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Probabilistic solutions in geotechnics

by

László Rétháti

**Institute for Geodesy
and Geotechnics, Budapest**



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Preface

Probability theory was first applied to geotechnics some two decades ago. This breakthrough was to be expected, given the fact that several other soil mechanics disciplines (road construction, hydraulic engineering, structural design) had already used the methods of probability theory by that time. However, the soil being a heterogeneous material, the precision of its characterization and of the prediction of its behaviour by deterministic methods was limited of necessity. The safety factor expressing the measure of uncertainty had to be introduced in order to make it possible to estimate the occurrence of individual events.

Probability theory being a branch of mathematical sciences, its methodology can be applied to the resolution of practical tasks with some modifications only. The various disciplines have to adapt it to their specific purposes. The present book wishes to help those experts of the geotechnical sciences who are facing problems of this kind.

I would like to express my grateful thanks to Elsevier Science Publishers and to Akadémiai Kiadó, the Publishing House of the Hungarian Academy of Sciences, to the staff of the Printing House and to Mr. BARS who translated the text for their valuable work in assisting the publication of this book.

Budapest, 1987

LÁSZLÓ RÉTHÁTI

Introduction

The theoretical foundations of combinatorique and the related “probability calculus” owe their development to an interest in games of chance. The first studies on dice games to appear were by CARDANO (1501–76) and GALILEI (1564–1642). The principal founders of the new scientific discipline, who studied the related problems with mathematical precision, were PASCAL (1623–62) and FERMAT (1601–65), but the name of HUYGHENS (1629–95) can also be added to this list.

JACQUES BERNOULLI (1654–1705) further generalized and perfected the principles of game theory. Significant work in this field was also done by MOIVRE (1667–1751), who was the first to deal with the normal distribution, which is still of outstanding importance. The first textbook on the theory of probability was published by LAPLACE (1749–1827), laying the foundations of the theory of least squares, among others. In the 19th century, the new discipline boasted of a line of famous mathematicians, including POISSON, GAUSS, CHEBISHEV, MARKOV, POINCARÉ and LYAPUNOV.

The first half of the 20th century was characterized by the rapid development of the theory (FRÉCHET, BOREL, CANTELLI, MISES) and by enhanced emphasis on practical problems (PEARSON, GOSSET, FISHER). The most important event in this respect was the publication of KOLMOGOROV’s “Basic concepts of probability calculus” (“Die Grundbegriffe der Wahrscheinlichkeitsrechnung”, 1933). The axiomatic, strictly mathematical foundation of the theory of probability and several important new ideas, such as the concept of stochastic processes, are connected with his name.

The subject matter of probability theory is *random phenomena*, and its goal is to discover the objective rules which govern this behaviour (RÉNYI, 1954). Most natural phenomena, or those brought about by man, may be described by *causal* models, meaning that if we know the cause-effect relationships, we can describe the processes using (“unequivocally”) deterministic methods. For studying random phenomena we must create a stochastic model, either because we do not understand the totality of the causal relationships, or because we do not regard their determination as necessary. There exist phenomena—as we shall see later in several examples—which can be described by both models, but care must be taken because in stochastic models only those circumstances whose individual effects are small may be left unconsidered (PRÉKOPA, 1962).

The wider practical application of the new discipline has developed in the past 50 years. Among the first to use the theory were the *chemical industry* and the *textile industry*, where applications—today regarded as indispensable—were developed in the field of quality control and later for the “statistical control” of production. In the United States, quality control of *structural materials* has been regulated since the early nineteen-thirties (Manual of Presentation of Data, 1933). The solution of numerous practical problems has also been provided by similar research activities in the field of *biometry*.

In the engineering professions, the theory of probability, especially one of its branches, mathematical statistics, started to gain ground in geodesy (error calculation), in structural design (including material testing), in hydrology, and in road and bridge construction. Safety factors in dimensioning structures of concrete, reinforced concrete and steel have been revised (FREUDENTHAL, 1947; JOHNSON, 1953; LÉVI, 1953; IKEDA, 1956). In hydrology, probabilistic research in connection with *forecasting* is most typical, but today numerous other problems are also being solved with the help of *stochastic models*. In road construction, the *statistical specifications*—using methods developed for industrial product control—and the *continuous control of construction* was introduced in the late fifties.

In the field of *geotechnics*, the first studies (LÉVI, 1958; LAZARD, 1961; LANGEJAN, 1965; WU and KRAFT, 1967) dealt only with the modern concept of the safety factor, and even in this respect only the initial steps were made. At the same time, the statistical investigation of the variability of soils (THORNBURN and LARSEN, 1959; LUMB, 1966) began, and the question of the number of samples to be tested was raised (ZLATAREV, 1965). In road construction, the necessity of continuous control by measurements made during the compaction of earthworks was emphasized (DAVIS, 1953; TURNBULL, COMPTON and AHLVIN, 1966; PETTITT, 1967; BEATON, 1968).

The international conference on “Applications of Statistics and Probability in Soil and Structural Engineering” held in 1971 in Hong Kong, represented a milestone in geotechnics. The published proceedings of this conference dealt with the following problems: spatial distribution (variability) of the soil physical characteristics and its effects on the risks, safety factors and costs; the number of samples (tests) required; correlation between the results of direct and indirect measurements, and between calculated and measured values; stability of foundations and earthworks; problems of quality control.

At the second international conference (Aachen, 1975), LUMB reported that from among 12,500 articles recorded between 1953 and 1973 in *Geodex and Geotechnical Abstracts*, 320 dealt with problems of applied mathematical statistics. Their distribution among various fields was as follows: soil classification 9%, distribution functions of physical characteristics 9%, sampling 6%, reliability improvement 13%, regression among the physical characteristics 54% and quality control 9%. From the

statistical data it also appears that only about 11% of the total were published during the first half of the period in question.

The situation outlined above shows that the synthesis of geotechnics with the theory of probability has only a very short history. The author and the readers of this book are in the favourable position of being able to utilize a well-founded, but at the same time easily surveyable, methodical body of knowledge.

1. Statistical parameters of the soil physical characteristics

The primary aim of the examination of soils is to reveal the layers in the subsoil and determine their physical (and eventually chemical) characteristics. Such results are reported in the form of bore-hole profiles and cross sections of the layers in coordinate systems used for the evaluation of laboratory and/or on-site tests and written interpretation by experts.

Since the birth of soil mechanics as a scientific discipline in 1925 it has been practice—and as is generally the case even today—that professional reports present only the mean and extreme (or the predicted least favourable) values of the physical characteristics. As the use of statistical methods became widespread, an increasing number of experts called attention to the fact that this attitude was untenable.

The modern requirements can be formulated very simply: the properties of the critical soil layers must be characterized by the *totality* of the test results. This means that for each layer, the “elementary” statistical parameters describing the distribution of every physical characteristic and eventually with the help of these parameters even the empirical distribution function must be determined. This applies not only to investigations of residential and industrial settlements, but also to processing of archive data on greater settlements, or individual characteristic layers (e.g. London Clay).

1.1. Basic conceptions

The theory of probability deals with the evaluation of natural *phenomena* and artificial *experiments* which exhibit some random behaviour, i.e. their outcome cannot be totally foreseen. The observation of the phenomenon, or the experiment may be *repeated* as many times as necessary, but because the result is affected by a great number of factors whose individual effects are small but *unaccountable*, the phenomenon (experiment) has a *random* character.

A possible outcome of a given experiment is called an *elementary event* (ω), the totality of these is the *event space* (Ω) and any partial sum of them is called a combined event, or simply an *event* ($A, B, \dots$).

Let A be a particular event connected with some experiment. Let us carry out the experiment independently n times—and denote by k the number of the occurrences of

the event among the experiments. The number k is called the *frequency* of event A , the quotient k/n is called the *relative frequency*. If an experimental series consisting of n elements is repeated several times, we obtain for k/n a series which has a random fluctuation. These numbers fluctuate around a definite—theoretical—value, which is called $P(A)$, the probability of event A .

If the outcome of the experiment is the numerical value of some quantity, then this “quantity” is called a *random variable*. This is not a concrete numerical value, but a quantity conceived in a more general sense, carrying in it all the values which may be obtained as the result of every possible realization of the experiment (VINCZE, 1968). If the random variable related to an experiment is ξ and during the experiment we observed the value x , then we say that the event $\xi = x$ was realized. If the variable falls at any arbitrary point of the continuum of real numbers, we speak of a *continuous* random variable; if it may assume only a finite or countable number of elements, we speak of a *discrete* random variable.

Every random variable establishes a probability distribution on the continuum of real numbers. The function $F(x)$, which expresses the probability of the occurrence of the event $\xi < x$, is called the *distribution function* of the random variable. This is a monotonically not decreasing function of x , for which

$$\lim_{x \rightarrow -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} F(x) = 1 ,$$

so its value falls between 0 and 1.

Using this function we can give the probability $F(b) - F(a)$ of the value of ξ falling into the interval (a, b) . If the random variable ξ has a *continuous* distribution, then its distribution function $F(x)$ can be differentiated; the function

$$F'(x) = f(x)$$

is called the density function. Then the probability related to the interval (a, b) may be determined as follows:

$$F(b) - F(a) = P(a < \xi < b) = \int_a^b f(x) dx .$$

Following similar reasoning, the distribution function of the *discrete* variable is

$$F(x) = \sum_{x_i \leq x} p_i ,$$

which appears graphically as a step-function.

The distribution functions and the density functions may be described with the help of one or more *parameters*. In practice these are the necessary (and generally sufficient) numerical data which provide a description of the distribution. The normal distribution can be described by the function

$$P(\xi < x) = F(x; m, \sigma) ,$$

where m (the expected value) and σ (the standard deviation) are two parameters of the distribution.

The random variable ξ forms a so-called *population*. To study its properties we perform a series of experiments consisting of a finite number of elements. The elements

$$x_1, x_2, \dots, x_n$$

of this *sample* establish an *empirical* distribution; from these empirical parameters can be determined. This operation is called estimation.

The parameter Θ of the population is a definite value, while ϑ , supplying the estimation is a random variable which may vary from sample to sample. For this random variable the following requirements apply:

- ϑ as a random variable fluctuates around the value of Θ ,
- its fluctuations should not be great (i.e. its dispersion should be small),
- the estimation should improve with the increasing number of the elements of the sample (it should be consistent), i.e. ϑ must converge stochastically to 0.

If the expected value of ϑ is Θ itself, we speak of an unbiased estimation.

1.2. Determination of the statistical parameters

The simplest statistical characteristics of the data- and time series may be determined using the so-called *central moments*. The k -th central moment of the population to be determined by sampling may be defined by the following expression:

$$\mu_k = M[\xi - M(\xi)]^k, \quad (1.1)$$

where

ξ — random variable,

M — symbol for the expected value.

1.2.1. Expected value

The expected value μ of the random variable of a *discrete* distribution is the weighted arithmetic mean of the possible values ($x_1, \dots, x_k$) of ξ :

$$\mu = M(\xi) = \sum_k x_k p_k, \quad (1.2)$$

where the weights p_k are the probabilities belonging to the individual values.

The expected value of the random variable of a *continuous* distribution is

$$\mu = M(\xi) = \int_{-\infty}^{+\infty} x f(x) dx, \quad (1.3)$$

where $f(x)$ is the density function of the variable.

In the overwhelming majority of the practical cases—for example in the course of the determination of the soil physical characteristics—there is no possibility of knowing the whole population, so its statistical parameters can only be determined by sampling. If n tests are performed to determine some physical parameter of a given soil layer, and their numerical results are $x_1, x_2, \dots, x_n$, then the *unbiased estimation* of the expected value of the population μ is:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i. \quad (1.4)$$

In the interpretation of the expected value the following must be taken into account:

a) Its numerical value and the reliability of this value is a function of the number of tests (see also *Fig. 1.2* and Section 7.4).

b) The set of any physical characteristics may exhibit a *trend* and may also contain *periodic* elements as a consequence of the variability in time and space (Section 4.2 deals with eliminating these, and taking them into account).

The expected value as a characteristic of the central trend of the random variable, is one of the calculation bases both in deterministic and stochastic models. (E.g., in the conventional processes the safety factor is the quotient of the expected value of the resistance by that of the load). But all this does not mean that the substitution of the expected value gives always the correct solution. A convincing example for it is presented by MOSTYN (1983), who asked 12 experts to answer the following question: “You have permeability tested a large homogeneous mass and have obtained the following results: 1.1–9.5–450–900–1100–1200 (the units are not important). All test results are assumed to be correct and no reasons are known to explain the scatter other than random variation. What single value of permeability would you ascribe to this mass to accurately and quickly determine the flow through it?”

Various answers were given to the question. According to MOSTYN the correct answer can be obtained by solving the differential equation

$$\frac{\partial}{\partial x} \left[k(x, y) \frac{\partial H}{\partial x} \right] + \frac{\partial}{\partial y} \left[k(x, y) \frac{\partial H}{\partial y} \right] = 0$$

by the finite difference method. The geometry used was a rectangular grid of 11 by 21 nodes. After performing 100 simulations each, he obtained that the characteristic permeability was close to the *geometric mean*, while the arithmetic mean may deviate from the effective value even by 300% in unfavourable cases.

1.2.2. Variance and standard deviation

If the soil layer to be studied is regarded as the totality of a collection of small elementary bodies for which the given physical characteristic (e.g. the void ratio) can still be defined, then the set of these bodies may be regarded, for practical purposes, as

infinitely great, relative to the total volume of the samples taken from it. Due to the heterogeneity of the soil, the measured physical characteristic varies from point to point, producing a scattering around the expected value. The index of this variability is the variance, defined by the expression:

$$D^2(\xi) = \sigma^2 = M[(\xi - M(\xi))^2] = M(\xi^2) - M^2(\xi) \quad (1.5)$$

and its positive square root, the *standard deviation* $D(\xi) = \sigma$.

The expression after the second equal sign is equivalent to the parallel-axis theorem or STEINER's theorem in statics, which relates the respective moments of inertia of a body about an arbitrary axis and about the central axis parallel to it (ELISHAKOFF, 1983).

The variance of a *discrete* variable may be determined by the following expression:

$$D^2(\xi) = \sum_i [x_i - M(\xi)]^2 p_i = \sum_i x_i^2 p_i - M^2(\xi), \quad (1.6)$$

and that of a *continuous* variable by the expression:

$$D^2(\xi) = \int_{-\infty}^{+\infty} [x - M(\xi)]^2 f(x) dx = \int_{-\infty}^{+\infty} x^2 f(x) dx - M^2(\xi). \quad (1.7)$$

If n samples are taken from the layer and their tests give the values $x_1, x_2, \dots, x_n$ for the actual physical property, then the *unbiased estimation* of the standard deviation of the population is

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^n x_i^2 - n\bar{x}^2}{n-1}}, \quad (1.8)$$

where $\bar{x}$ is the expected value defined by expression (1.4), i.e. the arithmetic mean. (Strictly speaking, only the variance will be an unbiased estimate, not the standard deviation.)

In connection with expression (1.8) the following must be noted:

1. The value s is usually called the *empirical standard deviation*, or the *corrected standard deviation* in order to distinguish it from the standard deviation of the population.

2. The reason that $n-1$ appears instead of n in the denominator is that the mean is also an estimated value, so the number of *degrees of freedom* has to be decreased by one (BESSEL's correction).

3. The calculation of s is simpler using the expression following the second equal sign.

4. The quantity s^2 is the *variance* of the measurements obtained by testing the samples.

If the samples can be assumed to come from a normal distribution, the standard deviation may be determined by approximation. Using another index of dispersion, the difference

$$R = x_{\max} - x_{\min},$$

(the *range* of x), the estimated value of the standard deviation is

$$s_R = A(n) \cdot R, \quad (1.9)$$

where $A(n)$ may be taken from the following table (Hungarian standard MSZ-256, 1973):

n	2	3	4	5	6	7	8	9	10
$A(n)$	0.89	0.59	0.49	0.43	0.40	0.37	0.35	0.34	0.32

In connection with the dispersion, the question of *rounding off* the measured values arises. If the interval of rounding is d , then we must add to the rounded value a random variable $\mathfrak{g}$, whose value falls between the spacing $-d/2 \dots +d/2$, which may be assumed to have a uniform distribution. The variance of this random variable is (see Section 1.3.2):

$$D^2(\mathfrak{g}) = \frac{d^2}{12}. \quad (1.10)$$

The fluctuation caused by rounding off is independent of fluctuations due to other causes. According to the rule of error propagation, the variances are additive, so we may write that

$$\sigma_k^2 = \sigma^2 + \frac{d^2}{12}, \quad (1.11)$$

where σ is the original ("effective") standard deviation and σ_k is the standard deviation after rounding off.

In soil mechanics the question of rounding off arises mainly during the determination of characteristics related to the water content (w, w_L, w_P). It is frequently proposed, with reference to the heterogeneity of the soil and the inherent errors of soil tests, that these be expressed without decimal values. In this case the rounding interval is $d=1.0$ and the standard deviation of the water content is $\sigma=1-2$ according to Section 1.3.1, thus the original standard deviation is increased, according to formula (1.11), by

$$\frac{\sigma_k - \sigma}{\sigma} 100 = \frac{1.041 - 1}{1} 100 = 4.1\% \quad \text{for} \quad \sigma = 1,$$

or

$$\frac{2.021 - 2}{2} 100 = 1.1\% \quad \text{for} \quad \sigma = 2.$$

According to the quoted standard the permitted increase is 1.5%; this means that it is only permissible to round off to the nearest integer if the original standard deviation σ exceeds 1.67 ($\sigma > 1.67$). (So according to Section 1.3.2 this is permitted for the consistency limits, though this is not always favourable for statistical processing of the data).

The dispersion (the uncertainty appearing in the physical characteristics) may be characterized also by the *entropy*. If a random variable x has n possible outcomes x_i ($i = 1, 2, \dots, n$) and the possibility assigned to each outcome is P_i , the entropy associated with x is (SHANNON, 1948):

$$H(x) = - \sum_{i=1}^n P_i \ln P_i. \tag{1.12}$$

Like the variance, this function is another measure of the dispersion. However, while $D^2(x)$ measures the dispersion about the mean of the random variable, the entropy measures the *overall uncertainty* associated with the random variable. The possible extreme values of the entropy are

$$H_{\min} = 0 \quad \text{and} \quad H_{\max} = \ln n;$$

the minimum is where one output is 1 (all the rest are zero), and the maximum is where $P_1 = P_2 = \dots = P_n = 1/n$.

Figure 1.1 shows the grain-size curves of three soils and Table 1.1 the density distribution of the grains (HARROP-WILLIAMS, 1983). The characteristics of the dispersion are as follows

	1	2	3
variance (mm ²)	0.0	149.0	18.7
entropy	0.0	0.846	1.774

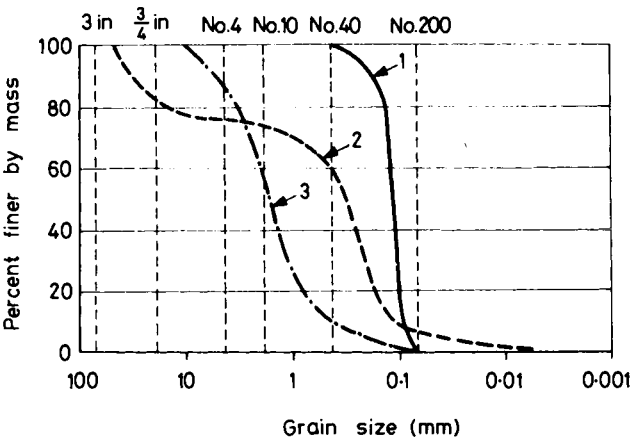


Fig. 1.1. Three grain size distribution curves for the numerical example of HARROP-WILLIAMS (1983)

Table 1.1. Frequency distribution of grain sizes (HARROP-WILLIAMS, 1983)

Intervals (mm)	Frequency, P_i		
	Curve 1	Curve 2	Curve 3
0– 1	1.000	0.720	0.230
1– 2	0.000	0.020	0.370
2– 3	0.000	0.000	0.140
3– 4	0.000	0.005	0.110
4– 5	0.000	0.005	0.050
5– 6	0.000	0.003	0.030
6– 7	0.000	0.002	0.010
7– 8	0.000	0.002	0.010
8– 9	0.000	0.013	0.020
9–10	0.000	0.010	0.010
10–50	0.000	0.220	0.020

As it is seen, the variance for curve 2 is larger than for curve 3, while the entropy of curve 3 exceeds that of curve 2. Curve 2 is gap-graded with most of the grain sizes falling in two distinct ranges 0–1 and 10–50 mm. Thus, there is less uncertainty (entropy) as to where a value lies than in curve 3. The variance is a measure of the spread about the mean and in this bimodal case may not accurately reflect the uncertainty of a particular value (HARROP-WILLIAMS, 1983).

1.2.3. Coefficient of variation

The coefficient of variation (C_v) is the quotient of the empirical standard deviation and the expected value, i.e. the estimation of the quotient σ/μ :

$$C_v = \frac{s}{\bar{x}}. \quad (1.13)$$

Its value expressed in percentages is the relative scatter:

$$s_r(\%) = 100 C_v = 100 \frac{s}{\bar{x}} \quad (1.14)$$

(this index is used less frequently today than C_v).

If samples taken from a population are tested continuously, then if after reaching an arbitrary number of samples k , we calculate the mean and the standard deviation of the measurements $k, k+1, k+2, \dots$, we see that the C_v values formed from their quotients show an increasingly smaller fluctuation. This *stabilization trend* can be observed clearly in the case presented in Fig. 1.2. The property tested was the unconfined compression strength of the typical sweet-water sediment of town Szeged (see also Section 1.4). From the data for 85 tests, which are available in the archives, the following statistical parameters can be determined:

$$\bar{q}_u = 184 \text{ kN/m}^2,$$

$$s = 90 \text{ kN/m}^2,$$

$$C_v = 0.49.$$

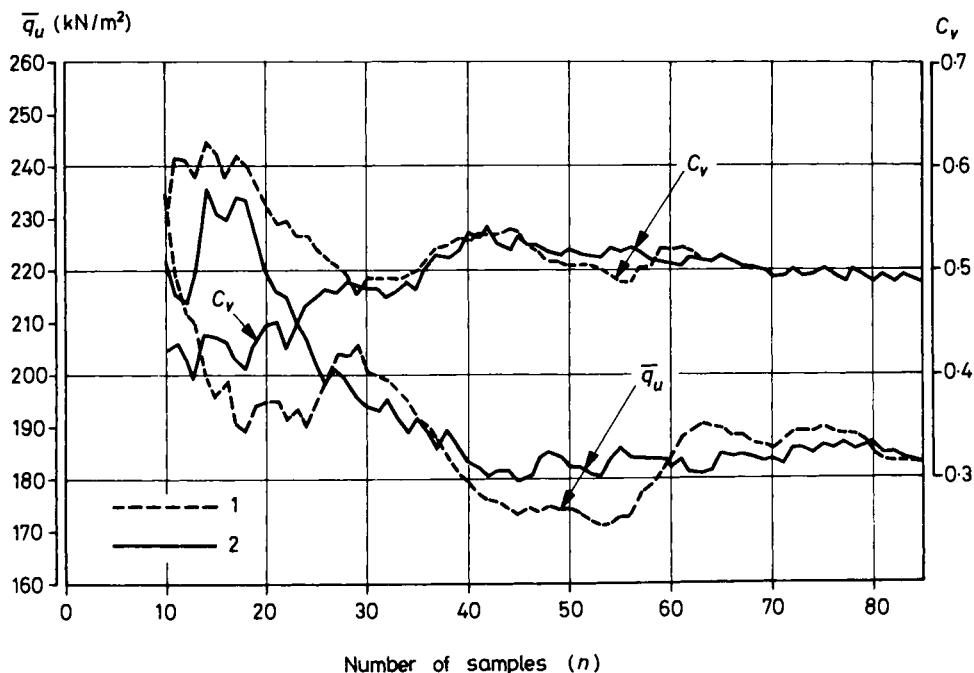


Fig. 1.2. Mean ($\bar{q}_u$) and coefficient of variation (C_v) of the unconfined compression strength of the soil S5 versus the number of the samples (n); 1 — samples in the original sequence, 2 — sample in random sequence

The continuous variations of the values $\bar{q}_u$ and s with increasing n may be examined in two ways: by taking the tests (a) in the sequence, the data were collected, or (b) in a random sequence (e.g. by drawing lots of serial numbers). Whichever method is applied, the coefficient of variation becomes stable when $n > 30$, as shown in Fig. 1.2 ($\bar{q}_u$ becomes stable somewhat later).

Similar results are obtained if we examine not a collection of sample data, but the *time series* of some natural event. Let us take groundwater as an example which plays an important role in geotechnics and consider its relationship to the atmospheric precipitation which supplies it. The mean value of the annual total precipitation (C) in Budapest for the period 1871–1975 is 629 mm, its coefficient of variation is 0.188. As shown in Fig. 1.3, the C_v values did not leave the range 0.172...0.182 after $n > 30$, i.e. they may be regarded to have been constant ($C_v \sim 0.18$) during the last 75 years. From this observation we may also draw interesting conclusions concerning the *reliability of the mean value* (RÉTHÁTI, 1980b).

The coefficient of variation, as a *measure of the reliability of the central tendency*, plays an important role in the statistical evaluation of the physical properties of soils and in the solution of geotechnical problems. This may be explained, like the stabilization trend, by the fact that its numerical value varies for *each physical property* of the soil, even within the same soil layer. The coefficient of variation of the same

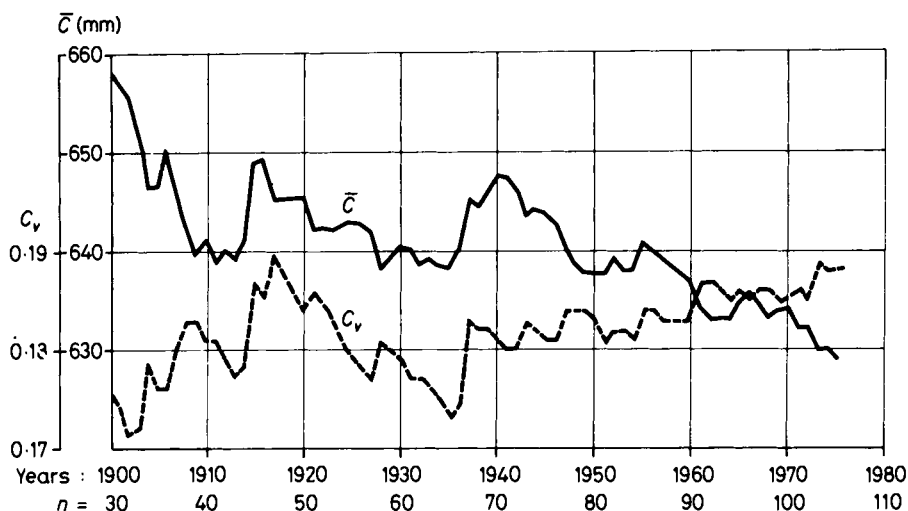


Fig. 1.3. The continuously formed mean ($\bar{C}$) and coefficient of variation (C_v) of the yearly sum of precipitations measured in Budapest versus the time (RÉTHÁTI, 1980b)

physical properties of soils studied in many parts of the world vary within a relatively narrow range. According to KRAHN and FREDLUND (1983) they are independent of the geological age of the soil. Thus, we have reliable data on the expected standard deviation of a given physical characteristic of a soil even before beginning a study or performing laboratory tests.

If we express the uncertainty by the entropy instead of the standard deviation (or the variance), even then we may speak of a relative value. According to Section 1.2.2 the two possible values of the entropy are zero and $\ln n$, respectively. So it is advised to define the relative entropy by the quotient

$$H_r(x) = H(x)/H_{\max}. \quad (1.15)$$

According to HARROP-WILLIAMS (1983) H_r characterize the grain size curve better than the uniformity coefficient U . Both characteristics assume the following values in the case of the three soils shown in Fig. 1.1:

	1	2	3
U	1.5	4.0	6.67
H_r	0	0.351	0.739

1.2.4. Coefficient of skewness

The coefficient of skewness (C_s) is the quotient of the third central moment and the standard deviation raised to the third power. Its unbiased estimation, with the number of the degrees of freedom also taken into account, may be determined by the following expression:

$$C_s = \frac{n \sum_{i=1}^n (x_i - \bar{x})^3}{(n-1)(n-2)s^3} \quad (1.16)$$

The coefficient of skewness—unlike the standard deviation—is a quantity with a sign, since the sum appearing in the numerator of expression (1.16) may be either positive or negative. Consequently, the signed value of C_s is the index of the direction and the magnitude of the *asymmetry*: if $C_s > 0$, the density function (frequency curve) is extended toward larger numbers and vice versa (Fig. 1.4).

As the differences $x_i - \bar{x}$ appear to the third power in expression (1.16), the coefficient of skewness is very sensitive to the *outlying* (extreme) values. From this fact it follows that

- the extreme values of the data series must be revised carefully by the method described in Section 7.1;
- calculating the coefficient of skewness for a sample of less than 40 elements ($n < 40$) is not useful, or the results—if calculated—must be accepted with reservation.

The asymmetry found in a data set representing the physical characteristics of a soil may be accidental, due to random causes, but it can also be caused by faulty sampling, or errors committed during testing, or may result from the drying of some samples (see Section 7.1.3). In addition there exist cases where the property studied has a *physical limit* in one direction. If, for example, the degree of saturation (S_r) of layers lying under groundwater level is investigated, the mean of the measurement results will be slightly below $S_r = 1$ (but close to it). Since $S_r > 1$ is not possible, the distribution shows an asymmetry in the negative direction (see Section 1.4). The situation is similar in the case of the void ratio: the conditions of formation of natural soils determine a lower limit,

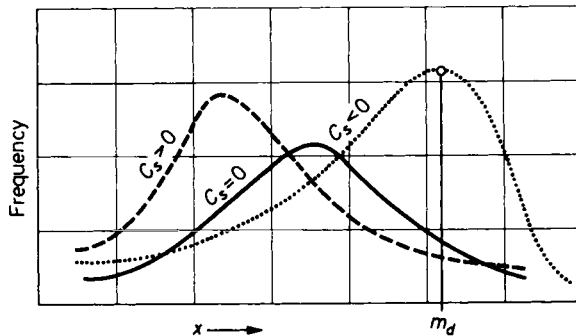


Fig. 1.4. Relationship between the asymmetry and the coefficient of skewness (C_s) of the density function

while the number of possible arrangements of the grains leading to a loose structure is great.

REIMANN gives the following formula for the *approximation* of skewness based on the theory of ordered samples:

$$f = \frac{(X_{10} - X_M) - (X_M - X_{90})}{(X_{90} - X_M) + (X_M - X_{10})}, \quad (1.17)$$

where X_{10} and X_{90} are values for $P = 10\%$, resp. $P = 90\%$ in the frequency curve and X_M is the median Me ($P = 50\%$).

Another approximation is PEARSON's skewness:

$$S_k = \frac{\bar{x} - m_d}{s}, \quad (1.18)$$

where m_d is the mode of the distribution (*Fig. 1.4*), where the density function has its maximum.

The knowledge of the coefficient of skewness is necessary for determining the *type of the distribution function* and for estimating whether or not the distribution of the data set is normal (see Sections 2.1.1 and 2.2.1).

1.2.5. Kurtosis

The kurtosis (C_k) is the quotient of the fourth central moment by the fourth power of the standard deviation minus 3. Its unbiased estimation is

$$C_k = \frac{n \sum_{i=1}^n (x_i - \bar{x})^4}{(n-1)(n-2)s^4} - 3. \quad (1.19)$$

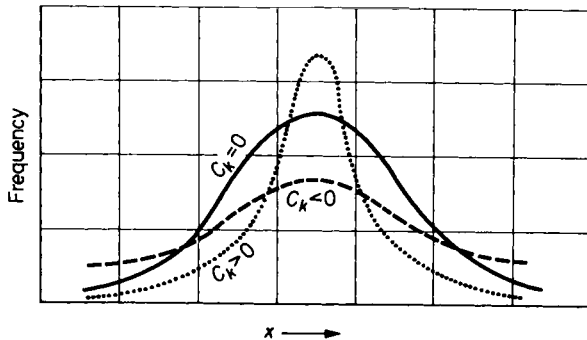


Fig. 1.5. Relationship between the peaked character and the kurtosis (C_k) of the density function

The kurtosis is the index of the “peaked” nature of the distribution: if $C_k > 0$, then the central section of the density function is higher than in a normal distribution and vice versa (Fig. 1.5). The additive term (-3) in expression (1.19) reflects the fact that for the normal distribution the value of the fraction is 3.

As with the coefficient of skewness, C_k may also be used for determining the type of the distribution function (see Sections 2.1.1 and 2.2.1). Since the differences $x_i - \bar{x}$ are raised to the fourth power, C_k is even more sensitive to the *outlying* values, than C_s , so its value calculated for $n < 100$ must be regarded only as an approximation.

1.2.6. Standard deviation of indirectly determined physical characteristics

If a physical characteristic of a soil is not determined by direct measurement (e.g. by laboratory tests) but is calculated, we must make a distinction based on whether or not the independent variable(s) used in the calculation are random; and whether or not the expression used for the calculation is an empirical relationship or a definition.

For example, if the angle of friction is determined from the number of blows N required for driving the probe 20 cm deep, then N is not a random variable (for practical purposes it contains no errors), but the relationship is empirical.

If the compression index is calculated from the liquid limit, then w_L in the empirical formula expressing the relationship is a random variable.

The consistency index is calculated from the formula

$$I_c = \frac{w_L - w}{I_p},$$

i.e. it is defined by a function containing three random variables.

The necessity of using an empirical relationship for calculating the actual physical characteristic increases the magnitude of the dispersion. This applies even more where the independent variable may also be error-laden. But if we choose for some reason the indirect determination, then the statistical parameters (standard deviation, coefficient of variation, etc.) may be calculated by the methods used for direct measurements.

1.3. Literature data on the statistical parameters

During the last 15 years numerous publications have appeared which present statistical parameters for the physical characteristics of various soil layers studied by the authors. The majority of these investigations were carried out for technical rather than scientific purposes, so these data provide a realistic *comparative* basis for evaluating the physical characteristics of the field studied by us and are suited simultaneously also for the

a priori determination of the statistical parameters. Nevertheless when using data found in literature, the following points must be carefully considered:

1. For cases involving a relatively small number of samples the coefficient of variation may not be sufficiently stable (see *Figs 1.2 and 1.3*).
2. The uncritical acceptance of *extreme values* may strongly bias the values of the statistical parameters (see Section 7.1.4).
3. For the purpose of statistical evaluation, the *distribution* of the sampling through time is also important (see Section 4.2.3). If the samples were collected over a time period of 1–2 weeks, the physical characteristics may be regarded as time invariant. Over longer periods a part of the dispersion must be regarded as having been caused by changes occurring through time. (This applies above all to the upper soil layers, exposed to meteorological effects and is also typical of the zone of fluctuations in the groundwater level and the capillary range connected with it.)
4. The standard deviation (and the coefficient of variation) increases generally with the *size of the area surveyed* and the *thickness* of the soil layer. In some cases the spatial variability is so great that doubt arises as to whether the measured properties apply to a single layer (these questions are dealt with in Section 7.3).
5. The statistical parameters assigned to the physical characteristics depend also on the *sampling and testing methods* (see Section 6.2).

In the following sections, data found in literature for various soil physical properties are surveyed (Sections 1.3.1–1.3.6), then an example of the statistical processing of archival materials of greater settlements is presented (Section 1.4).

1.3.1. Water content and saturation

FREDLUND and DAHLMAN (1971) evaluated the physical characteristics of the fat clay soil of the city of Edmonton, Canada, on the basis of the data from about 400 soil tests performed at various times. The statistical parameters related to *the water content* are

Table 1.2. Statistical parameters of the water content of the Edmonton clay (FREDLUND and DAHLMAN, 1971)

Mean depth $\bar{z}$ (m)	Number of samples n	$\bar{w}$ (%)	s	C_v	C_s	C_k
0.6	392	27.4	6.0	0.22	−0.01	−0.45
1.2	430	28.9	5.1	0.18	−0.22	−0.30
1.8	439	31.2	4.1	0.13	−0.41	0.69
2.4	422	33.0	4.1	0.12	−0.57	0.11
3.0	415	32.8	3.9	0.12	−0.27	0.37
3.6	392	34.7	3.5	0.10	−0.38	0.12
4.2	362	35.0	4.3	0.12	−0.73	0.15
4.8	307	35.5	4.3	0.12	−0.12	0.47
5.4	249	35.5	4.6	0.13	−0.76	0.76
6.0	177	34.8	4.5	0.13	−0.89	0.59

summed up in *Table 1.2*. For the 2 m-thick upper layer C_v varies from 0.13 to 0.22, for the subsequent layers the range is only 0.10–0.13 (see also the data given in Section 1.4). The coefficient of skewness varies between $-0.01 \dots -0.89$, the kurtosis between $-0.45 \dots +0.76$. For the degree of saturation the following values were obtained:

$\bar{z}(m)^*$	n	$\bar{S}_r$	s	C_v	C_s	C_k
1.55	111	0.92	0.121	0.13	-0.89	1.59
3.00	90	0.96	0.096	0.09	-1.71	4.21
4.30	38	0.98	0.084	0.09	-2.32	10.60
6.65	28	0.93	0.080	0.09	-1.30	1.84

* z = depth of samples.

The sampled layer was relatively homogeneous, as is concluded from the following facts:

- a) $\bar{w}$ is nearly constant with depth,
- b) the coefficient of variation of the liquid limit within each 1.5 m-thick layer is between 0.17 and 0.22.

SCHULTZE (1971) analysed the soil physical characteristics of the typical silt of the Rhine region and a clay originating from an open mining site there. In the case of the silt he notes that although the samples came from a relatively large area, the soil may be characterized as homogeneous; while for the clay the same statement may be made with regard to the small extension of the surveyed area. For the *water content* of the soils he obtained the following values:

for the silt:	$\bar{w} = 20.6\%$	$s = 4.7\%$	$C_v = 0.23,$
	$C_s = 0.25,$	$C_k = -1.00$	$(n = 406);$
for the clay:	$\bar{w} = 21.7\%$	$s = 2.0\%$	$C_v = 0.13,$
	$C_s = -0.21,$	$C_k = -0.04$	$(n = 29).$

(As explained in Section 1.2.4, the value of the coefficient of skewness cannot be regarded as realistic, due to the small number of the samples.)

The *degree of saturation* was evaluated by SCHULTZE only for the silt. He found the following values: $\bar{S}_r = 0.85$, $s = 0.162$, $C_v = 0.19$, $C_s = -1.09$ and $C_k = 0.78$ ($n = 334$). Unfortunately, he did not specify whether the samples were taken from below or above the groundwater level (the former seems likely at least for a part of the samples).

UNGÁR (1967) evaluated the physical characteristics of the subsoil layers of the Tarján residential quarter in the city of Szeged on the basis of 71 boreholes drilled there (he does not mention the number of the samples). His results may be summed up as

follows: 1. Humic clay from above -0.8 m depth: $\bar{w}=25\%$, $C_v=0.41$. The same from below -0.8 m: $\bar{w}=24\%$, $C_v=0.14$. 2. Infusion loess (with no indication of the groundwater level position): $\bar{w}=23\%$, $C_v=0.18$. 3. Lake clay lying below the groundwater level: $\bar{w}=25\%$, $C_v=0.31$.

HOLTZ and KRIZEK (1971) published the results of soil tests performed for various objects. Their results (e.g. $C_v=0.22-0.24$ for the water content) cannot be used for practical purposes, first because they neglected to distinguish the individual layers from one another (for one of these, e.g., the coefficient of variation of the plasticity index is $C_v=0.47!$), and secondly because a part of them is related to a rather long (linear) body. The same is even more applicable to the data of COROTIS *et al.* (1975) for various parts of Greece and the United States (so it is no wonder that they obtained values for the coefficient of variation of the plasticity index C_v of $0.40-0.45$, even after assigning the soils to three separate groups).

1.3.2. Consistency limits

One of the *identifying characteristics* of cohesive soils is the plasticity index (I_p), defined as the difference between the liquid limit (w_L) and the plastic limit (w_p). For example, according to Hungarian standards, cohesive soils are classified on the basis of their plasticity index as follows:

$I_p(\%)$	Name of the soil
0 – 5	Mo
5 – 10	silty Mo
10 – 15	silt
15 – 20	lean clay
20 – 30	medium clay
> 30	fat clay

This role gives a different interpretation to the standard deviation (and the coefficient of variation) of I_p , compared to the statistical parameters of the water content, or other physical characteristics. This may be shown as follows.

Let us assume that I_p lies between 15 and 20 in all parts of the layer under examination, i.e. it is a homogeneous lean clay. Let us denote the range of I_p by R and let us assume that in this range the distribution of I_p is uniform. Then according to Fig. 1.6—as the hatched part equals unity—the density function is:

$$f(x) = \text{Const} = \frac{1}{R}.$$

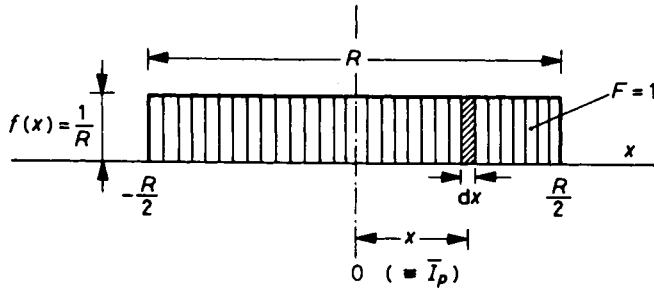


Fig. 1.6. Calculation of the standard deviation of the uniform distribution; R — range, F — integral of the density function $f(x)$

The variance of the random variable $x(=I_p)$ is

$$s^2 = \int_{-R/2}^{+R/2} x^2 \frac{1}{R} dx = \frac{R^2}{12} \quad (1.20)$$

and its standard deviation is

$$s = \frac{R}{\sqrt{12}} = \frac{R\sqrt{3}}{6}. \quad (1.21)$$

Returning to the lean clay mentioned in the above example, the coefficient of variation of the plasticity index of the layer is:

$$C_v = \frac{s}{\bar{x}} = \frac{(20-15)\sqrt{3}}{17.5 \cdot 6} = 0.08.$$

Figure 1.7 shows the relationship $C_v = f(R, \bar{x})$. Thus for the medium clay ($R = 10$, $\bar{x} = 25$) we see that $C_v = 0.12$.

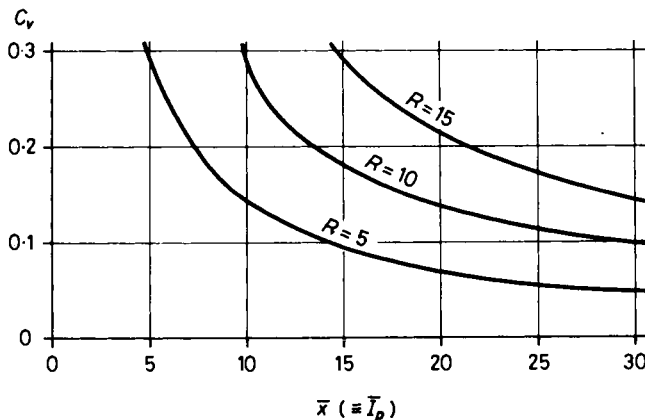


Fig. 1.7. Coefficient of variation of the uniform distribution versus the expected value ($\bar{x}$) and the range (R)

From this it follows that the coefficient of variation of I_p (and also, to a lesser extent, of w_L , and thus of w_p) is determined, to a certain extent, *ab ovo* by the classification system of the soils. So this statistical parameter tells us above all to what extent the soil can be regarded as a *single layer*; from which it follows that the variability of I_p also affects the variance of the other physical characteristics of the soil, especially those which are correlated with I_p (see, e.g., *Figs 5.8 and 5.10*).

After these preliminaries, we now consider examples of numerical values for the statistical parameters of the consistency limits.

Table 1.3. Statistical parameters of the liquid limit and the plastic limit of the Edmonton clay (FREDLUND and DAHLMAN, 1971)

	$\bar{z}(m)$	n	Mean	s	C_v	C_s	C_k
w_L	1.5	202	64.1	10.7	0.17	0.06	-0.47
	3.0	212	62.8	11.4	0.18	0.21	-0.54
	4.5	143	59.2	13.1	0.22	0.13	-0.85
	6.0	40	53.0	11.2	0.21	0.47	-0.86
w_p	1.5	200	25.6	3.4	0.13	0.10	-0.24
	3.0	212	25.4	3.2	0.13	0.31	-0.53
	4.5	144	25.1	3.8	0.15	-0.24	-0.11
	6.0	42	24.4	3.2	0.13	-0.52	-0.46

FREDLUND and DAHLMAN (1971) obtained for the liquid and the plastic limits of the soil mentioned in Section 1.3.1 the values given in *Table 1.3*. SCHULTZE (1971) calculated the following parameters for the Rhine clay he studied:

	n	Mean	s	C_v	C_s	C_k
w_L	75	55.1	12.2	0.22	-0.11	-0.30
w_p	75	21.7	3.4	0.16	-0.05	-0.02
I_p	75	33.9	9.6	0.28	-0.03	-0.20

THORNBURN and LARSEN (1959) examined four characteristic soils in the state of Illinois, USA (*Table 1.4*). The results must be accepted with reservation, since they used very few samples.

PAÁL (1974a, b) carried out a comprehensive statistical evaluation of the upper soil layers of the city of Buda. His results are contained in *Table 1.5*. In addition to the numerical values presented, the following facts deserve attention:

a) the altered (yellow coloured) layer of the Kiscell Clay is less homogeneous than the unaltered layer;

b) the overlying layers (the soils marked by C and E) are essentially more heterogeneous than the outcropping rock formations.

Table 1.4. Statistical parameters of four characteristic soils of the state of Illinois, (USA); *A* and *B* — various levels of the soils (THORNBURN and LARSEN, 1959)

Symbol	Level of the soil	Liquid limit			Plasticity index		
		Mean	<i>s</i>	<i>C_v</i>	Mean	<i>s</i>	<i>C_v</i>
1.	<i>A</i>	37.4	5.2	0.14	11.9	3.8	0.32
	<i>B</i>	40.6	6.4	0.16	16.2	5.5	0.34
2	<i>A</i>	41.3	4.7	0.11	13.9	3.6	0.26
	<i>B</i>	44.8	5.2	0.12	18.9	4.7	0.25
3.	<i>A</i>	54.6	3.8	0.07	26.1	3.5	0.13
	<i>B</i>	52.2	5.5	0.11	27.9	5.2	0.19
4.	<i>A</i>	42.9	4.1	0.10	16.6	2.7	0.16
	<i>B</i>	51.3	3.9	0.08	25.1	3.9	0.16

Table 1.5. Statistical parameters of the w_L , w_p and I_c values of the Buda soils; *A* — grey Kiscell Clay; *B* — yellow Kiscell Clay; *C* — layer found over the Kiscell Clay; *D* — Buda Marl; *E* — layer found over the marl (PAÁL, 1974a, b)

Soil*	Liquid limit			Plasticity index			Consistency index		
	Mean	<i>s</i>	<i>C_v</i>	Mean	<i>s</i>	<i>C_v</i>	Mean	<i>s</i>	<i>C_v</i>
<i>A</i>	54.7	7.2	0.13	29.1	6.2	0.21	1.29	0.16	0.12
<i>B</i>	58.3	8.6	0.15	32.5	7.2	0.22	1.17	0.12	0.10
<i>C</i>	52.1	11.2	0.21	28.8	8.4	0.29	1.14	0.15	0.13
<i>D</i>	58.6	10.5	0.18	32.3	8.1	0.25	1.25	0.17	0.14
<i>E</i>	53.0	14.6	0.28	28.6	11.5	0.40	1.12	0.15	0.13

* The numbers of the tested samples were: 444 (*A*), 476 (*B*), 113 (*C*), 229 (*D*), resp. 146 (*E*)

Table 1.5 shows also that the liquid limit of the Kiscell Clay exceeds that of the grey clay by 4%. PAÁL explains this by the fact that pyrite is present in the latter in its crystalline form, while in the former it occurs in the form of ferrous hydroxide, which readily forms a colloid.

UNGÁR (1967) obtained for the soils of Szeged, mentioned in Section 1.3.1, the following parameters:

Soil	w_L		w_p		I_p		I_c	
	$\bar{w}_L$	<i>C_v</i>	$\bar{w}_p$	<i>C_v</i>	$\bar{I}_p$	<i>C_v</i>	$\bar{I}_c$	<i>C_v</i>
Humous clay ($z < 0.8$)	43	0.11	18	0.08	25	0.23	0.69	0.26
Humous clay ($z \geq 0.8$)	47	0.16	19	0.06	28	0.26	0.79	0.19
Infusion loess	35	0.09	19	0.10	15	0.21	0.78	0.27
Lake clay	56	0.11	21	0.12	34	0.18	0.87	0.20

According to the examples found in literature (omitting the data of THORNBURN and LARSEN) the coefficients of variation of the measured physical characteristics have the following typical values: for w_L : $C_v=0.09-0.28$; for w_p : $0.06-0.16$; for I_p : $0.18-0.40$; for I_c : $0.10-0.27$.

1.3.3. Characteristics of the grain size distribution curve

Since the classification and identification of granular soils is based on the grain size distribution curve, the statements made in connection with the consistency limits apply also to the statistical parameters for granular soils. So the following additional published data are given only for the sake of completeness.

On the basis of a study of the grain size distribution in samples taken from upper soil layers of the Tarján residential quarter in the city of Szeged, UNGÁR (1967) determined a coefficient of variation of 0.18 for the dominant grain size (d_m) and $C_v=0.20$ for the uniformity coefficient (U).

SCHULTZE (1971) found $C_v=0.89$ for the value of U in the case of the Rhine Silt. THORNBURN and LARSEN (1959) calculated values between 0.05 and 0.19 for the coefficient of variation of the mass percent $S_d < 0.002$ mm.

1.3.4. Void ratio and bulk density

The statistical parameters given for the *void ratio* by the authors discussed in Sections 1.3.1–1.3.3 are summarized in *Table 1.6*. It may be seen from this table that the coefficient of variation of e is generally in the range 0.15–0.22. The deviations from this may be explained as follows:

Table 1.6. Literature data for the statistical parameters of the void ratio

Author	Name of soil	n	$\bar{e}$	s	C_v	C_s	C_k
FREDLUND and DAHLMAN (1971)	$\bar{z}=1.55$ m	120	0.90	0.157	0.18		
	$\bar{z}=3.00$ m	96	0.91	0.138	0.15		
	$\bar{z}=4.55$ m	47	0.91	0.186	0.20		
	$\bar{z}=6.65$ m	21	0.75	0.236	0.32		
SCHULTZE (1971)	Rhine Silt	327	0.63	0.135	0.22	0.24	–0.33
UNGÁR (1967)	humous clay ($z < 0.8$)		0.74		0.14		
	humous clay ($z > 0.8$)		0.64		0.09		
	infusion loess		0.63		0.09		
	lake clay		0.79		0.15		
PAÁL (1974a, b)	A	483	0.54	0.11	0.20		
	B	522	0.66	0.11	0.17		
	C	97	0.72	0.13	0.18		
	D	258	0.62	0.18	0.29		
	E	183	0.70	0.15	0.21		

a) The Canadian clay—as the authors have shown—is inhomogeneous in depth. In the frequency curve of the void ratio of samples taken from -6.0 m, three modes appear, which results from the fact that these samples do not belong to the same population. In addition, the fact that there are only 21 samples in this category must be considered.

b) The large variance of the void ratio of the Buda Marl (symbol *D*) is explained by PAÁL (1974a) by the fact that in some areas the marl is rocky ($e = 0.2$), while in others it is very fragmentary ($e = 1.0$), but the sampling procedure may also have had a role in producing this variation.

c) The low values obtained for the soils of Szeged may be explained partly by the fact that the subsoil of the town—with the exception of the upper layer—is homogeneous (see Section 1.4) and partly by the relatively small size of the area surveyed.

The density of the soil grains, which must be known for the determination of the void ratio and the degree of saturation, has a relatively low variability. According to SCHULTZE (1971, 1975a) its coefficient of variation, C_v , ranges from 0.01 to 0.02, which is reassuring, as in practice the tabulated values are nearly always used for calculations, and laboratory determinations (by pycnometer) are rather scarce.

According to SCHULTZE (1975a) the coefficient of variation of the bulk density of the soil is 0.04 for clay and about 0.09 for sand and gravel. PAÁL (1974a, b) obtained the following values for the soils listed in Table 1.5:

Symbol of the soil	A	B	C	D	E
Coefficient of variation	0.05	0.05	0.07	0.07	0.06

The coefficient of variation of the soils of Szeged studied by UNGÁR (1967) were $C_v = 0.04 - 0.06$ (the maximum was found in the top layer of the humous clay).

As the bulk density is a function of e and S_r and the latter is sensitive to variations in the proportion of colloidal grains (see Fig. 5.8)—primarily for genetic reasons—it is justified to assume that the expected value of γ is also a function of I_p . This hypothesis was fully supported by the statistical evaluation of about 3200 samples, taken from various areas of Hungary (RÉTHÁTI, 1968a). According to Fig. 1.8, the mean bulk density of the layers lying below groundwater level decreases monotonically with I_p (because the void ratio increases continuously according to Fig. 5.8), while the bulk density of the layers found above groundwater level varies with I_p according to a maximum curve (because the degree of saturation of the soils with low plasticity may be essentially less than unity).

The relationship shown in Fig. 1.8 is presented numerically in Table 1.7 including the values for the standard deviation, allowing us to establish the following.

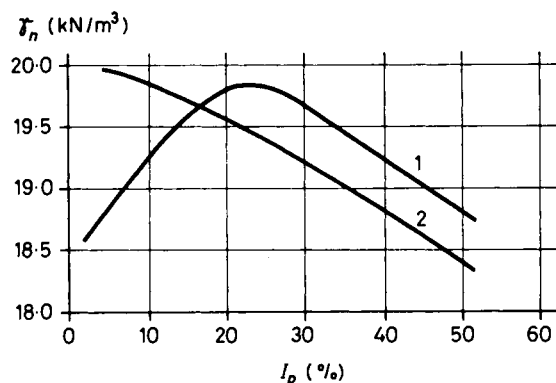


Fig. 1.8. Average bulk density (γ_n) of the Hungarian cohesive soils versus the plasticity index (I_p); 1 — above the groundwater level, 2 — below the groundwater level

Table 1.7. Statistical parameters of the bulk density of Hungarian inorganic cohesive soils (γ_n , kN/m^3)

$I_p(\%)$	Below groundwater level			Above groundwater level		
	Mean	s	C_v	Mean	s	C_v
0–5	19.92	0.67	0.033	18.73	1.52	0.081
5–10	19.89	0.76	0.038	19.05	1.24	0.065
10–15	19.75	0.77	0.039	19.46	1.18	0.061
15–20	19.66	0.90	0.046	19.67	1.11	0.056
20–25	19.50	1.14	0.058	19.82	1.13	0.057
25–30	19.54	0.88	0.045	19.73	1.00	0.051
30–35	19.10	1.10	0.058	19.59	1.02	0.052
35–40	18.92	1.04	0.055	19.31	0.96	0.050
>40	18.53	1.12	0.060	18.94	1.16	0.061

a) The mean value of C_v is 0.048 for layers below groundwater level and 0.059 for those found above groundwater level. The difference between these values results from the fact that both the void ratio and the degree of saturation are controlled by the water saturation.

b) The coefficients of variation of the bulk density values of layers below groundwater level increase continuously with I_p , because the variability of the void ratio is also increasing. The situation is just the opposite for those above groundwater level, due to the low dispersion of the degree of saturation of the soils with high plasticity, partly for genetic reasons and partly because S_r remains practically unchanged during the shrinking process (as long as $w > w_s$).

From the figure and the table we may also conclude that the coefficient of variation of γ_n is very small, if we regard samples from an area of 90,000 km^2 as a single population and take into account only the position of the groundwater level and the plasticity index of the soil.

1.3.5. Deformation characteristics of the soil

The literature contains very few data on the statistical parameters of the deformation characteristics of the soil and these, too, are mostly compression (oedometer) test data. FREDLUND and DAHLMAN (1971) calculated the coefficient of variation of the compression index (C_c) appearing in equation

$$e = e_0 - C_c \log \frac{p + p_0}{p_0} \quad (1.22)$$

of the compression curve for the Edmonton Clay, mentioned in Section 1.3.1. In addition, they evaluated the so-called recompression ratio defined by the expression

$$\frac{C_r}{1 + e_0} \quad (1.23)$$

(C_r is the bend of the hysteresis loop in the coordinate system $e, \log p$; e_0 is the initial void ratio). The results of their investigations are contained in Table 1.8.

Table 1.8. Statistical parameters of the compression index (C_c) and the recompression ratio (C_r) of the Edmonton Clay (FREDLUND and DAHLMAN, 1971)

	$\bar{z}(\text{m})$	n	Mean	s	C_v	C_s	C_k
C_c	1.55	108	0.184	0.047	0.26	0.75	1.62
	3.00	95	0.167	0.048	0.29	0.38	-0.15
	4.30	40	0.159	0.048	0.30	1.14	1.00
	6.65	20	0.110	0.052	0.47	0.90	-0.48
C_r	1.55	27	0.065	0.017	0.26	-0.16	-0.64
	3.00	39	0.065	0.020	0.31	0.35	0.01
	4.30	27	0.065	0.034	0.51	0.69	-0.63
	6.65	14	0.032	0.017	0.53	0.83	-0.57

LUMB (1966) obtained $C_v = 0.26$ for the compression index of the sandy clay characterizing the surroundings of Hong Kong.

