\,x_0'(\tilde{x})\frac{z^{n+1}}{(n+1)!}\Big)d\tilde{x}\Big]\\
=&\frac{1}{2}\int_{\hat{x}}^{\tilde{x}}\Big[\Big(\sum_{n\ge 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0(\tilde{x}),\eta)\frac{z^n}{n!}x_0'(\tilde{x})\Big)\\
&-\Big(\sum_{m\ge 1}\frac{\partial^m S_{\mathrm{odd}}}{\partial x^m}(x_0(\tilde{x}),\eta)\frac{z^m}{m!}x_0'(\tilde{x})\Big)\Big]d\tilde{x}\\
&+\sum_{n\ge 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0(\tilde{x}),\eta)\frac{z^{n+1}}{(n+1)!}\\
=&\frac{1}{2}\int_{\hat{x}}^{\tilde{x}}S_{\mathrm{odd}}(x_0(\tilde{x}),\eta)\,x_0'(\tilde{x})d\tilde{x}+\sum_{n\ge 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0(\tilde{x}),\eta)\frac{z^{n+1}}{(n+1)!}\\
=&\frac{1}{2}\int_{x_0(\hat{x})}^{x_0(\tilde{x})}S_{\mathrm{odd}}(x,\eta)dx+\sum_{n\ge 0}\frac{\partial^n S_{\mathrm{odd}}}{\partial x^n}(x_0(\tilde{x}),\eta)\frac{z^{n+1}}{(n+1)!}\,. \qquad (1.1.57)
\end{aligned}
$$

If we set

$$R(x,\eta)=\int_0^x S_{\mathrm{odd}}(x,\eta)dx, \qquad (1.1.58)$$

it is then clear, again by the Taylor expansion, that the rightmost side of (1.1.57) is equal to

$$R(x,\eta)\Big|_{x=x(\tilde{x},\eta)}. \qquad (1.1.59)$$

Hence, combining (1.1.53) and (1.1.57), we obtain

$$\int_0^{\tilde{x}}\tilde{S}_{\mathrm{odd}}(\tilde{x},\eta)d\tilde{x}=\Big(\int_0^x S_{\mathrm{odd}}(x,\eta)dx\Big)\Big|_{x=x(\tilde{x},\eta)}. \qquad (1.1.60)$$

This completes the proof of Theorem 1.1.3.

Thus we can describe the relation between $\tilde{\psi}_\pm(\tilde{x},\eta)$ and $\psi_\pm(x,\eta)$ in all orders of η^{-1} with the help of the formal coordinate transformation constructed in Theorem 1.1.1. But, probably it is difficult for the reader to imagine how the relation (1.1.51) is related to Stokes phenomena. As a matter of fact, application of the Borel transformation to the both sides of (1.1.51) is an essential step in our reasoning. As we will see below, the treatment of the factor $(\partial x/\partial\tilde{x})^{-1/2}$ is a straightforward one, we first concentrate our attention on the computation of the second factor in the right-hand side of (1.1.51), i.e., $\psi_+(x(\tilde{x}, \eta), \eta)$. Then by the Taylor expansion we find

$$\psi_\pm(x(\tilde{x}, \eta), \eta) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{j \geq 1} x_j(\tilde{x}) \eta^{-j} \right)^n \left(\frac{\partial^n}{\partial x^n} \psi_\pm(x, \eta) \right) \Big|_{x = x_0(\tilde{x})}. \tag{1.1.61}$$

Using (1.1.36) we define the inverse function $g(x)$ of $x_0(\tilde{x})$, i.e.,

$$g(x_0(\tilde{x})) = \tilde{x}, \quad x_0(g(x)) = x. \tag{1.1.62}$$

We also let $\tilde{x}_j(x)$ denote

$$x_j(g(x)). \tag{1.1.63}$$

Then the right-hand side of (1.1.61) assume the form

$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{j \geq 1} \tilde{x}_j(x) \eta^{-j} \right)^n \frac{\partial^n}{\partial x^n} \psi_\pm(x, \eta). \tag{1.1.64}$$

As the multiplication operator η turns out to be $\partial/\partial y$ via the Borel transformation, it is reasonable to imagine η^{-1} will turn out to be an integral operator $(\partial/\partial y)^{-1} = \int^y dy$. Thus, at least formally, the Borel transform of (1.1.64) is expressed as

$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{j \geq 1} \tilde{x}_j(x) \left(\frac{\partial}{\partial y} \right)^{-j} \right)^n \frac{\partial^n}{\partial x^n} \psi_{\pm, B}(x, y). \tag{1.1.65}$$

Let C denote the integro-differential operator

$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{j \geq 1} \tilde{x}_j(x) \left(\frac{\partial}{\partial y} \right)^{-j} \right)^n \frac{\partial^n}{\partial x^n} \tag{1.1.66}$$

and let $\tilde{C}(\tilde{x}, \partial/\partial \tilde{x}, \partial/\partial y)$ denote the operator written with $(\tilde{x}, y)$. Then by denoting by $\tilde{B}(\tilde{x}, \partial/\partial \tilde{x}, \partial/\partial y)$ the operator

$$\left(\frac{\partial x(\tilde{x}, \partial/\partial y)}{\partial \tilde{x}} \right)^{-1/2}, \tag{1.1.67}$$

we find that the Borel transformed version of (1.1.51) is

$$\tilde{\psi}_{\pm, B}(\tilde{x}, y) = \tilde{A}(\tilde{x}, \partial/\partial \tilde{x}, \partial/\partial y) \psi_{\pm, B}(x_0(\tilde{x}), y), \tag{1.1.68}$$

where

$$\tilde{A} = \tilde{B}\tilde{C}. \tag{1.1.69}$$

Furthermore, as [AKT5, Appendix C] shows, the estimation (1.1.38) guarantees that the operator $\tilde{A}$ is an integro-differential operator of a special kind; it is called a microdifferential operator in microlocal analysis [SKK] and it does not change the location of the singularities of the operand. (See Fig. 1.3.) Once we obtain such an operator $\tilde{A}$, we can locate the singularity of $\tilde{\psi}_{+,B}(\tilde{x}, y)$ in a neighborhood ω of the origin $(0, 0)$. Since $\tilde{A}$ does not change the singularity of the operand $\psi_{+,B}(x_0(\tilde{x}), y)$, and since the singular points of $\psi_{+,B}(x, y)$ are

$$y = \pm\left(\frac{2}{3}x^{\frac{3}{2}}\right) \tag{1.1.70}$$

by Fact B about $\psi_{+,B}(x, y)$, they are confined to

$$\pm\frac{2}{3}x_0(\tilde{x})^{\frac{3}{2}} \quad \text{with} \quad \tilde{x} \neq 0 \tag{1.1.71}$$

in ω. Thus the singularity $2x_0(\tilde{x})^{3/2}/3$ hits the path of integration which defines the Borel sum of $\psi_+(x, \eta)$, i.e.,

$$\left\{(\tilde{x}, y);\ \operatorname{Im} y = \operatorname{Im}\left(-\frac{2}{3}x_0(\tilde{x})^{3/2}\right),\ \operatorname{Re} y \geq \operatorname{Re}\left(-\frac{2}{3}x_0(\tilde{x})^{3/2}\right)\right\}. \tag{1.1.72}$$

This means that we observe Stokes phenomena in general if

$$\operatorname{Im}\left(-\frac{2}{3}x_0(\tilde{x})^{3/2}\right) = 0, \tag{1.1.73}$$

whereas (1.1.43) together with (1.1.41) implies

$$\begin{aligned}\int_0^{\tilde{x}} \sqrt{\tilde{Q}(\tilde{x})}\,d\tilde{x} &= \int_0^{\tilde{x}} \frac{dx_0(\tilde{x})}{d\tilde{x}}\sqrt{x_0(\tilde{x})}\,d\tilde{x} \\ &= \int_0^{x_0(\tilde{x})} \sqrt{x}\,dx = \frac{2}{3}x_0(\tilde{x})^{3/2}.\end{aligned} \tag{1.1.74}$$

Thus we may observe Stokes phenomena if

$$\operatorname{Im}\int_0^{\tilde{x}} \sqrt{\tilde{Q}(\tilde{x})}\,d\tilde{x} = 0, \tag{1.1.75}$$

i.e., if $\tilde{x}$ hits a Stokes curve defined by Definition 1.1.1. Furthermore (1.1.68) enables us to find a local counterpart of (1.1.27), a discontinuity formula for $\tilde{\psi}_{+,B}(\tilde{x}, y)$ along

$$\tilde{l}_0 = \left\{ (\tilde{x}, y) \in \omega;\ \mathrm{Im}\, y = \mathrm{Im}\left(\frac{2}{3}x_0(\tilde{x})^{3/2}\right),\ \mathrm{Re}\, y \geq \mathrm{Re}\left(\frac{2}{3}x_0(\tilde{x})^{3/2}\right)\right\} \tag{1.1.76}$$

in the following manner. Combining (1.1.68) and (1.1.27) we find

$$\begin{aligned}\Delta_{\tilde{l}_0}\tilde{\psi}_{+,B}(\tilde{x}, y) &= \Delta_{\tilde{l}_0}\tilde{A}\psi_{+,B}(x_0(\tilde{x}), y) \\ &= \tilde{A}\Delta_{\tilde{l}_0}\psi_{+,B}(x_0(\tilde{x}), y) \\ &= \tilde{A}i\psi_{-,B}(x_0(\tilde{x}), y) \\ &= i\tilde{\psi}_{-,B}(\tilde{x}, y).\end{aligned} \tag{1.1.77}$$

This local discontinuity formula means that, barring the possible relevance of singularities of $\tilde{\psi}_{+,B}(\tilde{x}, y)$ other than $y = 2x_0(\tilde{x})^{3/2}/3$, the exponentially small term that appears at the Stokes phenomenon is given by $i\tilde{\psi}_-(\tilde{x}, \eta)$ (or $-i\tilde{\psi}_-(\tilde{x}, \eta)$ depending on the direction of the analytic continuation). Thus we have a clean picture of Stokes phenomena along the Stokes curve defined in Definition 1.1.1. Making this illustration as the primary aim, we have so far used $\tilde{x}$ as the target variable. However, as the switchover of the variable x to $\tilde{x}$ in (1.1.66), the computation in $(\tilde{x}, y)$ coordinate is somewhat messy. Hence we now summarize basic results in the comparison of the Airy equation and a Schrödinger equation with a simple turning point that is made in the same variable. In order to fix our notation let $\tilde{L}$ denote

$$\frac{d^2}{d\tilde{x}^2} - \eta^2\tilde{Q}(\tilde{x}) \tag{1.1.78}$$

with the origin being its simple turning point, and let $\tilde{L}_B$ denote its Borel transform. Then, using the inverse function $g(x)$ of $x_0(\tilde{x})$ given by (1.1.62), we find

$$\tilde{L}_B\Big|_{\tilde{x}=g(x)} = \left(\frac{dg}{dx}\right)^{-2}\left[\frac{\partial^2}{\partial x^2} - \frac{d^2g/dx^2}{dg/dx}\frac{\partial}{\partial x}\right] - \tilde{Q}(g(x))\frac{\partial^2}{\partial y^2}. \tag{1.1.79}$$

Let us define $\mathscr{L}$ by

$$\mathscr{L} = \frac{\partial^2}{\partial x^2} - \frac{d^2g/dx^2}{dg/dx}\frac{\partial}{\partial x} - \left(\frac{dg}{dx}\right)^2\tilde{Q}(g(x))\frac{\partial^2}{\partial y^2}, \tag{1.1.80}$$

and let $\mathscr{X}$ denote the operator $\tilde{A}$ in (1.1.63) written down in (x, y) coordinate. Then the result we have obtained tells us that

$$\mathscr{L}\mathscr{X}\psi_{\pm,B}(x, y) = 0 \tag{1.1.81}$$

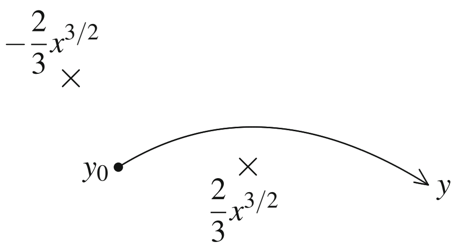


Fig. 1.3 The path of integration used in (1.1.84)

holds near the origin except for $\{x = 0\}$, i.e., the line corresponding to the turning point. Further, by using the same computation as in [AKT5], we can concretely describe the structure of $\mathscr{X}$ as follows: let U denote

$$\{(x, y) \in \mathbb{C};\ |x|, |y| < \delta\}, \tag{1.1.82}$$

where δ is a sufficiently small positive number, and let U^* denote

$$U - \Big(\{(x, y) \in U;\ x = 0\} \cup \{(x, y) \in U;\ y = 2x^{3/2}/3\} \cup \{(x, y) \in U;\ y = -2x^{3/2}/3\}\Big). \tag{1.1.83}$$

Then, for a multi-valued analytic function $\varphi(x, y)$ defined on U^*, we find

$$(\mathscr{X}\varphi)(x, y) = \int_{y_0}^{y} K(x, y - y', \partial/\partial x)\varphi(x, y')dy', \tag{1.1.84}$$

where $K(x, y, \partial/\partial x)$ is a differential operator that is defined on $\{(x, y) \in \mathbb{C}^2;\ |x| < C$ and $|y| < C'\}$ for some positive constants C and C' and $y = y_0$ is a reference point that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator. (See Fig. 1.3.)

Remark 1.1.3 The differential operator K is the so-called differential operator of infinite order. An important property of such an operator is that it does not change the location of singularities of the operand. To illustrate this point let us give a simple example of such an operator;

$$\cosh\big(\sqrt{\partial/\partial x}\,\big) = \sum_{n \geq 0} \frac{\partial^n/\partial x^n}{(2n)!}. \tag{1.1.85}$$

It is easy to confirm that the map

$$\cosh\big(\sqrt{\partial/\partial x}\,\big) : \mathscr{O}(U) \longrightarrow \mathscr{O}(U) \tag{1.1.86}$$

is well-defined for any open set U, where $\mathscr{O}(U)$ stands for the space of holomorphic functions on U.

Since the meaning of a Borel transformed WKB solution at a turning point is obscure because of the strong singularity it presents there, we have excluded the set $\{x = 0\}$ from our consideration. However microdifferential operator $\mathscr{X}$ itself is well-defined in a full neighborhood of the origin. Hence we can obtain Theorem 1.1.4, which asserts the equivalence of the operator $\mathscr{L}$ and the Borel transform $\mathscr{M}$ of the Airy operator.

Theorem 1.1.4 *Let $\mathscr{M}$ stand for*

$$\frac{\partial^2}{\partial x^2} - x\,\frac{\partial^2}{\partial y^2}. \tag{1.1.87}$$

Then there exist invertible microdifferential operators $\mathscr{X}$ and $\mathscr{Y}$ which satisfy

$$\mathscr{L}\mathscr{X} = \mathscr{Y}\mathscr{M} \tag{1.1.88}$$

near the origin. Further the operator $\mathscr{Y}$ enjoys an integral representation similar to (1.1.84)*, that is, there exists a differential operator $\tilde{K}$ for which*

$$(\mathscr{Y}\varphi)(x, y) = \int_{y_0}^{y} \tilde{K}(x, y - y', \partial/\partial x)\varphi(x, y')dy' \tag{1.1.89}$$

holds.

We refer the reader to [AKT5] for the proof. We only note that the invertibility of operators $\mathscr{X}$ and $\mathscr{Y}$ are the counterpart of the fact that the reasoning in Theorem 1.1.3 can be reversed thanks to (1.1.36), i.e., the fact that $\psi_+(x, \eta)$ can be represented as $(\partial\tilde{x}/\partial x)^{-1/2}\tilde{\psi}_\pm(\tilde{x}(x, \eta), \eta)$.

Remark 1.1.4 In (1.1.88) we cannot expect $\mathscr{X} = \mathscr{Y}$, and this fact makes a clear contrast to a similar and general result known in microlocal analysis; the general result asserts that we can choose an appropriate operator $\mathscr{Z}$ that intertwines $\mathscr{L}$ and $\mathscr{M}$, i.e., $\mathscr{L}\mathscr{Z} = \mathscr{Z}\mathscr{M}$. This fact indicates that the operators $\mathscr{X}$ and $\mathscr{Y}$ are "WKB-theoretic transformations", and it is also clear from our reasoning in this section. In this sense we often refer to the contents of Theorems 1.1.1 and 1.1.4 by saying "the WKB-theoretic canonical form of the Schrödinger equation near its simple turning point is given by the Airy equation".

In ending this section we summarize the results explained so far in a form of the connection formula for the Borel resummed WKB solutions of (1.1.1), following the presentation of A. Voros [V]; here we accept the Borel summability of WKB solutions, leaving its proof to [KoS]. In what follows, an open set U in $\mathbb{C}$ is said to be a Stokes region if its boundary consists of Stokes curves $\{\varGamma_j\}_{j=1,2,\dots,N}$ emanating respectively from turning points $\{a_j\}_{j=1,2,\dots,N}$ of (1.1.1).

Theorem 1.1.5 *Assume that every turning point of* (1.1.1) *is simple. Suppose further that for each pair* (a, b) *of turning points of* (1.1.1) *we have*

$$\mathrm{Im} \int_a^b \sqrt{Q(x)}\, dx \neq 0 \tag{1.1.90}$$

by appropriately fixing the branch of $\sqrt{Q}$*. Let us consider the situation where two Stokes regions* U_1 *and* U_2 *share in their boundaries a Stokes curve* Γ *that emanates from a turning point* a*, and let* $\psi_\pm^j = \psi_\pm^j(x, \eta)$ *denote the Borel sum on* U_j *of*

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_a^x S_{\mathrm{odd}}\, dx\right). \tag{1.1.91}$$

The $\psi_\pm^1$ *can be analytically continued to* U_2 *and they are related to* $\psi_\pm^2$ *as in either* (1.1.92a) *or* (1.1.92b)*, depending on the geometric situations explained below:*

$$\begin{cases} \psi_+^1 = \psi_+^2 \\ \psi_-^1 = \psi_-^2 \pm i\psi_+^2 \end{cases} \quad \textit{if } \mathrm{Re} \int_a^x \sqrt{Q(x)}\, dx < 0 \textit{ on } \Gamma, \tag{1.1.92a}$$

$$\begin{cases} \psi_+^1 = \psi_+^2 \pm i\psi_-^2 \\ \psi_-^1 = \psi_-^2 \end{cases} \quad \textit{if } \mathrm{Re} \int_a^x \sqrt{Q(x)}\, dx > 0 \textit{ on } \Gamma. \tag{1.1.92b}$$

Here the sign $\pm$ *in the right-hand side of* (1.1.92a) *and* (1.1.92b) *is chosen as follows: the sign* $+$ *is chosen when the path of analytic continuation from* U_1 *to* U_2 *crosses* Γ *anticlockwise (seen from the turning point* a*), and the sign* $-$ *is chosen when it crosses* Γ *clockwise.*

Remark 1.1.5 The assumption (1.1.90) might be somewhat puzzling, as we have not so far discussed its origin. The point is as follows: although in (1.1.71) we discussed the location of singularities of $\tilde{\psi}_{+,B}(\tilde{x}, y)$ locally in ω, the general theory of linear partial differential equations [SKK, Chap. II] guarantees that the statement there holds globally, as we will discuss later in Sect. 1.4. To be more concrete, we find that $\{(\tilde{x}, y);\ y = \int_0^{\tilde{x}} \sqrt{\tilde{Q}(\tilde{x})}\, d\tilde{x}\}$ is contained in the singularities of $\tilde{\psi}_{+,B}(\tilde{x}, y)$ even outside ω. Hence, in particular, we see that $\tilde{\psi}_{+,B}(\tilde{x}, y)$ is singular at $y = \int_0^b \sqrt{\tilde{Q}(\tilde{x})}\, d\tilde{x}$, where b is a turning point different from 0. It is then evident from the discussion concerning Fig. 1.1 that, if

$$\mathrm{Im} \int_0^b \sqrt{\tilde{Q}(\tilde{x})}\, d\tilde{x} = 0, \tag{1.1.93}$$

then, in general, we cannot find an appropriate path of integration to define the Borel sum of $\tilde{\psi}_{+,B}(\tilde{x}, \eta)$. This is the background of the assumption (1.1.90).

Remark 1.1.6 It immediately follows from the definition of a Stokes curve that

$$\operatorname{Im} \int_a^b \sqrt{Q(x)}\, dx = 0 \tag{1.1.94}$$

implies

$$\begin{aligned}&\text{the turning point } b \text{ (resp., } a) \text{ is contained in the Stokes curve}\\ &\text{emanating from } a \text{ (resp., } b).\end{aligned} \tag{1.1.95}$$

The geometric situation described as in (1.1.95) is usually referred to as

$$a \text{ and } b \text{ are connected by a Stokes segment.} \tag{1.1.96}$$

We also say that the Stokes geometry of (1.1.1) is degenerate, if the phenomenon (1.1.96) is observed. The **degeneration of the Stokes geometry** will play an important role in Chap. 2.

1.2 WKB Analysis of Higher Order Differential Equations in the Small

We have briefly reviewed in Sect. 1.1 the basic part of the exact WKB analysis of differential equations of the second order. Having the results in mind, we prepare some basic notions for differential equations of higher order, and show that the local properties of Borel transformed WKB solutions are deduced from the results for the second order equations.

The equation we want to study is the following equation with a large parameter η:

$$P\psi = \left(\frac{d^m}{dx^m} + q_1(x)\eta \frac{d^{m-1}}{dx^{m-1}} + \cdots + q_m(x)\eta^m\right)\psi = 0, \tag{1.2.1}$$

where $q_j(x)$ is a polynomial and η is a positive large number. If we set

$$\psi = \exp\left(\int^x S(x,\eta)dx\right), \tag{1.2.2}$$

then S satisfies a non-liner differential equation of order $(m-1)$, which we call the higher-order version of the Riccati equation. Further, if we assume

$$S(x,\eta) = \eta S_{-1}(x) + S_0(x) + \eta^{-1}S_1(x) + \cdots \tag{1.2.3}$$

with $S_{-1}(x)$ satisfying

$$\big(S_{-1}(x)\big)^m + q_1(x)\big(S_{-1}(x)\big)^{m-1} + \cdots + q_m(x) = 0, \tag{1.2.4}$$

then $S_j(x)$ ($j \geq 0$) are recursively determined in a unique way through the higher-order version of the Riccati equation. In what follows we call ψ given in the form of (1.2.2) (or possible its constant multiple as (1.2.5) below)

a **WKB solution** of (1.2.1).

In order to make the computation of the Borel transform of a WKB solution run smoothly we consider

$$\psi = \eta^{-1/2} \exp\left(\int_{x_*}^{x} S(x,\eta)dx\right) \tag{1.2.5}$$

where x_* is an appropriately fixed point, and in parallel with (1.1.12) we further expand it as

$$\exp\left(\eta \int_{x_*}^{x} S_{-1}(x)dx\right)\left(\sum_{j\geq 0} \psi_j(x)\eta^{-j-1/2}\right). \tag{1.2.6}$$

Thanks to the extra factor $\eta^{-1/2}$ in (1.2.5), its Borel transform is cleanly given by

$$\psi_B(x,y) = \sum_{j\geq 0} \frac{\psi_j(x)}{\Gamma(j+1/2)}\left(y + \int_{x_*}^{x} S_{-1}(x)dx\right)^{j-1/2}, \tag{1.2.7}$$

and our central issue is to study the analytic properties of $\psi_B(x,y)$. For this purpose we first introduce the notion of the characteristic polynomial $p(x,\xi,\eta)$ of Eq. (1.2.1); it is, by definition, given by

$$\xi^m + q_1(x)\xi^{m-1}\eta + \cdots + q_m(x)\eta^m. \tag{1.2.8}$$

For the algebraic manipulation of the characteristic polynomial, it is often convenient to introduce ζ and $\tilde{p}(x,\zeta)$ given by the following:

$$\zeta = \xi/\eta, \tag{1.2.9}$$

$$\tilde{p}(x,\zeta) = \zeta^m + q_1(x)\zeta^{m-1} + \cdots + q_m(x). \tag{1.2.10}$$

It is clear that

$$\tilde{p}(x,\zeta) = \eta^{-m}p(x,\xi,\eta). \tag{1.2.11}$$

In what follows we assume for the sake of simplicity that the polynomial $\tilde{p}(x, \zeta)$ has the simple multiplicity, that is, it does not have the form $(\tilde{q}(x, \zeta))^l$ $(l \geq 2)$ with some polynomial $\tilde{q}(x, \zeta)$. We also note that condition (1.2.4) is nothing but

$$\tilde{p}(x, S_{-1}(x)) = 0. \tag{1.2.12}$$

With some abuse of language we often call $\tilde{p}(x, \zeta)$ as the characteristic polynomial of (1.2.1).

To proceed further, we introduce appropriate cuts in the x-plane so that the solutions $\zeta_j(x)$ $(j = 1, 2, \ldots, m)$ of

$$\tilde{p}(x, \zeta) = 0 \tag{1.2.13}$$

may be single-valued on the cut plane. Then (1.2.12) means that we can choose j so that

$$S_{-1}(x) = \zeta_j(x) \tag{1.2.14}$$

holds, and with this choice of $\zeta_j(x)$, we let $\psi_j(x, \eta)$ denote the WKB solution ψ given in (1.2.5). Moreover, by using thus defined single-valued solutions $\zeta_j(x)$ $(j = 1, 2, \ldots, m)$ we introduce the notion of turning points and Stokes curves for Eq. (1.2.1) by the following

Definition 1.2.1 (i) A point $x = a$ is said to be a turning point of (1.2.1) if (1.2.13) has a multiple solution in ζ there. Further, if

$$\zeta_j(a) = \zeta_k(a) \quad (j \neq k) \tag{1.2.15}$$

holds, the turning point is said to be of type (j, k).
(ii) Let $x = a$ be a turning point of type (j, k). Then a Stokes curve of type (j, k) that emanates from a is, by definition, the curve given by

$$\operatorname{Im} \int_a^x \big(\zeta_j(x) - \zeta_k(x)\big) dx = 0. \tag{1.2.16}$$

Remark 1.2.1 If $m = 2$, the determiner "of type (j, k)" is not necessary. It is then clear that Definition 1.2.1 coincides with Definition 1.1.1 for the Schrödinger equation.

Remark 1.2.2 We will later introduce the notion of a "**virtual turning point**", and after then we sometimes refer to the turning point in Definition 1.2.1 as the "ordinary turning point". A Stokes curve emanating from a virtual turning point is referred to in some references (e.g. [AKT2, AKoT], etc.) a "new Stokes curve", and then it is usual to refer to a Stokes curve emanating from an "ordinary turning point" as an "ordinary Stokes curve". We use in this article the wording a "**new Stokes curve**" only in the discussion made from the historic viewpoint; in our later discussions, a Stokes curve normally means either a new one or an ordinary one.

Definition 1.2.2 Let $x = a$ be a turning point of type (j, k). Then each segment of the Stokes curve emanating from $x = a$ is labelled as $(j > k)$ or $(j < k)$, according as

$$\mathrm{Re} \int_a^x \big(\zeta_j(x) - \zeta_k(x)\big) dx > 0 \tag{1.2.17}$$

or

$$\mathrm{Re} \int_a^x \big(\zeta_j(x) - \zeta_k(x)\big) dx < 0 \tag{1.2.18}$$

holds on the segment.

Definition 1.2.3 If exactly two solutions $\zeta_j(x)$ and $\zeta_k(x)$ $(j \neq k)$ of (1.2.13) coalesce at $x = a$, and if

$$\frac{\partial \tilde{p}(x, \zeta)}{\partial x}\Big|_{x=a, \zeta=\zeta_j(a)} \neq 0, \tag{1.2.19}$$

then the point $x = a$ is said to be a **simple turning point**.

Remark 1.2.3 It is clear that the above definition of a simple turning point coincides with that given in Sect. 1.1 for the Schrödinger equation.

In analyzing the local structure of WKB solutions with the help of these notions, we make essential use of Theorem 1.2.1. The theorem is basically a variant of the Späth-type division theorem for holomorphic functions of several complex variables, and we omit its proof here. To illustrate its content, we just note that a differential operator P of WKB type on an open set $U (\subset \mathbb{C}_x)$ is, in an intuitive description, an operator whose total symbol $\sigma(P)$ (in the sense in microlocal analysis) is of the following form:

$$\sigma(P) = \sum_{j \geq 0} \eta^{-j} P_j(x, \xi/\eta), \tag{1.2.20}$$

where $\{P_j(x, \zeta)\}_{j \geq 0}$ are holomorphic in x in U and entire in ζ (actually polynomials of ζ in our current context where the target operator P is an mth order differential operator), and they satisfy the following growth order condition.

There exists a constant $C_0 > 0$ for which the following holds: for each compact set K in $U \times \mathbb{C}_\zeta$ we can find another constant M_K so that we have

$$\sup_K |P_j(x, \zeta)| \leq M_K j! C_0^j. \tag{1.2.21}$$

Remark 1.2.4 To avoid the possible confusion of the reader, we note that the above growth order condition is more restrictive than that used in [AKKoT]. Concerning the technicalities at this point we refer the reader to [AKT5, Remark C.1]. The main reason we have strengthened the condition is that the strengthened version enables us to obtain the concrete expression of a differential operator of WKB type as an integro-differential operator given in (1.1.84). We also note that the target operator P in [AKKoT] is more general than the operator we are discussing here; a linear differential operator of infinite order with a large parameter η such as $x - \cosh(\sqrt{\eta^{-1}d/dx}) = x - \sum_{n\geq 0}(\eta^{-1}d/dx)^n/(2n)!$ (cf. (1.1.85)) is also covered by [AKKoT, Theorem 5.1].

Theorem 1.2.1 ([AKKoT, Theorem 5.1]) *Let P be the differential operator with a large parameter η given in* (1.2.1)*, and assume that $x = a$ is a simple turning point of* (1.2.1) *that is of type* (j, k)*. Then in a sufficiently small neighborhood of $x = a$ we can find differential operators Q and R of WKB type which satisfy the following conditions:*

$$\eta^{-m}P = QR, \tag{1.2.22}$$

$$Q = \sum_{j\geq 0}\eta^{-j}Q_j(x, \eta^{-1}d/dx) \text{ and } R = \sum_{j\geq 0}\eta^{-j}R_j(x, \eta^{-1}d/dx)$$
$$\text{are differential operators respectively of order } (m-2) \text{ and of order } 2 \text{ in } d/dx, \tag{1.2.23}$$

$$Q_0(a, \zeta_j(a)) \neq 0, \tag{1.2.24}$$

$$R_0(x, \zeta) = (\zeta - \zeta_j(x))(\zeta - \zeta_k(x)), \tag{1.2.25}$$

where $Q_0(x, \zeta)$ (resp., $R_0(x, \zeta)$) denotes the principal symbol of the operator Q (resp., R), that is, $Q_0(x, \xi/\eta)$ (resp., $R_0(x, \xi/\eta)$) with ξ/η being denoted by ζ.

Let us first show how Theorem 1.2.1 enables us to clarify the local structure of WKB solutions of (1.2.1) near its simple turning point: we denote by T_j a solution of the Riccati equation associated with $R\varphi = 0$ whose top degree part is $\zeta_j(x)$, and let φ_j denote the WKB solution of $R\varphi = 0$ that is determined by T_j. Then it follows from (1.2.22) that

$$P\varphi_j = \eta^m QR\varphi_j = 0. \tag{1.2.26}$$

Hence φ_j coincides with ψ_j given by (1.2.5) up to a multiplicative constant. Therefore we find

$$T_j = S \tag{1.2.27}$$

for S satisfying (1.2.14). Thus the study of local structure of WKB solutions of (1.2.1) is reduced to the study of WKB solutions of

$$R\varphi = \left(\eta^{-2}\frac{d^2}{dx^2} + A(x,\eta)\eta^{-1}\frac{d}{dx} + B(x,\eta)\right)\varphi = 0, \tag{1.2.28}$$

where the top degree part of A and B are respectively given by $-(\zeta_j + \zeta_k)$ and $\zeta_j \zeta_k$. Although (1.2.28) is not of the form of equations studied in Sect. 1.1, basically the same reasoning as in Sect. 1.1 applies to the study of (1.2.28) as is briefly described below. We first note that, if we define

$$T_{\text{odd}} = \frac{1}{2}(T_j - T_k) \tag{1.2.29}$$

$$T_{\text{even}} = \frac{1}{2}(T_j + T_k), \tag{1.2.30}$$

where T_k is a solution of the Riccati equation whose top degree part is ζ_k, then we have

$$T_{\text{even}} = -\frac{1}{2T_{\text{odd}}}\frac{\partial T_{\text{odd}}}{\partial x} - \frac{1}{2}\eta A, \tag{1.2.31}$$

and we can obtain the well-normalized WKB solution φ_j and φ_k:

$$\varphi_j = \frac{1}{\sqrt{T_{\text{odd}}}} \exp\left(\int_a^x \left(T_{\text{odd}} - \frac{1}{2}\eta A\right) dx\right) \tag{1.2.32}$$

and

$$\varphi_k = \frac{1}{\sqrt{T_{\text{odd}}}} \exp\left(-\int_a^x \left(T_{\text{odd}} - \frac{1}{2}\eta A\right) dx\right). \tag{1.2.33}$$

Furthermore we can construct a formal coordinate transformation that transforms (1.2.28) to the Airy equation with the additional gauge transformation

$$\varphi \mapsto \left(\exp\left(\frac{1}{2}\int_a^x \eta A dx\right)\right)\varphi, \tag{1.2.34}$$

which eliminates the first order part in d/dx in (1.2.28). Hence, by using the well-normalized solutions φ_j and φ_k given above, we find the following structure theorem of singularities of Borel transformed WKB solutions $\varphi_{j,B}$ and $\varphi_{k,B}$. To state the theorem we introduce functions $y_j(x)$ and $y_k(x)$ which stand for respectively $-\int_a^x \zeta_j(x)dx$ and $-\int_a^x \zeta_k(x)dx$.

Theorem 1.2.2 *Let $x = a$ be a simple turning point of P of type (j, k). Then in a sufficiently small neighborhood ω of $(x, y) = (a, 0)$, we find the following properties on ω:*
(i) *$\varphi_{j,B}(x, y)$ and $\varphi_{k,B}(x, y)$ are singular only along $\Gamma_j \cup \Gamma_k$ outside $\{x = a\}$, where*

$$\Gamma_j = \{(x, y) \in \omega;\ y = y_j(x)\} \tag{1.2.35}$$

and

$$\Gamma_k = \{(x, y) \in \omega;\ \ y = y_k(x)\}. \tag{1.2.36}$$

(ii) *The singular part of* $\varphi_{j,B}(x, y)$ (*resp.,* $\varphi_{k,B}(x, y)$) *along* Γ_k (*resp.,* Γ_j) *coincides with* $\sqrt{-1}\varphi_{k,B}/2$ (*resp.,* $\sqrt{-1}\varphi_{j,B}/2$).

Theorem 1.2.2 shows that the geometric situation is basically the same as that observed for the Schrödinger equations, as far as x is confined to a neighborhood U of a simple turning point. To make full use of this we now posit the following Property [AC] as a guiding principle; it is a counterpart of Facts A and C observed for the Borel transformed WKB solutions of the Airy equation.

Property [AC] The Borel transformed WKB solutions $\varphi_{l,B}(x, y)$ $(l = 1, 2, \ldots, m)$ of (1.2.1) can be analytically continued endlessly, i.e., without encountering the natural boundaries. Further its growth order near $y = \infty$ is tame.

Remark 1.2.5 The above statement is too crude and more elaboration is certainly needed. For example, no statement corresponding to Fact B is included. Hence, in what follows, we content ourselves with using Property [AC] only as a guiding principle in our reasoning. At the same time we note that a good supporting evidence of Property [AC] is given when the integral representation of solutions of (1.2.1) is available (Appendix A.1).

Accepting the above posit, we can define, except for particular values of x, the Borel sum of $\varphi_j(x, \eta)$, and hence that of $\psi_j(x, \eta)$, by fixing the path γ_j of integration to define it:

$$\gamma_j = \{(x, y) \in U \times \mathbb{C};\ \ \mathrm{Im}\, y = \mathrm{Im}\, y_j(x), \mathrm{Re}\, y \geq \mathrm{Re}\, y_j(x)\}. \tag{1.2.37}$$

Barring the possible relevance of singularities other than $y = y_j(x)$ or $y = y_k(x)$ to the path γ_j, we find the Borel sum $\varphi_j(x, \eta)$ given by

$$\int_{\gamma_j} \exp(-y\eta)\varphi_{j,B}(x, y)dy \tag{1.2.38}$$

is well-defined unless $y_k(x)$ lies on γ_j. We clearly observe that the configuration of the path γ_j and the location of singularities of the integrand is the same as that in Fig. 1.1. Therefore we find that the analytic continuation of the Borel sum $\varphi_j(x, \eta)$ acquires or loses the exponentially small term $\sqrt{-1}\varphi_k(x, \eta)$ when x satisfies

$$\mathrm{Im}\, y_j(x) = \mathrm{Im}\, y_k(x), \quad \mathrm{Re}\, y_j(x) < \mathrm{Re}\, y_k(x), \tag{1.2.39}$$

that is, when x hits the Stokes curve of type $(j > k)$. Here we have used the fact that φ_j and φ_k are well-normalized as in (1.2.32) and (1.2.33). Thus we have confirmed that the WKB analysis of higher order equations "in the small" is essentially the same as the WKB analysis of the Schrödinger equation.

1.3 The Impact of the Work [BNR] of Berk, Nevins and Roberts

Berk, Nevins and Roberts published a decisively important paper [BNR] in 1982. In this paper they studied where the Stokes phenomena of WKB solutions of

$$\left(\frac{d^3}{dx^3} + 3\eta^2 \frac{d}{dx} + 2ix\eta^3\right)\psi = 0 \tag{1.3.1}$$

are observed. Parenthetically we note that Eq. (1.3.1) is now called **BNR equation** after Berk, Nevins and Roberts. Their discovery can be summarized as follows:

The totality of Stokes curves is not enough to describe the Stokes phenomena in the large but the addition of some new Stokes curves is necessary for the concrete description.

Let us briefly explain their reasoning in what follows.

One can readily confirm that $x = \pm 1$ are the simple turning points of (1.3.1), and with the appropriately labelled solutions $\zeta_j(x)$ of the equation

$$\zeta^3 + 3\zeta + 2ix = 0 \tag{1.3.2}$$

we find its Stokes curves as in Fig. 1.4. The wiggly lines designate the cuts to fix the branches of $\zeta_j(x)$. Here we observe two crossing points C_1 and C_2 of Stokes curves. It is clear that such crossing does not appear in the study of equations of the second order.