SCHULTZE (1971) characterizes the compression modulus (E , MN/m²) of the previously mentioned Rhine Silt, based on the tests of 785 samples, by the following statistical parameters:

$\bar{E}_s$	s	C_v	C_s
9.57	3.55	0.37	0.02

The statistical parameters of the compression modulus of the Buda Clay are, according to PAÁL (1974a, b), as follows:

	A	B	C	D	E
n (pieces)	137	211	22	55	57
$\bar{E}_s$ (MN/m ²)	14.3	12.5	9.1	15.8	12.0
C_v	0.26	0.32	0.25	0.49	0.40

(In this connection we must note that if the relationship E_s, z is taken into account, dispersion decreases.)

In a later study SCHULTZE (1975a) compared the compression modulus determined in the laboratory with the values obtained with *in situ* investigations. His experiments with a silt gave the following results:

Mode of determination	n	$\bar{E}_s$	s	C_v
Oedometer test	48	7.5	1.74	0.23
Standard penetration test	84	7.0	2.07	0.29
Static probing	65	5.8	1.48	0.25

In another experimental series he compared the compression modulus obtained by surface loading tests with the laboratory value. In the former case he had $\bar{E}_s = 7.7$ MN/m², $C_v = 0.21$; in the latter $\bar{E}_s = 7.7$ MN/m², $C_v = 0.40$ (see Section 6.2.4).

His experiments, performed with granular soil, for characterizing the reliability of the *in situ* tests and for comparing the various types of tests gave the following results:

Type of tests	Measured quantity	n	C_v
Standard penetration test	number of blows	59	0.27
Static penetration test	peak resistance	348	0.30
Heavy hitting test	number of blows	1207	0.50

Although the available data for the deformation characteristics of the soil are insufficient, it may be established that their *variability is rather great*.

1.3.6. Shear and compression strength

From among the physical characteristics of the soil it is the shear strength for which the most data can be found in the literature, because numerous geotechnical tasks can be solved using this property.

SCHULTZE (1975a) published data for the angle of friction, specifically its tangent for granular soils. According to his studies the coefficient of variation of $\tan \phi$ is 0.05 – 0.14

Table 1.9. Statistical parameters of the $\tan \varphi$ value of granular soils according to the data of SCHULTZE (1975a)

Soil type	U	D^*	n	$\tan \varphi$	s	C_v
Gravelled sand	> 3	$0.3-0.5$	6	0.794	0.042	0.05
	> 3	> 0.5	10	0.920	0.049	0.05
Sand (without fine sand fractions)	< 3	≤ 0.3	43	0.650	0.056	0.09
	< 3	$0.3-0.5$	28	0.716	0.080	0.11
	< 3	> 0.5	76	0.826	0.119	0.14
Sand (with fine sand fractions)	< 3	≤ 0.3	66	0.652	0.053	0.08
	< 3	$0.3-0.5$	42	0.715	0.075	0.10
	< 3	> 0.5	111	0.826	0.109	0.13
Fine sand	< 3	< 0.3	23	0.657	0.046	0.07
	< 3	$0.3-0.5$	14	0.712	0.055	0.08
	< 3	> 0.5	35	0.826	0.084	0.10

* Degree of density (interpreted as by DIN).

(Table 1.9). The mean values and the coefficients of variations of the angle of friction in gravelly sand ($n=29$) obtained by direct shear, resp. calculated by the formula based on the initial void ratio (e_a)

$$\cot \varphi = 3.36e_a + 0.005, \quad (1.24)$$

resp. 0.10 were nearly identical (37.8° and 37.7° , resp. 0.11 and 0.10). In an earlier study (1971) using also gravelly sand ($n=97$) he found a coefficient of variation $C_v=0.05$ for φ and 0.07 for $\tan \varphi$ (the coefficient of skewness was 0.02).

LUMB (1966) found a coefficient of variation $C_v=0.14$ for $\tan \varphi$ of a silty sand. HARR (1977) published—with reference to oral information from prof. HOLTZ—the following values for the angle of friction of granular soils:

Soil	n	$\bar{\varphi}$	s	C_v
Gravel	38	36.2	2.16	0.06
Sand	73	38.8	2.80	0.07
Sand	136	36.4	4.05	0.11
Sand	30	40.5	4.56	0.11

The coefficient of variation of φ for *cohesive* and *semi-cohesive* soils is somewhat higher than that for granular soils. SCHULTZE (1975a) found $C_v=0.10$ for a silt and $C_v=0.15$ for a clay on the basis of relatively few samples ($n=19$, resp. 6). For the coefficient of variation of the angles of friction in clay calculated by the formula

$$\varphi = 9.3 - 16.07 \log I_p \quad (1.25)$$

(substituting a relative number for I_p) he obtained $C_v=0.11$. (From the decrease of the standard deviation as compared to the direct experiments no far-reaching conclusions can be drawn, as the author used here 79 samples instead of 6.)

LUMB (1970) obtained the following statistical parameters for two cohesive soils examined by him:

Symbol of soil	$\overline{\tan \varphi}$	s	C_v
2.	0.54	0.035	0.06
3.	0.62	0.054	0.09

He also established by his experiments that the dam prepared from the studied soils proved to be more homogeneous from the viewpoint of the angle of friction than the natural soil (the variability of the cohesion was identical).

WEBER and GEHRISCH (1980) obtained a somewhat higher coefficient of variation than those mentioned before, investigating a cohesive soil forming a slope ($\tan \varphi = 0.54$, $C_v = 0.19$).

As to the cohesion, two cases must be mutually distinguished, depending on whether $\varphi \neq 0$, or $\varphi = 0$.

LUMB (1970) and WEBER and GEHRISCH (1980) studied cases of $\varphi \neq 0$. Their results may be summed up as follows:

Authors	Soil No.	$\bar{c}$ (kN/m ²)	s	C_v
LUMB (1970)	1.	4.58	3.13	0.68
	2.	6.17	1.63	0.26
	3.	8.55	3.72	0.44
WEBER and GEHRISCH (1980)	—	32.06	16.64	0.52

The data of LADD *et al.* (1971) refer to the case of $\varphi = 0$. (The mean water content of the Bangkok Clay studied by them was 70%, so the assumption of $\varphi = 0$ is indeed realistic.) They have determined the cohesion by three methods: the vane test gave values between $C_v = 0.25$ and 0.35, the undrained triaxial, resp. the unconfined compression test gave values between $C_v = 0.27$ and 0.41. SCHULTZE (1975a) also determined by three different methods the cohesion of the "Fischbach" Clay: by undrained triaxial test, by unconfined compression test ($c = q_u/2$) and from an empirical relationship $c = f(w)$. His results were as follows:

Test	n	$\bar{c}$	s	C_v
Triaxial	80	400	104	0.26
Unconfined compression	127	417	168	0.40
$c = f(w)$	125	390	82	0.21

(An interesting point of the results is that the smallest dispersion was obtained for the empirical relationship, in spite of the fact that the correlation coefficient characterizing it was merely $r = 0.50$.)

Table 1.10. Literature data for the statistical parameters of the unconfined compression strength (q_u , kN/m²)

Author	Type of soil	n	$\bar{q}_u$	C_v	C_s	C_k
FREDLUND and DAHLMAN (1971)	clay, $\bar{z} =$ 1.5 m	279	224	0.49	1.24	1.46
	3.0 m	295	181	0.41	1.40	2.72
	4.5 m	187	160	0.40	0.69	0.14
	6.0 m	53	140	0.62	1.21	2.90
SCHULTZE (1971)	Rhine Clay	25	374	0.53	-0.03	-2.96
		36	575	0.52	-0.21	-2.93
		123	839	0.39	-0.06	-2.90
MORSE (1971)	glacial clay	1.	22	412	0.15	
		2.	21	308	0.15	
		3.	18	481	0.31	
		4.	25	231	0.20	
MATSUO-KURODA (HARR 1977)	clay	231	104	0.29		

The low variability of the *unconfined compression strength* (q_u) was supported also by the previously described tests (SCHULTZE, 1975a; LADD *et al.*, 1971). An additional proof is given by the values shown in Table 1.10. KRAHN and FREDLUND (1983) testing a relatively small area obtained $C_v = 0.51 - 0.55$.

The *cube strength of concrete* corresponds to the q_u of soils. According to the 1965 specifications of the *American Concrete Institute*, the concrete is to be evaluated depending on the coefficient of variation of the cube strength in the following way:

$$\begin{aligned}
 C_v < 0.10 & : \text{excellent} \\
 C_v = 0.10 - 0.15 & : \text{good} \\
 C_v = 0.15 - 0.20 & : \text{satisfactory} \\
 C_v > 0.20 & : \text{bad}
 \end{aligned}$$

The specifications of Cuba (TORRI, 1975) contain one further distinction, namely: if $C_v > 0.25$, the quality of the concrete is very inferior. (According to the Hungarian standards $C_v = 0.15$ must be taken into account in the design.) These numerical values show that the variability of the concrete as an artificially homogenized material is hardly lower than that of a soil layer, though its strength may be defined better due to the brittle fracture.

The shear strength data of the soil may be summed up as follows:

1. For the angle of friction (resp. its tangent) of granular soils C_v is 0.05 - 0.14. The literature data permit to conclude that the coefficient of variation increases with the decreasing grain size, and for the compact state it is higher than for the loose state.
2. The variability of ϕ of cohesive soils ($C_v = 0.06 - 0.19$) exceeds somewhat that of granular soils.
3. The variability of the cohesion is very high ($C_v = 0.25 - 0.68$, the mean value is 0.37), which serves as a warning, pointing to the decisive role of variance of the cohesion in dimensioning.

4. The coefficient of variation of the unconfined compression strength varies between 0.15 – 0.62; the mean value is identical with that of the C_v values given for the cohesion (0.37).

1.4. Comprehensive statistical evaluation of large areas

On examining great residential and industrial settlements, every possibility is given for evaluating the soil layers by statistical methods. Useful additional information is also supplied for this work by *earlier* soil tests in the given area (or in its vicinity). Their importance is especially high where extensive and relatively homogeneous layers are involved which can be well delimited geologically (“London Clay”, “Buda Marl”, etc.).

The comprehensive statistical study of the subsoil of a Hungarian town *Szeged*, was based on similar considerations (RÉTHÁTI and UNGÁR, 1978a, b, c). In selecting the site the following considerations played a role:

- a very great number of laboratory tests (about 11,000 tests from 2600 samples);
- the random distribution of the tests in time (boreholes for 155 expert opinions were drilled in various months of 25 calendar years);
- the physical characteristics did not change with the depth within a layer (because the layers were not thick) and their spatial variability was also insignificant;
- the possibility of comparing parts of the same layer lying below and above the groundwater level was given.

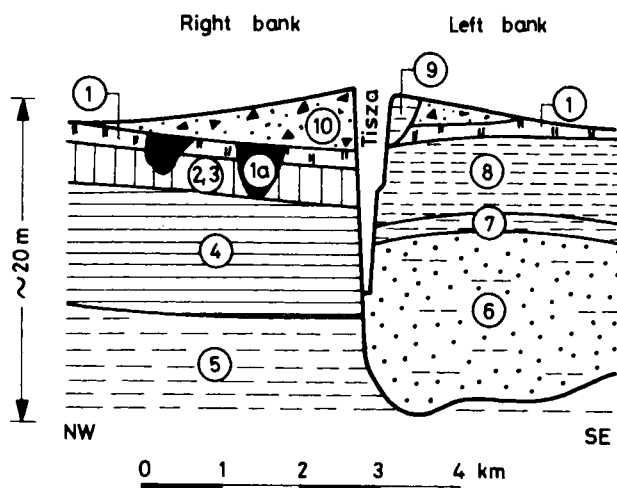


Fig. 1.9. Geological section of the town Szeged. Pleistocene: infusio loess above (2) and below (3) the groundwater level, lake clay (4), limnogenic sediment (5). Holocene: covering layer (1) and its organic variant (1a), sand with silt lens (6), silt (7), clay (8), silt and sand layers (9), fill (10)

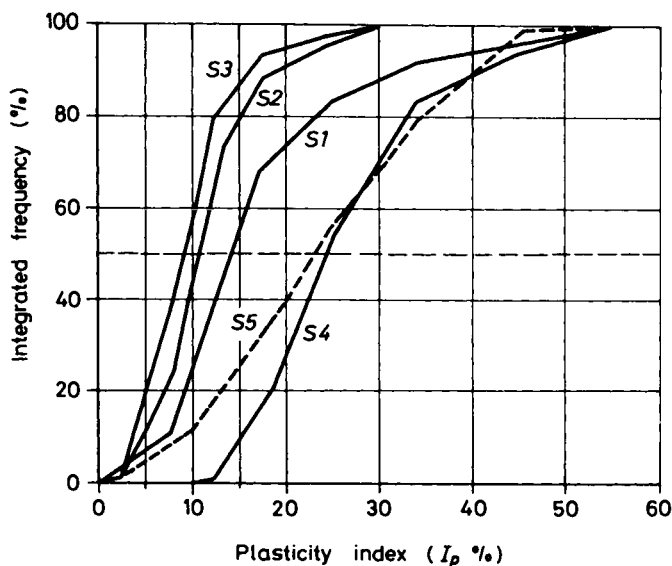


Fig. 1.10. Distribution functions of I_p for the soils S1–S5 of Szeged

The river Tisza crossing the town separates very different geological formations from each other: the subsoil of the town on the right bank of the river is formed mostly by *Pleistocene* layers, while on the left bank to a depth of about 20 m, by *Holocene* layers (Fig. 1.9).

The part of the town lying on the *right bank* was developed originally on a separate island-type area, surrounded by marshes and water currents. After the flood catastrophe, which devastated the town in 1879, about 1.5 million m³ earth was refilled (to a thickness of 2–3 m in the town centre), under which the original soil, the humous (or organic) silt and clay (in the following S1) can be found even today. The first layer of a satisfactory load-bearing capacity, the 2–2.5 m thick infusion loess accumulated in an area of high groundwater, or lying temporarily under water, so it is not macroporous (S2 for samples taken from above groundwater level, S3 for those taken from below groundwater level). Its loess character is proved also by the CaCO₃ content (30% approximately). Its subjacent layer is a yellow lake-clay (S4), whose CaCO₃ content is merely about 5%. Under it the 20 m-deep boreholes revealed sweet-water sediment layers consisting of bluish-grey clay and silt (S5). Figure 1.10 shows the distribution of the plasticity index for each layer.

On the left bank Holocene sediment layers of the rivers Tisza and Maros are found. These layers were not included in the statistical processing.

The distribution of the laboratory tests according to soil types and physical characteristics is shown in Table 1.11. Column 2 of the table indicates in more detail if

Table 1.11. Distribution of the laboratory tests of soils at Szeged by sample types and physical characteristics

Symbol of soil	Sample type	Characteristic tested			
		w	I_p, I_c	e, S_r, γ_n	q_u
S1	z	78	64		
	K	56	37	54	19
	N	24	8	25	21
S2	z	50	47		
	K	85	44	85	29
	N	9	7	9	6
S3	z	371	340		
	K	531	281	538	227
	N	46	27	46	35
S4	z	375	356		
	K	512	197	513	299
	N	129	70	129	108
S5	z	189	186		
	K	52	9	52	32
	N	108	57	108	85

Table 1.12. Statistical parameters of the physical characteristics of the soils of Szeged (RÉTHÁTI and UNGÁR, 1978c)

Parameter	Symbol of soil	Soil physical characteristics								
		w* (%)	w _L (%)	w _p (%)	I_p (%)	I_c^*	e	S_r	γ_n (kN/m ³)	q_u (kN/m ²)
M(x)	S1	31.1	44.5	24.2	20.6	0.62	0.866	0.917	18.8	144
	S2	22.3	32.3	19.4	13.0	0.82	0.674	0.871	19.7	125
	S3	24.5	32.2	20.7	11.5	0.63	0.697	0.902	19.7	173
	S4	28.3	54.0	24.2	29.9	0.86	0.821	0.901	19.3	219
	S5	28.6	52.8	25.4	27.4	0.82	0.823	0.905	19.2	184
s	S1	9.5	17.2	8.3	11.8	0.31	0.225	0.090	1.4	64
	S2	4.1	4.3	2.4	4.8	0.30	0.087	0.102	0.7	47
	S3	3.7	4.3	2.2	4.1	0.29	0.092	0.102	0.8	102
	S4	4.9	11.1	3.4	10.3	0.17	0.093	0.088	0.6	117
	S5	4.8	14.3	3.8	12.1	0.26	0.099	0.090	0.6	100
C _v	S1	0.30	0.39	0.34	0.57	0.49	0.26	0.10	0.07	0.45
	S2	0.18	0.13	0.12	0.37	0.37	0.13	0.12	0.04	0.38
	S3	0.15	0.13	0.11	0.36	0.47	0.13	0.11	0.04	0.59
	S4	0.17	0.21	0.14	0.34	0.20	0.11	0.10	0.03	0.53
	S5	0.17	0.27	0.15	0.44	0.32	0.12	0.10	0.03	0.54
C _s	S1	1.70	2.03	2.61	1.39		2.55	-2.81	-1.13	0.14
	S2	0.48	0.86	-0.58	0.70		-0.12	-0.93	-0.30	0.97
	S3	-0.04	0.98	0.37	0.92	-0.39	1.14	-1.78	-1.70	2.25
	S4	0.78	0.58	0.36	0.80	-0.51	0.36	-1.79	-1.03	2.19
	S5	0.77	0.13	1.71	0	-1.54	1.14	-1.70	-0.32	1.65

* From "partially disturbed" samples.

the tests were made on partly disturbed samples placed in metal boxes (z), or on core samples taken from small diameter (K) or large diameter (N) boreholes.

The statistical parameters of the nine physical characteristics examined in detail are shown in *Table 1.12*. The set of data reflects the results of the first, rough processing, inasmuch as (a) the samples taken from the small- and the large-diameter boreholes are not separated and (b) the outlying values are not screened out (we return to these questions in Sections 6.2 and 7.1).

The data of *Table 1.12* permit to establish the following:

1. The coefficients of variation obtained for the *covering layer* are generally 1.5–2 times as high as those found in other soils.

2. The *coefficient of skewness* deviates essentially from zero for each physical characteristic (the values of the heterogeneous covering layer exceed 2–3-fold those of the other layers also in this case). The *degree of saturation* is highly asymmetric because the mean value is close to the physically possible upper limit ($S_r = 1$); thus, the maximum of C_s for the covering layer ($\bar{S}_r = 0.92$) and its minimum for the layer S2 ($\bar{S}_r = 0.87$) is not accidental. The unfavourable statistical parameters of the *unconfined compression strength* are due to the structural and the state changes during the sampling, the transport and the storage (see Section 7.1), as well as to the still numerous uncertainties in the test methods.

3. On the basis of comparison with literature data, the following statements may be made. The variance of the consistency limits and the plasticity index is—with the exception of the covering layer—close to the upper limit of the intervals described in Section 1.3.2. This explains that the same applies also to the coefficient of variation of the other physical characteristics. The relatively low values obtained for the void ratio

Table 1.13. Statistical parameters of the water content of the soils of Szeged by sample types (RÉTHÁTI and UNGÁR, 1978a)

Symbol of soil	Type of sample	$\bar{w}$	s	C_v	C_s	C_k
S1	w_z	31.11	9.45	0.30	1.70	4.98
	w_m	29.21	7.67	0.26	1.58	4.58
S2	w_z	22.27	4.06	0.18	0.48	0.64
	w_m	21.74	3.74	0.17	−0.24	−0.50
S3	w_z	24.50	3.74	0.15	−0.04	3.60
	w_m	23.40	3.85	0.16	0.13	3.13
S4	w_z	28.27	4.86	0.17	0.78	5.65
	w_{mK}	26.69	4.13	0.15	−0.41	0.82
	w_{mN}	27.75	3.22	0.12	−0.33	0.92
S5	w_z	28.63	4.82	0.17	0.77	1.28
	w_{mK}	26.44	4.37	0.16	−0.56	−0.11
	w_{mN}	27.63	4.71	0.17	0.20	0.52

may be explained by permanent saturation of the capillaries of layers S2–S5, or by their being below groundwater level respectively.

Interesting conclusions may be drawn from the statistical parameters of the *water content* determined by types of samples (*Table 1.13*):

- the variants S2 and S3 of the infusion loess show only a difference of $\Delta w = 1.7\%$, which refers to a considerable capillary saturation;
- the mean and the standard deviation of the partly disturbed samples is greater than that of the core samples, which fact refers to the water absorption of the sample while lifting it through the water column (mainly in the case of soils of a low plasticity);
- there is hardly any difference between the water content of the partly disturbed and the undisturbed samples of the soil S2 ($\Delta w = 0.53\%$), while in the variant lying below the groundwater level the difference is twofold ($\Delta w = 1.10\%$), confirming the water absorption stated in the preceding paragraph.

The evaluation of the other statistical properties of the soils may be found in Section 2.3.

2. Description of the soil physical characteristics by distribution functions

The statistical parameters discussed in Chapter 1 cannot supply a complete image of the statistical structure of the data series. The role and the significance of these characteristics is similar to that of the maximum, the ruling and the effective grain sizes, the uniformity coefficient of the grain-size composition of the soils. Though these are also valuable information carriers, they are nothing, but parameters of the *grain-size curve*. This is the reason, why e.g., for calculating the coefficient of permeability the method of ZAMARIN, taking the whole distribution into account, is usually preferred to the empirical relationships determined for individual points of the curve (d_m, d_e).

In mathematical statistics the *density*, resp. the *distribution functions* correspond by analogy to the grain-size curve, permitting to determine not alone the physical characteristics, but also the loads and the effects calculated from them (earth pressure, load-bearing capacity of the soil, etc.). So they form one of the bases for determining the realistic factor of safety and in this way of modern dimensioning, in addition to the determination of the required number of samples (tests) and the reliability of the calculated mean values of the physical characteristics.

2.1. Distribution types used in geotechnics

The number of the distribution types used for solving mathematical statistical tasks is rather great. In the course of the gradual spreading of the probability theory it was proved that relatively few types must be known, resp. used for considering a given problem, as the possibility of the choice is a priori delimited by the nature of the studied phenomenon. This explains also the fact that every professional discipline has its own statistical methodology, including also the types of the distribution functions.

These statements apply also to geotechnics (e.g., the problems arising here seldom require the use of most of the discrete, resp. extreme distributions), but naturally this does not exclude the possibility of extending the present methodical system now or in the future.

In connection with the function types discussed in this Section the following are to be noted:

- from among the problems implied by the groundwater only those connected with the effective geotechnical tasks—taken in a narrow sense—are discussed;
- the Appendix gives the tabulated values of those functions which cannot or can hardly be determined by the simplest computers;
- this book deals only with the fundamental knowledge material; further details may be found in the quoted literature and in mathematical textbooks (e.g., FELLER, 1957; BURY, 1975; RAIFFA and SCHLAIFER, 1961, etc.).

2.1.1. Basic conceptions

The distribution functions may be classified according to several viewpoints, from which the following three are of particular importance.

1. If the possible values of ξ form a finite, or an infinite series $x_1, x_2, \dots, x_n$, we speak of a *discrete* variable, resp. distribution, while the distribution is *continuous*, if the function $f(x)$ exists (see Section 1.1). We may assign to the discrete distribution a stepped density function (*histogram*), but in cases where n is sufficiently great, the distribution may be described also by a continuous function fitted to it (see Fig. 2.3). The contrary case occurs also frequently, namely when we construct a histogram for a variable of continuous distribution, possibly with the purpose of describing the continuous variable by a discrete density function (see Figs 2.3 and 2.5).

2. For certain distribution types the random variable has a lower and/or an upper *limit*, which follows in the given case from the definition of the studied characteristic, or from the physical background of the phenomenon.

3. Some distribution functions are *symmetric*, others are not.

A considerable part of the continuous function types belongs to the so-called PEARSON family. The common property of their density functions is that they satisfy the following differential equation

$$\frac{y'}{y} = \frac{D + Ex}{A + Bx + Cx^2}, \quad (2.1)$$

where the capital letters are real constants.

PEARSON (1930; ELDERTON and JOHNSON, 1969) distinguishes seven types of curves (I–VII), depending on the value of the so-called criterion, defined by the expression

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)}, \quad (2.2)$$

where β_1 characterizing the skewness and β_2 characterizing the peakiness may be calculated from the characteristics given by expressions (1.15) and (1.19) in the following way:

$$\beta_1 = C_s^2 \quad \text{and} \quad \beta_2 = C_k + 3. \quad (2.3)$$

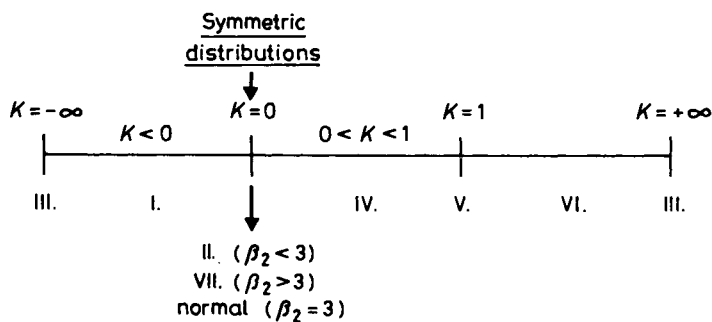


Fig. 2.1. PEARSON-system of the distribution functions according to the criterion "K"

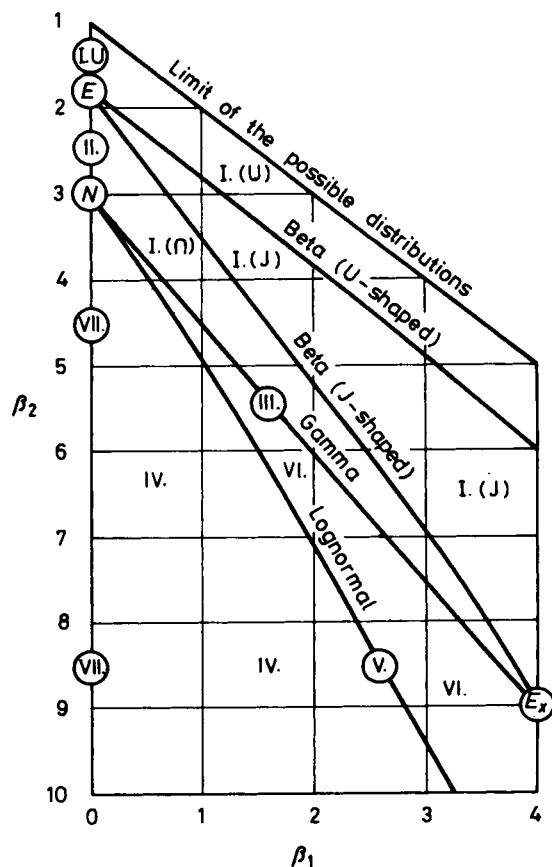


Fig. 2.2. Function types belonging to the PEARSON-family in the system of β_1, β_2 coordinates; E — uniform distribution ($\beta_1 = 0, \beta_2 = 1.8$), N — normal distribution, Ex — exponential distribution

Figure 2.1 shows the classification of the function types depending on the K value.

Type I is the so-called *beta* distribution (see Section 2.1.5), with fixed a and b limits. Type IV is unlimited in both directions and is rarely used for dimensioning constructions. Between these we find the case $K=0$, containing a great part of the symmetric distributions. We may assign hereto the *normal* distribution (Section 2.1.3), type II ($\beta_2 < 3$) and type VII ($\beta_2 > 3$); to the last category we may assign also the *STUDENT*-distribution discussed in Section 2.1.3. A representative of type V is the *lognormal* distribution (Section 2.1.3). Type VI is rarely used, but even more so type III, generalizing the *gamma*-function (see Section 2.1.4), to which the χ^2 - and the exponential distributions also belong (see Sections 2.1.3 and 2.1.4).

Every *PEARSON*-distribution is determined completely by the first four moments; in special cases even less, e.g., for the normal distribution two moments are sufficient (PRÉKOPA, 1962).

The values β_1 and β_2 defined by expression (2.3) determine a point in the coordinate system of Fig. 2.2. The position of this point gives valuable information concerning the type of the function by which we may try to describe the distribution of the studied characteristic (see Section 2.3).

2.1.2. The binomial and the POISSON-distributions

If an experiment can have only two outputs and the probability of the occurrence of one of them is p and of the other one is $q = 1 - p$, then by denoting the number n of the experiments in which the first output occurs by ξ , its probability is given by the so-called *binomial distribution*, whose form is

$$P_k = P(\xi = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \dots, n. \quad (2.4)$$

The possibilities P_k increase for a while with k , then they decrease; the greatest probability belongs to the index

$$k = [(n+1)p]. \quad (2.5)$$

The expected value and the variance are

$$M(\xi) = np \quad \text{and} \quad D^2(\xi) = npq \leq \frac{n}{4}. \quad (2.6)$$

Its practical use is illustrated by the following two examples:

1. Let us assume that the foundations are damaged with a probability of $p = 0.01$; if 5 foundations out of 100 were damaged, what can be said about the work of the contractor (HARR, 1977).

As $n = 100$, $p = 0.01$, $q = 1 - 0.01 = 0.99$ and $k = 5$, the probability that five foundations are damaged is according to formula (2.4)

$$P_k = \binom{100}{5} 0.01^5 \cdot 0.99^{95} = 2.9 \cdot 10^{-3}.$$

So we may state that the contractor "realized" an event, whose probability was merely 0.003.

2. Let us examine the probability that a high groundwater level of 10 years frequency does not occur during 10 years, then let us examine the same for 20 years and for 100 years.

In the first case $n = 10$, $p = 0.1$ and $k = 0$, so

$$P(k=0) = \binom{10}{0} 0.1^0 \cdot 0.9^{10} = 0.350.$$

As for $n = 20$ P is 0.360 and for $n = 100$ P is 0.367, it can be seen that P_k depends hardly at all on n , and on the other hand its value is so high that the maximum groundwater level defined in this way cannot be accepted as the basis of the design (RIGGS, 1961).

Further possibilities of the geotechnical utilization of the binomial distribution are discussed in Sections 3.2, 8.3, 10.1 and 12.4.4.

If n increases beyond all limits, while P tends to zero, so that the product $np = \lambda > 0$ remains constant, the probability belonging to the fixed k is

$$P_k = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \dots, n. \quad (2.7)$$

This limit value of the binomial distribution is called POISSON-distribution. It assumes its maximum value in the case of $k = [\lambda]$; its expected value and its variance agree and are

$$M(\xi) = D^2(\xi) = \lambda. \quad (2.8)$$

This function type can be used well for describing so-called *random processes of events*; its truncated form (the ERLANG-distribution) is used in the telephone technique (PRÉKOPA, 1962).

Expression (2.7) is suited also for describing the *soil physical characteristics*, if they are interpreted as discrete variables. The course of the calculation is illustrated by the following example.

The mean value of the water content of soil S4 mentioned in Section 1.4 is $\bar{w} = 28.3\%$, its variance is $s^2 = 24.0\%$ (see Table 1.12). As these values are nearly identical, a good fit of the POISSON-distribution may be expected to the data series. Omitting a few outlying results let us assign the remaining 372 values to 20 intervals ($\Delta w = 1\%$) in a way that the values between $w = 20.55$ and 21.55 should correspond to $k = 0$ (Table 2.1).

The parameter λ of the POISSON-distribution may be calculated from the expected value as

$$\lambda = \frac{\sum_{k=0}^{19} n_k \cdot k}{\sum_{k=0}^{19} n_k} = 7.26,$$

where n_k is the frequency of the water content falling into the individual categories. (As $\bar{w} = 21.05\%$ belongs to $k = 0$, the water content corresponding to $\lambda = 7.26$ is $w = 28.31 \cong 28.3$.) So the function describing the distribution is

$$P_k = \frac{7.26^k}{k!} e^{-7.26}.$$

Table 2.1. Description of the water content of the soil S4 by Poisson-distribution

Category limits	k	n_k	Effective	Calculated
			probability	
20.55 – 21.55	0	5	0.013	0.001
21.55 – 22.55	1	7	0.019	0.005
22.55 – 23.55	2	17	0.046	0.019
23.55 – 24.55	3	28	0.075	0.045
24.55 – 25.55	4	22	0.059	0.081
25.55 – 26.55	5	33	0.089	0.118
26.55 – 27.55	6	55	0.148	0.143
27.55 – 28.55	7	44	0.118	0.148
28.55 – 29.55	8	42	0.113	0.135
29.55 – 30.55	9	41	0.110	0.109
30.55 – 31.55	10	22	0.059	0.080
31.55 – 32.55	11	12	0.032	0.052
32.55 – 33.55	12	12	0.032	0.031
33.55 – 34.55	13	11	0.030	0.018
34.55 – 35.55	14	5	0.013	0.009
35.55 – 36.55	15	4	0.011	0.004
36.55 – 37.55	16	3	0.008	0.002
37.55 – 38.55	17	2	0.005	0.001
38.55 – 39.55	18	4	0.011	0
39.55 – 40.55	19	3	0.008	0

The probabilities P_k calculated for the various categories ($k=0, 1, \dots, 19$) and those determined from the actual values are given in columns 5 and 4 of Table 2.1. The agreement is very good, and can be verified also graphically (Fig. 2.3).

If the product $\lambda=np$ is great, then both the binomial and the Poisson-distributions can be well approximated by the normal distribution.

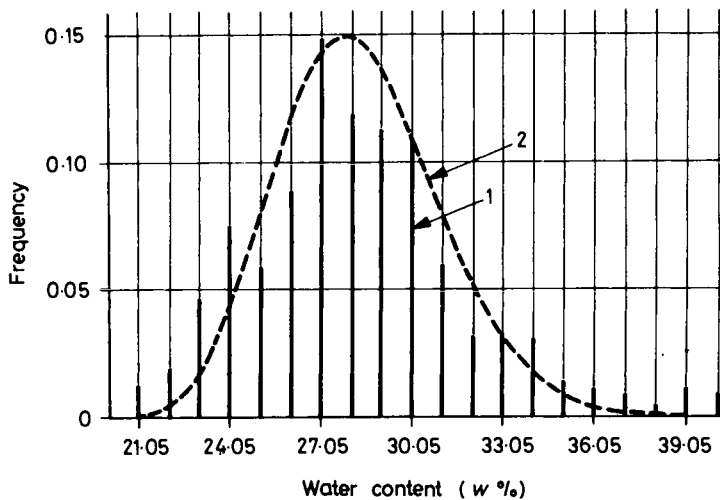


Fig. 2.3. Empirical distribution of the water content of soil S4 (1) and the Poisson-distribution fitted to it (2)

2.1.3. Normal-, lognormal-, χ^2 - and STUDENT-distributions

One of the most frequent and most important distributions in probability theory and in technical calculations is the so-called normal distribution, which may be described by the following density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}. \quad (2.9)$$

The distribution function is obtained by the integration of the same:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx. \quad (2.10)$$

This function may be written also in the following form:

$$F(x) = \frac{1}{2} + \operatorname{erf} \frac{x-m}{\sigma}, \quad (2.11)$$

where erf is the symbol of “error function”, which may be written in its standardized variation as

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{u^2}{2}\right) du. \quad (2.12)$$

Its frequent use is justified by the theorems of the *central limit distribution*. According to one of the latter the density function of the series

$$\eta_n = \xi_1 + \xi_2 + \dots + \xi_n, \quad (2.13)$$

formed from the independent random variables $\xi_1, \xi_2, \dots, \xi_n$ originating from an identical distribution approximates ever more with the increasing number of the members the density function defined by expression (2.9).

In practice the condition of the identical distribution is usually neglected. This explains the fact that the normal distribution fits generally well with the empirical distribution of the characteristics determined by repeated measurements in every case where the random variable is created by the *summation* of numerous slight random effects.

The *events (results)* $x_1, x_2, \dots, x_n$ may be regarded as individual realizations of the independent variables $\xi_1, \xi_2, \dots, \xi_n$ of an identical distribution and of an identical expected value x_0 .

The best approximation is the x_0 -value, for which the function

$$f(x_1 - x_0)f(x_2 - x_0) \dots f(x_n - x_0) \quad (2.14)$$

is maximum. At the same time the best estimation that can be obtained from measurements is the empirical mean defined by the expression

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}. \quad (2.15)$$

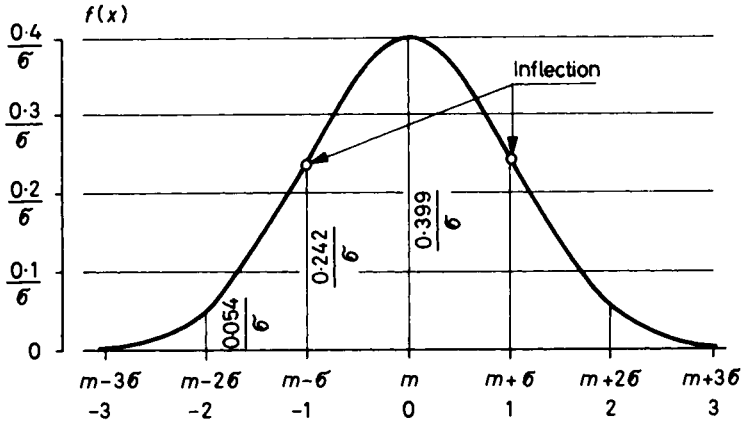


Fig. 2.4. Density function of the normal distribution

It can be proved (RÉNYI, 1954) that these types of definitions agree with each other exclusively in the case of the normal distribution.

The density function of the normal distribution is unimodal and symmetric around the point m , so m is simultaneously the expected value, the median and the mode of the distribution (Fig. 2.4). It may be proved by partial integration that σ is nothing else, but the standard deviation of the distribution.

Instead of the distribution function $F(x)$ interpreted as the integral of expression (2.9), the so-called *standard normal distribution function* is used in practice, utilizing the circumstance that the probabilities of the intervals, which are symmetric to point m and whose length is proportional with σ are independent of m and σ :

$$P(m - \lambda\sigma \leq \xi < m + \lambda\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{+\lambda} e^{-\frac{u^2}{2}} du = 2\Phi(\lambda) - 1. \quad (2.16)$$

The new variable defined by the quotient

$$u = \frac{x - m}{\sigma} \quad (2.17)$$

is called the standardized value of the original variable. The standardized density function $\phi(\lambda)$ is interpreted also similarly.

By utilizing expression (2.16) we may go over from the distribution $N(m, \sigma)$ to the distribution $N(0, 1)$, whose expected value is zero and its standard deviation is 1. The argument of $\Phi(\lambda)$ belonging to the given x value is

$$\lambda = \frac{x - m}{\sigma}.$$

Between the value of the function belonging to this and the probability $\Phi(x)$ belonging to the interval $m \pm \lambda\sigma$ the following relationship exists according to expression (2.16):

$$\Phi(x) = 2\Phi(\lambda) - 1. \quad (2.18)$$

Tables F1 and F2 of the Appendix give the standardized density function $\varphi(\lambda)$, resp. the distribution function $\Phi(\lambda)$ of the normal distribution.

As an example let us establish the empirical density function of some physical characteristic of the soils of Szeged mentioned in Section 1.4. As for the normal distribution

$$C_s = 0 \quad \text{and} \quad C_k = 0,$$

it seems advised to use the I_c -values of soil S3 ($C_s = 0$, $C_k = 0.31$) for this purpose. In the first step let us determine the number f of the values falling into the categories of the width $\Delta I_c = 0.10$, then the values divided by $n (= 341)$ of the same, namely the relative frequencies f_R (columns 2 and 3 of Table 2.2). The values of the function of the normal distribution belonging to the centre of the intervals are obtained by expressing the former in their standardized form. So, e.g., the centre of the category $I_c = 0.895 - 0.995$ is 0.945, whose standardized value according to expression (2.17) is

$$u = \lambda = \frac{0.945 - 0.63}{0.29} = 1.09,$$

Table 2.2. Description of the consistency index of the soil S3 by normal distribution

Category limits	f	f_R	$\frac{f_R \cdot s}{\Delta I_c}$	λ	$\varphi(\lambda)$
-0.405 ÷ -0.305	1	0.003	0.009	-3.40	0.001
-0.305 ÷ -0.205	2	0.006	0.017	-3.05	0.004
-0.205 ÷ -0.105	0	0.000	0.000	-2.71	0.010
-0.105 ÷ -0.005	2	0.006	0.017	-2.36	0.025
-0.005 ÷ 0.095	7	0.021	0.061	-2.02	0.052
0.095 ÷ 0.195	10	0.029	0.084	-1.67	0.099
0.195 ÷ 0.295	25	0.073	0.212	-1.33	0.165
0.295 ÷ 0.395	30	0.088	0.255	-0.98	0.247
0.395 ÷ 0.495	25	0.073	0.212	-0.64	0.325
0.495 ÷ 0.595	50	0.147	0.426	-0.29	0.383
0.595 ÷ 0.695	48	0.141	0.409	0.05	0.398
0.695 ÷ 0.795	44	0.129	0.374	0.40	0.368
0.795 ÷ 0.895	48	0.141	0.409	0.74	0.303
0.895 ÷ 0.995	22	0.065	0.188	1.09	0.220
0.995 ÷ 1.095	13	0.038	0.110	1.43	0.144
1.095 ÷ 1.195	8	0.023	0.067	1.78	0.082
1.195 ÷ 1.295	2	0.006	0.017	2.12	0.042
1.295 ÷ 1.395	0	0.000	0.000	2.47	0.019
1.395 ÷ 1.495	0	0.000	0.000	2.81	0.008
1.495 ÷ 1.595	2	0.006	0.017	3.16	0.003
1.595 ÷ 1.695	1	0.003	0.009	3.50	0.001
1.695 ÷ 1.795	1	0.003	0.009	3.84	0.000
Total:	341	1.001			

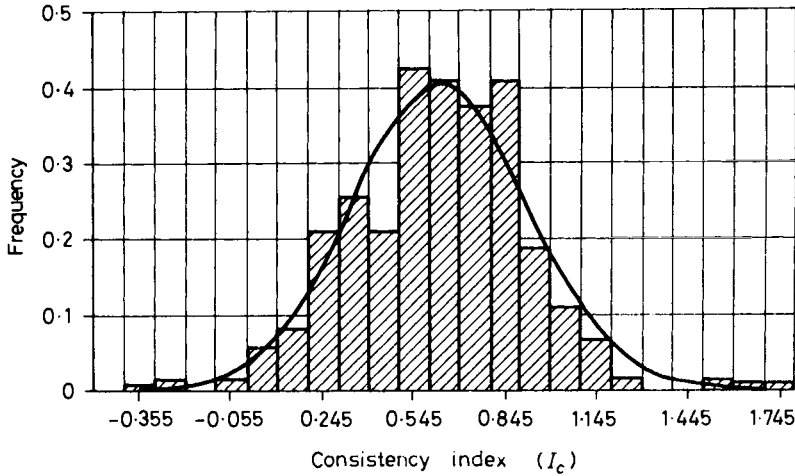


Fig. 2.5. Histogram of the consistency indexes of soil S3 and the density function of the normal distribution fitted to it

where 0.63 is the mean of the measuring series and 0.29 is its standard deviation. According to Table F1 $\varphi(x)=0.22$ (column 6) belongs to $\lambda=1.09$ (column 5). We can compare the empirical and the theoretical distributions only if we multiply the values of the former appearing in column 3 of the table by $s/\Delta I_c$ (column 4). The two distributions obtained in this way are presented in Fig. 2.5.

The normal distribution is called usually also GAUSS-distribution, but we must note that it was first determined by MOIVRE and it was LAPLACE who used this function type for solving concrete statistical problems.

If the variable

$$\eta = \ln \xi \quad (2.19)$$

is of a normal distribution, then the random variable ξ is said to be of the *lognormal distribution*. The density function of the random variable η is

$$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{(\ln x - m)^2}{2\sigma^2}}, \quad x > 0 \quad (2.20)$$

and its distribution function is

$$F(x) = P(\eta < \ln x) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\ln x} e^{-\frac{(\ln x - m)^2}{2\sigma^2}} dx. \quad (2.21)$$

If we start from the distribution of the variable η , the expected value of ξ is

$$M(\xi) = e^{m + \frac{\sigma^2}{2}} \quad (2.22)$$

and its variance is

$$D^2(\xi) = e^{2m + \sigma^2} (e^{\sigma^2} - 1), \quad (2.23)$$

where m is the expected value of the variable η and σ is its standard deviation.

This type of distribution has a *lower limit* (because the density function is not interpreted for the values $x \leq 0$) and is asymmetric in the direction of the increasing x -values.

The lognormal (GALTON-) distribution is generally characteristic for processes brought about by the *multiplication* of numerous random effects of the random variable. It is often met with in the discussion of *divisioning, cutting up processes*, as a distribution type which is suited for characterizing some dimension of the end product. This is the reason why the *grain size curve* of well graduated soils assumes in the semilogarithmic coordinate system ($\log d, S$) the form of the normal distribution function in numerous cases (see also Section 5.1.1).

Let us examine in the following, if the unconfined compression strength (q_u) of soil S5 can be described with the help of the lognormal distribution. The most important initial data are the following:

n	Minimum value (kN/m ²)	Maximum	Mean	s	C_s	C_k
117	51	600	184	99.6	1.65	3.74

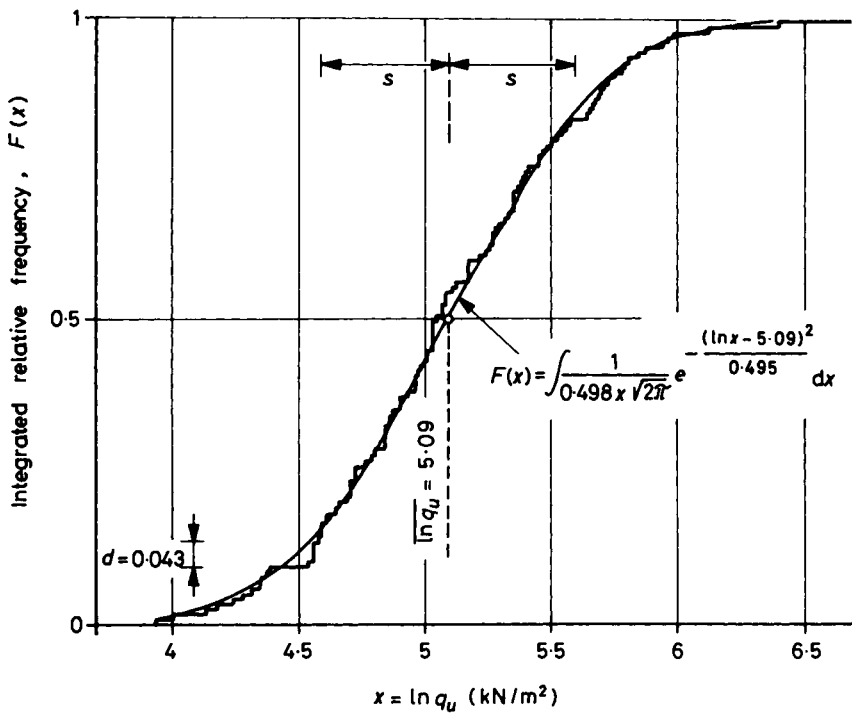


Fig. 2.6. Empirical distribution function of the unconfined compression strength of soil S5 and the lognormal distribution fitted to it

Accordingly the abscissae of the empirical distribution assume values between $\ln 51 = 3.932$ and $\ln 600 = 6.397$ (Fig. 2.6). On the basis of the mean (5.090) and the standard deviation (0.498) of the $\ln q_u$ values, the distribution function belonging to these parameters can be determined. E.g., the ordinate belonging to the abscissa $x = m + 2s = 5.090 + 0.996 = 6.086$ may be obtained by finding in Table F2 the $\Phi(\lambda)$ value ($= 0.9772$) belonging to $\lambda = 2$. According to Fig. 2.6 the distribution function obtained in this way agrees well with the empirical distribution.

Expressions (2.22) and (2.23) call attention to the fact that the expected value and the standard deviation of the original distribution are not identical with the retrieved values m and s obtained for the series $\ln q_u$. So, e.g.,

$$\bar{q}_u \neq \text{num } \ln 5.090,$$

which would correspond to 162.4 kN/m^2 , but it may be calculated in the following way:

$$\bar{q}_u = e^{5.09 + 0.124} = 183.8 \text{ kN/m}^2.$$

The distribution of the quadratic sums ($\xi_1^2 + \xi_2^2 + \dots + \xi_n^2$) of n independent random variables of $N(0, 1)$ distribution is called χ^2 (chi square)-distribution with n degrees of freedom. Its density function is

$$k_n(x) = \frac{x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}, \quad x > 0 \quad (2.24)$$

where Γ is the symbol of the so-called gamma function (see in Section 2.1.4).

The expected values and the variance of the distribution are:

$$M(\chi^2) = n \quad \text{and} \quad D^2(\chi^2) = 2n. \quad (2.25)$$

The density functions for the cases $n = 1, 2$ and 6 are shown in Fig. 2.7. The values of the distribution functions belonging to various degrees of freedom are contained in Table F3 of the Appendix.

The χ^2 -distribution may be used for solving the following tasks: fitting tests (Section 2.2.2), statistical tests (Section 6.2.1), determination of confidence intervals of the standard deviation (Section 7.4.2), homogeneity and independence tests.

If $\eta, \xi_1, \xi_2, \dots, \xi_n$ are independent random variables of $N(0, 1)$ distribution, then the distribution of the random variable

$$t = \frac{\eta \sqrt{n}}{\sqrt{\xi_1^2 + \xi_2^2 + \dots + \xi_n^2}} \quad (2.26)$$

is called STUDENT-, or t -distribution of n degrees of freedom. Its density function is:

$$s_n(x) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{2}}}. \quad (2.27)$$

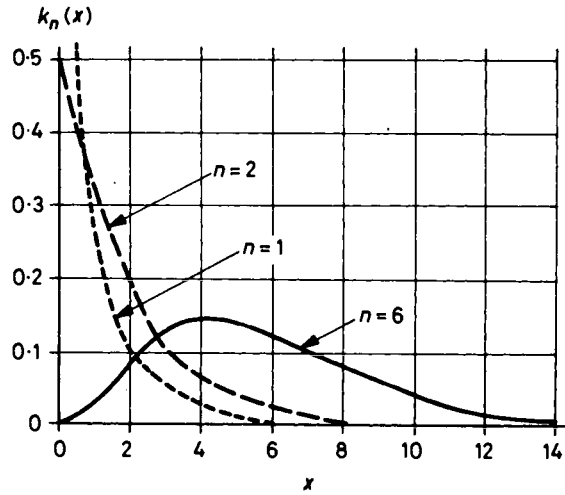


Fig. 2.7. Density function of the χ^2 -distribution for three different n -values (PRÉKOPIA, 1962)

The expected value of the distribution is zero (if $n \geq 2$) and its standard deviation is

$$D(t) = \sqrt{\frac{n}{n-2}}, \quad n \geq 3. \quad (2.28)$$

The t -distribution is symmetric and “bell-shaped”, but it is flatter than the density function of the normal distribution; if $n \rightarrow \infty$, then expression (2.27) converges to the density function of normal distribution.

The STUDENT-distribution can be used for solving the following tasks: determination of the required number of samples (tests) (Section 4.4); calculating the confidence intervals of the expected value (Section 7.4.1); determining the closeness of regression (Section 3.3); statistical tests for comparing soil layers and physical characteristics (Section 7.3). The values of the function belonging to different degrees of freedom are contained in *Table F4* of the Appendix.

STUDENT is the pen-name of the English mathematician W. S. GOSSET, used in his publications compiled on the request of his employer, the owner of a brewery in Dublin, to avoid competitors finding out that mathematical statistics could also be applied successfully in this field.

2.1.4. Gamma-distribution

The distribution with the density function

$$f(x) = \frac{\lambda^k}{\Gamma(k)} (x - x_0)^{k-1} e^{-\lambda(x-x_0)}, \quad x > x_0 \quad (2.29)$$

is called *gamma-distribution*.

The interpretation of the quantity in the denominator is

$$\Gamma(k) = (k-1)!, \quad (2.30)$$

where k is subject to a single restriction, namely that it must not be a negative integer.

The so-called gamma-function defined by expression (2.30) is essentially the generalization of the factorial conception. If k is not an integer, it must be produced as a sum:

$$k = a + j + 1,$$

where $1 \leq a \leq 2$ and j is an integer. The requested gamma-function is

$$\Gamma(k) = a(a+1)\Gamma(a),$$

where the value of $\Gamma(a)$ may be read off from Table 2.3.

As the expression (2.29) contains three constants (k , λ and x_0), the distribution defined in this way is called *gamma-distribution of three parameters*. If the density function starts from point $x=0$ and not from $x=x_0$, we speak of a *gamma-distribution of two parameters* ($x_0=0$).

The expected value and the variance may be calculated as follows:

$$M(\xi) = \frac{k}{\lambda} + x_0 \quad \text{and} \quad D^2(\xi) = \frac{k}{\lambda^2}. \quad (2.31)$$

Table 2.3. Gamma-function for the values $1 \leq k \leq 2$

k	$\Gamma(k)$	k	$\Gamma(k)$	k	$\Gamma(k)$	k	$\Gamma(k)$
1.00	1.00000	1.25	0.90640	1.50	0.88623	1.75	0.91906
1.01	0.99433	1.26	0.90440	1.51	0.88659	1.76	0.92137
1.02	0.98884	1.27	0.90250	1.52	0.88704	1.77	0.92376
1.03	0.98355	1.28	0.90072	1.53	0.88757	1.78	0.92623
1.04	0.97844	1.29	0.89904	1.54	0.88818	1.79	0.92877
1.05	0.97350	1.30	0.89747	1.55	0.88887	1.80	0.93138
1.06	0.96874	1.31	0.89600	1.56	0.88964	1.81	0.93408
1.07	0.96415	1.32	0.89464	1.57	0.89049	1.82	0.93685
1.08	0.95973	1.33	0.89338	1.58	0.89142	1.83	0.93969
1.09	0.95546	1.34	0.89222	1.59	0.89243	1.84	0.94261
1.10	0.95135	1.35	0.89115	1.60	0.89352	1.85	0.94561
1.11	0.94740	1.36	0.89018	1.61	0.89468	1.86	0.94869
1.12	0.94359	1.37	0.88931	1.62	0.89592	1.87	0.95184
1.13	0.93993	1.38	0.88854	1.63	0.89724	1.88	0.95507
1.14	0.93642	1.39	0.88785	1.64	0.89864	1.89	0.95838
1.15	0.93304	1.40	0.88726	1.65	0.90012	1.90	0.96177
1.16	0.92980	1.41	0.88676	1.66	0.90167	1.91	0.96523
1.17	0.92670	1.42	0.88636	1.67	0.90330	1.92	0.96877
1.18	0.92373	1.43	0.88604	1.68	0.90500	1.93	0.97240
1.19	0.92089	1.44	0.88581	1.69	0.90678	1.94	0.97610
1.20	0.91817	1.45	0.88566	1.70	0.90864	1.95	0.97988
1.21	0.91558	1.46	0.88560	1.71	0.91057	1.96	0.98374
1.22	0.91311	1.47	0.88563	1.72	0.91258	1.97	0.98768
1.23	0.91075	1.48	0.88575	1.73	0.91467	1.98	0.99171
1.24	0.90852	1.49	0.88595	1.74	0.91683	1.99	0.99581
						2.00	1.00000

The coefficients of skewness and kurtosis may also be determined with the help of the distribution parameters:

$$\beta_1 = C_s^2 = \frac{4}{k} \quad \text{and} \quad \beta_2 = C_k + 3 = \frac{3(k+2)}{k}. \quad (2.32)$$

The gamma-distribution may be described also by other distribution types depending on the magnitude of the constant k , namely

- a) by the exponential distribution with a density function $f(x) = \lambda e^{-\lambda x}$, if $k = 1$;
- b) by the χ^2 -distribution of n degrees of freedom, if $k = n/2$, $x_0 = 0$ and $\lambda = 0.5$, and
- c) by the normal distribution, if $k \rightarrow \infty$.

The gamma-distribution belongs to group III Of PEARSON's system (see Fig. 2.1). FOSTER (1924) and later RIBKIN (in hydrology) dealt with its practical application by plotting a graph for determining the p -values of the distribution function. In this respect numerous authors have pointed out that these aids—and even the gamma-distribution itself—may be used only in cases, where

$$C_s = 2C_v. \quad (2.33)$$

Indeed, if we determine the coefficient of skewness and the coefficient of variation for the case $x_0 = 0$ from expressions (2.31) and (2.32), the correctness of expression (2.33) can be proved:

$$C_s = \frac{2}{\sqrt{k}} = 2C_v = 2 \frac{\sqrt{k}}{\lambda} \frac{\lambda}{k} = \frac{2}{\sqrt{k}}.$$

The “over-determination”, i.e. the contradiction that we have four equations for two parameters of the distribution, may be met with also at other distribution types (e.g., at the normal distribution). This fact in itself does not mean that our data series cannot be described by the involved theoretical distribution function. In such cases it is advised to calculate the constants from the *lower order* central moments, as in this way the distorting effect of the outlying (and eventually erroneous) values may also be decreased.

The values of the gamma-distribution function for the cases of $\lambda = 0.5$ and $x_0 = 0$ are given in Table F5 of the Appendix. If $\lambda \neq 0.5$ and $x_0 \neq 0$, the value x_t belonging to the given probability must be calculated from the following expression:

$$x_t = (x - x_0)2\lambda. \quad (2.34)$$

The parameters of the density function defined by expression (2.29) may be calculated as follows:

$$x_0 = \bar{x} - 2 \frac{s}{C_s}; \quad (2.35)$$

$$\lambda = \frac{\bar{x} - x_0}{s^2}; \quad (2.36)$$

$$k = \frac{(\bar{x} - x_0)^2}{s^2}. \quad (2.37)$$

If x_0 is nearly identical with the actually observed minimum, the task may be regarded as solved by the calculation of the three parameters; otherwise we may proceed as suggested by SZIGYÁRTÓ (1966) in the following way:

(1) If $x_0 < 0$ and simultaneously $\xi \geq 0$, it is advised to accept zero for the ruling value of x'_0 .

(2) If x_0 exceeds the observed minimum (ξ_1) and $C_s > 0$, it is advised to calculate with $x'_0 = 0.5\xi_1$.

(3) If $x_0 > \xi_1$ and $C_s \leq 0$, the approximation by gamma-distribution must be rejected.

In case (2) x'_0 may be calculated also in the following way (CSOMA and SZIGYÁRTÓ, 1975):

$$x'_0 = x_{0f} + \frac{\xi_1 - x_{0f}}{15} (15 - k), \quad (2.38)$$

where x_{0f} is the lower limit of the random variable determined from the physical side.

For studying the possibility of the approximation by gamma-distribution we shall use again the data series of the q_u values of the soil S5. In the first step we verify if the selection of the function type on the basis of the statistical parameters (see Section 2.1.3) is apparently realistic. As $C_v = 99.6/184 = 0.541$, $C_s (= 1.65)$ is not equal with $2C_v$, but the deviation is insignificant. We obtain by the formulae (2.32)

$$\frac{C_k}{C_s^2} = 1.5 \quad (2.39)$$

for the gamma-distribution; in our case this quotient is $3.74/2.72 = 1.37$, i.e. the deviation is insignificant also here.

Let us determine now the value of x_0 ; according to expression (2.35)

$$x_0 = 184 - 2 \frac{99.6}{1.65} = 63.3.$$

As the actually observed minimum value was $\xi_1 = 51 \text{ kN/m}^2$ and the physically possible lower limit of q_u is $x_{0f} = 0$, the corrected $x_0 = x'_0$ value is according to formula (2.38)

$$x'_0 = 0 + \frac{51 - 0}{15} (15 - 1.467) = 46.0,$$

where the second term in the parenthesis is the k -value calculated by expression (2.37), with the assumption of $x_0 = 63.3$.

The parameters calculated with x'_0 will be according to expressions (2.36) and (2.37) as follows:

$$\lambda = \frac{184 - 46}{99.6^2} = 0.0139 \quad \text{and} \quad k = \frac{(184 - 46)^2}{99.6^2} = 1.92.$$

In the knowledge of the three parameters the values of the distribution function may be found in *Table F5*. So, e.g., x_i belonging to the probability $p = 0.95$ will be according to expression (2.34):

$$x = 46 + \frac{9.23}{0.02786} = 377,$$

where the numerator of the fraction is the value x_i belonging to $p = 0.95$ and $k = 1.92$, interpolated to k .

Figure 2.8 shows the empirical distribution functions and those determined from the statistical parameters of the data series. The agreement is very good, which cannot be said of the normal distribution $N(184, 100)$, marked by a dashed line in the figure.

In the following two *gamma type* distributions are discussed, which were used successfully for the statistical characterization of the strength parameters of certain

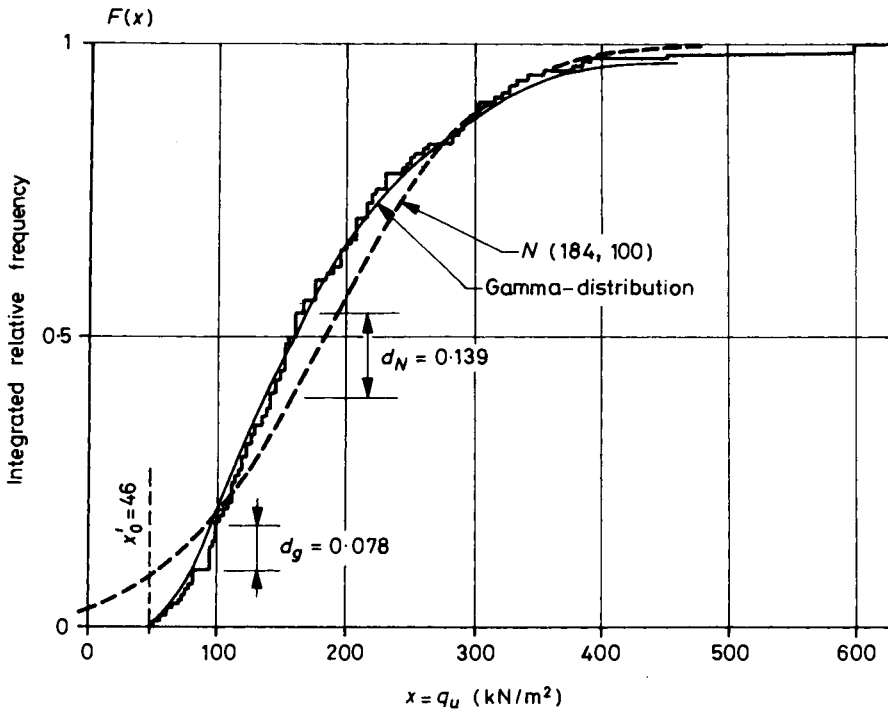


Fig. 2.8. Empirical distribution function of the unconfined compression strength of the soil *S5* and the three-parameter gamma-distribution fitted to it; *N* — approximation by normal distribution

building materials, so they may be suitable for examining the physical characteristics of the soils, in the first place their unconfined compression strength and their cohesion.

The so-called *gamma-sum* function (KORDA, 1972) may be described by the density function of the following form:

$$f(x) = \sum_{k=1}^l C_k \frac{b^{a_k}}{\Gamma(a_k)} (x-A)^{a_k-1} \cdot e^{-b(x-A)}, \quad x \geq A, \quad (2.40)$$

which deviates from the function (2.29)—apart from the symbols ($A = x_0$, $b = \lambda$, $a_k = k$)—by weighting the latter function by various c_k -values and summarizing the function values obtained in this way.

Between the c_k coefficients the relationship

$$\sum_{k=1}^l c_k = 1$$

exists and the number of free parameters is $2l + 1$.

The case $l = 1$ corresponds to the gamma-distribution with three parameters. If $l = 2$, we have five parameters: A , a_1 , a_2 , b and c , for whose determination we need also the fourth and fifth-order central moments. Consequently, good results may be expected from the gamma-sum function only if data series consisting of *several thousand elements* are examined.

The so-called *gamma-power* distribution (TUSNÁDY and ROCKNICH, 1969) approximates the probability variable ξ by a suitable t -power of the variable ξ_1 of gamma-distribution. In the first step the logarithms of the sample elements (so the new variable $y = \ln x$), then their mean, their standard deviation and their coefficient of skewness must be calculated.

2.1.5. Beta-distribution

The distribution with the density function

$$f(x) = \frac{(b-a)^{-1-\alpha-\beta}}{B(\alpha+1, \beta+1)} (x-a)^\alpha (b-x)^\beta \quad (2.41)$$

is called beta-distribution, where

$$a \leq x \leq b, \quad \beta > -1 \quad \text{and} \quad \alpha > -1.$$

The so-called EULER-type *beta-function* appearing in the denominator may be defined as follows:

$$B(\alpha+1, \beta+1) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, \quad (2.42)$$

where Γ is the gamma-function mentioned in Section 2.1.4.

The density function of the beta-distribution may assume surprisingly numerous forms depending on the parameters α and β (HARR, 1977), namely

- a rectangular, or uniform distribution, if $\alpha = \beta = 0$;
- a triangular distribution, if one parameter is zero, the other one is 1;
- a symmetrical distribution, if $\alpha = \beta$;
- skewed right, if $\beta > \alpha$ (and vice versa);
- U-shaped distribution, if α and β are both negative;
- J-shaped distribution, if $\alpha \geq 0$ and $\beta > 0$;
- unimodal and bell-shaped (but generally skew) distribution, if α and β have the same sign, with a maximum, resp. a minimum appearing in the position

$$x = a + \frac{\alpha(b-a)}{\alpha + \beta}$$

(depending on whether both are positive or negative).

The parameters α and β may be determined as follows:

$$\alpha = \frac{x_r^2}{V_r} (1 - x_r) - (1 + x_r); \quad (2.43)$$

$$\beta = \frac{\alpha + 1}{x_r} - (\alpha + 2), \quad (2.44)$$

where

$$x_r = \frac{\bar{x} - a}{b - a} \quad \text{and} \quad V_r = \left(\frac{s}{b - a} \right)^2. \quad (2.45)$$

The expected value and the variance of the distribution are as follows:

$$\bar{x} = a + \frac{\alpha + 1}{\alpha + \beta + 2} (b - a); \quad (2.46)$$

$$s^2 = \frac{(b - a)^2 (\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2 (\alpha + \beta + 3)}. \quad (2.47)$$

The coefficient of skewness and the kurtosis of the distribution may be calculated by the following expressions:

$$C_s = \frac{2(\beta - \alpha)}{(\alpha + \beta + 4)} \sqrt{\frac{\alpha + \beta + 3}{(\alpha + 1)(\beta + 1)}} \quad (2.48)$$

$$C_k = \frac{3(\alpha + \beta + 3)[2(\alpha - \beta)^2 + (\alpha + 1)(\beta + 1)(\alpha + \beta + 4)]}{(\alpha + 1)(\beta + 1)(\alpha + \beta + 4)(\alpha + \beta + 5)} - 3. \quad (2.49)$$

For an example let us study the data series S_r obtained for the soil S5. The statistical parameters are $\bar{x} = 0.905$, $s = 0.090$, $C_s = 1.70$, $C_k = 3.50$ and the two extreme values are $S_r = 0.54$, resp. $S_r = 1.00$; so according to expressions (2.45) we have

$$x_r = \frac{0.905 - 0.54}{1 - 0.54} = 0.793 \quad \text{and} \quad V_r = \left(\frac{0.09}{1 - 0.54} \right)^2 = 0.0383,$$

as in this case $a=0.54$ and $b=1.00$. The two parameters of the distribution are according to (2.43) and (2.44):

$$\alpha = \frac{0.793^2}{0.0383} (1 - 0.793) - (1 + 0.793) = 1.606$$

and

$$\beta = \frac{1.606 + 1}{0.793} - (1.606 + 2) = -0.320.$$

So we have to do expectably with a J-shaped distribution (because $\alpha > 0$ and $\beta < 0$), skewed to the left (because $\alpha > \beta$).

The beta-function appearing here is according to expression (2.42):

$$B(\alpha + 1, \beta + 1) = \frac{\Gamma(2.606)\Gamma(0.68)}{\Gamma(3.286)}.$$

According to Section 2.1.4:

$$\Gamma(2.606) = 1.606 \cdot 0.894 = 1.436$$

$$\Gamma(3.286) = 1.286 \cdot 2.286 \cdot 0.900 = 2.646$$

Table 2.4. Description of the S_r -values of the soil S5 by beta-distribution; f_r – empirical frequency, $f(x)$ – calculated frequency

Category	f	f_r	$f(x)$
0.53 – 0.54	2	0.013	0.000
0.55 – 0.56	0	0.000	0.000
0.57 – 0.58	1	0.006	0.001
0.59 – 0.60	0	0.000	0.002
0.61 – 0.62	1	0.006	0.003
0.63 – 0.64	0	0.000	0.005
0.65 – 0.66	0	0.000	0.007
0.67 – 0.68	1	0.006	0.009
0.69 – 0.70	1	0.006	0.011
0.71 – 0.72	2	0.013	0.014
0.73 – 0.74	2	0.013	0.017
0.75 – 0.76	3	0.019	0.021
0.77 – 0.78	2	0.013	0.025
0.79 – 0.80	2	0.013	0.029
0.81 – 0.82	5	0.031	0.034
0.83 – 0.84	12	0.075	0.039
0.85 – 0.86	5	0.031	0.045
0.87 – 0.88	7	0.044	0.053
0.89 – 0.90	15	0.094	0.061
0.91 – 0.92	15	0.094	0.071
0.93 – 0.94	25	0.157	0.085
0.95 – 0.96	17	0.107	0.103
0.97 – 0.98	15	0.094	0.134
0.99 – 1.00	26	0.164	0.241
Total:	159	0.999	1.010

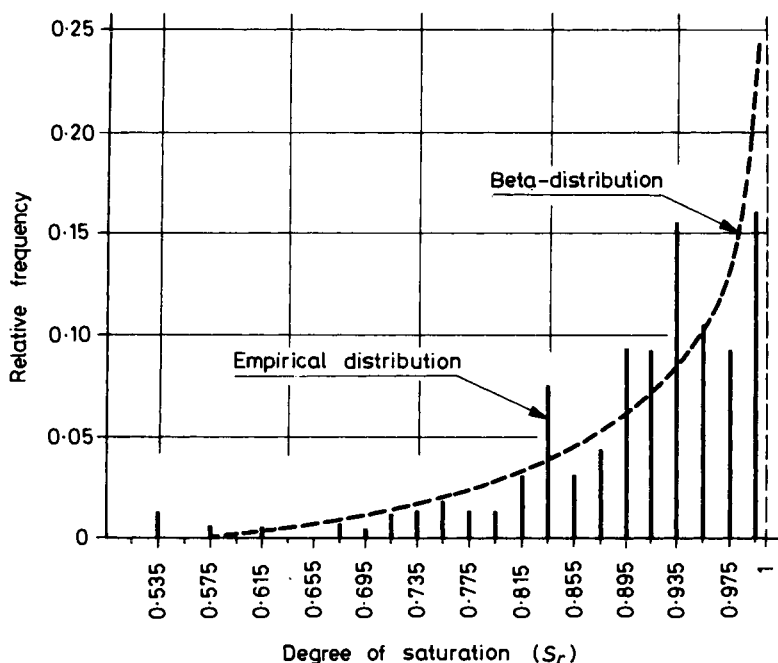


Fig. 2.9. Approximation of the histogram constructed for the degrees of saturation of the soil S5 by the density function of the beta distribution

and the second term of the numerator is $\Gamma(1.86) : 0.68 = 1.33$. So the density function defined by expression (2.41) is

$$f(x) = \frac{0.46^{-2.286}}{0.722} (x - 0.54)^{1.606} (1 - x)^{-0.32} =$$

$$= \frac{1}{0.46 \cdot 0.722} \left(\frac{x - 0.54}{0.46} \right)^{1.606} \left(\frac{1 - x}{0.46} \right)^{-0.32}.$$

The empirical distribution of the S_r -values, namely their frequency (column 2) and their relative frequency (column 3) are given in Table 2.4. The probabilities assigned to the interval centres by the beta distribution (column 4) may be calculated by substituting 24 in the denominator of the first term of the density function instead of 0.46, because 24 intervals correspond to the difference $a - b$. So, for example, the probability belonging to $S_r = 0.915$ is

$$f(x) = \frac{1}{24 \cdot 0.722} \cdot 0.815^{1.606} \cdot 0.185^{-0.32} = 0.071.$$

Figure 2.9 shows the empirical and the theoretical distribution.

2.1.6. Eliminating the asymmetry of data series

As the handling of series with a skewed distribution is inconvenient, we prefer often solutions which are suited for the elimination of the asymmetry. The best known methods used for this purpose may be summed up as follows.

a) Instead of the physical characteristic some modified *variant* of the same is considered. Maybe, for example, that the distribution of the void ratio (e) is asymmetric, while that of the porosity (n) is not. The same result may be obtained, if we chose $\tan \varphi$ (or $\cotan \varphi$) instead of the numerical value of the angle of friction φ .

b) In favourable cases the asymmetry may be eliminated by using the *square root* or the *logarithm* of the elements of the data series for the calculation. (If the latter solution is selected, the data series is approximated by lognormal distribution.)

c) We may try to treat separately the parts lying to the right and to the left of the *mode*, with the assumption that the density function is composed of two normal distributions, which may be described by different statistical parameters.

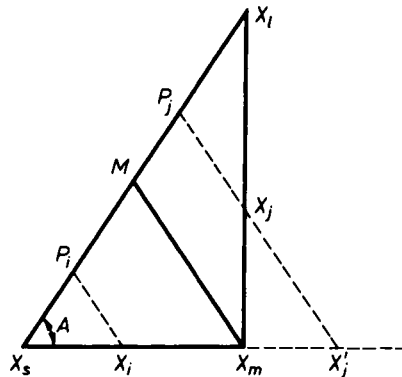


Fig. 2.10. Basic principle of the SMEMAX-method

In the following three methods are discussed, which ensure the elimination of the asymmetry, independently of the statistical structure of the data series.

The so-called SMEMAX-method (BETHLAHMY, 1977) obtained its name from the three statistical parameters used for transforming the original data series (S = smallest, ME = median, MAX = maximum). The process may be understood from Fig 2.10. The minimum value (X_s), the maximum value (X_l) and the median (X_m) determine a rectangular triangle, whose basis and height divide the data series into two groups of elements, under and over the median (X_i , resp. X_j). The transformed values may be read off the hypotenuse by drawing a parallel with the straight MX_m from the points X_i , resp. X_j . (M is the centre of the hypotenuse.)

For the numerical solution we must calculate the angle A at point X_s :

$$A = \arctan \frac{X_i - X_m}{X_m - X_s} \quad (2.50)$$

With the help of this angle the transformed $P_i(X_i < X_m)$ and $P_j(X_j > X_m)$ values may be calculated as follows:

$$P_i = \frac{X_i - X_s}{2 \cos A}; \quad P_j = \frac{(X_m - X_s) + (X_i - X_m) \cotan A}{2 \cos A} \quad (2.51)$$

BETHLAHMY (1977) demonstrated the applicability of the method proposed by him on the example of time series formed from the maximum yearly water-flow rates of 65 rivers of the United States. For the coefficients of skewness and kurtosis of the original and the transformed series the following values were obtained:

	C_s			C_k		
	min.	max.	mean	min.	max.	mean
Original series	0.06	3.62	0.75	0.03	15.26	1.55
Transformed series	0.01	0.56	0.12	0.00	1.83	0.52

The data published in the original paper show also that the C_{st} - and the C_{kt} -values of the transformed series are generally the higher, the higher the original C_s - and C_k -values were. The involved relationships are as follows:

$$C_{st} = 0.054C_s + 0.028,$$

$$C_{kt} = 0.124C_k - 0.33.$$

The correlation coefficients indicating the closeness of the relationship are $r = 0.31$ and 0.60 , which refers to the fact again that it is more difficult to ensure the condition $C_k = 0$, than the condition $C_s = 0$.

Table 2.5 presents the original (a) and the transformed (b) values of the unconfined compression strength of the soil S4, determined by the SMEMAX-method (on the basis of samples taken from large diameter drillings). According to the original data

$$X_s = 0.24, \quad X_m = 1.52 \quad \text{and} \quad X_i = 3.70,$$

so from expression (2.50)

$$A = \arctan \frac{3.70 - 1.52}{1.52 - 0.24} = 59.58^\circ.$$

So for example the following values correspond according to the expressions (2.51) to $X_i = q_u = 0.98 \text{ kN/m}^2$ (under the median) and to $X_j = q_u = 2.86 \text{ kN/m}^2$ (over the median) respectively:

$$P_i = \frac{0.98 - 0.24}{2 \cos 59.58^\circ} = 0.731 \text{ kN/m}^2$$

Table 2.5. Original values (a) of the unconfined compression strength of the soil S4, its transformed values by the SMEMAX-method (b) and by the alpha-method (c)

a	b	c	a	b	c	a	b	c	a	b	c
98	0.731	5.454	150	1.244	6.385	85	0.602	5.175	143	1.175	6.273
286	2.041	8.107	243	1.792	7.633	121	0.958	5.897	80	0.553	5.060
61	0.365	4.577	359	2.464	8.818	66	0.415	4.712	140	1.145	6.224
260	1.890	7.826	183	1.444	6.872	92	0.671	5.328	40	0.158	3.915
159	1.305	6.524	175	1.397	6.760	113	0.879	5.750	124	0.987	5.951
92	0.671	5.328	170	1.368	6.688	90	0.652	5.285	191	1.490	6.982
362	2.482	8.845	224	1.681	7.406	139	1.136	6.207	183	1.444	6.872
234	1.739	7.527	279	2.000	8.033	111	0.859	5.712	215	1.629	7.295
273	1.966	7.969	130	1.047	6.056	100	0.750	5.495	255	1.861	7.770
283	2.024	8.075	95	0.701	5.392	98	0.731	5.454	167	1.351	6.644
215	1.629	7.295	205	1.571	7.167	143	1.175	6.273	168	1.357	6.658
136	1.106	6.158	205	1.571	7.167	127	1.017	6.004	300	2.122	8.252
168	1.357	6.658	240	1.774	7.598	112	0.869	5.731	151	1.254	6.401
152	1.264	6.416	185	1.455	6.900	198	1.531	7.076	261	1.896	7.837
63	0.385	4.632	370	2.528	8.917	139	1.136	6.207	192	1.496	6.996
168	1.357	6.658	88	0.632	5.242	335	2.325	8.595	204	1.565	7.154
55	0.306	4.405	148	1.224	6.353	183	1.444	6.872	162	1.322	6.569
110	0.849	5.693	85	0.602	5.175	169	1.363	6.673	302	2.134	8.272
145	1.195	6.305	115	0.899	5.787	163	1.328	6.584	152	1.264	6.416
115	0.899	5.787	111	0.859	5.712	203	1.560	7.141	232	1.728	7.503
280	2.006	8.044	199	1.537	7.089	159	1.305	6.524	72	0.474	4.866
120	0.948	5.879	128	1.126	6.021	159	1.305	6.524	260	1.890	7.826
100	0.750	5.495	113	0.879	5.750	175	1.397	6.760	207	1.583	7.193
103	0.780	5.556	188	1.473	6.941	183	1.444	6.872	120	0.948	5.879
143	1.175	6.273	85	0.602	5.175	166	1.345	6.629	95	0.701	5.392
271	1.954	7.947	150	1.244	6.385	127	1.017	6.004	140	1.145	6.224
351	2.418	8.745	110	0.849	5.693	24	0.000	3.241	64	0.395	4.659

and

$$P_j = \frac{1.52 - 0.24 + (2.86 - 1.52) \cotan 59.58^\circ}{2 \cos 59.58^\circ} = 2.041 \text{ kN/m}^2.$$

The coefficients of skewness and kurtosis of the transformed series (b) are $C_{st} = 0.20$ and $C_{kt} = 0.22$, respectively.

VENUGOPAL (1980) simplified this process as follows. Let us denote the difference $X_m - X_s$ by R_1 , the difference $X_i - X_m$ by R_2 , the distance $P_i X_s$ by P_i , the distance $P_j X_s$ by P_j and be

$$f_1 = a/R_1, \quad \text{resp.} \quad f_2 = a/R_2,$$

where $2a$ is the subtense of the triangle ($= X_i - X_s$).

Now

$$\left. \begin{aligned} P_i &= f_1(X_i - X_s); & X_i < X_m; \\ P_j &= f_2(X_j - X_m); & X_j > X_m. \end{aligned} \right\} \quad (2.52)$$

With the help of these equations the transformation can be performed to arrive at the transformed values. Eqs (2.52) are readily solved for X_i and X_j , when their transformed equivalents are known. Thus

$$\left. \begin{aligned} X_i &= \frac{P_i}{f_1} + X_s; & P_i < a; \\ X_j &= \frac{P_j - a}{f_2} + X_m; & P_j > a \end{aligned} \right\} \quad (2.53)$$

The second method was proposed by Box and Cox (1964) under the name *power transformation*. The transformed values of the variable y_i are as follows:

$$\left. \begin{aligned} z_i &= \frac{y_i^\lambda - 1}{\lambda} & (\lambda \neq 0); \\ z_i &= \log y_i & (\lambda = 0). \end{aligned} \right\} \quad (2.54)$$

As C_s varies continuously with λ , the value of λ belonging to $C_s = 0$ can be easily determined (generally it is between -1 and $+1$). Alternatively, the likelihood function (see in Section 2.2.1) can be used to estimate λ . If a single random variable, z , is related to an other random variable y , such that $\alpha_i = (y_i - 1)/\lambda$ and if the z 's are normally distributed:

$$P \frac{y_1 y_2 \cdots y_n}{\lambda} = P(z_1 z_2 \cdots z_n) \left| \frac{dz}{dy} \right|^n. \quad (2.55)$$

If

$$J = \left| \frac{dz}{dy} \right|^n = y_1^{\lambda-1} \cdot y_2^{\lambda-1} \cdots y_n^{\lambda-1} = \prod_{i=1}^n y_i^{\lambda-1}, \quad (2.56)$$

$$\begin{aligned} P \frac{y_1 y_2 \cdots y_n}{\lambda} &= \frac{J}{(\sqrt{2\pi}\sigma_z)^2} \\ &\exp \left[\sum_{i=1}^n \frac{(z_i - \bar{z})^2}{2\sigma_z^2} \right]. \end{aligned} \quad (2.57)$$

Therefore, the likelihood function for a given series can be written as:

$$l\left(\frac{\lambda}{y}\right) \propto \frac{J}{\sigma_z^n}. \quad (2.58)$$

Eq. (2.58) can be used to arrive at the maximum likelihood estimate of λ . As SINGH (1980) remarks, the solution of the maximum likelihood function and the condition $C_s = 0$ do not lead always to the same λ .

$C_s \sim 0$ can be attained with the help of the power transformation, but for C_k this is generally not satisfied. CHANDER *et al.* (1978) proposed a method for taking this into account.

The essence of the third method (RÉTHÁTI, 1979b, 1983b) is that every element of the data series is raised to a power $\alpha = \alpha_{cr}$, making the coefficient of skewness of the transformed series zero.

Figure 2.11 shows the C_s -values belonging to various α -powers of an artificially prepared series of normal distribution (consisting of $n = 39$ elements), whose expected

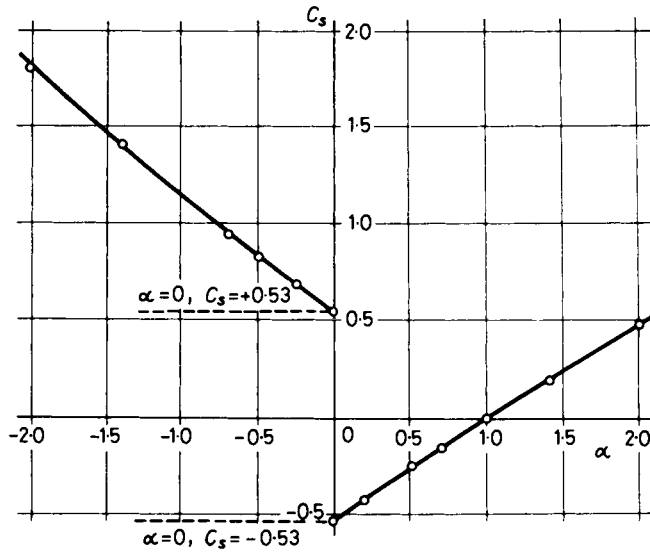


Fig. 2.11. Variation of the coefficient of skewness (C_s) of an artificially prepared series of normal distribution with the power (α) of the elements

value and standard deviation are 20, resp. 21.12. An interesting feature of the curve is that at $\alpha=0$ there is a break and the value belonging to it is $C_s = -0.53$, resp. $C_s = +0.53$, on approximating it from the right, resp. from the left. It may be proved with the help of the limit transition

$$\lim_{\alpha \rightarrow 0} \frac{\zeta^\alpha - 1}{\alpha} = \ln \zeta$$

that the absolute value of the coefficient of skewness belonging to $\alpha=0$ agrees with the absolute value of C_{sL} , i.e. the coefficient of skewness of the lognormal series. With this fact and the initial assumptions, as well as the form of the curve taken into account we may approximate the relationship $C_s(\alpha)$ by the following equation:

$$C_s(\alpha) = \pm (1 - \alpha) C_{sL} \cdot e^{m\alpha}, \quad (2.59)$$

where the positive sign applies to the case of $\alpha > 0$, the negative sign to the case of $\alpha < 0$. (The constant m for the series studied is -0.064 .)

The calculations performed with natural data and time series prove that expression (2.59) is applicable also to these series. As in this case the original coefficient of skewness belonging to $\alpha=1$ (C_{s1}) is not zero, the corresponding form of Eq. (2.59) will be

$$C_s(\alpha) = \left(1 - \frac{\alpha}{\alpha_{cr}}\right) C_{sL} \cdot e^{m\alpha}, \quad (2.60)$$

where α_{cr} is the exponent, for which $C_s=0$.

The two constants of Eq. (2.60) are determined purposefully in the following way. By extracting the square root from the elements of the series C_{sg} then from the logarithmic series the coefficient of skewness C_{sL} may be calculated. With the help of these we may write:

$$C_{sg} = \left(1 - \frac{0.5}{\alpha_{cr}}\right) C_{sL} \cdot e^{0.5m}$$

and

$$C_{sL} = \left(1 - \frac{1}{\alpha_{cr}}\right) C_{sL} \cdot e^m.$$

From the second equation

$$e^m = \frac{C_{s1}}{C_{sL}} \frac{\alpha_{cr}}{\alpha_{cr} - 1}. \quad (2.61)$$

On substituting this into the first equation we obtain a mixed quadratic equation for α_{cr} , whose solution is

$$\alpha_{cr} = 0.5 \pm \sqrt{\frac{A+1}{4A}}, \quad (2.62)$$

where

$$A = \frac{C_{sg}^2}{C_{s1} \cdot C_{sL}} - 1.$$

For examining the given time series we must proceed as follows: in the first step we determine the value of α_{cr} from Eq. (2.62) (the true root is selected on the basis of the signs of the three values $C_{s1} - C_{sg} - C_{sL}$), then we substitute this into Eq. (2.61) and calculate the constant m .

If also zero, or negative numbers (or only these) are present in the data series, we must add a positive number to every element of the series for making them greater than zero (as namely C_{sL} cannot be interpreted for $\xi < 0$). In this way s , C_s and C_k are not changed (but C_r naturally is), but in this case the "retransformation" must be carried out twice. If the coefficient of variation of the examined variable is a low value, it is advised to carry out the calculation by subtracting a constant d from every term of the original series.

For an example let us use again the q_u data series of the soil *S4*. For the original series $C_{s1} = 0.77$ and $C_k = 0.32$. On taking the square root, resp. the natural logarithm of the individual elements, the coefficient of skewness of the new series produced in this way will be $C_{sg} = 0.181$, resp. $C_{sL} = -0.628$. Using these values we have

$$A = \frac{0.181^2}{0.77(-0.628)} - 1 = -1.068$$

and from expression (2.62):

$$\alpha_{cr} = 0.5 \pm \sqrt{\frac{-0.068}{-4.272}} = 0.63 \quad \text{and} \quad 0.37,$$

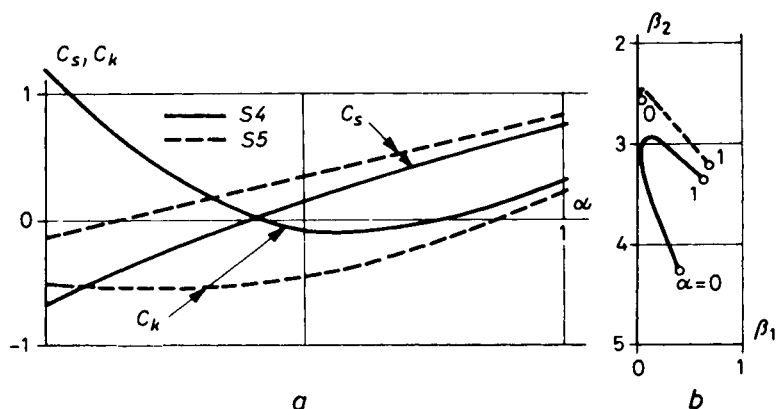


Fig. 2.12. (a) Variation of C_s and C_k , respectively (b) of β_1 and β_2 with the power of the elements of the data series (calculated from the q_u data series of the soils S4 and S5)

As C_s changes its sign between $\alpha=0.5$ and $\alpha=0$, so $\alpha_{cr}=0.37$; according to Eq. (2.60) $m=0.312$. The random variable $\eta=\xi^{0.37}$ is given in columns c of Table 2.5. For the transformed series $C_{st}=0.005$, $C_{kt}=0.08$. Table 2.6 shows also the results of calculations performed with four additional physical characteristics.

Eliminating the asymmetry we produce series of a “quasi-normal” distribution. This means that generally the transformed series does not satisfy perfectly the criteria of the normal distribution, namely because $C_k \neq 0$. The reason of it is that the zero point of the curve $C_k(\alpha)$ is not in the position $C_s=0$ (a in Fig. 2.12); this explains also why the mean value of the “residual” coefficient kurtosis (C_{kt}) is 0.52 in the case of the SMEMAX-transformation, and 0.63 in the case of the alpha-transformation (see Table 2.6). Part b of the figure illustrates how the pair of values β_1 and β_2 moves in PEARSON’s coordinate system with the variation of α , bringing about ever different types of distributions.

If $C_k < 0.2$, the distribution of the series may be regarded with a good approximation as normal. The practically acceptable upper limit depends on the nature of the actual task; it may be determined generally between 0.6–0.8. The reason is not alone the smaller weight of C_k as compared to C_s during the subsequent calculations with the distribution function, but also the fact that this statistical parameter may be unrealistically distorted by individual outlying values (more exactly by their fourth power).

Table 2.6. Alpha-transformation of the distribution of soil physical characteristics

Symbol of soil	Physical characteristic	Borehole diameter	Original		α_{cr}	m	C_{st}	C_{kt}
			C_s	C_k				
S4	q_u	N	0.77	0.32	0.37	-0.312	0.005	0.07
S3	q_u	K	2.25	7.44	0.03	-0.018	0.005	1.31
S3	e	K	1.19	2.73	-1.39	0.069	0.026	0.58
S4	e	K	0.41	1.14	0.12	0.035	0.000	0.68
S5	q_u	N	0.84	0.20	0.13	-0.081	0.000	-0.52

2.2. Selecting the function type and controlling the fitting

The data and time series occurring in practice are in the rarest case so regular as to permit to determine some known type of distribution unambiguously. So the first task, after arranging the data, is the selection of the function type which approximates the empirical distribution best. (This operation is called *fitting*.) Then the merit of the *fit* must be characterized by mathematical–statistical methods. A special case of this fitting test is the examination of whether the data series can be described by the normal distribution (normality test).

2.2.1. Selecting the function type

For solving the task we must start from the *mathematical* and the *physical* properties of the data series.

For studying the mathematical parameters it is advised to construct in the first step a *histogram* of the involved characteristic, whose direct inspection reveals if the distribution is symmetric and if not, how great the degree of the asymmetry is. The histogram supplies a good picture about the form of the density function as well (e.g., it shows the difference between the J- and U-shaped beta distributions) and on whether there are suspiciously outlying values in the data series.

The physical contents of the data assist in the first place to determine the *limits of the distribution* (e.g., the degree of saturation cannot exceed the unity, the unconfined compression strength cannot be less than zero). Often conclusions can be drawn concerning the type of the function describing the distribution from some physical property of the studied characteristic, e.g. if a diminution process is involved, as in the case of the grain size distribution curve, the lognormal distribution will fit expectably best.

After the preliminary survey of the given conditions we must determine the *statistical parameters* of the data series. The coefficients of skewness and kurtosis have a primary role in the selection of the function type. The parameters β_1 and β_2 calculated from them with the help of expression (2.3) determine a point in PEARSON's coordinate system (see Section 2.1.1 and Fig. 2.2). The requested function type can be found with a sufficient safety on the basis of the position of this point and the given mathematical and physical properties, mentioned before.

The fitting operation is finished by determining the parameters of the distribution. In the probability theory two solutions of this task are known:

- the method of the central moments and
- the so-called maximum likelihood process.

The method of the *central moments* is based on the fact that unequivocally definable relationship can be established between these and the distribution parameters. This is how we proceeded on elaborating the numerical examples in Section 2.1 (e.g. under 2.1.4 we have determined the parameters x_0 , λ and k of the gamma-distribution from the three empirical values $\bar{x} - s - C_s$).

FISHER's *maximum likelihood* method gives the optimum for the *common* estimation of the parameters. With the help of the so-called likelihood function (resp. its partial differential quotients, the so-called likelihood equations) it may be proved for example that the best estimation of the parameters of the normal distribution is the empirical mean and standard deviation (PRÉKOPA, 1962).

The essence of the process is as follows (REIMANN, 1976). Be the sample related to the variable ξ

$$\xi_1, \xi_2, \dots, \xi_n.$$

The likelihood function may be defined as

$$L(x_1, x_2, \dots, x_n; \Theta) = f(x_1; \Theta) \cdot f(x_2; \Theta) \dots f(x_n; \Theta),$$

where Θ is the unknown parameter of the distribution.

Accordingly, the likelihood-function is the common density (in the case of a discrete distribution the common probability) of the sample elements of the variable $(\xi_1, \xi_2, \dots, \xi_n)$ at point $(x_1, x_2, \dots, x_n)$ of the n -dimensional space. The essence of the method is the determination of the Θ -value, at which $L(x_1, x_2, \dots, x_n; \Theta)$, as the function of the parameter Θ is maximum and this is accepted as the estimation of the parameter Θ . The maximum is supplied by the root of the equations:

$$\frac{\partial L}{\partial \Theta} = 0, \quad \text{or} \quad \frac{\partial \ln L}{\partial \Theta} = 0.$$

The applicability of the method is limited by the complexity of the solution of the equations. So, e.g., for the parameter k of the gamma distribution of two parameters the equation

$$N \ln N + N \ln k - N \ln \sum x_i - N\psi(k-1) + \sum \ln x_i = 0$$

can be written up, where

$$\psi(k-1) = \frac{d}{dk} \ln \Gamma(k)$$

(a tabulated value, which may be read off a table) can be determined only by iteration (KORDA, 1972); λ may be calculated from the following expression:

$$\lambda = \frac{N \cdot k}{\sum x_i}.$$

If limited distribution functions are used, the lower and/or upper limits must be determined very carefully in every case, when they are not unambiguously defined quantities. So, e.g., it may be imagined that for the

beta-distribution not alone the parameters α and β are regarded as unknown, but also the lower (a) and the upper (b) limits. The four unknowns may be determined by Eqs (2.46)–(2.49), but the parameters a and b must be revised on the basis of soil-mechanical considerations, because the higher order powers used for their evaluation may lead to unrealistic results.

The selection of the function type is influenced also by the *nature* of the actual *task*. As it is proved in the subsequent chapters of this book, only one arm of the distribution function plays a role in the solution of practical problems (e.g., the range of the *low* shear strengths in calculating the resistance appearing on the slip surface). Consequently the reality must be described as best as possible mainly by this arm of the function type, selected by us.

2.2.2. Fitting test

PEARSON's χ^2 -test, which is suitable for testing both discrete and continuous distributions, is based on the fact that if in a sample (with r variables and n elements) the variable assumes various i values with a probability of p_i and the value i appears n_i times, the sum

$$\chi^2 = \sum_{i=1}^r \frac{n_i^2}{np_i} - n \quad (2.63)$$

shows—if the number of the elements is sufficiently large—a χ^2 -distribution of $f=r-1$ degrees of freedom. The course of the fitting test is now the following.

The elements of the data series are assigned to intervals (whose number is r), then the frequency n_i of the measured values falling into the individual intervals is determined. Then we calculate the p_i probability values belonging to $i=1 \dots r$ of the theoretical (hypothetical) distribution, then we form the sum χ^2 defined by expression (2.63). In *Table F3* of the Appendix we find the probability p corresponding to the degree of freedom $f=r-s-1$ and to χ^2 (s is the number of the estimated parameters of the distribution). If $p \geq 5\%$, the fitting may be qualified as satisfactory, but more or less strict levels may also be chosen.

According to FISHER—for counterbalancing the too great weight of the low probabilities—the condition

$$np_i \geq 5 \quad (2.64)$$

must be ensured for every category (interval), even at the cost that the process becomes only approximative in a certain measure.

Let us now follow up the course of the fitting test over the series of the plasticity index data of the soil S5. As $\beta_1 = 0$ and $\beta_2 = 2.17$, we may count on a reversed U-shaped beta distribution according to *Fig. 2.2*. But as β_2 is close to the value (to $\beta_2 = 1.8$) characterizing the uniform distribution, we may try to describe the empirical distribution by it. The results of the calculation are contained in *Table 2.7*. If the whole data set is taken into account, $n = 206$, $r = 11$, $p_i = 1/11 = 0.0909$ and the product np_i is for every category $206 \cdot 0.0909 = 18.73 > 5$, so no reductions are necessary.

The requested sum is

$$\chi^2 = 266.77 - 206 = 60.77$$

and the number of degrees of freedom is $f = r - 1 = 10$ (here it was not necessary to estimate the parameters of the distribution). In the row belonging to $n = f = 10$ of Table F3 even the greatest value is less than 60.77, so $p < 0.1$, which means that the hypothesis of the uniform distribution must be rejected. Columns 5–7 of the table contain the results of the calculations performed with truncated data sets; the details of the four fitting tests are summed up in the table below:

Category	n	r	f	χ^2	p
1–11.	206	11	10	60.77	<0.1%
2–10.	200	9	8	23.94	0.5%
2–9.	194	8	7	9.70	20.7%
3–9.	182	7	6	2.48	86.8%

Accordingly we can state that the distribution of the plasticity indices can be approximated by the uniform distribution in the category $I_p = 5–45$ well and in the category $I_p = 10–45$ very well. In similar cases we may go over even to the beta distribution, but this requires more calculations and is more difficult to treat. (Column 3 of Table 2.7 seems indeed to verify our preliminary assumption based on PEARSON's coordinate system.)

The fitting test of the *continuous distribution* may be carried out by KOLMOGOROV's method. According to one of its theorems, $F(x)$ is approximated by the empirical distribution function of the sample of n elements taken from the population of $F(x)$

Table 2.7. Examination of the hypothesis of the uniform distribution by the χ^2 -method (plasticity indexes of the soil S5)

Category		n_i	n_i^2/np_i			
No.	limits		cat. 1–11	cat. 2–10	cat. 2–9	cat. 3–9
1.	0 – 5.05	4	0.85			
2.	5.05 – 10.05	12	7.69	6.48	5.94	
3.	10.05 – 15.05	23	28.25	23.81	21.81	20.35
4.	15.05 – 20.05	27	38.93	32.81	30.06	28.04
5.	20.05 – 25.05	22	25.84	21.78	19.96	18.62
6.	25.05 – 30.05	30	48.06	40.50	37.11	34.62
7.	30.05 – 35.05	23	28.25	23.81	21.81	20.35
8.	35.05 – 40.05	29	44.91	37.85	34.68	32.35
9.	40.05 – 45.05	28	41.86	35.28	32.33	30.15
10.	45.05 – 50.05	6	1.92	1.62		
11.	50.05 – 55.05	2	0.21			
Total:			266.77	223.94	203.70	184.48

distribution in such a way that the product $d\sqrt{n}$ —where d is the greatest difference between the functions—supplies a random variable whose distribution can be characterized by KOLMOGOROV's function (Table F6 of the Appendix) with a good approximation.

The course of the calculations may be followed up on the basis of Figs 2.6 and 2.8. Both illustrate the empirical distribution function of the unconfined compression strength of the soil S5, with three different theoretical distribution functions fitted to it. In Fig. 2.6 the maximum difference between the ordinates is $d=0.043$ and as $n=117$

$$z = d\sqrt{n} = 0.043\sqrt{117} = 0.4651,$$

to which $L(z)=0.018$ corresponds in Table F6, so

$$p = 100[1 - L(z)] = 98.2\%,$$

which represents a very close fit. In Fig. 2.8

$d=0.078$ for the gamma-distribution and

$d=0.139$ for the normal distribution,

so $z=0.8437$, resp. 1.5035 and $L(z)=0.525$, resp. 0.978 . Accordingly we may state that the gamma-distribution fits well ($p=47.5\%$), the normal distribution does not ($p=2.2 < 5\%$).

It is often experienced that contradictory information is supplied on the goodness of the fit by the χ^2 -method and by the KOLMOGOROV-test. The reason is according to COROTIS *et al.* (1975) that the individual observations are treated separately by the KOLMOGOROV-test, so—contrary to the χ^2 -test—no information is lost due to the selection of the categories. In addition, in the case of a small number of elements (and of reductions performed for satisfying the criterion $np_i > 5$) the χ^2 -test is less reliable. On the other hand, KOLMOGOROV's test—strictly taken—is realistic only if we assume that the model is independent of the data. In the opposite case we cannot know how to “adjust” the distribution, as compared to the deduction of the number of the degrees of freedom in the case of the χ^2 -test. (We must add also that in practice the test of KOLMOGOROV proved to be less rigorous.)

A practical method of the fitting test is to calculate the limit value, which is determining from the viewpoint of the dimensioning and to count the elements of the data series which lie under and over it. Let us examine again the case discussed in connection with Fig. 2.8. The value of the gamma distribution belonging to $p=5\%$ (see Section 2.1.4) is

$$X_5 = 46 + 0.66/0.02786 = 69.7 \text{ kN/m}^2.$$

The number of the lower values (51, 55, 63, 65) is four and the frequency corresponding to $p=5\%$ belonging to $n=117$ is $117 \cdot 0.05 = 5.85$; so the agreement is relatively good.

On fitting the normal distribution to the void ratios of the soil S4 we obtain $100\bar{e}=81.82$ and $100s=9.45$. So the values belonging to $p=95\%$ and $p=5\%$ (see

Table F2) are

$$X_{9.5} = 81.82 + 1.645 \cdot 9.45 = 97.4, \text{ resp.}$$

$$X_5 = 81.82 - 1.645 \cdot 9.45 = 66.3.$$

From among the actual values 23 were greater than 97.4 and 18 were smaller than 66.3, as against the value expected on the basis of the theoretical distribution, namely $512 \cdot 0.05 = 26$. For the series modified by the alpha-transformation, $100 \bar{e} = 1.698$ and $s = 0.0235$, so $X_{19.5} = 1.737$ and $X_{1.5} = 1.659$. As $\alpha_{cr} = 0.1204$ (see Table 2.6)

$$X_{9.5} = 1.737 \frac{1}{70.12} = 98.1 \text{ and } X_5 = 67.2.$$

The two "counted" values belonging to them are 17 and 26, meaning that the transformed distribution fits better the small values and the normal distribution the great values.

The original skewness of the data series used in the second example was relatively slight ($C_s = 0.41$). A greater correction may be expected from the α -transformation on examining the void ratios of the soil S3 ($C_s = 1.19, n = 535$). Indeed the number of the counted values in the case of the normal distribution are 12 and 38, while in the case of the transformed series 25 and 28, i.e. the latter are considerably closer to the theoretical value ($535 \cdot 0.05 = 27$). Attention: on calculating the upper limit the sign of λ_s is negative (and vice versa), because α is negative.

The fitting of samples consisting of a small number of elements may be tested by FISHER's so-called *e*-method, elaborated for the exponential distribution.

2.2.3. Normality test

The possibility of describing the empirical distribution by the normal distribution is verified in the same way as in the case of other function types. The difference is on the one hand that the fitting can be tested also by a graphical method and on the other hand that for short series a special process is available.

Short data series may be tested by the SHAPIRO-WILK test. In the first step we order the measured values by their magnitude ($x_1, x_2, \dots, x_n$), then we calculate the following quantities:

$$d_j = x_{n-1+j} - x_j \quad (j = 1, 2, \dots, k), \quad (2.65)$$

where

$$k = \begin{cases} n/2, & \text{if } k \text{ is even,} \\ \frac{n-1}{2}, & \text{if } k \text{ is odd.} \end{cases} \quad (2.66)$$

Then we determine the following quantity:

$$N = \frac{\left(\sum_{j=1}^k a_{nj} d_j \right)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad (2.67)$$

where a_{nj} is a multiplication factor, which depends on n and j (Table F7 of the Appendix). If $W > W_p$, we maintain our hypothesis related to the normality on the given probability level, while if $W < W_p$, we reject the hypothesis. (The critical W_p values belonging to the probability levels $p=0.95$ and 0.99 are contained in Table 2.8.)

Table 2.8. W_p -values belonging to $p=0.95$ and $p=0.99$ (SHAPIRO-WILK test)

n	$p=$		n	$p=$		n	$p=$		n	$p=$	
	0.95	0.99		0.95	0.99		0.95	0.99		0.95	0.99
10	0.842	0.781	20	0.905	0.868	30	0.927	0.900	40	0.940	0.919
11	0.850	0.792	21	0.908	0.873	31	0.929	0.902	41	0.941	0.920
12	0.859	0.805	22	0.911	0.878	32	0.930	0.904	42	0.942	0.922
13	0.866	0.814	23	0.914	0.881	33	0.931	0.906	43	0.943	0.923
14	0.874	0.825	24	0.916	0.884	34	0.933	0.908	44	0.944	0.924
15	0.881	0.835	25	0.918	0.888	35	0.934	0.910	45	0.945	0.926
16	0.887	0.844	26	0.920	0.891	36	0.935	0.912	46	0.945	0.927
17	0.892	0.851	27	0.923	0.894	37	0.936	0.914	47	0.946	0.928
18	0.897	0.858	28	0.924	0.896	38	0.938	0.916	48	0.947	0.929
19	0.901	0.863	29	0.926	0.898	39	0.939	0.917	49	0.947	0.929
									50	0.947	0.930

According to archive data $n=21$ large-diameter samples were sampled from the soil *S1* for determining the unconfined compression strength. The test results were as follows:

i	q_u	i	q_u	i	q_u	i	q_u
1	40	6	80	11	121	16	192
2	50	7	80	12	143	17	205
3	57	8	80	13	150	18	220
4	58	9	80	14	160	19	220
5	78	10	95	15	175	20	230
						21	260

The calculation of the sum appearing in the numerator of expression (2.67) is shown in Table 2.9. As n is odd, $k=10$ according to expression (2.66), so the product $j=10a_{nj} \cdot d_j$ must be determined. Column 2 of the table contains the values taken from Table F7 ($n=1, j=1 \dots 10$). Column 3 contains the values $x_n, x_{n-1}, \dots, x_{n-k}$, column 4 the values $x_1, x_2, \dots, x_k$ and column 5 is formed on the basis of expression (2.65). As the denominator in expression (2.67) is nothing else, but $s^2(n-1)=21,241$, so

$$W = \frac{88,280}{21,241} = 4.156.$$

According to Table 2.8 $W_p=0.908 < W$, so we maintain our hypothesis related to the normality on the probability level $p=0.95$.

Table 2.9. Numerical example of the SHAPIRO-WILK test

j	a_{nj}	x_{n-j+1}	x_j	d_j	$a_{nj} \cdot d_j$
1	0.4643	260	40	220	102.146
2	0.3285	230	50	180	59.130
3	0.2578	220	57	163	42.021
4	0.2119	220	58	162	34.328
5	0.1736	205	78	127	22.047
6	0.1399	192	80	112	15.669
7	0.1092	175	80	95	10.374
8	0.0804	160	80	80	6.432
9	0.0530	150	80	70	3.710
10	0.0263	143	95	48	1.262
Total:					297.119

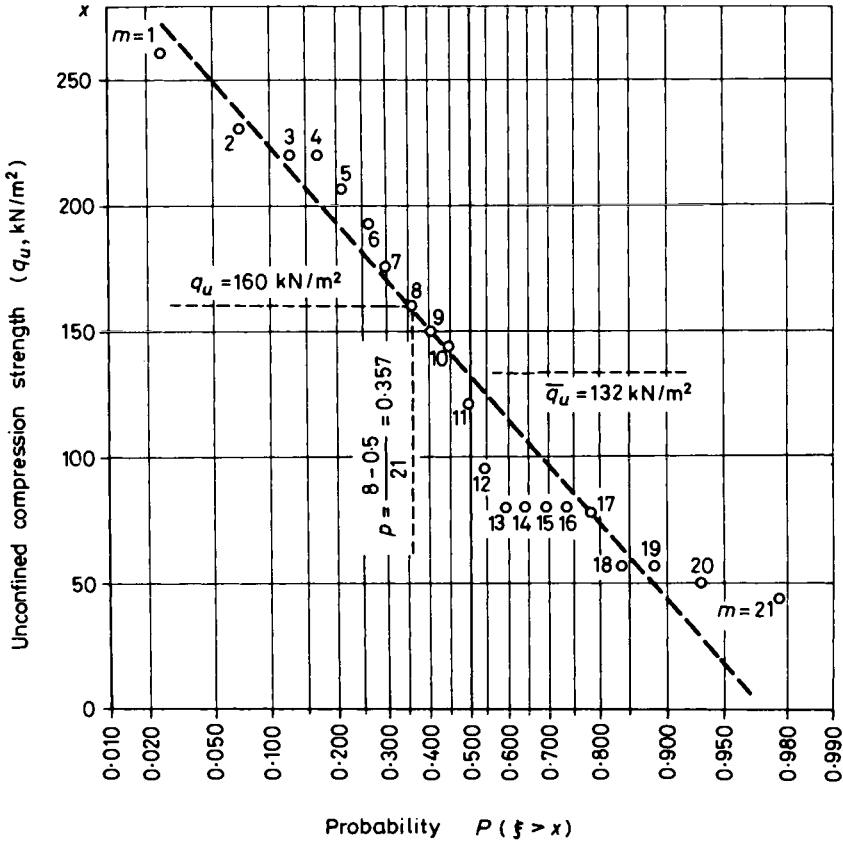


Fig. 2.13. Normality test with the help of the probability coordinate system of HAZEN for q_u values of the soil $S1$ determined from large-diameter boreholes

The essence of the *graphic method* is that the elements of the data series are plotted in a so-called probability coordinate system p, x , in which the points determine a straight line if the fitting is good. The “empirical probability” belonging to the m -th element of the sample written up in magnitudinal order may be determined by the formulae

$$p = \frac{m}{n+1}, \quad \text{resp.} \quad p = \frac{m-0.5}{n} \quad (2.68)$$

depending on whether n is higher or lower than 25–30. For the pairs of the values p_m, x_m determined in this way we apply an equalizing straight line and if we find that the fitting is satisfactory, we use this straight line, representing the relationship P, x for the subsequent calculations.

Figure 2.13 shows the pairs p_m, x_m of the compression strength values tested by the SHAPIRO–WILK test. The empirical probability

$$p_8 = \frac{8-0.5}{21} = 0.357$$

Table 2.10. Abscissa-values of the HAZEN-scale

p (%)	50	45	40	35	30	25	20	15	10
x (mm)	0	5.3	10.9	16.4	22.5	29.0	36.2	44.6	55.1
p (%)	8	6	5	4	3	2	1	0.5	0.1
x (mm)	60.4	66.8	70.7	75.3	80.9	88.3	100	110.8	133

belonging to the 8th value in the magnitudinal order ($x_8 = 160 \text{ kN/m}^2$) means that the *empirical probability* of $q_u > 160$ is about 36%. It is advised to determine the equalizing straight line of the 21 value pairs in a way that the mean value ($\bar{q}_u = 132 \text{ kN/m}^2$) should belong to $p = 0.5$. According to the figure the dispersion is considerable even so, due to the small number of the elements. (The severally repeating elements bring about a priori a large dispersion, as also the point belonging to $m = 21$, which refers to the physical limit $q_u = 0$.)

The abscissae of the probability coordinate system of the normal distribution may be plotted according to the so-called HAZEN-scale, shown in Table 2.10. The hypothesis of the *lognormal* distribution may be examined by plotting the logarithms of the observed quantities.

Similar systems may be constructed also for other function types. All these processes serve primarily for making possible to test the distribution *by sections* (e.g., the extreme ranges of decisive importance for dimensioning). The graphic representation has the same advantages as KOLMOGOROV’s process over the numerical χ^2 -test.

2.3. Examination of the distribution of soil physical characteristics

In the literature relatively few data can be found today on which *distribution functions* give the best fitting to the various soil physical characteristics. The evaluation is aggravated also by the fact that the statements contained in the publications can be generalized only with certain reservations; the *limits* mentioned in the introduction of Section 1.3 are valid also here. Thus, for example, far-reaching conclusions cannot be drawn even from one of the seemingly most valuable studies (COROTIS *et al.*, 1975), because the soils examined by the authors came from various regions of the world.

After these preliminary notions the statements related to the distribution functions describing the soil physical characteristics may be summed up as follows.

SCHULTZE (1971, 1975a) studied which physical characteristics can be approximated by the normal distribution. On the basis of his calculations he established that w_L , e (in 80%) and γ_s may be assigned to this category, but the normality test gave a negative result for the following characteristics: w_L , w_p , I_p , S_r , q_u , E_s , A and U . For the angle of friction (ϕ) he established the normal distribution for one of the soils (1975a), but not for the second soil (1971) nor for the $\tan \phi$ values of both soils. In the case of the latter soil he studied also the data series of $\cotan \phi$ and found a normal distribution.

COROTIS *et al.* (1975) tried to find out whether the physical characteristics of three groups of soils could be described by the normal, resp. lognormal distribution. The percentage distribution was as follows:

	Normal	Lognormal	Other types
w	67%	33%	—
w_L	67%	33%	—
w_p	—	100%	—
e	67%	—	33%
c_v	67%	33%	—

(We must note here again that the goodness of the fit was determined by KOLMOGOROV's test; the χ^2 -test verified only the normality of e and c_v , but not that of w and w_L .)

LUMB (1966, 1970) found that w_L , w_p , I_p and c were of a normal distribution, as well as the parameters characterizing the median, the slope and the asymmetry of the grain size distribution curve:

$$Md = \log d_{50}; \quad (2.69)$$

$$Dv = 0.5 \log (d_{75}/d_{25}); \quad (2.70)$$

$$Sk = 0.5 \log (d_{75}/d_{50}) / (d_{50}/d_{25}). \quad (2.71)$$

Table 2.11. β_1 and β_2 values calculated for the physical characteristics of the soils of Szeged (with combined samples from the small and large diameter boreholes)

Soil		w	w_L	w_p	I_p	I_c	e	S_r	γ	q_u	Mean
S1	β_1	2.76	4.12	6.81	1.93	0.13	6.50	7.90	1.28	0.02	3.49
	β_2	7.39	8.10	12.30	4.92	3.34	11.19	14.14	3.98	1.86	7.47
S2	β_1	0.01	0.74	0.34	0.49	0.02	0.01	0.86	0.09	0.94	0.39
	β_2	3.45	3.43	3.27	2.69	3.17	2.67	3.39	3.87	3.93	3.32
S3	β_1	0.03	0.96	0.14	0.85	0	1.30	3.17	2.89	5.06	1.60
	β_2	7.62	5.13	4.32	4.46	3.31	5.52	7.38	11.59	10.72	6.67
S4	β_1	0.05	0.34	0.13	0.64	0.03	0.13	3.20	1.06	4.80	1.15
	β_2	7.17	3.19	3.47	3.57	4.61	4.03	7.35	8.14	10.95	5.83
S5	β_1	1.10	0.02	2.92	0	0.98	0.36	2.89	0.10	2.72	1.23
	β_2	6.70	2.31	9.15	2.17	5.13	4.14	6.50	4.94	6.74	5.31
Mean*:	β_1	0.30	0.52	0.88	0.50	0.26	0.45	2.53	1.04	3.38	
	β_2	6.24	3.52	5.05	3.22	4.06	4.09	6.16	7.14	8.09	

* Without the soil S1.