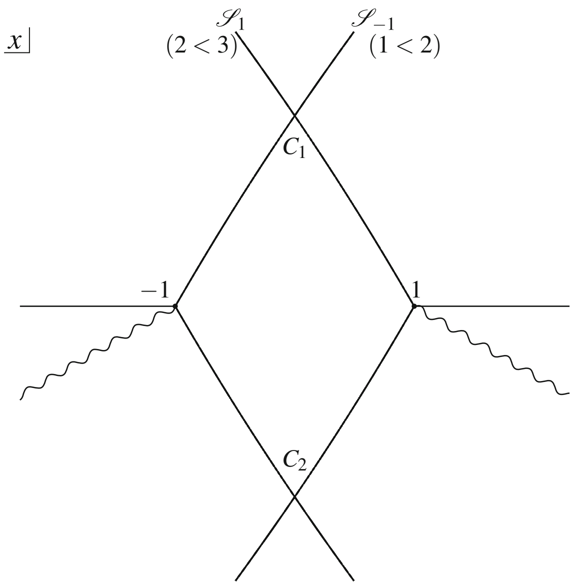


Fig. 1.4 The Stokes curves for BNR equation

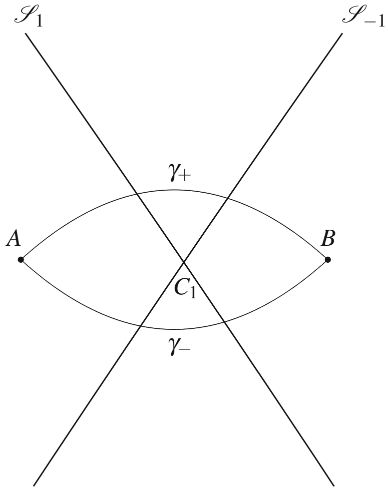


Fig. 1.5 Paths of analytic continuation near C_1

Let us consider the problem near C_1 and fix the notations as in Fig. 1.5. Here γ_+ and γ_- are paths of continuation from point A to point B as are shown there. Assuming the Borel summability of WKB solutions we let α and β respectively denote the Stokes multiplier, or what we call the **connection constant**, across Stokes curves $\mathscr{S}_1$ and $\mathscr{S}_{-1}$; that is,

$$\psi_3 \mapsto \psi_3 + \alpha\psi_2 \quad \text{across } \mathscr{S}_1, \tag{1.3.3}$$

$$\psi_2 \mapsto \psi_2 + \beta\psi_1 \quad \text{across } \mathscr{S}_{-1}. \tag{1.3.4}$$

Hence by the analytic continuation of ψ_3 along γ_+ we obtain

$$\psi_3 + \alpha(\psi_2 + \beta\psi_1) \tag{1.3.5}$$

near B, whereas, by the analytic continuation along γ_-, we obtain

$$\psi_3 + \alpha\psi_2 \tag{1.3.6}$$

near B. Since the Eq. (1.3.1) does not have any singularity near C_1, this is a contradiction if $\alpha\beta \neq 0$. To resolve this paradoxical problem [BNR] proposed to introduce a **new Stokes curve** $\mathscr{S}_{C_1}$ of type $(1 < 3)$ that emanates from C_1 with the connection coefficient $-\alpha\beta$ attached, after making a detailed study of the structure of solutions of (1.3.1) represented in the integral form by the method of the steepest descent. (Cf. Appendix A.1.) Their proposal is summarized as in Fig. 1.6.

Then we can readily find that the paradoxical problem observed in Fig. 1.5 is cleanly resolved in Fig. 1.6.

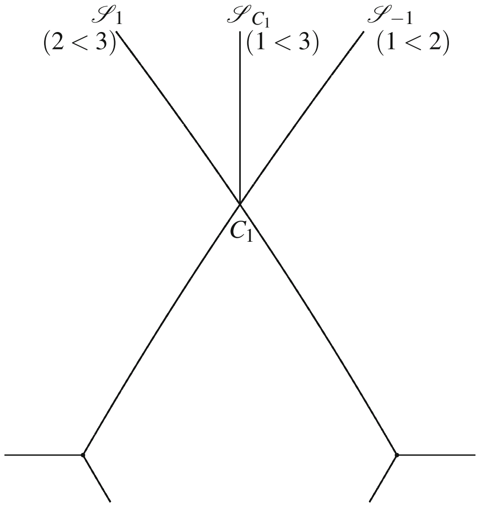


Fig. 1.6 Introduction of the new Stokes curve $\mathscr{S}_{C_1}$

1.4 A Virtual Turning Point—a Gift of Microlocal Analysis to the Exact WKB Analysis

As is explained in Sect. 1.1, the notion of Stokes curves in the exact WKB analysis is determined by the relative location of the singularities of the Borel transformed WKB solutions and the path of integration that determines the Borel sum, which is dependent on the choice of arg η (cf. Remark 1.1.1(i)). Hence a crossing point of Stokes curves is highly dependent on the way how the Borel resummation is performed. Therefore we should be happier if we could find something more intrinsically related to the operator P in (1.2.1) as the starting point of a new Stokes curve.

To think over this problem, let us first recall the situation discussed in Sect. 1.1. The Stokes phenomena for WKB solutions of the Schrödinger equation are observed through the interplay of two "cognate" singularities of the Borel transformed WKB solutions, which merge at a turning point. Here "cognate singularities" mean that they are mutually tied up with, as is observed in Fact B for the Airy equation and shown to be so in general by Theorem 1.1.4. To find out analogous "cognate singularities" in the Borel transform ψ_B of WKB solution ψ of the Eq. (1.2.1), we employ the following basic fact in microlocal analysis [SKK].

Fact D: The most elementary carrier of the singularities of solutions of the equation $P_B u = 0$, i.e., the Borel transform of (1.2.1), is a bicharacteristic strip, if the operator P_B is with simple characteristics. Further P_B is with simple characteristics near a simple turning point as is noted in Remark 1.4.3. For the convenience of the reader let us recall the definition of a **bicharacteristic strip**.

Definition 1.4.1 A bicharacteristic strip associated with the operator $P_B(x, \partial/\partial x, \partial/\partial y)$ is, by definition, a curve $\{(x(t), y(t); \xi(t), \eta(t))\}$ in the cotangent bundle $T^*\mathbb{C}^2_{(x,y)}$ that is determined by the following Hamilton-Jacobi equation:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial\sigma}{\partial\xi} & (1.4.1.a) \\ \dfrac{dy}{dt} = \dfrac{\partial\sigma}{\partial\eta} & (1.4.1.b) \\ \dfrac{d\xi}{dt} = -\dfrac{\partial\sigma}{\partial x} & (1.4.1.c) \\ \dfrac{d\eta}{dt} = -\dfrac{\partial\sigma}{\partial y} & (1.4.1.d) \\ \sigma(x, \xi, \eta) = 0 & (1.4.1.e), \end{cases} \tag{1.4.1}$$

where σ denotes the symbol of the operator $P_B(x, \partial/\partial x, \partial/\partial y)$, i.e., $P_B(x, \xi, \eta)$.

Remark 1.4.1 As σ is free from y, (1.4.1.d) implies η is a constant, which we later choose to be 1.

Remark 1.4.2 When $P_B = \partial^2/\partial x^2 - x\partial^2/\partial y^2$, the bicharacteristic strip b of P_B that emanates from $(x, y;\ \xi, \eta) = (0, 0;\ 0, 1)$ is given by

$$(x(t), y(t);\ \xi(t), \eta(t)) = (t^2, -2t^3/3;\ t, 1). \tag{1.4.2}$$

Hence the projection $\pi(b)$ of b to the base manifold $\mathbb{C}^2_{(x,y)}$ is given by

$$\{(x, y) \in \mathbb{C}^2;\ y = y_-(x)\} \cup \{(x, y) \in \mathbb{C}^2;\ y = y_+(x)\}, \tag{1.4.3}$$

where $y_\pm(x) = \pm\int_0^x \sqrt{x}\, dx$ given by (1.1.14) with $x_* = 0$. Thus we find that the "cognate singularities" in the Borel transformed WKB solutions for the Airy equation, which are described in Fact B, are the two portions of one curve that is a projection of a non-singular curve in $T^*\mathbb{C}^2$, which is a carrier of singularities of solutions of the equation $P_B u = 0$. We also find that the turning point is nothing but a kink in the projection of a non-singular curve. We note that the geometric situation for a general Schrödinger equation with a simple turning point $x = a$ is basically the same, at least near $x = a$, as the above statement for the Airy equation; the projected bicharacteristic strip (often referred to as a **bicharacteristic curve** in classical analysis) is contained in

$$\{(x, y);\ y = \pm\int_a^x \sqrt{Q}\, dx\}, \tag{1.4.4}$$

as (1.4.1.a) and (1.4.1.b) entail

$$\left(\frac{dy}{dx}\right)^2 = Q(x). \tag{1.4.5}$$

(See (1.4.21) below, which gives a precise version of (1.4.5) for the higher order operator P.)

Remark 1.4.3 It follows from Definition 1.2.3

$$\frac{\partial \sigma}{\partial x}\Big|_{x=a,\xi=\xi_j(a),\eta=1} \neq 0 \tag{1.4.6}$$

at a simple turning point $x = a$ of type (j, k). Hence (1.4.1.c) guarantees that the bicharacteristic strip emanating from $(x, y;\ \xi, \eta) = (a, 0;\ \xi_j(a), 1)$ is locally non-singular in $T^*\mathbb{C}^2$.

Thus we have seen that a turning point is a singular point of the projection to the base manifold $\mathbb{C}^2$ of a non-singular curve in $T^*\mathbb{C}^2$, which is the most elementary carrier of singularities of solutions, that is, a bicharacteristic strip of P_B. Otherwise stated, a turning point appears as a confluent point of the loci of two cognate singularities—or rather, one singularity in their origin whose projection only looks like having two confluent components.

Then a natural question is:

Are there any other similar singular points in a bicharacteristic curve?

Fortunately the answer is:

Yes!

The most basic one among such singularities is a self-intersection point of a bicharacteristic curve, to which two distinct points in a non-singular bicharacteristic strip correspond in general. As we explain below, the x-component of such a point plays a similar role in the exact WKB analysis as the turning point defined in Definition 1.2.1 (i); in particular, a new Stokes curve found by [BNR] is a portion of a Stokes curve emanating from such a point. We coin the term a "**virtual turning point**" to designate the x-component of a self-intersection point of a bicharacteristic curve. We note a virtual turning point was called a "new turning point" when it was first found in [AKT2].

Since the notion of a virtual turning point does not find any similar—or even related—precedents in the traditional asymptotic analysis, we first illustrate the situation concretely for BNR equation. Although it may be redundant from the logical viewpoint, we believe it to be useful for the reader to grasp the characteristic feature of this novel notion.

Example 1.4.1 Let P_B stand for the Borel transformed BNR operator, that is,

$$P_B = \frac{\partial^3}{\partial x^3} + 3\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2} + 2ix\frac{\partial^3}{\partial y^3}. \tag{1.4.7}$$

Then its symbol σ is

$$\sigma = \xi^3 + 3\xi\eta^2 + 2ix\eta^3. \tag{1.4.8}$$

Let us now study the global behaviour of the bicharacteristic strip $b_{(x_0,\xi_0)}$ that emanates from

$$(x, y;\ \xi, \eta) = (x_0, 0;\ \xi_0, 1) \tag{1.4.9}$$

with

$$x_0 = 1, \xi_0 = -i. \tag{1.4.10}$$

Then (1.4.1.e) is satisfied, and we can find that a global solution $(x(t), y(t);\ \xi(t), \eta(t))$ of (1.4.1) satisfying (1.4.9) is given as follows:

$$\begin{cases} x(t) = -4(t + 1/2)(t^2 + t - 1/2) \\ y(t) = -6it^2(t + 1)^2 \\ \xi(t) = -2it - i \\ \eta(t) = 1. \end{cases} \tag{1.4.11}$$

Then a straightforward computation shows that the relation

$$x(t) = x(t'),\ y(t) = y(t'),\ t \neq t' \tag{1.4.12}$$

entails

$$t^2 + t = t'^2 + t' = 1/2\,; \tag{1.4.13}$$

hence the self-intersection point of the bicharacteristic curve, i.e., the projection of $b_{(x_0,\xi_0)}$, is given by

$$x = 0,\ y = -3i/2, \tag{1.4.14}$$

as is shown in Fig. 1.7.

Therefore it follows from the definition that $x = 0$ is the virtual turning point of BNR equation. Further, by using the same labelling of the characteristic roots $\{\xi_j\}$ as in Sect. 1.3, we find

$$\operatorname{Im} \int_0^x (\xi_1 - \xi_3)dx = 0 \tag{1.4.15}$$

passes through C_1 and C_2. (Cf. Proposition 1.4.1.) Otherwise stated, a Stokes curve emanating from the virtual turning point $x = 0$ gives the new Stokes curves that [BNR] proposed to introduce. Moreover, we can confirm that no Stokes phenomena are observed in a neighborhood of the virtual turning point, as we show in Proposition 1.4.2. To emphasize this fact we designate the segment C_1C_2 of the Stokes curve by a dotted line. Thus we obtain the following Fig. 1.8 to describe the Stokes geometry of BNR equation, i.e., the location of the turning points, both ordinary and

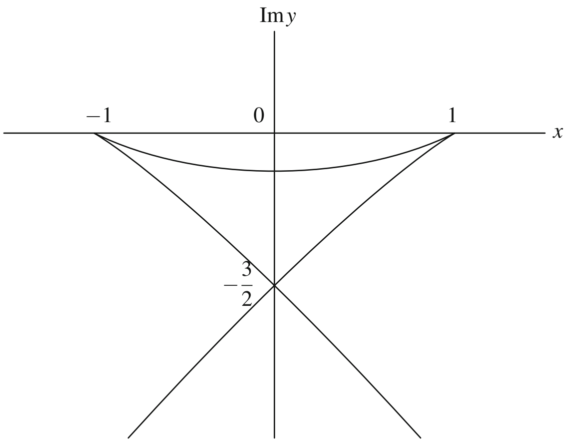


Fig. 1.7 The self-intersection point of the bicharacteristic curve in question

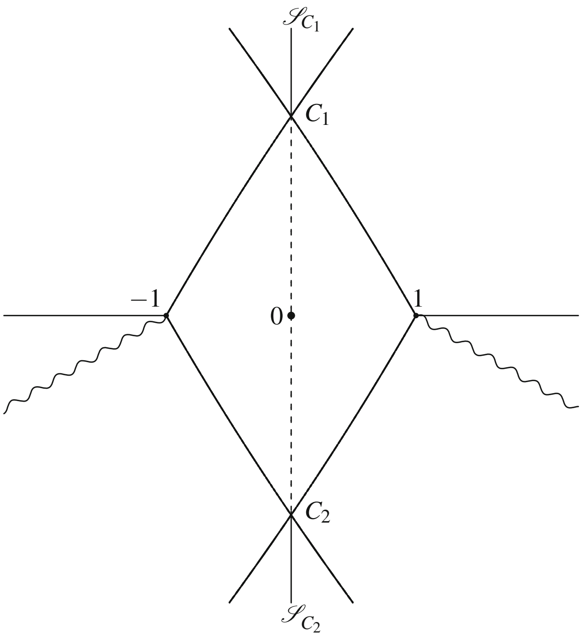


Fig. 1.8 The complete Stokes geometry for BNR equation

virtual, and the Stokes curves emanating, either from an ordinary turning point or from a virtual turning point.

Before proceeding further, we show some basic properties concerning a Stokes curve that emanates from a virtual turning point. To state the results, we prepare some notations.

Let P_B denote the Borel transform of the operator P in (1.2.1), and assume that its symbol σ has the form

$$\prod_{r=1}^{m}(\xi - \zeta_r(x)\eta), \tag{1.4.16}$$

where $\zeta_r(x)$ is defined on an appropriately cut plane. The suffix r of a characteristic root $\zeta_r(x)$ is used to designate the type of a turning point and a Stokes curve in what follows. We also denote the characteristic variety of P_B by V, i.e.,

$$V = \{(x, y;\ \xi, \eta) \in T^*\mathbb{C}^2;\ \sigma(x, \xi, \eta) = 0,\ \eta \neq 0\}. \tag{1.4.17}$$

Then we find the following

Proposition 1.4.1 *Let s_1 (resp., s_2) be a simple turning point of equation (1.2.1) of type (j, k) (resp., (k, l)). Let $\mathscr{S}_1$ (resp., $\mathscr{S}_2$) designate the Stokes curve that emanates from s_1 and with type (j, k) (resp., from s_2 with type (k, l)), and suppose that $\mathscr{S}_1$ and $\mathscr{S}_2$ cross at a point C. Assume further that x_* satisfies the relation (1.4.18) below for a triplet of mutually distinct suffixes (j, k, l):*

$$\int_{s_1}^{x_*} \zeta_j dx = \int_{s_1}^{s_2} \zeta_k dx + \int_{s_2}^{x_*} \zeta_l dx. \tag{1.4.18}$$

Then x_* is a virtual turning point of (1.2.1), and the Stokes curve $\mathscr{S}$ emanating from x_* with type (j, l) passes through the point C.

Proof Let us first consider the part V_i of the characteristic variety V that is defined by $\{\xi = \zeta_i(x)\eta\}$. Then we find

$$\left.\frac{\partial\sigma}{\partial\xi}\right|_{V_i} = \Big(\prod_{r\neq i}(\zeta_i - \zeta_r)\Big)\eta^{m-1} \tag{1.4.19}$$

and

$$\left.\frac{\partial\sigma}{\partial\eta}\right|_{V_i} = -\zeta_i\Big(\prod_{r\neq i}(\zeta_i - \zeta_r)\Big)\eta^{m-1}. \tag{1.4.20}$$

Hence it follows from (1.4.1.a) and (1.4.1.b) that

$$\left.\frac{dy}{dx}\right|_{V_i} = -\zeta_i(x). \tag{1.4.21}$$

Therefore the bicharacteristic curves passing through $(x, y) = (s_1, 0)$ are given either by

$$y = -\int_{s_1}^{x} \zeta_j(x)dx \tag{1.4.22}$$

or by

$$y = -\int_{s_1}^{x} \zeta_k(x)dx. \tag{1.4.23}$$

We note that the union of these curves is, in a neighborhood of $(x, y) = (s_1, 0)$, the projection of a bicharacteristic strip of P_B passing through

$$B_1 \underset{\text{def}}{=} \{(x, y;\ \xi, \eta) = \big(s_1, 0;\ \zeta_j(s_1)\eta\ (= \zeta_k(s_1)\eta), \eta\big)\}, \tag{1.4.24}$$

which is a non-singular curve in $T^*\mathbb{C}^2_{(x,y)}$. We further extend the bicharacteristic strip so that it may pass through

$$B_2 \underset{\text{def}}{=} \left\{(x, y;\ \xi, \eta) = \left(s_2, -\int_{s_1}^{s_2} \zeta_k(x)dx;\ \zeta_k(s_2)\eta\ (= \zeta_l(s_2)\eta), \eta\right)\right\}. \tag{1.4.25}$$

It is clear that such an extension is possible because s_2 is a simple turning point. Then, after passing through B_2, the bicharacteristic strip is described by

$$(x, y;\ \xi, \eta) = \big(x, -\int_{s_1}^{s_2} \zeta_k dx - \int_{s_2}^{x} \zeta_l dx;\ \zeta_l(x)\eta, \eta\big). \tag{1.4.26}$$

Thus its projection crosses the curve (1.4.22) at (x_*, y_*) if x_* satisfies (1.4.18) and y_* is given by

$$y_* = -\int_{s_1}^{x_*} \zeta_j(x)dx. \tag{1.4.27}$$

Hence it follows from the definition that $x = x_*$ is a virtual turning point of (1.2.1).

Let us next confirm that the Stokes curve emanating from $x = x_*$ with type (j, l) passes through the point C. For this purpose let us consider the following integral I:

$$I = \int_{s_1}^{x_*} (\zeta_j - \zeta_k)dx. \tag{1.4.28}$$

Then it follows from the definition of x_* that we find

$$\begin{aligned} I &= \int_{s_1}^{s_2} \zeta_k dx + \int_{s_2}^{x_*} \zeta_l dx - \int_{s_1}^{x_*} \zeta_k dx \\ &= -\int_{s_2}^{x_*} \zeta_k dx + \int_{s_2}^{x_*} \zeta_l dx. \end{aligned} \tag{1.4.29}$$

We further rewrite I as

$$\int_{s_1}^{C} (\zeta_j - \zeta_k)dx + \int_{C}^{x_*} (\zeta_j - \zeta_k)dx. \tag{1.4.30}$$

It follows from (1.4.29) that it can be also written as

$$\int_{s_2}^{C} (\zeta_l - \zeta_k)dx + \int_{C}^{x_*} (\zeta_l - \zeta_k)dx. \tag{1.4.31}$$

Combining (1.4.30) and (1.4.31) we obtain

$$\int_{x_*}^{C} (\zeta_j - \zeta_l)dx = \int_{s_1}^{C} (\zeta_j - \zeta_k)dx + \int_{s_2}^{C} (\zeta_k - \zeta_l)dx. \tag{1.4.32}$$

On the other hand, C is a crossing point of Stokes curves $\mathscr{S}_1$ and $\mathscr{S}_2$; hence we have

$$\operatorname{Im} \int_{s_1}^{C} (\zeta_j - \zeta_k)dx = \operatorname{Im} \int_{s_2}^{C} (\zeta_k - \zeta_l)dx = 0. \tag{1.4.33}$$

Therefore (1.4.32) entails

$$\operatorname{Im} \int_{x_*}^{C} (\zeta_j - \zeta_l)dx = 0. \tag{1.4.34}$$

Thus we have confirmed the required result.

Remark 1.4.4 Using the labelling of the characteristic roots $\{\zeta_r\}_{r=1}^{m}$ given by (1.4.16), we say the point x_* in (1.4.18) is a **virtual turning point of type** (j, l).

Remark 1.4.5 As $\zeta_r(x)$ does not depend on η, it is clear that the notion of a virtual turning point is independent of η.

Remark 1.4.6 The geometric situation in Proposition 1.4.1 is in exceptional harmony. In order to show this point, let us suppose the operator P has another simple turning point $\tilde{s}_1$ $(\neq s_1)$ of type (j, k). Then by tracing the bicharacteristic strip, we find

$$\int_{s_1}^{\tilde{x}_*} \zeta_j dx = \int_{s_1}^{\tilde{s}_1} \zeta_k dx + \int_{\tilde{s}_1}^{s_1} \zeta_j dx + \int_{s_1}^{s_2} \zeta_k dx + \int_{s_2}^{\tilde{x}_*} \zeta_l dx \tag{1.4.35}$$

will give another virtual turning point $\tilde{x}_*$. But, barring the case

$$\operatorname{Im} \int_{s_1}^{\tilde{s}_1} (\zeta_j - \zeta_k)dx = 0, \tag{1.4.36}$$

we cannot expect the relation (1.4.34).

Remark 1.4.7 The condition (1.4.36) indicates its relevance to the analysis of the so-called fixed singularities of WKB solutions of the Schrödinger equation, i.e., the study of the periodic structure of the singularities of Borel transformed WKB solutions. (Cf. [AKT5] and references cited therein.) Here we content ourselves only with noting that we can formulate the periodic structure of singularities of Borel transformed WKB solution for higher order equations also, by using the notion of a bicharacteristic strip.

Let ϖ denote the period integral $\oint_\gamma \zeta dx$ for a closed path γ in the Riemann surface associated with $\tilde{p}(x, \zeta) = 0$. (Parenthetically we note that the Riemann surface associated with BNR equation is simply connected, i.e., ϖ vanishes then.) In this setting it is known [AKT2, (2.4)] that there exists a constant c for which

$$\varpi = \int_0^c \xi(t) \frac{dx(t)}{dt} dt \tag{1.4.37}$$

holds for a bicharacteristic strip $\{(x(t), y(t);\ \xi(t), 1)\}$ which emanates from $(x(0), 0;$ $\xi(0), 1)$ with $x(0)$ being a turning point and with $\xi(0)$ being the corresponding double root of $\sigma = 0$. It then follows from (1.4.37) and the definition of a bicharacteristic strip that we find

$$\begin{aligned} \varpi &= \int_0^c \xi(t) \frac{\partial \sigma}{\partial \xi} dt \\ &= \int_0^c \left(-\eta \frac{\partial \sigma}{\partial \eta} \right) dt \\ &= -\int_0^c \frac{dy(t)}{dt} dt = -y(c). \end{aligned} \tag{1.4.38}$$

Hence in what follows we assume

$$\operatorname{Im} \varpi \neq 0, \tag{1.4.39}$$

which is a generalized version of (1.4.36).

Now, by using the reasoning in Sect. 1.3 with the usual assumption of Borel summability of WKB solutions, we find the following Proposition 1.4.2. It explains the psychological background of coining the name "**virtual turning point**"; we cannot detect it by the local study of Stokes phenomena.

Proposition 1.4.2 *Let x_* be a virtual turning point of type (j, l) which is described in Proposition 1.4.1. Assume that no other (ordinary or virtual) turning point coincides with x_*. Then no Stokes phenomena are observed for WKB solutions ψ_j and ψ_l on the Stokes curve*

$$\operatorname{Im} \int_{x_*}^x (\zeta_j - \zeta_l) dx = 0, \tag{1.4.40}$$

if x is sufficiently close to x_*.

Proof Let $y_j(x)$ (resp., $y_l(x)$) stand for $-\int_{s_1}^{x}\zeta_j(x)dx$ (resp., $-\int_{s_2}^{x}\zeta_l(x)-\int_{s_1}^{s_2}\zeta_k(x)$). Since x_* is supposed not to be coincident with a turning point, we find

$$\frac{dy_j}{dx}=-\zeta_j(x)\neq-\zeta_l(x)=\frac{dy_l}{dx} \tag{1.4.41}$$

holds at $x=x_*$. Hence $\mathrm{Re}(y_j(x)-y_l(x))$ changes its sign when x passes through x_* in moving in the curve given by (1.4.40). (Note that $y_j(x_*)=y_l(x_*)$ holds thanks to (1.4.18).) Then the analytic continuation of $\psi_j(x)$ across the portion of the curve (1.4.40) where $\mathrm{Re}\, y_j(x)>\mathrm{Re}\, y_l(x)$ remains intact, whereas that across the portion where $\mathrm{Re}\, y_l(x)>\mathrm{Re}\, y_j(x)$ assumes the form $\psi_j+\gamma\psi_l$ for some constant γ. Since x_* is a non-singular point of Eq. (1.2.1), γ should be 0.

Remark 1.4.8 Let us now investigate the relevance of Propositions 1.4.1 and 1.4.2 from the viewpoint of the relative location of singularities of Borel transformed WKB solutions. Let us consider the problem in the situation described in Proposition 1.4.1, and suppose that no turning point of (1.2.1) lies on the portion $(x_*, C]$ in the Stokes curve emanating from x_*. In order to fix the situation we assume

$$\mathrm{Re}\, y_j(x)<\mathrm{Re}\, y_l(x) \tag{1.4.42}$$

holds on $(x_*, C]$, where $y_j(x)$ and $y_l(x)$ are the functions given in the proof of Proposition 1.4.2. As is shown by Proposition 1.4.2, the singular point $y=y_l(x)$ does not cause any Stokes phenomena even when x lies in (x_*, C) despite the fact that it lies on the path of integration defining the Borel sum $\psi_j(x,\eta)$. However, when x reaches the point C, the situation suddenly changes in general, as another singular point $y=y_k(x)=-\int_{s_1}^{x}\zeta_k dx$ $\left(=-\int_{s_2}^{x}\zeta_k dx-\int_{s_1}^{s_2}\zeta_k dx\right)$ may intervene between $y_j(x)$ and $y_l(x)$ on $\{\mathrm{Im}\,(y_j-y_l)=0\}$ as is shown in Fig. 1.9, depending on the ordering of $\{\mathrm{Re}\, y_j, \mathrm{Re}\, y_k, \mathrm{Re}\, y_l\}$. Let us first consider the case where

$$\mathrm{Re}\, y_j(x)<\mathrm{Re}\, y_k(x)<\mathrm{Re}\, y_l(x) \tag{1.4.43}$$

holds at C. Since the situation along the Stokes curves $\mathscr{S}_1$ and $\mathscr{S}_2$ are the same as is observed in Fig. 1.1, when x reaches the point C the singularity $y=y_l(x)$ causes a Stokes phenomenon for $\psi_k(x,\eta)$ and the singularity $y=y_k(x)$ causes a Stokes phenomenon for $\psi_j(x,\eta)$. Otherwise stated, $y=y_l(x)$ causes a Stokes phenomenon for $\psi_j(x,\eta)$ via the intermediator $\psi_k(x,\eta)$. Hence the Stokes curve (1.4.40), which

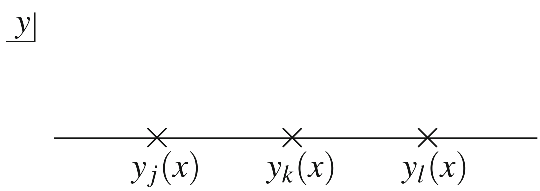


Fig. 1.9 The location of $\{y_j(x), y_k(x), y_l(x)\}$ at $x=C$

is "**inert**" on the portion (x_*, C), turns out to be "**active**" after it passes over C. This is the mechanism how the effect of a Stokes curve emanating from a virtual turning point becomes visible. Next we consider the case where

$$\operatorname{Re} y_j(x) < \operatorname{Re} y_l(x) < \operatorname{Re} y_k(x) \tag{1.4.44}$$

holds at $x = C$. In this case $y = y_k(x)$ may cause a Stokes phenomenon for $\psi_l(x, \eta)$, but that is irrelevant to $\psi_j(x, \eta)$. Hence the Stokes curve (1.4.41) remains inert after it passes over C.

Comparing (1.4.43) and (1.4.44) we are led to the following

Definition 1.4.2 Let a Stokes curve $\mathscr{S}_1$ of type (j, k) and a Stokes curve $\mathscr{S}_2$ of type (k', l) cross at a point C. Suppose either

$$k = k' \quad \text{and} \quad j > k > l \tag{1.4.45}$$

or

$$k = k' \quad \text{and} \quad j < k < l \tag{1.4.46}$$

holds. Then, following [BNR], we say C is an **ordered crossing point**. Even $k = k'$, if k is not sandwiched between j and l, we say C is a non-ordered crossing point. We also say C is a non-ordered crossing point when (j, k, k', l) are mutually distinct (and, in particular, $k \neq k'$).

In describing the Stokes geometry involving virtual turning points, we usually employ the following convention.

Definition 1.4.3 When no Stokes phenomena are observed near a point x_0 in a Stokes curve, we say that the Stokes curve is **inert** near x_0 and we describe the Stokes curve near x_0 by a dotted line.

Now, having Remark 1.4.8 in mind, we summarize Propositions 1.4.1 and 1.4.2 in an explicit manner as follows:

Proposition 1.4.3 *Let us consider the situation discussed in Propositions 1.4.1 and 1.4.2, and assume*

$$j > k > l. \tag{1.4.47}$$

Let α, β and γ respectively denote the connection coefficient attached to each Stokes curve near the ordered crossing point C. Suppose that their signs are chosen so that they are consistent with the directions of the paths in Fig. 1.10 along which WKB solutions are analytically continued. Then we find

$$\alpha\beta + \gamma = 0. \tag{1.4.48}$$

The proof is given by the same reasoning as in Sect. 1.3.

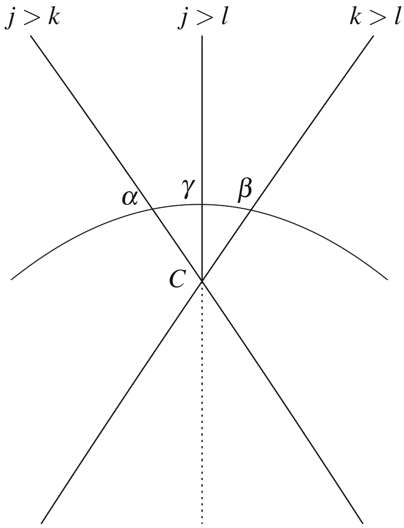


Fig. 1.10 The path of analytic continuation near the ordered crossing point C

Remark 1.4.9 The reasoning in Remark 1.4.8 indicates that the Stokes curve of type (j, l) emanating from the virtual turning point $\tilde{x}_*$ in Remark 1.4.6 remains inert in a domain where the topological configuration of Stokes curves in Proposition 1.4.1 is the only relevant one. Actually it is often the case that a Stokes curve emanating from a virtual turning point is inert at any point in the curve. In such a case we say that the virtual turning point is **redundant**. The simplest example is given by a virtual turning point when the operator P is of the second order [AKT2, Example 2.4], as there is no-crossing point of Stokes curves. Probably this is the reason why the importance of virtual turning points has not been recognized in the traditional asymptotic analysis.

1.5 The Relevance of Virtual Turning Points and the Connection Formula for WKB Solutions of a Higher Order Differential Equation

In this section we summarize the observations made in Sect. 1.4 as a recipe for finding the **Stokes geometry** for the Eq. (1.2.1), that is, the figure that describes the location of turning points, both ordinary and virtual, and Stokes curves emanating from them. We use a dotted line to emphasize that the part of the Stokes curve in question is inert. In what follows we assume that all ordinary turning points are simple. We also assume (1.4.36) does not hold, that is, we assume

$$\operatorname{Im} \int_{s_1}^{s_2} (\zeta_j - \zeta_k) dx \neq 0 \tag{1.5.1}$$

for any pair of turning points (s_1, s_2) of the same type, say (j, k). We note that this assumption is a counterpart of the assumption (1.1.90) in Theorem 1.1.5.

Recipe 1.5.1 We describe the Stokes geometry of Eq. (1.2.1) by the following procedures:
(R.i) Draw all Stokes curves that emanate from ordinary turning points.
(R.ii) Draw the Stokes curve that emanates from a virtual turning point.
(R.iii) The Stokes curve in (R.ii) is drawn by a dotted line until it hits an ordered crossing point.
(R.iv) When the Stokes curve in (R.ii) is of type $(j > l)$ and it hits an ordered crossing point C that is formed by a Stokes curve of type $(j > k)$ and a Stokes curve of type $(k > l)$, we use a solid line to draw the portion of the Stokes curve in (R.ii) after passing over C.
(R.v) If three Stokes curves of type $(j > k)$, type $(k > l)$ and type $(j > l)$ meet at a point C, then we stipulate that the connection coefficients attached to them near C should satisfy the following:
(R.v.a) The connection constant α (resp., β) attached to the Stokes curve of type $(j > k)$ (resp., of type $(k > l)$) are kept intact when the Stokes curves pass through the point C.
(R.v.b) The connection constant γ (resp., γ') attached to the portion of the Stokes curve of type $(j > l)$ before hitting the point C (resp., after passing over the point C) may be different.
(R.v.c) The constants $(\alpha, \beta, \gamma, \gamma')$ should satisfy

$$\alpha\beta + \gamma = \gamma', \tag{1.5.2}$$

when they are assigned to each portion of the Stokes curves in accordance with the directions (designated by the arrows) of analytic continuation of WKB solutions, which are described in Fig. 1.11. We note that the relative location of Stokes curves in question near C is either (a) or (b) given there.

Remark 1.5.1 If in Fig. 1.11b the portion of the Stokes curve to which γ' is attached contains a virtual turning point, then (1.5.2) reduces to (1.4.48).

Remark 1.5.2 In Recipe 1.5.1 (R.v) the change of the connection coefficient is observed in a neighborhood of an ordered crossing point concerning those attached to a Stokes curve of **non-adjacent type** $(j > l)$; this is consistent with Remark 1.4.8.

Remark 1.5.3 In Recipe 1.5.1 (R.v) the three Stokes curves meeting at C may be either Stokes curves emanating from ordinary turning points or the solid line part of

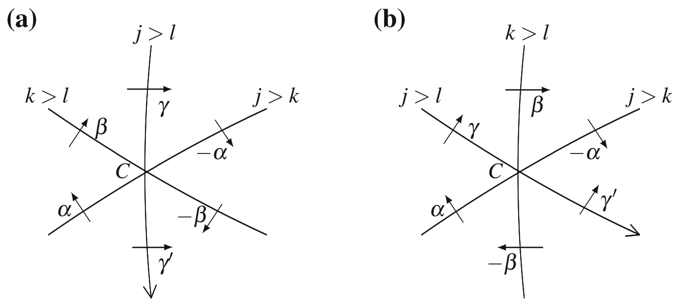


Fig. 1.11 Two possibilities of the relative location of Stokes curves

Stokes curves emanating from virtual turning points. See [AKoT, Sect. 2] for some illuminating example, which shows some portion of a Stokes curve emanating from an ordinary turning point may turn out to be inert.

Remark 1.5.4 Practically speaking, we can normally find the global structure of WKB solutions by Recipe 1.5.1; however, no rigorous algorithm for drawing the complete Stokes geometry is yet available. See [H3, ShI] for some details. We will come back to this point in Chap. 2.

1.6 How to Locate a Virtual Turning Point with the Help of a Computer

In order to put Recipe 1.5.1 into practice we present a method of locating a virtual turning point with the help of a computer. We consider the situation in (R.v), supposing that the Stokes curve of type $(j > k)$ (resp., $(k > l)$) emanates from an ordinary turning point s_1 (resp., s_2). Assume further that neither of these Stokes curves crosses any cut introduced in Sect. 1.2 to make the solutions $\zeta_j(x)$ $(j = 1, 2, \ldots, m)$ of the Eq. (1.2.13), i.e., $\tilde{p}(x, \zeta) = 0$, to be single-valued on the cut plane. Assuming all these, we try to reverse the reasoning in the proof of Proposition 1.4.1. For this purpose we evaluate

$$\rho(x) \underset{\text{def}}{=} \mathrm{Re} \int_x^C (\zeta_j(x) - \zeta_l(x))dx \tag{1.6.1}$$

along

$$\mathrm{Im} \int_x^C (\zeta_j(x) - \zeta_l(x))dx = 0. \tag{1.6.2}$$

Since $\rho(x)$ is the real part of a holomorphic function, it monotonically decreases or increases. On the other hand, it follows from the definition of the point C that

$$\mathrm{Im} \int_{s_1}^{C} (\zeta_j(x) - \zeta_k(x))dx = \mathrm{Im} \int_{s_2}^{C} (\zeta_k(x) - \zeta_l(x))dx = 0. \tag{1.6.3}$$

Hence we normally find a point x_* in the curve (1.6.2) where

$$\rho(x_*) = \int_{s_1}^{C} (\zeta_j(x) - \zeta_k(x))dx + \int_{s_2}^{C} (\zeta_k(x) - \zeta_l(x))dx \tag{1.6.4}$$

holds. Then, by comparing (1.6.4) with (1.4.32), we find that the point x_* thus found is a virtual turning point of type (j, l).

When some cuts cross the Stokes curves in question, we should take their effect into account and the reasoning becomes more involved. But the core idea in the computation is described by the above reasoning.