The coefficient of consolidation (c_v) proved to be of a lognormal distribution, c and $\tan \varphi$ of a beta-distribution (for c : $C_s=0.50$ and $C_k=-0.27$; for $\tan \varphi$: $C_s=0.13$ and $C_k=-0.72$). In the first quoted study the cohesion of the soil examined there was of a normal distribution, but this must be regarded as an exceptional case, as indicated also by the fact that the distribution of the unconfined compression strength (related closely to the cohesion) is always asymmetric (SCHULTZE, 1971; WU and KRAFT, 1967; RÉTHÁTI and UNGÁR, 1978c).

The relatively homogeneous data set of a large volume related to the soils of Szeged contains additional valuable information. The initial basis of the evaluation is formed by the β_1 - and β_2 -values, calculated by the expressions (2.6) for 9 various physical characteristics of the soils S1 . . . S5 (Table 2.11). If we plot the value pairs determined by the latter in PEARSON's coordinate system, we may establish without any further calculation the type of the distribution which fits the various physical characteristics expectably best. By the position of the value pairs even the committed faults can be easily judged, if we use instead of the determined type another distribution (uniform, normal, etc.), which can be handled more easily.

Figure 2.14, constructed on the basis of Table 2.11—omitting the extraordinarily heterogeneous soil S1—permits to establish the following.

The *liquid limit* shows generally a beta-distribution of a reversed U-shape, but is close to the normal distribution (in the case of the soil S3 to the lognormal distribution).

The type applying to the *plastic limit* varies by soils; the possibility of the approximation by the normal distribution is worse than in the case of w_L .

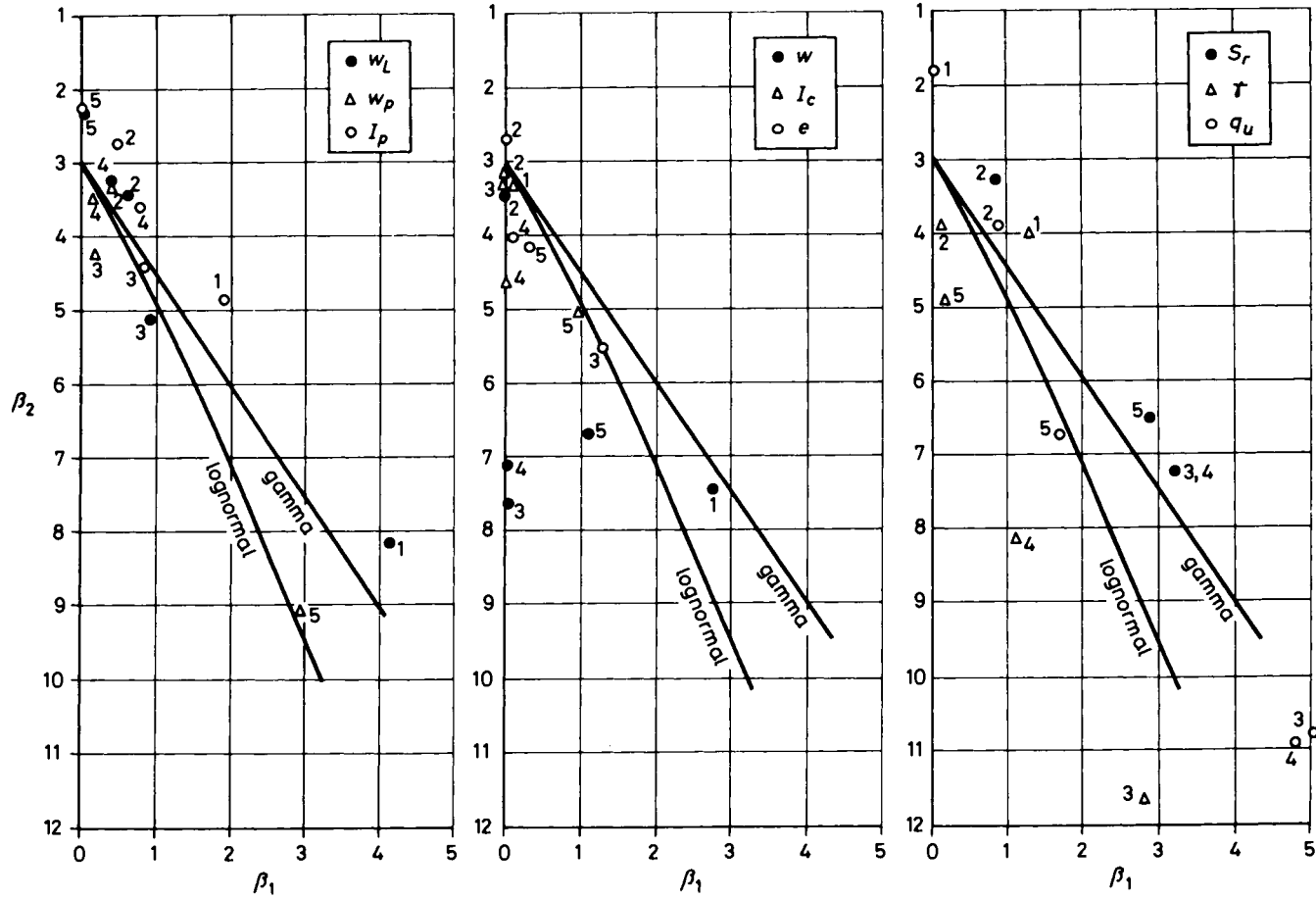


Fig. 2.14. β_1, β_2 value pairs determined for various physical characteristics of the soils of Szeged plotted in PEARSON's coordinate system (with the samples originating from small and large diameter boreholes combined)

As to the *plasticity index*, the same statements which were given for the liquid limit apply.

The distribution of the *water content* is of the type VII of PEARSON (symmetric and in the case of the soil S2 it is close to the normal distribution), with the exception of the soil S5, which follows the type IV distribution of PEARSON.

The *consistency index* excels among the physical characteristics by its highly symmetric distribution, so that it can be approximated in the case of three soils by the normal distribution.

For the *void ratio* a type of general validity cannot be given; the approximation by the normal distribution is permissible at the most only for the soil S2.

The *degree of saturation* has a high asymmetry and may be approximated by a *J*-shaped and a reversed *U*-shaped beta-distribution.

The examination of the bulk density is practically of no interest, due to its insignificant coefficient of variation; we state only for the sake of completeness that its distribution is very far from the normal distribution.

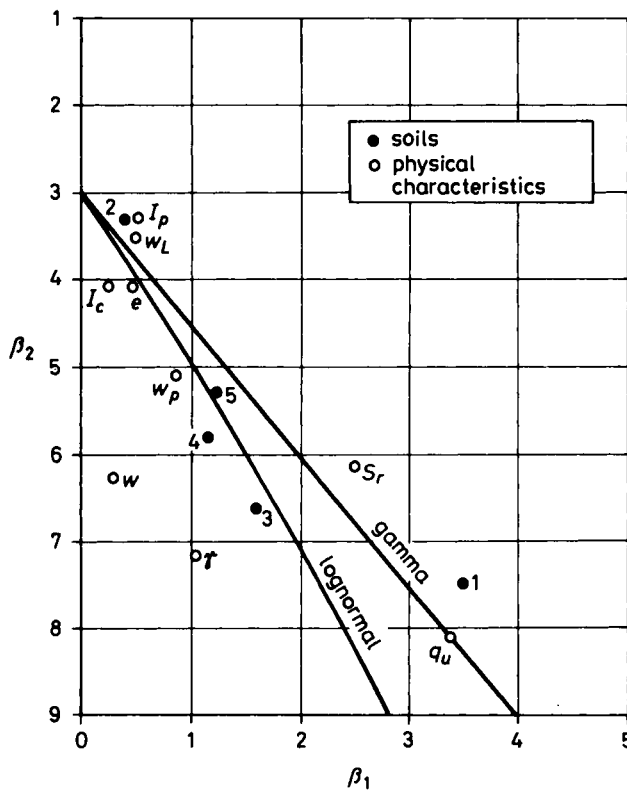


Fig. 2.15. Average β_1 and β_2 values calculated for the physical characteristics (soils of Szeged, 1 . . . 5) plotted in PEARSON's coordinate system

The value pair β_1, β_2 of the *unconfined compression strength* moves over the widest scale. In three cases it is close to the gamma-distribution, in one case to the lognormal distribution, while the data set *S1* consisting of a relatively small number of elements is of a uniform distribution.

Figure 2.14 shows also the reason of the contradictory data appearing in literature, while simultaneously we may establish that the range of the distribution types by physical characteristics can be delimited rather well. This latter fact can be illustrated more unambiguously by plotting the means appearing in both last rows of *Table 2.11* in PEARSON's coordinate system (*Fig. 2.15*).

The means found in the two last columns of the table and their graphical equivalents point to the differences between the individual *soil types*. This apparently strange formation of the means is of interest especially for illustrating the difference between the infusion loess varieties lying above (*S2*) and below (*S3*) groundwater level, due probably to different conditions of the sampling.

Figure 2.14 (apart from its first part) is only of an orientating character even for Szeged, as it treats the samples originating from small and large diameter boreholes together and on the other hand, the data set accepted as basis contains also outlying values (we return to these questions in Sections 6.2 and 7.1).

3. Correlation (regression) analysis

The totality of the soil physical characteristics forms a *system* whose members may be correlated with one or more other members of it. Revealing these relationships may supply very useful information, especially in the following cases.

a) The number of the complex and/or expensive investigations may be reduced considerably, if a correlation between them and some simple test (or a physical characteristic) can be found.

b) The relationships may reveal some formerly unknown law, which may be utilized possibly in other fields as well.

c) The empirical relationships may permit the *extrapolation* of the phenomena (e.g., to determine the capillary moisture distribution at levels which could never be reached during the experiments).

d) By utilizing earlier tests we may calculate such physical characteristics, which *were not determined* for some reason before.

e) The closeness of the relationship supplies an information on how far the involved two variables may be regarded as independent (the importance of this problem is dealt with in Section 8.5).

f) The empirical function describing the relationship screens out the random deviations (at the same time it gives a numerical value of their magnitude), so it *improves the precision* and the possibility of surveying the phenomenon (see e.g., Section 3.7.4).

In addition to the relationships between the soil physical characteristics there are numerous geotechnical fields, where the determination and the use of empirical correlations is indispensable. Such are, e.g.—to mention only a few—the calculation of the shear strength and the deformation characteristics (and from these the bearing capacity, the expected settlements, etc.) from the parameters obtained by *probing*; the determination of the compression modulus by *pressiometer tests*; judging the risk of frost (frost heave) by the capillary-transferred water quantity, etc.

3.1. Nature of the relationship between the variables

As mentioned already in the introduction of this book, part of the natural and the artificial processes (phenomena) may be described by *causal*, others only by *stochastic* models. This applies—properly interpreted—also to the relationships existing between the variables (e.g., between the physical characteristics). In the first case all the components of the causal relationship are known, so the variables are in a *functional* (*deterministic*) relationship with each other. We may assign here, e.g., the relationship

$$\frac{Q}{t} = k \cdot A \frac{h}{l} \quad (3.1)$$

stating that the water quantity flowing through the sample of cross-section area A is proportional with the hydraulic gradient h/l (DARCY's law). The situation is different if the causes determining the relationship are unknown (or not all of them are known). In such cases we say that there is a *correlation*-, or a *regression*-type relationship between the variables.

In practice we cannot draw a sharp limit between the two types of relationships. The value pairs of the relationship $Q/t = f(h)$ defined by Eq. (3.1) will not fall exactly on a straight line, when we measure the water quantities by varying h . The reason is that we can measure h and Q/t only with a limited precision, with the consequence that both variables must be regarded as *random variables* (see Section 3.7.9). The reverse case may

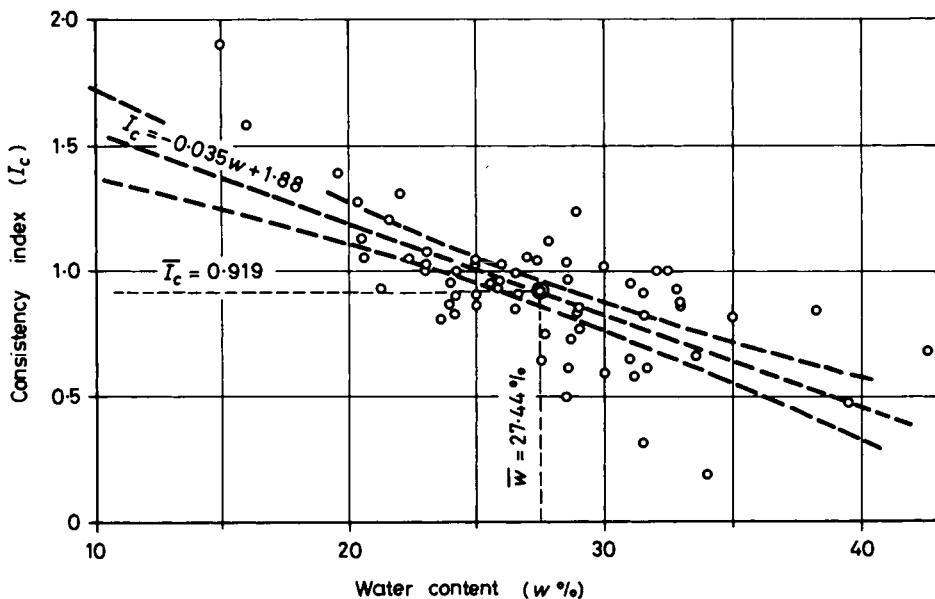


Fig. 3.1. Relationship between the water content and the consistency index of the soil S5

also occur, namely when under favourable circumstances we reveal by determining the correlations also their *physical causes*, or verify our preliminary assumption.

There exist physical characteristics which we define by a formula. For example, the consistency index is calculated by the expression

$$I_c = \frac{w_L - w}{w_L - w_p},$$

whose interpretation could be that there exists an “unequivocally” determined linear relationship between I_c and w . But in reality this is not a deterministic function, as w_L and w_p vary even within a layer qualified as homogeneous (quite apart from the random faults of the tests). Consequently, the two “constants” of the function $I_c = A - Bw$ vary from sample to sample, so we may speak also here of a correlation-type relationship (Fig. 3.1).

The requirement mentioned in connection with the distribution functions that the mathematical–statistical evaluation of the data must start from the *physical* background (or more generally from the known *causal* relationships) applies even more increasingly to the correlation (regression) analysis. By observing this principle the selection of the variables is facilitated and the risk of contradictory statements with the physical laws due to accidental circumstances is eliminated. The examination of the physical background indicates also the expectable closeness of the relationship between two variables. So, e.g., the physical basis of the relationship $w = f(w_L)$ is the fact that the more fine grains there are in the soil, the more water it can bind, as expressed both by the natural water content and the numerical value of the liquid level.

3.2. Conception and classification of the correlation

If we plot the measured value pairs x, y of two random variables (ξ and η) in a coordinate system, the points obtained are located around a curve. The function

$$y = M(\eta|\xi) = x \quad (3.2)$$

is called the regression of the random variable η related to ξ (the symbol $|$ refers to conditional probability). It may be proved that in the case that η and ξ are of a normal distribution, the regression of η is a straight line, whose equation is

$$\eta = m_2 + \rho \frac{\sigma_2}{\sigma_1} (\xi - m_1), \quad (3.3)$$

where $m_1 = M(\xi)$, $m_2 = M(\eta)$, $\sigma_1 = D(\xi)$, $\sigma_2 = D(\eta)$ and ρ is the so-called *correlation coefficient*. The expression (3.3) is called *regression line* and the quotient $\rho \cdot \sigma_2/\sigma_1$ *regression coefficient*.

The correlation coefficient may be calculated by the following expression:

$$\rho = \frac{M[(\xi - m_1)(\eta - m_2)]}{D(\xi) \cdot D(\eta)}, \quad (3.4)$$

where the numerator is the *covariance* (central mixed moment) of the variables ξ and η .

The determination of the theoretical regression is generally very complicated (often not even the type of the distribution is known), therefore, in practice, we are contented with finding the *best linear estimation* of the form

$$y = ax + b \quad (3.5)$$

of the variable η . The expression (3.3) is valid also in this case; its name—for distinguishing it from the theoretical regression—is *linear regression*. The constants of expression (3.5) may be determined as follows:

$$a = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad (3.6)$$

and

$$b = \bar{y} - a\bar{x}. \quad (3.7)$$

The *empirical correlation coefficient* r may be calculated by the following expression:

$$r = a \frac{s_1}{s_2} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}, \quad (3.8)$$

where s_1 and s_2 represent the empirical standard deviation of the variables x and y , respectively.

The same result is obtained if we start from the following expression:

$$r = \sqrt{a \cdot a'}, \quad (3.9)$$

where

$$a' = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (y_i - \bar{y})^2}. \quad (3.10)$$

(The sign of the correlation coefficient is identical with the sign of a).

Figure 3.2 illustrates the relationship between the directional tangents and the correlation coefficient. The straight line of the slope $a = \tan \alpha$ satisfies the condition

$$\sum [y_i - y(x)]^2 = \sum \Delta y_i^2 = \text{minimum},$$

and the straight line of the slope $a' = \tan \alpha'$ satisfies the condition

$$\sum [x_i - x(y)]^2 = \sum \Delta x_i^2 = \text{minimum}.$$

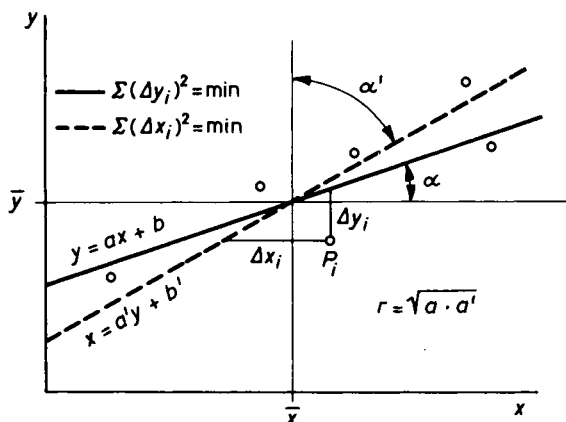


Fig. 3.2. Interpretation of the correlation coefficient

If $\alpha + \alpha' = 90^\circ$, then $a \cdot a' = 1$ and $r = 1$, which corresponds to a *function relationship*. If the lines are normal to each other, then $r = 0$; in this case the variables are—linearly—*uncorrelated*. So the correlation coefficient is the index of the closeness of the assumably linear correlation (see Section 3.3). The quantity

$$\sigma_f = \sqrt{\frac{\sum_{i=1}^n [(y_i - y(x))^2]}{n-2}}$$

is called the standard deviation of the function (or *residual standard deviation*) and may be calculated as follows:

$$\sigma_f = \sqrt{\frac{n-1}{n-2}} \sigma_y \sqrt{1-r^2}, \quad (3.11)$$

where $\sigma_y = \sigma_2$ is the original standard deviation of the dependent variable. (Here we have taken into account even the number of the degrees of freedom, with regard to the samples of finite elements.) So the correlation coefficient informs us also on how far the dispersion of y may be reduced by bringing it into correlation with another variable. If, e.g., $r = 0.8$, then

$$\frac{\sigma_f}{\sigma_y} = \sqrt{1-0.8^2} = 0.6,$$

which means that we have managed to reduce the original dispersion by 40%.

$\Sigma \Delta y/n$ is called *mean deviation* (its symbol is M. D.). It is falling increasingly into the background, partly because it fails to correspond to the basic principles of regression analysis, and partly because σ_f can be determined without any difficulty even by the simplest computers.

If any of the constants a and b are determined by the given physical conditions, we may proceed as follows.

If we fix the intercept ($b = b_0$), the regression coefficient may be determined by the expression

$$a = \frac{\sum x_i y_i - b_0 \sum x_i}{\sum x_i^2}, \quad (3.12)$$

while if we fix the regression coefficient ($a = a_0$), then the intercept may be determined by the expression

$$b = \frac{\sum y_i - a_0 \sum x_i}{n}. \quad (3.13)$$

Before deciding on fixing any constant, the physical background of the problem must be examined carefully (see Section 3.7.4).

On calculating the correlation of the 65 value pairs I_c, w of the soil S5 (see Fig. 3.1) the following partial results were obtained:

$$\begin{aligned} \bar{I}_c &= 0.919, & a &= -0.035, \\ \bar{w} &= 27.44, & a' &= -13.056, \\ \sigma_1 &= 0.266, & b &= 0.919 - (-0.035)27.44 = 1.88, \\ \sigma_2 &= 5.141, \end{aligned}$$

So the correlation coefficient is

$$r = \sqrt{(-0.035)(-13.056)} = -0.676$$

and the residual standard deviation is

$$\sigma_f = 0.266 \sqrt{1 - 0.676^2} = 0.196.$$

This means that in the knowledge of $\bar{w}$ we can estimate I_c with the help of the relationship

$$I_c = -0.035 w + 1.88$$

up to about 68% of the cases, with an error of $\Delta I_c < 0.2$.

In certain disciplines (e.g. in geodesy) the so-called *compensation calculation* is used widely even today. Its basic principle is identical with that of the regression analysis (based also on the method of least squares) and the straight line obtained as a result of this calculation is identical with the regression straight line. Its disadvantage is that it gives no information on the closeness of the relationship and the determination of the residual standard deviation requires much more time.

We distinguish between the following calculation modes of the regression (correlation).

a) If it is expected that the relationship between the variables is not described by a straight line (as concluded from the graphic image of the point pairs, or from earlier studies, or from known physical relationships), we have to do with the so-called *nonlinear (curvilinear) correlation*.

b) We may rely on *approximative processes* depending on the nature of the actual task, the requirements of the precision (or for a preliminary orientation).

c) If the number of the random variables is three or more, we speak of a *multivariate (multiple) correlation*.

d) If on examining two time-series we correlate not the simultaneous elements, but the elements shifted by 1, 2, . . . , k time-intervals, then we calculate a so-called *cross-correlation* (there is no literary evidence of its geotechnical utilization).

e) If the elements of a data series are functions of the time or the distance and we study their closeness with the preceding 1., 2., . . . , k -th element, the answer is given by the *autocorrelation* (-regression).

The processes a)–c) are discussed in the following Sections, the process e) is dealt with in Section 4.2.1. But there exists a variety of the relationships which must be treated here.

In numerous cases we study associations between variables which cannot be measured (e.g., the occurrence or non-occurrence of an event, the sign of a change, etc.). In this case the association, the closeness of the relationship may be characterized by the *squared contingency mean*. If only two events (two outputs) of both variables are possible, the requested quantity equals χ^2/n ; this index is similar to r^2 (zero means independence, 1 means a function relationship). The essence of the method is illustrated by the following example.

Let us assume that the subsoil of a residential quarter of single-floor buildings consists of fat, volume-changing clay. The buildings damaged are grouped in two groups, depending on whether the groundwater level is higher than $z=4$ m, resp. if there is a cellar under the building. Accordingly, the following table may be constructed:

z	Cellar	
	yes	no
< 4 m Total number of buildings	19	66
Number of damaged buildings	2	26
> 4 m Total number of buildings	75	42
Number of damaged buildings	0	2

If we mark the damaged state by (+), the sound state by (–), the following matrix may be written up for examining the role of the *groundwater*:

	(+)	(–)
$z < 4$	28	57
$z > 4$	2	115

from which

$$\frac{\chi^2}{n} = \frac{(28 \cdot 115 - 2 \cdot 57)^2}{(28 + 57) \cdot (2 + 115) \cdot (28 + 2) \cdot (57 + 115)} = 0.188$$

and $r=0.43$. The relationship between the presence or absence of a *cellar* and the damaged state is characterized by the following matrix:

	(+)	(-)
cellar present	2	92
cellar absent	28	80

For this relationship $r=0.33$, i.e. the association is looser than in the preceding case, the position of the groundwater level plays a *primary* role. Both effects can be studied together (*combined*) by assigning the buildings with no cellar and a groundwater level of $z < 4$ to group *A*, those with a cellar and $z > 4$ groundwater level to group *B*. In this case the matrix is as follows:

	(+)	(-)
<i>A</i>	26	40
<i>B</i>	0	75

The measure of the association increased ($r=0.51$), as it could be expected on the basis of the above said. The examination may be continued, e.g., by grouping the buildings according to the material of their walls, or depending on whether they were built with reinforced concrete cornices.

The role of the cellar is supported also by practical experiences. In the statistics of building damages in Hungary 26 buildings (resp. groups of buildings) are recorded which were damaged by the volume change of the soil and whose cellar conditions are also known (RÉTHÁTI, 1977a). Only two of these buildings had cellars, 24 had none, or only partial cellars. As in this case *no value pairs could be formed*, the association may be studied with the help of the *binomial* distribution. If we assume that the presence of a cellar plays no role (so $p=0.5$ and $c=0.5$), then according to Eq. (2.7) the probability that only two buildings with cellars fall to the first category is

$$P_k = \binom{26}{2} 0.5^2 \cdot 0.5^{24} = 4.8 \cdot 10^{-6}.$$

(If the ratio of the probabilities p and q are taken to be 1:3, the actual observation corresponds still to a probability of not more than $P_k=0.02$.)

The regression can be calculated not only with the assumption that

$$\Sigma [y_i - y(x)]^2 = \text{minimum}$$

but also on the basis of other conditions. RIZKALLAH and MASCHWITZ (1983) discuss the possibility of four such conditions, i.e.:

$$\frac{1}{n} \sum \left| \frac{y_i - y(x)}{y_i} \right| = \text{minimum}; \quad (3.14)$$

$$\max \left| \frac{y_i - y(x)}{y_i} \right| = \text{minimum}; \quad (3.15)$$

$$\frac{1}{n} \sum |y_i - y(x)| = \text{minimum}; \quad (3.16)$$

$$\max |y_i - y(x)| = \text{minimum}. \quad (3.17)$$

A “coefficient of optimization” belongs to each of the above conditions (w_2 , w_3 , w_4 and w_5); these can be defined as follows:

$$w_2 = \frac{\frac{1}{n} \sum \left| \frac{y_i - \bar{y}}{y_i} \right| - \frac{1}{n} \sum \left| \frac{y_i - y(x)}{y_i} \right|}{\frac{1}{n} \sum \left| \frac{y_i - \bar{y}}{y_i} \right|}, \quad (3.18)$$

$$w_3 = \frac{\max \left| \frac{y_i - \bar{y}}{y_i} \right| - \max \left| \frac{y_i - y(x)}{y_i} \right|}{\max \left| \frac{y_i - \bar{y}}{y_i} \right|}, \quad (3.19)$$

$$w_4 = \frac{\frac{1}{n} \sum |y_i - \bar{y}| - \frac{1}{n} \sum |y_i - y(x)|}{\frac{1}{n} \sum |y_i - \bar{y}|}, \quad (3.20)$$

$$w_5 = \frac{\max |y_i - \bar{y}| - \max |y_i - y(x)|}{\max (y_i - \bar{y})}. \quad (3.21)$$

The coefficients w_i correspond approximately to r^2 , their value is generally between 0 and +1. The calculation process is as follows (RIZKALLAH and MASCHWITZ, 1983). A possible basis for the optimization is the result of the ordinary regression analysis (step 1). By this step the coefficient of regression a_0 to a_k in the equation

$$y = a_0 + a_1 x_1 + \dots + a_k x_k$$

and r^2 are derived. Beside that the respective average and maximum relative error of estimation [Eqs (3.14) and (3.15)], as well as the average and the maximum absolute errors of estimation [Eqs (3.16) and (3.17)] are calculated. Depending on the aim of the

analysis these errors will be optimized iteratively in additional calculations (steps 2, 3, 4, or 5) using a comprehensive computer program. Each step of calculation will lead to respective coefficients a_0 to a_k of the preelected estimation formula and to the “coefficients of optimization” w_2 to w_5 .

3.3. Interpretation and reliability of the relationships

We have stated in Section 3.2 that the absolute value of the correlation coefficient measures the closeness of the correlation between two variables. This must be complemented by the statement that the low (or even zero) values of r do not refer *unconditionally* to the absence of a relationship between the two variables (i.e. that for example in the case of $r=0$ they are uncorrelated). If, e.g., the value pairs x_i, y_i are on the perimeter of a circle at identical distances from each other, the regression analysis according to Section 3.2 would refer to the complete absence of a correlation ($r=0$), though there exists a well-defined function between x and y . The reason for this apparent contradiction is that the correlation coefficient is primarily the *index of the degree of linearity* and indicates within this framework the closeness of the correlation.

The properties of PEARSON's *correlation quotient* (RÉNYI, 1954)

$$K(\eta|\xi) = \frac{D[M(\eta|\xi)]}{D(\eta)} \quad (3.22)$$

are better. The expression in the numerator may be interpreted as follows: Let us divide the range of the values ξ to intervals and let us determine the arithmetic mean of the η_i -values falling into the 1st, 2nd . . . , i -th intervals, i.e. the expected conditional $M(\eta|\xi)$ -values. By the deviations of the same from the expected value of $M(\eta)$ we may calculate the standard deviation, which corresponds numerically to the numerator of expression (3.22) exactly. It is easy to admit that the correlation quotient is not unconditionally zero, if $\rho=0$.

By plotting the value pairs in a coordinate system we may verify easily if the hypothesis of the linearity is realistic. On the basis of this we may decide also if we should follow the method described in Section 3.2, or rely on another process (see later).

On *interpreting* the content of the correlation we may have committed other errors of a different nature as well. A rather frequent case is the existence of a more or less close regression between two variables without any *physical basis* for it. This “pseudo-correlation” is illustrated by the following example from the field of geotechnics. In the range of the high plasticity soils q_u decreases gradually with the increase of I_p (see Fig. 5.10). The reason is not the *direct* decrease of the strength by the increased proportion of colloidal particles (on the contrary: the sticking effect is enhanced thereby), but the fact that the structure of the soil becomes ever looser with the increase of plasticity, as is shown unambiguously by another relationship (see Fig. 5.8).

This risk may be eliminated partly by clearing up the physical background and partly by correlating the physical characteristic with as many others as possible (see Section 3.5).

Let us return now to the characterization of the closeness of the correlation. The regression straight line is only one realization of the—possibly not even linear—correlation, characterizing the population. For gaining information on the position of the straight line, resp. the reliability of the constants a and b we must start from the dispersion of the latter.

The standard deviation of the regression coefficient is

$$s_a = \sqrt{\frac{s_f^2}{(n-1)s_x^2}}, \quad (3.23)$$

where s_f is the residual standard deviation defined by Eq. (3.11), n is the number of the point-pairs and s_x is the standard deviation of the independent variable.

The standard deviation of the intercept is

$$s_b = s_f \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{(n-1)s_x^2}}. \quad (3.24)$$

The standard deviation characterizing the dependent variable varies with the value of x according to the following relationship:

$$s_{y_i} = s_f \sqrt{\frac{1}{n} + \frac{(x - \bar{x})^2}{(n-1)s_x^2}}. \quad (3.25)$$

Accordingly the dispersion is minimal in the position $x = \bar{x} = s_f/\sqrt{n}$ and increases with the distance on receding from it, which refers to the risks of the extrapolation. If $x = 0$, the expression (3.25) gives the standard deviation of the intercept, i.e., it will be identical with the expression (3.24).

The formula permits to determine the band of the extension p of the regression straight line. The ordinate Y_i of the limiting curve may be calculated by the following expression:

$$Y_i = ax + b \pm \lambda s_{y_i}, \quad (3.26)$$

where λ is the value of the normal distribution belonging to the probability p . (if $n < 50$, it is advised to use the STUDENT-distribution.)

The probability band of the regression straight line of the relationship $I_c = f(w)$ discussed in Section 3.2 is shown in Fig 3.1. As in the case of a probability of $p = 0.95$

$$\Phi(\lambda) = 1.95/2 = 0.975$$

according to expression (2.16), to which $\lambda = 1.96$ belongs in Table F2, so, e.g., Y_i corresponding to $x = 15$ may be calculated as follows:

$$Y_i = -0.035 \cdot 15 + 1.88 \pm 1.96 \cdot 0.196 \sqrt{\frac{1}{65} + \frac{(x - 27.44)^2}{64 \cdot 5.141^2}},$$

which corresponds to 1.42, resp. 1.29.

The *reliability of the dependent variable* determined by the regression straight line may be characterized by the so-called confidence intervals (see Section 7.4). These intervals are limited by pairs of straight lines, which may be characterized by the equation

$$y = ax + b \pm \lambda s_f, \quad (3.27)$$

i.e. which are parallel with the regression straight line, where λ is the argument belonging to the probability p of normal distribution (or to the STUDENT-distribution, if the number of the elements is small).

If in the preceding numerical example we look, e.g., for the I_c -value belonging to $w = 20\%$, inferior values of which may occur only with a probability of 0.05, we must proceed as follows. As a one-tailed probability is involved, we must find the λ -value belonging to $p = 0.90$ (1.64 according to Table F2); the requested value is:

$$I_c = -0.035 \cdot 20 + 1.88 - 1.64 \cdot 0.196 = 0.86.$$

On solving technical tasks the question often arises whether the numerical value of the constants a and b , resp. of the correlation coefficient itself does not deviate accidentally from a value assumed by us (e.g. from zero). The process used for establishing this—similarly to that applied in statistical tests (Section 6.2.1)—is called *significance test* and is carried out as follows.

In the case of the *correlation coefficient* we examine usually if it differs significantly (i.e. not randomly) from zero. This may be established by the following three methods.

a) As the standard deviation of the empirical correlation coefficient is

$$s_r = \frac{1 - r^2}{\sqrt{n}} \quad (3.28)$$

the quotient

$$h = \frac{|r| \sqrt{n}}{1 - r^2} \quad (3.29)$$

expresses the multiple of the standard deviation, which must be added to (or deduced from) r in order to attain that it changes its sign. $h = 3$ may be regarded as a critical value, but the use of the stricter criterion $h \geq 4$ may also be justified.

b) If $n \geq 5$, the distribution of r follows the symmetric type II PEARSON-distribution; if $n \geq 50$, it follows the normal distribution with a good approximation. In the latter case the standardized variable t is

$$t = \frac{r \sqrt{n-2}}{\sqrt{1-r^2}}, \quad (3.30)$$

which must be compared with the function value of STUDENT's distribution.

c) If we have an estimated value (which may be also zero) of the correlation coefficient ρ of the population and the variables are normally distributed, then

according to FISHER we must perform the test with the following standardized variable:

$$t = (z - \bar{z})\sqrt{n-3}, \quad (3.31)$$

where

$$z = 0.5 \ln \frac{1+r}{1-r} \quad \text{and} \quad \bar{z} = 0.5 \ln \frac{1+\rho}{1-\rho}. \quad (3.32)$$

The subsequent course of the test is identical with the process described under *b*).

In our numerical example $r=0.676$ and $n=65$, so $h=10$, i.e. r differs very significantly from zero according to criterion (a). As the value of the standardized variable t is 7.281 according to expression (3.30), the probability that r deviates only randomly from zero is less than even 0.001 according to Table F4 (the number of the degrees of freedom is $65-1=64$). Now we use FISHER's process for determining the confidence intervals of $|r|$. On substituting $\rho=0.676$ into the second expression of (3.32) we obtain $\bar{z}=0.822$ and as $t=2$ belongs approximately to $p=0.05$, we have according to Eq. (3.31)

$$z_{1,2} = 0.822 \pm \frac{2}{\sqrt{62}} = 1.076 \quad \text{and} \quad 0.568.$$

On substituting these two values into the first equation of (3.32) we obtain $r_1=0.792$ and $r_2=0.514$. (The latter value refers again to the great distance of r from zero.)

The constants of the regression straight line may be examined also similarly. The standardized variable of the *regression coefficient* is

$$t = \frac{r(a-a_0)}{a} \sqrt{\frac{n-2}{1-r^2}} \quad (3.33)$$

and that of the intercept is

$$t = \frac{r(b-b_0)}{a} \sqrt{\frac{n-2}{(1-r^2)(s_x^2 + \bar{x}^2)}}, \quad (3.34)$$

where a_0 and b_0 are values assumed by us. We perform the significance test again with the help of the STUDENT table. By the formula given for the coefficient a we usually examine if the *trend* is not of a random nature and by the formula given for b , how realistic the assumption of the *direct proportionality* ($b_0=0$) of the variables is.

Perhaps it is not unnecessary to note that if we assign the observed (measured) data to *intervals* and perform the regression analysis for the means obtained in this way, the constants of the regression straight line stay unchanged, but the absolute value of the correlation coefficient will be higher. The "new" value of r may be misleading for the user, therefore, we must emphasize this circumstance by all means.

The reliability of the forecast (estimation) may be judged according to the suggestion of MINCER and ZARNOWITZ (1969) with the help of the following equation:

$$A_i = a_0 + a_1 F_1 \quad (3.35)$$

in which F_i is the estimate corresponding to the actual observation of A_i ; a_0 and a_1 are constants. The estimates are deemed accurate if, on regressing the actual observations on the corresponding estimates $a_0 \rightarrow 0$ and $a_1 \rightarrow 1$. The attainment of this may be controlled with the help of formulae (3.33) and (3.43).

3.4. Nonlinear correlation

As it was mentioned before, the nonlinear regression demands a very complex computation work. This may be avoided by linearizing the selected function by a suitable transformation and then calculating the correlation with the new variables as described in Section 3.2. Examples of a few possibilities of the transformation are given in Table 3.1.

Table 3.1. Transformation of nonlinear functions

Original function	Transformed function
$y = ax^b$	$\log y = \log a + b \log x$
$y = a \cdot \exp(-bx)$	$\ln y = \ln a - bx$
$y = a/x$	$\log y = \log a - \log x$
$y = \frac{a}{x} + b$	$y = a \cdot z + b; \quad z = 1/x$
$y = a \sin^b x$	$\log y = \log a + b \log \sin x$
$y = ax^2 + bx$	$z = ax + b; \quad z = y/x$
$y = \frac{a}{x} + bx$	$z = a + bv; \quad z = xy, \quad v = x^2$

The regression analysis of polynomes of the form

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

can be solved also very simply as a multivariable function obtained by the substitutions $X_1 = x, X_2 = x^2, \dots, X_n = x^n$ (see Section 3.5).

The closeness of the regression between the transformed variables is measured by the *correlation index* (I). This parameter is calculated in the same way as the correlation coefficient r of the linear regression, taken in a narrower sense. The difference of the names refers in the first place to the fact that the standard deviation

$$s_f = \sqrt{\frac{n-1}{n-2}} s_{y_i} \sqrt{1-I^2} \tag{3.36}$$

corresponding to Eq. (3.11) is not the residual standard deviation of the original variable but that of the transformed variable (y_i).

The selection of the suitable function form may be facilitated by the earlier determined theoretical or empirical *relationship*, but much is revealed also by the

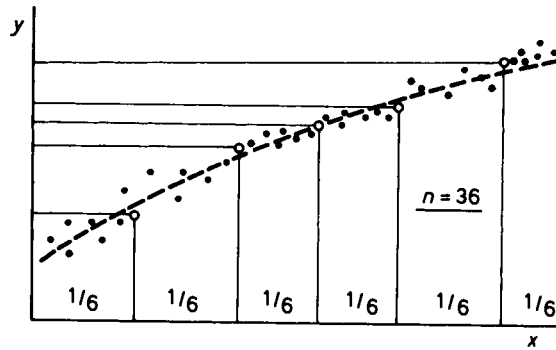


Fig. 3.3. Approximate determination of the equalizing straight line with the help of the quantiles of the empirical distribution

graphical plot of the value pairs. We may proceed also by determining the *quantiles* of the variables and forming the value pairs from them (REIMANN, 1975).

The “quantile” is the value of the variable, whose smaller values are assumed with a lower probability than a given $p\%$. Figure 3.3 shows how the value pairs formed from identical quantiles of the variables x and y give the appropriate form of the regression curve.

Columns 1 and 2 of Table 3.2 contain the w and q_u values determined from samples taken from large diameter boreholes of the soil S4. In the first approximation let us assume that the relationship between both physical characteristics is described by a straight line of the form

$$q_u = aw + b.$$

In this case the calculated result of the correlation according to Section 3.2 is

$$a = -11.56, \quad b = 489, \quad r = -0.423, \quad s_f = 70.0.$$

According to the experiments of JÁKY (1944) performed probably with remoulded soils

$$q_u = ae^{bw},$$

where e is the basis of the natural logarithm. Taking the logarithms of both sides of the equation:

$$\ln q_u = \ln a + bw.$$

So the transformed variables are in this case: $x = w$ and $y = \ln q_u$. According to the regression analysis

$$a = 1040, \quad b = -0.0693, \quad I = -0.398.$$

The residual standard deviation may be calculated by substituting the w values into the exponential expression (which may be described by the constants yet known) and

Table 3.2. Numerical example for calculating the nonlinear correlation

w	q_u	ae^{bw}	w	q_u	ae^{bw}	w	q_u	ae^{bw}	w	q_u	ae^{bw}
24.5	286	190.56	25.5	359	177.81	27.8	85	151.63	31.0	143	121.48
32.2	61	111.79	26.0	183	171.76	28.5	121	144.45	29.0	80	139.53
22.1	260	225.02	29.5	175	134.78	28.2	66	147.48	33.0	140	105.77
26.2	92	169.39	30.0	170	130.20	24.0	92	197.28	36.0	40	85.92
21.1	362	241.16	31.0	224	121.48	28.0	113	149.54	31.5	124	117.35
27.8	234	151.63	26.0	279	171.76	26.0	90	171.76	23.0	191	211.42
25.6	273	176.58	30.4	130	126.64	24.0	139	197.28	29.0	183	139.53
25.0	283	184.07	35.8	95	87.12	29.8	111	132.01	26.5	215	165.91
28.5	215	144.45	29.3	205	136.66	27.0	100	160.26	27.5	255	154.81
26.5	136	165.91	30.0	205	130.20	29.4	98	135.72	32.0	167	113.35
25.5	168	177.81	28.5	240	144.45	31.0	143	121.48	30.5	168	125.76
28.0	152	149.54	28.9	185	140.50	28.5	127	144.45	25.5	300	177.81
28.5	63	144.45	26.0	370	171.76	31.0	112	121.48	28.0	151	149.54
27.5	168	154.81	30.0	88	130.20	28.0	198	149.54	27.5	261	154.81
31.5	55	117.35	27.5	148	154.81	23.9	139	198.65	27.1	204	159.16
29.0	110	139.53	25.7	85	175.36	26.0	335	171.76	24.0	162	197.28
31.0	145	121.48	25.7	115	175.36	30.0	183	130.20	25.0	302	184.07
31.0	115	121.48	27.7	111	152.68	31.0	169	121.48	27.0	152	160.26
28.0	280	149.54	24.5	199	190.56	29.0	163	139.53	20.0	232	149.54
32.0	120	113.35	33.2	128	104.31	26.5	203	165.91	30.0	72	130.20
29.0	103	139.53	27.0	113	160.26	29.0	159	139.53	28.0	260	143.45
31.0	143	121.48	23.6	188	202.82	21.5	175	234.57	29.6	207	133.85
27.5	271	154.81	26.5	85	165.91	27.5	183	154.81	30.0	120	130.20
27.0	351	160.26	27.4	150	155.88	23.0	166	211.42	28.5	95	144.45
32.5	150	109.50	25.0	110	184.07	24.0	127	197.28	26.0	140	171.76
29.5	243	134.78	28.0	98	149.54	30.0	24	130.20	30.2	64	128.40

determining the ordinates y_{si} of the curve (column 3 of Table 3.2). The residual standard deviation is now:

$$s_f = \sqrt{\frac{\sum (y_{si} - y_i)^2}{n-2}} = \sqrt{\frac{(190.56 - 286)^2 + \dots + (128.4 - 64)^2}{102}} = 71.5.$$

Thirdly we examine the merit of the compensation by the power parabola of the form

$$q_u = aw^b.$$

The result of the calculation will be as follows:

$$a = 80,419, \quad b = -1.89, \quad r = -0.391$$

and the residual standard deviation is $s_f = 72.9$.

On comparing the three function types we may conclude that there is *no essential difference* between them. Similar results were obtained in earlier tests with samples taken from various parts of Hungary (RÉTHÁTI, 1968a).

In given cases also transformations of another character may be tried, e.g., by using the porosity instead of the void ratio and instead of the angle of friction its tangent, or cotangent as random variables (see Section 2.1.6).

3.5. Evaluation of multivariable relationships

For improving the reliability of the estimation there is still an obvious possibility beside the methods mentioned so far (increasing the number of the measurements, transforming the variables), namely testing two or more variables instead of only one. In this case the task may be performed by the so-called multivariable (multiple) regression analysis (Sections 3.5.1–3.5.2), or the factor analysis using the results of the latter (Section 3.5.3).

3.5.1. The multivariable regression

For finding a relationship among three or more variables we must determine the constants of the equation

$$x_1 = b_1 + a_{12}x_2 + \dots + a_{1n}x_n. \quad (3.37)$$

Writing up the equation in this form refers to the necessity of evaluating the relationship of the variables in *all* their *combinations*.

The constants of Eq. (3.37) for three variables may be determined as follows (PRÉKOPA, 1962).

In the first step we calculate the so-called *total correlation coefficients* r_{ik} similarly to the “simple” correlation coefficients defined in Section 3.2 for two variables. (The numerical value of the total correlation coefficient measures the strength of the *total* correlation between the variables marked by i and k .) In their knowledge we write up the so-called empirical correlation matrix of the form:

$$R = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}, \quad (3.38)$$

then we calculate its determinant $|R|$ and its R_{ik} subdeterminants, supplied with a sign.

The regression coefficients a_{ik} may be calculated by the expression

$$a_{ik} = - \frac{s_i}{s_k} \frac{R_{ik}}{R_{ii}} \quad (3.39)$$

and the additive constant b_i of the equation by

$$b_i = m_i - \sum_{k=1}^n a_{ik} m_k, \quad (3.40)$$

where m_i is the expected value of the variable x_i .

The following step is the calculation of the *direct correlation indexes* between the variables, the so-called *partial correlation coefficients* by the expression

$$\rho_{ik} = - \frac{R_{ik}}{R_{ii}R_{kk}}. \tag{3.41}$$

The index of the closeness of the regression, the so-called *multiple correlation coefficient* is determined by the expression

$$\rho_i = \sqrt{1 - \frac{|R|}{R_{ii}}} \tag{3.42}$$

and the residual standard deviation of the variable x_i by

$$s_{fi} = \sqrt{\frac{n-1}{n-2}} s_i \sqrt{1 - \rho_i^2}. \tag{3.43}$$

The following example shows the course of the calculation. The unconfined compression strength of the soil S5 has been determined by $n = 56$ large-diameter core samples ($q_u = x_1$); q_u decreases with the increasing water content ($w \equiv x_2$), as explained in Section 3.4. As q_u increases with the increasing ratio of the colloidal-size clay minerals (though this effect is partly compensated by the increase of the void ratio), a more or less close correlation may be expected between the unconfined compression

Table 3.3. Numerical example for calculating the terivariate correlation

q_u	w	I_p	q_u	w	I_p	q_u	w	I_p
296	22.3	32.4	118	27.6	17.6	139	39.5	24.4
98	23.7	26.0	97	31.6	13.7	195	31.5	37.3
381	21.2	26.0	94	26.5	26.6	72	31.5	12.1
284	25.0	40.7	65	26.7	26.3	159	29.5	8.5
354	31.0	16.9	109	24.0	26.9	159	20.5	41.1
55	27.5	7.2	106	24.2	16.7	175	23.0	40.9
143	31.0	54.5	81	28.7	17.7	247	20.0	43.7
160	33.0	45.8	242	32.1	34.7	262	28.5	23.6
283	27.0	42.6	148	25.5	26.8	255	20.5	29.3
207	19.5	6.7	219	24.2	39.5	175	23.5	44.1
203	29.0	28.8	148	25.8	28.5	223	21.5	32.2
127	33.5	27.8	196	27.3	38.3	166	29.0	40.0
136	29.0	26.0	95	26.5	25.6	112	34.0	29.1
303	25.0	16.1	121	31.2	20.1	147	35.5	23.4
205	28.9	47.7	390	26.0	34.8	77	27.5	4.3
260	38.2	38.0	454	28.5	49.7	51	28.5	15.0
136	23.0	34.6	215	29.0	38.0	160	32.8	32.4
113	24.1	11.0	295	26.0	41.1	207	30.0	34.2
144	25.0	17.0	290	25.0	42.4			

strength and the plasticity index ($I_p = x_3$). The numerical values of the variables selected in this way are contained in *Table 3.3*.

On correlating the variables by pairs we obtain the following total correlation coefficients:

$$r_{12} = -0.177, \quad r_{13} = 0.444 \quad \text{and} \quad r_{23} = -0.020,$$

and the correlation matrix is

$$R = \begin{vmatrix} 1 & -0.177 & 0.444 \\ -0.177 & 1 & -0.020 \\ 0.444 & -0.020 & 1 \end{vmatrix}.$$

The values of the subdeterminants are

$$R_{12} = -[(-0.177)1 - 0.444(-0.020)] = 0.1681,$$

$$R_{13} = +[(-0.177)(-0.020) - 0.444 \cdot 1] = -0.4405,$$

$$R_{23} = 0.0986.$$

The further values are:

$$R_{11} = 0.9996, \quad R_{22} = 0.8029 \quad \text{and} \quad R_{33} = 0.9687.$$

On expanding the determinant according to the first row we have:

$$|R| = 0.9996 - 0.177 \cdot 0.1681 - 0.444 \cdot 0.4405 = 0.7743.$$

The constants of the equation are:

$$a_{12} = -\frac{90.20}{4.41} \frac{0.1681}{0.9996} = -3.44,$$

$$a_{13} = -\frac{90.20}{12.01} \frac{-0.4401}{0.996} = 3.31 \quad \text{and}$$

$$b_1 = 184.9 + 3.44 \cdot 27.5 - 3.31 \cdot 29.04 = 183.4,$$

so the equation of the requested regression plane is

$$q_u = 183.4 - 3.44w + 3.31I_p.$$

The partial correlation coefficients are:

$$\rho_{12} = -\frac{0.1681}{\sqrt{0.9996 \cdot 0.8029}} = -0.188,$$

$$\rho_{13} = 0.448,$$

$$\rho_{23} = -0.066.$$

The multiple correlation coefficient related to the variable x_1 is

$$\rho_1 = \sqrt{1 - \frac{0.7743}{0.9996}} = 0.475.$$

The residual standard deviation calculated with the help of the latter is

$$s_{f1} = \sqrt{\frac{55}{54}} 90.2 \sqrt{1 - 0.475^2} = 80.1.$$

From the numerical example the following conclusions may be drawn:

a) The multiple correlation coefficient ($\rho_1 = 0.475$) exceeds hardly at all the total correlation coefficient characterizing the relationship of the variables x_1 and x_2 ($r_{13} = 0.444$) due to the very loose correlation of the new variable (w) with q_u ($r_{12} = -0.177$).

b) For the same reason the residual standard deviation (80.1) is hardly lower than the value which may be calculated from correlation of the two variables q_u and I_p (81.6).

The result of the calculations is to be plotted purposefully in a way that the variable of the lower weight should be the parameter (Fig. 3.4).

In the general case the role of the new (third) variable may be evaluated as follows: if the independent variables x_2 and x_3 are uncorrelated ($r_{23} = 0$), the correlation matrix will be

$$R = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & 0 \\ r_{13} & 0 & 1 \end{vmatrix}.$$

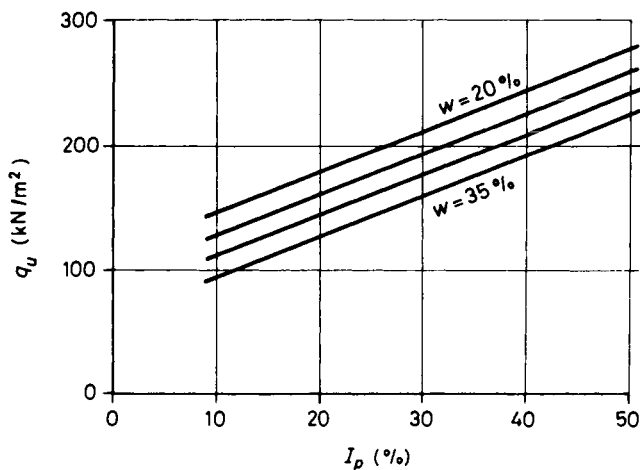


Fig. 3.4. The relationship $q_u(I_p, w)$ determined for the soil S5 by terivariate regression

The subdeterminants are $R_{11}=1$, $R_{22}=1-r_{13}^2$, $R_{33}=1-r_{12}^2$, $R_{12}=-r_{12}$, $R_{13}=-r_{13}$ and $R_{23}=r_{12}r_{13}$; the determinant is $|R|=1-r_{12}^2-r_{13}^2$, so the multiple correlation coefficient is

$$\rho_1 = 1 - \frac{1-r_{12}^2-r_{13}^2}{1} = r_{12}^2 + r_{13}^2.$$

If there is a function-like linear relationship between x_2 and x_3 ($r_{23}=1$), then $r_{12}=r_{13}$ and

$$\rho_1 = r_{12} = r_{13},$$

i.e. the reliability of the estimation is not improved by the introduction of the new variable.

The presence of multicollinearity in the explanatory variables causes the estimates from the ordinary regression analysis to become too large in absolute values and some of these estimates may even have the wrong sign. Such variables tend to represent a similar effect on the dependent variable and are prone to shave the regression coefficient for this effect; this makes it difficult to determine the influence attributable to each explanatory variable (KOPPULA, 1983). These disadvantages may be eliminated by the Ridge regression (HOERL and KENNARD, 1970).

3.5.2. Correlation matrices of soil physical characteristics

If given soil layers are investigated by several laboratory tests, the closeness of the relationships between the physical characteristics may be determined for all the relations. The calculation results may be expressed most clearly by writing up the correlation coefficients in matrix form.

Table 3.4 shows the matrix of the soil *S1*. The individual r -values (whose 100-fold values are given in the Table) are the results of bivariate linear regression analysis performed according to Section 3.2. The values *deviating significantly from zero* ($h \geq 3$)

Table 3.4. Correlation matrix of the soil *S1*

	w	w_L	w_p	I_p	I_c	e	S_r	γ_n
w	100	61	76	33	-23	85	-4	-44
w_L	61	100	77	89	45	39	4	-5
w_p	76	77	100	39	36	40	14	-9
I_p	33	89	39	100	39	29	-2	-2
I_c	-23	45	36	39	100	-36	5	-30
e	85	39	40	29	-36	100	-29	-58
S_r	-4	4	14	-2	5	-29	100	34
γ_n	-44	-5	-9	-2	30	-58	34	100
Mean*	47	46	42	33	30	45	13	26

* Mean of the absolute values, without the values 100.

according to Eq. (3.29) appear in bold print. According to the table the closest correlation was obtained for $w_L(I_p)$ defined by CASAGRANDE, while surprisingly the value of r for the relation $w_p(I_p)$ was only 0.39. As the soil was sampled from below groundwater level, S_r is practically independent of all the other variables and for the same reason the w, e correlation is rather close ($r=0.85$). As the examined covering layer consists of cohesive soils over a great part of the town, the water content is in a relatively close correlation with the consistency limits ($r=0.61$ and 0.76).

Table 3.5. Correlation matrix of the soil S2 (above the diagonal) and of the soil S3 (below the diagonal)

	w	w_L	w_p	I_p	I_c	e	S_r	γ_n	Mean*
w	100	6	36	-13	- 59	72	62	-33	40
w_L	26	100	8	86	20	1	0	14	19
w_p	35	33	100	- 44	33	0	22	3	21
I_p	7	87	-12	100	2	1	-12	11	24
I_c	- 69	29	11	24	100	- 55	38	26	33
e	65	5	0	5	- 52	100	-4	- 85	31
S_r	46	17	18	9	- 30	- 26	100	46	26
γ_n	- 11	8	-3	10	10	- 44	33	100	31
q_u	-24	-14	15	-10	23	-13	-33	-26	
Mean**	37	29	16	22	32	28	26	17	

* Mean of the $|r|$ values obtained for the soil S2.

** Mean of the $|r|$ values obtained for the soil S3, without q_u .

For saving place and for surveying the varieties of the same layer below and above groundwater level better, we may include both in a single matrix, as shown in Table 3.5 prepared for the infusion loess of Szeged, i.e. for the soils S2 and S3. On the basis of this table it may be established that

- the natural water content is here also closer correlated with w_p from among both consistency limits, but the numerical value of r is considerably lower;
- the correlation w, S_r is here closer, because the soil is considerably more homogeneous, whereby the standard deviation of γ_s and e is significantly lower than in the case of the covering layer (e.g. for e we obtained $s=0.092$, as compared to $s=0.225$ obtained for the Holocene layer);
- in both layers the mean of the r -values is highest for the *water content* from among all the various relations, verifying PECK's statement that the water content is one of the most characteristic index-properties of the soil.

Numerous correlation matrices were published in the paper of HOLTZ and KRIZEK (1971). If we select from them those containing also the physical characteristics appearing in Tables 3.4–3.5, we obtain the data given in Table 3.6. [The authors studied instead of the void ratio the dry bulk density, which is proportional—as it is known—

Table 3.6. Means (above the diagonal), resp. absolute maximum values (below the diagonal) of the correlation coefficients obtained for California soils (HOLTZ and KRIZEK, 1971)

	w	w_L	w_p	I_p	γ_d	q_u
w		68	63	58	-91	-13
w_L	91		79	90	-50	16
w_p	94	92		56		9
I_p	72	96	56		-43	38
γ_d	-96	-60		-55		13
q_u	-54	47	26	49	25	

Table 3.7. Relationship between the density and the shear strength parameters, as well as the grain size distribution characteristics for two California soils (HOLTZ and KRIZEK, 1971): G — gravel content (%), S' — sand content (%), No. 200 — sieve residue

	G	S'	No. 200	w_{opt}	γ_{dmax}	ϕ'	c'
G	100	-84	-97	-58	73	44	-13
S'	-95	100	69	31	-54	-55	40
No. 200	-97	85	100	64	-75	-35	0
w_{opt}	-84	70	89	100	-96	-47	-13
γ_{dmax}	81	-65	-88	-97	100	58	0
ϕ'	72	-63	-74	-79	82	100	-49
c'	37	-38	-34	-37	25	10	100

with $(1 + e)$ within the same layer.] From the table it may be established that the correlation coefficients obtained for the soils S2 and S3 are generally lower than the values of Table 3.6, which may be explained by the *larger sampling area* of the Szeged-samples.

The correlation matrix—as generally the regression—may be used for calculating physical characteristics—which can be determined otherwise by *time- and cost-demanding tests*—from simple identifying or state characteristics. An example for it is shown in Table 3.7. The tests were carried out in the designing phase of large *earthworks* with the aim of estimating from the characteristics of the grain-size distribution curve (from the gravel and sand contents, the sieve residues) the optimum water content, the maximum dry bulk density and the parameters of the shear strength.

3.5.3. Factor analysis

The correlation matrices presented in the preceding section prove that each soil physical characteristic is in a looser, or closer correlation with the others. This means that often even 6–8 independent variables can be assigned to a dependent variable. Obviously, we would not proceed correctly in doing so, because the *weak* relationships

cannot be expected to increase the closeness of the correlation, and on the other hand the correlation conceived in this way would be subject to *all the faults* of the experimental series.

The number of the *characteristic* components contained in the system of the physical characteristics may be established with the help of the so-called *component analysis*. In the first step we must write up the correlation matrix, then determine the *eigenvalues* (whose number is identical with that of the variables). By relating these to each other the number of the “main components” in the system can be established. Let us assume, e.g., that the correlation between the variables is characterized by the following matrix, completed by the inscription of λ (AGTERBERG, 1974):

$$R = \begin{vmatrix} 1 - \lambda & 0.32 & 0.41 & -0.31 \\ 0.32 & 1 - \lambda & 0.68 & -0.38 \\ 0.41 & 0.68 & 1 - \lambda & -0.36 \\ 0.31 & 0.38 & -0.36 & 1 - \lambda \end{vmatrix}.$$

After expanding the determinant and solving the obtained fourth-degree equation the four roots will be:

$$\lambda_1 = 2.26; \quad \lambda_2 = 0.74; \quad \lambda_3 = 0.70 \quad \text{and} \quad \lambda_4 = 0.31.$$

The first main component represents $100 \cdot 2.26/4 = 55\%$ of the total variance of the system, so the role of the rest is relatively inferior.

Factor analysis goes a step further. The process is based on the assumption that the measured variables are backed by factors which are free of any redundancy and the measured values are linear combinations of the same. Factor analysis screens out the parallelities between the variables and supplies a predicting equation which contains only the highest weighted independent variables.

The use of factor analysis in geotechnics is still in the initial phase, as is testified by the literature (BIAREZ and FAVRE, 1977; COROTIS *et al.*, 1975). This is due partly to the very lengthy computation work in the case of four or more variables (i.e. just where it is justified) and partly to the fact that the majority of the tasks can be solved well also by the multivariable correlation (KÁROLYI, 1975). GHINELLI and VANNUCCI (1983) state that factor analysis is of a great help in the geotechnical study of very large areas because it allows a thorough examination, storage and interpretation of a great quantity of data of various origins and significance, reducing it to a very compact form, without losing essential information. It also allows the simplification of procedures for estimating geotechnical characteristics of any kind of soil stratum, determining the internal relationships and reducing the influence of anomalies and causes of error.

3.6. Approximative regression analysis

With the perfection and the rapid spreading of the computers the application range of the approximative processes is gradually decreasing, yet they may still be useful for a *preliminary orientation* (estimation), or for observing phenomena which are “covered up” by the computer-aided process (the aim of the preliminary orientation may be among others the selection of variables supplying the best correlation from among those possible variables).

3.6.1. Numerical methods

The best known numerical process is *rank correlation*. The dependent variables ($y_1, \dots, y_n$) are arranged by their magnitude and given serial numbers ($1 \dots n$), to which the similarly established rank numbers of the independent variables are written. On deducing the first rank number from the second one, the second from the third, etc. the sum of the absolute values of the differences will be $n - 1$ for the dependent variables, but a greater number for the independent variables, because it is not a function relationship what is involved. It may be proved that the maximum value of the difference $P = S_1 - (n - 1)$ is for even, resp. odd n is

$$P_{\max} = \frac{n(n-2)}{2} \quad \text{and} \quad P_{\max} = \frac{n(n-2)-1}{2}. \quad (3.44)$$

In the full absence of a relationship $P = P_{\max}$, while in the case of a function relationship $P = 0$, so it is advised to characterize the closeness of the relationship by the following parameter:

$$r_B = 1 - \frac{P}{P_{\max}}. \quad (3.45)$$

The characteristic parameter of the *sign correlation* is

$$r_s = \frac{p-q}{p+q}, \quad (3.46)$$

where p and q are the number of the positive, resp. negative products of the deviations. Naturally, it is not necessary to determine the numerical values of the products $(x_i - \bar{x}) \cdot (y_i - \bar{y})$, only their sign, i.e. only two deductions for each value pair.

If the two-dimensional common distribution of ξ and η (resp. x and y) is normal, the correlation coefficient may be estimated as follows. In the first step we determine the quantiles $Q_{0.5}$ (belonging to $p = 0.5$) of the x_i - and y_i -values, then we divide by them the set of the points to four ranges (Fig. 3.5). By counting the number of the points falling into the range “A” we determine the empirical probability $P(A)$ by dividing the number of the points falling here by the number of the sample elements. After evaluating the quantity

$$\alpha = 4P(A) - 1, \quad (3.47)$$

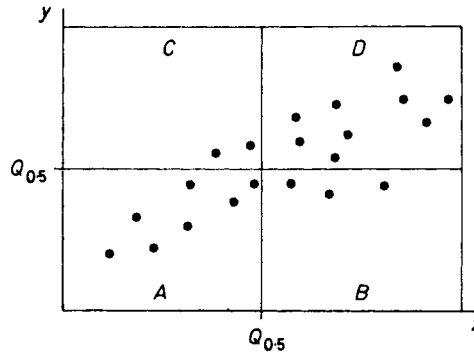


Fig. 3.5. Estimation of the correlation coefficient from the probability $P(A)$ belonging to the quantile $Q_{0.5}$

the correlation coefficient may be estimated from the following expression:

$$r = \sin \frac{\pi}{2} \alpha. \quad (3.48)$$

It may be easily admitted that

a) if $P(A)=0.25$, then $P(C)$ and $P(B)$ and so also $P(D)$ are 0.25, thus the value $\alpha=r=0$ reflects the reality well;

b) if $P(A)=0.5$, then a truly close relationship is involved, because $P(B)=0$ and $P(C)=0$, so the points fall only into the ranges A and D .

The *periodicity* may be tested by two additional numerical processes, namely the method of “median crossing” and that of “turning points”.

3.6.2. Graphical methods

The advantages of the graphical methods are as follows:

1. the direct inspection of the set of points may call attention to relationships (or anomalies, as described in Sections 3.7.1 and 3.7.4) which would not be noticed during the numerical calculation;
2. we are not bound to the known functions.

If a correlation between *two* variables is involved, the construction of the regression straight line is facilitated by the fact that it must pass through the points $\bar{x}$, $\bar{y}$. Trials with several variants may be made as well, accepting finally the one for which the graphically determined residual standard deviation is minimum. In the knowledge of the latter the approximate value of the correlation coefficient may be calculated by Eq. (3.11). If the straight line gives no satisfactory result, we may go over to another—e.g. single, or

double logarithmic—coordinate system, or we may try to approximate the set of the points by a curve (assisted also by the quantile method mentioned in Section 3.4).

In the case of *three or more variables* the process is naturally more complex. The best-known methods may be summed up as follows (ZSUFFA, 1963):

a) In the case of *three* variables we determine the relationships $y_1 = f(x_1)$ and $y_2 = f(x_2)$ independently. The requested relationship in the first approximation is

$$y = C_1 f(x_1) + C_2 f(x_2); \quad (3.49)$$

the proportionality factors C_1 and C_2 may be determined by error calculation.

b) In the case of *four or more* variables one of the following solutions may be applied.

1. If the independent variables can be ranked on the basis of the assumed closeness of their correlation with the dependent variable, it is advised to use the method proposed by EZEKIEL (*deviation method*) (SZESZTAY, 1954).

2. If the weights are nearly identical, the decomposition to tervariant relationships may lead to the best result; the final form of the requested relationship is:

$$y = \frac{p_1 y_1 + p_2 y_2 + \dots + p_n y_n}{p_1 + p_2 + \dots + p_n} \quad (3.50)$$

(here, too, the weights p_i must be determined by error calculation).

3. In numerous cases the so-called coaxial method may be useful (SZESZTAY, 1954).

3.7. Examples

The discipline of soil mechanics looks back on a past of six decades. During this time numerous *empirical formulae* were conceived, mainly in the field of soil physics. Most of them were results of simple compensation calculations, but in the past 10–15 years ever more relationships satisfying even the modern requirements by utilizing the advantages offered by the regression analysis are met with. The material accumulated in this field is so extensive that even its abstracted description is impossible in the framework of this book. The following examples serve only to give an idea about how and for what purposes regression analysis can be used.

3.7.1. Demonstration of the effect of groundwater moving in marl

PAÁL (1975) found on evaluating the test results of over 100 samples that the *consistency index* of the Buda Marl was a function of the depth z measured from the terrain, expressed as

$$I_c = 0.01z + 1.18.$$

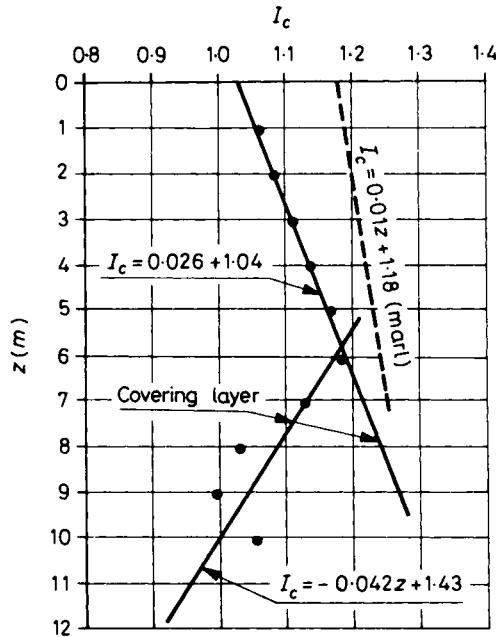


Fig. 3.6. Variation of the consistency index of the marl of Buda and its covering layer versus z (PAÁL, 1975)

The improvement of the consistency with the depth may be demonstrated unequivocally also in the case of the stand-up yellow and grey Kiscell Clay. This trend is similar also in the case of the *covering layers*, but only up to a certain depth z . The equation of the regression straight line applying to the covering layer of the marl between $z=0$ and $z=6$ m is

$$I_c = 0.026z + 1.04,$$

while for $z > 6$ m it is

$$I_c = -0.42z + 1.43,$$

i.e. from this depth downwards the consistency deteriorates rapidly (Fig. 3.6).

PAÁL (1975) explains this phenomenon as follows:

- a) in rock-like soils lying in their original position the water can flow only in smaller veins and ducts, so it can cause no deterioration in the state of the soil;
- b) in layers of a secondary position (generally close to the bedrock) a comprehensive gravitational mass of water is present, due to the looser frame structure.

3.7.2. Testing cohesive soils in the oedometer

TERZAGHI constructed the oedometer for determining the deformation characteristics of soft soils, but it is used even today frequently for testing *structured* and *hard, brittle* soils. Figure 3.7 points to a severe error source due to this practice. Accordingly, the relationship $E_s(I_c)$ may be described by two regression straight lines joining each other with a sharp break, so that for values of $I_c > 1.4$ the regression coefficient is *negative* (PAÁL, 1975), referring to the fact that the frame structure of the soil is ruined both by the sampling from the borehole (see Section 1.3.4) and by pressing the ring of the oedometer into the sample. (The relatively low values obtained for the compression modulus prove that the test results cannot be regarded as realistic even in the range of $I_c > 1.4$.)

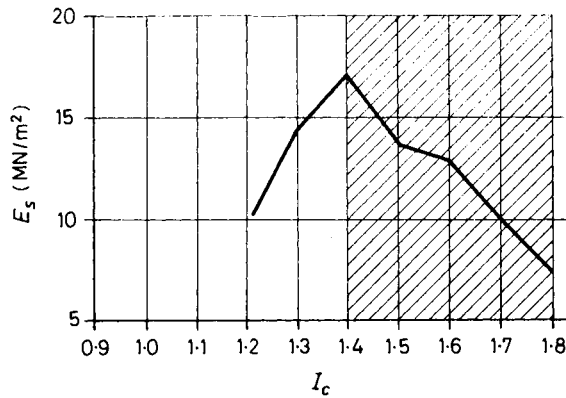


Fig. 3.7. Relationship between the consistency index of the marl of Buda and its modulus of compressibility determined in the laboratory (PAÁL, 1975)

3.7.3. Analytical description of PROCTOR's curve

The standard laboratory compaction test gives only 4–5 interrelated w, γ_d values, so the PROCTOR-curve is constructed often rather arbitrarily. An essentially better result could be expected from the evaluation if we knew the analytical form of the curve.

In the range $w < w_{\text{opt}}$ γ_d increases monotonously, then from here on it decreases monotonously, so that its form will be similar to the limiting curve, which is characterized by the equation

$$\gamma_d = \frac{\gamma_s}{1 + w\gamma_s}$$

(Fig. 3.8). The knowledge of the range around the extreme value, and within it, primarily the determination of w_{opt} and $\gamma_{d\text{max}}$, is of an outstanding importance.

BORUS *et al.* (1969) asked three different laboratories to make compaction tests on two different soils. The six experimental series supplied 20 value pairs each. They

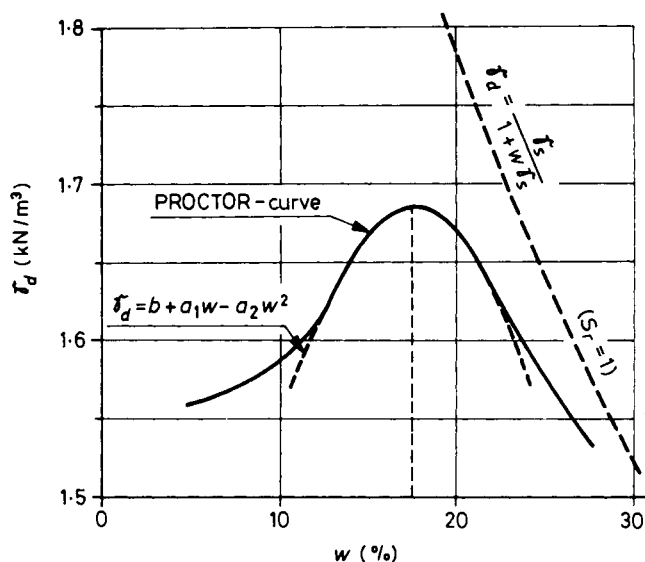


Fig. 3.8. Description of the central section of the PROCTOR-curve by a quadratic parabole

Table 3.8. Description of the PROCTOR-curve by a quadratic parabola (BORUS *et al.*, 1969)

Symbol of		Equation of the regression curve	Correlation index (<i>I</i>)	w_{opt}	$\gamma_{d\text{max}}$
soil	laboratory				
HL	I.	$\gamma_d = 1.46 + 0.0919w - 0.00446w^2$	0.919	10.3	1.932
	II.	$\gamma_d = 1.57 + 0.0671w - 0.00360w^2$	0.715	10.0	1.904
	III.	$\gamma_d = 1.28 + 0.1172w - 0.00550w^2$	0.966	10.6	1.904
A	I.	$\gamma_d = 1.61 + 0.0548w - 0.00251w^2$	0.851	10.9	1.912
	II.	$\gamma_d = 1.62 + 0.0482w - 0.00225w^2$	0.779	10.7	1.893
	III.	$\gamma_d = 1.38 + 0.0865w - 0.00352w^2$	0.967	12.3	1.908

approximated the central section of the PROCTOR-curve by a mixed quadratic function and found that the measure of the fitting was satisfactory (Table 3.8). They observed also that the dispersion was greater in the ascending arm and explained it by the friction of the grains against grains in this range, while at the same time the rammer compacts the layer lying under it, but loosens the layers around it.

3.7.4. Evaluation of the test with variable water pressure

On performing this test the coefficient of permeability of the soil may be determined by the following formula:

$$k = \frac{al}{A} \cdot \frac{\ln(h_0/h_i)}{t_i - t_0},$$

where a — cross-sectional area of the tube,
 A — cross-sectional area of the sample,
 l — length of the sample,
 h_0 — height of the water level (measured from the lower discharge level)
 belonging to the time moment $t = t_0$,
 h_i — the same at the time moment t_i .

The test is performed by reading off the difference h_0 between the upper and the lower water levels at the starting time moment (t_0), then repeating the measurement (the reading) at various t_i time moments. The k coefficient can be determined even from two h_i , t_i value pairs, but usually 4–5 readings are made at least to improve the precision.

The random errors may be eliminated by plotting the readings in the coordinate system $(t_i - t_0), \ln h_0/h_i$; the tangent of the compensation straight line gives namely just the second term on the right side of the formula. The question is now: how to perform the compensation. Let us assume that the value pairs seen in *Fig. 3.9* were obtained. The compensation (regression) straight line may be determined by one of the following three processes.

1. We fix the point 0,0 and determine the regression straight line with the help of Eq. (3.12) ($b_0 = 0$). The result of the calculation is:

$$\ln(h_0/h_i) = 0.008498(t_i - t_0).$$

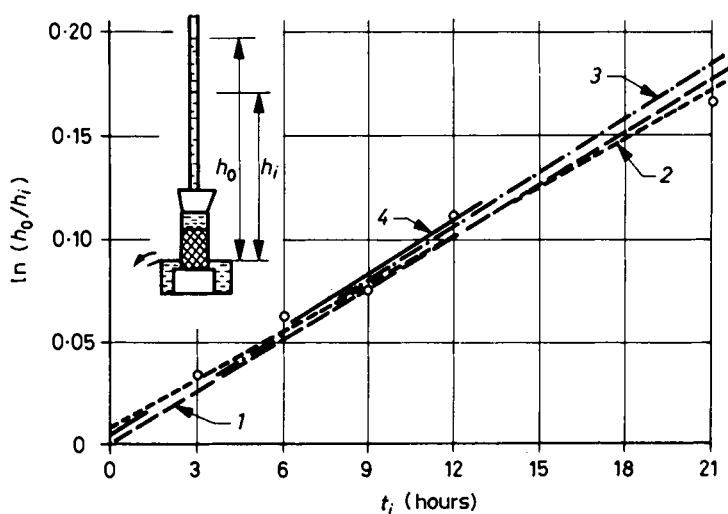


Fig. 3.9. Interpretation of the test with varying water pressure; 1 — with fixing the point 0,0; 2 — with the help of all the readings; 3 — with fixing the points 0,0 and $\bar{x}, \bar{y}$; 4 — with the omission of the value belonging to 21 h

The square sum of the deviations and the residual standard deviation is (as we have estimated a single parameter, the number of the degrees of freedom is $n - 1$):

$$\Sigma \Delta y_i^2 = 0.0004518 \quad \text{and} \quad s_f = 0.0095.$$

2. We do not fix the point 0, 0 (so we obtain $n = 6$ value pairs) and carry out the regression analysis with this assumption. The result of the calculation in this case is:

$$\ln(h_0/h_i) = 0.007867(t_i - t_0) + 0.0088,$$

$$r = 0.992 \quad \text{and} \quad s_f = 0.0083.$$

3. As the regression straight line must pass through the point $\bar{x} = 8.5$, $\bar{y} = 0.07567$, while at the same time $h_i = h_0$ belongs to $t_0 = 0$, from the coordinates of these points we have

$$\ln(h_0/h_i) = 0.07567/8.5 = 0.0089 \quad \text{and}$$

$$s_f = 0.0107.$$

On evaluating the results we must start from the fact that also *the reading of h_0 is not error-free*, so we cannot fix the 0, 0 point (1st method). Fixing two points (3rd method) is still less justified, because this process is contradicting the spirit of expression (3.12), as referred to also by the increase of the residual standard deviation. So the *2nd method* is left, which may be refined by omitting the reading belonging to $t_i - t_0 = 21$ h (if this can be justified, e.g., by the cooling of the laboratory room temperature at night). In this case

$$\ln(h_0/h_i) = 0.008867(t_i - t_0) + 0.004,$$

$$s_f = 0.0068$$

and the coefficient of permeability is

$$k = 3.2 \cdot 10^{-3} \text{ cm/h}, \quad \text{i.e.} \quad 8.8 \cdot 10^{-7} \text{ cm/s.}$$

3.7.5. Correlation between the consistency limits, the organic substance and the lime content

In the late fifties we have started an experimental series for determining the empirical correlation between the organic content and the liquid limit, resp. the plastic limit (RÉTHÁTI, 1966, 1968a). From the samples received from *various parts of Hungary* the laboratory determined first the liquid limit and the plastic limit, then from the little cylinders prepared for the latter test it determined the organic content (in the DENNSTEDT-furnace) and the CaCO_3 -content (by SCHEIBLER's method). From the 311 samples only 143 samples, resp. tests are evaluated in the following (as the lime content has not been determined in every case).

It has been known for a long time that the organic components raise not alone the water content of the soil, but also its liquid limit and its plastic limit. Similar

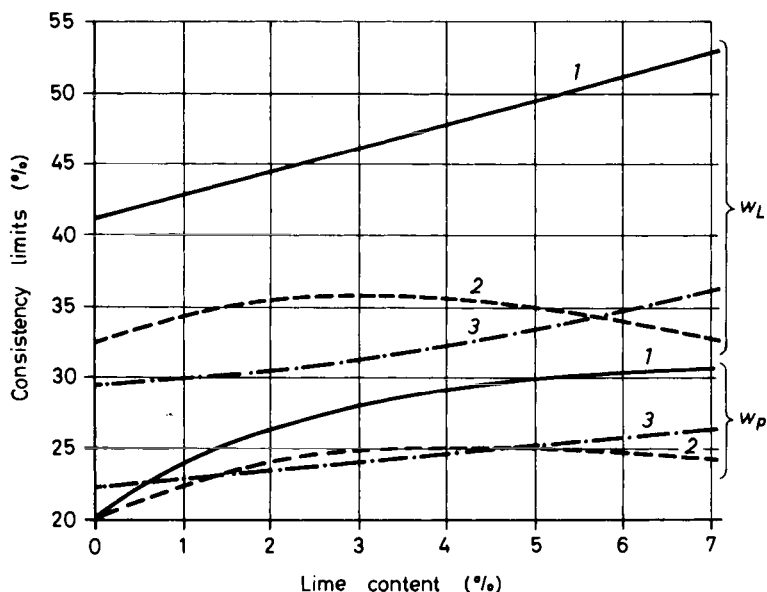


Fig. 3.10. Variation of the liquid limit and the plastic limit with the lime content according to FREUDENBERG (1960); 1 — Elbe Silt; 2 — black soil; 3 — loess

observations were made also in connection with the CaCO_3 -content (Fig. 3.10), but the relationship is not so unequivocal by far. In addition we had to study the role of the lime content also because the thermal decomposition of the carbonates occurs at various temperature (depending on the type of the carbonate and its interaction with the organic components), so the distribution of the CO_2 collected in the apparatus cannot be determined by its origin unambiguously.

Leaving the liquid limit unconsidered the following correlation matrix is obtained for the other three variables ($I_{om}=1$, $w_p=2$, $\text{CaCO}_3=3$):

$$R = \begin{vmatrix} 1 & 0.797 & 0.028 \\ 0.797 & 1 & 0.011 \\ 0.028 & 0.011 & 1 \end{vmatrix}.$$

Accordingly, the lime content is very loosely correlated to both other variables, so the multiple correlation coefficient (ρ_1) is practically identical with the correlation coefficient of the bivariate I_{om} , w_p correlation ($r_{12}=0.797$). As the standard deviation of I_{om} is $s=5.368$, the residual standard deviation is $s_f=3.26$.

Introducing the liquid limit is of no sense after all this, with regard to the very significant closeness of the w_L , w_p correlation ($r=0.87$).

Table 3.9. Determination of the organic content from the liquid limit and the plastic limit

Variables	CaCO ₃ (%)	Regression straight line	I_{om}	$\bar{x}$	r	s_f
I_{om}, w_L	0–56.5	$I_{om}=0.133w_L-2.62$	5.18	58.7	0.751	3.56
	< 2.8	$I_{om}=0.157w_L-4.80$	5.05	62.9	0.803	3.88
	≥ 2.8	$I_{om}=0.106w_L-0.45$	5.31	54.5	0.701	2.92
I_{om}, w_p	0–56.5	$I_{om}=0.253w_p-2.59$	5.18	30.8	0.797	3.26
	< 2.8	$I_{om}=0.335w_p-4.88$	5.05	29.6	0.851	3.42
	≥ 2.8	$I_{om}=0.182w_p-0.49$	5.31	31.8	0.784	2.54

The effect of the lime content may be “magnified” for the examination by dividing the test material into two parts by the median of the CaCO₃-content (2.8%). According to Table 3.9 the two categories obtained in this way show a rather great difference which may be utilized by estimating the value of I_{om} for the soils containing more lime than 2.8% from the new relationship. (In this way the residual standard deviation of the original relationship may be reduced by 18% in the case of w_L and by 22% in the case of w_p , according to the Table.)

The process applied here may be used successfully also in cases where *the third variable cannot be expressed by numerical data* (see the “quadratic contingency mean” mentioned in Section 3.2). In such cases the third variable is weighted—as the correlation coefficient is unknown—by a significance test performed for the two regression straight lines (see Section 3.3).

I_{om} may be calculated according to Table 3.9 more accurately from the plasticity limit than from the liquid limit. The reason is in all probability that the organic content has been determined from the small cylinders, obtained during the w_p test. As to the closeness of the correlation, the situation is not so favourable in practice, namely because only the soil suspected to contain organic substances was tested. As it is generally known, the consistency limits are influenced also by the clay-mineral content (the ratio of the colloidal-size grains) of the soil in the same sense, as by the organic content. A solution for separating both effects may be found in Section 5.3.

4. Determination of the site and the number of soil tests

The primary basic condition for the successful solution of geotechnical tasks is the knowledge of the soil and groundwater conditions, extending to all details, which may be of importance for the object. The aim of direct and indirect explorations and the subsequent laboratory tests is the recognition of the chemical and physical properties of a delimited *block of soil*, whose limiting surfaces are always determined according to the purpose of the actual task. If, for example, the foundation of a building is involved, we must explore the mass of the soil interacting with the building, while for a material extraction site the contours of the exploration are determined by the available area and the quantitative and qualitative requirements. The task is still more unambiguous if the density of an earthwork is to be controlled. In this case the soil block to be tested is identical with the mass of the earthwork.

The exploration of the subsoil conditions has considerable practical *limits*. The boreholes (probe holes, etc.) satisfy only the depth requirement, while horizontally they reveal only a very little part of the soil block (if, e.g., 110 mm diam. boreholes are drilled at every 30 m, the proportion of the explored and the unexplored masses is $1 : 1 \cdot 1 \cdot 10^{-5}$). The situation is aggravated by the fact that—with the exception of the so-called core drillings—only a part of the raised soil mass reaches the laboratory, where again only a part of the received samples is tested.

Drawing conclusions concerning the properties of the whole population on the basis of the samples is characteristically a task of mathematical statistics. In the following we examine the assistance which can be offered by the latter to the determination of the exploration sites, resp. the number of the tests (Sections 4.3 and 4.4), with special regard to the variability of the physical characteristics (Section 4.2) and economical considerations (Section 4.5).

4.1. Statistical interpretation of sampling

The totality of the individual elements forming the subject of the statistical investigation, including the corresponding numerical values, is called *statistical set* (*basic set, population*). In the general case several numerical values (e.g. reflecting various material properties) may be assigned to several populations. We distinguish between *finite* and *infinite* populations, depending on the number of the elements.

The aim of the statistical investigation is the recognition of certain parameters (expected value, standard deviation, distribution, etc.) of the population, *by samples*. In practice namely all the elements of the population, resp. the numerical values assigned to them cannot be recognized, only in very rare cases. The consequence of this—mostly economical—restriction is that on determining the *statistical parameters*, the *relationships* between the physical characteristics, we had to take into account always the existence of the population hidden behind the numerical values (Chapters 1–3).

A fundamental requirement of sampling—in the statistical sense—is *random selection of the samples*. This means that every element of the population has an equal chance of being sampled within the so-called *n*-element sample:

$$x_1, x_2, \dots, x_n$$

and of the sequence of getting into it (this selection mode is called usually also *representative sampling*).

An important question from numerous viewpoints is, how far the individual elements of the sample can be regarded as *independent* from the others. This property of the sample can be ensured unequivocally only by selecting the elements with *replacement*. If this is not so—and in the soil-mechanical practice this is the case always—the problem must be judged as follows (PRÉKOPA, 1962). As by the extraction of a sample the composition of the population is changed, the probability variables x_i can be regarded as independent only if

1. the population is of a continuous distribution (it has no two identical elements),
2. every value is present in the population in such a great number that the selection of a relatively small number of the elements is negligible from the aspect of the percentage composition of the individual numerical values.

In the first case the population is by necessity infinite, in the second case it *may be* regarded as such. The latter condition must be considered on the basis of the ratio of the sample by the number of the elements of the population.

We may speak of selecting the *place* of the samples mainly if the elements of the population can be characterized by geometrical coordinates. In this case *three sampling modes* may be distinguished:

- a) simple random sampling,
- b) systematic and
- c) stratified random sampling.

The essence of the three different methods is illustrated in *Fig. 4.1*. The left-hand part of the figure refers to the marking out of the drilling sites, the right one refers to the problems of the sampling within the boreholes. Numerous papers deal with the selection from among the three possibilities and its statistical consequences (e.g. QUENOUILLE, 1949; PETHÖ, 1974). The *comparison* of the methods is based on the

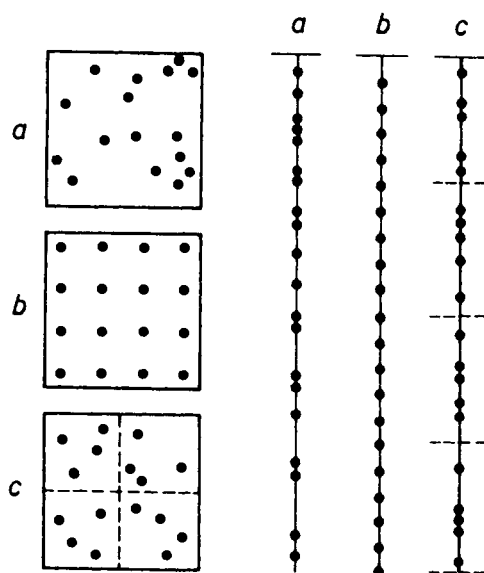


Fig. 4.1. Simple random sampling (a), systematic sampling (b) and stratified random sampling (c) in horizontal and vertical senses

dispersion of the expected value, but the calculations and the evaluation are aggravated by the circumstance that the independence of the elements plays a decisive role here.

With regard to the *sequence* of the sample elements three cases are distinguished:

- a) the sample elements are written up in the sequence of the selection;
- b) the elements, resp. the numerical values represented by them are ordered by their magnitude (this is the so-called ordered sample);
- c) the sequence is determined by the time, or the geometrical dimensions (e.g. distances measured from a given point).

The basic principles of mathematical statistics described before are asserted in *geotechnical investigations* in the following way.

1. The examined soil block is regarded as a population; the number of its elements is

$$N = \frac{V}{V_m}, \quad (4.1)$$

where

V — total volume of the soil block,
 V_m — volume of a sample of the soil.

2. As the quotient V_m/V is very small, the elements of the soil block may be treated as an *infinite* statistical population.

3. The sample of the soil is—according to its purpose—simultaneously a sample in statistical sense, but we shall call it so only if one or more *physical characteristics* can be assigned to it.

4. The sampling in a statistical sense has *three phases* in geotechnics: marking out the drilling sites, taking soil samples from the boreholes and marking out the samples to be tested from among the material which arrived to the laboratory.

5. As the soil samples are bound to *geometrical coordinates*, the sequence of the sampling (more exactly its position occupied in space) cannot be left unconsidered. This does not mean naturally that *ordered samples* cannot be formed from the soil physical characteristics (e.g. for determining the distribution function).

6. There are cases where the samples are *not selected* randomly; during the evaluation this circumstance must be taken into account carefully. We proceeded so, e.g., in Section 3.7.5 in considering the closeness of the relationship I_{om}, w_p ; namely only the samples suspected to contain organic substances were tested. Similarly on controlling the density of earthworks it is not indifferent if the testing sites are marked out randomly, or mostly where the compaction work was assumably insufficient.

4.2. Variation of the soil physical characteristics in time and space

4.2.1. Components of the data and time series

If we plot the tested variable (e.g. a soil physical characteristic) versus the time, or the distance, we obtain in the general case a series, which may be decomposed to the following three components:

$$y(x) = T(x) + P(x) + \varepsilon(x). \quad (4.2)$$

The first term of the right side represents the *trend*, the second term the *periodic* elements and the third term the stochastic component.

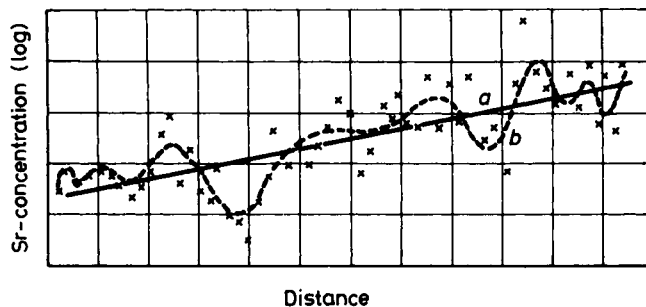


Fig. 4.2. Trend of the strontium-concentration versus the distance approximated (a) by a straight line and (b) by a 20th order polynomial (AGTERBERG, 1967)

The three components appearing on the right side of expression (4.2) cannot be separated unequivocally in every case. In the example shown in Fig. 4.2 the author approximated the trend (i.e. the systematic variation of the strontium concentration with the distance) in two ways: by a straight line and by a 20th degree polynomial. It can be established by simple inspection that the latter contains also periodic elements.

To avoid the difficulties illustrated in Fig. 4.2 the trend is described generally by a straight line, more rarely by a parabola or an exponential curve.

The *periodic* elements are examined in a new time-series, resp. data-series obtained by deducing the values, which may be calculated from the trend, from the values of the dependent variable. (This is identical with a transformation obtained by substituting the original axis x by the line of the trend.) Several methods are known for verifying the periodicity, out of which the *harmonic analysis* and the *autocorrelation* are used most frequently. The former is applied mostly for examining time series, especially where cyclic phenomena are involved (e.g. the variation of the water discharge within a year). In geology and in geotechnics the autocorrelation is preferred, so in the following this method is discussed.

The *autocorrelation* (by its other name: *serial correlation*) gives numerical values for the “memory” of the process and for the independence of the elements. Here a *single* set is examined—as the name also suggests—, from which two variables are obtained by correlating the individual elements with the elements preceding them by “steps” (“spacings”) of $k = 1, 2, \dots$. Let us assume, e.g., that the water content in a thick, homogeneous layer varies depending on the depth in the following way:

$z(m)$	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
$w(\%)$	25.4	27.3	28.1	26.4	28.7	29.3	29.8	27.5	26.9	26.3	24.8	23.4

If $k = 0.5$ m spacing is chosen, the following value pairs are obtained: $x_1 = 25.4$, $y_1 = 27.3$; $x_2 = 27.3$, $y_2 = 28.1$; $\dots$ $x_{11} = 26.0$, $y_{11} = 25.8$. The regression analysis according to Section 3.2 gives $r_1 = 0.50$ as *first order autocorrelation coefficient* belonging to the step $k = 1$.

By varying the number of the step intervals we obtain the plot of $r_k = f(k)$, the so-called *autocorrelogram*, whose zero point gives the step, at which the *memory is zero* (so these elements are fully *independent* of each other) and its extreme points give the the step distance at which the correlation between the elements is the closest. Various criteria serve for determining if the individual r_k values (e.g. the maximum r_k) differ significantly from zero. So, e.g., according to ANDERSON (1941), the confidence intervals may be calculated by the following formula:

$$CL = \frac{-1 \pm t \sqrt{n-1-k}}{n-1}, \tag{4.3}$$

where

n — number of elements in the series,

k — step spacing,

t — value belonging in the STUDENT-function table to the chosen probability p .

(Taken strictly the formula is only valid for the case $k/n < 0.1$; from here on its reliability decreases.)

Let us return now to expression (4.2). If the trend is zero (or can be regarded as zero according to the criterion described in Section 3.3), we speak of a *stationary series*, characterized by the fact that on increasing k the autocorrelation coefficient r_k tends to zero, contrarily to the non-stationary (or weakly stationary) series, where r_k fluctuates around a constant value. If the series contains also periodic (cyclic) elements, r_k may assume also negative values, even when the series is stationary.

In the simplest (and rarest) case, called usually also “white noise”, $T(x) = P(x) = 0$ and the elements of the series are uncorrelated. If only the first condition is satisfied, the series represents a so-called *first-order MARKOV* process, where every element depends only on the directly preceding element. In this case the autocorrelogram may be written up in the following form:

$$r_k = e^{-\alpha k}. \quad (4.4)$$

If $T(x) \neq 0$, the autocorrelogram is distorted, whereby the point pairs k, r_k may be compensated for by an exponential curve of the form

$$r_k = ce^{-\alpha k}, \quad (4.5)$$

where $c < 1$. So in this case not alone the “noise”, but also the “signal”, whose measure is $(1 - c)/c$, can be verified in the autocorrelogram. By deducing the trend we may verify if $P(x) = 0$ or not.

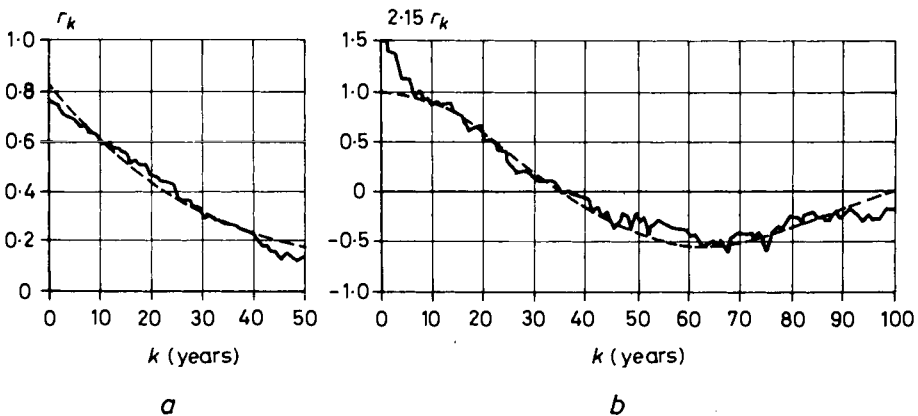


Fig. 4.3. Autocorrelogram of the original data series (a) and after releasing it from the trend (b), related to the thickness of clay layers deposited in a glacial lake (AGTERBERG and BANERJEE, 1969)

If the series contains also periodic elements, the process is a *second-order* process, which may be described by the following theoretical curve:

$$r_k = e^{-\alpha k} [\cos \omega_0 k + (\alpha/\omega_0) \sin k\omega_0], \quad (4.6)$$

where $\omega_0 = 2\pi/T$, T is the length of the period and α is the index of the memory.

AGTERBERG (1974) presented an interesting example of testing depth data series. In a bed of glacial origin of Lake Barlow, Ojibway (Canada), thin silt and clay lenses are alternating, in correspondence with the deviating weather conditions of the calendar half-years. He measured the thickness (v) of 537 silt lenses and 537 clay lenses. Their dimension increased with the depth so much that instead of the numerical thickness values he used their logarithms for the calculation. According to the autocorrelogram determined for the clay (Fig. 4.3, detail a) $c \Rightarrow 0.83$, which refers to the fact that the series contains a trend and/or periodic elements. Regarding the former as linear (corresponding to the exponential curve, because the logarithms of the thicknesses were used) and deducing its ordinates from the original $\ln v$ values, the autocorrelogram marked b was obtained. According to the calculation the length of the period is $T = 2\pi/\omega_0 = 2\pi/0.05 = 126$ years and $\alpha = 0.01$, which refers to a relatively good "memory". (The ordinates of the autocorrelogram are multiplied by 2.15 in order to have $r_k = 1$ in the position $k = 0$.)

4.2.2. Variability of the physical characteristics in space

The examples given in Sections 1.3 and 1.4 prove also the generally known fact that the natural soils are rather heterogeneous as compared, e.g., to the artificial building and other materials. The physical characteristics vary from point to point even within a relatively small area and their dispersion may be considerable even within a layer qualified as homogeneous. All this does not mean that the time and data series formed from the physical characteristics are characterized only by a random "noise". By this assumption we would commit a grave error already in the exploration and testing phase, apart from the consequences of this view in the evaluation of the basic data.

The variability may be examined in *one, two, or three* dimensions, by assigning a coordinate to each physical characteristic, but the *time* may also be introduced as fourth dimension into the evaluation. The primary aim of the data analysis is—in the sense of Section 4.2.1—the verification of the possible trend and of periodic elements.

In calculating the *trend* the one-dimensional case is most frequent, when the systematic variation of the physical characteristics is examined vertically along a borehole. In the variation by depth *gravitation* plays a primary role: the structure and the density of the soil (with this the water content and all the other characteristics depending on the latter) vary in the function of the overburden pressure, with the consequence that within the same layer the void ratio decreases, while the cohesion and the compression modulus increase with depth. At the same time the greatest variability due to *geological processes* appears also in this sense, e.g., the decomposition of the rocks, the soil formation in the water, or on the dry surface (see the example described in Section 4.2.1).

The evaluation is more unequivocal, if the trend is calculated by several ("parallelly connected") drillings instead of a single one. *Figure 4.4* plotted in this way shows the

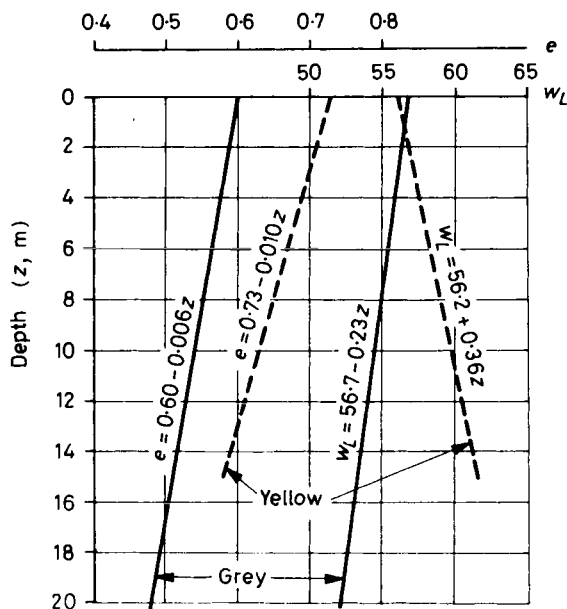


Fig. 4.4. Vertical trend of the void ratios, resp. liquid limits of the grey and the yellow clay of Kiscell (PAÁL, 1975)

variation of the liquid limit and the void ratio with the depth in Kiscell Clay. PAÁL (1975) made an interesting observation in the variation of the liquid limit: as the yellow Kiscell Clay is the oxidized variant of the grey Kiscell Clay (and lies always directly over the latter), the values of w_L should continue in the data of the grey layer recorded near the surface. (The discontinuity may have been caused perhaps by chemical changes due to oxidation.)

Most literary data concerning the variations by depth are related to the compression modulus. According to Table 4.1 the variation of the compression modulus of the involved soils expressed in MN/m^2 units varies between $0.52\text{--}1.22 \text{ m}^{-1}$

Table 4.1. Literature data for variation of the modulus of compressibility with depth

Soil	Author(s)	E_s , MN/m^2	r	b/a
Vienna Clay	BRANDL	$1.18z + 3.9$		0.30
Dunaújváros Loess	EGRI and RÉTHÁTI (1959)	$0.77z + 1.0$		0.77
Grey Kiscell Clay	PAÁL (1975)	$0.62z + 9.5$	0.56	0.07
Yellow Kiscell Clay		$0.37z + 10.4$	0.25	0.04
Buda Marl		$1.22z + 8.7$	0.45	0.14
Covering layer on the Kiscell Clay		$0.78z + 6.7$	0.39	0.12
Dunaújváros Loess	VARGA (1966)	$0.52z + 2.2$		0.24
Budapest clays		$0.88z + 5.6$		0.16

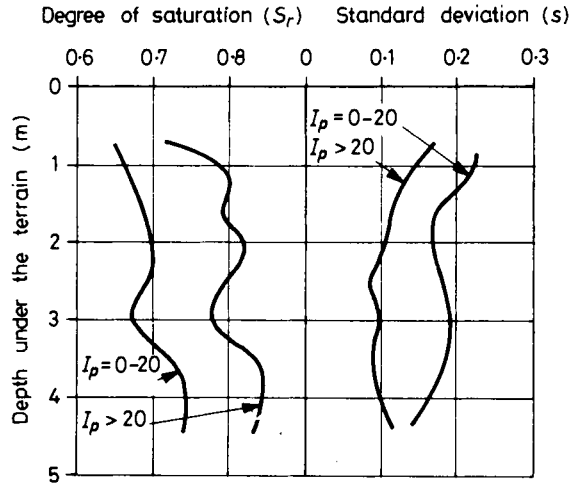


Fig. 4.5. Variation of the degree of saturation of cohesive soils lying above the capillary range, with depth

and the specific value of the increase (related to E_{s0}) between 0.04 – 0.77 m^{-1} . In connection with the E_s -values we must note that a) they were determined in the oedometer; b) their standard deviation is relatively large, similarly to that of the unconfined compression strength (Section 1.3.6).

The variation of the standard deviation with depth may also be defined in a similar sense. The determination of the trend is—in a strict sense—not so unequivocal as that of the mean, but if the depth interval is sufficiently small and a large set of data is available, this means no difficulty. The example shown in Fig. 4.5 illustrates (a) the variation of the degree of saturation of cohesive soils in the covering layer (lying above the capillary zone) and (b) the variation of the standard deviation by depth.

The 742 samples used for this determination were sampled in various parts of Hungary, but at different dates, so the conclusion that the effect of the meteorological factors extends to about 2 m (from where the dispersion becomes stationary) is sufficiently well founded. As S_r varies less than e and w due to the volume change of the cohesive soils, the data of FREDLUND and DAHLMAN (1971)—who stated that the standard deviation of the *water content* in clays which settled near Edmonton in Canada becomes stationary below -1.80 m —are more convincing. (This problem is of importance—among others—for foundations standing on volume-changing soils.)

If the examined physical characteristic varies with depth, the *coefficient of variation* is also a function of depth in the general case. If, e.g., a linear relationship exists for a physical characteristic, which may be expressed in the form of

$$\bar{u} = a + bz$$

and for the standard deviation in the form of

$$s = c + dz,$$

then the coefficient of variation defined by the expression

$$C_v = \frac{s}{\bar{u}} = \frac{a+bz}{c+dz}$$

is only constant if

$$\frac{a}{b} = \frac{c}{d}.$$

If the mean and the standard deviation are described by a power-parabola, the condition

$$C_v = \frac{a(z+z_{0s})^b}{c(z+z_{0u})^d} = \text{const.}$$

is satisfied only if $z_{0s} = z_{0u}$ and $b = d$. The individual details of the verification are seen in Fig. 4.6. The depth-independent coefficient of variation is in both cases $C_v = a/c$.

The *two-dimensional case* occurs when the variation of the studied physical characteristic is examined in the vertical plane led through two or more drillings (i.e. in soil profiles). The task may be solved by terivariate regression analysis. The regression plane expressed in the form

$$u = b + a_1x + a_2z$$

determines a vector whose horizontal and vertical projections give the gradients of the variation of the physical characteristic. The *significant deviation from zero* of the

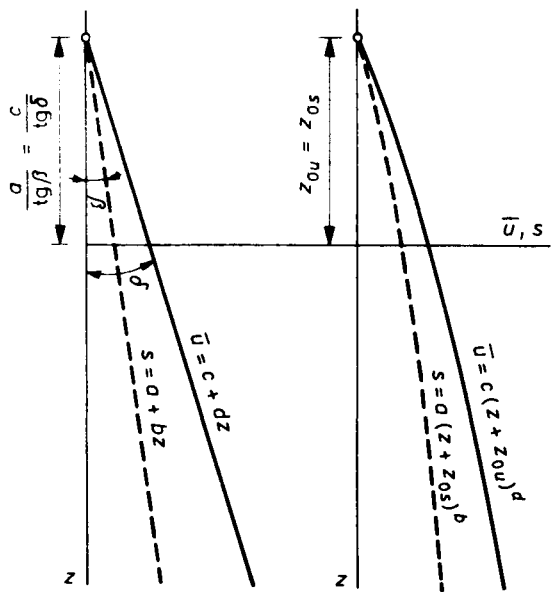


Fig. 4.6. Conditions of the satisfaction of equality $C_v(z) = \text{const.}$, if the mean and the standard deviation is described by a straight line, resp. a power parabola

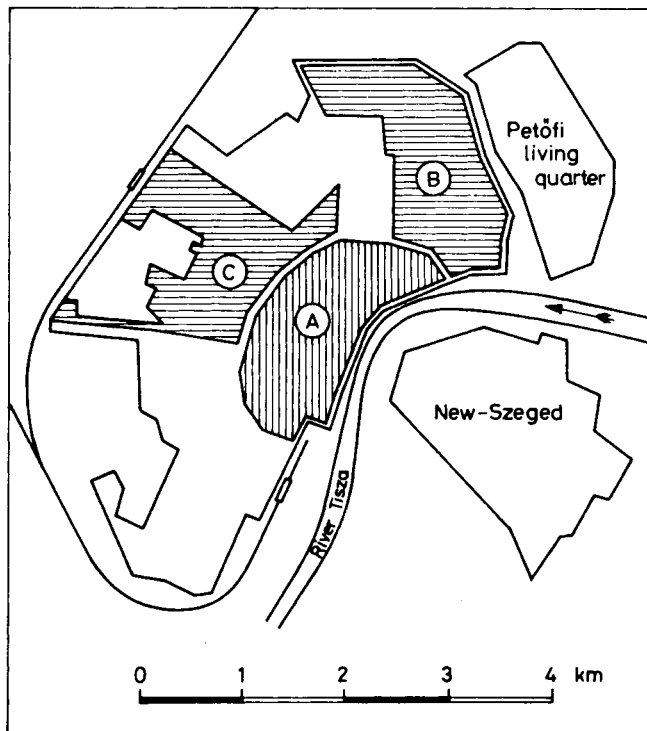


Fig. 4.7. Part areas of Szeged foreseen for examining the variability in horizontal sense (RÉTHÁTI and UNGÁR, 1978c)

coefficients a_1 and a_2 must be examined with the help of Eq. (3.33). If for example $a_1 \sim 0$, the horizontal variation may be neglected, the relationship $u(z)$ must be determined by a repeated (now two-dimensional) regression analysis.

The *three-dimensional case* may also be interpreted similarly, with the difference that the gradient is then a space vector.

The calculation of the trend is not the only method of investigating the variability in space. We may proceed also by dividing the surveyed area to *smaller units*, assigning to these a soil physical characteristic, then comparing these values with the help of the

Table 4.2. Mean and standard deviation of the void ratios of samples taken from small diameter boreholes (RÉTHÁTI and UNGÁR, 1978c)

Part of the area	Soil S3			Soil S4		
	n	$\bar{e}$	s	n	$\bar{e}$	s
A	136	0.6947	0.0874	101	0.8051	0.0858
B	69	0.6976	0.1142	46	0.7904	0.1220
C	111	0.7393	0.1019	117	0.8241	0.0904
A + B + C	316	0.7110	0.1006	264	0.8112	0.0954

Table 4.3. Results of statistical tests performed for the means and the standard deviations of the void ratios measured in various parts of the Szeged area (RÉTHÁTI and UNGÁR, 1978c): I — simultaneous test with *F*-test; II — comparison by pairs with the help of the modified *t*-test; III — *F*-test, IV — modified *t*-test for the whole set, V — *F*-test for the whole set

Partial area combinations	Method	Symbol of soils	
		<i>S3</i>	<i>S4</i>
<i>A, B, C</i>	I	—	+
<i>A — B</i>	II	+	+
<i>A — C</i>		—	—
<i>B — C</i>		—	+
<i>A — B</i>	III	—	—
<i>A — C</i>		—	+
<i>B — C</i>		+	—
<i>A — (A, B, C)</i>	IV	+	+
<i>B — (A, B, C)</i>		+	+
<i>C — (A, B, C)</i>		—	+
<i>A — (A, B, C)</i>	V	—	+
<i>B — (A, B, C)</i>		+	—
<i>C — (A, B, C)</i>		+	+

statistical tests described in Section 6.2.1. This method could be applied well in the case of the town Szeged, where the subsoil consists of thin layers, so the variation of the physical characteristics with the depth could be neglected (RÉTHÁTI and UNGÁR, 1978c). We have marked out for this purpose the districts shown in *Fig. 4.7* of the town, determining for these the mean and the standard deviation of the void ratios of the infusion loess lying below groundwater level (*S3*) and of the lake clay (*S4*) (*Table 4.2*). As these layers lie below groundwater level, we may be sure that the comparative analysis is not distorted by the shrinking process. The results of the statistical tests performed on the basis of the probability level $p = 95\%$ are shown in *Table 4.3* (where the + sign indicates that there is no significant difference between both part-areas, resp. between the surveyed part and the whole area). On the basis of the table the following statements may be made.

- a)* According to the simultaneous tests (I) the territorial variability of the soil *S3* is higher than that of the soil *S4*, as supported also by the fact that for the methods II–V the proportion of the minus signs is 6:4.
- b)* The results obtained by comparing the means (methods II–IV) prove that the *means* of area *C* differ significantly from those of the other areas.
- c)* The results obtained by the methods III and V comparing the standard deviations show that the areas *A* and *B* are less homogeneous than the area *C*.

d) The results of the statistical tests may be evaluated from a geotechnical viewpoint to the effect that the soil conditions of area C are slightly less favourable than those of both other areas, though the latter are also lagging behind the countrywide mean (see Sections 5.2.2 and 5.2.3).

The determination of the trend (or more generally the compensation of the dispersions) may offer considerable help for solving *practical tasks*: by filtering out the random errors (giving at the same time their numerical values), giving information on the spatial variation of the mean and the reliability of the physical characteristic used for the planning. In the knowledge of the regression plane determined for the load bearing bed and the confidence intervals we can specify, e.g., the required length of driven piles more accurately.

In the knowledge of the trend of the E_s values over depth we can calculate the settlement of foundations and on the basis of their horizontal distribution the expectable tilting direction in a simpler way (see in Section 10.4.5). In the knowledge of the descent line of the regression plane we may decide also if the heterogeneity of the subsoil of slab foundations should be taken into account in the dimensioning, with regard to the compensation effect of the rigid slab.

UNGÁR (1974, 1975a, b) describes an interesting example. On examining a building in town Szeged the suspicion arose that water gets into the soil in a concentrated way from a broken pipe led under the sanitary block. The drillings failed to show any water dome, at the same time the demolition was to be avoided, partly because of the expenses and partly in the interest of the undisturbed operation. Finally, the probability of a pipe break was proved by the gradually decreasing sulfate content in the water on approaching the sanitary block. The relationship describing this in form

$$\text{SO}_4^{--} = \frac{1700}{1 + \exp(2.2 + 0.00747l)}$$

(where l = distance in m) proved to be very close, the correlation index was $I = 0.98$. (The pipe break was found later indeed.)

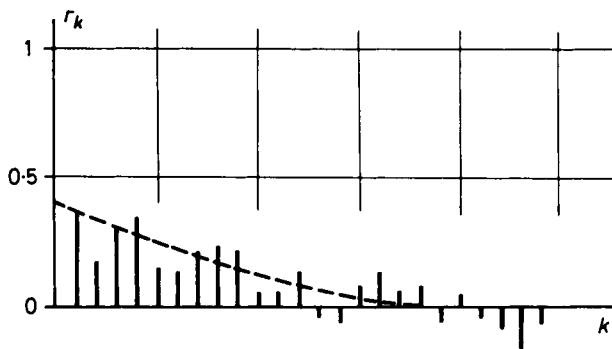


Fig. 4.8. Autocorrelogram of the liquid limit of soil samples taken along a straight line (BJERRUM, 1967)

A still outstanding question is the *spatial correlation* of the physical characteristics.

Several research workers have performed studies to plot autocorrelograms by determining the soil properties in horizontal spacings of 15–50 cm. BJERRUM (1967) obtained for the liquid limit the correlogram shown in Fig. 4.8, which may be described by the empirical formula

$$r_k = 0.42(1.44k)K_1(1.44k), \quad (4.7)$$

i.e. as a bilateral MARKOV-process (K_1 is the second type first-order BESSEL-function).