1.7 The Relevance of a Virtual Turning Point to the Bifurcation Phenomena of Stokes Curves

In Sect. 1.5 we have shown how a virtual turning point is relevant to the connection formula for WKB solutions. In this section we present a result which manifests the relevance of a virtual turning point to a geometric problem. The example discussed is the most “elementary” part of the phenomena to be discussed in Chap. 2.

The first example we study here is the Stokes geometry of BNR equation (1.3.1); so far we have assumed the large parameter η to be positive, but we now change $\arg \eta$; we study the Stokes geometry when $\arg \eta$ is close to $\pi/2$. The virtual turning point, together with ordinary turning points, remains unchanged, but Stokes curves move as $\arg \eta$ changes; we show their concrete configuration in Fig. 1.12. We then observe an interchange of the relative location of a Stokes curve emanating from a virtual turning point and that emanating from an ordinary turning point before and after the bifurcation of a Stokes curve occurs (cf. Fig. 1.12b). This clearly visualizes how natural and important to incorporate virtual turning points in Stokes geometry. But a careful reader might say: As Fig. 1.12b contains some degeneration (i.e., two turning points are connected by Stokes segments), the situation may not be so universal. To get rid of such concern, the best way is to consider the problem when the Eq. (1.2.1) contains some parameter t other than η (cf. [AHKKoNSShT, AKSShT]). Actually this is the situation discussed in Chap. 2.

Let us consider the situation when a Stokes curve of type $(1 > 2)$ hits a simple turning point $s(t)$ of type $(2, 3)$ for $t = t_1$. Then the characteristic root $\zeta = \zeta_2(x)$ has a square-root type singularity at $x = s(t_1)$, and the Stokes curve of type $(1 > 2)$ bifurcates there (cf. Fig. 1.13). We next suppose that the Stokes geometry for $t = t_2$

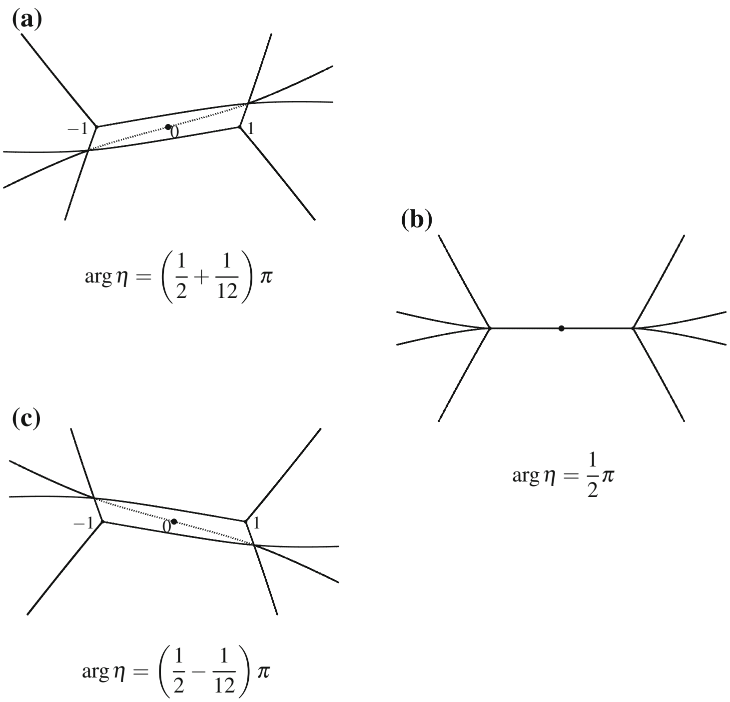
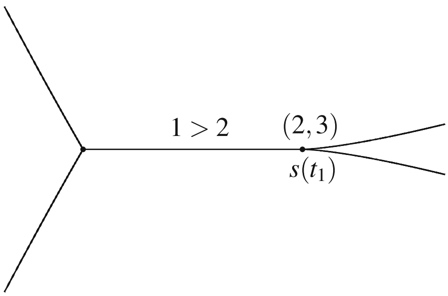


Fig. 1.12 The Stokes geometry of BNR equation when $\arg\eta$ is close to $\pi/2$

Fig. 1.13 Bifurcation of a Stokes curve



and t_3 (with $|t_j - t_1| \ll 1$ $(j = 2, 3)$) is given respectively as in Figs. 1.14a and 1.15a if we describe them ignoring virtual turning points. Then the Stokes curve of type $(1 > 2)$ abruptly changes its direction as t passes through t_1. But, if we incorporate a virtual turning point $v(t)$ as in Figs. 1.14b and 1.15b, then the topological structure of the Stokes geometry turns out to be stable. This observation is a starting point of the discussion in Chap. 2.

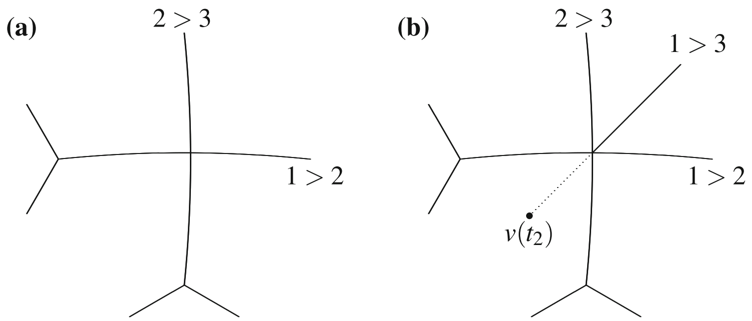


Fig. 1.14 Stokes geometry at $t = t_2$ with, **a** the virtual turning point ignored, and **b** the virtual turning point $v(t)$ added

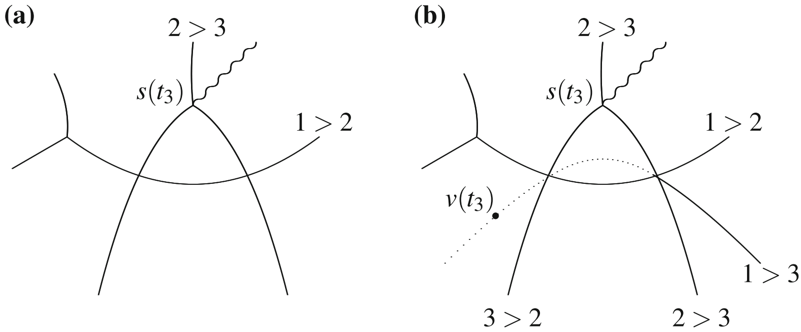


Fig. 1.15 Stokes geometry at $t = t_3$ with, **a** the virtual turning point ignored, and **b** the virtual turning point $v(t)$ added

1.8 *s*-Virtual Turning Points for Holonomic Systems

Since the structure theorem for (micro)differential equations in [SKK, Chap. II] covers not only single equations but also over-determined systems, the reasoning given so far can be applied to, for example, the system of differential equations which is satisfied by the **Pearcey integral**

$$\psi(x, \eta) = \psi(x_1, x_2, \eta) = \int \exp(\eta S(x, t))dt, \tag{1.8.1}$$

where

$$S(x, t) = S(x_1, x_2, t) = t^4 + x_2 t^2 + x_1 t. \tag{1.8.2}$$

In this case ψ satisfies [A] the so-called **Pearcey system**:

$$\begin{cases} \left(4\dfrac{\partial^3}{\partial x_1^3} + 2\eta^2 x_2 \dfrac{\partial}{\partial x_1} + \eta^3 x_1\right)\psi = 0 \\ \left(\dfrac{\partial^2}{\partial x_1^2} - \eta \dfrac{\partial}{\partial x_2}\right)\psi = 0. \end{cases} \tag{1.8.3}$$

If we fix the parameter η, then (1.8.3) consists of two differential equations in (x_1, x_2)-variables. Thus, for fixed η, (1.8.3) shares several properties with a linear ordinary differential equation such as the finite dimensionality of the space of its local solutions. Such a system is called a holonomic system, and hence we say that (1.8.3) is a **holonomic system with a large parameter** η. Here we do not give the definition of a holonomic system, but in a rough description it consists of n linear differential equations defined on an n-dimensional space; for example, $n = 2$ for the Pearcey system. Although too lengthy, the name originally used in [SKK, Chap. II, Sect. 4], i.e., a maximally over-determined system, abbreviated as **MOS**, is more appealing to the intuition of the reader. Naming issue apart, the structure theorem of [SKK] applies to the Borel transformed Pearcey system

$$\begin{cases} \left(4\dfrac{\partial^3}{\partial x_1^3} + 2x_2 \dfrac{\partial}{\partial x_1}\dfrac{\partial^2}{\partial y^2} + x_1 \dfrac{\partial^3}{\partial y^3}\right)\psi_B(x, y) = 0 \\ \left(\dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial}{\partial x_2}\dfrac{\partial}{\partial y}\right)\psi_B(x, y) = 0, \end{cases} \tag{1.8.4}$$

and then we can study the bicharacteristic flow in $T^*\mathbb{C}^3_{(x,y)}$ to define the notion of a virtual turning point. Similar holonomic systems with a large parameter η also appear [A] in analyzing the **Shudo integral**

$$\int \cdots \int \exp\left(\eta S(q_0, q_1, \ldots, q_{n-1}, q_n)\right) dq_1 \cdots dq_{n-1}, \tag{1.8.5}$$

where

$$S(q_0, q_1, q_2, \ldots, q_{n-1}, q_n) = \sum_{j=1}^{n} \frac{1}{2}(q_j - q_{j-1})^2 - \sum_{j=1}^{n-1} V(q_j) \tag{1.8.6}$$

for the potential V, say,

$$V(q) = -\frac{q^3}{3} - cq \tag{1.8.7}$$

with c being a real constant. The Shudo integral is a basic quantity in the study of the quantized Hénon map and Shudo [Sh] studied its analytic structure (for fixed q_0) using the notion of virtual turning points and new Stokes curves.

Now, after writing down the holonomic system with a large parameter η explicitly for the Shudo integral (with $n = 3$), Aoki [A, Sect. 3] comments that we can find another differential equation that it satisfies if η is also regarded as an independent variable, not just as a large parameter; that is, the Shudo integral, as a function of (x, η), satisfies a MOS. Inspired by this comment of Aoki, here we introduce the notion of s-virtual turning points for a holonomic system with a large parameter η that has a solution ψ which enjoys the integral representation (1.8.8) given below:

$$\psi = \int \exp(\eta S(x, t))dt \tag{1.8.8}$$

for a polynomial $S(x, t)$ that satisfies the following condition:

$$\begin{aligned} &\varpi : \left\{(x, t) \in \mathbb{C}^n_x \times \mathbb{C}^p_t \,;\ \frac{\partial S}{\partial t_1} = \cdots = \frac{\partial S}{\partial t_p} = 0\right\} \longrightarrow \mathbb{C}^n_x \\ &\text{is a finite proper map,} \end{aligned} \tag{1.8.9}$$

that is, ϖ maps a closed set to a closed set, and for each x_0 in $\mathbb{C}^n_x$ there exist an open neighborhood ω of x_0 and vectors $t^{(j)}(x)$ $(j = 1, 2, \ldots, N)$ whose components $t^{(j)}_k$ $(1 \le j \le N,\ 1 \le k \le p)$ are continuous functions on ω for which we find

$$\varpi^{-1}(\omega) \subset \bigcup_{j=1}^{N} \{(x, t) \in \omega \times \mathbb{C}^p; t = t^{(j)}(x)\}. \tag{1.8.10}$$

In contrast to the definition of a virtual turning point we introduce the notion of an s-virtual turning point by directly using the geometry of the characteristic variety of the MOS that is satisfied by the Borel transform $\psi_B(x, y)$ of ψ given by (1.8.8), that is,

$$\begin{aligned} \psi_B(x, y) &= \int \exp(-y\eta) \left(\int \exp\big(\eta S(x, t)\big)dt\right) d\eta \\ &= \int \delta\big(y - S(x, t)\big)dt. \end{aligned} \tag{1.8.11}$$

Actually the general theory of "integration (along fibers) of (over-determined) systems" [SKK, Chap. II] together with the conditions given by (1.8.9) and (1.8.10) tells us that the characteristic variety in question is contained (outside the zero-section of the cotangent bundle) in

$$\begin{aligned} V = \big\{&(x, y;\ \xi, \eta) \in T^*(\mathbb{C}^{n+1}_{(x,y)})\,;\ y = S(x, t) \text{ with } \mathrm{grad}_t S(x, t) = 0 \text{ and} \\ &(\xi, \eta) = c(-\mathrm{grad}_x S(x, t), 1) \text{ for a non-zero complex number } c\big\}. \end{aligned} \tag{1.8.12}$$

Hence its projection to the base manifold $\mathbb{C}^{n+1}_{(x,y)}$ is contained in

$$\begin{aligned}\mathscr{S} = \big\{&(x, y) \in \mathbb{C}^{n+1}; \text{there exists } t \text{ in } \mathbb{C}^p \text{ for which}\\ &y = S(x, t) \text{ and } \mathrm{grad}_t S(x, t) = 0 \text{ hold.}\big\}\end{aligned} \tag{1.8.13}$$

Since all points in (each connected component of) V, and hence those in $\mathscr{S}$ also, are thought to be "cognate", it is reasonable to introduce the notion of an s-virtual turning point as in Definition 1.8.1; it is the x-component of a confluent point of the loci of two "cognate" singularities, just like a virtual turning point introduced in Sect. 1.4.

Definition 1.8.1 A point x_0 is said to be an s-virtual turning point of the system that ψ in (1.8.8) satisfies, if there exist t and t' ($\neq t$) for which the following conditions are satisfied:

$$S(x_0, t) = S(x_0, t') \tag{1.8.14}$$

$$(\mathrm{grad}_t S)(x_0, t) = (\mathrm{grad}_t S)(x_0, t') = 0. \tag{1.8.15}$$

Remark 1.8.1 An ordinary turning point of the system that ψ satisfies is not an s-virtual turning point given above; an ordinary turning point is a point where

$$S(x, t^{(j)}(x)) = S(x, t^{(k)}(x)) \quad \big(t^{(j)}(x) \not\equiv t^{(k)}(x)\big) \tag{1.8.16}$$

holds for $t^{(j)}(x)$ and $t^{(k)}(x)$ used in (1.8.10), which are not holomorphic at the point in question.

Remark 1.8.2 In a neighborhood of an s-virtual turning point x_0 we can find $t^{(j)}(x)$ and $t^{(k)}(x)$ in (1.8.10) which satisfy

$$t^{(j)}(x_0) = t \text{ and } t^{(k)}(x_0) = t'. \tag{1.8.17}$$

Hence a (new) Stokes surface emanating from an s-virtual turning point x_0 is given by

$$\mathrm{Im}\ S(x, t^{(j)}(x)) = \mathrm{Im}\ S(x, t^{(k)}(x)) \tag{1.8.18}$$

near x_0.

Remark 1.8.3 We note that our study in this section only covers integrals of the form (1.8.8); the integrand is always a single-valued function. Although this is a very restricted situation, we still believe that analysis of such integrals is an important subject in application.

Remark 1.8.4 Although we believe the union of all the ordinary turning points and all the s-virtual turning points coincides with the totality of virtual turning points defined with the help of the study of the geometry of bicharacteristic flows when both notions are available, we have not confirmed the fact in full generality. Hence we have coined the name "an s-virtual turning point" (cf. Remark 1.8.5 for the background information concerning this naming) so that we may avoid possible confusions. We note, however, as we will see in Example 1.8.2, the definition of a virtual turning point for the Shudo integral with q_0 fixed coincides with the definition of an s-virtual turning point restricted to that particular value of q_0.

Remark 1.8.5 We have used the prefix "s-" of s-virtual turning points just because the theory of "integration of systems" is a counterpart of the stationary phase method in microlocal analysis. Fortunately enough, the prefix "s-" is consistent with the terminology "a saddle point", which is basic in the study of the Shudo integral [Sh].

In order to validate our belief stated in Remark 1.8.4, we study two important examples in what follows.

Example 1.8.1 (Pearcey integral) If we use $S(x, t)$ given by (1.8.2), then we have

$$\frac{\partial S}{\partial t} = 4t^3 + 2x_2 t + x_1. \tag{1.8.19}$$

Hence the finite proper mapping condition (1.8.9) is clearly satisfied. Further, a straightforward computation shows that (1.8.14) and (1.8.15) entail $x_2 \neq 0$ for $t \neq t'$ and that t and t' are solutions of the following equation in u:

$$4u^2 + 6\frac{x_1}{x_2}u + \left(9\frac{x_1^2}{x_2^2} + 2x_2\right) = 0. \tag{1.8.20}$$

Hence, by using (1.8.15) and (1.8.20), we further find that t and t' satisfy

$$6\frac{x_1}{x_2}u^2 + 9\frac{x_1^2}{x_2^2}u - x_1 = 0. \tag{1.8.21}$$

Thus it is clear that a point (x_1, x_2) with $x_1 = 0$ (and $x_2 \neq 0$) is an s-virtual turning point. On the other hand, if $x_1 \neq 0$, we find

$$4u^2 + 6\frac{x_1}{x_2}u - \frac{2}{3}x_2 = 0. \tag{1.8.21$'$}$$

Then the comparison of (1.8.20) and (1.8.21$'$) implies

$$27x_1^2 + 8x_2^3 = 0; \tag{1.8.22}$$

it is then clear that $t = t'$ holds in this case; that is, the point in question is an ordinary turning point (cf. Remark 1.8.1). These conclusions are consistent with the observation of Aoki [A, (23)].

Example 1.8.2 (Shudo integral) If we choose $S(q)$ in (1.8.6) as $S(x, t)$ with regarding (q_0, q_n) as x and $(q_1, \ldots, q_{n-1})$ as $t = (t_1, \ldots, t_{n-1})$, then the condition

$$\frac{\partial S}{\partial t_l} = 0 \quad (l = 1, 2, \ldots, n-1) \tag{1.8.23}$$

reduces to

$$q_{l+1} = 2q_l + q_l^2 - q_{l-1} + c \quad (l = 1, 2, \ldots, n-1). \tag{1.8.24}$$

Hence we can write down q_{l+1} $(1 \leq l \leq n-1)$ as a polynomial of (q_0, q_1); in particular, we find

$$q_n = q_1^{2^{n-1}} + Q_n(q_0, q_1), \tag{1.8.25}$$

where Q_n is a polynomial of (q_0, q_1) whose degree in q_1 is less than 2^{n-1}. Thus q_1 satisfies an algebraic equation whose coefficients depend only on $(q_0, q_n) = x$, and we locally (in x) find 2^{n-1} solutions $q_1^{(j)}(x)$ $(j = 1, 2, \ldots, 2^{n-1})$ of the algebraic equation (1.8.25). Therefore the finite proper mapping condition (1.8.9) is clearly satisfied; furthermore, by using $q_1^{(j)}(x)$ given above, we find a point (q_0, q_n) is an s-virtual turning of the MOS that the Shudo integral satisfies, if

$$\begin{aligned} &S\big(q_0, q_1^{(j)}(q_0, q_n), q_2(q_0, q_1^{(j)}(q_0, q_n)), \ldots, q_{n-1}(q_0, q_1^{(j)}(q_0, q_n)), q_n\big) \\ =&S\big(q_0, q_1^{(k)}(q_0, q_n), q_2(q_0, q_1^{(k)}(q_0, q_n)), \ldots, q_{n-1}(q_0, q_1^{(k)}(q_0, q_n)), q_n\big) \end{aligned} \tag{1.8.26}$$

holds for $j \neq k$. The point is located at the same point as the virtual turning point (for a fixed q_0) detected by Shudo [Sh].

Chapter 2
Application to the Noumi-Yamada System with a Large Parameter

2.1 Introduction

It is known that a traditional Painlevé equation (of the variable t) is obtained by the compatibility condition of a system of second order linear differential equations of the variables x and t. Here, when we focus upon the underlying linear system, the latter variable t is often called a deformation parameter. We can consider, with the appropriate introduction of a large parameter η into these systems, the Stokes geometry for both the linear and non-linear systems in the same way as that described in the previous chapter.

It is highly expected to have geometrical correspondence between the Stokes geometry of the t-space for the Painlevé equation and that of the x-space for the underlying linear differential equation. As a matter of fact, Kawai and Takei [KT1] have shown that, in the Stokes geometry of the x-space, i.e., the one for the underlying linear differential equation, a pair of ordinary turning points is directly connected by a Stokes curve if the deformation parameter t of the linear differential equation is located at a point in a Stokes curve of the t-space for the Painlevé equation. In other words, when t belongs to the Stokes curve of the non-linear differential equation, the Stokes geometry of the x-space becomes degenerate in the sense that two different Stokes curves emanating from each turning point of the pair accidentally coincide.

Several families of non-linear equations are recently found as a higher order extension of a traditional Painlevé equation. The Noumi-Yamada system $(\mathrm{NY})_m$ ($m = 2, 3, \ldots$) is one of such a family, and like a traditional Painlevé equation it can be obtained by the compatibility condition of a system of higher order linear differential equations. Therefore, by introducing a large parameter into these systems, the similar correspondence between the Stokes geometry of the t-space for $(\mathrm{NY})_m$ and that of the x-space for the underlying linear system is expected as that for a traditional Painlevé equation. In fact, Takei [T4] shows that, in the x-space, a pair of ordinary turning points is directly connected by a Stokes curve if the deformation parameter t is located in a Stokes curve $\mathscr{T}$ of the t-space and furthermore if t is sufficiently close to a turning point from which the Stokes curve $\mathscr{T}$ emanates.

N. Honda et al., *Virtual Turning Points*, SpringerBriefs in Mathematical Physics,
DOI 10.1007/978-4-431-55702-9_2

Since the underlying linear system is a higher order one unlike that of a traditional Painlevé equation, if $t \in \mathscr{T}$ is located far from the turning point, one cannot necessarily observe degeneration of the Stokes geometry of the x-space. That is, we often encounter a configuration of the Stokes geometry of the x-space which has no pair of turning points directly connected by a Stokes curve even if t is located at a point in a Stokes curve of the non-linear system.

The bifurcation phenomenon of a Stokes curve (Sect. 1.7) is one of the origins of such an unexpected situation; in some particular case of $(\mathrm{NYL})_{2m}$, a linear system that underlies $(\mathrm{NY})_{2m}$ (cf. (2.2.6) below), when a simple turning point s_0 hits, at $t = t^*$, a Stokes curve $\mathscr{S}$ which connects two ordinary turning points s and d, $\mathscr{S}$ then bifurcates in general (depending on the type of s_0 and that of $\mathscr{S}$) and after the bifurcation no pair of ordinary turning points is connected by a Stokes curve even if the parameter t lies in the Stokes curve of $(\mathrm{NY})_{2m}$ but each of the triplet $\{s_0,\ s,\ d\}$ of ordinary turning points is connected by a Stokes curve with one of the triplet $\{v_j\}_{j=1,2,3}$ of virtual turning points when the parameter t lies in the Stokes curve of $(\mathrm{NY})_{2m}$ [AHKKoNSShT, Fig. 10].

In order to systematically understand the repeated bifurcation phenomena and their relevance to virtual turning points, we introduce the notion of what we call a **bidirectional binary tree**, which connects several ordinary turning points in a manner specified later (Definition 2.4.3). The appearance of such a tree in the Stokes geometry of $(\mathrm{NYL})_{2m}$ is a counterpart of the appearance of a pair of ordinary turning points connected by a Stokes curve in the Stokes geometry of the underlying Schrödinger equation of a traditional Painlevé equation [KT2, Chap. 4]. Note that a bidirectional binary tree of degree other than 2 always contains, by definition, a part of a new Stokes curve as its edges, and hence, a virtual turning point is indispensable in finding a bidirectional binary tree. As we want to explain the core part of the problem in a concise manner, we do not discuss in this article the procedure to find finitely many virtual turning points needed for the description of the Stokes geometry; that is, we start with the model of the Stokes geometry in the terminology of [H3]. As a practical problem, finding the model of the Stokes geometry is an important step in obtaining a concrete figure, and it requires much computational effort, as is seen in [H1].

2.2 $(\mathrm{NY})_\ell$ and $(\mathrm{NYL})_\ell$ with a Large Parameter

The Noumi-Yamada system $(\mathrm{NY})_\ell$ ($\ell = 2, 3, \dots$) is a system of non-linear differential equations of unknown $(\ell + 1)$-functions $u(t) = (u_0(t), \dots, u_\ell(t))$ of the variable t, which was first introduced by Noumi and Yamada [NY]. It is well-known that the first member $(\mathrm{NY})_2$ and the second member $(\mathrm{NY})_3$ of $(\mathrm{NY})_\ell$ are equivalent to the traditional Painlevé equations (P_{IV}) and (P_{V}), respectively. The corresponding system with a large parameter was introduced in [T1] and intensively studied from the viewpoint of the exact WKB analysis. Let us first recall the explicit form of the Noumi-Yamada system with a large parameter η. As the structure of $(\mathrm{NY})_\ell$ depends on the parity of ℓ, we concentrate our attention, in most cases, on the case where $\ell = 2m$ ($m \in \mathbb{N}$), i.e., ℓ is even. The system $(\mathrm{NY})_{2m}$ with a large parameter η is of the following form:

$$\eta^{-1}\frac{du_j}{dt} = u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \widehat{\alpha}_j \quad (j = 0, 1, \ldots, 2m), \tag{2.2.1}$$

where each index j of u_j is considered to be an element of $\mathbb{Z}/(2m+1)\mathbb{Z}$, i.e., $u_{j+2m+1} = u_j$ and $\widehat{\alpha}_j$ is a formal power series of η^{-1} with constant coefficients, that is, $\widehat{\alpha}_j$ has the form

$$\widehat{\alpha}_j = \alpha_j^{(0)} + \eta^{-1}\alpha_j^{(1)} + \eta^{-2}\alpha_j^{(2)} + \cdots \qquad (j = 0, 1, \ldots, 2m)$$

with $\alpha_j^{(k)} \in \mathbb{C}$. We sometimes denote by α_j the leading term $\alpha_j^{(0)} \in \mathbb{C}$ of $\widehat{\alpha}_j$. In addition, we assume that these $\widehat{\alpha}_j$'s satisfy the condition

$$\widehat{\alpha}_0 + \widehat{\alpha}_1 + \cdots + \widehat{\alpha}_{2m} = \eta^{-1}, \tag{2.2.2}$$

which entails that the leading terms α_j's satisfy

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{2m} = 0. \tag{2.2.3}$$

Note that it follows from the condition (2.2.2) that, by summing up all the equations of $(\mathrm{NY})_{2m}$, we have

$$\frac{d}{dt}(u_0 + \cdots + u_{2m}) = 1. \tag{2.2.4}$$

Hence we also put the following additional equation into those of $(\mathrm{NY})_{2m}$ as a normalization condition:

$$u_0 + u_1 + \cdots + u_{2m} = t. \tag{2.2.5}$$

Summing up, the system $(\mathrm{NY})_{2m}$ consists of $(2m+2)$ equations, that is, Eqs. (2.2.1) and (2.2.5).

As it is well-known, the non-linear equation $(\mathrm{NY})_\ell$ describes the compatibility condition of a system of linear partial differential equations. In our case it consists of a linear differential equation $(\mathrm{NYL})_\ell$ in x-variable that depends on a parameter t (a deformation parameter) and another linear differential equation in t-variable that controls the isomonodromic deformation of $(\mathrm{NYL})_\ell$; the explicit form of $(\mathrm{NYL})_\ell$ is as follows.

$$\frac{d\psi}{dx} = \eta A_t(x)\psi, \tag{2.2.6}$$

where $\psi = {}^t(\psi_0(x), \ldots, \psi_\ell(x))$ and $A_t(x)$ is a square matrix of the size $\ell + 1$ with a parameter t defined by

$$A_t(x) = -x^{-1} \begin{pmatrix} \widehat{e}_0 & u_1(t) & 1 & & & & \\ & \widehat{e}_1 & u_2(t) & 1 & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & \widehat{e}_{\ell-2} & u_{\ell-1}(t) & 1 \\ x & & & & & \widehat{e}_{\ell-1} & u_\ell(t) \\ xu_0(t) & x & & & & & \widehat{e}_\ell \end{pmatrix}. \tag{2.2.7}$$

Here $u(t) = (u_0(t), u_1(t), \ldots, u_\ell(t))$ is a solution of $(\mathrm{NY})_\ell$ and $\widehat{e}_j$ $(j = 0, 1, \ldots, \ell)$ is a formal power series of η^{-1} with constant coefficients determined by the relations

$$\widehat{e}_0 + \cdots + \widehat{e}_\ell = 0, \quad \widehat{\alpha}_j = \widehat{e}_j - \widehat{e}_{j+1} + \eta^{-1}\delta_{j,0} \quad (j = 0, 1, \ldots, \ell). \tag{2.2.8}$$

Here $\delta_{j,0}$ denotes the Kronecker's symbol.

2.3 Stokes Geometry of $(\mathrm{NY})_{2m}$

Now we define the Stokes geometry of the non-linear system $(\mathrm{NY})_{2m}$. For this purpose, we first construct a formal solution $\widehat{u}(t) = (\widehat{u}_0(t), \widehat{u}_1(t), \ldots, \widehat{u}_{2m}(t))$ of $(\mathrm{NY})_{2m}$ in the form

$$\widehat{u}(t) = u^{(0)}(t) + u^{(1)}(t)\eta^{-1} + u^{(2)}(t)\eta^{-2} + u^{(3)}(t)\eta^{-3} + \cdots. \tag{2.3.1}$$

Here $u^{(k)}(t) = (u_0^{(k)}(t), \ldots, u_{2m}^{(k)}(t))$ and each $u_j^{(k)}(t)$ is a multi-valued holomorphic function over $\mathbb{C}$ except for a finite number of exceptional points. We say that $\widehat{u}(t)$ is a 0**-parameter formal solution** of $(\mathrm{NY})_{2m}$ if it satisfies $(\mathrm{NY})_{2m}$ as a formal power series of η^{-1}.

We briefly explain how to construct such a 0-parameter formal solution, which does not necessarily exist for an arbitrary parameter of $(\mathrm{NY})_{2m}$. We introduce some subsets of the space of parameters $(\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \in \mathbb{C}^{2m+1}$ to describe a condition which assures the existence of a 0-parameter solution. By taking (2.2.3) into account, let $A^{2m} \subset \mathbb{C}^{2m+1}$ denote the space of allowable parameters

$$\{(\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \in \mathbb{C}^{2m+1};\ \alpha_0 + \alpha_1 + \cdots + \alpha_{2m} = 0\}. \tag{2.3.2}$$

Then we define

$$E^{2m}_{\mathrm{cup}} := \bigcup_{\substack{0 \le i \le 2m, \\ 0 \le k \le 2m-1}} \{(\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \in A^{2m};\ \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k} = 0\}. \tag{2.3.3}$$

Note that the set E_{cup}^{2m} consists of a finite number of hypersurfaces in A^{2m}, and hence, $A^{2m} \backslash E_{\mathrm{cup}}^{2m}$ is an open dense subset in A^{2m}.

By putting (2.3.1) into (2.2.1) and (2.2.5), we find that the leading term $u^{(0)}(t)$ of (2.3.1) satisfies the system of algebraic equations

$$h_j(u^{(0)}(t)) = 0 \quad (j = 0, \ldots, 2m) \quad \text{and} \quad g(u^{(0)}(t)) = 0,$$

where $h_j(u)$ and $g(u)$ are the polynomials of $u = (u_0, \ldots, u_{2m})$ respectively defined by

$$\begin{aligned} h_j(u) &:= u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j \quad (j = 0, 1, \ldots, 2m), \\ g(u) &:= \sum_{j=0}^{2m} u_j. \end{aligned}$$

Then it follows from Theorem 6 in [AH] that, if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{2m}) \notin E_{\mathrm{cup}}^{2m}$, then $u^{(0)}(t)$ can be solved as a multi-valued holomorphic function of the variable t with a finite number of branching points of finite degree. Furthermore, it is bounded near the branching points and it is the unique solution as a multi-valued holomorphic function. Thus we can get the leading term $u^{(0)}(t)$ when $\alpha \notin E_{\mathrm{cup}}^{2m}$.

Next let $H(u)$ denote the Jacobian matrix

$$\frac{\partial(h_0, \ldots, h_{2m-1}, g)}{\partial(u_0, \ldots, u_{2m})} \tag{2.3.4}$$

of polynomials $h_0, h_1, \ldots, h_{2m-1}$ and g of the variables u_j's. Note that, since the sum of h_j's are identically zero, and thus, their Jacobian matrix is degenerate, we substitute g for the last h_{2m} of h_j's in the above definition. Then the same theorem says that $\det H(u^{(0)}(t))$ has an only finite number of zero points as a function of t, that is, $\det H(u^{(0)}(t))$ never vanishes identically if $\alpha \notin E_{\mathrm{cup}}^{2m}$.

Now we construct the lower order term $u^{(k)}(t)$ $(k \geq 1)$. By the normalization condition (2.2.5) and the differential equations except for one corresponding to $j = 2m$ in (2.2.1), we can obtain the following recursive relations:

$$H(u^{(0)}(t))u^{(k+1)} = R^{(k)}\left(t,\, u^{(0)}(t),\, \ldots,\, u^{(k)}(t),\, \frac{du^{(k)}}{dt}(t)\right) \quad (k = 0, 1, 2, \ldots). \tag{2.3.5}$$

Here $R^{(k)}$ consists of polynomials of the variables $t, u^{(0)}, \ldots, u^{(k)}$ and $\dfrac{du^{(k)}}{dt}$. Since $\det H(u^{(0)}(t))$ does not vanish except for a finite number of points as we have already noted, we can successively determine $u^{(k)}(t)$ by (2.3.5). Hence, for a generic parameter $\widehat{\alpha}$, we have obtained a 0-parameter formal solution $\widehat{u}(t)$ of $(\mathrm{NY})_{2m}$.

Let us now consider the linearized system of $(\mathrm{NY})_{2m}$ at the 0-parameter solution $u = \widehat{u}(t)$ thus obtained. By putting $u = \widehat{u}(t) + \widehat{U}(t)$ into the system (2.2.1) where $\widehat{U}(t) = (\widehat{U}_0(t), \widehat{U}_1(t), \ldots, \widehat{U}_{2m}(t))$ are new unknown functions and by taking its linear part with respect to $\widehat{U}$, we obtain the system of linear differential equations

$$\eta^{-1}\frac{d\widehat{U}}{dt} = \widehat{C}\,(t,\ \eta)\ \widehat{U}, \tag{2.3.6}$$

where $\widehat{C}\,(t, \eta)$ is the square matrix of size $2m+1$ of a formal power series of η^{-1} with coefficients in possibly multi-valued holomorphic functions of t, that is, $\widehat{C}\,(t,\ \eta)$ has the form

$$\widehat{C}\,(t,\ \eta) = C_0(t) + \eta^{-1}C_1(t) + \eta^{-2}C_2(t) + \cdots$$

with $C_k(t)$ being a matrix of multi-valued holomorphic functions. It is easy to see that the leading matrix $C_0(t)$ is given by the Jacobian matrix of the polynomials h_j's at $u = u^{(0)}(t)$, i.e.,

$$C_0(t) = \frac{\partial(h_0, \ldots, h_{2m})}{\partial(u_0, \ldots, u_{2m})}(u^{(0)}(t)). \tag{2.3.7}$$

Definition 2.3.1 A turning point and a Stokes curve of $(\mathrm{NY})_{2m}$ are, by definition, those of the linearized system (2.3.6) of $(\mathrm{NY})_{2m}$ at the 0-parameter solution $u = \widehat{u}(t)$.

Remark 2.3.1 In this section we use the words "a turning point" and "a Stokes curve" in the traditional sense. Since the linearized system (2.3.6) of $(\mathrm{NY})_{2m}$ is of size $(2m + 1) \times (2m + 1)$, the complete description of its Stokes geometry requires the introduction of "virtual turning points" and "new Stokes curves emanating from virtual turning points". Although no satisfactory study in this direction has yet been done for $(\mathrm{NY})_{2m}$, we believe that the study of the Stokes geometry of $(\mathrm{NYL})_{2m}$, particularly the introduction of the function $\Phi(T)$, which we will do in the subsequent sections, should play a basic role in such study. See [S2, H1] for supporting evidences of the belief. We also note that the study of the Stokes geometry of higher order Painlevé equations $(P_{\mathrm{I}})_m$ etc. also supports such a belief, although the underlying linear equations are of size 2×2 (cf. [KKoNT1, KKoNT2]).

Let $N(\nu,\ t)$ be a characteristic polynomial of the matrix $C_0(t)$, i.e.,

$$N(\nu,\ t) = \det\,(\nu I_{2m+1} - C_0(t))\,. \tag{2.3.8}$$

Then it follows from the definition of a turning point that a turning point of $(\mathrm{NY})_{2m}$ is a point $t^* \in \mathbb{C}$ at which a pair of roots of the polynomial $N(\nu,\ t)$ of ν merges. The characteristic polynomial $N(\nu,\ t)$ of $(\mathrm{NY})_{2m}$ has the following specific feature:

Lemma 2.3.1 ([T4]) *The* $\widetilde{N}(\nu,\ t) := \nu^{-1}N(\nu,\ t)$ *is a polynomial of* ν^2.

By the lemma, the roots of $N(\nu,\ t)$ consist of m-pairs $(\nu_k(t), -\nu_k(t))$ $(k = 0, 1, \ldots, m-1)$ and the extra root $\nu = 0$. The extra root comes from the fact that polynomials h_j's are linearly dependent, and hence, it is almost irrelevant to the

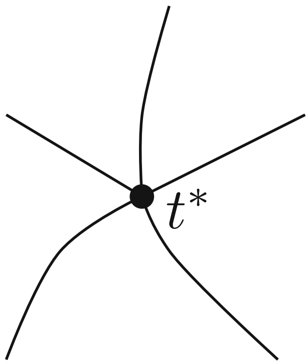


Fig. 2.1 Stokes curves emanating from a turning point of the first kind

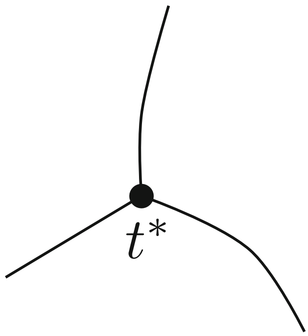


Fig. 2.2 Stokes curves emanating from a turning point of the second kind

Stokes geometry of $(\mathrm{NY})_{2m}$. In fact, if we eliminate the unknown function u_{2m} from the equations of (2.2.1) with $j = 0, 1, \ldots, 2m-1$ by the normalized condition (2.2.5) and apply the same argument to the system consisting of these $2m$-equations (where we forget the last equation of (2.2.1)), then the corresponding characteristic polynomial coincides with $\widetilde{N}(\nu,\ t)$ and the extra root never appears. Hence, in what follows, we ignore the extra root of $N(\nu,\ t)$.

Let $t^* \in \mathbb{C}$ be a turning point of $(\mathrm{NY})_{2m}$. Then, by these observations, we have the following two possibilities at t^*:

1. There exists k such that $\nu_k(t^*) = -\nu_k(t^*)$, that is, $\nu_k(t^*) = 0$. In this case, t^* is said to be a **turning point of the first kind**.
2. There exist $i \neq j$ such that either $\nu_i(t^*) = \nu_j(t^*)$ or $\nu_i(t^*) = -\nu_j(t^*)$ holds. We say that t^* is a **turning point of the second kind**.

It is known (Sect. 2.4 in [AH]) that the set of the branching points of the leading term $u^{(0)}(t)$ of the 0-parameter solution exactly coincides with that of turning points of the first kind.

Let t^* be a turning point of the first kind, that is, $\nu_k(t^*) = 0$ for some k. It is also known (Sect. 5.3 in [AH]) that, for generic α, the ramification degree of $u^{(0)}(t)$ at t^* is 2, that is, $u^{(0)}(t)$ has a Puiseux expansion of $(t-t^*)^{1/2}$ at $t = t^*$. Then, by (2.3.7),

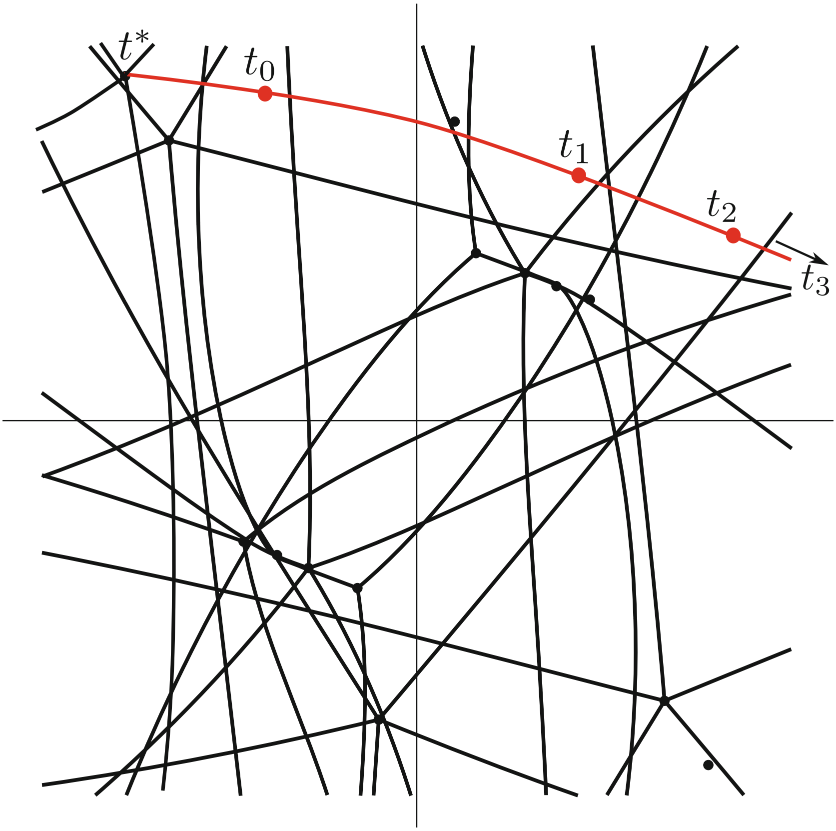


Fig. 2.3 The starting Stokes geometry of $(\mathrm{NY})_4$ (drawn with ordinary turning points and Stokes curves). The point t^* is, for example, a turning point of the first kind. For the other points t_0, t_1, t_2 and t_3 located in a Stokes curve emanating from t^*, see Example 2.4.3

the $\widetilde{N}(0,\ t)$ has also a Puiseux expansion of $(t-t^*)^{1/2}$ there, and its leading term is of order $1/2$ for generic α, i.e.,

$$\widetilde{N}(0,\ t) = c_{1/2}(t-t^*)^{1/2} + c_1(t-t^*) + \cdots \qquad (c_{1/2} \neq 0).$$

Therefore the root $\nu_k(t)$ has a Puiseux expansion of $(t-t^*)^{1/4}$ at $t = t^*$:

$$\nu_k(t) = d_{1/4}(t-t^*)^{1/4} + d_{1/2}(t-t^*)^{1/2} + \cdots \qquad (d_{1/4} \neq 0).$$

As a Stokes curve emanating from t^* is defined by

$$\mathrm{Im} \int_{t^*}^{t} (\nu_k(s) - (-\nu_k(s)))\, ds = 0 \iff \mathrm{Im} \int_{t^*}^{t} \nu_k(s) ds = 0,$$

which is equivalent to, by a Puiseux expansion of $\nu_k(t)$,

$$\mathrm{Im}\left(\frac{4d_{1/4}}{5}(t-t^*)^{5/4}+\frac{2d_{1/2}}{3}(t-t^*)^{3/2}+\cdots\right)=0,$$

we conclude that 5-Stokes curves emanate from a turning point of the first kind (cf. Fig. 2.1).

When t^* is a turning point of the second kind, then $u^{(0)}(t)$ is holomorphic near t^* and, for generic α, $\widetilde{N}(0,\ t)$ has a simple root at $t=t^*$. Hence, by the same argument as that for a turning point of the first kind, we see that 3-Stokes curves emanate from a turning point of the second kind in general (cf. Fig. 2.2).

Remark 2.3.2 The number of turning points of the first kind is less than or equal to $m2^{2m+1}$. That of the second kind is less than or equal to

$$2m(2m+1)_{2m}C_m - 3m2^{2m}.$$

These estimates are strict for generic α. See [AH, AHU] for details.

2.4 A Bidirectional Binary Tree

We now study the Stokes geometry of the linear system $(\mathrm{NYL})_{2m}$. Let t_1 be a point in a Stokes curve of $(\mathrm{NY})_{2m}$ which is different from a turning point. We denote by $G := G(t_1)$ the Stokes geometry of $(\mathrm{NYL})_{2m}$ with $t=t_1$. As stated before, when t lies in a Stokes curve of $(\mathrm{NY})_{2m}$, the corresponding Stokes geometry of $(\mathrm{NYL})_{2m}$ takes a specific configuration, in particular, there exists a so-called bidirectional binary tree in G whose definition is given now.

We first recall some conventions. Let $\mathscr{S}$ be a closed curve in $\mathbb{C}$, and let p_1, p_2, q_1 and q_2 be points in $\mathscr{S}$. We assume that, hereafter, a relevant curve does not form a loop. Then we denote by $[p_1, q_1]_{\mathscr{S}}$ or simply by $[p_1, q_1]$ the closed portion between p_1 and q_1 of the curve $\mathscr{S}$. Furthermore, we write $[p_1, q_1] \subset [p_2, q_2]$ if and only if p_2, p_1, q_1, q_2 are located in this order on $\mathscr{S}$. Hence the notation $[p_1, q_1] \subset [p_2, q_2]$ used in this article has the stronger meaning other than inclusion of sets.

Let $\mathscr{V}$ (resp. $\mathscr{W}$) be a Stokes curve emanating from a turning point v (resp. w) in G. Here, and in what follows, "a turning point" means either an ordinary one or a virtual one; when necessary, we always write so expressly. Assume that v and w are connected by both $\mathscr{V}$ and $\mathscr{W}$, that is, $[v,\ w]_{\mathscr{V}} = [v,\ w]_{\mathscr{W}}$ holds. Note that the Stokes curves $\mathscr{V}$ and $\mathscr{W}$ have the same type.

Definition 2.4.1 We say that a closed portion ℓ of $[v,\ w]$ is a **bidirectional segment** between turning points v and w in G if there exist points p and q in $[v,\ w]$ satisfying the following conditions:

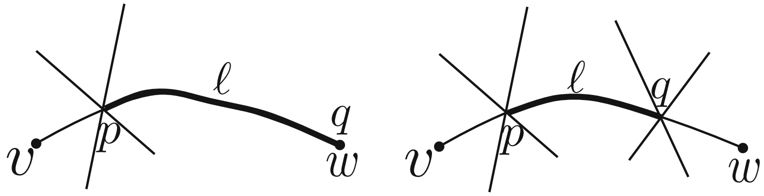


Fig. 2.4 Bidirectional segments

1. $\ell = [p, q] \subset [v, w]$.
2. The points p and q are either v or w or a point where another Stokes curve crosses.

We often write $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$ to describe this situation (Fig. 2.4), and we call p and q to be the end points of ℓ.

Note that $(p, q; v, w; \mathscr{V}, \mathscr{W})$ implies that, in particular, points v, p, q, w are located in this order on the curve $\mathscr{V}$ or $\mathscr{W}$. Now let us recall the definition of a binary tree in the graph theory.

Definition 2.4.2 A **binary tree** $T = (B, E, L)$ consists of E: a set of leaf nodes, B: a set of branching nodes and L: a set of edges whose end points are in $B \cup E$, which satisfy the following conditions.

1. The degree of each leaf node is one (the degree of a node p is the number of edges with p in their end points).
2. The degree of each branching node is three.
3. For any two nodes in $B \cup E$, they are connected by a path and such a path is unique. Here a path is, by definition, a subset of edges which forms a polygonal chain.

The **degree of a binary tree** T is, by definition, the number of leaf nodes. We also define the **depth of a binary tree** T to be the number of edges of a maximal path in the tree T.

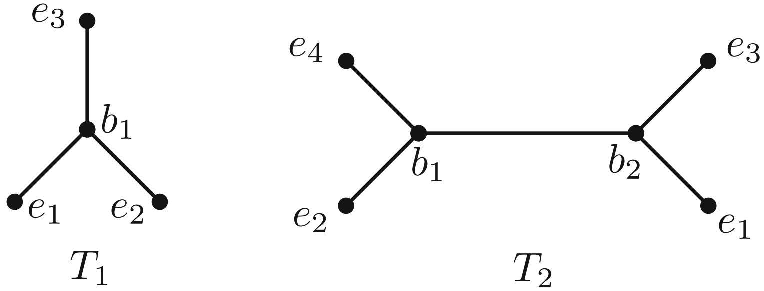


Fig. 2.5 Binary trees T_1 and T_2.

Example 2.4.1 Let us see the binary trees T_1 and T_2 in Fig. 2.5.

For examples, T_2 consists of 4-leaf nodes $\{e_1, e_2, e_3, e_4\}$, 2-branching nodes $\{b_1, b_2\}$ and 5-edges $\{e_1b_1, e_3b_1, e_2b_2, e_4b_2, b_1b_2\}$.

The degree of T_1 is 3 and that of T_2 is 4. The depth of T_1 is 2 and that of T_2 is 3.

Definition 2.4.3 A triplet $T = (B, E, L)$ is called a **bidirectional binary tree** in the Stokes geometry G if the following conditions are satisfied:

1. (B, E, L) is a binary tree where L consists of bidirectional segments in G whose end points are contained in $B \cup E$.
2. Let $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$ (cf. Definition 2.4.1 for the notation). Then $p \in E$ if $p = v$ and $p \in B$ otherwise, and similarly $q \in E$ if $q = w$ and $q \in B$ otherwise.
3. Let $b \in B$ and let ℓ be $(p, b; v, w; \mathscr{V}, \mathscr{W})$ in L with b in its end points. Suppose that $\ell_1 = (p_1, b; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1)$ and $\ell_2 = (p_2, b; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2)$ are the other two bidirectional segments in L with b in their end points. Then we have (cf. Fig. 2.6):

   (a) The Stokes curves $\mathscr{V}_1$ and $\mathscr{V}_2$ form an ordered crossing at b with the Stokes curve $\mathscr{W}$. That is, there exist mutually distinct indices i, j and k such that the type of $\mathscr{V}_1$ (resp. $\mathscr{V}_2$ and $\mathscr{W}$) near b is (i, j) (resp. (j, k) and (i, k)) and either

   $$\text{“}i < j \text{ on } \mathscr{V}_1 \text{ and } j < k \text{ on } \mathscr{V}_2\text{”}$$

   or

   $$\text{“}j < i \text{ on } \mathscr{V}_1 \text{ and } k < j \text{ on } \mathscr{V}_2\text{”}$$

   hold (see Definition 1.2.2 for the meaning of the labels $i < j$, etc. used here).

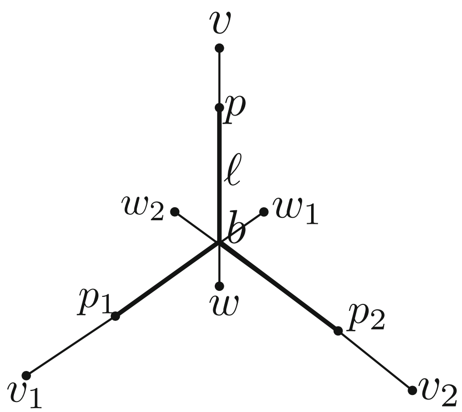


Fig. 2.6 The condition 3. of a bidirectional binary tree

(b) w is a turning point obtained from v_1 and v_2 by the method given in the proof of Proposition 1.4.1 (cf. (1.4.32)). That is, we have the integral relation

$$\int_b^{v_1} (\lambda_i(x) - \lambda_j(x))dx + \int_b^{v_2} (\lambda_j(x) - \lambda_k(x))dx = \int_b^{w} (\lambda_i(x) - \lambda_k(x))dx.$$

4. Each point in E is an ordinary turning point.

Remark 2.4.1 It follows from the conditions 3(a) and 3(b) of Definition 2.4.3 that, if $i < j$ on $\mathscr{V}_1$ and $j < k$ on $\mathscr{V}_2$ hold, we have $i < k$ on $\mathscr{W}$. Similarly, $j < i$ on $\mathscr{V}_1$ and $k < j$ on $\mathscr{V}_2$ imply $k < i$ on $\mathscr{W}$.

We will give a few examples of bidirectional binary trees.

Example 2.4.2 The simplest bidirectional binary tree is that of degree 2 (Fig. 2.7). It is nothing but a pair (s, d) of ordinary turning points connected by a Stokes curve $\mathscr{S}$

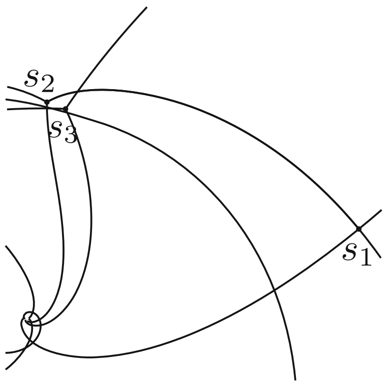


Fig. 2.7 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_0$, where only relevant Stokes curves and ordinary turning points are drawn for simplicity

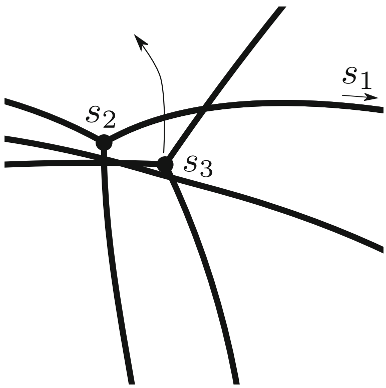


Fig. 2.8 The magnification of Fig. 2.7 near s_2

(resp. $\mathscr{D}$) emanating from s (resp. d). As a matter of fact, the bidirectional segment $(s,\ d;\ s,\ d;\ \mathscr{S},\ \mathscr{D})$ and its end points form a bidirectional binary tree of degree 2.

Example 2.4.3 We give some concrete examples of bidirectional binary trees observed in the Stokes geometry of $(\mathrm{NYL})_4$. Let t^* be a turning point of $(\mathrm{NY})_4$ and let $\mathscr{T}$ denote a Stokes curve of $(\mathrm{NY})_4$ that emanates from t^*. We take points t_0, t_1, t_2 and t_3 in $\mathscr{T}$ so that t^*, t_0, t_1, t_2 and t_3 are located in this order on $\mathscr{T}$ (cf. Fig. 2.3) .

Let us see the Stokes geometry of $(\mathrm{NYL})_4$ when $t = t_k$ $(k = 0, 1, 2, 3)$. Note that, in all figures, s_1, s_2, s_3 and s_4 are ordinary turning points and $v_1, v_2, \ldots, v_6$ are virtual turning points. Figures 2.7 and 2.8 describe the Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_0$. As t_0 in $\mathscr{T}$ is located quite near the turning point t^*, we can observe that ordinary turning points s_1 and s_2 are directly connected by a Stokes curve, and they form a bidirectional binary tree of degree 2 as it is explained in the previous example.

We also find another ordinary turning point s_3 close to the portion $[s_1,\ s_2]$. As a matter of fact, when t moves from t_0 to t_1, s_3 hits against $[s_1,\ s_2]$. As we already mentioned in Sect. 2.1, when s_3 crosses $[s_1,\ s_2]$, a bifurcation phenomenon occurs and consequently the tree of degree 2 becomes the higher one: The tree T_1 (resp. T_2) of Figs. 2.9 and 2.10 (resp. Figs. 2.11 and 2.12) is the bidirectional binary tree of degree 3. The tree T_1 consists of 3-leaf nodes $\{s_1, s_2, s_3\}$ (s_1 is a double turning point, and s_2, s_3 are simple turning points), 1-branching node $\{b_1\}$ and 3-bidirectional segments $\{s_1b_1,\ s_2b_1,\ s_3b_1\}$. For example, the bidirectional segment s_1b_1 lies in a common portion of two Stokes curves emanating from s_1 and v_1. In this way, each bidirectional segment of T_1 lies in a common portion of two Stokes curves. An important feature of T_1 is, for example, the Stokes curve emanating from s_2 and that from s_3 form an ordered crossing at b_1 and a virtual turning point v_1 is obtained from s_2 and s_3 (cf. (1.4.32)). In the same way, the Stokes curve emanating from s_1 and that from s_2 (resp. s_3) form an ordered crossing at b_1 and these turning points determine a virtual turning point v_3 (resp. v_2).

When t moves from t_1 to t_2, the tree T_1 changes its shape continuously and is deformed to the tree T_2 in Fig. 2.12. Note that, in the figure, the simple turning point s_4 is located quite near the edge s_3b_1 of T_2. Then, when t moves from t_2 to t_3 on $\mathscr{T}$, the turning point s_4 really crosses the edge s_3b_1 and the tree T_2 grows.

The degree of T_3 in Figs. 2.13 and 2.14 becomes 4 because the turning point s_4 joins in the tree as a new leaf node after s_4 hits against the edge of the tree. The tree T_3 consists of 4-leaf nodes $\{s_1, s_2, s_3, s_4\}$, 2-branching nodes $\{b_1, b_2\}$, and 5-edges $\{s_1b_1, s_2b_1, s_3b_2, s_4b_2, b_1b_2\}$. In particular, the segment b_1b_2 is in a common portion of two Stokes curves emanating from virtual turning points v_5 and v_6. Although it is not trivial, a careful study of the Stokes geometry G guarantees that branching nodes b_1 and b_2 satisfy the condition 3. of Definition 2.4.3.

For a bidirectional binary tree T in G, we can define its total integral value $\Phi(T)$ as in Definition 2.4.4 below. The value $\Phi(T)$ is closely tied up with the Stokes geometry of $(\mathrm{NY})_{2m}$, as Corollary 2.5.1 below shows. We also note that it is a counterpart of the function $\phi_J(t)$ $(J = \mathrm{I},\ \mathrm{II}, \ldots, \mathrm{VI})$ used in constructing instanton-type solutions

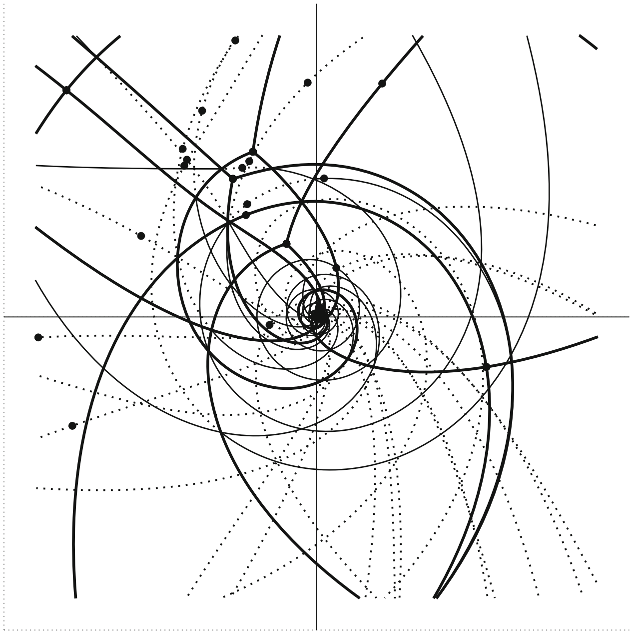


Fig. 2.9 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_1$ [H1, Fig. III-1–6]

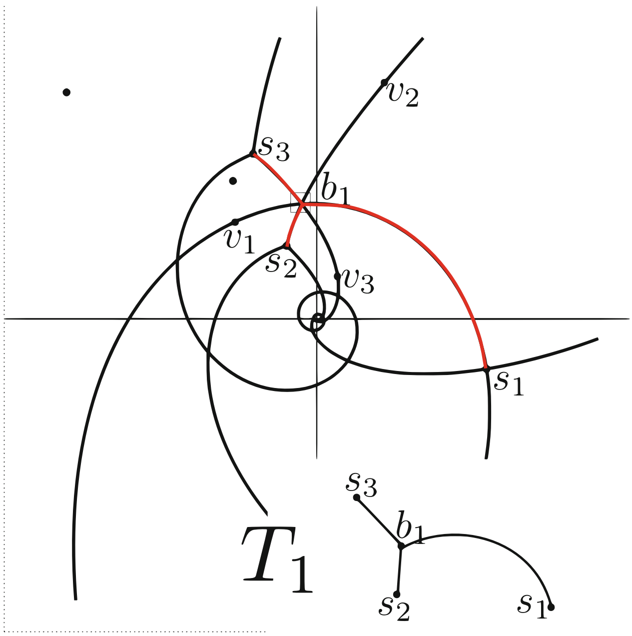


Fig. 2.10 Extract related Stokes curves and turning points of T_1 from Fig. 2.9 ($t = t_1$)

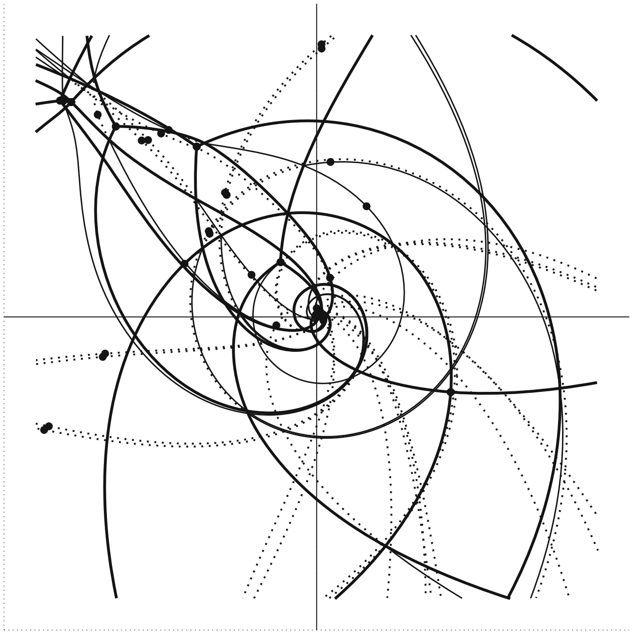


Fig. 2.11 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_2$ [H1, Fig. III-1–7]

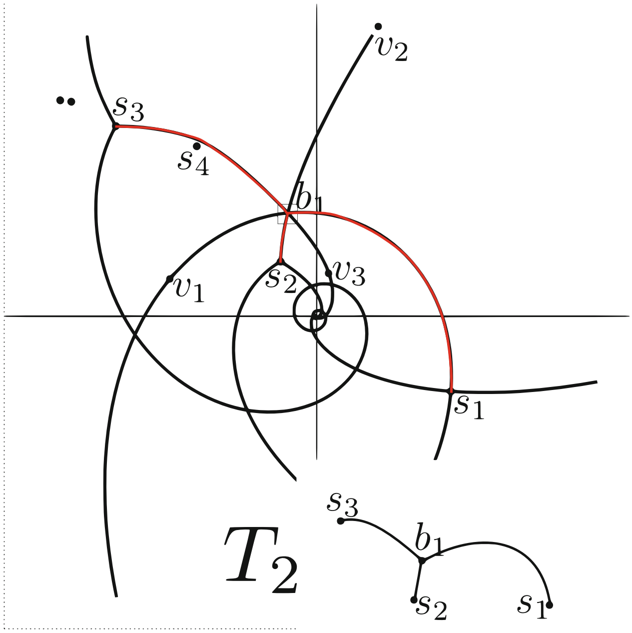


Fig. 2.12 Extract related Stokes curves and turning points of T_2 from Fig. 2.11 ($t = t_2$)

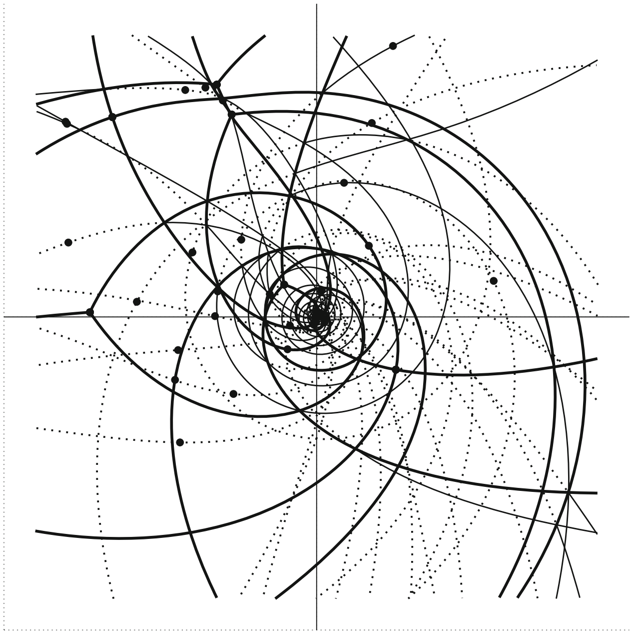


Fig. 2.13 The Stokes geometry of $(\mathrm{NYL})_4$ at $t = t_3$ [H1, Fig. III-1–9]

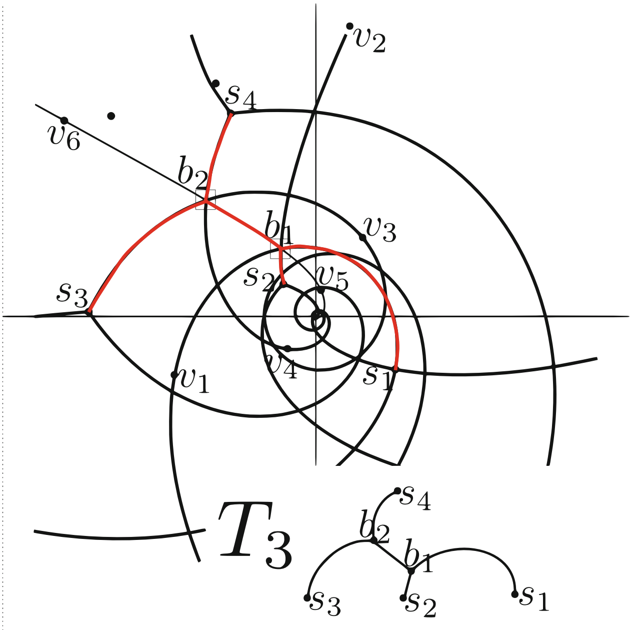


Fig. 2.14 Extract related Stokes curves and turning points of T_3 from Fig. 2.13 ($t = t_3$)

of the traditional Painlevé equation (P_J). (See [KT2, Chap. 4] for the construction of instanton-type solutions of (P_J).)

Definition 2.4.4 Let $T = (B, E, L)$ be a bidirectional binary tree in G. We define the **total integral value** $\Phi(T)$ of T as follows:

$$\Phi(T) = \sum_{\ell \in L} \left| \int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|, \tag{2.4.1}$$

where, for $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$, we set $[\ell] := [p, q]$ and denote by (i_ℓ, j_ℓ) the type of the Stokes curve $\mathscr{V}$ or $\mathscr{W}$.

Remark 2.4.2 Let $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. Then, as $[p, q]$ is a part of the Stokes curve $\mathscr{V}$ or $\mathscr{W}$ and $p \neq q$, each integral

$$\int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx$$

takes a non-zero real value. Hence, if we equip each edge in L with an appropriate orientation, we can write (2.4.1) in a much simpler form

$$\Phi(T) = \sum_{\ell \in L} \int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx.$$

The following lemma gives us the most basic property of a bidirectional binary tree.

Lemma 2.4.1 *For any edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$, we have*

$$\Phi(T) = \left| \int_v^w (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \right|. \tag{2.4.2}$$

In particular, the integral value of the right hand side of the above equality does not depend on the choice of the edge ℓ of the tree T.

Proof We consider, as a bidirectional binary tree, a tree $T = (B, E, L)$ which satisfies all the conditions in Definition 2.4.2 except for the last condition 4, that is, a virtual turning point is also allowed as a leaf node. Then we show the claim for such a tree. We prove the claim by the induction on the number of edges in such a T, which is denoted by $\#T$ in this proof.

When $\#T = 1$, the only edge has the form $(v, w; v, w; \mathscr{V}, \mathscr{W})$. Hence the claim is trivial.

Now assume that the claim is true for T with $\#T \leq k$ $(k \geq 1)$. We will show the claim for T with $\#T = k + 1$. Let $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. We assume that $p \neq v$ and $q \neq w$, that is, p and q are branching nodes, and we will show the claim only in this case because the other cases can be proved in the same way.

Suppose that

$$\ell_1 = (p, q_1; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1), \ \ell_2 = (p, q_2; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2) \in L$$

are the other two edges with p in their end points and suppose also that

$$\ell_3 = (p_3, q; v_3, w_3; \mathscr{V}_3, \mathscr{W}_3), \ \ell_4 = (p_4, q; v_4, w_4; \mathscr{V}_4, \mathscr{W}_4) \in L$$

are the other two edges with q in their end points. Note that $T \backslash \ell$ (the tree T without the edge ℓ) consists of 4-connected components. We denote by T_1 (resp. T_2, T_3 and T_4) the connected component of $T \backslash \ell$ containing the point q_1 (resp. q_2, p_3 and p_4). Now we define the bidirectional binary tree $\widetilde{T}_1$ by T_1 where the edge ℓ_1 is replaced with

$$\widetilde{\ell}_1 = (v_1, q_1; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1),$$

i.e., $\widetilde{\ell}_1$ is obtained by substituting v_1 for p in ℓ_1. In the same way, $\widetilde{T}_2$ is the tree T_2 where ℓ_2 is replaced with

$$\widetilde{\ell}_2 = (v_2, q_2; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2).$$

Then, as $\#\widetilde{T}_1 \leq k$ and $\#\widetilde{T}_2 \leq k$ by the induction hypothesis, we obtain

$$\Phi(\widetilde{T}_1) = \left| \int_{v_1}^{w_1} (\lambda_{i_{\widetilde{\ell}_1}}(x) - \lambda_{j_{\widetilde{\ell}_1}}(x))dx \right|$$

and

$$\Phi(\widetilde{T}_2) = \left| \int_{v_2}^{w_2} (\lambda_{i_{\widetilde{\ell}_2}}(x) - \lambda_{j_{\widetilde{\ell}_2}}(x))dx \right|.$$

Hence, by noticing that the path $[v_1, \ p]$ (resp. $[v_2, \ p]$) appears in both sides, we have

$$\sum_{\ell' \in T_1} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| = \left| \int_{p}^{w_1} (\lambda_{i_{\widetilde{\ell}_1}}(x) - \lambda_{j_{\widetilde{\ell}_1}}(x))dx \right|$$

and

$$\sum_{\ell' \in T_2} \left| \int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx \right| = \left| \int_{p}^{w_2} (\lambda_{i_{\widetilde{\ell}_2}}(x) - \lambda_{j_{\widetilde{\ell}_2}}(x))dx \right|,$$

where we write $\ell' \in T_1$ (resp. T_2) when ℓ' is an edge of T_1 (resp. T_2). The condition that $\mathscr{W}_1$ and $\mathscr{W}_2$ form an ordered crossing at p entails that pairs of indices $(i_{\widetilde{\ell}_1}, \ j_{\widetilde{\ell}_1})$ and $(i_{\widetilde{\ell}_2}, \ j_{\widetilde{\ell}_2})$ share one and only one common index, and thus, we may assume $j_{\widetilde{\ell}_1} = i_{\widetilde{\ell}_2}$ (denote it by k) and $i_{\widetilde{\ell}_1} \neq j_{\widetilde{\ell}_2} \neq k$. Furthermore, the same condition implies that both $\int_p^{w_1} (\lambda_{i_{\tilde{\ell}_1}}(x) - \lambda_k(x))dx$ and $\int_p^{w_2} (\lambda_k(x) - \lambda_{j_{\tilde{\ell}_2}}(x))dx$ have the same signature. Therefore we get

$$\begin{aligned}
&\left|\int_p^{w_1} (\lambda_{i_{\tilde{\ell}_1}}(x) - \lambda_k(x))dx\right| + \left|\int_p^{w_2} (\lambda_k(x) - \lambda_{j_{\tilde{\ell}_2}}(x))dx\right| \\
&\quad = \left|\int_p^{w_1} (\lambda_{i_{\tilde{\ell}_1}}(x) - \lambda_k(x))dx + \int_p^{w_2} (\lambda_k(x) - \lambda_{j_{\tilde{\ell}_2}}(x))dx\right| \\
&\quad = \left|\int_p^{v} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|,
\end{aligned}$$

where the last equality comes from the fact that v is a virtual turning point obtained from w_1 and w_2. Summing up, we have

$$\sum_{\ell' \in T_1 \cup T_2} \left|\int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx\right| = \left|\int_p^{v} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|.$$

By applying the same argument to T_3 and T_4, we also have

$$\sum_{\ell' \in T_3 \cup T_4} \left|\int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx\right| = \left|\int_q^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|.$$

Then, as v, p, q, w are located in this order on the Stokes curve, by noticing $\ell = [p,\ q]$, we have

$$\begin{aligned}
&\left|\int_p^{v} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right| + \left|\int_q^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right| + \left|\int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right| \\
&\quad = \left|\int_v^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|,
\end{aligned}$$

and hence, we obtain

$$\begin{aligned}
\Phi(T) &= \sum_{\ell' \in T \setminus \ell} \left|\int_{[\ell']} (\lambda_{i_{\ell'}}(x) - \lambda_{j_{\ell'}}(x))dx\right| + \left|\int_{[\ell]} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right| \\
&= \left|\int_v^{w} (\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|.
\end{aligned}$$

Therefore the claim is true for T with $\#T = k + 1$, and thus, it is true for any T by the induction. This completes the proof.

2.5 Growing and Shrinking of a Bidirectional Binary Tree

Let t^* be a turning point of $(\mathrm{NY})_{2m}$ and let $\mathscr{T}$ denote a Stokes curve emanating from t^*. We take the parameterization $t = t(\theta)$ $(\theta \geq 0)$ of $\mathscr{T}$ by the length θ of the curve from t^* to $t \in \mathscr{T}$. We denote by $G(t)$ the Stokes geometry of $(\mathrm{NYL})_{2m}$ for a fixed t. Let $\theta_2 > \theta_1 > \theta_0 > 0$ and set $t_k = t(\theta_k) \in \mathscr{T}$ $(k = 0, 1, 2)$. Assume that a bidirectional binary tree $T = (B, E, L)$ exists in $G(t_1)$. We first give sufficient conditions which guarantee that T is continuously deformed when θ moves in a neighborhood of θ_1. We denote by $\mathscr{S}_T$ the set of all the Stokes curves appearing in edges of T.

C-1. An ordinary turning point of non-disjoint type never hits against a Stokes curve in $\mathscr{S}_T$. Here a turning point is said to be of non-disjoint type when the type of the turning point and that of a Stokes curve which the turning point touches share a common index.

From the condition C-1, each Stokes curve in $\mathscr{S}_T$ continuously moves when θ moves. Hence $\mathscr{S}_T$ forms a continuously moving family of Stokes curves.

C-2. Each Stokes curve in $\mathscr{S}_T$ intersects transversally with other related Stokes curves when they have some point in common.

Here, and in what follows, "a related Stokes curve" or "a related turning point" means that it appears in an element $(p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. It follows from C-1 that all the related Stokes curves and turning points continuously move with θ, and C-2 makes it sure that, in particular, a branching node also continuously moves. As a result of these conditions, each point in $B \cup E$ is regarded as a continuous function of θ, for which we also assume:

C-3. Any pair of points in $B \cup E$ never merges.

As the origin is a regular singular point for the system $(\mathrm{NYL})_{2m}$, to simplify our consideration, we prevent a point in $B \cup E$ from falling into the origin. Hence, in what follows, we always assume the following condition:

(†) Each point in $B \cup E$ stays in a compact region of $\mathbb{C}\backslash\{0\}$.

Then we obtain:

Theorem 2.5.1 *Assume that the conditions C-1–C-3 hold for every $\theta_0 < \theta < \theta_2$. Then, for any $\theta \in (\theta_0, \theta_2)$, there exists a bidirectional binary tree T_θ in $G(t(\theta))$ satisfying that $T_{\theta_1} = T$ and T_θ is continuously deformed when θ moves in (θ_0, θ_2). Furthermore the total integral value $\Phi(T_\theta)$ of T_θ is an analytic function of $\theta \in (\theta_0, \theta_2)$.*

Proof As a related Stokes curve and a point in $B \cup E$ continuously move by the conditions C-1 and C-2, it suffices to show that bidirectionality of each segment is really preserved. Let $(p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$ where p, q, v and w are known to be continuous functions of θ. By the condition C-3, we have $p \neq q$ for any

$\theta \in (\theta_0, \theta_2)$. If q is a branching node, i.e., $q \neq w$ at $\theta = \theta_1$, then we will show $q \neq w$ for any $\theta \in (\theta_0, \theta_2)$. To confirm this, let $\ell_1 = (p_1, q; v_1, w_1; \mathscr{V}_1, \mathscr{W}_1)$ and $\ell_2 = (p_2, q; v_2, w_2; \mathscr{V}_2, \mathscr{W}_2)$ be the other two edges with q in their end points. Since $p_1 \neq q$ and $p_2 \neq q$ hold by the condition C-3 and since $\mathscr{V}_1$ and $\mathscr{V}_2$ form an ordered crossing at q, we get

$$\left|\int_{v_1}^{q}(\lambda_i - \lambda_j)dx + \int_{v_2}^{q}(\lambda_j - \lambda_k)dx\right| \geq \left|\int_{p_1}^{q}(\lambda_i - \lambda_j)dx + \int_{p_2}^{q}(\lambda_j - \lambda_k)dx\right| \neq 0,$$

where the type of $\mathscr{V}_1$ (resp. $\mathscr{V}_2$) near q is assumed to be $(i,\ j)$ (resp. $(j,\ k)$). As w is a virtual turning point obtained from v_1 and v_2, we have

$$\int_{w}^{q}(\lambda_i - \lambda_k)dx = \int_{v_1}^{q}(\lambda_i - \lambda_j)dx + \int_{v_2}^{q}(\lambda_j - \lambda_k)dx \neq 0.$$

Hence we have obtained $w \neq q$. As a conclusion, for any $\theta \in (\theta_0, \theta_2)$, the points v, p, q, w are located in this order on $\mathscr{V}$ or $\mathscr{W}$, and hence, they form a bidirectional segment in $G(t(\theta))$. Therefore we find a bidirectional binary tree T_θ at every $\theta \in (\theta_0, \theta_2)$ which is continuous deformation of T.