ALONSO (1976) describes the physical characteristics of soils examined by him with the help of expression (4.5); the α -values varied between 0.14–1.91 m⁻¹ (see Section 11.1.2). Certain authors (e.g. LUMB, 1975a) propose for the case, where r_k may assume also negative values, to use instead of Eq. (4.6) the expression corresponding to the unilateral YULE-process:

$$r_k = e^{-\alpha k} \cos(\omega_0 k). \quad (4.8)$$

WU (1974) calls the attention to the possibility of separating the individual geological formations with the help of autocorrelograms, by demonstrating that r_k remains positive within each of them.

LUMB (1975b) established, relying on data found in literature, that no great error is incurred in practical work, if the physical characteristics *are assumed to be uncorrelated*.

4.2.3. Variability of the physical characteristics in time

The physical characteristics of the soils may change considerably due to natural, or artificial effects.

From among the *natural* effects the precipitation and the evaporation play the most important role, including inundations by polders and floods and the transpiration of the trees. The effect of the frost appears in the diminution of the grains and the rearrangement of the water content, the activities of the animals living under the soil surface lead to the increased permeability of the soil. The intensity of the natural effects decrease with depth and below groundwater level it is practically zero.

Artificial effects are the periodical and the permanent groundwater lowering, the construction of drains, refilling, or covering the terrain and numerous other human activities, a part of which may cause considerable changes in the physical characteristics of even the mass of soil lying below groundwater level (e.g. shrinkage).

The water content is one of the most characteristic index-properties of the soil, as mentioned in Section 3.5.2, on discussing the correlation matrixes; so the variability of the physical characteristics may also be evaluated best and in the simplest way by the variation of w . The heat and water balance of the so-called *covering layer* varies in time not alone by the variation of the quantity and the intensity of the precipitation and the evaporation, but also because the position (the depth) of the groundwater level (and the capillary zones attached to it hydraulically) is not constant either. The time series of the

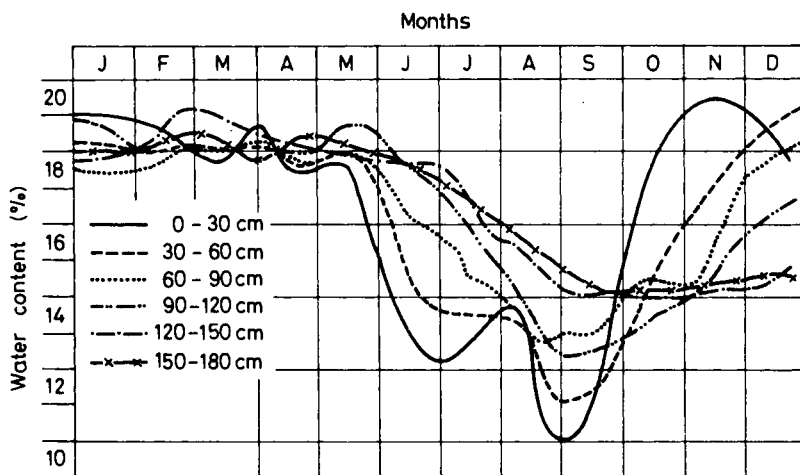


Fig. 4.9. Variation of the water content in the course of a year at various depths based on the measurements of BALLENEGGER (1938)

meteorological factors, and of the groundwater level along with them, have beside the random components and the temporary trend *cyclic* and *periodic* components too. The former are caused by the yearly rhythm of the weather, the latter—in all probability—by the periodicity of the solar activity.

Figure 4.9 shows an example for the yearly course of the water contents measured in various depths. The summer-end minimum (due to the enhanced evaporation and transpiration), then a subsequent rise can be observed well. The figure illustrates also that the amplitude of the variation of w decreases with the depth considerably.

Figure 4.10 shows the autocorrelogram of the mean groundwater levels recorded in September in an observation well. (The position in autumn is geotechnically important

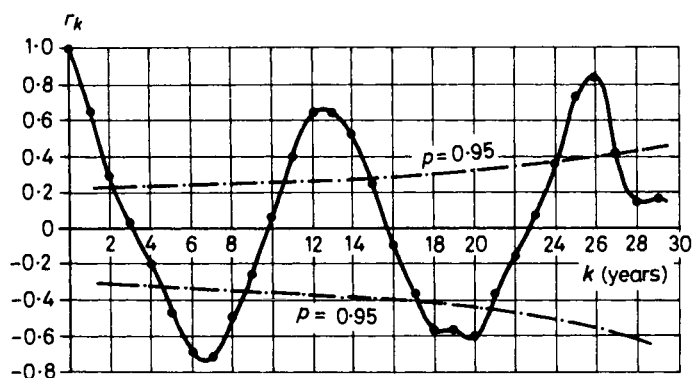


Fig. 4.10. Autocorrelogram of the average groundwater level recorded in September in an observation well, with the confidence limits of ANDERSON (RÉTHÁTI, 1977c)

from the viewpoint of the minimum “construction water level”, resp. for its prediction.) According to the figure the periodicity is very significant: the autocorrelation coefficient belonging to $k=13$ years is $r_k=0.64$, that belonging to $k=26$ years is $r_k=0.85$. According to ANDERSON’s (1941) criterion there are 16 r_k values which may be regarded as significant on the probability level of $p=95\%$.

The variability in time of the physical characteristics deserves special attention from the following three aspects.

1. If the surveyed area is extensive and the exploration of the soil takes several months, the measure of homogeneity of the layers lying close to the surface cannot be judged unequivocally. Similar difficulties arise also in calculating the horizontal projections of the trend. All this is especially inconvenient for the designer, when the upper layers have an important role from the viewpoint of the establishment.

2. If the statistical evaluation is based on archive data, we can obtain a realistic result only if the number of the individual tests is large and of a random distribution in time. If these conditions are not satisfied, the data series will not reflect the reality: if, e.g., the majority of the drillings was made in summer, the coefficient of skewness of the water content will be negative, contrarily to the rule appearing also in *Table 1.12*.

3. The physical characteristics of the covering layers reflect the meteorological, hydrological, etc. conditions, which predominated at the time of the drilling, but in the design also the variations expected during the life of the object must be taken into account. (The same applies, e.g., also to the exploitation of soils: the minability and the compactibility are also functions of the time.)

4.3. Marking out and performing the borings

The exploration plan is based generally on “systematic sampling”, i.e. the drillings are distributed in a quadratic network. (Naturally several other viewpoints must also be taken into account, e.g., in the case of sloped terrains we strive to plot soil profiles which are perpendicular to the contour lines.) In individual cases we may depart from this principle: on the site of buildings, which are very valuable or sensitive to settlement, or in the vicinity of settlement centres of damaged buildings, etc. more drillings are foreseen. As the time and cost expenditures are rather high, efforts must be made to gain *as much information as possible* by minimum exploration work. This principle can best be realized in the ideal case when the engineer is on the site at least before the drilling group moves in. Another relatively effective possibility is the exploration of the soil *in two steps*, in order to better recognize the disturbed or geotechnically significant parts of the area and to concentrate the sampling to a given layer. These possibilities may be exploited also in the case of the single-step exploration, if the final sites and the number of the drillings are selected and lowered after the evaluation of the first few drillings.

4.3.1. Marking out the borings (tests) on the basis of the entropy

The uncertainty of the data revealed by drilling may be characterized by the entropy (see in Section 1.2.2). In this case the following process is recommended (HARROP-WILLIAMS, 1983).

If in an exploration area a geotechnical property has m possibilities (for example, of the soil type being sand, silt, silty sand, clay, clayey silt, etc.) it can be divided into n regions and the *relative entropy* of the property in each region can be calculated. One would be most uncertain which of the possibilities would best represent the property in the region if each property is represented in equal proportions in the region; while, on the other hand, one would be certain when only one possibility existed in the region. The exploration area can then be mapped by regions of uncertainty concerning the property by making use of the calculated relative entropy value.

A relative entropy map developed by MILLER and KAHN (1962) for grain sizes at the south end of Lake Michigan is shown in Fig. 4.11. The dots illustrate the sample points, where grain size analyses were made. Figure 4.11 shows that below the berm the material is more random than above it. It follows that fewer tests (or interpolating borings) are necessary to predict the properties of the material above the berm than below it and the interpolation can be performed more accurately for intermediate points on the map.

The relative entropy maps, as opposed to the variance maps, provide a scaled parameter and can also be constructed for qualitative variables. The entropy reflects the uncertainty of the random variable and not its variation about the mean value (see

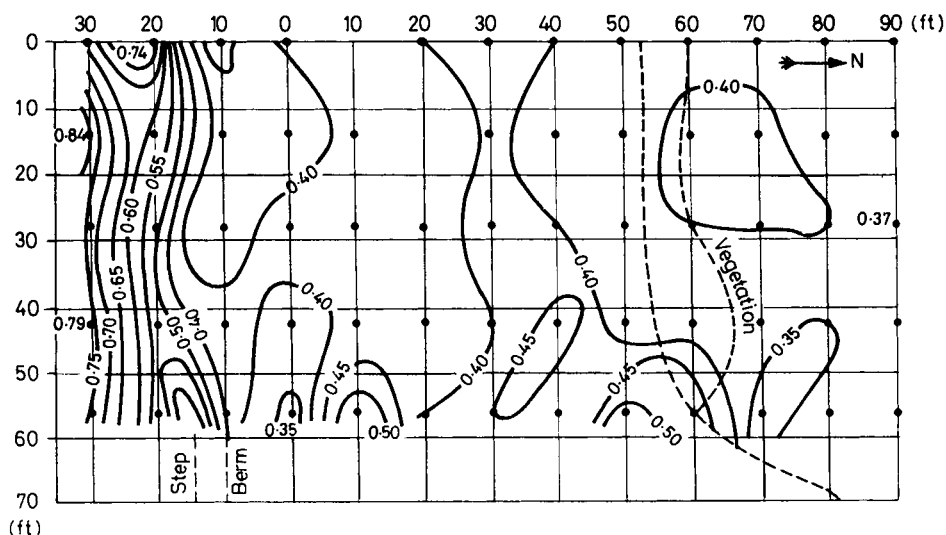


Fig. 4.11. Relative entropy map of grain size distribution at the south end of lake Michigan; HARROP-WILLIAMS (1983), after MILLER and KAHN (1962)

in Section 1.2.2). For the geotechnical engineer a relative entropy map would be a useful guide in constructing a sample program; decisions on the relative amounts of the samples that should be taken from regions with different entropies can be facilitated (HARROP-WILLIAMS, 1983).

4.3.2. Utilizing the methods of geostatistics for marking out the borings (tests)

The conception of geostatistics and its test methodics were created by MATHERON (1967). The aim of his research was the solution of certain geological and mining tasks. His method is used today in numerous other fields as well, e.g., in hydrology, meteorology and forestry (BOGÁRDI *et al.*, 1978).

MATHERON started from the fact that the geological processes and their results contained both deterministic and stochastic factors. (If the spatial distribution of the given parameter is completely of a random nature then the geostatistical methods cannot be used, while if it is completely regular, there is no need for geostatistics.)

For characterizing the spatial variability a random variable $g(x)$ is assigned to each tested point. In the following four basic hypotheses are assumed, i.e.:

1. *hypothesis*: $g(x)$ and $g(x+h)$ are independent of each other (h is the distance between the test points);

2. *hypothesis*: (first-order stationarity): the distribution is shift-invariant, i.e., the distribution belonging to points $x_1+h, x_2+h, \dots$ is identical with the distribution belonging to the points $x_1, x_2, \dots$;

3. *hypothesis* (second order stationarity):

a) the expected value of $g(x)$ is identical for every x , i.e.

$$M[g(x)] = m; \quad (4.9)$$

b) the covariance of the random variables $g(x)$ and $g(x+h)$ exists for every x and every h and it does not depend on x , only on h , i.e.

$$\begin{aligned} C(h) = \text{Cov}[g(x), g(x+h)] &= M\{[g(x)-m][g(x+h)-m]\} = \\ &= M[g(x)g(x+h)-m^2] \end{aligned} \quad (4.10)$$

(It can be proved that if $g(x)$ is stationary on the first order, it is so also on the second order, but the reverse of it is not true.)

4. *hypothesis* (theory intrinsic): the variance of the variable $g(x+h)-g(x)$ exists and it does not depend on x , i.e.

$$M\{[g(x+h)-g(x)]^2\} = 2\gamma(h), \quad (4.11)$$

where the term on the right side (in practice its half) is the *variogram*.

Let us assume that the surface of the load bearing layer of piles appeared in the following depths in the boreholes drilled along a straight line at 15 m spacings from each other:

$$8.9 - 9.6 - 7.8 - 8.8 - 7.6 - 9.2 - 8.5.$$

In this case

$$\begin{aligned}\gamma(15) &= \frac{1}{2 \cdot 7} [(9.6 - 8.9)^2 + (9.8 - 9.6)^2 + (7.8 - 9.8)^2 + (8.8 - 7.8)^2 + \\ &\quad (7.6 - 8.8)^2 + (9.2 - 7.6)^2 + (8.5 - 9.2)^2] = 0.716 \text{ m}^2.\end{aligned}$$

On varying the value of h we experience that $g(x)$ and $g(x+h)$ become independent of each other beyond a certain distance a . This value is called *influence range*.

An example for the variogram is shown in Fig. 4.12. The curve spread over the polygon is the theoretical curve of the variogram, for whose description the following possibilities exist.

According to the *spherical model*

$$\gamma(h) = c_0 + c \left(\frac{3h}{2a} - \frac{h^3}{2a^3} \right). \quad (4.12)$$

The value $c_0 + c$ is usually called threshold (palier, sill) value; the horizontal section of the variogram lying beyond the influence range assumes this value. The phenomenon that the intercept is not zero (but c_0) is called *nugget effect*. It may be caused by a measurement error, or a microstructure (spatial distribution law) of the parameter, whose spacings are shorter than the distance between the tested point pairs. So, e.g., in the case of gold-containing conglomerates the size of the individual gold grains (nuggets) is guiding, which is naturally much smaller than the spacing between the drillings.

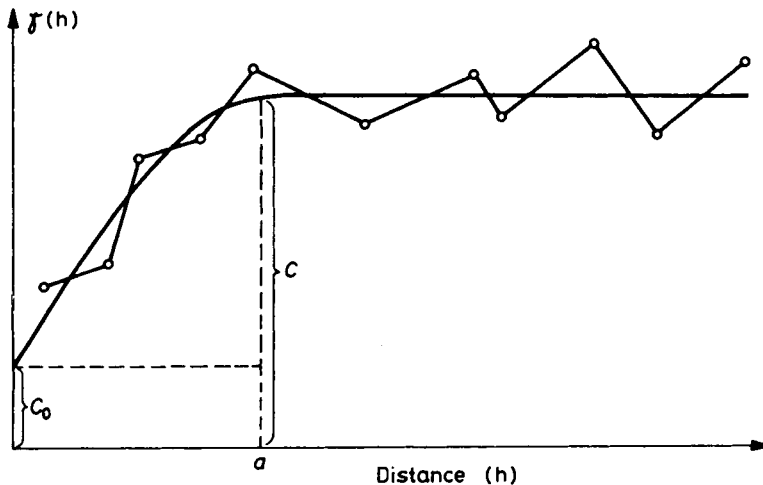


Fig. 4.12. Empirical and theoretical variogram

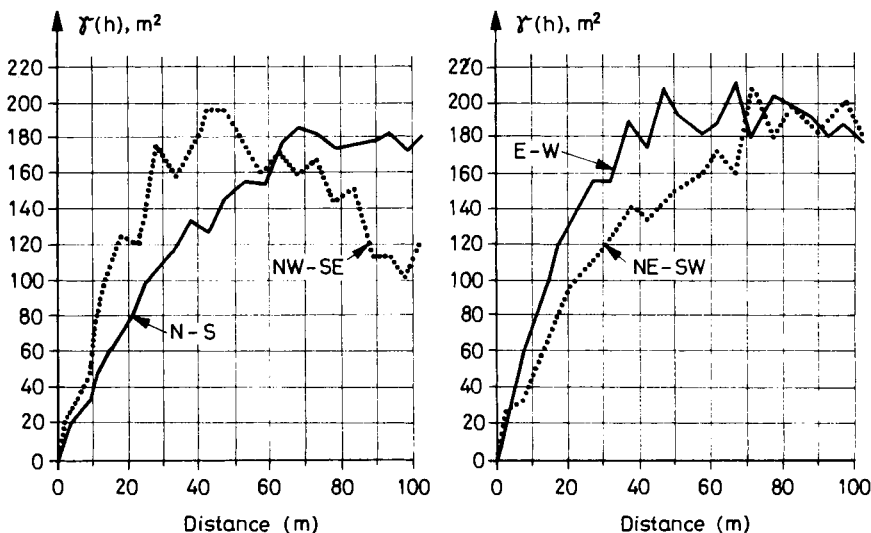


Fig. 4.13. Example for determining the variograms by discretion

The variogram may be vertical (within the drilling) or horizontal. It is advised to free the values of the tested parameter from the trend, especially if the trend can be calculated reliably (where the correlation coefficient is close to 1).

It is advised to prepare the variogram at least in the four main directions (Fig. 4.13). If an identical variogram is obtained in all the directions, the distribution of the given parameter is isotropic; if the variograms are different, the distribution is *anisotropic*. The anisotropy affects in the first place the influence range, but it may occur that also the sums $c_0 + c$ are different. The deviating influence ranges may be presented purposefully in the form of the so-called *anisotropy polygon*, around which an enveloping *ellipse* can be constructed generally (Fig. 4.14), whose major and minor axes correspond to the maximum, resp. to the minimum influence range.

With the help of the anisotropy polygon (or ellipse) we may determine the sites of the *drillings*, resp. mark out the interpolation boreholes. Planning a quadratic grid of drillings is advised only if the difference $a_{\max} - a_{\min}$ is not significant.

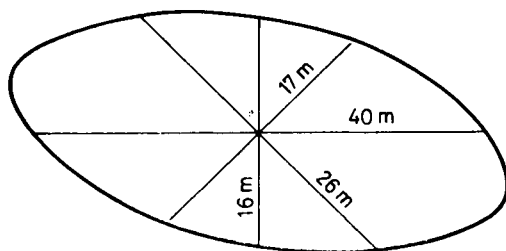


Fig. 4.14. Anisotropy polygon and ellipse

The value of the examined parameter (T) may be calculated also for points, where there are no drillings; this process is called *kriging* (after the name of Prof. D. G. KRIGE). Only the data of drillings (tests) which are at a distance of $d < a$ should be used for the calculations. The value of the parameter may be determined by weighting:

$$T = \lambda_1 \cdot g(x_1) + \lambda_2 \cdot g(x_2) + \dots + \lambda_n g(x_n), \quad (4.13)$$

with the following two principles:

- the estimate must be unbiased,
- the expected standard deviation of the estimate must be minimum.

For the weights a system of equations may be written up on the basis of the variograms. With the help of the λ_i values not alone the most probable value, but also the standard deviation can be calculated. A map may be constructed for the same as a newer basis for the elaboration of the exploration plan.

4.3.3. The probability of the exploration of thin, unfavourable layers

The greatest danger for a planned building is the failure if we are not informed about an *unfavourable layer*.

LUMB (1975b) remarks appropriately that the most significant information is often the one, that is missing, exactly as in Chinese painting. The question is the following: what is the probability that such a layer (lens) will not be revealed by the drillings?

Let us assume that the ground plan of the unfavourable layer is quadrangular (with sidelengths a and b) and the network of the drillings forms a square of the sidelength d (Fig. 4.15). The probability p of a drilling "hitting the mark" is

$$p = \frac{\text{number of favourable cases}}{\text{total number of the cases}}.$$

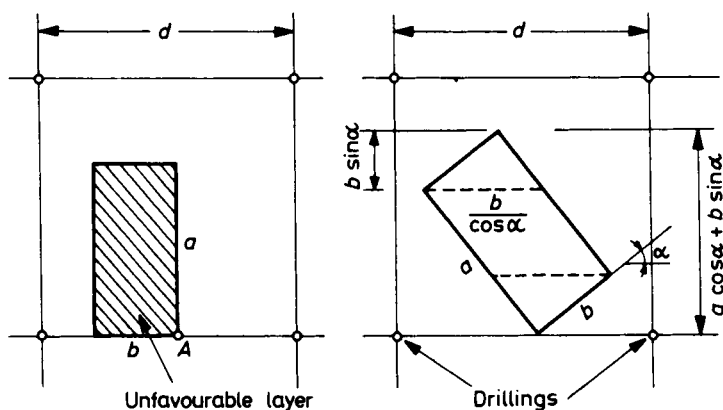


Fig. 4.15. The probability of not revealing an unfavourable layer of $a \cdot b$ basic area by a network of boreholes with grid-dimension d

Let us fix now the network of the drillings with the assumption that only the unfavourable layer changes its position. If point A passes over the straight line marked $I-I$, the probability that a drilling will fall in section b is:

$$p_1 = \frac{b}{d}. \quad (4.14)$$

As the probability that the straight line $I-I$ intersects the lens is a/d , the requested probability is:

$$p = \frac{b}{d} \cdot \frac{a}{d} = \frac{ab}{d^2}. \quad (4.15)$$

If the shorter side encloses the angle α with the horizontal, then the number of the favourable cases is as follows (see the figure):

$$b \sin \alpha \frac{b}{2 \cos \alpha} + (a \cos \alpha + b \sin \alpha - 2b \sin \alpha) \frac{b}{\cos \alpha} + b \sin \alpha \frac{b}{2 \cos \alpha} = ab,$$

which means that expression (4.14) is valid also for this case. The same formula may be used also if $2d > a > d$; in this case the number of the favourable cases is

$$(2d - a)b + 2b(a - d),$$

corresponding to the fact that now not only one, but even two drillings may hit the lens.

According to expression (4.15) the probability of exploring the unfavourable layer *increases proportionally with the horizontal extension of the layer*. For the one-dimensional case (e.g. where a linear object is involved) the expression (4.14) is valid.

4.3.4. The density of sampling

The minimum quantity of *samples* taken from the drilling is regulated by prescriptions. In addition the following two considerations must be observed.

1. The possibility of satisfying the requirements of Section 4.4 concerning the minimum number of the tests must be ensured.
2. The sites (and the density) of the sampling must be marked out in a way that the properties of the data series defined by Eq. (4.2) could be determined as unequivocally as possible.

The latter requirement deserves attention, because according to observations the components $T(x)$ and $P(x)$ are more significant in the vertical, than in the horizontal sense. The essence of this problem may be studied by *Fig. 4.16*. Let us assume that the unconfined compression strength of the studied layer varies as shown in part *a* of the figure, as the result of past geological processes (the length of the period $L=2$ m). If samples are taken from every $d=2$ m, then the following two cases for $\bar{q}_u$ may ensue:

- a) if the depth of the first sample is $z=0$ (or 1, 2, . . . m), the laboratory determines the effective $\bar{q}_u$ -value;

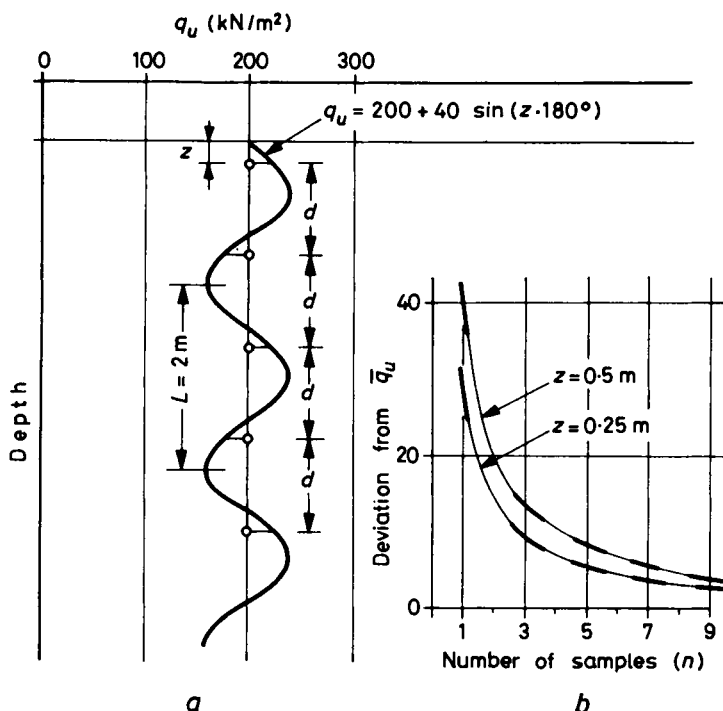


Fig. 4.16. Error incurred on determining the mean of soil physical characteristics (b) when the data vary according to a sine curve with depth (a)

b) in the opposite case the $\bar{q}_u$ -value determined by the laboratory varies according to the expression

$$\bar{q}_u = 200 + 40 \sin(z \cdot 180^\circ).$$

If $d = 1 \text{ m}$, the result depends also on the number n of the samples (with the exception of the case $z = 0, 1, 2, \dots$, where the effective $\bar{q}_u$ -value is obtained). If, e.g., $z = 0.25 \text{ m}$, then the deviation from the mean is zero, if n is an even number and $40/n\sqrt{2}$, if n is an odd number (part b of the figure), i.e. it decreases hyperbolically with the thickness of the layer. The case $L = d$ represents a relatively greater danger, because we get no information whatsoever about the presence of periodic elements even in the favourable case, when $\bar{q}_u$ corresponds to the effective value.

From the above we may draw the trivial conclusion that the samples must be taken as densely as possible. With regard to the real possibilities this may be modified as follows:

a) it is advised to increase the number of the core samples at least in one drilling to the maximum;

b) it is advised to take also the *water content* (i.e. also the partly disturbed samples) into account in the evaluation of the data series. In this connection also the probing may be useful.

4.4. Determination of the minimum number of samples (tests)

The number of the soil samples, resp. tests to be performed on the site and in the laboratory is determined by the following factors:

- the value of the planned object and
- its demand for soil tests,
- the required accuracy of the dimensioning,
- the measure of uniformity of the stratification,
- the homogeneity of the layers,
- the required cost and time expenditures of the explorations and tests, the available capacity and
- the measure of the reliability of these works.

These are generally valid viewpoints applying to the test of any material, with a proper interpretation. This means that the *basic principles* of the statistical processes elaborated for determining the required number of the samples are valid also for the soil. But in addition also the basic differences between the physical properties of the soil and the artificial building materials must be taken into account as follows (RÉTHÁTI, 1978a, 1979a).

a) During the sampling the physical characteristics, the structure and the stress state of the soil change due to its relatively *low strength* in a measure depending on the original state, the depth of the layer and the groundwater level, as well as the technology of the drilling (sampling).

b) The soil is a material consisting of *three phases*, as it contains not only air, but also a considerable quantity of water, with the consequences that

- during the transport and the storage it *dries* gradually, whereby its strength, its deformation and state characteristics change in a hardly controllable manner;

- the test results reflect also the *time factor*, whereby the unequivocal interpretation and the determination of the physical characteristics (shear strength, compression modulus, unconfined compression strength) is aggravated.

c) The soil in its natural state and position is in itself rather heterogeneous; measure of this may be increased by changes in meteorological conditions and the groundwater level in space and time (see Sections 4.2.2 and 4.2.3).

These circumstances refer unequivocally to the fact that correct sampling and testing methodology can be elaborated only by *statistical methods, adjusted to the given geotechnical conditions*.

The mathematical statistics start from the *reliability* of the expected value and the standard deviation determined on the basis of the samples (tests) (PRÉKOPA, 1962). With the help of the theses of the probability theory it may be proved that the standard

deviation of the expected value is

$$D(\bar{x}) = \frac{\sigma}{\sqrt{n}} \quad (4.16)$$

where n is the number of elements of the sample.

As according to a variant of the central limit distribution theorem $\bar{x}$ is approximately of a normal distribution if n is sufficiently large, the interval

$$\bar{x} \pm \lambda \frac{s}{\sqrt{n}} \sqrt{1 - \frac{n}{N}} \quad (4.17)$$

covers approximately the expected value of the population consisting of N elements with a probability of $2\Phi(\lambda) - 1$. In practice N can be regarded as infinite relative to n , while at the same time n is not sufficiently large for permitting to regard $\bar{x}$ as normally distributed. So expression (4.17) must be modified as follows:

$$\bar{x} \pm t_{\alpha} \frac{s}{\sqrt{n}}, \quad (4.18)$$

where t_{α} is the function value of the STUDENT-distribution belonging to the chosen probability level α and the degree of freedom $n - 1$.

If the expected value of the population is μ , then the condition that the difference $\bar{x} - \mu$ should be smaller with a probability of α than the value Δ determined by us according to the expression (4.18) is

$$\tau = \frac{t_{\alpha}}{\sqrt{n}} < \frac{\Delta}{s}. \quad (4.19)$$

As in the quotient standing on the left side t_{α} is itself a function of n , it is advised to perform the calculations with the help of *Table 4.4* instead of the gradual approximation.

There are two possibilities for a priori estimating the empirical standard deviation s appearing in formula (4.19):

- a) by determining its approximate value on the basis of the first 8–10 tests, or
- b) by accepting the values given in literature for the involved characteristics.

The advantage of the first method is that it can take into account the properties of the surveyed area (e.g. the natural heterogeneity of the layer) and the test methodics (sampling method, etc.) as well, while the latter method relies on considerably more data.

Let us assume that we wish to determine the mean water content of the soil S2 to be used for constructing an *embankment*, with an accuracy of $\Delta w = 1.5\%$, with a probability of 95%. As according to *Table 1.12* the standard deviation of the water content based on 144 tests is $s = 4.1$, so $\Delta/s = 0.366$, i.e. $n = 30$ samples at least must be sampled for attaining the required accuracy (see *Table 4.4*).

Table 4.4. n - and τ -value pairs for $\alpha=0.90, 0.95$ and 0.99

n	$\tau_{0.90}$	$\tau_{0.95}$	$\tau_{0.99}$	n	$\tau_{0.90}$	$\tau_{0.95}$	$\tau_{0.99}$
1	—	—	—	18	0.410	0.497	0.683
2	4.465	8.985	45.013	19	0.398	0.482	0.660
3	1.686	2.484	5.730	20	0.387	0.468	0.640
4	1.177	1.591	2.920	21	0.376	0.455	0.621
5	0.953	1.241	2.059	22	0.367	0.443	0.604
6	0.823	1.050	1.646	23	0.358	0.432	0.558
7	0.734	0.925	1.401	24	0.350	0.422	0.573
8	0.670	0.836	1.237	25	0.342	0.413	0.559
9	0.620	0.769	1.118	26	0.335	0.404	0.547
10	0.580	0.715	1.028	27	0.328	0.396	0.535
11	0.546	0.672	0.955	28	0.322	0.388	0.524
12	0.518	0.635	0.897	29	0.316	0.380	0.513
13	0.494	0.604	0.847	30	0.310	0.373	0.503
14	0.473	0.577	0.805	40	0.266	0.315	0.412
15	0.455	0.554	0.769	60	0.216	0.256	0.344
16	0.438	0.533	0.737	120	0.151	0.180	0.239
17	0.423	0.514	0.708	∞	0	0	0

If we divide the numerator and the denominator of the fraction standing on the right side of expression (4.19), we obtain the expression

$$\frac{t_{\alpha}}{\sqrt{n}} < \frac{\Delta_r}{C_v}, \quad (4.20)$$

where $\Delta_r = \Delta/\bar{x}$, the maximum error committed in the estimation of the mean and C_v is the coefficient of variation. This formula can be used in *geotechnics* better than (4.19), because according to Sections 1.2.3 and 1.3

— the coefficient of variation varies hardly at all with the mean (contrarily to the standard deviation),

— C_v values are given in literature more frequently, than s values for the various soil physical characteristics.

Let us assume that we wish to determine the void ratio of the soil *S4* with an error of 5% max. According to Table 1.12 $C_v = 0.11$, so $\Delta_r/C_v = 0.05/0.11 = 0.455$, to which $n = 21$ belongs in Table 4.4 (on the probability level of 95%).

The required number of the samples (tests) depends on several factors, even in the case of the *same layer*, namely on

- the geographical location of the investigation,
- the sampling technology and
- the test methodics.

Due to the effects of these factors significant differences may appear in the mean and the standard deviation, resp. in the coefficient of variation of the physical

characteristics. The consequence of the *territorial variability* may be evaluated on the basis of Table 4.2. If we wish to determine the void ratio with 5% maximum error, then in area *A* we must sample—on a probability level of 95%— $n = 27$ samples from the soil *S3* and $n = 20$ samples from the soil *S4* ($\Delta_r/C_v = 0.05/0.126 = 0.397$ and $0.05/0.107 = 0.467$, respectively). For the other parts of the area we obtain the following values:

	Area <i>B</i>		Area <i>C</i>	
	<i>S3</i>	<i>S4</i>	<i>S3</i>	<i>S4</i>
C_v	0.164	0.154	0.138	0.110
Δ_r/C_v	0.305	0.325	0.362	0.455
n	44	38	32	21

Table 4.5. Required minimum number of samples (tests) for the soils *S1* – *S5* ($\Delta_r = 0.10$, $\alpha = 0.95$)

Soil	Soil physical characteristics								
	w	w_L	w_p	I_p	I_c	e	S_r	γ_n	q_n
<i>S1</i>	38	63	48	115	100	28	7	5	79
<i>S2</i>	14	9	8	57	49	9	8	3	57
<i>S3</i>	11	9	8	52	84	9	7	3	105
<i>S4</i>	13	18	10	48	17	7	5	2	78
<i>S5</i>	13	32	11	76	42	9	6	2	97

According to the table in parts *B* and *C* of the area more tests (5 – 90%) must be performed than in part *A*, which proved to be more homogeneous. The role of the *sampling* and the *test methodics* is discussed in Section 6.2.6.

The statistical methods used for determining the required number of the samples and the data related to the dispersion of the tests call attention to the fact that the required number of the tests depends also on the *physical characteristics* of the soil to be determined. Table 4.5 points out the differences appearing in this respect.

4.5. Taking economic aspects into account

On determining the number of the tests two important viewpoints must be taken into consideration according to Section 4.4, each of which serves the purpose of designing the engineering objects on the basis of the *technical–economic optimum*.

On analysing the problem we must start from the fact that the soil physical characteristics forming the basis of the dimensioning cannot be specified by a *single numerical value*. Even if we neglect the variation of the physical characteristics in time and the random (eventually regular) errors appearing during the sampling and the tests, the *natural heterogeneity* must still be taken into account. The statistical methods

described in Section 4.4 serve to reduce the uncertainty caused by the latter. According to the inequalities (4.19) and (4.20) the uncertainty due to the heterogeneity of soils can be reduced successfully by

- a) increasing the number of the tests, and
- b) reducing the random dispersion by perfecting the sampling and testing methodology.

These solutions are *time and cost demanding*, i.e., they affect the economy of the design. On dimensioning the object we must always keep in mind that there is a very close relationship between the *factor of safety* and the *number of the tests*, permitting to determine the economic optimum.

The expressions (4.19) and (4.20) take into account from among the general considerations mentioned in the introduction of Section 4.4 the uniformity of the stratification, the degree of the homogeneity and the reliability of the exploration (tests). The required accuracy of the dimensioning, the value and the test requirements of the objects may be taken into account a priori—if no economic calculations are performed—by accepting a suitable *probability level*; the literature recommends $p = 0.90 - 0.95$; the lower limit is 0.85 (GOST-20522 1975).

So far we have dealt only with the number of the samples, resp. laboratory tests, with the tacit assumption that the position and the number of the explorations is given. In the following an example of taking the drilling costs into account in the economic calculations is shown.

Let us assume that we have to give a soil mechanical expert opinion for designing a road and the drillings reveal that the soil over a *section* of the track is exposed to the risk of frost. In such cases we can give no unequivocal proposition for delimiting the position of the protective layer against the frost (*Fig. 4.17, a*). The following two questions arise in this connection: (a) for how long a section should the protective layer be planned, and (b) how many interpolating drillings should be lowered for reducing the uncertainty to a minimum?

First we shall discuss the case, where the risk of frost cannot be evaluated by the *inspection* of the soil. Let us introduce the following notations:

- x_i — length of the section supplied with a protective layer,
- U — specific preparation cost of a 1 m long protective layer (with excavation of the mirror, exploitation, transport and compaction of the material),
- V — specific cost of a 1 m long demolition and rebuilding,
- A — volume/1 m length of the protective layer,
- p — probability of the demolition (damage).

The expected value of the *demolition cost* may be determined as follows. Be the probability of zero length of the section exposed to the risk of frost $p = 1$, that of a length d : $p = 0$; according to expression (4.14) p varies linearly between both extreme positions (part *b* of the figure). The risk of demolition exists for the length $d - x_i$ and its cost is

$$V \cdot A(d - x_i)p_i,$$

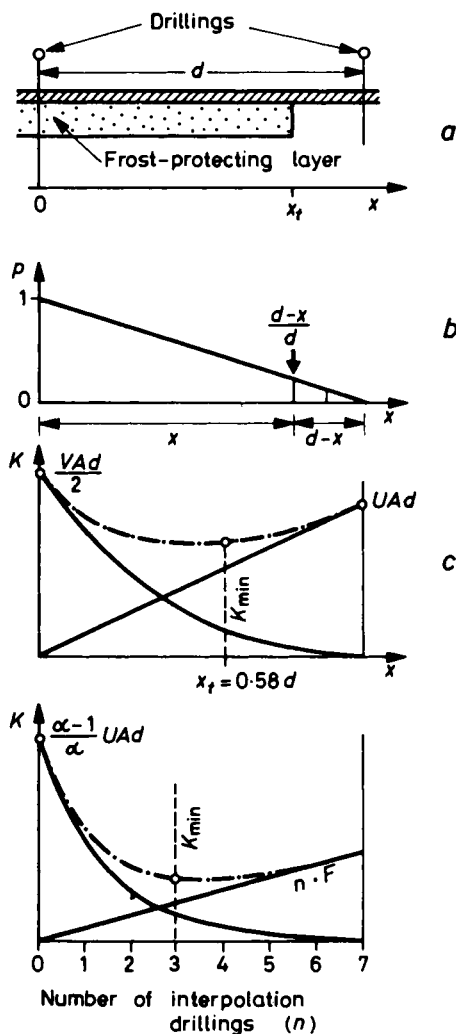


Fig. 4.17. Determining the length of a road section to be protected against frost and the number of the interpolating drillings, based on the principle of the economic optimum

where p_t may be calculated according to the figure by the following expression:

$$p_t = \frac{d - x_t}{2d}.$$

The total cost of the protection against the frost (K) varies with the length x of the built-in protective layer as follows (part c of the figure):

$$K = UAx + VA \frac{(d-x)^2}{2d}. \quad (4.21)$$

By introducing the notation

$$\alpha = V/U,$$

the derivative of expression (4.21) is

$$\frac{dK}{dx} = UA + \frac{\alpha UA}{2d} (-2d + 2x). \quad (4.22)$$

Making this equal to zero we obtain that the cost minimum may be attained by taking up

$$x_t = \frac{\alpha - 1}{\alpha} d. \quad (4.23)$$

(In part *c* of the figure $\alpha = 2.4$, so $x_t = 0.58d$.)

If we make an interpolation drilling at a distance of $x = d/2$, the cost of the frost protection over the section stretching between two drillings with different stratification will be

$$U \cdot A \frac{\alpha - 1}{\alpha} \frac{d}{2},$$

but the total cost is burdened also by the cost F of the drilling and the laboratory tests. It is easily admitted that the total cost varies with the number of the drillings (n) as follows:

$$K = nF + \frac{\alpha - 1}{\alpha} U \cdot A \frac{d}{2^n}. \quad (4.24)$$

The value n supplying the minimum cost may be determined graphically (part *d* of the figure), or analytically by the derivation of expression (4.24).

If the layer exposed to the risk of frost *can be recognized unequivocally* during construction, we may use the "learn as you go" method, proposed by TERZAGHI, i.e., in this case we delegate the responsibility for determining the required length of the protective layer to the contractor and the technical supervisor. In the cost estimate we must enter the sum

$$K = U \cdot A \frac{d}{2}$$

as the expected cost of frost protection.

5. Determination of the qualifying characteristics of the soil

The physical characteristics of the soil may be assigned to three groups (KÉZDI, 1974).

The first group is that of the *classifying (identifying) characteristics* (size, form and specific weight of the grains, quantity of the organic components, chemical properties). Their numerical values permit the objective characterization of the soils, with the exclusion of subjective elements.

The characteristics belonging to the second group (density, water content and consistency, degree of saturation) supply information on the *state* of the soil.

The third group contains the strength, the deformation and the hydraulic properties of the soil, which may be called by the common name *dynamical* characteristics.

This chapter deals with the classifying and the state characteristics, namely, how the mathematical statistical methods may be used in this field.

5.1. Granular soils

In the classification of granular soils the size and the distribution of the grains and of the pores play a primary role.

5.1.1. Analytical form of the grain size distribution curve

The soil is divided into fractions with the help of a series of sieves, assigning a mass percentage (S) to the various d grain diameters. The frequency curve (differentiating curve) plotted in the coordinate system $S, \log d$ is generally unimodal and of a relatively regular form; the area covered by it equals unity. On the basis of the above the assumption seems to be obvious that the integral curve (grain size distribution curve) obtained by continuous summation may be described by a suitably selected *probability distribution function*.

According to the investigations of JÁKY (1944) the grain size distribution curve may be approximated by the following function:

$$S = \exp \left[- \frac{1}{p^2} \ln^2 \left(\frac{d}{d_0} \right) \right], \quad (5.1)$$

where S — quantity of grains whose diameter is below d (expressed as a relative number),
 p — so-called grain-size distribution index ($= \ln d_0/d_{37}$), characterizing the stretching of the curve,
 d_0 — largest grain diameter.

In a suitable probabilistic coordinate system the point pairs S, d are positioned on a straight line, resp. can be equalized by such a line.

The majority of the granular soils was created by *diminution processes*. According to Section 2.1.3 the characteristic dimensions of the “end-product” which developed in this way, follow generally a *lognormal* distribution. As the derivative of expression (5.1) is

$$S' = \frac{2}{dp^2} \ln \frac{d}{d_0} \exp \left[-\frac{1}{p^2} \ln^2 \left(\frac{d}{d_0} \right) \right] \quad (5.2)$$

and this is not identical with the density function, defined by Eq. (2.20), we may state unambiguously that JÁKY did not assume a lognormal distribution. The difference between both function types may be evaluated by comparing the expressions (5.2) and (2.20). Figure 5.1 illustrates by an example their numerical deviations. The point pairs calculated with the constants $d_0 = 20$ mm, $p^2 = 4.5$ are positioned on a curve in the

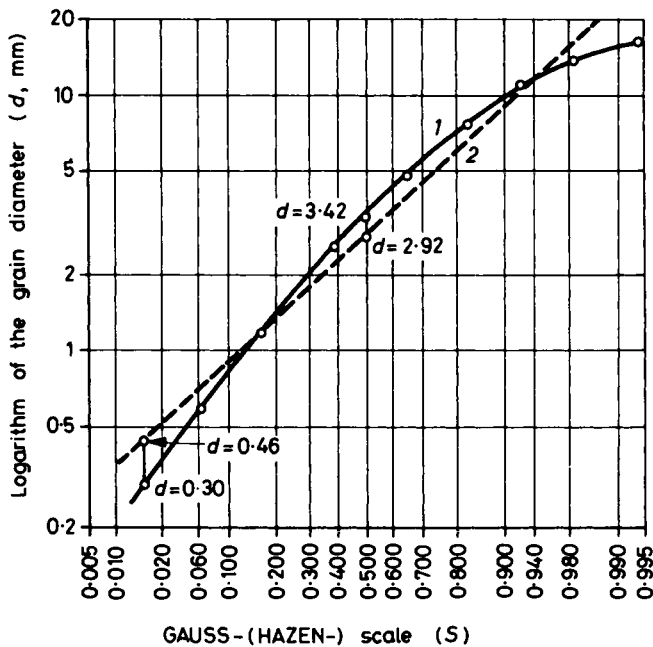


Fig. 5.1. Approximation of a grain size distribution curve described by JÁKY's relationship ($d_0 = 20$ mm, $p^2 = 4.5$) (1) and with lognormal distribution (2)

system of the GAUSS-HAZEN coordinates constructed with logarithmic ordinates, even though their deviation from the regression straight line is insignificant.

In connection with the analytical form of the grain size distribution curve we must clear up yet another question. It is usually assumed that the range of the small dimensions cannot be delimited unequivocally, because of the uncertainty appearing in the determination of the fine grain quantity. The probabilistic view suggests that also the size d_0 must not be assumed as known, though the largest grain of the sample would make this apparently possible. But this size is a random variable, similarly to the others, even more so, as the "sample" is only a part of the population, which may be regarded as infinite. This view means practically no difficulty whatever, because the mass percentage belonging to the size $d = d_0$ will be close to unity, whatever distribution type is selected.

5.1.2. Error sources of the determination of the grain size distribution

The difficulties connected with the determination of the grain size distribution curve begin with *sampling*. Apart from the fact, which is valid for all the physical characteristics, that the population can be characterized only approximately by the sample, also other disturbing factors aggravate the assertion of the principle that the sample tested in the laboratory must be *representative*. From among the most frequently occurring errors the following deserve special attention.

a) The grain size composition of the soils varies from point to point, as, e.g., the liquid limit (which depends also essentially on the grain size). Consequently the statements described in Section 4.4 are valid also for the *number of the samples*, resp. tests, similarly to the other physical characteristics.

b) From the material sent to the laboratory a *partial sample* must be separated. This operation may lead to the result—especially in the case of gravelly soils—that the tested material fails to represent even the submitted sample, not speaking of the population. The error due to this fact may be eliminated by carrying out the mechanical analysis of the *whole* material quantity which has been sent to the laboratory.

c) From the sample lifted through *groundwater* (the mass of the water found in the borehole) a part of the small diameter grains may be washed out. This can be avoided by preferring undisturbed samples to the partly disturbed samples, or by regarding the samples taken from the *capillary* range as ruling, not those which were sampled from below the groundwater level.

If numerous grain size distribution curves are available from the same layer, it is advised to perform their statistical evaluation graphically. KOTZIAS and STAMATOPOULOS (1983) evaluated the results of 491 tests for determining the suitability of the material for the filter layer of a dam. The results of their evaluation are presented in Fig. 5.2, which shows

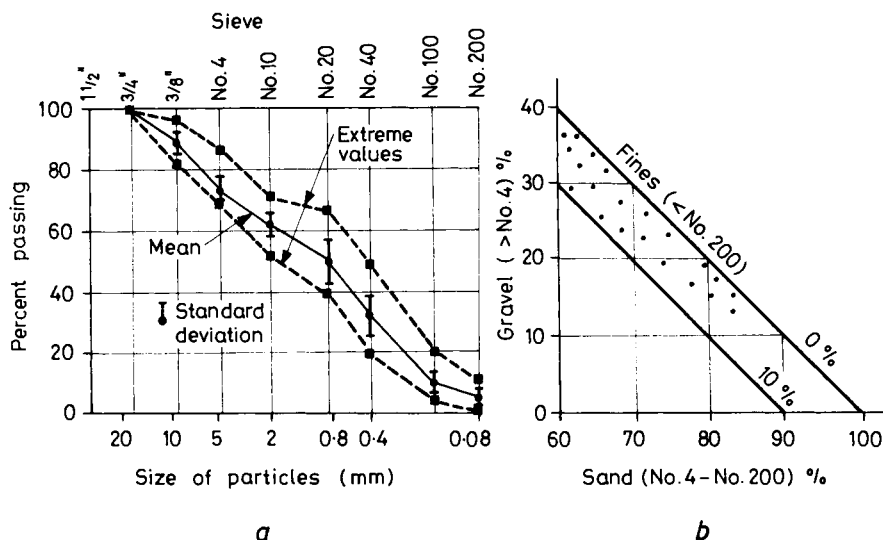


Fig. 5.2. Statistical parameters of grain size distribution curves determined for the same layer (KOTZIAS and STAMATOPOULOS, 1983)

a) the variations of the complete curve by explicitly stating the mean values, the standard deviations and the ranges of the grain sizes (part a of the figure);

b) the joint percentage distribution of gravel, sand and fines plotted in rectangular coordinates, the sand versus the gravel content for each curve (part b of the figure).

Figure 5.2 may be used for the following purposes (KOTZIAS and STAMATOPOULOS, 1983).

1. Where the overall features of numerous grain size distribution curves are of primary interest, as e.g. in filter design, in comparing different layers, or collating component parts of an earth/rock dam filling.

2. For choosing readily *representative* samples for performing specific tests, such as compaction, relative density, CBR, or strength based on gradation criteria encompassed by Fig. 5.2.

One of the greatest error sources of the tests is the fact that the grains are *not spherical*, as assumed in the evaluation of the screening and the hydrometer method. For example, the diameter of a slab-form grain may be even five-times as great as the diameter determined on the basis of the law of STOKES (KÉZDI, 1969). The “irregular” shape of the grains makes the separation of the fractions also uncertain, aggravating the process, which may be examined according to HARR (1977) as follows.

Be w the screen aperture, D the thickness of the wire and d the diameter of the assumably spherical grain (Fig. 5.3). For calculating the probability of the grain hitting against the wire we must start from the tangential position of the grain, i.e. from

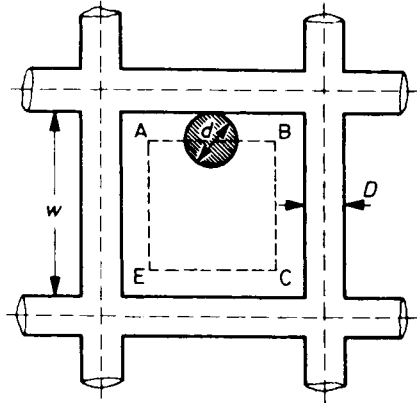


Fig. 5.3. Probability of a diameter d grain passing through a sieve of the aperture w (HARR, 1977)

the relative magnitude of the square marked by $ABCE$. By comparing the “favourable” and the “unfavourable” events it is easily seen that the probability that the grain will fall through the screen aperture is

$$P = \frac{(w-d)^2}{(w+D)^2}. \quad (5.3)$$

If, e.g., $w=0.074$ mm, $D=0.0021$ mm and $d=0.067$ mm, the probability that the grain hits against the wire, is $1-0.0085=0.9915$. By vibrating the screen N -times, the probability of the repeated impact is decreased to $(0.9915)^N$. The condition of the grain falling through the screen aperture with a probability of 90% may be calculated by the equation

$$(0.9915)^N = 0.10.$$

The solution of this equation ($N=270!$) shows that the accuracy of the sieving—especially of the manual sieving—is rather limited.

The *shape of the grains* is rarely determined in the soil-mechanical practice, so its significance (e.g. its effect on the compactibility) cannot be established as yet. So much is certain that the form characteristics of sandy gravel and of crushings of rocks influence the cube strength, the tensile strength and the optimal cement dosage of the concrete greatly (KAUSAY, 1966).

Using the hydrometer method, the presence of colloidal size grains causes additional difficulties. The settling process and along with it the determination of the grain-size distribution curve may be disturbed by two opposing effects: the *coagulation* of the grains and the *Brownian motion*. The coagulation may be prevented by adding chemicals, while the *Brownian motion* cannot be stopped, resp. it can be stopped just by bringing about the coagulation. Nevertheless it is worthwhile to deal with this phenomenon in more detail for its remarkable science-historical significance.

The phenomenon was named after a Scottish botanist, who published in 1828 his observations on the fast oscillating motion of pollen grains spread into a liquid.

According to the first scientific explanation by EINSTEIN (1905) the *random* motion of the grains is caused by *random differences* of the molecular stresses acting on the various surfaces of the grain. His theory has been generalized and developed by SMOLUKOWSKI, founding the highly important *random walk theory* of the probability theory. (An example of its utilization in geotechnics is given in Section 10.1 of the book.)

5.1.3. Description of the network of pores in the soil

The pores of the soil form a very complex system. The *shape* of the void spaces is irregular, their size is varying, their interconnections are also different. So the hypotheses concerning the movement of the liquid (e.g. the models assembled of parallel tubes) can be accepted only as very rough approximations. An independent chapter of physics deals with the so-called polycapillary systems and the phenomena taking place in porous media.

Very few researchers dealt with the characterization of the void system of the soil, due to the difficulties of the experimental work. This is very regrettable, because in the knowledge of the *size distribution* (diameter, cross section area, volume) of the pores we could describe better the movement of the water, the deformations caused by static load and compaction, or, e.g., the injection (stabilization) process.

At present the following possibilities are available for studying the void space system of the soil.

1. In the case of rough grained soils we may saturate the pores by a substance which is thinly flowing at high temperatures and solid at low temperatures, i.e. on cooling down.

2. The method based on pressing a non-wetting liquid into the soil (KÉZDI, 1979) may be used for examining finer grained materials as well. Under the effect of the pressure p the liquid enters only the pores of diameter d , for which the following relationship applies:

$$d = \frac{4\sigma}{p} \cos \Theta, \quad (5.4)$$

where σ —surface tension,

Θ —wetting angle.

The distribution of the pores may be determined by varying p .

3. Conclusions concerning the distribution of the pores by diameters may be drawn from the vertical distribution of the water content, resp. the degree of saturation of the capillary zones formed over the gravitational water level (RÉTHÁTI, 1957).

MANEGOLD (1937) established already in the nineteen-thirties the “complete disorder” of the voids and their extraordinarily great number contained in the unit volume permits in numerous cases to use the laws of the *probability theory*. They are

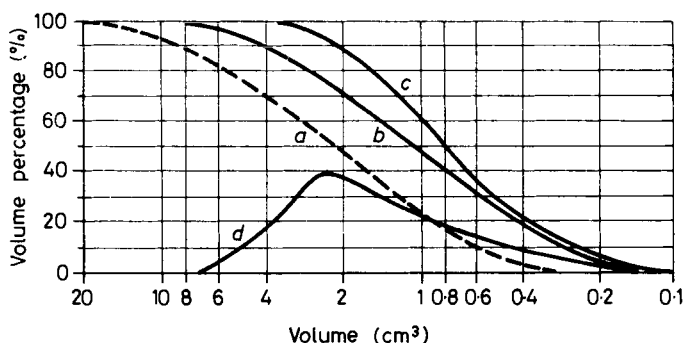


Fig. 5.4. Distribution of the grains and the pores according to their volume; *a* — solid parts, *b* and *c* — pores of the loose and the dense variants, *d* — ordinate differences of the two last curves (KÉZDI, 1979)

applicable in the first place to the geometrical dimensions of the pores, as the statistical description of the form and communication characteristics is much more difficult. In the following some research works related to the distribution of the pores are compared.

KÉZDI (1979) carried out experiments according to the first method with gravel and limestone crushings. Figure 5.4 presents the volumetric distribution curves of the solid components (*a*), resp. the pores formed in the loose state (*b*) and in the dense state (*c*) in one of the soils. Curve *d* illustrates the differences in the ordinates of curves *b* and *c*. Accordingly, the amount of the medium sized pores varies most during the compaction. KÉZDI explains this by the fact that there are few large pores and the small pores cannot be compacted sufficiently.

By the construction shown in Fig. 28 of the quoted book we can determine the diameter of the grains whose inferiors may be washed out. This example proves also that the knowledge of the pore distribution curve may be utilized in very large fields of geotechnics.

KOVÁCS (1977) used his results obtained by the first method mainly for determining the type of the pore distribution curve. By relating the original variable f , the cross-section area of the pores, to their mean value f_0 , he came to the conclusion that the new random variable ($x = f/f_0$) may be described by the *single parameter gamma distribution*: $x_0 = 0$, and as $\bar{x} = 1$, so $k = \lambda$ according to Eq. (2.31). The value of the parameter λ in the soil examined by him was about 1 ± 0.2 in the range $0.30 < n < 0.44$ of porosity, so according to Section 2.1.4, the *exponential distribution* seems to be an acceptable hypothesis.

The third method is based on the evaluation of w -, resp. S_r -profiles of the “open” capillary range of a saturation decreasing with height. In this range the force promoting the capillary rise increases gradually, so ever more pores, in which the tension required at the given height cannot develop, fall out from the water transport. The hydraulic gradient is very low here, the “lifting force” and the geometrical height are nearly

identical, so when we speak of water passages, we may assign to them pore diameters decreasing nearly hyperbolically with the height. As the grain size distribution has a determining role in forming the shape and the diameter of the passages and their distribution, the best agreement may be expected from the distribution functions written up on the analogy of the *grain size distribution curve*. In Section 5.1.1 we have examined two function types; that of JÁKY may be written up here in the following form (RÉTHÁTI, 1957):

$$S_r = S_{re} \exp \left[-\frac{1}{R^2} \ln^2 \left(\frac{z}{h_c} \right) \right], \quad (5.5)$$

where S_{re} — degree of saturation developing in the lower capillary range,
 R^2 — characteristic constant of the soil ($\neq p^2$),
 z — vertical distance measured from the gravitational water level,
 h_c — height of the lower capillary range.