Let us show that $\Phi(T_\theta)$ is an analytic function of θ. Take an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W}) \in L$. Then, by Lemma 2.4.1, we have

$$\Phi(T_\theta) = \left|\int_{v}^{w}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right|,$$

and hence, it suffices to show $\displaystyle\int_{v}^{w}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx$ to be an analytic function of θ. Since roots λ_{i_ℓ} and λ_{j_ℓ} analytically depend on θ outside ordinary turning points, we may consider the problem only near v and w, that is,

$$\int_{v'}^{v}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx \quad \left(\text{resp. } \int_{w'}^{w}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx\right)$$

is analytic for a fixed v' (resp. w') sufficiently close to v (resp. w).

We only show that $\displaystyle\int_{v'}^{v}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx$ is an analytic function of θ. If v is a simple turning point, then we have

$$\int_{v'}^{v}(\lambda_{i_\ell}(x) - \lambda_{j_\ell}(x))dx = \int_{C}\lambda_{i_\ell}(x)dx,$$

where C is a closed path which starts from v' and turns around v once with an appropriate orientation. Hence the integral is an analytic function of θ in this case.

If v is a double turning point or a virtual one, then λ_{i_ℓ} and λ_{j_ℓ} analytically depend on θ near v. Therefore it is enough to show that v itself analytically depends on θ.

When v is a double turning point, v is a simple root of the equation $\frac{dD}{dx}(x) = 0$ where $D(x)$ is the discriminant of the polynomial $\det(\lambda I_{2m+1} - A_t(x))$ of λ, from which analyticity of v on θ follows. When v is a virtual turning point, v is, by definition, a solution of the equation of x defined by

$$F(x) = \int_{x^*}^{x} (\lambda_{i_\ell}(z) - \lambda_{j_\ell}(z))dz + h(\theta) = 0,$$

where x^* is a fixed point near v and $h(\theta)$ is some analytic function of θ. As we have

$$\frac{dF}{dx} = \lambda_{i_\ell}(x) - \lambda_{j_\ell}(x)$$

which does not vanish near $x = v$ because v is not an ordinary turning point, we know that v also analytically depends on θ. This completes the proof.

Assume that $\theta_1 > 0$ is an exceptional point in the sense that either C-1 or C-2 or C-3 does not hold in $G(t(\theta_1))$. That is, one of the following cases happens in $G(t(\theta_1))$:

A. For an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$, an ordinary turning of non-disjoint type hits against $[v, w]$.
B. For an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$, the end points p and q of ℓ merge.
Z. At a branching node b, some edges with b in their end points become tangent at b; to be more precise, the Stokes curves containing these edges are tangent at b each other.

By taking the above theorem into account, we find that continuous deformation of T may fail at θ_1. We will investigate discontinuity of the Stokes geometry for the major cases which we often encounter in the study of the Stokes geometry of $(\mathrm{NYL})_{2m}$.

Remark 2.5.1 We have not observed Case Z alone in our concrete computations of the Stokes geometry of $(\mathrm{NYL})_{2m}$, although there is a theoretical possibility of an occurrence of such a case. Hence, in subsequent observations, we focus only on Cases A and B.

Case A: Let us consider a situation where an ordinary turning point s of non-disjoint type hits against the interior of $[p, q]$ for an edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$ in $G(t(\theta_1))$. We assume that s is not a leaf node of T_θ for $\theta < \theta_1$ and that s hits transversally against $\mathscr{V}$ (or $\mathscr{W}$) when θ tends to θ_1 from below. Case A is classified into the following 3 sub-cases A-1–A-3.

A-1. The type of ℓ and that of s are the same.
This is the most destructive case where the bidirectional segment ℓ is snapped by s (see Fig. 2.15). As a result, the corresponding bidirectional binary tree T_θ does not exist for $\theta > \theta_1$. The situation is a counterpart of the Nishikawa

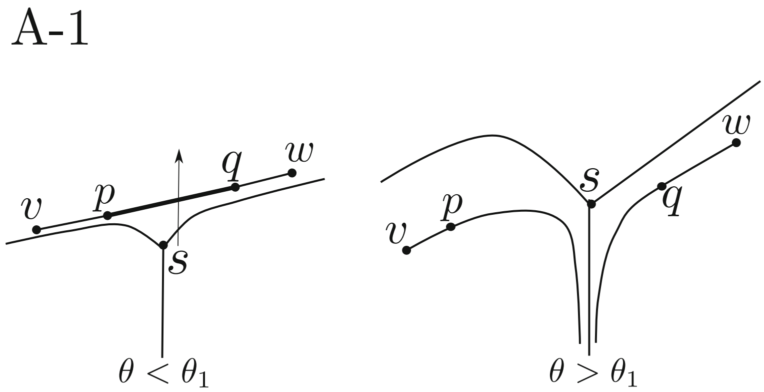


Fig. 2.15 Case A-1

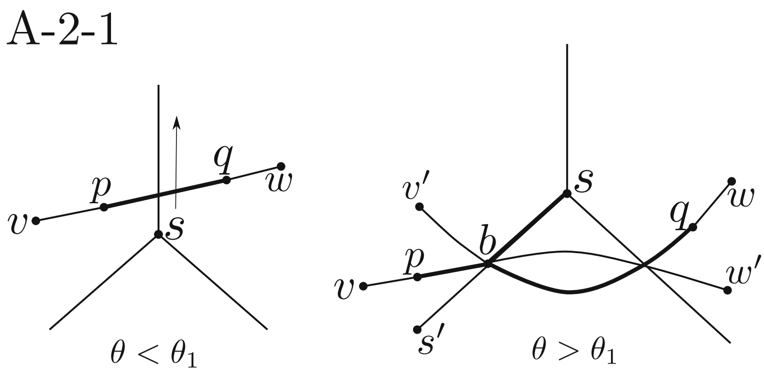


Fig. 2.16 Case A-2-1

phenomenon [KKoNT1] in the Noumi-Yamada system and it is intensively studied by Sasaki [S1, S2]. An important consequence of the vanishing of the bidirectional binary tree is that the relevant Stokes curve of $(\mathrm{NY})_{2m}$ becomes inert on the portion $\{t = t(\theta);\ \theta > \theta_1\}$; no Stokes phenomena for the solutions of $(\mathrm{NY})_{2m}$ are anticipated there, although it has not yet been proved.

A-2. The ordinary turning point s is simple, and the type of s and that of ℓ have one and only one common index.
When s touches ℓ, a bifurcation phenomenon described in Sect. 1.7 occurs, and thus, T_θ has a discontinuous change at $\theta = \theta_1$. Let us consider the case A-2 in detail. When s is sufficiently close to ℓ, the geometry near s, in a generic situation, becomes graphically equivalent to A-2-1 or A-2-2 described respectively in Fig. 2.16 or Fig. 2.17, that is, either one or two Stokes curves emanating from s intersect with ℓ in a sufficiently small neighborhood of s for $\theta < \theta_1$. (In Fig. 2.16 for $\theta < \theta_1$, we have omitted, for the sake of simplicity, a

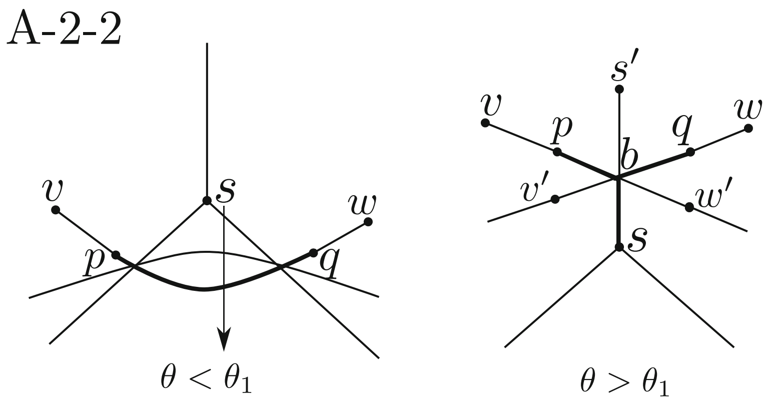


Fig. 2.17 Case A-2-2

new Stokes curve irrelevant to the structure of the tree in question. However, in Fig. 2.17 for $\theta < \theta_1$, we have included its counterpart to make the figure look well-balanced.) After s hits against ℓ, as shown in the same figure, still the bidirectional binary tree T_θ continues to exist where s becomes a new leaf node of T_θ and the degree of T_θ increases. That is, the tree T_θ ($\theta > \theta_1$) is obtained from the tree T_θ ($\theta < \theta_1$) in which the edge $\ell = (p, q; v, w; \mathscr{V}, \mathscr{W})$ is replaced with the following 3-edges ℓ_1, ℓ_2 and ℓ_3 having the same branching node b as their end points:

$$\ell_1 = (s, b; s, s'; \mathscr{S}, \mathscr{S}'), \quad \ell_2 = (p, b; v, w'; \mathscr{V}, \mathscr{W}'),$$
$$\ell_3 = (q, b; w, v'; \mathscr{W}, \mathscr{V}').$$

A-3. The geometric situation is similar to that of A-2 for $\theta < \theta_1$, but s is a double turning point.
The bidirectional binary tree T_θ continues to exist near $\theta = \theta_1$ where degree of T_θ is unchanged. This is because the vector field which defines the related Stokes curve of ℓ is non-degenerate near s, and hence, no bifurcation phenomenon happens and the tree T_θ is continuously deformed when θ moves near θ_1.

Case B: Let us consider a situation where end points of an edge $\ell = (v, b; v, w; \mathscr{V}, \mathscr{W})$ merge in $G(t(\theta_1))$, that is, b coincides with v. Here we only consider the case when one of the end points of ℓ is a leaf node. See [H2] for the other cases.

B-1. v is a double turning point d.
The ℓ has the form $(d, b; d, d'; \mathscr{D}, \mathscr{D}')$ (cf. Fig. 2.18). Since all the roots are holomorphic near d, related Stokes curves also continuously move when θ moves near $\theta = \theta_1$. Therefore no discontinuous phenomenon happens in this case. Note that, when $\theta = \theta_1$, that is, b and d coincide, the other two edges with

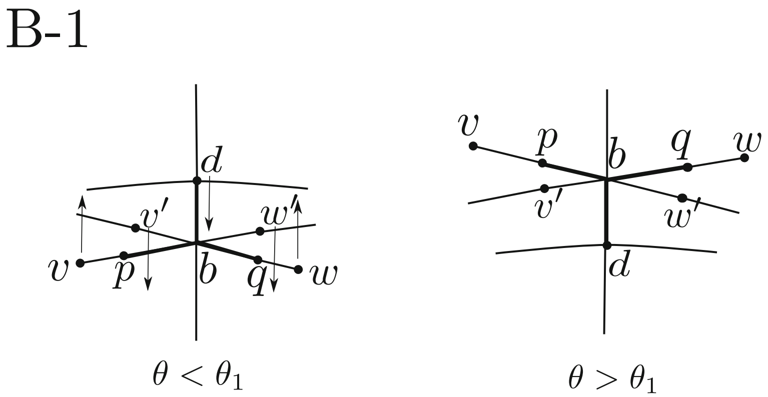


Fig. 2.18 Case B-1

b in their end points, i.e., $[p,\ b]$ and $[q,\ b]$ in Fig. 2.18, become tangent at b. To be more precise, the Stokes curves containing these two edges are tangent at b. Therefore the configuration of the relevant part of T_θ becomes the one described in Fig. 2.18, and hence, the bidirectional binary tree T_θ continues to exist after θ_1 and the degree of T_θ remains constant.

B-2. v is a simple turning point s.
This is the reverse case of A-2, that is, configurations at $\theta > \theta_1$ in Figs. 2.16 and 2.17 are the initial ones in this case, and the resulting configurations are those at $\theta < \theta_1$ in the same figures. Hence the leaf node v will be removed from the bidirectional binary tree T_θ, and the degree of T_θ decreases when $\theta > \theta_1$.

As an immediate consequence of these observations, we have the following proposition.

Proposition 2.5.1 *Assume that, for each exceptional point $\theta > 0$, only one of the cases A-2, A-3, B-1 or B-2 occurs in $G(t(\theta))$. Then there exists a bidirectional binary tree T_θ $(\theta > 0)$ satisfying that T_θ is continuously deformed when θ moves and its total integral value $\Phi(T_\theta)$ is an analytic function of θ.*

Proof When $\theta > 0$ is sufficiently small, it follows from Theorems 2.1 and 2.2 in [T4] that there exists a pair of ordinary turning points which are connected by a Stokes curve. Hence a bidirectional binary tree of degree 2 exists for a sufficiently small $\theta > 0$. Then the first part of the proposition follows from Theorem 2.5.1 and the above observations. Hence it suffices to show $\Phi(T_\theta)$ to be an analytic function of θ at $\theta = \theta_1$ where one of the cases A-2, A-3, B-1 or B-2 occurs in $G(t(\theta_1))$. Here we show the claim for A-2. The other cases can be proved by modifying the path of integration in the same way as that of A-2.

Suppose that a simple turning point s hits on an edge $\ell = (p,\ q;\ v,\ w;\ \mathscr{V},\ \mathscr{W}) \in L$ at $t = t(\theta_1)$. Further we assume that the type of s is $(i,\ j)$, the type of $\mathscr{V}$ is (j, k) and $j > k$ holds on $\mathscr{V}$. By Lemma 2.4.1, we have, for $\theta < \theta_1$,

$$\begin{aligned}\Phi(T_\theta) &= \left|\int_v^w (\lambda_j(x) - \lambda_k(x))dx\right| = \int_v^w (\lambda_j(x) - \lambda_k(x))dx \\ &= \int_{C_j} \lambda_j(x)dx - \int_{C_k} \lambda_k(x)dx,\end{aligned}$$

where the paths C_j and C_k are $[v, w]$ with orientation from v to w. The last expression is still valid for $\theta > \theta_1$ by modifying the path C_j so that the ordinary turning point s avoids hitting against C_j.

On the other hand, when $\theta > \theta_1$, the tree T_θ is obtained from the tree T_θ $(\theta < \theta_1)$ in which the edge ℓ is replaced with the following 3-edges having the same branching node b (cf. Figs. 2.16 and 2.17):

$$\begin{aligned}&\ell_1 = (s, b; s, s'; \mathscr{S}, \mathscr{S}'), \quad \ell_2 = (p, b; v, w'; \mathscr{V}, \mathscr{W}'), \\ &\ell_3 = (q, b; w, v'; \mathscr{W}, \mathscr{V}').\end{aligned}$$

Here we may assume that the type of $\mathscr{S}$ (resp. $\mathscr{V}$ and $\mathscr{W}$) near b is (i, j) (resp. (j, k) and (i, k)). Then it follows from Lemma 2.4.1 and the fact $j > k$ on $\mathscr{V}$, and hence, $k > i$ on $\mathscr{W}$ and $j > i$ on $\mathscr{S}'$ ($\iff$ $i > j$ on $\mathscr{S}$) that we have

$$\begin{aligned}\Phi(T_\theta) &= \left|\int_v^b (\lambda_j - \lambda_k)dx\right| + \left|\int_s^b (\lambda_i - \lambda_j)dx\right| + \left|\int_w^b (\lambda_k - \lambda_i)dx\right| \\ &= \int_v^b (\lambda_j - \lambda_k)dx + \int_s^b (\lambda_i - \lambda_j)dx + \int_w^b (\lambda_k - \lambda_i)dx \\ &= \int_v^b (\lambda_j - \lambda_k)dx + \int_C \lambda_j dx + \int_w^b (\lambda_k - \lambda_i)dx\end{aligned}$$

for $\theta > \theta_1$, where the path C is a closed path which starts from b and turns around s once. Then it is equal to

$$\left(\int_v^b \lambda_j dx + \int_C \lambda_j dx + \int_b^w \lambda_i dx\right) - \left(\int_v^b \lambda_k dx + \int_b^w \lambda_k dx\right).$$

As λ_j is changed to λ_i after the analytic continuation along C, we find

$$\left(\int_v^b \lambda_j dx + \int_C \lambda_j dx + \int_b^w \lambda_i dx\right) = \int_{C_j} \lambda_j(x)dx$$

and

$$\left(\int_v^b \lambda_k dx + \int_b^w \lambda_k dx\right) = \int_{C_k} \lambda_k(x)dx.$$

Hence we have obtained, for $\theta > \theta_1$,

$$\Phi(T_\theta) = \int_{C_j} \lambda_j(x)dx - \int_{C_k} \lambda_k(x)dx,$$

which entails that $\Phi(T_\theta)$ is analytic at $\theta = \theta_1$. This completes the proof.

Let ν_1 and ν_2 be a pair of roots which defines the turning point t^* of $(\mathrm{NY})_{2m}$, that is, $\nu_1(t)$ and $\nu_2(t)$ merge at $t = t^*$. Then, as a corollary of Proposition 2.5.1, we have:

Corollary 2.5.1 *Assume the same conditions as those in Proposition 2.5.1 hold. Then we have*

$$\Phi(T_\theta) = \frac{1}{2}\left|\int_{t^*}^{t(\theta)} (\nu_1(s) - \nu_2(s))ds\right| \qquad (\theta > 0). \tag{2.5.1}$$

Proof When $\theta > 0$ is sufficiently small, T_θ is of degree 2, that is, two ordinary turning points are connected by a Stokes curve. In this case, the equality was shown in Theorems 2.1 and 2.2 in [T4]. Then both sides of (2.5.1) are analytic functions of θ. Hence the equality holds for any $\theta > 0$.

Chapter 3
Exact WKB Analysis of Non-adiabatic Transition Problems for 3-Levels

As we have observed so far, virtual turning points and (new) Stokes curves emanating from them play a crucially important role in discussing the Stokes geometry of a higher order ordinary differential equation and/or a system of ordinary differential equations of size greater than two. Once all the non-redundant virtual turning points are provided, then we can explicitly calculate the analytic continuation of solutions of an ordinary differential equation in view of its complete Stokes geometry and connection formulas discussed in Sect. 1.4. Adopting this approach, we consider the non-adiabatic transition problem for three levels and compute transition probabilities of solutions in this chapter. This is a good application of the exact WKB analysis for a higher order ordinary differential equation to a physical problem, illuminating the role of virtual turning points in the calculation of analytic continuation of solutions.

3.1 Non-adiabatic Transition Problems for Three Levels—Generalization of the Landau-Zener Model

The argument in this chapter mainly follows that employed in [AKT4].

We first explain the **non-adiabatic transition problem** for three levels in a more specific manner. Let $H(t,\eta)$ be a 3×3 matrix with polynomial entries that depends on a large parameter η in the following manner:

$$H(t,\eta) = H_0(t) + \eta^{-1/2} H_{1/2}, \tag{3.1.1}$$

where $H_0(t)$ is a diagonal matrix of the form

$$H_0(t) = \begin{pmatrix} \rho_1(t) & & \text{\Large 0} \\ & \rho_2(t) & \\ \text{\Large 0} & & \rho_3(t) \end{pmatrix} \tag{3.1.2}$$

N. Honda et al., *Virtual Turning Points*, SpringerBriefs in Mathematical Physics,
DOI 10.1007/978-4-431-55702-9_3

whose diagonal entries $\rho_j(t)$ $(j = 1, 2, 3)$ are real polynomials, and $H_{1/2}$ is a constant Hermite matrix of the form

$$H_{1/2} = \begin{pmatrix} 0 & c_{12} & c_{13} \\ \overline{c_{12}} & 0 & c_{23} \\ \overline{c_{13}} & \overline{c_{23}} & 0 \end{pmatrix}. \tag{3.1.3}$$

In what follows we assume the following condition:

$$(\rho_1 - \rho_2)(\rho_2 - \rho_3)(\rho_3 - \rho_1) = 0 \text{ has only real and simple zeros.} \tag{3.1.4}$$

A non-adiabatic transition problem for three levels is then to consider

$$i\frac{d}{dt}\psi = \eta H(t, \eta)\psi \tag{3.1.5}$$

for a 3-vector $\psi = \psi(t, \eta)$ and, in particular, to relate the behavior of $\psi(t)$ for $t \to -\infty$ and that for $t \to +\infty$, that is, to compute the transition probabilities.

Usually a non-adiabatic transition problem (3.1.5) with $\rho_j(t)$ being a linear function of t is called the **Landau-Zener model** and (3.1.5) is its generalization. The original Landau-Zener problem is that for two levels of the form

$$i\frac{d}{dt}\varphi = \eta\left[\begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} + \eta^{-1/2}\begin{pmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{pmatrix}\right]\varphi \tag{3.1.6}$$

with a 2-vector φ and a complex constant $\mu \in \mathbb{C}$ (cf. [L, Z]). Since the pioneering work of Landau and that of Zener, many researches have been done not only for two level problems but also for three level problems from both physical and mathematical viewpoint (cf., e.g., [CH, H, BE, J1, J2, CLP, JP], and references cited therein). In our approach we use WKB solutions of (3.1.5) to describe asymptotic behaviors of $\psi(t)$ for $t \to \pm\infty$ and then apply the connection formulas given in [AKT4] to relate the behavior of WKB solutions near $t = -\infty$ with that near $t = +\infty$.

Remark 3.1.1 The particular η-dependence of $H(t, \eta)$ described by (3.1.1) and (3.1.6) guarantees the non-adiabatic character of the problem. See [H] for an intriguing discussion that explains the non-adiabatic character in the case of two level problems.

Here we briefly review the construction of WKB solutions of (3.1.5). One way of constructing WKB solutions for a system of ordinary differential equations like (3.1.5) is to use the formal diagonalization (cf. [W]). That is, we formally transform unknown functions ψ of (3.1.5) by

$$\psi = R(t, \eta)\varphi \tag{3.1.7}$$

with

$$R(t,\eta) = R_0(t) + \eta^{-1/2}R_{1/2}(t) + \eta^{-1}R_1(t) + \cdots \tag{3.1.8}$$

in such a way that a differential equation for φ becomes a diagonalized system. A straightforward calculation tells us that φ should satisfy

$$i\frac{d}{dt}\varphi = \eta\tilde{H}(t,\eta)\varphi, \tag{3.1.9}$$

where

$$\tilde{H}(t,\eta) = R(t,\eta)^{-1}(H_0(t) + \eta^{-1/2}H_{1/2})R(t,\eta) - i\eta^{-1}R(t,\eta)^{-1}\frac{\partial}{\partial t}R(t,\eta). \tag{3.1.10}$$

Since the top order part $H_0(t)$ of (3.1.5) with respect to η is already diagonal, we can take $R_0(t) = \mathrm{Id}$. Then, using the expression

$$\begin{aligned} R(t,\eta)^{-1} &= (\mathrm{Id} + \eta^{-1/2}R_{1/2}(t) + \eta^{-1}R_1(t) + \cdots)^{-1} \\ &= \mathrm{Id} - (\eta^{-1/2}R_{1/2}(t) + \eta^{-1}R_1(t) + \cdots) \\ &\qquad + (\eta^{-1/2}R_{1/2}(t) + \eta^{-1}R_1(t) + \cdots)^2 - \cdots, \end{aligned} \tag{3.1.11}$$

we find that

$$\tilde{H}(t,\eta) = H_0 + \eta^{-1/2}\left([H_0, R_{1/2}] + H_{1/2}\right) + \cdots + \eta^{-n/2}\tilde{H}_{n/2} + \cdots \tag{3.1.12}$$

where $[A, B] = AB - BA$ denotes the commutator of two matrices A and B, and $\tilde{H}_{n/2}$ has the form

$$\tilde{H}_{n/2} = [H_0, R_{n/2}] + (\text{terms depending only on } H_0, H_{1/2}, R_{1/2}, \ldots, R_{(n-1)/2}) \tag{3.1.13}$$

($n = 1, 2, \ldots$). Note that the (j,k)-entry of $[H_0, R_{n/2}]$ is $(\rho_j - \rho_k)(R_{n/2})_{jk}$, where $(R_{n/2})_{jk}$ is the (j,k)-entry of $R_{n/2}$ ($j, k = 1, 2, 3$). Hence, if we define each diagonal entry $(R_{n/2})_{jj}$ of $R_{n/2}$ to be zero and also determine the other off-diagonal entries $(R_{n/2})_{jk}$ ($j \neq k$) of $R_{n/2}$ so that all the off-diagonal entries of $\tilde{H}_{n/2}$ may vanish, Eq. (3.1.9) for φ becomes a diagonal system. For example, the first few terms of the transformation $R(t,\eta)$ and those of the coefficient matrix $\tilde{H}(t,\eta)$ of the diagonalized system (3.1.9) are given as follows:

$$R_0 = \mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1.14}$$

$$R_{1/2} = \begin{pmatrix} 0 & \frac{c_{12}}{\rho_2-\rho_1} & \frac{c_{13}}{\rho_3-\rho_1} \\ \frac{\overline{c_{12}}}{\rho_1-\rho_2} & 0 & \frac{c_{23}}{\rho_3-\rho_2} \\ \frac{\overline{c_{13}}}{\rho_1-\rho_3} & \frac{\overline{c_{23}}}{\rho_2-\rho_3} & 0 \end{pmatrix}, \tag{3.1.15}$$

$$R_1 = \begin{pmatrix} 0 & \frac{\overline{c_{23}}c_{13}}{(\rho_2-\rho_1)(\rho_2-\rho_3)} & \frac{c_{12}c_{23}}{(\rho_3-\rho_1)(\rho_3-\rho_2)} \\ \frac{c_{23}\overline{c_{13}}}{(\rho_1-\rho_2)(\rho_1-\rho_3)} & 0 & \frac{\overline{c_{12}}c_{13}}{(\rho_3-\rho_1)(\rho_3-\rho_2)} \\ \frac{\overline{c_{12}}\ \overline{c_{23}}}{(\rho_1-\rho_2)(\rho_1-\rho_3)} & \frac{c_{12}\overline{c_{13}}}{(\rho_2-\rho_1)(\rho_2-\rho_3)} & 0 \end{pmatrix}, \tag{3.1.16}$$

$$\tilde{H}_0 = H_0 = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \tag{3.1.17}$$

$$\tilde{H}_{1/2} = 0, \tag{3.1.18}$$

$$\tilde{H}_1 = \begin{pmatrix} \frac{|c_{12}|^2}{\rho_1-\rho_2} + \frac{|c_{13}|^2}{\rho_1-\rho_3} & 0 & 0 \\ 0 & \frac{|c_{12}|^2}{\rho_2-\rho_1} + \frac{|c_{23}|^2}{\rho_2-\rho_3} & 0 \\ 0 & 0 & \frac{|c_{13}|^2}{\rho_3-\rho_1} + \frac{|c_{23}|^2}{\rho_3-\rho_2} \end{pmatrix}. \tag{3.1.19}$$

As system (3.1.9) is now diagonalized, it can be readily solved and consequently we obtain a WKB solution

$$\psi^{(j)} = \eta^{-1/2} \exp\left(\frac{\eta}{i} \int_{t_0}^{t} \rho_j(t)\, dt + \frac{1}{i} \int_{t_0}^{t} \left(\frac{|c_{jk}|^2}{\rho_j - \rho_k} + \frac{|c_{jl}|^2}{\rho_j - \rho_l} \right) dt \right) \left(e^{(j)} + O(\eta^{-1/2}) \right) \tag{3.1.20}$$

($j = 1, 2, 3$) of (3.1.5) by substituting a solution of (3.1.9) into the right-hand side of (3.1.7), where t_0 is an appropriately fixed reference point, $e^{(j)} = {}^t(e_1^{(j)}, e_2^{(j)}, e_3^{(j)})$ is a unit vector satisfying

$$e_k^{(j)} = \delta_{jk} \quad (j, k = 1, 2, 3), \tag{3.1.21}$$

and $\{j, k, l\}$ is a permutation of $\{1, 2, 3\}$ (i.e., $\{j, k, l\} = \{1, 2, 3\}$ holds as sets). Here and in what follows, in parallel with the case of higher order scalar equations, we add an extra factor $\eta^{-1/2}$ to a WKB solution so that its Borel transform is cleanly defined.

Although the top term (with respect to η)

$$\frac{1}{i} \int_{t_0}^{t} \rho_j(t)\, dt \tag{3.1.22}$$

of a WKB solution (3.1.20) is single-valued and pure-imaginary, the next term

$$\frac{1}{i} \int_{t_0}^{t} \left(\frac{|c_{jk}|^2}{\rho_j - \rho_k} + \frac{|c_{jl}|^2}{\rho_j - \rho_l} \right) dt \tag{3.1.23}$$

and lower order terms are in general multi-valued and not necessarily pure-imaginary near $t = \pm\infty$. Taking this situation into account, we introduce additional normalization factors $N^{\pm,(j)}$ to define new fundamental systems of solutions

$$\psi^{\pm,(j)} = N^{\pm,(j)}\psi^{(j)} \tag{3.1.24}$$

of (3.1.5) so that the following asymptotic conditions may be satisfied near $t = \pm\infty$:

$$\lim_{t\to\pm\infty} \left|\psi^{\pm,(j)}(t)\right| = e^{(j)} \quad (j = 1, 2, 3). \tag{3.1.25}$$

Under these notations the S**-matrix** for Eq. (3.1.5) is described as

$$S = \begin{pmatrix} N^{+,(1)} & 0 & 0 \\ 0 & N^{+,(2)} & 0 \\ 0 & 0 & N^{+,(3)} \end{pmatrix}^{-1} M \begin{pmatrix} N^{-,(1)} & 0 & 0 \\ 0 & N^{-,(2)} & 0 \\ 0 & 0 & N^{-,(3)} \end{pmatrix}, \tag{3.1.26}$$

where M is the connection matrix for the WKB solutions $\psi^{(j)}$ of (3.1.5) from $t = -\infty$ to $t = +\infty$:

$$(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}) \ \text{near}\, t = -\infty \quad \underset{\substack{\text{analytic}\\ \text{continuation}}}{\longmapsto} \quad (\psi^{(1)}, \psi^{(2)}, \psi^{(3)})\, M \ \text{near}\, t = +\infty. \tag{3.1.27}$$

In particular, the square of the modulus of each entry S_{jk} of S represents the **transition probability** for (3.1.5).

Our task is then to compute the connection matrix M by applying the exact WKB analysis to Eq. (3.1.5), i.e., by using the connection formula given in [AKT4]. A key step is to find the complete Stokes geometry of (3.1.5) based on the fundamental results explained in Chap. 1 to obtain the correct connection formula. Note that, since all the turning points of the non-adiabatic transition problem (3.1.5) are not simple but double turning points, we need some modifications for the basic definitions and results of Chap. 1. Having in mind this different character of the problem in question, we consider several concrete examples of complete Stokes geometries of (3.1.5) in Sect. 3.2 and then calculate transition probabilities in Sect. 3.3.

3.2 Examples of Complete Stokes Geometries for Non-adiabatic Transition Problems

In this section, using some concrete examples, we discuss how to obtain complete Stokes geometries of non-adiabatic transition problems (3.1.5).

In the case of (3.1.5) the characteristic equation $\det(\zeta + iH_0(t))$ of $-iH_0(t)$ and its diagonal entries $-i\rho_j(t)$ $(j = 1, 2, 3)$ play the role of the characteristic polynomial (1.2.13) and its roots $\zeta_j(x)$ for a higher order differential Eq. (1.2.1), respectively. Thus turning points and Stokes curves for (3.1.5) are defined as follows:

Definition 3.2.1 (i) A point $t = \tau$ is said to be a turning point (of type (j, k)) of (3.1.5) if

$$\rho_j(\tau) = \rho_k(\tau) \tag{3.2.1}$$

holds for some j and k with $j \neq k$.
(ii) A Stokes curve (of type (j, k)) is the curve given by

$$\operatorname{Im}(-i)\int_{\tau}^{t}\big(\rho_j(t) - \rho_k(t)\big)dt = 0, \tag{3.2.2}$$

where $t = \tau$ is a turning point of type (j, k).

Thanks to the assumption (3.1.4) all the turning points of (3.1.5) are located on the real axis. Furthermore every turning point $t = \tau$ of (3.1.5) is a double turning point in the sense that τ is a double zero of the discriminant (with respect to ζ) of the characteristic equation $\det(\zeta + iH_0(t))$. (Note that a simple turning point of (1.2.1) is a simple zero of the discriminant of the characteristic polynomial (1.2.13)). Virtual turning points of (3.1.5) can be also defined similarly to those of a higher order Eq. (1.2.1) with slight modifications.

Remark 3.2.1 In Sect. 1.4, taking into account the fact that the Borel transform of (1.2.1) is with simple characteristics and that the most elementary carrier of singularities of solutions of a differential equation with simple characteristics is a bicharacteristic strip, we define a virtual turning point of (1.2.1) to be the x-component of a self-intersection point of a bicharacteristic curve, i.e., the projection to the base manifold $\mathbb{C}^2$ of a bicharacteristic strip, of (1.2.1). In the case of (3.1.5) in question, however, it has double turning points and hence its Borel transform is not with simple characteristics but rather with multiple characteristics. As is shown in, e.g., [KKO], singularities of solutions of such an equation propagate along the so-called **bicharacteristic chain**. Here a bicharacteristic chain $b(\kappa, T)$ is, by definition, the collection of bicharacteristic curves defined by

$$b(\kappa, T) = b_1(k_1; t^{(1)}) \cup \left(\bigcup_{j=2}^{n-1} b_j(k_j; t^{(j-1)}, t^{(j)})\right) \cup b_n(k_n; t^{(n-1)}), \tag{3.2.3}$$

where $\kappa = (k_1, \ldots, k_n)$ is a multi-index with $k_l \in \{1, 2, 3\}$ $(l = 1, \ldots, n)$ such that $k_l \neq k_{l+1}$ $(l = 1, \ldots, n-1)$, $T = (t^{(1)}, \ldots, t^{(n-1)})$ is a set of turning points satisfying $\rho_{k_l}(t^{(l)}) = \rho_{k_{l+1}}(t^{(l)})$ $(l = 1, \ldots, n-1)$, $b_1(k_1; t^{(1)})$ (resp., $b_n(k_n; t^{(n-1)})$) is a bicharacteristic curve in $\mathbb{C}^2_{(t,y)}$ defined by

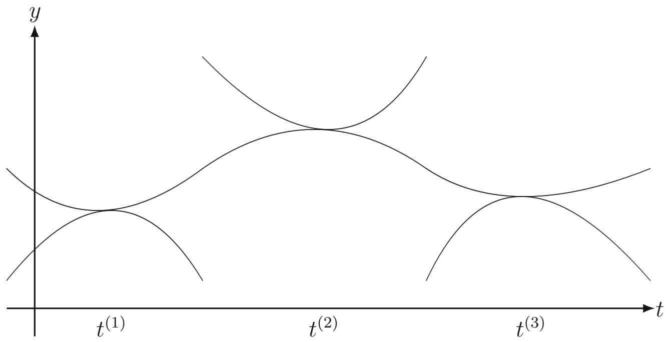


Fig. 3.1 Schematic figure of a bicharacteristic chain

$$\frac{dy}{dt} = i\rho_{k_1} \quad \left(\text{resp., } \frac{dy}{dt} = i\rho_{k_n}\right) \tag{3.2.4}$$

emanating from $t = t^{(1)}$ (resp., $t = t^{(n-1)}$), and $b_j(k_j; t^{(j-1)}, t^{(j)})$ ($j = 2, \ldots, n-1$) are bicharacteristic curves defined by

$$\frac{dy}{dt} = i\rho_{k_j} \tag{3.2.5}$$

that connects two turning points $t = t^{(j-1)}$ and $t = t^{(j)}$ such that b_j and b_{j+1} are tangent at $t = t^{(j)}$ ($j = 1, \ldots, n-1$) (cf. Fig. 3.1). In other words, singularities of solutions of the Borel transform of (3.1.5) bifurcate along two mutually tangent bicharacteristic curves at a double turning point, i.e., at a point where the simple characteristic condition is violated.

Thus virtual turning points of (3.1.5) are defined similarly to those of (1.2.1) with a bicharacterisitc curve being replaced by a bicharacteristic chain in their definition. For more details see Sect. 3 of [AKT4].

Remark 3.2.2 Although it is needed to replace a bicharacteristic curve by a bicharacteristic chain in defining virtual turning points of (3.1.5), the important properties similar to Proposition 1.4.1 does hold also for virtual turning points of (3.1.5). That is, if a Stokes curve $\mathscr{S}_1$ of type (j,k) of (3.1.5) that emanates from a (double) turning point $t^{(1)}$ crosses at a point C with another Stokes curve $\mathscr{S}_2$ of type (k,l) emanating from a (double) turning point $t^{(2)}$, and further if t_* satisfies the relation

$$\int_{t^{(1)}}^{t_*} \rho_j \, dt = \int_{t^{(1)}}^{t^{(2)}} \rho_k \, dt + \int_{t^{(2)}}^{t_*} \rho_l \, dt \tag{3.2.6}$$

for a triplet of mutually distinct suffixes (j,k,l), then t_* is a virtual turning point of (3.1.5) and the Stokes curve $\mathscr{S}$ emanating from t_* with type (j,l) passes through the crossing point C. More generally, if $\mathscr{S}$ is a new Stokes curve of type (k, l) of (3.1.5) that emanates from a virtual turning point t_* determined (by repeating this procedure successively) by the relation

$$\int_{t^{(1)}}^{t_*} \rho_k \, dt = \sum_{l=1}^{n-1} \int_{t^{(l)}}^{t^{(l+1)}} \rho_{k_l} \, dt + \int_{t^{(n)}}^{t_*} \rho_l \, dt, \tag{3.2.7}$$

and if $\mathscr{S}$ crosses at a point C with a Stokes curve $\widetilde{\mathscr{S}}$ of type (j, k) emanating from a double turning point $t^{(0)}$, then a point t_{**} satisfying

$$\int_{t^{(0)}}^{t_{**}} \rho_j \, dt = \int_{t^{(0)}}^{t^{(1)}} \rho_k \, dt + \sum_{l=1}^{n-1} \int_{t^{(l)}}^{t^{(l+1)}} \rho_{k_l} \, dt + \int_{t^{(n)}}^{t_{**}} \rho_l \, dt \tag{3.2.8}$$

is a virtual turning point of (3.1.5) and the (union of) Stokes curve emanating from t_{**} with type (j, l) passes through C. (The situation where $\widetilde{\mathscr{S}}$ is also a new Stokes curve can be discussed in a similar manner.) These properties can be verified by replacing bicharacteristic curves by bicharacteristic chains in the proof of Proposition 1.4.1. For more details see [AKT4, (3.5)] and its proof in pages 2413–2414 of [AKT4].