Table 5.1. Example for the determination of the constants of Eq. (5.5) by error calculation

z (cm)	Measured S_r	$h_c = 10 \text{ cm}, R^2 =$				$h_c = 12 \text{ cm}, R^2 =$				$h_c = 13 \text{ cm}, R^2 =$		
		1.2	1.05	1.02	1.0	0.67	0.64	0.60	0.58	0.50	0.48	0.45
14.9	0.89	0.876	0.859	0.856	0.853	0.932	0.929	0.925	0.922	0.963	0.962	0.959
16.9	0.79	0.795	0.769	0.763	0.759	0.839	0.833	0.823	0.817	0.871	0.866	0.858
18.9	0.75	0.713	0.680	0.672	0.667	0.735	0.724	0.709	0.701	0.756	0.747	0.733
20.9	0.66	0.636	0.596	0.587	0.581	0.632	0.618	0.599	0.588	0.637	0.625	0.606
23.0	0.53	0.561	0.516	0.507	0.500	0.532	0.516	0.494	0.482	0.522	0.508	0.485
25.0	0.40	0.497	0.450	0.439	0.432	0.448	0.431	0.407	0.395	0.425	0.410	0.387
26.9	0.33	0.442	0.394	0.383	0.376	0.378	0.361	0.338	0.325	0.347	0.332	0.309
28.9	0.24	0.391	0.342	0.331	0.324	0.316	0.299	0.276	0.264	0.279	0.265	0.242
$s_f =$		0.083	0.062	0.061	0.062	0.047	0.040	0.039	0.042	0.046	0.044	0.047

The constants R^2 and h_c of the equation may be determined most simply by error calculation. Table 5.1 presents the evaluation of an experiment performed with a sand fraction consisting of diam. $0.22 \text{ mm} < d < 0.29 \text{ mm}$ grains. Column 2 of the Table contains degrees of saturation assigned to the given height, as calculated from the water quantity absorbed by the dry soil; columns 3–14 show the S_r values belonging to various combinations of h_c and R^2 . On calculating standard deviations by the deviations from the measured values, we obtain that the value pair

$$h_c = 12 \text{ cm}, \quad R^2 = 0.60$$

gives the best fit; the residual standard deviation in this case is $s_f = 0.039$.

On plotting the value pairs in a *lognormal* probability coordinate system (Fig. 5.5) we find that the point pairs determine the regression straight line very accurately ($r = -0.993$, $s_f = 0.025$). A closer correlation exists with the other set of the points, shown in the figure, which is related to a sand fraction of diam. $0.08 \text{ mm} < d < 0.50 \text{ mm}$

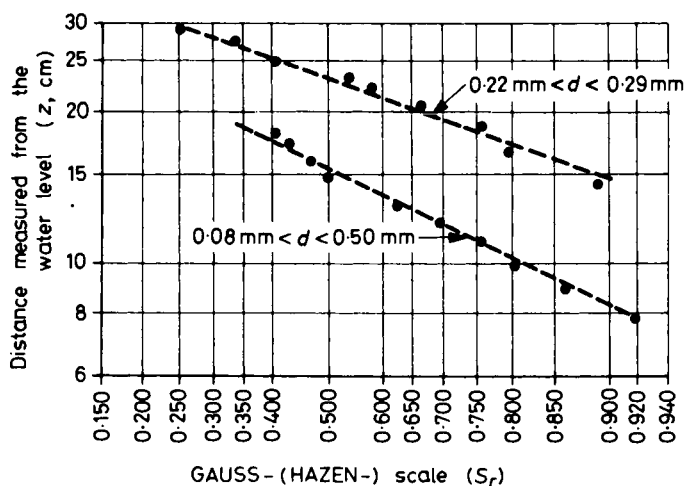


Fig. 5.5. Variation of the degree of saturation of two sand fractions in the open capillary range versus the distance measured from the gravitational water level

grains; the correlation coefficient is $r = -0.997$ and the residual standard deviation is merely 0.013. From these results we must draw the conclusion that the distribution of the pores by diameters can be described with the help of the lognormal function.

The slopes of the straight lines seen in Fig. 5.5 are very different, in spite of the fact that the grain diameter belonging to $S = 0.5$ is nearly identical for both soils ($d_{50} = 0.24$ and 0.26 mm). The cause of the difference may be found in the different stretching of the grain size distribution curves: for the first soil the uniformity coefficient is $U = 1.13$, for the second one it is $U = 1.62$. In a soil of a mixed grain size distribution the pore distribution curve is also more stretched and *small diameter passages* may also develop, which occur in soils of a uniform grain size distribution only in an essentially lower number.

From the numerical values of the saturation, developing in the lower capillary range, we had to draw the conclusion that the shape and the connection of the passages, the characteristics of the capillary spaces *are similar in the loose and the dense states* (RÉTHÁTI, 1957). For example, the result of 14 experiments gave for a set consisting of diam. $0.08 \text{ mm} < d < 0.50 \text{ mm}$ grains that $\bar{S}_{re} = 0.849 \pm 0.015$, i.e. the dispersion is very small, in spite of the fact that the initial void ratio varied between $e = 0.60$ and 0.87 . For four other fractions the following values were obtained:

	1	2	3	4
$\bar{S}_{re}$	0.979	0.990	0.975	0.977
e	0.59–0.80	0.56–0.80	0.73–0.96	0.56–0.78
s	0.017	0.021	0.021	0.017

5.2. Cohesive soils

According to the practice of several decades, the cohesive soils are *classified* on the basis of their plasticity index, but for the genetic distinction, the identification of layers and for estimating the expected values of the physical characteristics, the liquid limit and the plastic limit may be evaluated also independently (or in relation to I_p).

The *state* of the soil is judged usually by the numerical values of the consistency index and the unconfined compression strength. Though the order of magnitude of the strength and the deformation parameters may be deduced from these characteristics, their use as state characteristics is objectionable for several reasons. The *consistency index* is unsuited for this purpose, because

- a) it can be determined only by remoulding the soil, so we must compare the disturbed (w_L , w_p) and the undisturbed (w) states;
- b) though the strength of every soil is nearly identical at the liquid limit (see the investigations of JÁKY, 1944), for the plastic limit this is not even approximately true;
- c) it gives no information on the saturation, i.e.—indirectly—on the void ratio.

The *unconfined compression strength* and the *void ratio* alone are unsuited for characterizing the state of the soil, as they vary considerably with the clay mineral content. The natural soil cannot be characterized by the degree of compaction (T_r) used for evaluating the density of earthworks, while the relative density cannot be used because of the uncertainty in the determination of e_0 and $e_{\max}$, so *the density of the cohesive soils cannot be evaluated realistically*.

The solution of the problem can be approximated by mathematical statistics. During practice every engineer acquires a feeling, by which he can judge, if the conditions of the investigated case are *better or worse than average*. If this guesswork can be assisted by numerical values, the subjective judgment is supported by objective data. But this aim can be attained only, if we determine the empirical distributions of the involved characteristics.

5.2.1. Identification and classification on the basis of the consistency limits

According to the examples given in Section 3.5.2 a surprisingly close linear relationship exists between the liquid limit and the plasticity index of the soil. CASAGRANDE observed (still in the thirties) that from the constants of the straight line, i.e. from their position occupied in the coordinate system w_L , I_p , conclusions may be drawn as to the organic content and the deformation characteristics of the soil. According to his observations the straight line, equalizing the point pairs w_L , I_p of soils of an identical origin, is parallel with the so-called *A line*, which may be described in the form:

$$I_p = 0.73(w_L - 20) = 0.73w_L - 14.6. \quad (5.6)$$

Table 5.2. The relationship w_L , I_p for various soils

Soil	Author	n	$I_p =$	r
Cohesive soils of Hungary	RÉTHÁTI (1968a)	1937	$0.836 w_L - 15.5$	
Grey Kiscell Clay	PAÁL (1975)	444	$0.77 w_L - 12.9$	> 0.9
Yellow Kiscell Clay		476	$0.77 w_L - 12.0$	> 0.9
Buda Marl		229	$0.72 w_L - 9.6$	> 0.9
Pleistocene	SZABÓ (1976)		$0.85 w_L - 11.9$	
Pannon			$0.73 w_L - 11.7$	
Helvetan			$0.79 w_L - 14.2$	
Szeged, soil S1	RÉTHÁTI and UNGÁR (1978c)	81	$0.576 w_L - 5.1$	0.837
soil S2		67	$0.959 w_L - 18.0$	0.861
soil S3		461	$0.829 w_L - 15.2$	0.863
soil S4		442	$0.876 w_L - 17.4$	0.943
soil S5		207	$0.823 w_L - 16.1$	0.972

The observations of CASAGRANDE were basically verified by the research of the last years, except for his statements concerning the directional tangent of the "classifying" straight lines, as the straight lines representing the *inorganic soils* are generally steeper than the *A line*, as proved by the data of Table 5.2. The mean and the standard deviation of the constants of the involved soils are

$$a = 0.81 \pm 0.07 \quad \text{and} \quad b = -14.1 \pm 2.5,$$

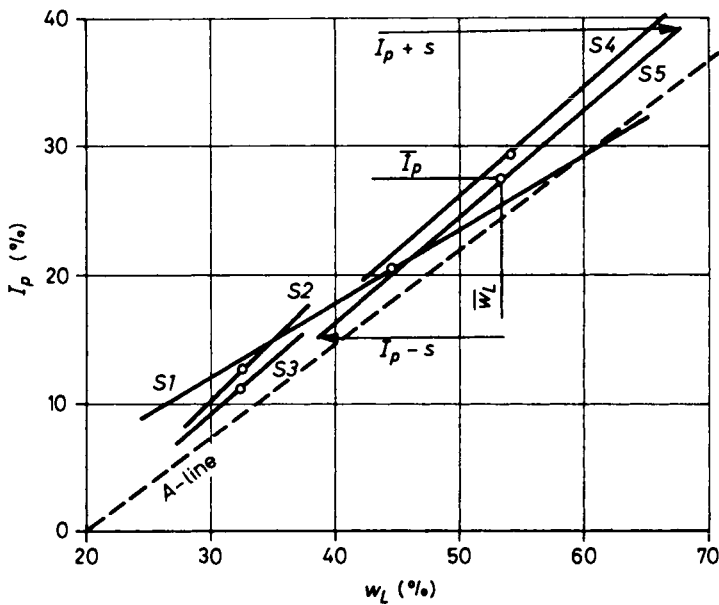


Fig. 5.6. The relationship (w_L , I_p) characterizing the soils of Szeged, indicating also the standard deviation (s) of I_p

Table 5.3. Mean and standard deviation of the void ratio by I_p -categories and its correlation with w_L , w_p and Δ on the basis of samples from various areas of Hungary (RÉTHÁTI, 1971a)

I_p (%)	Above groundwater level $r =$					Below groundwater level $r =$				
	$\bar{e}$	s	$e(w_L)$	$e(w_p)$	$e(\Delta)$	$\bar{e}$	s	$e(w_L)$	$e(w_p)$	$e(\Delta)$
0 – 4.9	0.655	0.112	0.644	0.569	0.513	0.661	0.099	0.842	0.800	0.768
5 – 9.9	0.665	0.105	0.369	0.417	0.428	0.676	0.154	0.467	0.419	0.376
10 – 14.9	0.665	0.138	0.405	0.439	0.422	0.700	0.119	0.458	0.462	0.458
15 – 19.9	0.679	0.129	0.109	0.131	0.134	0.740	0.146	0.226	0.167	0.122
20 – 24.9	0.679	0.145	0.356	0.390	0.389	0.775	0.199	0.524	0.537	0.536
25 – 29.9	0.708	0.137	0.421	0.401	0.382	0.776	0.149	0.298	0.296	0.286
30 – 34.9	0.738	0.168	0.628	0.645	0.643	0.870	0.224	0.553	0.546	0.534
35 – 39.9	0.782	0.149	0.400	0.372	0.353	0.917	0.221	0.509	0.483	0.466
> 40	0.867	0.249	0.580	0.545	0.407	0.986	0.231	0.617	0.588	0.436
Mean:		0.169	0.435	0.435	0.408		0.171	0.499	0.478	0.442

referring to the fact that the totality of the straight lines determine a *beam of rays* rather than a set of parallel straight. Also Fig. 5.6 constructed for the soils of Szeged seems to verify this if we disregard the extraordinarily heterogeneous soil *S1* with a variable organic content. (The length of the straight lines in the figure refers to the *heterogeneity* of the layers, as namely the end-points correspond to the $I_p \pm s$ values.)

From the consistency limits we may conclude on the magnitude of the *void ratio*, confirming the observation of CASAGRANDE, that the position of the point pairs w_L , I_p refers also to the deformation properties. This can be proved by fixing the value of I_p (as required according to Section 5.2.2), then establishing correlations within each I_p category between e and Δ , w_L and w_p (the interpretation of Δ see in Section 5.3). Table 5.3 contains the results of 3245 samples processed in this way, originating from various

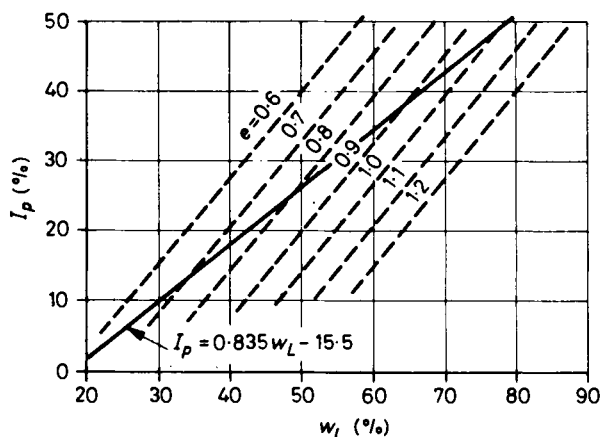


Fig. 5.7. Estimation of the natural void ratio of soils lying below the groundwater level in the knowledge of the liquid limit (w_L) and the plasticity index (I_p) (RÉTHÁTI, 1971a)

regions of Hungary, permitting to make the following statements concerning the empirical relationships:

- a) for the estimation it is indifferent which of the three independent variables is used for calculating the void ratio;
- b) the correlation—as expected—is somewhat closer in the case of samples taken from below groundwater level.

On selecting the relationships with the highest correlation coefficients from *Table 5.7* we may plot the straight lines of $e = \text{const.}$ in the system of the w_L, I_p coordinates (*Fig. 5.7*). Naturally the estimation based on this serves only for a preliminary orientation, but—for want of something better—it may be used for the evaluation of old expert opinions which gave only the consistency limits of the soil.

The relationship between the organic content and the consistency limits of the soil was dealt with in Section 3.7.5; further details are given in Section 5.3.

5.2.2. The void ratio as a state characteristic

According to *Table 5.3* the mean void ratio of cohesive soils is practically constant up to $I_p \sim 15\%$ and from there on it increases rapidly. These statistical data conform with the observation that the cellular and floccular structures of the clays settled in water are the looser, the more colloidal grains they contain. The table reflects also the well-known phenomenon that clays lying below water are looser than the layers found closer to the surface, which are exposed to shrinkage. (The difference appearing in the density increases also with the plasticity index, reflecting the quantity of the clay mineral content.)

From the above it follows unequivocally that the density of the soil can be evaluated realistically on the basis of the void ratio only if we assign to it also the plasticity index. Let us assume, e.g., that the tested soil lies below groundwater level, its plasticity index is $I_p = 28\%$ and its void ratio is $e = 0.81$. This means according to *Table 5.3* that the involved soil is looser than the mean; as according to expression (2.17):

$$u = \lambda = \frac{0.81 - 0.776}{0.149} = 0.2282,$$

to which $\Phi(\lambda) = 0.5903$ belongs in *Table F2*, we may state that in this I_p category 59% of the soils in Hungary are more dense than the soil tested by us.

Figure 5.8, a has been prepared by using this basic principle, with the following differences: (a) the position of the groundwater level was disregarded; (b) the probabilities characterizing the density were determined by “counting down” (see Section 2.2.2), not from the normal (or other) distribution function. So five ranges of the density may be assigned to every I_p category, whose names and frequency values

are as follows:

very loose	$f_r = 1/12$;
loose	$f_r = 3/12$;
medium dense	$f_r = 4/12$;
dense	$f_r = 3/12$;
very dense	$f_r = 1/12$.

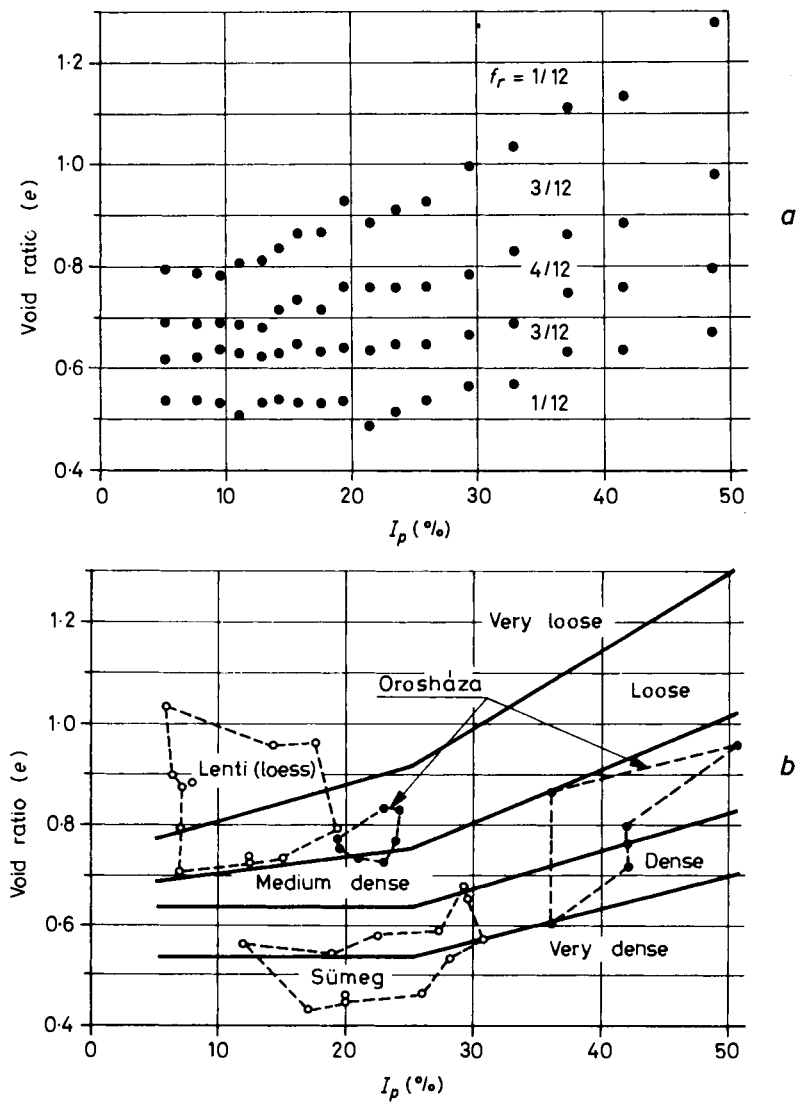


Fig. 5.8. Characterization of the state of cohesive soils by the void ratio (RÉTHÁTI, 1968a, 1973); *a* — designating the categories of density on the basis of the relative frequencies; *b* — denomination of the categories and examples

On equalizing the points by straight line sections we obtain *Fig. 5.8 b* (accepted also in the Hungarian standard). The figure shows also the point pairs " I_p , e " of four different layers. From their position the following conclusions may be drawn:

a) it is advised to examine the layers *individually* even within the same site (e.g. the leaner variant of the Orosháza Clay is loose, its more plastic variant is dense, resp. medium dense);

b) additional information may be gained for the qualification of the soil by relating the samples to the depth of the *groundwater level* (e.g. the clay of Sümeg above groundwater level could be assigned certainly to the category "very dense").

The representation in the system of coordinates " I_p , e " points out also the *differences in the formation and the mineral composition* of the layers, completing in this way the tests described in Section 5.2.1.

Figure 5.9 shows the density of the soils of Szeged (points 1...5 correspond to the mean void ratio and plasticity index calculated for the layers *S1*...*S5*). According to the figure

the soil *S1* is loose—very loose,
the soils *S2* and *S3* are medium dense, while
the soils *S4* and *S5* are loose—medium dense.

The loose structure is due mainly to the organic content in the case of soil *S1*, while in the case of soils *S4* and *S5* to their position below the groundwater level. The loess of a low plasticity is not loose even under the groundwater level, because its swelling capacity is very low (this is why the points 2 and 3 are so close to each other).

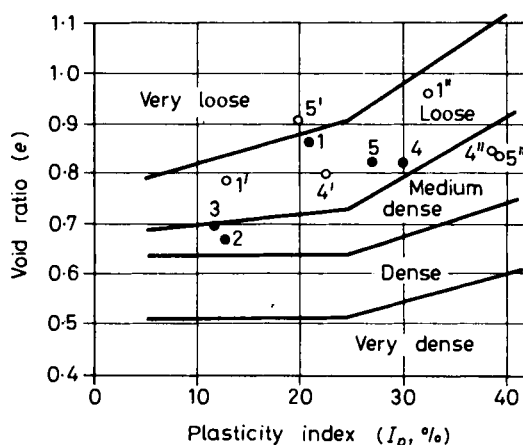


Fig. 5.9. Qualification of the soils of Szeged on the basis of their average void ratio; 1...5 for all the samples taken from the soils *S1*–*S5*, 4' and 5' for samples whose plasticity index is less than the median, 4'' and 5'' for samples whose plasticity index exceeds the median (RÉTHÁTI and UNGÁR, 1978c)

An interesting result is obtained if within each layer the *samples of low and high plasticity* are separated by a boundary line at the median of I_p . According to Fig. 5.9:

- both the low and the high plasticity variants of the soil $S1$ ($1'$, resp. $1''$) proved to be loose;
- the low plasticity variant ($4'$) of the soil $S4$ are looser, the high plasticity variant ($4''$) of the same are more dense than the mean value characterizing the soil;
- the low plasticity variant of the soil $S5$ ($5'$) qualified as loose is very loose, its high plasticity variant ($5''$) is moderately dense.

5.2.3. The unconfined compression strength as a state characteristic

The statistical processing of about 2000 samples from various regions of Hungary (RÉTHÁTI, 1968a, 1973) proved that q_u varies in the function of the plasticity index according to a *maximum curve* (Fig. 5.10). This may be explained by the fact that though the *adhesive effect* increases gradually with the increasing clay mineral content, yet from $I_p = 15-20\%$ on, this effect is counteracted ever more by the looser structure of the soil (see Table 5.3 and Fig. 5.8). The essentially lower mean of the soils lying below groundwater level, than of those found above this level is partly a consequence of the different void ratios as well.

We have said that the state of the soil can be judged by the void ratio only with the “mediation” of the plasticity index. According to Fig. 5.10 this applies still more to q_u . The category limits may be based on the same principles as in the case of e , but now the categories characterize the consistency of the soil and not its density (Fig. 5.11).

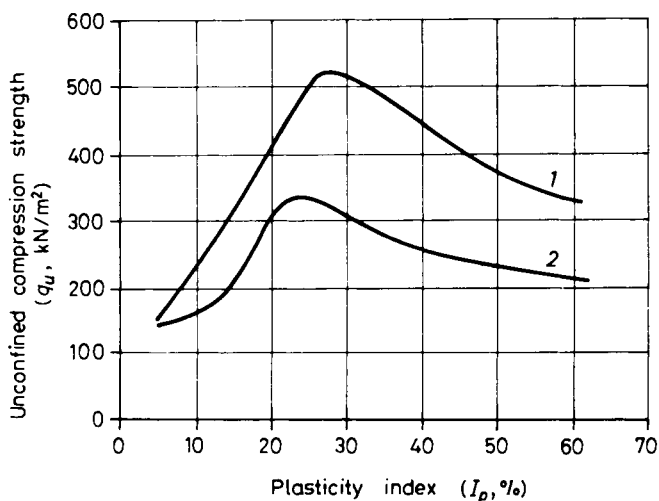


Fig. 5.10. Variation of the unconfined compression strength of the Hungarian cohesive soils with the plasticity index (RÉTHÁTI, 1968a, 1973); 1 — for soils found above the groundwater level, 2 — for soils lying below the groundwater level

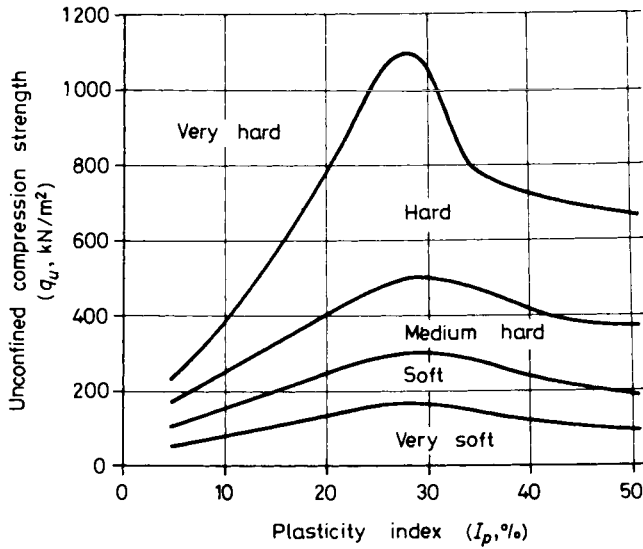


Fig. 5.11. Characterization of the state of cohesive soils by q_u and I_p (RÉTHÁTI, 1968a, 1973)

The *differentiating test* described in Section 5.2.2 may be performed also for q_u . The curve shown in Fig. 5.12 presents the category means of the unconfined compression strength versus I_p of 673 samples taken from below groundwater level in various regions of Hungary; the zero circles show the means of combined S3–S5 soils according to I_p . On the basis of the position of the zero-circles we may establish that

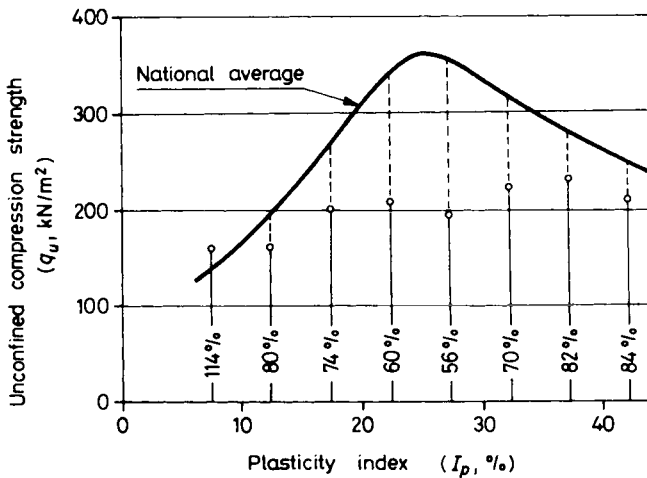


Fig. 5.12. Unconfined compression strength values of Szeged soil samples by I_p -categories, sampled from below groundwater level, in percentages of the countrywide mean (RÉTHÁTI and UNGÁR, 1978c)

a) the compression strength of the samples belonging to the low plasticity range ($I_p < 10\%$) attains the countrywide mean value, that of the fat clays ($I_p > 35\%$) approximates it;

b) the medium clays lag behind the mean value most;

c) the average ratio of the values of Szeged to those of the countrywide values is 0.78.

The soils of the town Szeged must be assigned to the “soft” category according to Fig. 5.11, except for the infusion loess (whose more favourable behaviour may be ascribed to its considerable CaCO_3 content).

The statistical data of the void ratio and the unconfined compression strength are in *complete harmony* and it can be proved also numerically that the soil conditions of Szeged are less favourable than the countrywide mean (especially the soils of intermediate plasticity).

5.3. Organic soils

We have established in Section 3.7.5 that there exists a close relationship between the organic content and the liquid, resp. the plastic limit. At the same time a doubt in the realistic value of the related calculations arose, because the research dealt with soils which were suspected a priori to contain organic substances. This disturbing circumstance may be excluded by substituting for the consistency limits their relative values in the calculation to each other, resp. related to I_p . Two possibilities offer themselves for doing so:

- choosing the distance measured from the A-line, as an independent variable, or
- setting up a tervariate correlation.

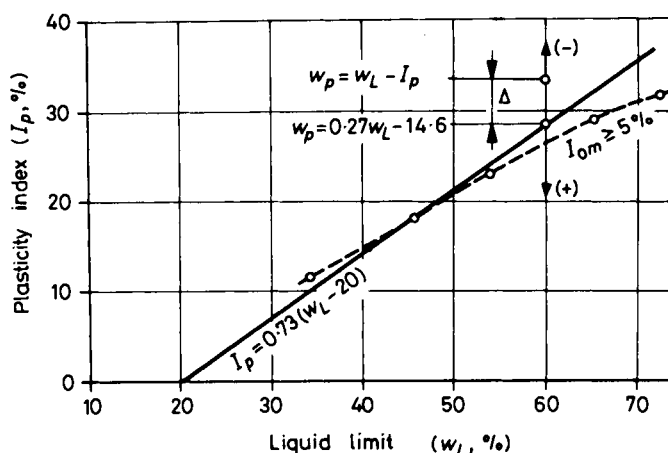


Fig. 5.13. Value pairs w_L , I_p by I_p -categories of soils with $I_{om} = 5\%$ organic content and interpretation of the difference $\Delta = w_p - w_{pc}$

The vertical distance (Δ) measured from the A line of CASAGRANDE with the correct sign may be calculated by the following expression (Fig. 5.13):

$$\Delta = w_p - w_{pC} = w_p - 0.27w_L - 14.6, \quad (5.7)$$

where w_{pC} corresponds to the plastic limit, at which the point representing the soil falls exactly on the A line (at the given w_L).

According to the regression analysis I_{om} is for the first 143 samples (see Section 3.7.5):

$$I_{om} = 0.355\Delta + 5.1 \quad (r = 0.697, s_f = 3.87), \quad (5.8)$$

and for the totality of the samples ($n = 311$):

$$I_{om} = 0.402\Delta + 5.8 \quad (r = 0.757, s_f = 4.32). \quad (5.9)$$

According to both relationships the points falling to the A line ($\Delta = 0$) represent soils of $I_{om} = 5.1$ – 5.8% organic content (RÉTHÁTI, 1966, 1968a), in harmony with the fact that the straight lines representing inorganic soils are above the A line (compare Table 5.2).

The relationship $I_{om} = f(w_L, w_p)$ determined for the first 143 samples by the tervariate correlation is:

$$I_{om} = 0.040w_L + 0.189w_p - 3.0 \quad (\rho_1 = 0.805, s_f = 3.20)$$

and for the totality of the samples:

$$I_{om} = -0.017w_L + 0.295w_p - 1.9 \quad (\rho_1 = 0.785, s_f = 4.10).$$

Accordingly, the relationship is closer (and the residual standard deviation is lower) than the relationships determined in the function of Δ . The explanation of this is that the tested organic soils give no parallel straight line with the A line in the coordinate system w_L, I_p (see Fig. 5.13).

Table 5.4. Residual standard deviation of the organic content for various independent variables

Variables	CaCO ₃ (%)			For the whole set
	< 2.8	≥ 2.8	0–56.5	
w_L	3.88	2.92	3.56	4.86
w_p	3.42	2.54	3.26	4.11
Δ	4.35	2.69	3.87	4.32
w_L, w_p	3.39	2.54	3.20	4.10

With the use of all the preceding variables and the data of Table 3.9 we obtain for the residual standard deviation of I_{om} the values given in Table 5.4, permitting to draw the following conclusions in addition to those established so far:

a) the relationships are closer for the first 143 samples and within them for soils of a high lime content than for the complete set, but we have no real basis for exploiting the advantages offered by this fact;

b) the organic content can be evaluated better by the plastic limit than by the liquid limit.

Certain laboratories characterize the organic content of the soil by the *ignition loss* (I), assuming—on the basis of empirical data—that the ignition loss belonging to the critical organic content ($I_{om} = 5\%$) is 10% . This approximation may be evaluated, on the basis of the available 309 value pairs, as follows. From 141 samples qualified as organic, the ignition loss of 8 samples was less than $I = 10\%$, so the approximation seems to be acceptable from this side (the probability level is $133/141 = 0.94$). Simultaneously in 30% of the tests assigned to the category of $I \geq 10\%$, the organic content was less than 5% (in 7.5% of the tests even less than 3%). This means that though this qualification is on the safe side, it is not sufficiently economical (if, e.g., material for an embankment must be transported from a great distance, after even an excess load of the excavating work, to replace the soil qualified erroneously as organic).

6. Comparison of tests performed with the same soil

On determining the statistical parameters of the physical characteristics in chapters 1 and 2 the *methods* used for this purpose were not dealt with. But we have stated that the variability of the physical characteristics depended on the distribution of the sampling in time, the magnitude of the area, the thickness of the layers and the methodics of testing (see Section 1.3). From among these factors we have discussed in detail the variability in space and time (Section 4.2), but the role and the significance of the test method was pointed out only through some examples.

In this chapter we shall try to find an answer first of all to the question of the variability caused by *repeated* experiments on testing a homogenized mass of soil originating from an identical layer. These numerical values supply information on the “eigen-dispersion” of the used apparatus (instrument) and the test methodics, as well as the reproducibility of the tests.

In the geotechnical practice we must often find out, if *two* apparatuses, or *two* test methodics may be regarded as equivalent. This is typically a mathematical–statistical task, on whose solution important technical and economical interests may depend.

The third part of this chapter deals with the case when the physical characteristics of a given soil are determined by *several laboratories*. The evaluation of similar test series is an essentially more complex task, as reflected also by the much larger ranges of the values of the physical characteristics, than in both preceding cases, as here the apparatuses and the persons performing the tests are not identical (and possibly not even the test methods).

6.1. The reproducibility of the tests

6.1.1. Definition and limits of the reproducibility

Tests repeated under identical conditions supply information on the accuracy and reliability of the apparatus (instrument) and the methodology chosen during the use of the latter. As we assumed that the standard deviation is of a random character, we can characterize the accuracy by very simple statistical methods, resp. parameters (see Section 7.4). This is the case also, where, e.g., a distance, or an angle is determined by geodetical instruments, repeating the measurements *in a number chosen arbitrarily*.

For investigating materials this process is modified significantly in a part of the cases, because the measurements cannot be repeated *on the same sample*. This is the situation also in the field of soil mechanics, as proved by the following examples. During the unconfined compression and the triaxial tests and during the compression tests the sample is deformed (in the first two cases broken) in a measure that its original state cannot be restored any more. Also the cases, where the samples are dried in a furnace, are similar: the repeated measurement of the water content is ab ovo impossible and the consistency limits cannot be determined repeatedly either, because their value may change by rewetting. The situation with the granular soils is not more favourable either, as, e.g., during the tests of the grain size distribution or the permeability, the original grain size composition may change.

On examining the measure of the reproducibility the implied errors can be reduced to the minimum only by homogenizing the mass of the soil to be tested very carefully. On evaluating the test series we must always consider how successful the homogenization was, or could be (e.g. forming equivalent partial samples from organic soils, or sandy gravel is a difficult task).

For repeated tests the *same apparatus* must be used (using the same type is not sufficient), controlling frequently, if its characteristic dimensions, etc. have not changed in the meantime. The same applies also to the *methodics of the tests*, where considerable subjective elements must be taken into account. (A source of error is, e.g., the variation of the vibration work of hand sieves from test to test.) Great care must be devoted to ensure that every circumstance, which may affect the test results, should be identical. This means, e.g., in the case of the permeability, or the capillary tests that the temperature of the liquid must be measured (it is more advised to ensure a constant temperature).

Accepting without control the *nominal* geometrical dimensions of the cylinders (rings) used for holding, or for "piercing" the sample may be the source of important errors; the constant value of these dimensions must be checked from time to time. The magnitude of the implied error may be controlled by forming the *differentials* of the physical characteristics. If, e.g., an error of ∂V is made in the measurement of the volume, the following deviation will appear in the degree of saturation (RÉTHÁTI, 1957):

$$\partial S_r = - \frac{w S_r^2 \cdot G_0}{(V \rho_s - G_0)^2} \partial V, \quad (6.1)$$

where in addition to the notations used so far

G_0 — dry mass of the sample,
 V — volume of the sample.

If, e.g., $w = 24\%$ ($= 0.24$), $G_0 = 120$ g, $\rho_s = 2.66$ g/cm³ and $V = 75$ cm³, then an error of $\partial V = -0.62$ cm³, i.e. of 0.8% made in the measurement of the volume leads to a deviation of 0.02 between the real and the measured value of S_r .

With regard to the subjective errors loading the tests, the individual properties of the laboratory assistant, his habits developed over the years, etc., the repeated tests must be performed always by the same person. This means also that the statistical parameter characterizing the reproducibility will reflect not alone the objective features of the apparatus and the test methodics, but also subjective elements.

6.1.2. Literature data on the reproducibility of soil physical characteristics

HAMMITT (1966) obtained the following values for the standard deviation and the coefficient of variation of the *water content* of the same soil, tested by several laboratories:

	<i>H</i>	<i>M</i>	<i>L</i>
<i>s</i>	2.7	0.82	0.92
<i>C_v</i>	13.1	6.3	6.7

(*H* = high plastic, *M* = medium plastic, *L* = low plastic).

The results of HAMMITT (1966) are summarized in *Table 6.1*, showing that the coefficient of variation of the liquid and plastic limits increases with *I_p*, while that of the plasticity index itself decreases with the increase of *I_p*.

BORUS and RÉV (1970) examined also the reproducibility of the water content determination. The first two rows of *Table 6.2* refer to artificially homogenized samples, the rest to nearly homogeneous samples taken from the same open-cast mine. This

Table 6.1. Dispersion in soil properties for "identical" soil samples (HAMMITT, 1966); *H* – high plastic, *M* – medium plastic, *L* – low plastic

		Mean	<i>s</i>	<i>C_v</i>
Liquid limit	<i>H</i>	54.3	5.4	0.099
	<i>M</i>	32.7	2.3	0.070
	<i>L</i>	27.0	1.7	0.063
Plastic limit	<i>H</i>	22.2	3.4	0.153
	<i>M</i>	22.2	2.8	0.125
	<i>L</i>	23.6	2.4	0.101
Plasticity index	<i>H</i>	32.0	5.7	0.178
	<i>M</i>	10.4	3.6	0.346
	<i>L</i>	3.8	2.1	0.552
Water content	<i>H</i>	20.6	2.7	0.131
	<i>M</i>	13.1	0.82	0.063
	<i>L</i>	13.1	0.92	0.067

Table 6.2. Reproducibility of the water content (BORUS and RÉV, 1970)

Soil	<i>n</i>	$\bar{w}$	<i>s</i>	<i>C_v</i>
Infusion loess	30	38.92	0.41	0.011
Kiscell Clay	30	47.78	1.10	0.023
Medium clay	27	28.52	1.02	0.036
Silt	28	12.70	1.11	0.087
Silty sand	24	19.37	1.97	0.102
Fine sand	28	2.81	0.54	0.192
Fine sand interwoven with silt veins	19	14.58	4.16	0.285

difference is reflected also in the test results: for the first two soils $C_v = 0.011-0.025$, for the rest $C_v = 0.036-0.285$ (the maximum belongs to the seemingly most heterogeneous soil, interwoven with clay veins). According to their earlier studies (BORUS and RÉV, 1968b) the error made in the determination of the water content in the vicinity of the liquid limit was

$$s_{rp} = 0.016w_L - 0.1,$$

so it is a function of the liquid limit as well.

According to the investigations of the *liquid limit* by the same authors the standard deviation increases linearly within w_L . On the basis of 30 tests from the same soil performed by each of two laboratory assistants the following relationship were obtained:

$$s_{rp} = 0.037w_L + 0.18$$

and

$$s_{rp} = 0.036w_L + 0.17, \text{ respectively.}$$

The correlation coefficients were $r = 0.926$, resp. $r = 0.962$. On taking the mean values of the constants appearing in the equations ($a = 0.0365$ and $b = 0.175$) $C_v = 0.042$ for $w_L = 30\%$ and $C_v = 0.039$ for $w_L = 60\%$ are obtained, so it is practically independent of the liquid limit.

According to the examination of the *plastic limit* (BORUS and RÉV, 1968b, 1970) the coefficient of variation characterizing the reproducibility varied between 0.002 and 0.093; the mean was $\bar{C}_v = 0.057$. This is somewhat higher than the value obtained for the liquid limit, which refers to a greater subjectivity in this test; a similar statement was made also by LUMB (1971). According to the observation of the authors the maximum of the standard deviation was at $w_p = 25\%$ as proved evidently also by their tests performed with a soil of a variable (artificially varied) sand content.

As the plasticity index is not a directly measured, but a derived quantity, its standard deviation is, according to the law of error propagation (compare with Section 1.2.2)

$$s = \sqrt{s_L^2 + s_p^2}, \quad (6.2)$$

where

s_L — standard deviation of the liquid limit,

s_p — standard deviation of the plastic limit.

The standard deviation characterizing the reproducibility of the *consistency index* may be calculated by the following expression:

$$s = \sqrt{\frac{(1 - I_c)^2 + s_L^2 + s_w^2 + I_c^2 s_p^2}{I_p^2}}, \quad (6.3)$$

where s_w is the standard deviation characterizing the reproducibility of the water content determination.

Another part of the literature data refers to the density of the soil. The coefficient of variation of the *dry bulk density* according to the repeated tests of BORUS and RÉV (1970) varied between 0.013 and 0.047, the mean value was $\bar{C}_v = 0.025$. During their research they have tested 13 different soils by using a diam. 40 mm sampler for taking samples from open-cast mines (the average number of the tested samples was 28). From the results obtained for γ_d they have derived the standard deviation characterizing the reproducibility of the *void ratio*, as

$$s_{rp} = (1 + e)0.03.$$

Table 6.3 shows the test results of various soils. The samples were taken by an identical technology from an open-cast mine within a 50 cm diam. circle under the supervision of an engineer, or a technician. The authors drew the following conclusions from the data shown in the Table: a) the dry bulk density of the cohesive soils can be determined more accurately than that of the granular soils; b) the reliability of the test depends greatly on the skill of the driller (in the given case the first individual had more practice).

Table 6.4 gives an information on the reproducibility of the *modified PROCTOR-test*. The maximum dry bulk density can be determined with a surprisingly high accuracy (e.g. relative to the optimum water content). Consequently, the density of earthworks can be evaluated correctly if the PROCTOR-apparatus is satisfactory according to the prescriptions and we can distinguish between the individual soil types well. It is also reassuring in this connection that the coefficient of variation of the dry bulk density is very low, about 0.01–0.04 according to the above.

Table 6.3. Reproducibility of the dry bulk density (BORUS and RÉV, 1968a)

Type of soil	Driller	<i>n</i>	γ_d (kN/m ³)	<i>s</i> (kN/m ³)	<i>C_v</i>
Silt	1.	31	17.2	0.25	0.015
Silt		30	13.8	0.20	0.014
Sand		30	15.6	0.34	0.022
Clay		26	16.2	0.16	0.011
Sandy silt	2.	29	14.5	0.48	0.033
Humous sand		25	14.7	0.41	0.028
Clay		30	15.5	0.23	0.015

Table 6.4. Reproducibility of the PROCTOR-test

Authors	Type of soil	Laboratory	n	$\gamma_{d\max}$ (kN/m ³)		w_{opt}	
				Mean	C_v	Mean	C_v
BORUS <i>et al.</i> (1969)	Silty sand	1	20	19.58	0.009	10.48	0.072
		2	20	19.17	0.017	10.66	0.067
	Kiscell Clay	1	20	19.30	0.012	12.21	0.104
		2	20	19.08	0.012	11.51	0.097
BORUS and RÉV (1970)	Clay ($I_p = 20\%$)	2	26	18.1	0.014	14.3	0.088
	Silty sand	2	24	20.5	0.012	10.1	0.109

Several authors dealt with the determination of the *shear strength*, but the results of their research are not of real value, because they did not use artificially homogenized samples.

LUMB (1971) tested transitional soils by three different methods, using undisturbed and compacted samples. For the coefficient of variation of the repeated tests he obtained the following values:

Type of test	Undisturbed		Compacted	
	c	$\tan \phi$	c	$\tan \phi$
Drained triaxial	0.135	0.016	0.240	0.021
Consolidated undrained triaxial	0.199	0.098	0.269	0.068
Undrained triaxial	0.188	0.223	0.255	0.056

WEBER and GEHRISCH (1980) report the test results of 30 samples taken from a cohesive soil. For the standard deviation and the coefficient of variation they obtained the following values:

	Mean	s	C_v
c (kN/m ²)	32.06	13.33	0.42
$\tan \phi$	0.54	0.094	0.17

PÁRDÁNYI and Mrs. VÁGÓ (1972, 1973) came to the conclusion that

- the standard deviation of the angle of friction and the cohesion is independent of both the void ratio and the degree of saturation;
- the standard deviation is higher for sand and silty sand than for transition soils.

Data on the reproducibility of the *unconfined compression strength* were published by BORUS and RÉV (1970), who sampled 40 mm diam. samples from open-cast mines. All the various soil types were tested by the same laboratory assistant (using different apparatuses).

Table 6.5. Reproducibility of the unconfined compression strength (BORUS and RÉV, 1970)

Type of soil	n	q_u (kN/m ²)		w(%)	
		mean	C_v	mean	C_v
Silt	26	275	0.24	12.7	0.09
Medium clay	27	191	0.24	28.5	0.04
Silt	33	173	0.28	20.8	0.05
Medium clay	30	209	0.23	21.7	0.04
Silty sand	24	65	0.23	19.4	0.10
Fat clay	30	412	0.17	19.8	0.04

They determined also the standard deviation of the water content of every group of samples for characterizing the degree of their homogeneity. According to Table 6.5, compiled for evaluating the test results, the mean value of the coefficient of variation of q_u was 0.23. According to the authors the standard deviation of the unconfined compression strength may be estimated by the following expression:

$$s_{rp} = (100 + q_u)0.15.$$

The concept of the reproducibility may be interpreted not alone for the generally used physical characteristics of the soil, but also for any test result obtained in the laboratory, or in situ. This is how we proceeded essentially in evaluating the test series performed for determining the saturation of the lower capillary range (see Section 5.1.3).

6.1.3. Determination of the natural heterogeneity of the soil

The standard deviation (s) of any physical characteristic of a given soil layer is composed of two main parts: the component s_{rp} characterizing the reproducibility and s_h characterizing the natural heterogeneity. The latter can be calculated according to the law of error propagation as

$$s_h = \sqrt{s^2 - s_{rp}^2}. \quad (6.4)$$

The value of s_h calculated in this way is somewhat higher than its real value, because the standard deviation appearing in the final result of the test has also other components, whose majority is due to the fact that changes in the state of the samples caused by the sampling itself, as well as the subsequent transport and storage are different. Nevertheless, the expression (6.4) can be used well as a first approximation (WU, 1974; RÉTHÁTI, 1974, 1978a).

Table 6.6 gives, with the use of s_{rp} values described in Section 6.1.2, numerical values of the natural heterogeneity of five characteristic soils of Szeged. On the basis of these values the following statements may be made.

a) The mean of the quotient s_h/s is for physical characteristics appearing in the Table: 0.97–0.89–0.79–0.36–0.83. This means that—apart from the void ratio—the

Table 6.6. Determination of the natural heterogeneity of the soils of Szeged (RÉTHÁTI, 1978a)

Symbol of the soil	Statistical characteristic	Soil physical characteristic					$\bar{C}_{vh}^{**}$
		w	w_L	w_p	e^*	q_u^*	
S1	s	9.45	17.2	8.3	0.225	64	0.31
	s_{rp}	1.10	3.6	2.2	0.093	37	
	s_h	9.39	16.8	8.0	0.205	52	
	C_{vh}	0.30	0.38	0.33	0.24	0.36	
S2	s	4.06	4.3	2.4	0.087	47	0.10
	s_{rp}	1.10	2.5	1.6	0.084	34	
	s_h	3.91	3.5	1.8	0.023	32	
	C_{vh}	0.18	0.11	0.09	0.03	0.26	
S3	s	3.74	4.3	2.2	0.092	102	0.10
	s_{rp}	1.10	2.5	1.7	0.085	41	
	s_h	3.57	3.5	1.4	0.035	93	
	C_{vh}	0.15	0.11	0.07	0.05	0.54	
S4	s	4.86	11.1	3.4	0.094	74	0.13
	s_{rp}	1.10	4.5	2.2	0.091	40	
	s_h	4.73	10.1	2.6	0.024	62	
	C_{vh}	0.17	0.19	0.11	0.03	0.37	
S5	s	4.82	14.3	3.8	0.077	89	0.14
	s_{rp}	1.10	4.4	2.2	0.091	43	
	s_h	4.69	13.6	3.1	0	80	
	C_{vh}	0.16	0.26	0.12	0	0.43	
$\bar{C}_{vh}$:		0.19	0.21	0.14	0.07	0.39	

* From samples of large diameter boreholes.

** Without the values obtained for q_u .

overwhelming part of the standard deviation described in Section 1.3 is not due to objective and subjective faults of the tests but to the natural heterogeneity of the layers.

b) The mean of the coefficients of variation defined by the quotient $C_{vh} = s_h/\bar{x}$ is an essentially higher value for the Holocene covering layer than for the Pleistocene layers. In this respect again the void ratio excels by its rather low coefficient of variation.

WEBER and GEHRISCH (1980) found on examining the cohesive soil mentioned in Section 6.1.2 that the standard deviation due to the natural heterogeneity was lower than the laboratory one (they obtained for the cohesion $s_h/s_{rp} = 0.75$ and 0.46 for $\tan \varphi$ 0.46).

6.2. Comparison of two test series

Generally two test series performed by two different methods (Section 6.2.3), or two different apparatuses (Section 6.2.4), resp. two different persons (Section 6.2.5) are comparatively analyzed.

An especially great attention must be devoted to the sampling technology (Section 6.2.2), as basically the true reflection of the reality by the determined physical characteristics depends on it.

The task mentioned above may be solved with the help of the so-called statistical tests (Section 6.2.1). Generally the question arises, if there exists any significant difference between both mean values and the two standard deviations, or more generally: if both series originate from the same distribution (population). In addition we shall examine also, how the deviation between the test series appears, i.e.

- in the determination of the required number of the samples (Section 6.2.6),
- in the empirical distribution of the physical characteristics (Section 6.2.7), and/or
- in the correlations between the physical characteristics (Section 6.2.8).

6.2.1. Statistical tests

The process, by which a statistical hypothesis is evaluated on the basis of samples, is called *statistical test*. A statistical *hypothesis* is understood always as an assumption on the distribution $f(x; \Theta)$ of a random variable ξ . The aim of the test is to make possible for us to decide as to the acceptance, or the rejection of the involved hypothesis. This decision may be correct or erroneous; the statistical test is constructed in a way that the latter case should occur very rarely when we apply it many times.

The initial assumption is called *zero-hypothesis* (its symbol is H_0). If we assume, e.g., that the expected value m of the sample elements is identical with m_0 , then the zero-hypothesis is

$$H_0 : m = m_0.$$

The statistical test informs us also about the probability of the zero-hypothesis coming true, i.e. in the above example it tells if the difference $m - m_0$ is significant and if so, on which *level of significance* it is. This level must always be fixed (the deviation may be significant, e.g., on the 95% level, but insignificant on the 99% level).

In mathematical statistics we distinguish between *parametric* and *non-parametric* processes. In the first case the types of the distributions are assumed to be known and the aim is to control the hypotheses for the unknown parameters of the distribution. The tests described in this Section—with the exception of the homogeneity test—belong here. In the latter case the type of the distribution is hypothesized, with disregard of the parameters, as in the case of the *fitting test*, described in Section 2.2.2.

In the following the statistical tests are discussed in three groups, depending on their comparing the expected values, the standard deviations, or the total distribution.

The process used for comparing the *expected values* may be summed up as follows.

Let us assume that the population is of a normal distribution and the numerical values of the standard deviation (σ) and the expected value (M_0) are known. If we have a sample consisting of n elements ($x_1, x_2, \dots, x_n$) and the hypothesis $\bar{x} = M_0$ is correct,

then the random variable

$$u = \sqrt{n} \frac{\bar{x} - M_0}{\sigma} \quad (6.5)$$

is of a distribution $N(0, 1)$. We find u_p in the Table of normal distribution, for which the hypothesis

$$P(|u| > u_p) = p \quad (6.6)$$

is true, selecting u_p so low that the random event p should be practically negligible. If now $|u| \leq u_p$, we accept the zero hypothesis, establishing that there is *no significant difference* between $\bar{x}$ and M_0 on the $100(1-p)\%$ level.

Two types of error may be incurred in every statistical test: the rejection of a true hypothesis (first type) and the acceptance of an untrue hypothesis (second type). The probability of committing an error of the first type is just the selected significance level (p). The probability of committing an error of the second type is (PRÉKOPA, 1962)

$$\Phi\left(\sqrt{n} \frac{M_0 - \bar{x}}{\sigma} + u_p\right) - \Phi\left(\sqrt{n} \frac{M_0 - \bar{x}}{\sigma} - u_p\right), \quad (6.7)$$

where Φ is the symbol of the standardized normal distribution.

If the distribution of the population is normal, but its standard deviation is unknown, the zero hypothesis must be controlled by the random variable

$$t = \sqrt{n} \frac{\bar{x} - M_0}{s}, \quad (6.8)$$

i.e. the STUDENT-distribution of $n-1$ degrees of freedom. (The critical value in this case is t_p , belonging to the probability level p .)

The expressions (6.5) and (6.9), the so-called u - and t -tests may be used for analyzing two test series, if the number of the elements in one of them is very high, or if it is regarded as a reference basis, e.g. for its enhanced reliability. A more frequent case is that neither test series is preferred in advance, but we wish only to establish, if there is a significant difference between them. In this case the expression corresponding to (6.5) is the WELCH-expression:

$$u' = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}, \quad (6.9)$$

where n_i is the number of the elements, $\bar{x}_i$ is the expected value of the series and σ_i is the standard deviation of the populations. (A conclusion to $\sigma_1 \neq \sigma_2$ may be derived from the F -test.)

If the standard deviation of the population is unknown (but $\sigma_1 = \sigma_2$), then we must calculate with the combined (weighted) empirical standard deviation, as to which one may be evaluated from both samples. In this case the random variable is the following

analogon of the variable defined by expression (6.8):

$$t' = \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}} \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{n_1 s_1^2 + n_2 s_2^2}}, \quad (6.10)$$

where s_i is the empirical standard deviation of the individual series. The t' -value defined by expression (6.10) is a STUDENT-variable of $n_1 + n_2 - 2$ degrees of freedom. According to PITMAN this test may be used also in cases, where no normal distribution is assumable, but the variables show an identical type of distribution.

The second group of the statistical tests contains processes used for comparing the *standard deviations*. If the samples come from populations whose distributions are $N(M_1, \sigma_1)$, resp. $N(M_2, \sigma_2)$, the standard deviations can be compared by FISHER's *F-test*. For this purpose we calculate first the quotient

$$F = \frac{s_1^2}{s_2^2}, \quad (s_1 > s_2) \quad (6.11)$$

where s_1 and s_2 are the empirical standard deviations of the samples. Then we select the significance level p and find in the table of the *F-distribution* (Table F8 in the Appendix) the F_p -values belonging to the degrees of freedom $n_1 - 1$ and $n_2 - 1$ (n_1 is the number of elements belonging to s_1 , i.e. the set of data whose standard deviation is greater). If

$$F < F_p,$$

we accept the zero hypothesis (of $\sigma_1 = \sigma_2$), in the opposite case we reject it.

If we wish to compare the standard deviation of the sample to a given value (e.g. the known standard deviation of the population), we must use the χ^2 -distribution.

The processes belonging to the third group of the statistical tests are used for comparing the *distribution functions* of two sets, i.e. for *testing homogeneity*, as described in Section 6.2.1.

The first of these processes is a variant of the fitting test method of KOLMOGOROV, described in Section 2.2.2. The process is based on the thesis of SMIRNOV, saying that if the elements of two samples, consisting of k , resp. l elements, taken from the same population are independent of each other, then the product (z) of the maximum difference (d_{kl}) between their empirical distribution functions, taken in the positive sense, by the value

$$\sqrt{n} = \sqrt{\frac{k \cdot l}{k + l}}$$

supplies a random variable whose distribution approximates well the distribution function of KOLMOGOROV, if k and l are great numbers. The difference $0 < d_{kl} < 1$ is equal to the maximum ordinate difference of the two histograms; the further process is the same as in the case of KOLMOGOROV's fitting test.

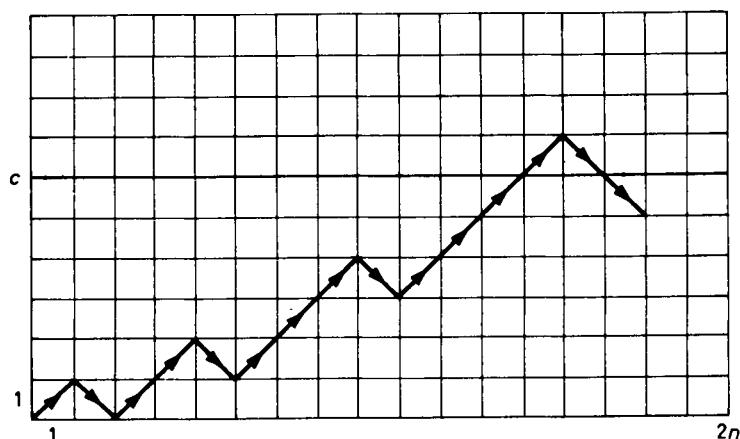


Fig. 6.1. Homogeneity test by the GNEDENKO and KOROLYUK process

The second process is connected with the names of GNEDENKO and KOROLYUK (REIMANN, 1976). The test is started by forming from the samples $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ a single *ordered* sample (written up in an increasing sequence) $z_1, \dots, z_{2n}$ and writing $+1$, resp. -1 under its elements, depending on their origin from the sample x , or y . This series may be plotted in an orthogonal coordinate system by drawing a vector of $\sqrt{2}$ unit length from the origin to the right upwards in the case of $+1$ and to the right downwards in the case of -1 (Fig. 6.1). Depending on the random formation of the sample elements

$$\binom{2n}{n}$$

such trajectories could be obtained. With the help of a purposefully applied reflection system it can be proved that there exist

$$\binom{2n}{n} - \binom{2n}{n-c} \quad (6.12)$$

trajectory variants, which do not reach an $y=c$ straight line selected by us. So the probability that the D_n^+ maximum difference—in the positive sense—between the empirical distribution functions $F_n(x)$ and $G_n(x)$ will be lower than a value $c = z\sqrt{2n}$ is

$$P(nD_n^+ < c) = 1 - \frac{\binom{2n}{n-c}}{\binom{2n}{n}}. \quad (6.13)$$

The limit value of the fraction on the right side is, when $n \rightarrow \infty$

$$\frac{e^{-c^2}}{e^n}$$

so the height c_0 , unattained by the trajectory with α probability, may be calculated by the expression

$$c_0 = \sqrt{-n \ln \alpha}. \quad (6.14)$$

So if the trajectory constructed by us intersects this height c_0 , then the hypothesis that the two distributions originate from the same population is rejected.