Thus, practically speaking, we can deal with virtual turning points and the Stokes geometry of the non-adiabatic transition problem (3.1.5) in a completely parallel way with those of a higher order differential Eq. (1.2.1) in view of Remark 3.2.2. In what follows we discuss several concrete examples to illustrate how to obtain a complete Stokes geometry of (3.1.5).

Example 3.2.1 First we consider a simple example discussed in [AKT4, Sect. 2]:

$$i\frac{d}{dt}\psi = \eta \left[\begin{pmatrix} b_1 t + a & 0 & 0 \\ 0 & b_2 t & 0 \\ 0 & 0 & b_3 t \end{pmatrix} + \eta^{-1/2} \begin{pmatrix} 0 & c_{12} & c_{13} \\ \overline{c_{12}} & 0 & c_{23} \\ \overline{c_{13}} & \overline{c_{23}} & 0 \end{pmatrix} \right] \psi, \tag{3.2.9}$$

that is, the three-level Landau-Zener problem with

$$\rho_1(t) = b_1 t + a, \quad \rho_2(t) = b_2 t, \quad \rho_3(t) = b_3 t. \tag{3.2.10}$$

Here we assume that a and b_j $(j = 1, 2, 3)$ are real constants satisfying

$$0 < b_1 < b_2 < b_3 \quad \text{and} \quad 0 < a. \tag{3.2.11}$$

Equation (3.2.9) is a straightforward generalization to three levels of the original Landau-Zener model (3.1.6) for two levels.

Let t_{jk} denote the unique solution of the equation $\rho_j(t) = \rho_k(t)$, i.e., t_{jk} is a turning point of type (j, k) of (3.2.9). It then follows from assumption (3.2.11) that

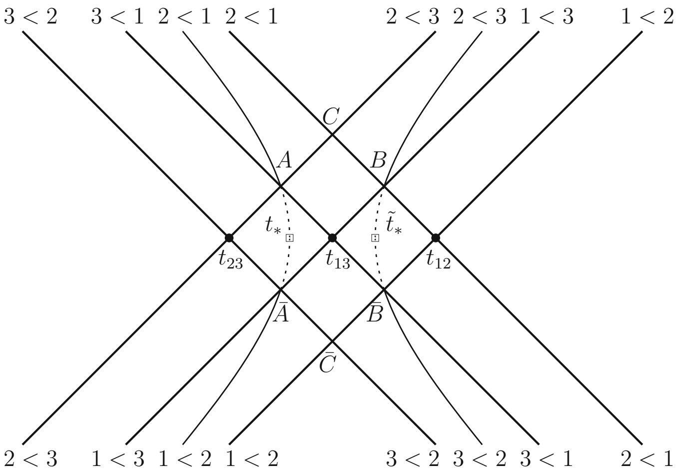


Fig. 3.2 Complete Stokes geometry of system (3.2.9)

$$\begin{array}{ccccc} t_{23} & < & t_{13} & < & t_{12} \\ \| & & \| & & \| \\ 0 & & \dfrac{a}{b_3 - b_1} & & \dfrac{a}{b_2 - b_1} \end{array}. \tag{3.2.12}$$

Each Stokes curve of (3.2.9) emanating from a turning point t_{jk}

$$\mathrm{Im}\,(-i) \int_{t_{jk}}^{t} \big(\rho_j(t) - \rho_k(t)\big)\, dt = \mathrm{Re}\, \frac{(b_k - b_j)(t - t_{jk})^2}{2} = 0 \tag{3.2.13}$$

is a straight line, as is shown in Fig. 3.2. In Fig. 3.2 a turning point is designated by a small dot and the type of each Stokes curve is also specified.

As Fig. 3.2 clearly visualizes, there are four ordered crossing points (A, B and their mirror images $\bar{A}$, $\bar{B}$) and two non-ordered crossing points (C and $\bar{C}$) for Eq. (3.2.9). Among them the ordered crossing point A (together with its mirror image $\bar{A}$) should be resolved by a new Stokes curve emanating from a virtual turning point that satisfies (3.2.6) corresponding to the crossing point A, that is,

$$\int_{t_{13}}^{t_*} \rho_1\, dt = \int_{t_{13}}^{t_{23}} \rho_3\, dt + \int_{t_{23}}^{t_*} \rho_2\, dt. \tag{3.2.14}$$

The explicit form of (3.2.14) is

$$(b_2 - b_1)(t_*)^2 - 2a\, t_* + \frac{a^2}{b_3 - b_1} = 0. \tag{3.2.15}$$

We then find that a new Stokes curve emanating from

$$t_* = \frac{a}{b_2 - b_1}\left(1 - \sqrt{\frac{b_3 - b_2}{b_3 - b_1}}\right), \tag{3.2.16}$$

which is one of the solutions of (3.2.15), resolves the ordered crossing points A and $\bar{A}$ simultaneously.

Similarly, the ordered crossing points B and $\bar{B}$ are resolved by a new Stokes curve emanating from a virtual turning point

$$\tilde{t}_* = \frac{a}{\sqrt{(b_2 - b_1)(b_3 - b_1)}} \tag{3.2.17}$$

which is a solution of

$$\int_{t_{12}}^{\tilde{t}_*} \rho_2\, dt = \int_{t_{12}}^{t_{13}} \rho_1\, dt + \int_{t_{13}}^{\tilde{t}_*} \rho_3\, dt. \tag{3.2.18}$$

Thus, adding two virtual turning points t_* and $\tilde{t}_*$ and (new) Stokes curves emanating from them, we resolve all the ordered crossing points of ordinary Stokes curves and obtain a complete Stokes geometry of (3.2.9), where there are no further ordered crossing points. See Fig. 3.2 for a complete Stokes geometry of (3.2.9). (The virtual turning points t_* and $\tilde{t}_*$ are designated by small rectangles there.)

Remark 3.2.3 In the above discussion we have obtained the complete Stokes geometry of (3.2.9) by locating relevant virtual turning points through the integral relations (3.2.14) and (3.2.18). On the other hand, the complete Stokes geometry of (3.2.9) can be obtained also by pursuing Recipe 1.5.1 more faithfully. As a matter of fact, in the case of Eq. (3.2.9), all possible virtual turning points are analytically determined and it turns out that there exist infinitely many virtual turning points and they form a discrete subset of $\mathbb{C}_t$. Then the discreteness of the set of virtual turning points enables us to pursue Recipe 1.5.1 and consequently we obtain the same complete Stokes geometry of (3.2.9) as Fig. 3.2. For more details about the application of Recipe 1.5.1 to Eq. (3.2.9) see Remark 2.1 of [AKT4].

Example 3.2.2 Next we consider a more complicated example, which was discussed in [AKT4, Example 3.1]:

$$i\frac{d}{dt}\psi = \eta\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & t/2 & 0 \\ 0 & 0 & t^2 \end{pmatrix} + \eta^{-1/2}\begin{pmatrix} 0 & c_{12} & c_{13} \\ \overline{c_{12}} & 0 & c_{23} \\ \overline{c_{13}} & \overline{c_{23}} & 0 \end{pmatrix}\right]\psi, \tag{3.2.19}$$

that is, the example with $\rho_1 = 1$, $\rho_2 = t/2$ and $\rho_3 = t^2$. This example has the following five turning points:

$$\begin{array}{ll} t = 2 & \text{of type (1,2)}, \\ t = -1, 1 & \text{of type (1,3)}, \\ t = 0, 1/2 & \text{of type (2,3)}, \end{array}$$

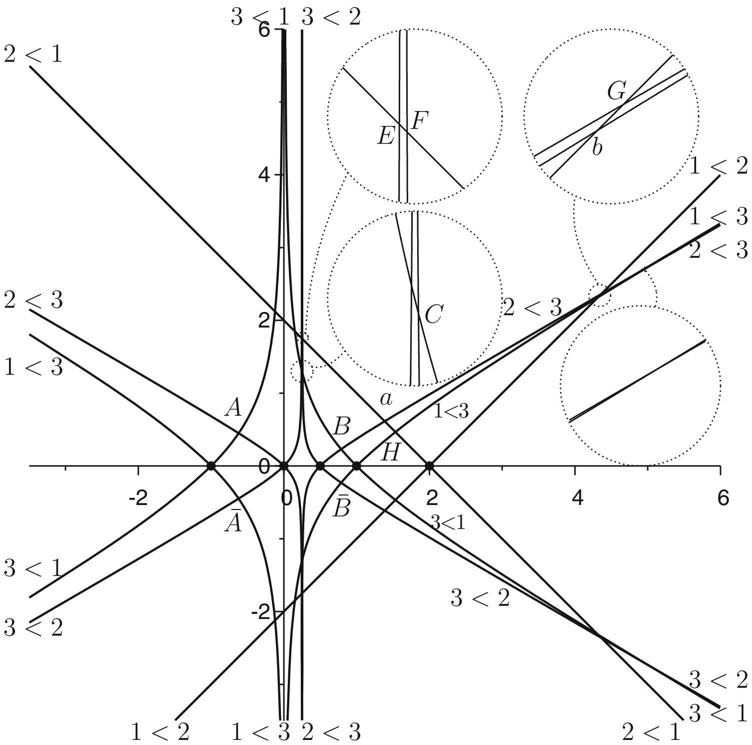


Fig. 3.3 Ordinary Stokes curves of system (3.2.19)

and the configuration of ordinary Stokes curves is given by Fig. 3.3. (As was pointed out by Dr. Sasaki, the types of Stokes curves in [AKT4, Example 3.1] were misprinted and all the inequalities used there in describing the types of Stokes curves should be replaced by opposite ones. For example, "type $1 < 3$" of [AKT4, Example 3.1] should be replaced by "type $3 < 1$", etc. In what follows we use the correct symbols.) From Fig. 3.3 we find that the Stokes geometry is symmetric with respect to the real axis (because of the reality of ρ_j) and that there exist several ordered crossing points of ordinary Stokes curves in the upper half-plane in Fig. 3.3. In what follows we explain how to find relevant virtual turning points and new Stokes curves by using the relations (3.2.6) and (3.2.8) to obtain a complete Stokes geometry of (3.2.19).

Let us start with an ordered crossing point A in the second quadrant. This is a crossing point of a Stokes curve of type $3 < 1$ emanating from $t = -1$ and a Stokes curve of type $2 < 3$ emanating from $t = 0$. Hence in this case (3.2.6) reads as

$$\int_{-1}^{t_*} \rho_1 \, dt = \int_{-1}^{0} \rho_3 \, dt + \int_{0}^{t_*} \rho_2 \, dt, \tag{3.2.20}$$

that is,

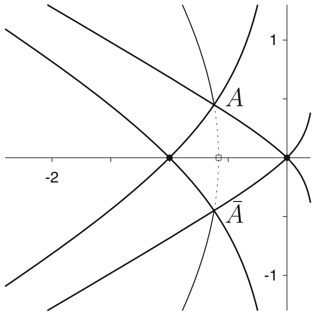


Fig. 3.4 A new Stokes curve resolving ordered crossing points A and $\bar{A}$

$$t_* + 1 = \frac{1}{3} + \frac{1}{4} t_*^2. \tag{3.2.21}$$

Thus, adding a new Stokes curve of type $(1, 2)$ emanating from a root $t_A = 2 - \sqrt{20/3}$ of (3.2.21) to Fig. 3.3, we find that the ordered crossing point A together with its mirror image $\bar{A}$ (complex conjugate of A) is resolved simultaneously (cf. Fig. 3.4).

Similarly, an ordered crossing point B of a Stokes curve of type $2 < 3$ emanating from $t = 1/2$ and that of type $3 < 1$ emanating from $t = 1$ in the first quadrant is resolved, together with its mirror image $\bar{B}$, by a new Stokes curve emanating from a virtual turning point of type $(1, 2)$ defined by

$$\int_{1/2}^{t_*} \rho_2 \, dt = \int_{1/2}^{1} \rho_3 \, dt + \int_{1}^{t_*} \rho_1 \, dt, \tag{3.2.22}$$

or,

$$\frac{1}{4} t_*^2 - \frac{1}{16} = \left(\frac{1}{3} - \frac{1}{24} \right) + (t_* - 1), \tag{3.2.23}$$

that is, $t_* = t_B = 2 - \sqrt{17/12}$. See Fig. 3.5 for the configuration in the upper half-plane of the new Stokes curve of type $2 < 1$ emanating from the virtual turning point $t_B = 2 - \sqrt{17/12}$.

As is observed in Fig. 3.5, this newly added new Stokes curve passes through a crossing point C of a Stokes curve emanating from $t = 1/2$ and that emanating from $t = 1$, and also creates an ordered crossing point D with a Stokes curve of type $3 < 2$ emanating from $t = 0$. A new Stokes curve that resolves this ordered crossing point can be specified by using (3.2.8) in the following manner: As D is a crossing point of a new Stokes curve emanating from a virtual turning point $t_B = 2 - \sqrt{17/12}$ defined by (3.2.22) and a Stokes curve emanating from $t = 0$, the explicit form of (3.2.8) in this case reads as

$$\int_{0}^{t_{**}} \rho_3 \, dt = \int_{0}^{1/2} \rho_2 \, dt + \int_{1/2}^{1} \rho_3 \, dt + \int_{1}^{t_{**}} \rho_1 \, dt, \tag{3.2.24}$$

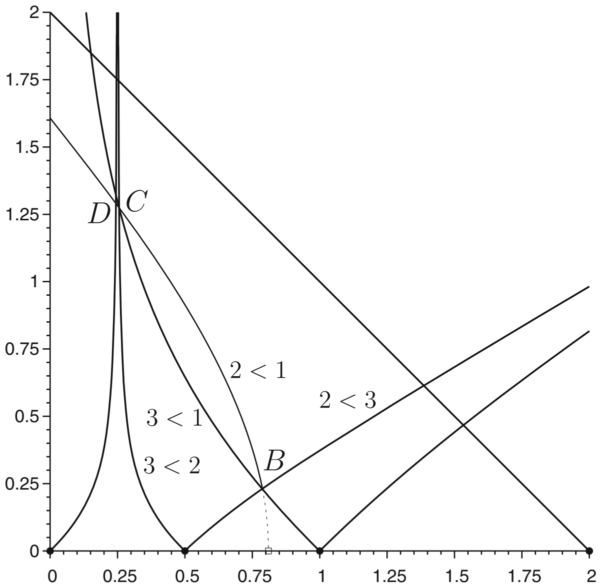


Fig. 3.5 A new Stokes curve resolving an ordered crossing point B

that is,

$$\frac{1}{3}(t_{**})^3 = \frac{1}{16} + \left(\frac{1}{3} - \frac{1}{24}\right) + (t_{**} - 1). \tag{3.2.25}$$

Thus, if we add a new Stokes curve emanating from a root $t_D = 1 - \varepsilon$ ($\varepsilon > 0$) of (3.2.25), we find that the ordered crossing point D is successfully resolved (cf. Fig. 3.6) and, further, these new Stokes curves do not create any more ordered crossing points.

Let us now study other ordered crossing points E and F in the first quadrant. The point E is a crossing point of a Stokes curve of type $3 < 2$ emanating from 0 and that of type $2 < 1$ emanating from 2. Hence it should be resolved by a new Stokes curve emanating from a virtual turning point defined by

$$\int_0^{t_*} \rho_3 \, dt = \int_0^2 \rho_2 \, dt + \int_2^{t_*} \rho_1 \, dt, \tag{3.2.26}$$

that is,

$$\frac{1}{3}t_*^3 = 1 + (t_* - 2). \tag{3.2.27}$$

As a matter of fact, we can check that a new Stokes curve that emanates from a solution t_E of (3.2.27) (near $1.1 + 0.7\sqrt{-1}$) passes through E, resolving ordered crossing (cf. Fig. 3.7).

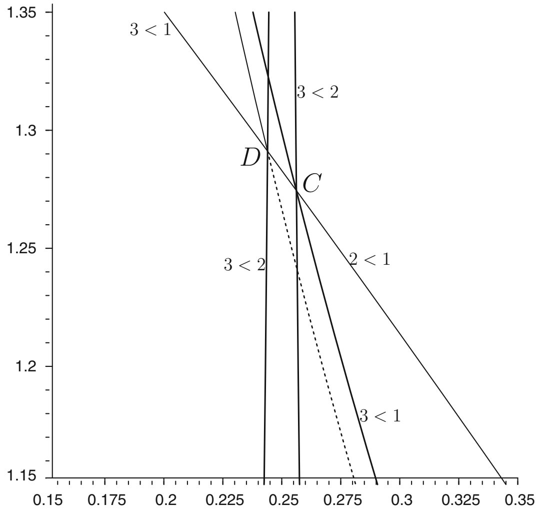


Fig. 3.6 A new Stokes curve resolving an ordered crossing point D

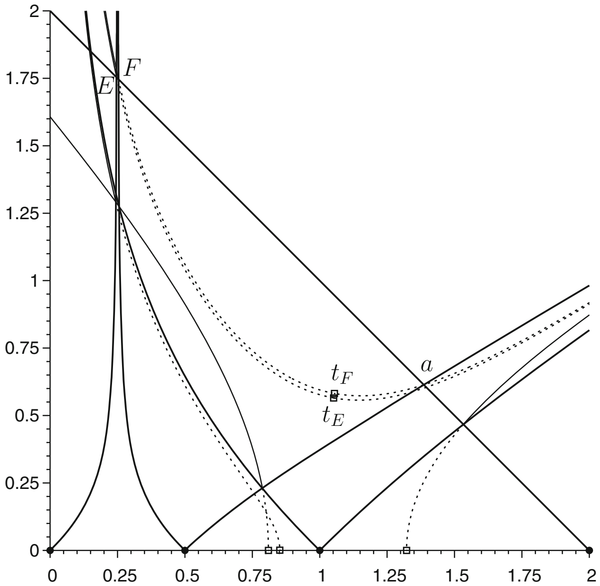


Fig. 3.7 A new Stokes curve resolving ordered crossing points F and G

Similarly, solving

$$\int_{1/2}^{t_*} \rho_3 \, dt = \int_{1/2}^{2} \rho_2 \, dt + \int_{2}^{t_*} \rho_1 \, dt, \tag{3.2.28}$$

i.e.,

$$\frac{1}{3}t_*^3 - \frac{1}{24} = \left(1 - \frac{1}{16}\right) + (t_* - 2), \tag{3.2.29}$$

we can confirm that a new Stokes curve emanating from a solution t_F of (3.2.29) (near t_E) resolves the ordered crossing point F (cf. Fig. 3.7). Note that the defining equation for a virtual turning point needed to resolve another ordered crossing point G is the same as (3.2.28). In fact, a new Stokes curve emanating from t_F also passes through G (and a non-ordered crossing point $t = a$; cf. Fig. 3.7). Thus both F and G are simultaneously resolved by one new Stokes curve emanating from t_F.

Finally, the ordered crossing point H, together with its mirror image $\bar{H}$, in Fig. 3.3 is resolved by a new Stokes curve emanating from a virtual turning point defined by

$$\int_1^{t_*} \rho_3\, dt = \int_1^2 \rho_1\, dt + \int_2^{t_*} \rho_2\, dt, \tag{3.2.30}$$

that is,

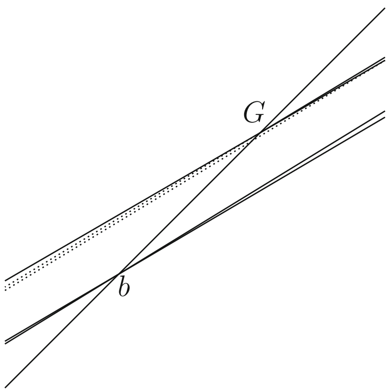
$$\frac{1}{3}t_*^3 - \frac{1}{3} = (2 - 1) + \left(\frac{1}{4}t_*^2 - 1\right). \tag{3.2.31}$$

The new Stokes curve emanating from a real root t_H of (3.2.31) (near 1.3) also passes through a (non-ordered) crossing point $t = b$ of an ordinary Stokes curve emanating from 2 and that emanating from 1 (cf. Fig. 3.8). Hence the new Stokes curve emanating from t_H creates no more ordered crossing points.

We have thus obtained a complete Stokes geometry of (3.2.19) described by Fig. 3.9, resolving all the ordered crossing points of ordinary Stokes curves and all the newly created ones by the addition of new Stokes curves.

In this way, starting from the configuration of ordinary Stokes curves and adding virtual turning points and new Stokes curves necessary to resolve the ordered crossing points of Stokes curves based on the use of (3.2.6) and (3.2.8), we obtain a complete Stokes geometry of a non-adiabatic transition problem (3.1.5). In the case of

Fig. 3.8 Three Stokes curves meeting at b



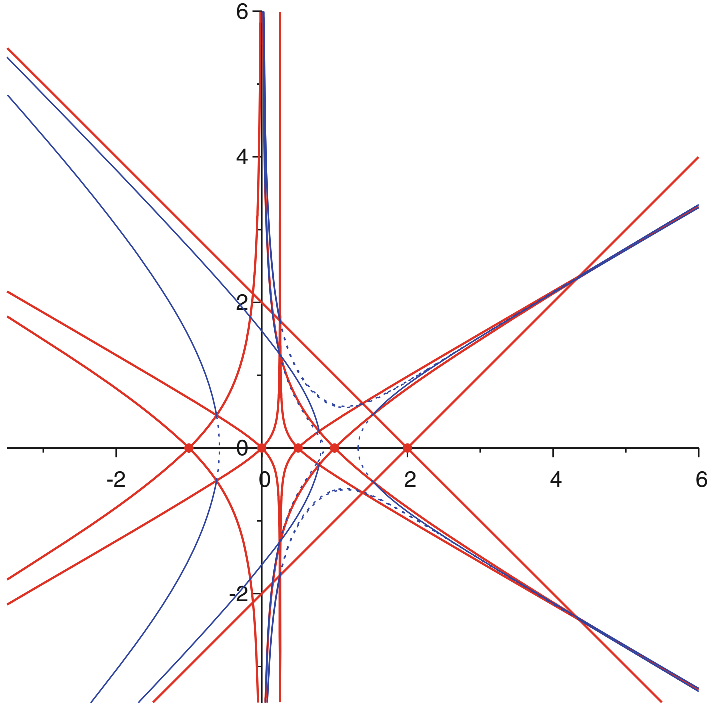


Fig. 3.9 Complete Stokes geometry of system (3.2.19). (Here ordinary Stokes curves are designated in *red* while new Stokes curves are designated in *blue*)

Examples 3.2.1 and 3.2.2 we can observe in Figs. 3.2 and 3.9 that all the new Stokes curves are inert (i.e., expressed by dotted lines) near the real axis, although several virtual turning points and new Stokes curves emanating from them have been added along the above procedure. This property does hold for any non-adiabatic transition problem (3.1.5) thanks to the 'reality' assumption (3.1.4). As a matter of fact, we have the following

Proposition 3.2.1 *Suppose the assumption* (3.1.4). *Then a new Stokes curve emanating from a non-real virtual turning point for a non-adiabatic transition problem* (3.1.5) *never crosses the real axis.*

Proof Let t_{**} be a non-real virtual turning point of type (j, l) determined by (3.2.6) or, more generally, (3.2.8). We rewrite (3.2.8) in the following manner:

$$\int_0^{t_{**}} (\rho_j - \rho_l)\, dt = \int_0^{t^{(0)}} \rho_j\, dt + \int_{t^{(0)}}^{t^{(1)}} \rho_k\, dt + \sum_{l=1}^{n-1} \int_{t^{(l)}}^{t^{(l+1)}} \rho_{k_l}\, dt + \int_{t^{(n)}}^{0} \rho_l\, dt. \quad (3.2.32)$$

Note that, since all $\rho_j(t)$ $(j = 1, 2, 3)$ are real polynomials and (3.1.4) is supposed, the right-hand side of (3.2.32) is a real number. Hence (3.2.32) is an algebraic equation for t_{**} with real coefficients and consequently the mirror image $\overline{t_{**}}$ of t_{**} also becomes a virtual turning point.

Let $\mathscr{S}$ be a new Stokes curve emanating from t_{**}:

$$\mathrm{Im}\,(-i)\int_{t_{**}}^{t}\big(\rho_j-\rho_l\big)dt=0, \tag{3.2.33}$$

that is,

$$\mathrm{Re}\int_{t_{**}}^{t}\big(\rho_j-\rho_l\big)dt=0. \tag{3.2.34}$$

The reality of the polynomials ρ_j and ρ_l entails that the mirror image $\bar{\mathscr{S}}$ of $\mathscr{S}$ is a new Stokes curve emanating from $\overline{t_{**}}$.

We now assume that $\mathscr{S}$ meets with the real axis. Then $\bar{\mathscr{S}}$ meets $\mathscr{S}$ on the real axis. We further assume that $\mathscr{S}\cup\bar{\mathscr{S}}$ is a non-singular curve as t ranges from t_{**} to $\overline{t_{**}}$ along it. Under these assumptions we find

$$\mathrm{Im}\int_{t_{**}}^{t}\big(\rho_j-\rho_l\big)dt \text{ is monotonically increasing or decreasing} \tag{3.2.35}$$

along the portion of $\mathscr{S}\cup\bar{\mathscr{S}}$ under consideration. Since both t_{**} and $\overline{t_{**}}$ are solutions of (3.2.32) and hence

$$\int_{t_{**}}^{\overline{t_{**}}}\big(\rho_j-\rho_l\big)dt=0 \tag{3.2.36}$$

holds, (3.2.35) implies $t_{**}=\overline{t_{**}}$, contradicting with the non-reality assumption for t_{**}. Thus $\mathscr{S}\cup\bar{\mathscr{S}}$ should contain a singular point, which will be designated by α in what follows.

The singular point α lies on the curve $\mathscr{S}\cup\bar{\mathscr{S}}$ defined by (3.2.34) (and its complex conjugate). Then it follows from the analyticity of the integral $\int_{t_{**}}^{t}(\rho_j-\rho_l)\,dt$ that $\rho_j(\alpha)=\rho_l(\alpha)$ should hold. Hence α is a turning point of (3.1.5) and located on the real axis thanks to the assumption (3.1.4). In particular,

$$\int_{t_{**}}^{\alpha}(\rho_j-\rho_l)\,dt=\int_{0}^{\alpha}(\rho_j-\rho_l)\,dt-\int_{0}^{t_{**}}(\rho_j-\rho_l)\,dt \tag{3.2.37}$$

becomes a real number, while it is pure imaginary as α is a point on the new Stokes curve $\mathscr{S}\cup\bar{\mathscr{S}}$. Thus the integral $\int_{t_{**}}^{\alpha}(\rho_j-\rho_l)\,dt$ must be zero and hence, by the same reasoning as above, α should coincide with t_{**}. This contradicts again with the non-reality assumption for t_{**}, as α lies on the real axis. It concludes that $\mathscr{S}$ does not cross the real axis, completing the proof of Proposition 3.2.1.

Proposition 3.2.2 *Under the assumption* (3.1.4) *it holds that a new Stokes curve of* (3.1.5) *emanating from a real virtual turning point does not cross the real axis except at the virtual turning point where it emanates.*

Proof Let t_{**} be a real virtual turning point of type (j, l) defined by (3.2.32) and $\mathscr{S}$ a new Stokes curve emanating from t_{**}. We readily find that $\mathscr{S}$ is symmetric with respect to the real axis, i.e., $\mathscr{S} = \bar{\mathscr{S}}$. Hence, if $\mathscr{S}$ crosses the real axis at a point α $(\neq t_{**})$ and $\mathscr{S}$ is non-singular there, $\mathscr{S}$ forms a loop. Then similar reasoning as in the proof of Proposition 3.2.1 verifies (3.2.35) along $\mathscr{S}$. This contradicts with

$$\int_{\text{along}\,\mathscr{S}} (\rho_j - \rho_l)dt = 0. \tag{3.2.38}$$

(Note that (3.2.38) follows from the single-valuedness of $\int_{t_{**}}^{t} (\rho_j - \rho_l)\,dt$.) Thus $\mathscr{S}$ is singular at $t = \alpha$ and hence $\rho_j(\alpha) = \rho_l(\alpha)$ holds. Again, similar reasoning as in the proof of Proposition 3.2.1 verifies that α should coincide with t_{**}, that is, $\mathscr{S}$ does not cross the real axis except at $t = t_{**}$. This completes the proof of Proposition 3.2.2.

Combining Propositions 3.2.1 and 3.2.2 together with the fact that a new Stokes curve is inert near a virtual turning point (cf. Recipe 1.5.1, (R.iii)), we can conclude that, under the assumption (3.1.4), all the new Stokes curves are inert near the real axis for any non-adiabatic transition problem (3.1.5). This property prevents the analysis of non-adiabatic transition problems (3.1.5) from becoming too complicated. For example, as we will see in the subsequent Sect. 3.3, it assures that all virtual turning points are irrelevant to the calculation of transition probabilities for (3.1.5) along the real axis.

In contrast with this, if the reality assumption (3.1.4) is violated, virtual turning points and new Stokes curves play a more important role. The following example, which is discovered by Sasaki [Sa], clearly shows that virtual turning points are inevitable in the calculation of transition probabilities for (3.1.5) in general situations (that is, without assuming the reality assumption (3.1.4)).

Example 3.2.3 Let $\rho_1(t) = t^3$, $\rho_2(t) = -t$ and $\rho_3(t) = -t + c + c^3$ with a non-zero complex constant c. That is, we consider

$$i\frac{d}{dt}\psi = \eta\left[\begin{pmatrix} t^3 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & -t+c+c^3 \end{pmatrix} + \eta^{-1/2}\begin{pmatrix} 0 & c_{12} & c_{13} \\ \overline{c_{12}} & 0 & c_{23} \\ \overline{c_{13}} & \overline{c_{23}} & 0 \end{pmatrix}\right]\psi. \tag{3.2.39}$$

Figure 3.10 shows the configuration of ordinary Stokes curves of (3.2.39) with $c = 0.4$.

In this case (3.2.39) has three turning points of type $(1, 2)$ at $t = 0$ and $t = \pm\sqrt{-1}$, three turning points of type $(1, 3)$ (one of which is $t = c$), and no turning points of type $(2, 3)$. Note that the assumption (3.1.4) is not satisfied for (3.2.39). Now, to avoid possible degeneracies, we add a small perturbation and let $c = 0.4 + i\varepsilon$ with a sufficiently small positive number ε. (Here we are taking $\varepsilon = 0.01$.) Then, starting from Fig. 3.10 and following similar procedures as in Examples 3.2.1 and 3.2.2, we obtain Fig. 3.11 which describes a complete Stokes geometry for (3.2.39) with $c = 0.4 + i\varepsilon$. (In Fig. 3.11 a virtual turning point is designated by a square. To

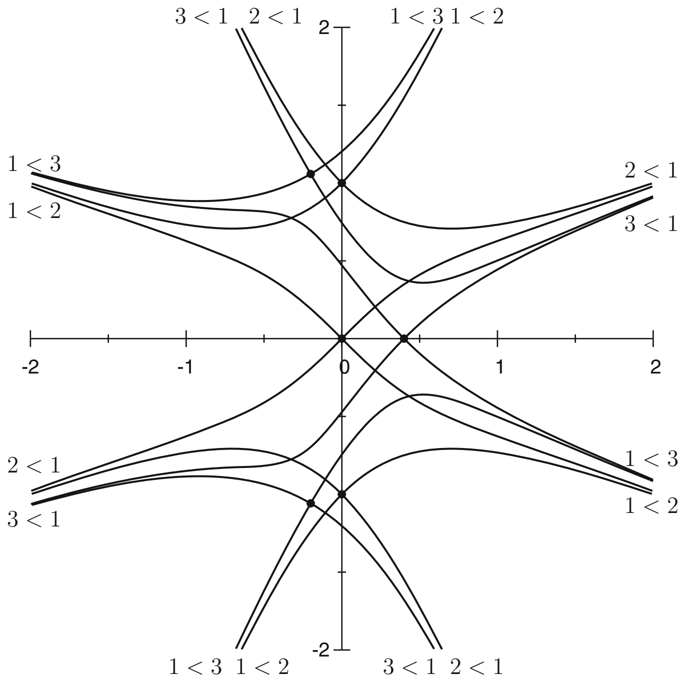


Fig. 3.10 Ordinary Stokes curves of system (3.2.39)

distinguish ordinary Stokes curves from new Stokes curves, we designate ordinary Stokes curves by thick (solid) lines there.)

Although Fig. 3.11 is a little bit complicated, it can be observed that we cannot go from $t = -\infty$ to $t = \infty$ along the real axis without crossing active portions (i.e., portions expressed by solid lines) of new Stokes curves. This implies that we need to take into account the effect of virtual turning points and new Stokes curves emanating from them to calculate transition probabilities along the real axis for (3.2.39).

3.3 Computation of Transition Probabilities

In the preceding section we discussed how to obtain a complete Stokes geometry for a non-adiabatic transition problem (3.1.5). In this section, based on the complete Stokes geometry thus obtained and using the connection formula for Borel resummed WKB solutions of (3.1.5), we compute the S-matrix and transition probabilities along the real axis for a non-adiabatic transition problem.

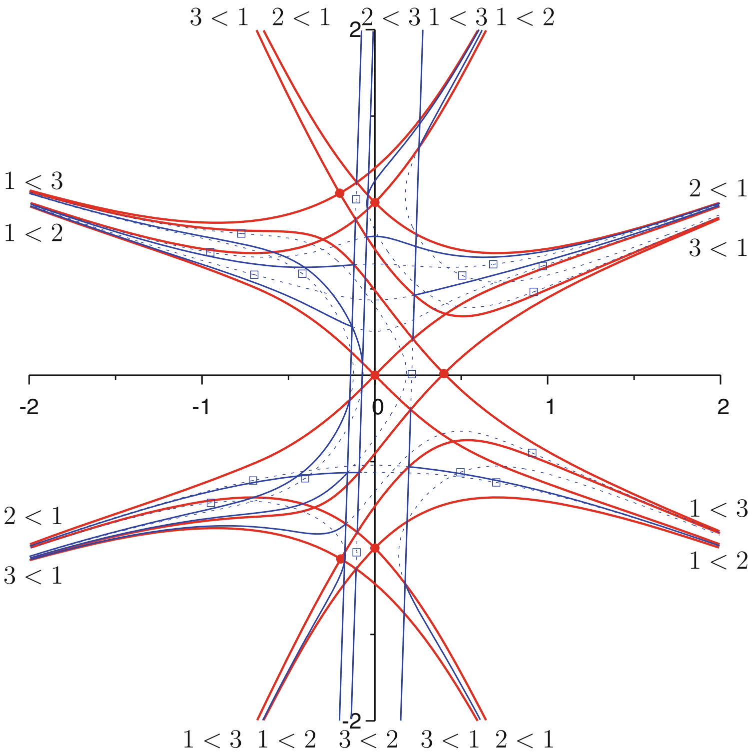


Fig. 3.11 Complete Stokes geometry of system (3.2.39). (As in Fig. 3.9, ordinary Stokes curves and new Stokes curves are designated in *red* and in *blue*, respectively)

In Sect. 3.1 we outlined a recipe for computing transition probabilities for (3.1.5); our task is to calculate the connection matrix M from $t = -\infty$ to $t = +\infty$ (cf. (3.1.27)) for WKB solutions $\psi^{(j)}$ given by (3.1.20). Since, as we have observed in Sect. 3.2, all the new Stokes curves are inert near the real axis for a non-adiabatic transition problem (3.1.5) under the assumption (3.1.4), it then suffices to calculate the connection matrix across an ordinary Stokes curve emanating from a double turning point of (3.1.5) located on the real axis.

To obtain the connection formula on an ordinary Stokes curve emanating from a double turning point, we make use of the technique of "block-diagonalization":

Theorem 3.3.1 ([W, Theorem 25.2], [T3]) *Suppose* (3.1.4) *and let* $t = \tau$ *be a turning point of* (3.1.5) *with type* (j, k)*, that is,* $\rho_j(\tau) = \rho_k(\tau) \neq \rho_l(\tau)$ *holds for a permutation* $\{j, k, l\}$ *of* $\{1, 2, 3\}$. *Then, near* $t = \tau$, (3.1.5) *is decomposed as*

$$i\frac{d}{dt}\varphi = \eta K(t, \eta)\varphi \tag{3.3.1}$$

with

$$K(t,\eta) = \begin{pmatrix} \rho_j(t) & & 0 \\ & \rho_k(t) & \\ 0 & & \rho_l(t) \end{pmatrix} + \eta^{-1/2} \left(\begin{array}{c|c} \tilde{K}^{(j,k)}(t,\eta) & 0 \\ \hline 0 & \tilde{K}^{(l)}(t,\eta) \end{array}\right) \tag{3.3.2}$$

by a formal transformation

$$\psi = S(t,\eta)\varphi = \left(\mathrm{Id} + \eta^{-1/2} S_{1/2}(t) + \eta^{-1} S_1(t) + \cdots\right)\varphi, \tag{3.3.3}$$

where $\tilde{K}^{(j,k)}(t,\eta) = \sum_n \eta^{-n/2} \tilde{K}^{(j,k)}_{n/2}(t)$ *is a formal power series of* $\eta^{-1/2}$ *with* 2×2 *matrix coefficients,* $\tilde{K}^{(l)}(t,\eta) = \sum_n \eta^{-n/2} \tilde{K}^{(l)}_{n/2}(t)$ *is a scalar formal power series, and all the coefficients* $\tilde{K}^{(j,k)}_{n/2}(t)$*,* $\tilde{K}^{(l)}_{n/2}(t)$ *and* $S_{n/2}(t)$ *are holomorphic functions in a neighborhood of* $t = \tau$*.*

Remark 3.3.1 Theorem 3.3.1 is a formal counterpart of Theorem 1.2.1. The formal series S and K, together with μ, ν, etc. below, are actually symbols of microdifferential operators, that is, they satisfy some appropriate growth order conditions like (1.2.21) so that we may discuss the problem exactly on the Borel plane. Although here we do not go into the details of this issue, we refer the reader interested in this point to [T3, KKoT] for the basic ideas used in discussing such a problem.

Thanks to Theorem 3.3.1 the connection problem near an ordinary turning point of type (j,k) for (3.1.5) is reduced to that for a 2×2 system

$$i\frac{d}{dt}\psi = \eta\left[\begin{pmatrix} \rho_j(t) & 0 \\ 0 & \rho_k(t) \end{pmatrix} + \eta^{-1/2}\sum_{n=0}^{\infty} \eta^{-n/2} \tilde{K}^{(j,k)}_{n/2}(t)\right]\psi. \tag{3.3.4}$$

Furthermore, for a 2×2 system of the form (3.3.4) the following theorem, which is a counterpart of Theorem 1.1.1 for (3.1.5), does hold:

Theorem 3.3.2 *Let*

$$i\frac{d}{dt}\psi = \eta H(t,\eta)\psi, \quad H(t,\eta) = \begin{pmatrix} \rho_1(t) & 0 \\ 0 & \rho_2(t) \end{pmatrix} + \eta^{-1/2}\sum_{n=0}^{\infty}\eta^{-n/2} H_{n/2}(t) \tag{3.3.5}$$

be a 2×2 *system for a* 2*-vector* $\psi = \psi(t,\eta) = {}^t(\psi_1(t,\eta), \psi_2(t,\eta))$ *and* $t = \tau$ *a simple zero of* $\rho_1(t) - \rho_2(t)$*, that is,* $t = \tau$ *is a turning point of* (3.3.5)*. Then, in a neighborhood of* τ*,* (3.3.5) *is transformed to a Landau-Zener model for two levels*

$$i\frac{d}{dz}\varphi = \eta\left[\begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} + \eta^{-1/2}\sum_{n=0}^{\infty}\eta^{-n/2}\begin{pmatrix} 0 & \mu_{n/2} \\ \nu_{n/2} & 0 \end{pmatrix}\right]\varphi \tag{3.3.6}$$

by a formal transformation

$$\psi = \exp\left(\frac{\eta}{2i}\int^t (\rho_1+\rho_2)\,dt\right)\sum_{n=0}^{\infty}\eta^{-n/2}T_{n/2}(t)\varphi \tag{3.3.7}$$

and a change of variables

$$2z\frac{dz}{dt} = \rho_2(t)-\rho_1(t), \quad i.e., \quad z = \left(\int_\tau^t (\rho_2-\rho_1)\,dt\right)^{1/2}. \tag{3.3.8}$$

Here $T_{n/2}(t)$ $(n = 0, 1, 2, \ldots)$ *is a* 2×2 *matrix each entry of which is holomorphic near* $t = \tau$ *with* $\det T_0(\tau) \neq 0$*, and* $(\mu_{n/2}, \nu_{n/2})$ $(n = 0, 1, 2, \ldots)$ *is a pair of constants that are uniquely determined by the original system* (3.3.5).

In view of Theorems 3.3.1 and 3.3.2 we find that at each turning point a system (3.1.5) can be reduced to a Landau-Zener model for two levels

$$i\frac{d}{dz}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \eta\left[\begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix} + \eta^{-1/2}\begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}\right]\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \tag{3.3.9}$$

with two 'invariants' $\mu = \mu_0 + \eta^{-1/2}\mu_{1/2} + \cdots$ and $\nu = \nu_0 + \eta^{-1/2}\nu_{1/2} + \cdots$. For the proof of Theorem 3.3.2 see, for example, [AKT4, Appendix]. It also follows from the construction of the transformation (3.3.7) that, in the case of a turning point $t = \tau$ of (3.1.5) with type (j, k), the top order part (μ_0, ν_0) of the invariants is explicitly given by

$$\mu_0 = \sqrt{\frac{2}{\lambda}}c_{jk}, \quad \nu_0 = \sqrt{\frac{2}{\lambda}}\overline{c_{jk}} \quad \text{with} \quad \lambda = \frac{d}{dt}\left(\rho_k-\rho_j\right)\Big|_{t=\tau}. \tag{3.3.10}$$

It is readily confirmed that (3.3.9) is equivalent to the Weber equation for the first component φ_1 of (3.3.9):

$$\frac{d^2}{dw^2}\varphi_1 = \eta^2\left(\frac{w^2}{4} - \eta^{-1}\sigma\right)\varphi_1, \tag{3.3.11}$$

where

$$w = \sqrt{2}e^{i\pi/4}z \quad \text{and} \quad \sigma = -\frac{1}{2}(1+i\mu\nu). \tag{3.3.12}$$

The structure of Borel transforms and Borel sums of WKB solutions of (3.3.11) is studied in, e.g., [T2, Sect. 3]. In particular, if we adopt the following WKB solution

$$\phi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \left(\eta^{1/2} w\right)^{\mp\sigma} \exp \pm \left\{ \eta \frac{w^2}{4} + \int_{\infty}^{w} \left(S_{\mathrm{odd}} - \eta \frac{w}{2} + \frac{\sigma}{w} \right) dw \right\} \quad (3.3.13)$$

(where we choose the branch of $w^{\mp\sigma} = \exp(\mp\sigma \log w)$ so that $-3\pi/4 < \mathrm{Im}\,(\log w) < 5\pi/4$ holds, that is, we place a cut on $\{\arg\, w = -3\pi/4\}$ and take the principal branch of $\log w$) as a fundamental system of solutions of (3.3.11), the connection formula on each Stokes curve of (3.3.11) can be described as follows:
On $\{\, w \in \mathbb{C} \mid \arg w = 0\}$ we have

$$\begin{cases} \phi_+ = \phi_+ + C_0 \phi_- \\ \phi_- = \phi_- \end{cases} \quad \text{with} \quad C_0 = \frac{i\sqrt{2\pi}}{\Gamma(\sigma + 1/2)}, \quad (3.3.14)$$

on $\{\, w \in \mathbb{C} \mid \arg w = \pi/2\}$ we have

$$\begin{cases} \phi_+ = \phi_+ \\ \phi_- = \phi_- + C_1 \phi_+ \end{cases} \quad \text{with} \quad C_1 = \frac{i\sqrt{2\pi}}{\Gamma(-\sigma + 1/2)} e^{i\pi\sigma}, \quad (3.3.15)$$

on $\{\, w \in \mathbb{C} \mid \arg w = \pi\}$ we have

$$\begin{cases} \phi_+ = \phi_+ + C_2 \phi_- \\ \phi_- = \phi_- \end{cases} \quad \text{with} \quad C_2 = \frac{i\sqrt{2\pi}}{\Gamma(\sigma + 1/2)} e^{-2i\pi\sigma}, \quad (3.3.16)$$

and on $\{\, w \in \mathbb{C} \mid \arg w = -\pi/2\}$ we have

$$\begin{cases} \phi_+ = \phi_+ \\ \phi_- = \phi_- + C_{-1} \phi_+ \end{cases} \quad \text{with} \quad C_{-1} = \frac{i\sqrt{2\pi}}{\Gamma(-\sigma + 1/2)} e^{-i\pi\sigma}, \quad (3.3.17)$$

where it is assumed that we cross each Stokes curve in an anticlockwise manner (viewed from the origin $w = 0$). As a consequence, letting

$$\begin{aligned} \varphi^{(+)} &= \eta^{-1/2} \left\{ \begin{pmatrix} 1 \\ \frac{-\eta^{-1/2}\nu}{2z} \end{pmatrix} + O(\eta^{-1/2}) \right\} e^{i\eta z^2/2} z^{i\mu\nu/2} (1 + O(\eta^{-1/2})), \\ \varphi^{(-)} &= \eta^{-1/2} \left\{ \begin{pmatrix} \frac{\eta^{-1/2}\mu}{2z} \\ 1 \end{pmatrix} + O(\eta^{-1/2}) \right\} e^{-i\eta z^2/2} z^{-i\mu\nu/2} (1 + O(\eta^{-1/2})), \end{aligned} \quad (3.3.18)$$

which correspond to (a common constant multiple of) solutions

$$(2i\eta)^{\sigma/2} \phi_+ \quad \text{and} \quad \frac{1}{2} \eta^{-1/2} \mu (2i\eta)^{-\sigma/2} \phi_- \quad (3.3.19)$$

of the Weber equation (3.3.11), respectively, we obtain the following connection formula for a Landau-Zener model (3.3.9) for two levels:

When we cross a Stokes curve $\{ z \in \mathbb{C} \mid \arg z = \pi/4 \}$ of (3.3.9) anticlockwise, we have

$$\varphi^{(+)} \longmapsto \varphi^{(+)}, \quad \varphi^{(-)} \longmapsto \varphi^{(-)} + (2\eta)^{i\mu\nu/2} \frac{\mu\sqrt{\pi}}{\Gamma(i\mu\nu/2+1)} e^{\pi(i+\mu\nu)/4} \varphi^{(+)}, \tag{3.3.20}$$

and when we cross a Stokes curve $\{ z \in \mathbb{C} \mid \arg z = 3\pi/4 \}$ anticlockwise, we have

$$\varphi^{(+)} \longmapsto \varphi^{(+)} + (2\eta)^{-i\mu\nu/2} \frac{\nu\sqrt{\pi}}{\Gamma(-i\mu\nu/2+1)} e^{3\pi(i-\mu\nu)/4} \varphi^{(-)}, \quad \varphi^{(-)} \longmapsto \varphi^{(-)}. \tag{3.3.21}$$

Note that the Stokes curves $\{ w \in \mathbb{C} \mid \arg w = \pi/2 \}$ and $\{ w \in \mathbb{C} \mid \arg w = \pi \}$ of (3.3.11) correspond to the Stokes curves $\{ z \in \mathbb{C} \mid \arg z = \pi/4 \}$ and $\{ z \in \mathbb{C} \mid \arg z = 3\pi/4 \}$ through (3.3.12), respectively. Hence, if we consider the analytic continuation of solutions of (3.3.9) from the left to the right (i.e., clockwise) in the upper half-plane across the two Stokes curves $\{ z \in \mathbb{C} \mid \arg z = \pi/4 \}$ and $\{ z \in \mathbb{C} \mid \arg z = 3\pi/4 \}$, the following connection formula holds:

$$\begin{aligned} \varphi^{(+)} &\longmapsto e^{-\pi\mu\nu} \varphi^{(+)} - (2\eta)^{-i\mu\nu/2} \frac{\nu\sqrt{\pi}}{\Gamma(-i\mu\nu/2+1)} e^{3\pi(i-\mu\nu)/4} \varphi^{(-)}, \\ \varphi^{(-)} &\longmapsto \varphi^{(-)} - (2\eta)^{i\mu\nu/2} \frac{\mu\sqrt{\pi}}{\Gamma(i\mu\nu/2+1)} e^{\pi(i+\mu\nu)/4} \varphi^{(+)}. \end{aligned} \tag{3.3.22}$$

Here we have used the relation

$$1 - \frac{\pi\mu\nu}{\Gamma(i\mu\nu/2+1)\Gamma(-i\mu\nu/2+1)} e^{-\pi\mu\nu/2} = e^{-\pi\mu\nu}. \tag{3.3.23}$$

Let us now return to the calculation of connection matrices for the original non-adiabatic transition problem (3.1.5). Theorems 3.3.1 and 3.3.2 imply that (3.1.5) can be reduced to a Landau-Zener model for two levels (3.3.9) by composition of the two WKB type formal transformations (3.3.3) and (3.3.7) near a turning point of (3.1.5). Furthermore, the connection formula (3.3.22) holds for (3.3.9). Thus, by the same reasoning as in Sect. 1.1, we can expect that the same connection formula as (3.3.22) also holds for (3.1.5) near a turning point when we adopt WKB solutions corresponding to (3.3.18) as (part of) a fundamental system of solutions of (3.1.5) and consider their analytic continuation from the left to the right in the upper half-plane across the two Stokes curves that emanate from a turning point in question. To be more specific, let $t = t_{jk}$ be a turning point of (3.1.5) with type (j, k). Here we assume that the indices j and k are chosen in such a way that

$$\lambda_{jk} \stackrel{\text{def}}{=} \frac{d}{dt} \left(\rho_k - \rho_j \right) \Big|_{t=t_{jk}} > 0 \tag{3.3.24}$$

holds. We further let $\psi_0^{(j)}$ and $\psi_0^{(k)}$ be WKB solutions of (3.1.5) of the following form:

$$\begin{aligned}
\psi_0^{(j)} &= \eta^{-1/2} \exp\left[\frac{\eta}{i} \int_{t_{jk}}^{t} \rho_j \, dt + \frac{1}{i} \int_{t_{jk}}^{t} \left\{ |c_{jk}|^2 \left(\frac{1}{\rho_j - \rho_k} + \frac{1}{\lambda_{jk}(t - t_{jk})} \right) \right. \right. \\
&\quad \left. \left. + \frac{|c_{jl}|^2}{\rho_j - \rho_l} \right\} dt \right] \left(\frac{\lambda_{jk}(t - t_{jk})^2}{2} \right)^{i|c_{jk}|^2/(2\lambda_{jk})} \left(e^{(j)} + O\left(\eta^{-1/2}\right) \right), \\
\psi_0^{(k)} &= \eta^{-1/2} \exp\left[\frac{\eta}{i} \int_{t_{jk}}^{t} \rho_k \, dt + \frac{1}{i} \int_{t_{jk}}^{t} \left\{ -|c_{jk}|^2 \left(\frac{1}{\rho_j - \rho_k} + \frac{1}{\lambda_{jk}(t - t_{jk})} \right) \right. \right. \\
&\quad \left. \left. + \frac{|c_{kl}|^2}{\rho_k - \rho_l} \right\} dt \right] \left(\frac{\lambda_{jk}(t - t_{jk})^2}{2} \right)^{-i|c_{jk}|^2/(2\lambda_{jk})} \left(e^{(k)} + O\left(\eta^{-1/2}\right) \right),
\end{aligned} \tag{3.3.25}$$

where $\{j, k, l\}$ is a permutation of $\{1, 2, 3\}$. Note that $(\rho_j - \rho_k)^{-1}$ has a simple pole at $t = t_{jk}$ while $(\rho_j - \rho_k)^{-1} + (\lambda_{jk}(t - t_{jk}))^{-1}$ is holomorphic and can be integrated from $t = t_{jk}$. In fact, $\psi_0^{(j)}$ and $\psi_0^{(k)}$ are 'well-normalized' WKB solutions of (3.1.5) in the sense that they respectively correspond to the WKB solutions $\varphi^{(+)}$ and $\varphi^{(-)}$ of (3.3.9) through composition of the WKB type transformations (3.3.3) and (3.3.7) that reduces (3.1.5) to (3.3.9). In terms of $\psi_0^{(j)}$ and $\psi_0^{(k)}$, when we consider the analytic continuation from the left to the right in the upper half-plane across the two Stokes curves of (3.1.5) emanating from t_{jk}, it is expected that the following connection formula holds:

$$\begin{aligned}
\psi_0^{(j)} &\longmapsto e^{2i\pi\kappa_{jk}}(1 + O(\eta^{-1/2}))\psi_0^{(j)} \\
&\qquad - (2\eta)^{-\kappa_{jk}} \sqrt{\frac{2\pi}{\lambda_{jk}}} \frac{\overline{c_{jk}}}{\Gamma(1 - \kappa_{jk})} e^{3i\pi(1+2\kappa_{jk})/4}(1 + O(\eta^{-1/2}))\psi_0^{(k)}, \\
\psi_0^{(k)} &\longmapsto \psi_0^{(k)} - (2\eta)^{\kappa_{jk}} \sqrt{\frac{2\pi}{\lambda_{jk}}} \frac{c_{jk}}{\Gamma(1 + \kappa_{jk})} e^{i\pi(1-2\kappa_{jk})/4}(1 + O(\eta^{-1/2}))\psi_0^{(j)},
\end{aligned} \tag{3.3.26}$$

where $\kappa_{jk} = i|c_{jk}|^2/\lambda_{jk}$. Note that, using the relation (3.3.10), we have replaced μ and ν by $\sqrt{2/\lambda_{jk}}\, c_{jk}$ and $\sqrt{2/\lambda_{jk}}\, \overline{c_{jk}}$, respectively, in deriving (3.3.26).

Remark 3.3.2 (i) Being different from the transformation to the Airy equation discussed in Sect. 1, the transformation to the Landau-Zener model for two levels discussed here contains the invariants $\mu = \mu_0 + \eta^{-1/2}\mu_{1/2} + \cdots$ and $\nu = \nu_0 + \eta^{-1/2}\nu_{1/2} + \cdots$ which are, in general, formal power series of $\eta^{-1/2}$. Very rigorously speaking, since in [T2, Sect. 3] only the Weber equation (3.3.11) with σ being a genuine constant is studied, the current situation is not completely covered by the above argument. On the other hand, it is known that these invariants are given in the form of contour integrals of WKB solutions and the result for the Borel summability of such contour integrals for second order equations (cf. [KoS]) strongly suggests

that these invariants should be Borel summable. Once the Borel summability of the invariants are established, by the general theory for the Borel resummation we can expect that the connection formula (3.3.26) should hold even when the invariants are infinite series. It is desirable that, related to these problems, the general theory for the Borel resummation should be refined and become more accessible, especially in the setting of singular perturbations, that is, in the case where divergent series in question depends on a parameter.
(ii) We also remark that the connection formula at a double turning point discussed here is also studied in [DP, Sects. 4 and 5] from the viewpoint of the theory of resurgent functions à la Ecalle.

In conclusion we now obtain the following recipe for computing transition probabilities for a problem (3.1.5) under the assumption (3.1.4):

Recipe 3.3.1.

(R.i) As a fundamental system of solutions we take WKB solutions $\psi^{(j)}$ given by (3.1.20). We also choose normalization factors $N^{\pm,(j)}$ so that $\psi^{\pm,(j)} = N^{\pm,(j)}\psi^{(j)}$ may satisfy (3.1.25).
(R.ii) We list up all the turning points of (3.1.5). Thanks to the assumption (3.1.4), they are all located on the real axis. We arrange them in an increasing order like $\{t^{[n]}_{j_n,k_n}\}_{n=1,\dots,N}$, that is, we number the turning points as $t^{[1]}_{j_1,k_1}, t^{[2]}_{j_2,k_2}, \dots$ from the left (i.e., from $t=-\infty$), where $t^{[n]}_{j_n,k_n}$ is of type (j_n,k_n). Here we assume that

$$\lambda^{[n]}_{j_n,k_n} = \frac{d}{dt}\left(\rho_{k_n} - \rho_{j_n}\right)\Big|_{t=t^{[n]}_{j_n,k_n}} > 0 \tag{3.3.27}$$

holds for any n. (We may assume (3.3.27) without loss of generality by exchanging the indices j_n and k_n if necessary.)
(R.iii) At each turning point $t^{[n]}_{jk}$ (in what follows we abbreviate (j_n,k_n) to (j,k) if there is no fear of confusions) we introduce 'well-normalized' WKB solutions defined by

$$\begin{aligned}
\psi^{[n],(j)} &= \eta^{-1/2}\exp\left[\frac{\eta}{i}\int_{t^{[n]}_{jk}}^{t}\rho_j\,dt + \frac{1}{i}\int_{t^{[n]}_{jk}}^{t}\left\{|c_{jk}|^2\left(\frac{1}{\rho_j-\rho_k}+\frac{1}{\lambda^{[n]}_{jk}(t-t^{[n]}_{jk})}\right)\right.\right.\\
&\qquad\left.\left.+\frac{|c_{jl}|^2}{\rho_j-\rho_l}\right\}dt\right]\left(\frac{\lambda^{[n]}_{jk}(t-t^{[n]}_{jk})^2}{2}\right)^{i|c_{jk}|^2/(2\lambda^{[n]}_{jk})}\left(e^{(j)}+O\left(\eta^{-1/2}\right)\right),\\
\psi^{[n],(k)} &= \eta^{-1/2}\exp\left[\frac{\eta}{i}\int_{t^{[n]}_{jk}}^{t}\rho_k\,dt + \frac{1}{i}\int_{t^{[n]}_{jk}}^{t}\left\{-|c_{jk}|^2\left(\frac{1}{\rho_j-\rho_k}+\frac{1}{\lambda^{[n]}_{jk}(t-t^{[n]}_{jk})}\right)\right.\right.\\
&\qquad\left.\left.+\frac{|c_{kl}|^2}{\rho_k-\rho_l}\right\}dt\right]\left(\frac{\lambda^{[n]}_{jk}(t-t^{[n]}_{jk})^2}{2}\right)^{-i|c_{jk}|^2/(2\lambda^{[n]}_{jk})}\left(e^{(k)}+O\left(\eta^{-1/2}\right)\right),
\end{aligned}\tag{3.3.28}$$

where $e^{(j)}$ is a unit vector satisfying (3.1.21) and $\{j, k, l\}$ is a permutation of $\{1, 2, 3\}$. Comparing WKB solutions $\psi^{(j)}$ given by (3.1.20) with (3.3.28), we compute the constants $\beta_{jk}^{[n]}$, $\gamma_j^{[n]}$ and $\gamma_k^{[n]}$ satisfying

$$\psi^{[n],(j)} = \gamma_j^{[n]}\psi^{(j)}, \quad \psi^{[n],(j)} = \gamma_k^{[n]}\psi^{(k)} \quad \text{and} \quad \beta_{jk}^{[n]} = \frac{\gamma_j^{[n]}}{\gamma_k^{[n]}}. \tag{3.3.29}$$

Then $\psi^{(j)}$ $(j = 1, 2, 3)$ should satisfy the following connection formula when they are analytically continued from the left to the right across the two Stokes curves emanating from $t_{jk}^{[n]}$ in the upper half-plane:

$$\begin{aligned} &\psi^{(j)} \longmapsto (1 + \alpha_{jk}^{[n],-}\alpha_{jk}^{[n],+})\psi^{(j)} - \alpha_{jk}^{[n],-}\psi^{(k)}, \quad \psi^{(k)} \longmapsto \psi^{(k)} - \alpha_{jk}^{[n],+}\psi^{(j)}, \\ &\psi^{(l)} \longmapsto \psi^{(l)}, \end{aligned} \tag{3.3.30}$$

where

$$\alpha_{jk}^{[n],\pm} = (2\eta)^{\pm\kappa_{jk}^{[n]}} \sqrt{\frac{2\pi}{\lambda_{jk}^{[n]}}} \frac{c_{jk}^{\pm}}{\Gamma(1 \pm \kappa_{jk}^{[n]})} e^{i\pi(1/2\mp1)(\kappa_{jk}^{[n]}\mp1/2)} (\beta_{jk}^{[n]})^{\pm1} \left(1 + O\left(\eta^{-1/2}\right)\right) \tag{3.3.31}$$

with

$$c_{jk}^{+} = c_{jk}, \quad c_{jk}^{-} = \overline{c_{jk}}, \quad \kappa_{jk}^{[n]} = \frac{i|c_{jk}|^2}{\lambda_{jk}^{[n]}}. \tag{3.3.32}$$

(The constant $\kappa_{jk}^{[n]}$ is called the **Landau-Zener parameter** at $t = t_{jk}^{[n]}$.)
(R.iv) The connection formula (3.3.30) can be described with a 3×3 matrix $M^{[n]}$ in the following form:

$$\left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right) \longmapsto \left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right) M^{[n]}. \tag{3.3.33}$$

Then

$$\begin{pmatrix} N^{+,(1)} & 0 & 0 \\ 0 & N^{+,(2)} & 0 \\ 0 & 0 & N^{+,(3)} \end{pmatrix}^{-1} M^{[N]} \cdots M^{[1]} \begin{pmatrix} N^{-,(1)} & 0 & 0 \\ 0 & N^{-,(2)} & 0 \\ 0 & 0 & N^{-,(3)} \end{pmatrix} \tag{3.3.34}$$

gives the S-matrix for a system (3.1.5).

Remark 3.3.3 By the same reasoning as in the proof of Proposition 3.2.2, we can confirm that a Stokes curve emanating from an ordinary turning point on the real axis never crosses the real axis again under the assumption (3.1.4). This fact together with Propositions 3.2.1 and 3.2.2 validates the above recipe.

Example 3.2.1 (*revisited*) Making use of Recipe 3.3.1, we compute the S-matrix for the system

$$i\frac{d}{dt}\psi = \eta\left[\begin{pmatrix} b_1 t + a & 0 & 0 \\ 0 & b_2 t & 0 \\ 0 & 0 & b_3 t \end{pmatrix} + \eta^{-1/2}\begin{pmatrix} 0 & c_{12} & c_{13} \\ \overline{c_{12}} & 0 & c_{23} \\ \overline{c_{13}} & \overline{c_{23}} & 0 \end{pmatrix}\right]\psi \tag{3.3.35}$$

discussed in Example 3.2.1. For (3.3.35) we can take the following WKB solutions $\psi^{(j)}$ $(j = 1, 2, 3)$ as a fundamental system of solutions:

$$\psi^{(j)} = \eta^{-1/2}\exp\left(\frac{\eta}{i}\int_0^t \rho_j(t)dt\right)(\rho_k - \rho_j)^{-\kappa_{kj}}(\rho_l - \rho_j)^{-\kappa_{lj}}\left(e^{(j)} + O\left(\eta^{-1/2}\right)\right), \tag{3.3.36}$$

where $\{j, k, l\}$ is a permutation of $\{1, 2, 3\}$ and

$$\kappa_{\alpha\beta} = \frac{i|c_{\alpha\beta}|^2}{b_\beta - b_\alpha} \tag{3.3.37}$$

$(\alpha, \beta = 1, 2, 3)$ denotes the Landau-Zener parameter. Here we assume that the branch of multi-valued analytic functions $(\rho_\alpha - \rho_\beta)^{-\kappa_{\alpha\beta}}$ is determined in such a way that

$$\text{As } t \to -\infty \quad \arg(\rho_\alpha - \rho_\beta) = 0 \quad \text{for } \alpha < \beta, \qquad \arg(\rho_\alpha - \rho_\beta) = \pi \quad \text{for } \alpha > \beta. \tag{3.3.38}$$

$$\text{As } t \to +\infty \quad \arg(\rho_\alpha - \rho_\beta) = -\pi \quad \text{for } \alpha < \beta, \qquad \arg(\rho_\alpha - \rho_\beta) = 0 \quad \text{for } \alpha > \beta. \tag{3.3.39}$$

This choice of the branch immediately implies (3.1.25) if we define normalization factors $N^{\pm,(j)}$ by

$$\begin{aligned} N^{-,(1)} &= e^{-i\pi(\kappa_{12}+\kappa_{13})} + O\left(\eta^{-1/2}\right), \\ N^{-,(2)} &= e^{-i\pi\kappa_{23}} + O\left(\eta^{-1/2}\right), \\ N^{-,(3)} &= 1 + O\left(\eta^{-1/2}\right), \end{aligned} \tag{3.3.40}$$

$$\begin{aligned} N^{+,(1)} &= 1 + O\left(\eta^{-1/2}\right), \\ N^{+,(2)} &= e^{-i\pi\kappa_{12}} + O\left(\eta^{-1/2}\right), \\ N^{+,(3)} &= e^{-i\pi(\kappa_{23}+\kappa_{13})} + O\left(\eta^{-1/2}\right). \end{aligned} \tag{3.3.41}$$

Now, as we have already observed in Sect. 3.2, system (3.3.35) has three turning points. We number them as follows:

$$t_{23}^{[1]} = 0 < t_{13}^{[2]} = \frac{a}{b_3 - b_1} < t_{12}^{[3]} = \frac{a}{b_2 - b_1}. \tag{3.3.42}$$

We first compute the connection formula at $t_{23}^{[1]} = 0$. At $t = t_{23}^{[1]} = 0$ the well-normalized WKB solutions of (3.3.35) are defined by

$$\psi^{[1],(2)} = \eta^{-1/2} \exp\left(\frac{\eta}{2i} b_2 t^2\right) \times \left(\frac{(b_3 - b_2)t^2}{2}\right)^{\kappa_{23}/2} \left(1 - \frac{b_2 - b_1}{a} t\right)^{-\kappa_{12}} \left(e^{(2)} + O\left(\eta^{-1/2}\right)\right),$$
$$\psi^{[1],(3)} = \eta^{-1/2} \exp\left(\frac{\eta}{2i} b_3 t^2\right) \times \left(\frac{(b_3 - b_2)t^2}{2}\right)^{-\kappa_{23}/2} \left(1 - \frac{b_3 - b_1}{a} t\right)^{-\kappa_{13}} \left(e^{(3)} + O\left(\eta^{-1/2}\right)\right). \tag{3.3.43}$$

In what follows we omit the symbol $(1 + O(\eta^{-1/2}))$ for the sake of simplicity. Comparing these WKB solutions with (3.3.36), we find

$$\psi^{[1],(2)} = (2(b_3 - b_2))^{-\kappa_{23}/2} a^{\kappa_{12}} \psi^{(2)}, \quad \psi^{[1],(3)} = e^{-i\pi\kappa_{23}} (2(b_3 - b_2))^{\kappa_{23}/2} a^{\kappa_{13}} \psi^{(3)}, \tag{3.3.44}$$

that is,

$$\gamma_2^{[1]} = (2(b_3 - b_2))^{-\kappa_{23}/2} a^{\kappa_{12}}, \quad \gamma_3^{[1]} = e^{-i\pi\kappa_{23}} (2(b_3 - b_2))^{\kappa_{23}/2} a^{\kappa_{13}} \tag{3.3.45}$$

and

$$\beta_{23}^{[1]} = e^{i\pi\kappa_{23}} (2(b_3 - b_2))^{-\kappa_{23}} a^{\kappa_{12} - \kappa_{13}}. \tag{3.3.46}$$

Hence the connection formula (3.3.30) at $t_{23}^{[1]} = 0$ reads as

$$\psi^{(1)} \longmapsto \psi^{(1)}, \quad \psi^{(2)} \longmapsto (1 + \alpha_{23}^{[1],-} \alpha_{23}^{[1],+}) \psi^{(2)} - \alpha_{23}^{[1],-} \psi^{(3)},$$
$$\psi^{(3)} \longmapsto \psi^{(3)} - \alpha_{23}^{[1],+} \psi^{(2)}, \tag{3.3.47}$$

where

$$\alpha_{23}^{[1],\pm} = (2\eta)^{\pm\kappa_{23}} \sqrt{\frac{2\pi}{b_3 - b_2}} \frac{c_{23}^{\pm}}{\Gamma(1 \pm \kappa_{23})} e^{i\pi(1/2 \mp 1)(\kappa_{23} \mp 1/2)} (\beta_{23}^{[1]})^{\pm 1},$$
$$\beta_{23}^{[1]} = e^{i\pi\kappa_{23}} (2(b_3 - b_2))^{-\kappa_{23}} a^{\kappa_{12} - \kappa_{13}}. \tag{3.3.48}$$

That is, we have

$$\left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right) \longmapsto \left(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right) M^{[1]} \tag{3.3.49}$$

with

$$M^{[1]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \alpha_{23}^{[1],-} \alpha_{23}^{[1],+} & -\alpha_{23}^{[1],+} \\ 0 & -\alpha_{23}^{[1],-} & 1 \end{pmatrix}. \tag{3.3.50}$$

In a similar manner, at $t_{13}^{[2]}$ we have

$$\begin{aligned}
&\psi^{(1)} \longmapsto (1+\alpha_{13}^{[2],-}\alpha_{13}^{[2],+})\psi^{(1)} - \alpha_{13}^{[2],-}\psi^{(3)}, \quad \psi^{(2)} \longmapsto \psi^{(2)},\\
&\psi^{(3)} \longmapsto \psi^{(3)} - \alpha_{13}^{[2],+}\psi^{(1)},
\end{aligned} \tag{3.3.51}$$

where

$$\begin{aligned}
\alpha_{13}^{[2],\pm} &= (2\eta)^{\pm\kappa_{13}}\sqrt{\frac{2\pi}{b_3-b_1}}\frac{c_{13}^{\pm}}{\Gamma(1\pm\kappa_{13})}e^{i\pi(1/2\mp 1)(\kappa_{13}\mp 1/2)}(\beta_{13}^{[2]})^{\pm 1},\\
\beta_{13}^{[2]} &= e^{i\pi(-\kappa_{12}+\kappa_{23}+\kappa_{13})}(2(b_3-b_1))^{-\kappa_{13}}\left(\frac{b_3-b_2}{b_3-b_1}a\right)^{-\kappa_{12}-\kappa_{23}} e^{i\eta a^2/(2(b_3-b_1))}.
\end{aligned} \tag{3.3.52}$$

That is, we have

$$\left(\psi^{(1)},\psi^{(2)},\psi^{(3)}\right) \longmapsto \left(\psi^{(1)},\psi^{(2)},\psi^{(3)}\right) M^{[2]} \tag{3.3.53}$$

with

$$M^{[2]} = \begin{pmatrix} 1+\alpha_{13}^{[2],-}\alpha_{13}^{[2],+} & 0 & -\alpha_{13}^{[2],+} \\ 0 & 1 & 0 \\ -\alpha_{13}^{[2],-} & 0 & 1 \end{pmatrix}. \tag{3.3.54}$$

Furthermore, at $t_{12}^{[3]}$ we have

$$\begin{aligned}
&\psi^{(1)} \longmapsto (1+\alpha_{12}^{[3],-}\alpha_{12}^{[3],+})\psi^{(1)} - \alpha_{12}^{[3],-}\psi^{(2)}, \quad \psi^{(2)} \longmapsto \psi^{(2)} - \alpha_{12}^{[3],+}\psi^{(1)},\\
&\psi^{(3)} \longmapsto \psi^{(3)},
\end{aligned} \tag{3.3.55}$$

where

$$\begin{aligned}
\alpha_{12}^{[3],\pm} &= (2\eta)^{\pm\kappa_{12}}\sqrt{\frac{2\pi}{b_2-b_1}}\frac{c_{12}^{\pm}}{\Gamma(1\pm\kappa_{12})}e^{i\pi(1/2\mp 1)(\kappa_{12}\mp 1/2)}(\beta_{12}^{[3]})^{\pm 1},\\
\beta_{12}^{[3]} &= e^{i\pi\kappa_{12}}(2(b_2-b_1))^{-\kappa_{12}}\left(\frac{b_3-b_2}{b_2-b_1}a\right)^{\kappa_{23}-\kappa_{13}} e^{i\eta a^2/(2(b_2-b_1))}.
\end{aligned} \tag{3.3.56}$$

That is, we have

$$\left(\psi^{(1)},\psi^{(2)},\psi^{(3)}\right) \longmapsto \left(\psi^{(1)},\psi^{(2)},\psi^{(3)}\right) M^{[3]} \tag{3.3.57}$$

with

$$M^{[3]} = \begin{pmatrix} 1+\alpha_{12}^{[3],-}\alpha_{12}^{[3],+} & -\alpha_{12}^{[3],+} & 0 \\ -\alpha_{12}^{[3],-} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3.58}$$

Thus we conclude that (the top order part with respect to η^{-1} of) the S-matrix for system (3.3.35) is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\pi\kappa_{12}} & 0 \\ 0 & 0 & e^{-i\pi(\kappa_{23}+\kappa_{13})} \end{pmatrix}^{-1} M^{[3]}M^{[2]}M^{[1]} \begin{pmatrix} e^{-i\pi(\kappa_{12}+\kappa_{13})} & 0 & 0 \\ 0 & e^{-i\pi\kappa_{23}} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} e^{i\pi(\kappa_{12}+\kappa_{13})} & \alpha_{23}^{-}\alpha_{13}^{+}e^{i\pi(2\kappa_{12}-\kappa_{23})} - \alpha_{12}^{+}e^{i\pi\kappa_{23}} & -\alpha_{13}^{+}e^{2i\pi\kappa_{12}} + \alpha_{12}^{+}\alpha_{23}^{+} \\ -\alpha_{12}^{-}e^{i\pi\kappa_{13}} & -\alpha_{12}^{-}\alpha_{23}^{-}\alpha_{13}^{+}e^{i\pi(\kappa_{12}-\kappa_{23})} + e^{i\pi(\kappa_{12}+\kappa_{23})} & (\alpha_{12}^{-}\alpha_{13}^{+} - \alpha_{23}^{+})e^{i\pi\kappa_{12}} \\ -\alpha_{13}^{-}e^{i\pi(-\kappa_{12}+\kappa_{23})} & -\alpha_{23}^{-}e^{i\pi\kappa_{13}} & e^{i\pi(\kappa_{23}+\kappa_{13})} \end{pmatrix}. \tag{3.3.59}$$

(Here we have abbreviated $\alpha_{23}^{[1],\pm} = \alpha_{23}^{\pm}$ etc.)

In this way we can calculate the S-matrix and transition probabilities for a problem (3.1.5) by using the exact WKB analysis.

Remark 3.3.4 Our recipe for the computation of the S-matrix is based on the reduction to a Landau-Zener model for two levels at a turning point. In this computation the Landau-Zener parameters naturally appear as (the top order part of) the invariants at a turning point.

Under the assumption (3.1.4) virtual turning points are irrelevant to the calculation of transition probabilities along the real axis thanks to Propositions 3.2.1 and 3.2.2. However, when (3.1.4) is violated, virtual turning points and new Stokes curves emanating from them also play an essentially important role in the calculation of the S-matrix and transition probabilities.

Example 3.2.3 (revisited) To see the effect of virtual turning points to the calculation of transition probabilities for (3.1.5) in a more general situation, let us consider the system

$$i\frac{d}{dt}\psi = \eta\left[\begin{pmatrix} t^3 & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & -t+c+c^3 \end{pmatrix} + \eta^{-1/2}\begin{pmatrix} 0 & c_{12} & c_{13} \\ \overline{c_{12}} & 0 & c_{23} \\ \overline{c_{13}} & \overline{c_{23}} & 0 \end{pmatrix}\right]\psi \tag{3.3.60}$$

discussed in Example 3.2.3. As was already observed in Sect. 3.2, some new Stokes curves emanating from virtual turning points cross the real axis for (3.3.60). Figure 3.12 shows the configuration of Stokes curves of (3.3.60) near the real axis; only active Stokes curves are written and inert Stokes curves are all omitted there. There are eight ordinary Stokes curves, which emanate from two turning points near the real axis, and additional two Stokes curves that cross the real axis in Fig. 3.12. The latter two are new Stokes curves emanating from non-real virtual turning points (cf. Fig. 3.11).

Let α_j ($j = 1, \ldots, 4$) be the Stokes constants attached to the four ordinary Stokes curves specified in Fig. 3.12 and β_k ($k = 1, 2$) the Stokes constants attached to the two new Stokes curves. Then, if we consider the analytic continuation of $\psi^{(3)}$ from $t = -\infty$ to $t = +\infty$ along the real axis through the upper half-plane, we readily find that $\psi^{(3)}$ should be changed as

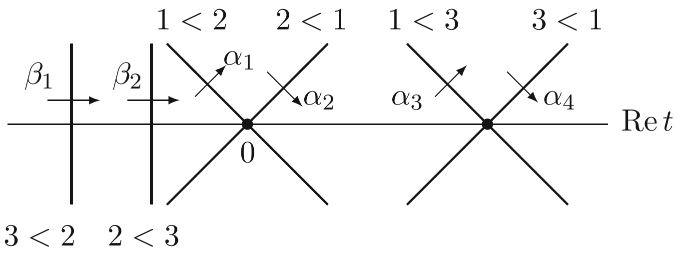


Fig. 3.12 Active Stokes curves of (3.3.60) near the real axis

$$
\begin{aligned}
\psi^{(3)} &\mapsto \psi^{(3)} \\
&\mapsto \psi^{(3)} + \beta_2 \psi^{(2)} \\
&\mapsto \psi^{(3)} + \beta_2(\psi^{(2)} + \alpha_1 \psi^{(1)}) \\
&= \psi^{(3)} + \beta_2 \psi^{(2)} + \alpha_1 \beta_2 \psi^{(1)} \\
&\mapsto \psi^{(3)} + \beta_2 \psi^{(2)} + \alpha_1 \beta_2 (\psi^{(1)} + \alpha_2 \psi^{(2)}) \\
&= \psi^{(3)} + \beta_2 (1 + \alpha_1 \alpha_2) \psi^{(2)} + \alpha_1 \beta_2 \psi^{(1)} \\
&\mapsto (\psi^{(3)} + \alpha_3 \psi^{(1)}) + \beta_2 (1 + \alpha_1 \alpha_2) \psi^{(2)} + \alpha_1 \beta_2 \psi^{(1)} \\
&= \psi^{(3)} + \beta_2 (1 + \alpha_1 \alpha_2) \psi^{(2)} + (\alpha_1 \beta_2 + \alpha_3) \psi^{(1)} \\
&\mapsto \psi^{(3)} + \beta_2 (1 + \alpha_1 \alpha_2) \psi^{(2)} + (\alpha_1 \beta_2 + \alpha_3)(\psi^{(1)} + \alpha_4 \psi^{(3)}) \\
&= (1 + \alpha_3 \alpha_4 + \alpha_1 \alpha_4 \beta_2) \psi^{(3)} + \beta_2 (1 + \alpha_1 \alpha_2) \psi^{(2)} + (\alpha_1 \beta_2 + \alpha_3) \psi^{(1)}.
\end{aligned}
\tag{3.3.61}
$$

In particular, the transition probability from $\psi^{(3)}$ near $t = -\infty$ to $\psi^{(2)}$ near $t = +\infty$ is described by $\beta_2(1 + \alpha_1\alpha_2)$, to which the Stokes constant β_2 attached to the new Stokes curve of type $2 < 3$ (i.e., the second one from the left in Fig. 3.12) gives the main contribution. We can thus conclude that the effect of new Stokes curves and virtual turning points is inevitable for (3.3.60).

Remark 3.3.5 System (3.3.60) has six ordinary turning points: two near the real axis, two in the upper half-plane and two in the lower half-plane. Let us denote them by $t_j^{(0)}$, $t_j^{(+)}$ and $t_j^{(-)}$ $(j = 1, 2)$, respectively. It is observed in Fig. 3.11 that the new Stokes curve of type $2 < 3$ crossing the real axis, i.e., the second one from the left in Fig. 3.12, is a new Stokes curve passing through ordered crossing points of Stokes curves emanating from $t_1^{(-)}$ and $t_2^{(-)}$. In view of (R.iii) and (R.v) of Recipe 1.5.1, we then find that the Stokes constant β_2 in question is completely determined by the Stokes constants attached to Stokes curves emanating from $t_1^{(-)}$ and $t_2^{(-)}$. Hence it can be said that the new Stokes curve of type $2 < 3$ in question and the Stokes constant β_2 attached to it represent an effect of the turning points $t_j^{(-)}$ $(j = 1, 2)$ to the transition probabilities. Note that any Stokes curve emanating from $t_j^{(-)}$ does not cross the real axis and hence $t_j^{(-)}$ does not directly affect the transition probabilities. We can thus conclude that the new Stokes curve in question and the virtual turning point it emanates from visualize the indirect effects of the non-real turning points $t_j^{(-)}$ $(j = 1, 2)$ to the transition probabilities along the real axis.

Appendix A
Integral Representation of Solutions and the Borel Resummed WKB Solutions

A.1 The Case of Laplace Type Equations

In the case of higher order ordinary differential equations with a large parameter, the Borel summability of WKB solutions and the endless continuability of their Borel transforms (i.e., Property [AC]) are not established yet. However, when an integral representation of solutions is available, these important properties can be confirmed by making full use of the integral representation. In this Appendix, considering a higher order differential equation of Laplace type

$$P\psi = \left(\frac{d^m}{dx^m} + (c_{m-1}x + d_{m-1})\eta\frac{d^{m-1}}{dx^{m-1}} + \cdots + (c_0x + d_0)\eta^m\right)\psi = 0, \tag{A.1.1}$$

where $c_j, d_j \in \mathbb{C}$ are constants, we explain this fact and give a characterization of (ordinary and new) active Stokes curves of (A.1.1) in terms of the integral representation.

The (inverse) Fourier-Laplace transform with a large parameter

$$\psi(x) = \int \exp(\eta x\xi)\hat{\psi}(\xi)d\xi \tag{A.1.2}$$

transforms a Laplace type equation (A.1.1) into a first order equation

$$\widehat{P}\hat{\psi} = \eta^m\left[-C(\xi)\eta^{-1}\frac{\partial}{\partial\xi} - \eta^{-1}C'(\xi) + D(\xi)\right]\hat{\psi} = 0 \tag{A.1.3}$$

with

$$C(\xi) = c_{m-1}\xi^{m-1} + \cdots + c_0 \text{ and } D(\xi) = \xi^m + d_{m-1}\xi^{m-1} + \cdots + d_0. \tag{A.1.4}$$

N. Honda et al., *Virtual Turning Points*, SpringerBriefs in Mathematical Physics,
DOI 10.1007/978-4-431-55702-9

Hence, by solving (A.1.3) explicitly, we obtain the following integral representation of solutions for (A.1.1):

$$\psi(x) = \int_{\Gamma} e^{\eta f(x,\xi)} \frac{1}{C(\xi)} d\xi, \tag{A.1.5}$$

where

$$f(x,\xi) = x\xi + g(\xi), \quad g(\xi) = \int^{\xi} \frac{D(\xi)}{C(\xi)} d\xi. \tag{A.1.6}$$

In what follows we take a steepest descent path of Re f passing through a saddle point of f as an integration path Γ. For the sake of reader's convenience we here recall the definition of a saddle point and a steepest descent path.

Definition A.1.1 (i) A **saddle point** of $f(x,\xi)$ (or a saddle point of (A.1.5)) is a (non-degenerate) critical point of $f(x,\xi)$, that is, a simple zero of $\partial f/\partial\xi = 0$.
(ii) A **steepest descent path** of Re f (or a steepest descent path of (A.1.5)) is a solution curve of the vector field $-\mathrm{grad}_{(s,t)}\,\mathrm{Re}\, f(x,\xi)$, where s and t denote the real part and the imaginary part of ξ.

Let Σ_{s} and Σ_{∞} denote the set of saddle points of $f(x,\xi)$ and that of singular points of (A.1.5), respectively. It is clear that

$$\Sigma_{\infty} = \{\text{zeros of } C(\xi)\} \cup \{\infty\}, \tag{A.1.7}$$

while a saddle point of f is a zero of

$$\frac{\partial f}{\partial \xi} = x + \frac{dg}{d\xi} = x + \frac{D(\xi)}{C(\xi)}. \tag{A.1.8}$$

Hence we readily obtain the following

Lemma A.1.1

$$\Sigma_{\mathrm{s}} = \{\zeta_1(x), \ldots, \zeta_m(x)\}, \tag{A.1.9}$$

where $\zeta_j(x)$ $(j = 1, \ldots, m)$ *is a root of the characteristic polynomial*

$$\zeta^m + (c_{m-1}x + d_{m-1})\zeta^{m-1} + \cdots + (c_0 x + d_0) = 0 \tag{A.1.10}$$

of (A.1.1).

Here and in what follows, unless specifically mentioned, we consider the problem in a generic situation, that is, we assume that the zeros of $C(\xi)$ and those of $D(\xi)$ are mutually distinct and further that every $\zeta_j(x)$ is a simple root of (A.1.10) (i.e., x does not belong to the set of turning points of (A.1.1)).

Let $\zeta_j(x)$ $(j = 1, \ldots, m)$ be a saddle point of $f(x, \xi)$ and consider a solution of (A.1.1) given by

$$\Psi_j(x) = \int_{\Gamma^{(j)}} e^{\eta f(x,\xi)} \frac{1}{C(\xi)} d\xi, \tag{A.1.11}$$

where $\Gamma^{(j)}$ is a steepest descent path of $\mathrm{Re}\, f(x, \xi)$ passing through $\zeta_j(x)$. If we apply the saddle point method to (A.1.11), we obtain the asymptotic expansion of (A.1.11) for $\eta \to \infty$ of the form

$$\psi_j(x) = \eta^{-1/2} e^{\eta f(x,\zeta_j(x))} \sum_{n=0}^{\infty} \psi_{j,n}(x)\eta^{-n}. \tag{A.1.12}$$

(One way of deriving (A.1.12) from (A.1.11) will be discussed later.) The main claim of this appendix is then the following

Theorem A.1.1 (i) *The right-hand side of* (A.1.12) *is a* (*suitably normalized*) *WKB solution of* (A.1.1) *with the top term* $S_{-1}(x) = \zeta_j(x)$.
(ii) *Suppose that the steepest descent path* $\Gamma^{(j)}$ *does not pass any other saddle point and can be extended to a singular point of* (A.1.5) (*i.e., a point in* Σ_∞). *Then the WKB solution* (A.1.12) *is Borel summable and the integral* (A.1.11) *provides the Borel sum of* (A.1.12).

To prove Theorem A.1.1, we first rewrite the integral (A.1.11) by using a change of variable $y = -f(x, \xi) = -(x\xi + g(\xi))$ as follows:

$$\begin{aligned}
\Psi_j(x) &= \int_{\Gamma^{(j)}} e^{\eta f(x,\xi)} \frac{1}{C(\xi)}\, d\xi \\
&= \int_{\tilde{\Gamma}^{(j)}} e^{-\eta y} \left[\left(C(\xi)\frac{dy}{d\xi} \right)^{-1} \Bigg|_{\Gamma_+^{(j)}} - \left(C(\xi)\frac{dy}{d\xi} \right)^{-1} \Bigg|_{\Gamma_-^{(j)}} \right] dy \\
&= \int_{\tilde{\Gamma}^{(j)}} e^{-\eta y} \left[- \frac{1}{C(\xi)x + D(\xi)} \Bigg|_{\xi=\xi_+(x,y)} + \frac{1}{C(\xi)x + D(\xi)} \Bigg|_{\xi=\xi_-(x,y)} \right] dy,
\end{aligned} \tag{A.1.13}$$

where $\tilde{\Gamma}^{(j)}$ is a path emanating from $-y_j(x) = -f(x, \zeta_j(x))$ and running parallel with the positive real axis (cf. Fig. A.1). Note that, since $\zeta_j(x)$ is a non-degenerate critical point of $f(x, \xi)$, two steepest descent paths (denoted respectively by $\Gamma_\pm^{(j)}$ in Fig. A.1) emanate from $\zeta_j(x)$ and the inverse change of $y = -f(x, \xi)$ has two branches $\xi = \xi_\pm(x, y)$ with the following Puiseux expansion there:

$$\xi_\pm(x, y) = \zeta_j(x) + \sum_{n=1}^{\infty} (\pm 1)^n \Xi_{n/2}(x) \big(y + y_j(x)\big)^{n/2}. \tag{A.1.14}$$

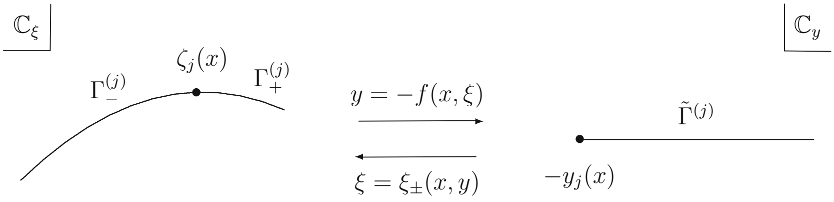


Fig. A.1 Steepest descent paths $\Gamma_\pm^{(j)}$ and a change of variable $y = -f(x,\xi)$

Let $\varphi_\pm(x, y)$ denote

$$\varphi_\pm(x, y) = -\frac{1}{C(\xi)x + D(\xi)}\bigg|_{\xi=\xi_\pm(x,y)}. \tag{A.1.15}$$

Then it follows from (A.1.14) that at $y = -y_j(x)$

$$\varphi_+(x, y) - \varphi_-(x, y) = \big(y + y_j(x)\big)^{-1/2}\chi(x, y), \tag{A.1.16}$$

where $\chi(x, y)$ is consisting only of terms with non-negative integral powers of $(y + y_j(x))$ (and hence holomorphic in (x, y)), as $\varphi_+ - \varphi_-$ becomes $-(\varphi_+ - \varphi_-)$ after the analytic continuation along a tiny circle around $y = -y_j(x)$. Hence, applying the so-called Watson's lemma (cf., e.g., [C, Sect. 3.4]) to the Laplace integral of the form

$$\Psi_j(x) = \int_{\tilde{\Gamma}^{(j)}} e^{-\eta y}\big(y + y_j(x)\big)^{-1/2}\chi(x, y)\,dy, \tag{A.1.17}$$

we obtain an asymptotic expansion for $\eta \to \infty$ of the form (A.1.12). This is one of the explicit derivations of (A.1.12) from the integral (A.1.11) or, equivalently, (A.1.17). Note that only the local behavior of $\chi(x, y)$ near $y = -y_j(x)$ contributes to the asymptotic expansion (A.1.12).

Now a key step to prove Theorem A.1.1 is to show the following

Proposition A.1.1 *Let*

$$P_B = \frac{\partial^m}{\partial x^m} + (c_{m-1}x + d_{m-1})\frac{\partial^m}{\partial x^{m-1}\partial y} + \cdots + (c_0 x + d_0)\frac{\partial^m}{\partial y^m} \tag{A.1.18}$$

be the Borel transform of P (with respect to η). Then $\varphi_\pm(x, y)$ satisfy

$$P_B\varphi_\pm(x, y) = 0. \tag{A.1.19}$$

Proof As the argument below runs in the same manner for both $\varphi_\pm$, we omit the suffix $\pm$ in this proof. Using Cauchy's integral formula, we write $\varphi(x, y)$ as

$$\begin{aligned}\varphi(x, y) &= -\frac{1}{2\pi i}\oint \frac{1}{C(\xi)x + D(\xi)}\bigg|_{\xi=\xi(x,y')} \frac{dy'}{y'-y} \\ &= -\frac{1}{2\pi i}\oint \frac{1}{C(\xi)(y + x\xi + g(\xi))} d\xi, \end{aligned} \tag{A.1.20}$$

where we have employed a change of variable $y' = -f(x, \xi) = -(x\xi + g(\xi))$ to obtain the second equality and hence a path of the second integration is a tiny circle around $\xi^\dagger$, the point corresponding to y through this change of variable.

If we let $\phi(\xi, z)$ denote $(C(\xi)(z + g(\xi)))^{-1}$, (A.1.20) can be written also as

$$\varphi(x, y) = -\frac{1}{2\pi i}\oint \phi(\xi, y + x\xi)\, d\xi. \tag{A.1.21}$$

We then find

$$\frac{\partial^m}{\partial x^{m-j}\partial y^j}\oint \phi(\xi, y + x\xi) d\xi = \oint \left(\frac{\partial}{\partial y}\right)^m \left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\bigg|_{\xi'=\xi} d\xi, \tag{A.1.22}$$

and

$$\begin{aligned} x&\frac{\partial^m}{\partial x^{m-j}\partial y^j}\oint \phi(\xi, y + x\xi) d\xi \\ &= \oint \left(\frac{\partial}{\partial y}\right)^m \left(x\xi^{m-j}\phi(\xi, y + x\xi)\right) d\xi \\ &= \oint \left(\frac{\partial}{\partial y}\right)^{m-1} \frac{\partial}{\partial \xi'}\left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\bigg|_{\xi'=\xi} d\xi \\ &= \oint \left(\frac{\partial}{\partial y}\right)^{m-1} \left(-\frac{\partial}{\partial \xi}\right)\left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\bigg|_{\xi'=\xi} d\xi. \end{aligned} \tag{A.1.23}$$

Here, to obtain the last equality, we have used the integration by parts based on the relation

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left(\xi^{m-j}\phi(\xi, y + x\xi)\right) \\ &= \frac{\partial}{\partial \xi'}\left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\bigg|_{\xi'=\xi} + \frac{\partial}{\partial \xi}\left(\xi^{m-j}\phi(\xi, y + x\xi')\right)\bigg|_{\xi'=\xi}. \end{aligned} \tag{A.1.24}$$

Hence

$$
\begin{aligned}
&P_B\varphi(x,y)\\
&= -\frac{1}{2\pi i}\oint \left(\frac{\partial}{\partial y}\right)^{m-1}\left[-C(\xi)\frac{\partial}{\partial \xi} - C'(\xi) + D(\xi)\frac{\partial}{\partial y}\right]\phi(\xi, y + x\xi')\Big|_{\xi'=\xi} d\xi\\
&= -\frac{1}{2\pi i}\oint \left(\frac{\partial}{\partial y}\right)^{m-1}\left[-\frac{\partial}{\partial \xi}\left(\frac{1}{z+g(\xi)}\right) + \frac{D(\xi)}{C(\xi)}\frac{\partial}{\partial z}\left(\frac{1}{z+g(\xi)}\right)\right]\Big|_{z=y+x\xi} d\xi\\
&= 0.
\end{aligned}
\tag{A.1.25}
$$

This completes the proof of Proposition A.1.1.

Once Proposition A.1.1 is verified, Theorem A.1.1 can be proved as follows: We first note that

$$
\frac{d}{dx}f(x,\zeta_j(x)) = \frac{d}{dx}\big(x\zeta_j(x) + g(\zeta_j(x))\big) = \zeta_j(x) + \big(x + g'(\zeta_j(x))\big)\,\zeta_j'(x) = \zeta_j(x).
\tag{A.1.26}
$$

Hence $f(x,\zeta_j(x))$ can be written as $\int^x \zeta_j(x)dx$, that is, the phase function of (A.1.12) has the same form as that of the WKB solution with the top term $S_{-1}(x) = \zeta_j(x)$. Furthermore, since (A.1.12) is obtained from (A.1.17) by Watson's lemma, the right-hand side of (A.1.12) coincides with the inverse Borel transform of $\varphi_+ - \varphi_- = (y + y_j(x))^{-1/2}\chi(x,y)$, i.e.,

$$
\eta^{-1/2}e^{\eta f(x,\zeta_j(x))}\sum_{n=0}^{\infty}\psi_{j,n}(x)\eta^{-n} = \mathscr{B}^{-1}\left(\varphi_+ - \varphi_-\right),
\tag{A.1.27}
$$

where $\mathscr{B}^{-1}$ stands for the inverse Borel transform. It then follows from Proposition A.1.1 that the right-hand side of (A.1.12) satisfies a differential equation (A.1.1). Hence, thanks to the uniqueness of WKB solutions of (A.1.1) (that is, the fact that the higher order terms $S_n(x)$ ($n \geq 0$) of WKB solutions are uniquely determined by the top term $S_{-1}(x)$), we can conclude that the right-hand side of (A.1.12) is a (suitably normalized) WKB solution of (A.1.1) with the top term $S_{-1}(x) = \zeta_j(x)$. This verifies the assertion (i) of Theorem A.1.1.

The above reasoning also entails that $\varphi_+ - \varphi_-$ is the Borel transform of (A.1.12). In view of the definition of $\varphi_\pm(x,y)$, i.e.,

$$
\varphi_\pm(x,y) = -\left.\frac{1}{C(\xi)x + D(\xi)}\right|_{\xi=\xi_\pm(x,y)} = \left.\left(C(\xi)\frac{dy}{d\xi}\right)^{-1}\right|_{\Gamma_\pm^{(j)}},
\tag{A.1.28}
$$

we readily find that $\varphi_\pm(x,y)$ is singular only at points in $\Sigma_{\mathrm{s}} \cup \Sigma_\infty$. Among them a point of Σ_∞, i.e., a zero of $C(\xi)$ (and the point at infinity), is a singular point of the integral representation (A.1.5) and corresponds to the point at infinity in the

y-variable, while a point of Σ_s, i.e., a saddle point of (A.1.5), corresponds to a finite singular point of $\varphi_{\pm}(x, y)$ in $\mathbb{C}_y$ under the change of variable $y = -f(x, \xi)$. Hence, if $\Gamma^{(j)}$ does not pass any other saddle point, then $\varphi_+ - \varphi_-$ (i.e., the Borel transform of (A.1.12)) can be analytically continued to $y = \infty$ along $\tilde{\Gamma}^{(j)}$. This together with the boundedness of $(C(\xi)x + D(\xi))^{-1}$ at each point of Σ_∞ assures that (A.1.12) is Borel summable under the above condition and that (A.1.11) (or, equivalently, (A.1.13)) provides its Borel sum. This completes the proof of Theorem A.1.1.

The proof of the assertion (ii) of Theorem A.1.1 immediately implies the following:

Criterion for detecting active Stokes curves of Laplace type equations
The Borel resummed WKB solution (A.1.12) *of a Laplace type equation* (A.1.1) *obtained through the integral representation* (A.1.5) *presents a Stokes phenomenon at x* (*that is, x belongs to an active Stokes curve of* (A.1.1)) *if and only if a steepest descent path of* (A.1.5) *connects two saddle points $\zeta_j(x)$ and $\zeta_{j'}(x)$.*

A.2 Exact Steepest Descent Method

As was explained in Sect. A.1, the steepest descent method for integral representations of solutions is closely related to the Borel resummation technique for WKB solutions and provides a criterion for their Borel summability and the detection of active Stokes curves. Although the extension of this approach to more general equations is far from being trivial, an intriguing generalization that is called "**exact steepest descent method**" is invented in [AKT3]. In this section we give an outline of this new method.

Let us consider the following differential equation with polynomial coefficients:

$$P\psi = \sum_{\substack{0\le j\le m \\ 0\le k\le n}} a_{jk} x^k \eta^{m-j} \left(\frac{d}{dx}\right)^j \psi = 0, \tag{A.2.1}$$

where n denotes the largest degree of the coefficients of $\eta^{m-j}(d/dx)^j$. In this case, if we apply the Fourier-Laplace transform (A.1.2) to (A.2.1), we obtain an nth order equation

$$\hat{P}\hat{\psi} = \sum_{j,k} a_{jk} \eta^{m-k} \left(-\frac{d}{d\xi}\right)^k (\xi^j \hat{\psi}) = 0, \tag{A.2.2}$$

which cannot be explicitly solved in general when $n \ge 2$. A key idea of the exact steepest descent method is to use a WKB solution $\hat{\psi}_k = \eta^{-1/2}\exp(\eta \int^{\xi} (-x_k(\xi)) d\xi + \cdots)$ $(k = 1, \ldots, n)$ of (A.2.2) and regard, instead of (A.1.5),

$$\int e^{\eta x\xi} \hat{\psi}_k \, d\xi = \eta^{-1/2} \int \exp\left[\eta\left(x\xi - \int^{\xi} x_k(\xi) d\xi\right) + \cdots\right] d\xi \tag{A.2.3}$$

as an integral representation of solutions of (A.2.1). Here $-x_k(\xi)$ $(k = 1, \ldots, n)$ denotes a root of the characteristic polynomial of $\hat{P}$. Note that, in view of (A.2.2), the characteristic polynomial of $\hat{P}$ is essentially the same as that of P and, in fact, $x_k(\xi)$ is a root of

$$p(x, \xi) = \sum_{j,k} a_{jk} x^k \xi^j = 0, \tag{A.2.4}$$

i.e., the characteristic polynomial of P, with respect to x. Ignoring the lower order terms (with respect to η), we find that (A.2.3) has the same form as (A.1.5) and that its phase function is given by

$$f_k(x, \xi) = x\xi - \int^{\xi} x_k(\xi) d\xi. \tag{A.2.5}$$

Thus, if we take a saddle point of $f_k(x, \xi)$, which is nothing but a root $\zeta_j(x)$ $(j = 1, \ldots, m)$ of $p(x, \xi) = 0$ with respect to ξ (and hence there exist m saddle points), and let $\Gamma_k^{(j)}$ denote a steepest descent path of Re $f_k(x, \xi)$ passing through $\zeta_j(x)$, it is expected that

$$\int_{\Gamma_k^{(j)}} e^{\eta x \xi} \hat{\psi}_k \, d\xi \tag{A.2.6}$$

should provide the Borel resummed WKB solution of (A.2.1) with the top term $S_{-1}(x) = \zeta_j(x)$. As a matter of fact, considering $\hat{\psi}_k$ in (A.2.6) to be the Borel sum of a WKB solution of (A.2.2) and rewrite (A.2.6) as

$$\begin{aligned}
\int_{\Gamma_k^{(j)}} e^{\eta x \xi} \hat{\psi}_k \, d\xi &= \int_{\Gamma_k^{(j)}} e^{\eta x \xi} \left(\int_{z = \int^{\xi} x_k(\xi) d\xi + v, v \geq 0} e^{-\eta z} \hat{\psi}_{k,B}(\xi, z) dz \right) d\xi \\
&= \iint \exp(-\eta y) \hat{\psi}_{k,B}(\xi, y + x\xi) d\xi dy
\end{aligned} \tag{A.2.7}$$

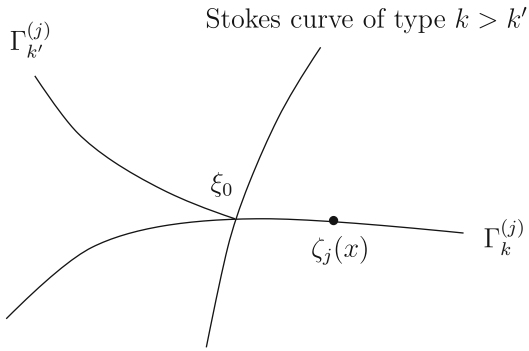
by using the definition of the Borel sum of $\hat{\psi}_k$ and employing a change of variable $y = z - x\xi$, where $\hat{\psi}_{k,B}(\xi, z)$ denotes the Borel transform of $\hat{\psi}_k(\xi, \eta)$, we can verify that

$$\chi(x, y) = \int \hat{\psi}_{k,B}(\xi, y + x\xi) d\xi \tag{A.2.8}$$

is the Borel transform of a (suitably normalized) WKB solution ψ_j of (A.2.1) when $y + \int^x \zeta_j(x) dx$ is sufficiently small. This implies that the assertion (i) of Theorem A.1.1 holds also for a differential equation (A.2.1) with polynomial coefficients.

However, the situation is much more complicated than the case of $n = 1$, i.e., the case of Laplace type equations. When $n \geq 2$, Stokes phenomena occur with Borel resummed WKB solutions $\hat{\psi}_k$ on a Stokes curve of (A.2.2) and we need to take the effect of Stokes curves and Stokes phenomena for (A.2.2) into account. For example, suppose that a steepest descent path $\Gamma_k^{(j)}$ crosses a Stokes curve of (A.2.2) with type

Fig. A.2 A new steepest descent path $\Gamma_{k'}^{(j)}$ bifurcates at a crossing point of $\Gamma_k^{(j)}$ with a Stokes curve of type $k > k'$



$k > k'$ at a point ξ_0, as shown in Fig. A.2. Then a Stokes phenomenon occurs at $\xi = \xi_0$ and the Borel sum of $\hat{\psi}_k$ picks up the Borel sum of another WKB solution $\hat{\psi}_{k'}$ of (A.2.2) after crossing the Stokes curve. As its consequence,

$$\int_{\Gamma_{k'}^{(j)}} e^{\eta x \xi}\, \hat{\psi}_{k'}\, d\xi \tag{A.2.9}$$

appears in the description of the Borel sum of a WKB solution ψ_j of (A.2.1) in question, where $\Gamma_{k'}^{(j)}$ denotes a steepest descent path of Re $f_{k'}(x, \xi)$ emanating from ξ_0 (cf. Fig. A.2).

Thus we are led to the following definition:

Definition A.2.1 We call the union $\Gamma_k^{(j)} \cup \Gamma_{k'}^{(j)}$ an **exact steepest descent path** passing through a saddle point $\zeta_j(x)$. More generally, if $\Gamma_{k'}^{(j)}$ (or $\Gamma_k^{(j)}$ again) crosses another Stokes curve of type $k' > k''$ (or of type $k > k''$), we further bifurcate another steepest descent path $\Gamma_{k''}^{(j)}$ of Re $f_{k''}$ at the crossing point in the same manner. We continue this procedure until no steepest descent path crosses a Stokes curve anymore. Then the union $\Gamma^{(j)} = \Gamma_k^{(j)} \cup \Gamma_{k'}^{(j)} \cup \Gamma_{k''}^{(j)} \cup \cdots$ of steepest descent paths thus obtained is called an **exact steepest descent path** passing through a saddle point $\zeta_j(x)$.

Then, as a generalization of the criterion for detecting active Stokes curves of Laplace type equations, we naturally obtain the following conjecture:

ESDP Ansatz (Exact Steepest Descent Path Ansatz).
A Stokes phenomenon for a Borel resummed WKB solution of (A.2.1) *occurs at x* (*that is, x belongs to an active Stokes curve of* (A.2.1)) *if and only if an exact steepest descent path connects two saddle points $\zeta_j(x)$ and $\zeta_{j'}(x)$.*

This is an outline of the exact steepest descent method, which is a natural generalization of the steepest descent method for Laplace type equations. For more details we refer the reader to [AKT3], where several concrete examples are discussed through the exact steepest descent method.

References

[A] Aoki, T.: Toward the exact WKB analysis of holonomic systems. RIMS Kôkyûroku **1433**, 1–8 (2005) (in Japanese)

[AH] Aoki, T., Honda, N.: Geometric properties of the Riemann surfaces associated with the Noumi-Yamada systems with a large parameter. J. Math. Soc. Japan **63**, 1085–1119 (2011)

[AHKKoNSShT] Aoki, T., Honda, N., Kawai, T., Koike, T., Nishikawa, Y., Sasaki, S., Shudo, A., Takei, Y.: Virtual turning points—a gift of microlocal analysis to the exact WKB analysis. Algebraic Analysis of Differential Equations, pp. 29–43. Springer (2008)

[AHU] Aoki, T., Honda, N., Umeta, Y.: On the number of the turning points of the second kind of the Noumi-Yamada systems with a large parameter. RIMS Kôkyûroku Bessatsu, B **37**, 1–30 (2013)

[AKKoT] Aoki, T., Kawai, T., Koike, T., Takei, Y.: On the exact WKB analysis of operators admitting infinitely many phases. Adv. Math. **181**, 165–189 (2004)

[AKSShT] Aoki, T., Kawai, T., Sasaki, S., Shudo, A., Takei, Y.: Virtual turning points and bifurcation of Stokes curves. J. Phys. A **38**, 3317–3336 (2005)

[AKT1] Aoki, T., Kawai, T., Takei, Y.: The Bender-Wu analysis and the Voros theory. Special Functions, pp. 1–29. Springer (1991)

[AKT2] Aoki, T., Kawai, T., Takei, Y.: New turning points in the exact WKB analysis for higher-order ordinary differential equations. Analyse algébrique des perturbations singulières, I, pp. 69–84. Hermann, Paris (1994)

[AKT3] Aoki, T., Kawai, T., Takei, Y.: On the exact steepest descent method: a new method for the description of Stokes curves. J. Math. Phys. **42**, 3691–3713 (2001)

[AKT4] Aoki, T., Kawai, T., Takei, Y.: Exact WKB analysis of non-adiabatic transition probabilities for three levels. J. Phys. A **35**, 2401–2430 (2002)

[AKT5] Aoki, T., Kawai, T., Takei, Y.: The Bender-Wu analysis and the Voros theory. II. Advanced Studies in Pure Mathematics, vol. 54, pp. 19–94. Mathematical Society of Japan, Tokyo (2009)

[AKoT] Aoki, T., Koike, T., Takei, Y.: Vanishing of Stokes curves. Microlocal Analysis and Complex Fourier Analysis, pp. 1–22. World Scientific, Singapore (2002)

[BW] Bender, C.M., Wu, T.T.: Anharmonic oscillator. Phys. Rev. **184**, 1231–1260 (1969)

[BNR] Berk, H.L., Nevins, W.M., Roberts, K.V.: New Stokes' line in WKB theory. J. Math. Phys. **23**, 988–1002 (1982)

[BE] Brundobler, S., Elser, V.: *S*-matrix for generalized Landau-Zener problem. J. Phys. A **26**, 1211–1227 (1993)

N. Honda et al., *Virtual Turning Points*, SpringerBriefs in Mathematical Physics,
DOI 10.1007/978-4-431-55702-9

[CH] Carroll, C.E., Hioe, F.T.: Transition probabilities for the three-level Landau-Zener model. J. Phys. A **19**, 2061–2073 (1986)

[CLP] Colin de Verdière, Y., Lombardi, M., Pollet, J.: The microlocal Landau-Zener formula. Annales I'Institut Henri Poincaré **71**, 95–127 (1999)

[C] Costin, O.: Asymptotics and Borel Summability, vol. 141. Monographs and Surveys in Pure and Applied Mathematics, CRC Press, Boca Raton (2009)

[DP] Delabaere, E., Pham, F.: Resurgent methods in semi-classical asymptotics. Annales I'Institut Henri Poincaré **71**, 1–94 (1999)

[F] Fedoryuk, M.V.: Asymptotic methods in analysis. Encyclopedia of Mathematical Sciences, vol. 13, pp. 83–191. Springer, New York (1989)

[H] Hagedorn, G.A.: Proof of the Landau-Zener formula in an adiabatic limit with small eigenvalue gaps. Commun. Math. Phys. **136**, 433–449 (1991)

[H1] Honda, N.: Some examples of the Stokes geometry for Noumi-Yamada systems. RIMS Kôkyûroku **1516**, 21–167 (2006)

[H2] Honda, N.: On the Stokes geometry of the Noumi-Yamada system. RIMS Kôkyûroku Bessatsu, B **2**, 45–72 (2007)

[H3] Honda, N.: The geometric structure of a virtual turning point and the model of the Stokes geometry. RIMS Kôkyûroku Bessatsu, B **10**, 63–117 (2009)

[J1] Joye, A.: Proof of the Landau-Zener formula. Asymptot. Anal. **9**, 209–258 (1994)

[J2] Joye, A.: Exponential asymptotics in a singular limit for n-level scattering systems. SIAM J. Math. Anal. **28**, 669–703 (1997)

[JP] Joye, A., Pfister, C.-E.: Complex WKB method for 3-level scattering systems. Asymptot. Anal. **23**, 91–109 (2000)

[KKO] Kashiwara, M., Kawai, T., Oshima, T.: Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics. I. Proc. Japan Acad. **49**, 420–425 (1974)

[KKoNT1] Kawai, T., Koike, T., Nishikawa, Y., Takei, Y.: On the Stokes geometry of higher order Painlevé equations. Astérisque **297**, 117–166 (2004)

[KKoNT2] Kawai, T., Koike, T., Nishikawa, Y., Takei, Y.: On the complete description of the Stokes geometry for the first Painlevé hierarchy. RIMS Kôkyûroku **1397**, 74–101 (2004)

[KKoT] Kawai, T., Koike, T., Takei, Y.: On the exact WKB analysis of higher order simple-pole type operators. Adv. Math. **228**, 63–96 (2011)

[KT1] Kawai, T., Takei, Y.: WKB analysis of Painlev'e transcendents with a large parameter. I. Adv. Math. **118**, 1–33 (1996)

[KT2] Kawai, T., Takei, Y.: Algebraic Analysis of Singular Perturbation Theory. American Mathematical Society, Providence (2005) (Translated from the Japanese edition published by Iwanami in 1998)

[KoS] Koike, T., Schäfke, R.: On the Borel summability of WKB solutions of Schrödinger equations with polynomial potentials and its application. In preparation

[KoT] Koike, T., Takei, Y.: The effect of new Stokes curves in the exact steepest descent method. Microlocal Analysis and Complex Fourier Analysis, pp. 186–199. World Scientific, Singapore (2002)

[L] Landau, L.: A theory of energy transfer, II. pp. 63–66. Collected Papers of L.D. Landau, Pergamon Press (1965)

[NY] Noumi, M., Yamada, Y.: Symmetry in Painlevé equations. Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, pp. 245–260. Kyoto University Press (2000)

[Sa] Sasaki, Shinji: On the inevitable influence of virtual turning points on the non-adiabatic transition probabilities for three-level systems. Preprint (RIMS-1818) (2015)

[S1] Sasaki, S.: On the role of virtual turning points in the deformation of higher order linear differential equations. RIMS Kôkyûroku **1433**, 27–64 (2005)

[S2] Sasaki, S.: On the role of virtual turning points in the deformation of higher order linear differential equations, II. RIMS Kôkyûroku **1433**, 65–109 (2005)

[SKK] Sato, M., Kawai, T., Kashiwara, M.: Microfunctions and pseudo-differential equations. Lecture Notes in Mathematics, vol. 287, pp. 265–529. Springer (1973)

[Sh] Shudo, A.: A role of virtual turning points and new Stokes curves in Stokes geometry of the quantum Hénon map. Algebraic Analysis of Differential Equations, pp. 251–264. Springer (2008)

[ShI] Shudo, A., Ikeda, K.: Stokes phenomena for quantized Hénon map. RIMS Kôkyûroku **1168**, 53–67 (2000) (in Japanese)

[S] Silverstone, H.J.: JWKB connection-formula problem revisited via Borel summation. Phys. Rev. Lett. **55**, 2523–2526 (1985)

[T1] Takei, Y.: Integral representation for ordinary differential equations of Laplace type and exact WKB analysis. RIMS Kôkyûroku **1168**, 80–92 (2000)

[T2] Takei, Y.: An explicit description of the connection formula for the first Painlevé equation. Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, pp. 271–296. Kyoto University Press (2000)

[T3] Takei, Y.: On a double turning point problem for systems of linear ordinary differential equations. Preprint

[T4] Takei, Y.: Toward the exact WKB analysis for higher-order Painlevé equations—the case of Noumi-Yamada systems. Publ. Res. Inst. Math. Sci. **40**, 709–730 (2004)

[V] Voros, A.: The return of the quartic oscillator. The complex WKB methods. Annales de l'Institut Henri Poincaré **39**, 211–338 (1983)

[W] Wasow, W.: Asymptotic Expansions for Ordinary Differential Equations. Interscience Publishers, New York (1965)

[Z] Zener, C.: Non-adiabatic crossing of energy levels. Proc. Roy. Soc. Lond. **137**, 696–702 (1932)

Index

N. Honda et al., *Virtual Turning Points*, SpringerBriefs in Mathematical Physics,
DOI 10.1007/978-4-431-55702-9