The process may also be generalized for the case, where the numbers of the sample elements are not identical (REIMANN and VINCZE, 1960).

The third process corresponds to the χ^2 test described in Section 2.2.2, again with the difference that we compare now two empirical distribution functions, so it serves here for testing the homogeneity, not the fitting.

Examples of statistical tests used for concrete geotechnical tasks are given in Sections 6.2.2–6.2.5.

6.2.2. Reflection of the sampling method in the physical characteristics

The material test methods may be assigned to two great groups: one contains the methods used for testing the material in its *original state*, the other one those testing it by *sampling*. The advantage of the first, so-called non-destructive testing is revealed by its name; its disadvantages are—in the majority of cases—the lower accuracy (reliability), or the higher cost. In geotechnical practice the physical characteristics of soils are determined without destroying the soil mass only in 5–10% max. of the cases (density and moisture measurements by radioisotopes, by seismic and geo-electric soil exploration, by loading tests, etc.). The other group of methods extends to a wider scale of tests, but their enhanced reliability can be ensured only by reducing the degree of the disturbance of the soil to the minimum by the sampling.

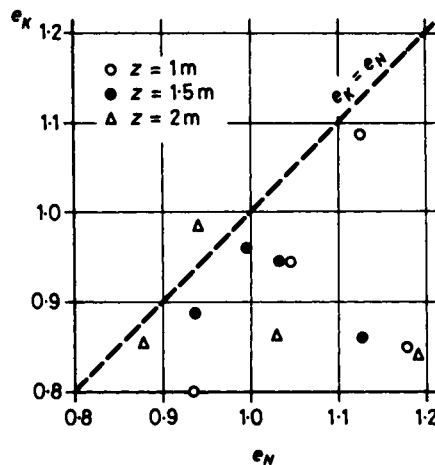


Fig. 6.2. Relationship between the void ratios of samples of a silty sand taken from small diameter (e_K) and large diameter (e_N) boreholes (Mrs. VÁGÓ, 1974)

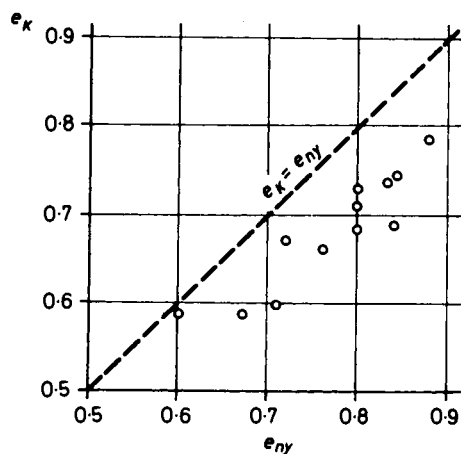


Fig. 6.3. Void ratios of an earthwork prepared of loess by samples taken from an open-cast mine (e_{ny}) and a small diameter borehole (e_k) (PÁRDÁNYI, 1962)

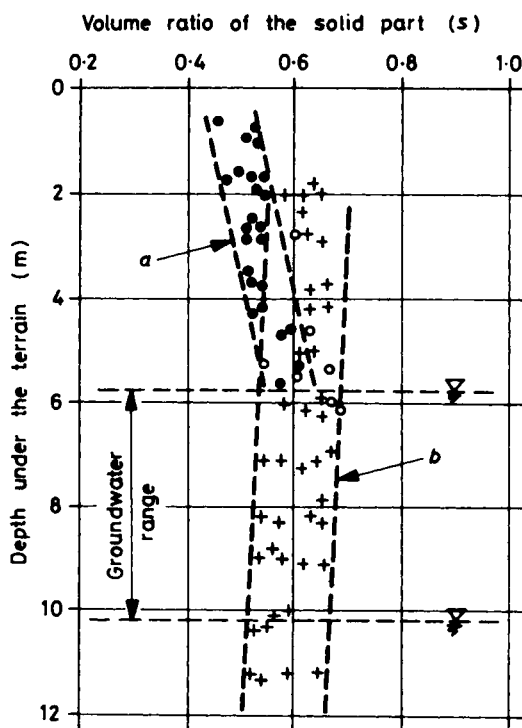


Fig. 6.4. Volume ratio of the solid part of loess samples taken from an open-cast mine (a) and a large diameter borehole (b) (KÉZDI, 1976a)

The sampling changes the *stress state*, the *structure* and the *phase composition* of the soil in a measure depending on the following factors:

- the sampling technology (introduction of the sampler by rotation, pressure or blows into the soil),
- the construction, and
- the geometrical dimensions of the sampler,
- the position of the groundwater,
- the shear strength and the compressibility of the soil, the adhesion of its surface to the metal parts,
- the phase composition of the soil (in the first place its density).

The effects of the above factors are illustrated by the following examples.

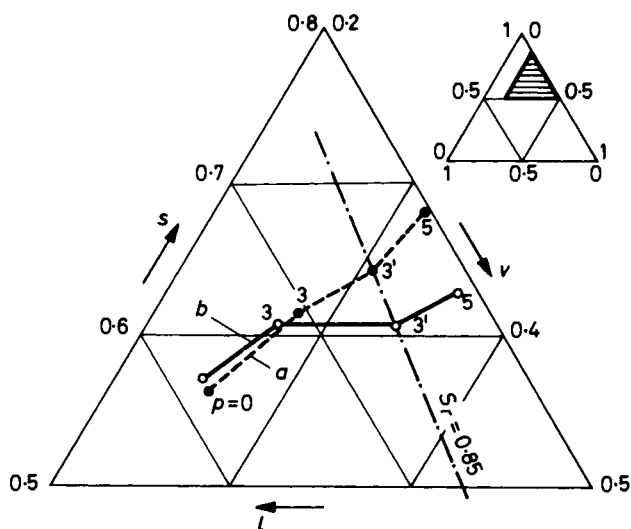
In *Fig. 6.2* the void ratios of silty sand samples (of $D \cong 100$, resp. 40 mm diam.) sampled from boreholes of large (> 100 mm) and small (65 mm) diameter, determined in the laboratory, are compared. According to the figure the mean difference between these types of samples is $\Delta e = 0.13$.

In *Fig. 6.3* the void ratios of samples taken by two different methods from an artificially consolidated earthwork of loess are compared. The average difference is $\Delta e = 0.09$ (the more reliable sample taken from the open-cast mine was looser by that much, than the sample taken from the small diameter borehole).

In *Fig. 6.4* the void ratios (the volume ratios of the solid part) of loess samples taken from the shaft and from a large diameter borehole are compared. The figure permits to establish that

- a) the standard deviation of the void ratios of the shaft samples is considerably lower than that of the samples taken from the large diameter borehole,
- b) the samples taken from the shaft are looser (the difference is especially great between 1 m and 5 m),
- c) the void ratios of the samples taken from the borehole increase with the depth, those of the samples taken from the shaft decrease with the depth. This phenomenon is explained by KÉZDI (1976a) in the following way: the small-diameter sampler loosens the soil ever increasingly with increasing depth.

Figure 6.5 deserves attention for another reason. The loess samples carved out from the shaft (a) suffered a considerable subsidence when submerged in water, leading to an increased volume ratio of the solid phase (s). In the sample taken from the core of the 100 mm diam. borehole only the water content changed with the submersion, its void ratio did not change. (On this basis the tested loess layer would not be qualified as macroporous, inclined to subsidence, which might lead to dangerous consequences.) According to the figure the deviating behaviour of both samples is due rather to the *destruction of the original soil structure* and less to the difference between the initial void ratios.



Different results may be obtained for the physical characteristics of the soil also if it is sampled from identical diameter boreholes but by different sampler types. *Figure 6.6* shows in addition that the magnitude of the deviation depends also on the numerical value of the physical characteristic to be measured.

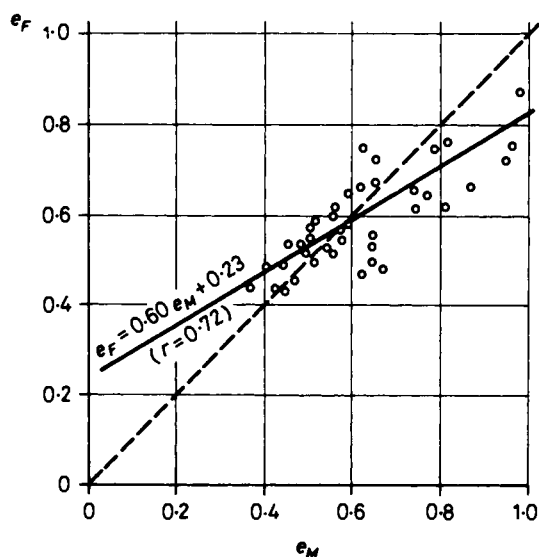


Fig. 6.6. Comparison of the void ratios of samples taken by MAZALÁN's sampler (e_M) and the double-walled sampler Mark F-62 (e_F) (KÉZDI, 1976)

The role of the *borehole diameter* can be studied on the basis of the archive material of Szeged, utilizing now also the statistical tests mentioned in Section 6.2.1 (RÉTHÁTI and UNGÁR, 1978a, b). Detailed data on the quantity of the available test results and their distribution by the various sample types are contained in *Table 1.11*, those of the statistical parameters in *Table 1.13*.

The statistical analysis is started with the water content. On testing this physical characteristic we can compare the water content of the core sample with that of the partly disturbed sample. The statistical tests will be performed on the 99.9% significance level. For the sake of completeness we shall use also the u - and the t -test, regarding the standard deviation of the undisturbed sample as "known", resp. where samples from two boreholes of different diameters are compared, the standard deviation of those sampled from the large-diameter borehole will be regarded as "known".

The number of the partly disturbed samples of the soil $S1$ is $n_1 = 78$, their mean water content is $\bar{w} = 31.11$, that of the core samples taken from the small + large diameter boreholes is $\bar{w}_2 = 29.91$; their standard deviation is $s_2 = 7.67$ (see *Table 1.13*). Then according to expression (6.5)

$$u = \frac{31.11 - 29.21}{7.67} \sqrt{78} = 2.19.$$

The value $\Phi(\lambda)$ forming the basis of the comparison is according to expression (2.18)

$$\Phi(\lambda) = \frac{1 + 0.999}{2} = 0.9995,$$

to which $\lambda = u_p = 3.29$ belongs according to *Table F2*. As $u(=2.19) < u_p(=3.29)$, our assumption about the identical mean of the water contents proved to be realistic. A similar conclusion is reached also in the case of the t -test, as namely $t_p = 3.44$ belongs to $p = 0.001$ and $n = 77$, according to *Table F4* (so we make our decision on the basis of the inequality $2.19 < 3.44$).

According to expression (6.9), ($n_2 = 80$)

$$u' = \frac{31.11 - 29.21}{\sqrt{\frac{9.45^2}{78} + \frac{7.67^2}{80}}} = 1.39 < 3.29$$

and according to expression (6.10)

$$t' = \sqrt{\frac{78 \cdot 80 \cdot 156}{158}} \frac{31.11 - 29.21}{\sqrt{78 \cdot 9.45^2 + 80 \cdot 7.67^2}} = 1.38 < 3.35,$$

so the deviation between the mean values may be regarded on the basis of both the u' -test and the t' -test as of a random nature. (Here the degree of freedom for the t' -test is 156.)

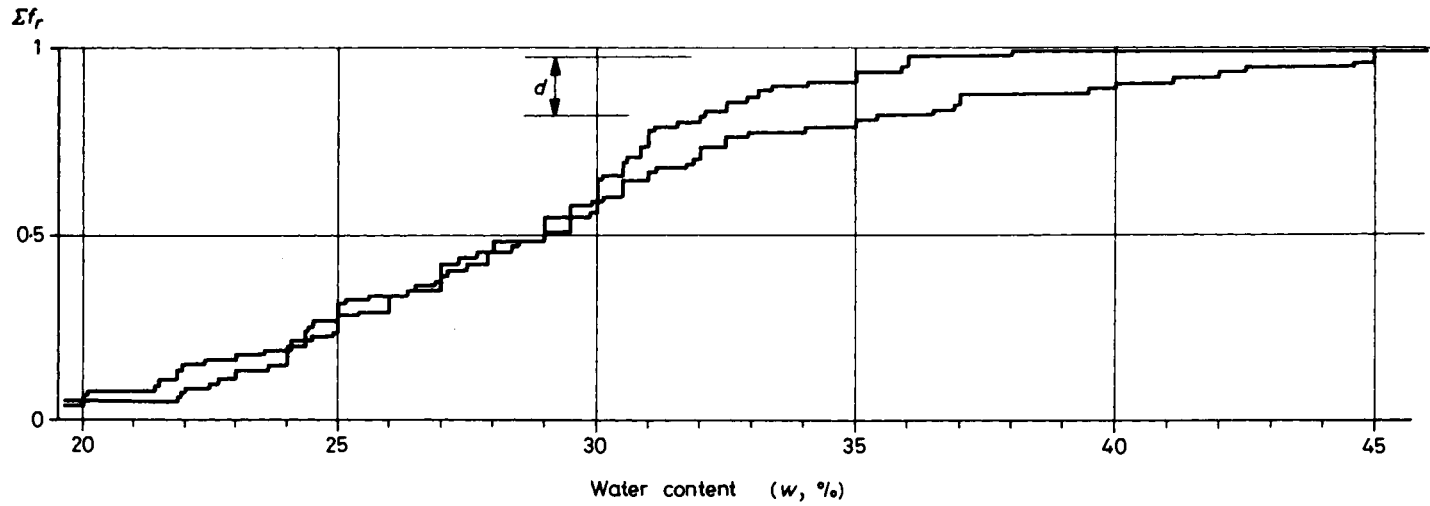


Fig. 6.7. Comparative KOLMOGOROV and SMIRNOV-test of partially disturbed (w_z) and small diameter core samples (w_{mk}) of the soil S1 (RÉTHÁTI and UNGÁR, 1978a)

As $s_1 > s_2$, s_1 gets into the numerator on evaluating F :

$$F = \frac{s_1^2}{s_2^2} = \frac{9.45^2}{7.67^2} = 1.52.$$

According to *Table F8* the critical value belonging to $f_1 = 77$ and $f_2 = 79$ is $F_p \cong 1.47 < 1.52$, so our hypothesis on the identity of the *standard deviations* on the significance level of 95% must be rejected.

Figure 6.7 shows the empirical distribution of the water content in both types of samples (for having $k=l$ we omit two of the partly disturbed samples). The maximum ordinate difference is $d=0.1282$ and the z -value defined by SMIRNOV is

$$z = d \sqrt{\frac{k \cdot l}{k+l}} = 0.1282 \sqrt{\frac{78 \cdot 78}{78+78}} = 0.80067,$$

to which $L(z)=0.457$ belongs in *Table F6*. With this

$$p = 100[1 - L(z)] = 54.3\% > 5\%,$$

so our hypothesis on the identity of the distribution is acceptable.

As the number of the elements in the data series is not identical, the sum χ^2 must be calculated by the following expression:

$$\chi^2 = mn \sum_{i=1}^r \frac{1}{\mu_i + v_i} \left(\frac{\mu_i}{m} - \frac{v_i}{n} \right)^2, \quad (6.15)$$

where

m — number of elements in the first data series (partly disturbed samples),

n — number of elements in the second data series (core samples),

r — number of categories,

μ_i and v_i — frequency of the first, resp. second variable by categories ($i = 1 \dots r$).

On dividing the w -values into $r = 17$ intervals ($< 20, 20-21.9, \dots 48-49.9, \geq 50$) and establishing their frequency within these, we obtain

$$\chi^2 = 12.19.$$

In *Table F3* we find that $p = 72.9\% > 5\%$ belongs to this value and to the degree of freedom $r-1 = 16$, so we may state that both distributions originate, with a high probability, from the same population.

After performing the calculations also for the soils $S2$ and $S3$ and marking the validity of the zero-hypothesis by a + sign, the results of the statistical test may be summed up as follows:

Soil	u -	u' -	t -	t' -	F -	$K-S$ -	χ^2 -
test							
$S1$	+	+	+	+	—	54.3%	72.9%
$S2$	+	+	+	+	+	50.4%	50.4%
$S3$	—	—	—	—	+	0.1%	0.5%

Accordingly six tests in the case of soil *S1* and all the seven tests in the case of soil *S2* gave positive results. With soil *S3* the situation is reversed: the hardly differing standard deviations (3.74 and 3.85) would refer to the same population, but the expected value and the total distribution proved to be different, supporting the correctness of the statement made in Section 1.4 that the partly disturbed sample absorbs water during its rise through the water column and the effect of this fact appearing in the numerical value of the water content is significant. (In the case of soil *S2* there is no significant difference between the water contents of the sample types according to the above data; this may be explained unequivocally by the fact that the layer lies above the groundwater level.)

For the soils *S4* and *S5* the considerably greater number of the core samples permits to distinguish between the samples taken from the small and the large diameter boreholes. So three combinations may be tested by correlating the water content of two from among the partly disturbed samples (w_z) and those sampled from the small, respectively the large diameter boreholes (w_{mK} , resp. w_{mN}). The results of the statistical tests obtained on the basis of this principle (Table 6.7) permit to draw the following conclusions:

a) In the relation $w_z - w_{mK}$ the calculations performed with soil *S4* prove that both the means and the standard deviations differ significantly from each other. In the case of soil *S5* only the u -test and the χ^2 -test gave a similar result, but $u' (= 3.13)$ and $t' (= 2.95)$ were also close to the critical value. The same applies also to soil *S1*, if we take into account only the 56 undisturbed samples taken from the small diameter borehole and not the totality of the core samples.

b) The tests in the relation $w_z - w_{mN}$ led to the unexpected result that these sampling methods may be regarded as identical (apart from the standard deviations of the soil *S4*).

c) In the relation $w_{mK} - w_{mN}$ a significant deviation was obtained for the soil *S4* (even the u' and the t' -values, whose tests gave positive results, were close to the critical value), but not so for the soil *S5*. On the basis of the *S4* sample of a considerably greater number of elements ($641 \gg 161$) it may be established that the water contents of the

Table 6.7. Statistical tests of the water contents determined from various sample types of the soils *S4* and *S5* (RÉTHÁTI and UNGÁR, 1978a)

Symbol of the soil	Tested relation	u -	u' -	t' -	F -	$K-S$ -	χ^2 -
		test					
<i>S4</i>	$w_z - w_{mK}$	—	—	—	—	<0.1%	<0.1%
	$w_z - w_{mN}$	+	+	+	—	71.8%	38.7%
	$w_{mK} - w_{mN}$	—	+	+	—	4.2%	9.5%
<i>S5</i>	$w_z - w_{mK}$	—	+	+	+	57.0%	7.9%
	$w_z - w_{mN}$	+	+	+	+	33.0%	20.9%
	$w_{mK} - w_{mN}$	+	+	+	+	73.4%	50.2%

Table 6.8. Statistical tests performed for five physical characteristics of the soils S3, S4 and S5 (RÉTHÁTI and UNGÁR, 1978b)

Symbol of the soil	Physical characteristic	Relation tested	test					χ^2 -
			u -	u' -	t' -	F -	$K-S$ -	
S3	I_c	$z-m(K)$	+	+	+	-	11.7%	42.1%
		$z-K$	+	+	+	+	10.2%	<0.1%
		$z-N$	+	+	+	+	8.0%	16.1%
S4	e	$K-N$	+	+	+	+	25.5%	23.1%
		$K-N$	+	+	+	-	83.3%	68.0%
		$K-N$	-	-	+	-	34.8%	1.8%
	S_r	$K-N$	+	+	+	-	21.3%	63.2%
	q_u	$K-N$	-	-	-	-	~0%	~0.5%
S5	I_c	$z-m(N)$	+	+	+	+	54.1%	26.9%
	e	$K-N$	+	+	+	-	29.1%	84.3%
	S_r	$K-N$	+	+	+	-	41.7%	7.9%
	γ	$K-N$	+	+	+	-	29.1%	0.7%
	q_u	$K-N$	+	+	+	-	83.0%	70.0%

large diameter samples are higher than those of the small diameter samples (see the discussion of the reason in Section 7.1.3).

d) On summarizing the number of the signs assigned to the statistical tests by sample types it may be established (RÉTHÁTI and UNGÁR, 1978a) that from the aspect of the reliability of the water content the large-diameter sample is followed by the partly disturbed (and not by the small-diameter/sample).

Table 6.8 shows the result of the statistical evaluation of *other soil physical characteristics*, permitting to state the following:

a) The type of sample used for determining the consistency index is practically indifferent, but we must note that the standard deviations of the small diameter samples are higher than those of the partly disturbed samples ($0.403 > 0.294$ and $0.185 > 0.169$).

b) There is no significant deviation between the means of the *void ratios* of the samples taken from small and large diameter boreholes, but there is between their standard deviation: this is lower in the large diameter samples of soil S4 and in the small diameter samples of soil S5. (The obtained results are more realistic in the case of the soil S4, with regard to the proportion 642: 160 of the number of the sample elements.)

c) The standard deviation of the *degree of saturation* is significantly greater for the samples taken from small diameter boreholes (compare with Section 7.1.3). The statistical tests related to the means proved to be not sufficiently "sharp", as shown by the data described in the above mentioned section.

d) The standard deviation of the bulk density was lower for the samples taken from the large diameter borehole; the deviation between the means was insignificant.

e) The same applies also to the *unconfined compression strength*, with the difference that a very significant deviation appeared also between the means in the case of soil S4 with a great number of sample elements.

6.2.3. Comparison of various test processes

The examples quoted in literature permit to draw useful conclusions concerning the applicability of the statistical tests, the evaluation of their results from technical viewpoints and their utilization in practice.

BORUS and RÉV (1970) examined the dependence of the liquid limit on the dropping height, resp. the velocity of rotation adjusted on the CASAGRANDE-apparatus.

They studied the effect of the velocity of rotation by determining five w_L -values for three soil types, applying 2 blows/s as prescribed in the standard, and 1 blow/s, respectively. The results are summarized in the Table below:

Symbol of the soil:	<i>I</i>	<i>B</i>	<i>O</i>
2 blows/s	97.3 ± 1.01	80.8 ± 1.03	53.0 ± 0.45
1 blow/s	98.5 ± 1.15	81.4 ± 0.91	55.2 ± 1.00

The t' - and F -values calculated from the above data are as follows: $t' = 1.57-0.87-4.01$ and $F = 1.30-1.28-4.94$. In the STUDENT-table $t_p = 3.355$ corresponds to $p = 0.01$ and $n = 8$ degrees of freedom, permitting one to establish that there exists a slight deviation between the two processes (at the significance level of 99%) only in the case of the soil marked Θ , but it is insignificant. The critical F_p -value (for $f_1 = 4$ and $f_2 = 4$) is 6.39, which means that on the significance level of 95% the standard deviation may be regarded as identical.

The authors studied also the role of the *dropping height* in the framework of the same test series. On setting 0.6 cm instead of 1.0 cm, as prescribed in the standard, they obtained the following values for w_L and its standard deviation:

<i>I</i>	<i>B</i>	<i>O</i>
110.9 ± 1.50	90.6 ± 1.20	63.1 ± 1.15

from which the following t' - and F -values may be calculated:

	<i>I</i>	<i>B</i>	<i>O</i>
t'	15.04	12.39	16.36
F	2.21	1.36	6.53

As t_p and F_p are identical with the values calculated for the first test series, we may state that a 40% reduction of the dropping height results in a very great error in w_L and also in a certain increase of the standard deviation characterizing the reproducibility of the test.

Table 6.9. Effect of the cylinder diameters on w_p , calculated from the data of BORUS and RÉV (1970)

Soil	n	Diameter	$w_p \pm s$	t'	F	$t_{0.01}$	$F_{0.05}$
I	9	3 mm	29.33 ± 1.6				
		2 mm	29.13 ± 1.9	0.23	1.41	2.92	3.44
		4 mm	30.67 ± 2.6	1.24	2.64	2.92	3.44
B	13	3 mm	25.22 ± 3.6				
		2 mm	25.17 ± 3.4	0.03	1.12	2.80	2.69
		4 mm	25.63 ± 2.7	0.32	1.78	2.80	2.69
O	10	3 mm	23.45 ± 1.6				
		2 mm	23.12 ± 2.1	0.25	1.72	2.88	3.18
		4 mm	24.35 ± 1.9	0.72	1.41	2.88	3.18
A	10	3 mm	16.95 ± 1.5				
		2 mm	16.87 ± 2.2	0.06	2.15	2.88	3.18
		4 mm	17.05 ± 1.9	0.08	1.60	2.88	3.18

The authors studied also the effect of the diameter of the small cylinders prepared for determining the *plastic limit*. The results of their test series are summed up in Table 6.9, permitting of establishing that

- a deviation of ± 1 mm from the prescribed cylinder diameter of 3 mm is permissible,
- the 2 mm diam. cylinders give somewhat better w_p -values than the 4 mm diam. cylinders ($\bar{r} = 0.14$ and 0.59).

LADD *et al.* (1971) evaluated the *field vane test* results for establishing the effect of the rotation velocity (v , degrees/min) on the measured shear strength (τ , kN/m²). One firm performed 9 measurements at each of the velocities specified by the ASTM-standard (5°/min.) and two other velocities (1°/min. and 30°/min.) with the following results:

v	$\bar{\tau}$	s	C_v
1°/min.	200	68	0.34
5°/min.	250	71	0.28
30°/min.	264	51	0.19

It may be verified easily that the deviation from the velocity specified by the standard causes no significant changes neither in the mean, nor in the standard deviation (even in the case of 30°/min. we obtain only $F = 1.94$, as against $F_p = 3.44$ belonging to the significance level of 95%). At the same time it is noteworthy that with the increase of v the value C_v decreases considerably, in spite of the increase of $\bar{\tau}$. (Usually the opposite case prevails, which may be regarded as a rule of rather general

validity, as explained in Section 1.2.3.) This means that by testing more soil types of higher element numbers not only the correctness of the equality $\tau_s \cong \tau_{30}$ could be proved probably, but also that of the inequality $s_{30} \ll s_s$, whereby the *reliability of the measured shear strength* would be increased considerably.

PETTITT (1967) compared two field density measurement methods: the non-destructive isotopic method (*i*) and the rubber membrane process (*g*) used for measuring the volume of manually excavated holes by a balloon. At one workplace the following mean values and standard deviations of the degree of density T_{ry} were obtained:

Method	<i>n</i>	$\bar{T}_{ry}(\%)$	<i>s</i>
i	146	93.5	10.9
g	86	90.6	6.1

There is no difference between the means at the significance level of 99% (but there is at the 95% level). The square of the quotient of the standard deviations ($F = 3.19$) exceeds the critical value ($F_p = 1.40$). The reason given by PETTITT is that the isotope instrument can sense finer the deviations appearing in the density, while the balloon-method gives erroneous values for both the much denser and looser soil masses, than the mean, due to the disturbance of the soil state.

This example illustrates that the *lower standard deviation means no higher reliability by necessity* (as, e.g., it did in the example related to the shear strength). Similar conclusions may be drawn from the test series performed for determining the relationship between the measured water content and the *drying temperature* of the sample (BORUS and RÉV, 1970). From the data of Table 6.10 it may be established that

a) with increasing temperature *w* increases the more, the higher the mineral clay content of the soil (because at a higher temperature even the crystal water is evaporated),

b) a *lower standard deviation* belongs to the higher *w*-values obtained at higher temperatures, but the advantages implied by this fact cannot be exploited.

SCHULTZE (1975a) determined the modulus of compressibility in three different ways (Table 6.11). On evaluating his tests performed with the silt *A* we find no difference between the oedometer test and the standard penetration test. For the relation between the oedometer and the static penetration test, $t' > t_p$ and $F < F_p$ as well as for the relation between both penetration tests, $t > t_p$ and $F > F_p$. The tests performed with the silt *B* may be evaluated in a way, that the oedometer and the surface loading test give an identical numerical value for E_s , but for the latter the standard deviation is considerably greater: $F (= 3.75) > F_{0.05} (= 2.98)$.

Laboratory and field *vane tests* were performed with a Louisiana clay. LUMB (1971) proves the equivalence of these methods with the fact that the $p = 95\%$ probability

Table 6.10. Relationship between the calculated value of the water content and the temperature of the drying oven (BORUS and Rév, 1970)

Soil	Temperature (°C)	$\bar{w}$	s
Infusion loess	105	38.92	0.41
	120	39.72	0.18
	160	40.31	0.14
Kiscell Clay	105	47.78	1.10
	120	48.63	0.45
	128	48.78	0.45
Bentonite	105	68.60	5.97
	120	87.21	2.20
	145	91.20	0.83

Table 6.11. Statistical tests for the modulus of compressibility determined by various methods (SCHULTZE, 1975a)

Symbol of the soil	Test method	n	$\bar{E}_s(\text{MN/m}^2)$	s
A	Oedometer	48	7.5	1.74
	Standard penetration test	84	7.0	2.07
	Static penetration test	65	5.8	1.48
B	Oedometer	11	7.73	1.59
	Surface loading test	11	7.69	3.08

range of the regression straight line determined for coherent value pairs (see Section 3.3) contains the straight line of 45° slope (Fig. 6.8). The same cannot be said of the unconfined compression strength (U), of the undrained triaxial test (UU) resp. of the laboratory vane test (LV), as namely according to the regression analysis

$$U, UU = 0.73LV.$$

We are facing a similar case, if we compare the *liquid limits* of CASAGRANDE and VASILEV (w_L and w_{LV}). Both w_L values determined by different processes deviate from each other considerably and the deviation between them is not even constant. For example, according to ŠKOPEK (1971), the constants a and b appearing in the relationship

$$w_{LV} = a + bw_L$$

depend also on the ratio of the $d < 0.002$ mm grains and the activity of the soil (Table 6.12). In this and in similar cases the statistical tests cannot be used for studying the differences between two processes.

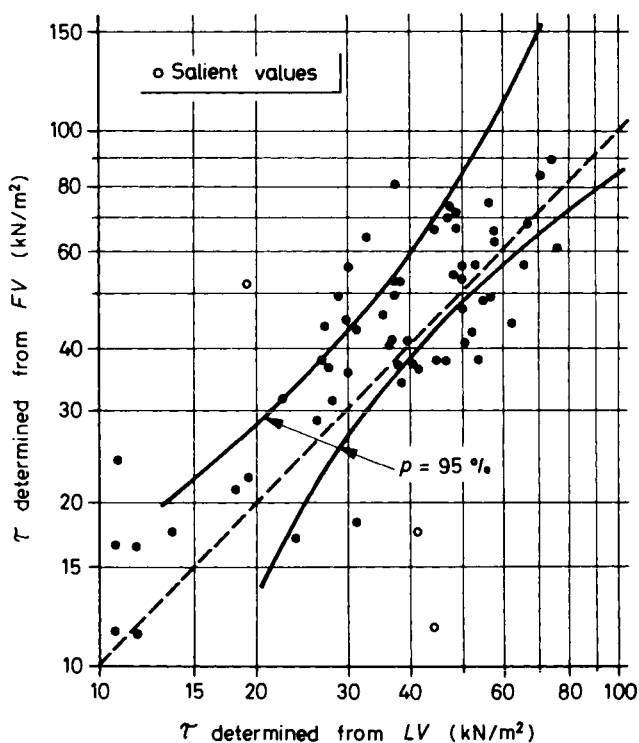


Fig. 6.8. Comparison of shear strengths determined by laboratory (LV) and in situ (FV) vane test (LUMB, 1971)

Table 6.12. Relationship between the liquid limit of CASAGRANDE (w_L) and that of VASILIEV (w_{LV}) versus the clay mineral content (S_a) and the activity (A) (ŠKOPEK, 1971)

Characteristic of the group	n	$w_{LV} =$	r	s_f
Total set	236	$0.72 w_L + 7.77$	0.93	3.96
$S_a < 30\%$	145	$0.72 w_L + 8.33$	0.95	3.27
$S_a > 30\%$	87	$0.81 w_L - 0.03$	0.72	9.42
$A < 0.75$	102	$0.64 w_L + 9.96$	0.96	3.41
$A = 0.75 - 1.25$	68	$0.68 w_L + 10.62$	0.96	3.86
$A > 1.25$	44	$0.63 w_L + 13.23$	0.91	6.57

6.2.4. Comparison of two apparatuses serving the same purpose

In the following a few examples are presented, where *only one characteristic* of the apparatuses to be compared is different, while the test methodics and the modes of their evaluation are identical.

Table 6.13. Comparison of liquid limits determined by two different CASAGRANDE-apparatuses I and II (BORUS and RÉV, 1970)

Soil	Apparatus	<i>n</i>	$\bar{w}_L \pm s$
<i>I</i>	I.	12	102.8 ± 6.70
	II.	10	96.5 ± 4.86
<i>B</i>	I.	13	87.2 ± 4.32
	II.	10	84.6 ± 3.60
<i>O</i>	I.	13	53.8 ± 4.30
	II.	12	51.5 ± 3.38
<i>A</i>	I.	13	35.5 ± 2.32
	II.	12	33.6 ± 2.17
<i>K</i>	I.	13	27.4 ± 1.35
	II.	7	27.7 ± 1.34

Table 6.13 shows the liquid limits of five soils and their standard deviations, determined by two CASAGRANDE-apparatuses showing the following differences:

	Mass of the cup	Base
Apparatus No. I:	153.5 ± 4.4 g	metal, lined by soft rubber
Apparatus No. II:	199.2 ± 2.7 g	timber, lined by hard rubber

According to the statistical tests there were no significant differences between the means and the standard deviations of the liquid limits determined by both apparatuses. This result must be accepted, though according to Table 6.13 the values of both means and standard deviations determined by apparatus No. I were consistently higher than those obtained by apparatus No. II in 90% of the cases. (If several soil types were available, value pairs could be formed from the test results for examining the significance of the intercept of the regression straight line, to see if it cannot be regarded as zero.)

At the Institute for Geodesy and Geotechnics (Budapest) a test series was performed to check if no erroneous results were supplied by two newly acquired CASAGRANDE-apparatuses (Nos II and III), which showed the following differences relative to No. I apparatus, manufactured according to the standard:

- the Shore-hardness of the base of apparatus No. II satisfied the standard specification, but its material was polyurethane, not hard rubber;
- the base of apparatus No. III seemed to be very soft.

The test series was performed by 13 laboratory assistants, each of whom determined the liquid limit of the same soil by all three apparatuses. The results were as follows:

Apparatus:	I	II	III
$\bar{w}_L$	53.40	53.78	57.13
s	2.447	2.760	1.762

According to the statistical tests there was no difference between apparatuses I and II ($t' = 0.35 \leq 3.75$ and $F = 1.24 < 2.69$), while the mean supplied by apparatus III showed a significant difference ($t' = 4.13 > 3.75$), as compared to that obtained by apparatus I, therefore its further use had to be abandoned. (The standard deviation was also different (lower)—though not significantly—from the values obtained by both other apparatuses, due probably to the soft base).

Interesting conclusions may be drawn from a Polish test series performed to investigate the role of the size of the loading test plate (RÉTHÁTI and UNGÁR, 1978b). The square-form loading plates of different widths ($B = 0.5 - 10$ m) gave similar E_s -values, but the *standard deviation* decreased gradually with the increasing size.

6.2.5. Qualifying the work of the person performing the test

If two persons are testing identical (homogenized) soils, using identical apparatuses and test methodics, the reliability of their work can be compared with the help of statistical tests.

The liquid limit of the previously mentioned soil *O* was determined by two laboratory assistants, who performed 30 tests each. In one series $w_L = 52.9 \pm 1.80$, in the other one $w_L = 57.5 \pm 2.20$ were obtained. The t' -test showed—at the significance level of 99%—a significant difference between the means ($8.71 > 2.66$), but not between the standard deviation ($1.49 < 1.88$). The mean value supplied by the first test series was closer to the values of *Table 6.13*, so the work of assistant I. could be regarded to be more reliable.

If the soils, the apparatuses, or the test methods are not identical, the comparative analysis cannot be carried out unequivocally. Such a situation arose, e.g., on evaluating the data of *Table 6.3*, where the significance of the deviation appearing between the quality of the work of two drillers could be assessed only by probability (see Section 6.1.2).

6.2.6. Required number of samples depending on the test method

The required number of soil samples (tests) may be determined by expressions (4.14) and (4.15) with the help of the standard deviation, resp. coefficient of variation (Section 4.4). We have also established there that n is different for each soil, for each part area and for each physical characteristic.

On the basis of Sections 6.2.2.–6.2.5. we may state that in addition to the above the minimum number of the samples (tests) depends also on the technical specifications of the apparatus, the test methodics and even on the person performing the test.

The importance of the problem may be studied by analyzing the role of the *borehole diameter*. Be the standard deviation of the involved physical characteristic, determined for the large diameter sample s and for the small diameter sample αs ; in this case the condition of an identical reliability of the means (see Section 7.4) is

$$\frac{s}{\sqrt{n_N}} = \frac{\alpha s}{\sqrt{n_K}}, \quad (6.16)$$

where

n_N — number of samples taken from large diameter boreholes,

n_K — number of samples taken from small diameter boreholes.

On transforming the above equation the following ratio of both sample types is obtained:

$$\frac{n_K}{n_N} = \alpha^2. \quad (6.17)$$

Table 6.14 shows the values obtained for the soil S4. The physical characteristics appearing here may be assigned to two groups, depending on whether the *means* obtained for the two types of samples are nearly identical, or differ significantly from each other. In the first case the expressions (6.16) and (6.17) may be used without any restriction. In the second case, where the measurement is laden with a *regular fault*, only the reliability of the mean for the given sample type may be improved by increasing the number of the samples, but the regular fault cannot be eliminated in this way. In such cases we may resort to one of the following processes (RÉTHÁTI, 1978a, 1979a):

a) we multiply the value of the characteristic determined on the basis of small diameter boreholes by a correction factor obtained from previous statistical processing;

b) we establish with the help of the large diameter samples relationships, which are valid for the given layer (e.g. between q_u and e , resp. w) and calculate the related characteristic (in the example: q_u) from the independent variable of the small diameter borehole (from e or w);

c) we exclude the “faulty” samples from the data set of the small diameter borehole (see Section 7.1) and accept the rest as the effective number of the samples.

Table 6.14. Comparison of samples taken from small and large diameter boreholes for determining the required number of the tests (soil S4) (RÉTHÁTI, 1978a, 1979a)

Physical characteristic	Sample type	$\bar{x}$	s	α^2
w	K	26.69	4.13	1.65
	N	27.75	3.22	
I_c	K	0.890	0.185	1.14
	N	0.922	0.173	
e	K	0.818	0.094	1.31
	N	0.832	0.082	
S_r	K	0.895	0.093	2.25
	N	0.922	0.062	
γ	K	1.925	0.064	1.78
	N	1.936	0.048	
q_u	K	2.38	1.23	2.76
	N	1.66	0.74	

If one of two test methods must be selected, the *technical* and the *economic aspects* must be studied in their interaction. This problem arises especially sharply if the process demanding more in time and cost expenditure supplies the soil physical characteristic with a lower standard deviation. In such cases the method (resp. the number of the tests) may be selected on the basis of economical calculations, taking the relationship between the factor of safety and the cost of the object into account (see e.g. Section 9.4).

Let us return now to one of the examples described in Section 6.2.3 (SCHULTZE, 1975a, determination of E_s). According to *Table 6.11* the standard deviation obtained by the standard penetration test was 2.07 MN/m², that obtained by the static penetration test was 1.48 MN/m², meaning according to expression (4.19) that the number of the standard penetration tests must exceed by 96% that of the static penetration tests for determining $\bar{E}_s$ with an identical reliability.

With the help of the mathematical statistical methods described here *the wages of the laboratory assistants* may also be rendered more realistic. On comparing, e.g., the work of two laboratory assistants, their wages must be determined in the proportion $1 : \alpha^2$, if both are doing conscientious work with an identical intensity. So are reconciled the concepts of "high quality work" and "economic work".

6.2.7. Distribution functions of physical characteristics determined by various methods

If a test method causes changes on one side of the distribution function, as compared to another method, then not only the standard deviation, but also the skewness and the kurtosis will change. This phenomenon may be studied by the comparative analysis of samples taken from small and large diameter boreholes, regarding that generally changes of an identical trend (sign) may be observed in the structure, the density and the water content.

The difference between the sample types may be appreciated by the values β_1 and β_2 defined by expressions (2.6). Table 6.15 showing the PEARSON-characteristics of the soils S3, S4 and S5 permits to state that

a) the β_1 - and β_2 -values of the *water content* and the *void ratio* are similar in both types of the samples,

b) the result obtained for the *degree of saturation* is somewhat surprising, namely inasmuch as the distribution function belonging to the large diameter samples is considerably farer from the normal than that of the small diameter samples. The reason is that this sampling type is just more reliable, as $\bar{S}_r$ is here closer to 1 (see Table 6.14), implying the increase of C_s and C_k (upper physical limit!),

c) the *unconfined compression strength* of both sample types shows a very significant difference, whereby the distribution of the values obtained from samples taken from large diameter borehole is close to the normal. This fact is of a high importance, because from q_u also the *cohesion* can be calculated with a good approximation and the knowledge of this is necessary for the solution of numerous geotechnical tasks.

The tabulated values are plotted graphically in Fig. 6.9.

Table 6.15. β_1 and β_2 values calculated for four physical characteristics of the soils S4 and S5 by sample types

Symbol of the soil	Physical characteristic	Small diameter		Large diameter	
		β_1	β_2	β_1	β_2
S4	w	0.17	3.82	0.11	3.92
	e	0.17	4.11	0.02	3.67
	S_r	2.82	6.72	4.67	11.21
	q_u	4.88	10.30	0.58	3.27
S5	w	0.31	2.89	0.04	3.52
	e	0.14	3.30	0.32	3.94
	S_r	1.14	3.18	3.42	8.92
	q_u	5.71	8.49	0.67	3.05

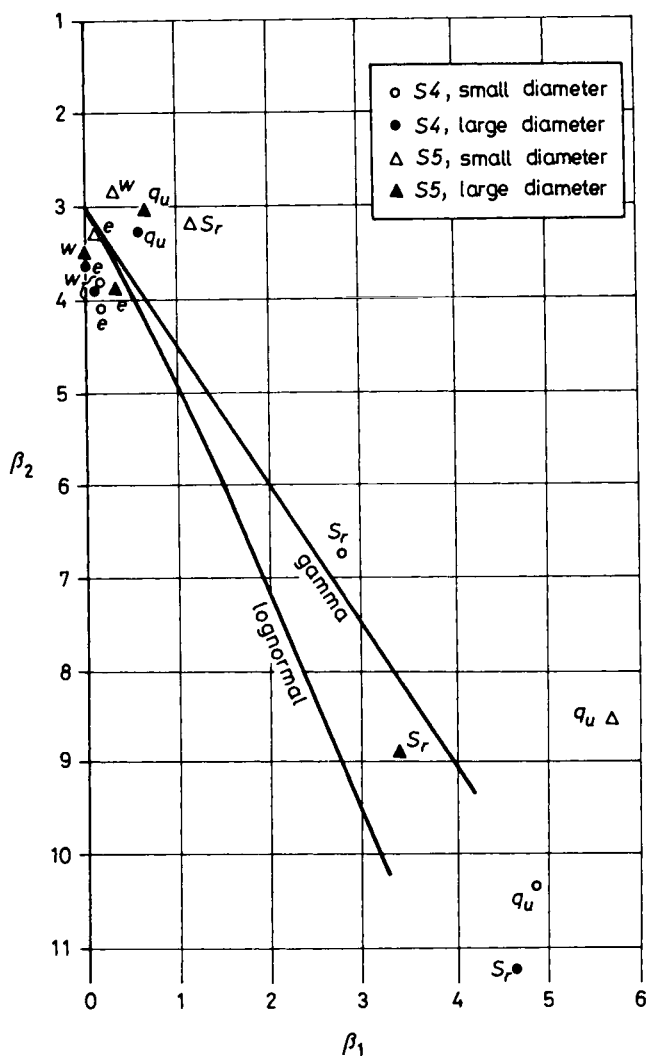


Fig. 6.9. Value pairs β_1, β_2 determined for samples of the soils S4 and S5 taken from small and large diameter boreholes

6.2.8. Reflection of the test methodology in the correlation matrix of the physical characteristics

As the test methodics (e.g. the boring technology) affect the various physical characteristics differently and in different degrees, a single correlation matrix cannot reflect two different test methods. This applies not only to the matrix itself, i.e., to the correlation coefficients, but also to the constants of the regression straight line.

Table 6.16. Correlation matrix for samples of the soil S4 taken from small (K) and large diameter boreholes (N)

		e	S_r	γ	q_u
w	N	0.72	0.63	-0.34	-0.18
	K	0.66	0.57	-0.21	-0.43
w_L	N	0.39	-0.13	-0.25	-0.18
	K	0.29	-0.02	-0.09	-0.01
w_p	N	0.37	-0.08	-0.35	0.24
	K	0.17	-0.02	-0.18	0.20
I_p	N	0.28	-0.12	-0.14	0.11
	K	0.24	-0.01	-0.03	-0.06
I_c	N	-0.20	-0.51	0.00	0.29
	K	-0.37	-0.51	0.01	0.46
e	N	1	0.01	-0.75	-0.21
	K	1	-0.09	-0.68	-0.22
S_r	N		1	0.37	0.03
	K		1	0.45	-0.55
γ	N			1	0.26
	K			1	-0.15

Table 6.16 shows the correlation matrix determined for the samples of soil S4 taken from large and small diameter boreholes. The mean of the absolute values of the correlation coefficients calculated for the former exceeds (if q_u is disregarded) by 0.05 the mean value calculated for the latter. The correlation of the unconfined compression strength with the other physical characteristics improved evidently, but the residual standard deviation increased considerably.

6.3. Evaluation of ring tests

Ring tests are carried out to determine generally

- the degree of homogeneity of the physical characteristics of artificially produced (or eventually natural) products, and
- the possibly existing significant differences between the test methods, resp. between
- the testing laboratories participating in the ring test.

The measurements (observations) may be evaluated by the so-called *variance analysis* (Sections 6.3.1 and 6.3.2), but also approximative processes may be used (Section 6.3.3). The ring tests permit also to judge the “compatibility” of two measurements (Section 6.3.4).

6.3.1. Variance analysis

Let us assume that a physical characteristic of a given soil block has been determined by k laboratories with the help of m different test methodology (instruments). In this case the measurement results may be tabulated in the following way:

$j \backslash i$	1	2	...	k
1				
$\vdots$				
m				

The general symbol of the individual elements is x_{ij} , where

i — serial number of the laboratory,
 j — serial number of the test methodology.

This general member may be written up in the following form:

$$x_{ij} = \mu + a_i + g_j + \varepsilon_{ij},$$

where

μ — expected value of the physical characteristic of the soil block regarded as homogeneous,
 a_i — deviation due to the *row-effect* (difference between the laboratories),
 g_j — deviation due to the *column-effect* (difference between the test methods),
 ε_{ij} — *random error*.

If we assume that both the row-effect and the column-effect play an important role in the standard deviation of the measured results, we may evaluate the measurements by the double-factor (“two-way”) variance analysis. The course of the calculation is as follows.

First we evaluate the so-called overall mean of the measurement $N = k \cdot m$:

$$\bar{X} = \frac{1}{km} \sum_{i=1}^k \sum_{j=1}^m x_{ij}. \quad (6.18)$$

The square sum

$$Q = \sum_{i=1}^k \sum_{j=1}^m (x_{ij} - \bar{X})^2 \quad (6.19)$$

determining the total variance consists of three component parts:

$$Q = Q_1 + Q_2 + Q_3, \quad (6.20)$$

Table 6.17. Dispersion resolving table of the double factor variance analysis (NEMESDY, 1979)

Cause of the dispersion	Square sum	Degree of freedom	Empirical variance	Expectable variance values
Row-effect (k)	Q_1	$k-1$	$s_1^2 = \frac{Q_1}{k-1}$	$\sigma_e^2 + ks_k^2$
Column-effect (m)	Q_2	$m-1$	$s_2^2 = \frac{Q_2}{m-1}$	$\sigma_e^2 + ms_m^2$
Random error	Q_3	$(k-1)(m-1)$	$s_3^2 = \frac{Q_3}{(k-1)(m-1)}$	σ_e^2
Total dispersion	Q	$km-1$		

where Q_1 reflects the row-effect, Q_2 the column-effect, Q_3 the random variation. These quantities may be calculated by the following expressions:

$$Q_1 = m \sum_{i=1}^k (\bar{x}_i - \bar{X})^2; \quad (6.21)$$

$$Q_2 = k \sum_{j=1}^m (\bar{x}_j - \bar{X})^2; \quad (6.22)$$

$$Q_3 = \sum_{i=1}^k \sum_{j=1}^m (x_{ij} - \bar{x}_i - \bar{x}_j + \bar{X})^2, \quad (6.23)$$

where $\bar{x}_i$ — mean of the i -th column,

$\bar{x}_j$ — mean of the j -th row.

The calculation results may be summarized purposefully in the so-called dispersion-resolving table (Table 6.17). The degrees of freedom appearing in column 3 of the Table correspond to the FISHER-COCHRAN-principle.

The row-effect can be separated from the column-effect with the help of the F -test. The significance of the *row-effect* is evaluated on the basis of the quotient

$$F_k = \frac{s_1^2}{s_3^2}. \quad (6.24)$$

The critical value F_p must be assigned, according to Table 6.17, to the degrees of freedom $f_1 = k-1$ and $f_2 = (k-1)(m-1)$. The *column-effect* may be examined with the help of the quotient

$$F_m = \frac{s_2^2}{s_3^2} \quad (6.25)$$

with the degrees of freedom $f_1 = m-1$ and $f_2 = (k-1)(m-1)$ taken into consideration.

If $F_k > F_p$, the row-effect (in our example the deviation due to the dissimilarity of the laboratories) is *significant*; if $F_k < F_p$, it is *insignificant*. (The column-effect may also be evaluated in the same way.)

The last column in Table 6.17 shows the expected variance values, permitting to

estimate the standard deviations characterizing the row-effect (s_k) and the column-effect (s_m):

$$s_k = \sqrt{\frac{s_1^2 - s_3^2}{k}}, \quad (6.26)$$

$$s_m = \sqrt{\frac{s_2^2 - s_3^2}{m}}. \quad (6.27)$$

The so-called single-factor ("one-way") variance analysis is used for examining the row-effect (or the column-effect) alone, i.e. essentially if the measurements originate from the same population. This is the case, e.g., when the same soil, regarded as homogeneous, is tested by k laboratories with the help of m repeated measurements, or when $k \cdot m$ samples, taken from k different points of an earthwork are tested by a single laboratory. (In the first case the degree of the "homogeneity" of the laboratories, in the second case that of the earthwork is assessed.)

In the case of the single-factor analysis the measured data are summarized again in a two-way table. As here the effect of a single factor is examined, the square sum Q characterizing the total standard deviation consists now of two (not three) component parts:

$$Q = Q_1 + Q_2. \quad (6.28)$$

The variance s_1^2 characterizing the effect of the *intergroup-factor* may be calculated by the expression

$$s_1^2 = \frac{m \sum_{i=1}^k (\bar{x}_i - \bar{X})^2}{k - 1} \quad (6.29)$$

and the variance s_2^2 characterizing the random error within the group by the expression

$$s_2^2 = \frac{\sum_{i=1}^k \sum_{j=1}^m (x_{ij} - \bar{x}_i)^2}{N - k}, \quad (6.30)$$

where the notations are the same as those used in the double-factor analysis.

The zero-hypothesis is now that s_1 , the intergroup standard deviation and s_2 , the standard deviation within the groups do not differ significantly from each other. So we must compare the quotient

$$F = \frac{s_1^2}{s_2^2} \quad (6.31)$$

with the F_p -value belonging to the degrees of freedom $f_1 = k - 1$ and $f_2 = N - k$ and the significance level p . If

$$F < F_p,$$

we may accept that the samples came from the same population, while in the opposite case they did not.

6.3.2. Density control of earthworks by ring tests

Let us assume that at $k = 10$ points of an important earthwork the density is determined by $m = 4$ different methods (which may be, e.g., sampling by trepanation, volumetry by sand blasting, isotopic measurement and probing). The numerical values of the degrees of density (T_{ry}) are shown in *Table 6.18*.

The analysis of the test series has the following two aims: a) establishing if the earthwork can be regarded as homogeneous, b) finding out if there exist significant differences between the measurement methods. This means that the evaluation must be performed by the double-factor variance analysis. The calculation results may be summed up as follows.

Sums according to Eqs (6.21), (6.22) and (6.23) are:

$$Q_1 = 41.252; \quad Q_2 = 6.654; \quad Q_3 = 49.602.$$

The total square sum Q , calculated by (6.19) is 97.65, so it agrees well with the sum of the three Q_i -values within the accuracy limits of the calculation. It may be easily proved that Q can be evaluated much more easily by determining the standard deviation s of 40 measurements, from which

$$Q = s^2(N - 1). \quad (6.32)$$

The empirical variances are:

$$s_1^2 = \frac{Q_1}{k-1} = \frac{41.252}{9} = 4.584;$$

$$s_2^2 = \frac{Q_2}{m-1} = \frac{6.654}{3} = 2.218;$$

$$s_3^2 = \frac{Q_3}{(k-1) \cdot (m-1)} = \frac{49.602}{9 \cdot 3} = 1.837.$$

The quotient F characterizing the row-effect is:

$$F_k = \frac{s_1^2}{s_3^2} = \frac{4.584}{1.837} = 2.495.$$

Table 6.18. Numerical example for the single and double factor variance analysis (density control of earthworks)

$j \backslash i$	1	2	3	4	5	6	7	8	9	10	$\bar{x}_j$
1	93.7	92.9	95.2	96.4	94.2	92.5	96.3	95.5	95.5	92.9	94.51
2	95.8	94.3	92.3	97.8	96.0	95.2	93.9	95.4	94.4	95.3	95.04
3	96.2	97.2	93.6	98.8	92.8	95.4	94.4	93.9	94.2	94.8	95.13
4	96.1	98.2	94.3	97.9	93.5	96.2	95.4	93.9	94.8	96.3	95.66
$\bar{x}_i$	95.45	95.65	93.85	97.73	94.13	94.83	95.00	94.68	94.73	94.83	

As $F_{0.95}=2.25$ belongs, according to *Table F8*, to the degrees of freedom $f_1 = k - 1 = 9$ and $f_2 = (k - 1)(m - 1) = 27$, so $F_k > F_p$, we may state that the *density of the tested road section is not homogeneous*. (If the data of *Table 6.18* are understood as related to k laboratories and not to k measurement points, then this statement applies to the work of the laboratories.)

The quotient F characterizing the column-effect is

$$F = \frac{s_2^2}{s_3^2} = \frac{2.218}{1.837} = 1.207.$$

As $F_{0.95}=2.96$ belongs according to *Table F8* to the degrees of freedom $f_1 = m - 1 = 3$ and $f_2 = 27$, so $F < F_p$, consequently for the qualification of the earthwork it is *indifferent, which method is used for measuring the density*.

We shall return in Section 12.4 to the technical evaluation of the heterogeneity appearing in the row-effect.

Let us interpret now *Table 6.18* as the density of a given soil mass tested by $k = 10$ laboratories, each of which has tested $m = 4$ samples. In this case we may examine by the *single-factor* variance analysis, if the measurement results originate from the same population.

The square sum Q_1 and the variance are identical with the values calculated before (41.252 and 4.584, respectively) and the variance s_2^2 is, according to expression (6.30)

$$s_2^2 = \frac{56.397}{40 - 10} = 1.880.$$

The quotient of the variances is

$$F = \frac{4.584}{1.880} = 1.56,$$

the critical value is $F_{0.95}=2.21$ ($f_1=9, f_2=30$), so the zero-hypothesis is not rejected, but naturally the measurement results may be refined by excluding one or two laboratories (see Section 7.1.2).

6.3.3. Approximative evaluation of the ring tests

According to Section 6.3.1 there are two basic conditions for the correct analysis of the ring tests: a) every column must contain m elements identically, b) every x_{ij} value must be known.

If these conditions are missing, an approximate image on the effect of the various factors can be obtained, as, e.g., in the case of $k=2$ we have compared a series of measurements on the basis of the means and the standard deviations with the help of *statistical tests*. This method may be generalized for the case of $k > 2$. For this purpose we may use the expression (6.10) directly, where now n_1 and s_1 are the number of

Table 6.19. Numerical results of some ring tests (BORUS and RÉV, 1970)

Physical characteristic	One laboratory		Several laboratories	
	n	$\bar{x} \pm s$	n	$\bar{x} \pm s$
w_L	65	99.5 ± 6.83	21	99.3 ± 13.60
	66	86.2 ± 5.58	21	84.5 ± 7.20
	64	53.9 ± 5.12	24	50.5 ± 4.87
	60	34.7 ± 2.77	23	33.9 ± 3.10
γ_d	28	1.575 ± 0.057	28	1.570 ± 0.119
	27	1.497 ± 0.019	27	1.524 ± 0.050
	33	1.539 ± 0.054	26	1.533 ± 0.067
	29	1.648 ± 0.027	26	1.646 ± 0.058
	24	1.571 ± 0.034	26	1.586 ± 0.084
	40	1.671 ± 0.030	15	1.681 ± 0.098
	19	1.649 ± 0.079	24	1.646 ± 0.096
	19	1.656 ± 0.042	21	1.586 ± 0.070
q_u	26	2.75 ± 0.67	21	2.31 ± 0.82
	27	1.91 ± 0.45	19	1.93 ± 0.95
	33	1.73 ± 0.48	19	2.13 ± 1.36
	30	2.09 ± 0.48	20	2.52 ± 1.14
	24	0.65 ± 0.15	15	0.44 ± 0.30

elements and the standard deviation of the tested group, respectively, n_2 and s_2 are the number of elements and the standard deviation of the whole data set. We may proceed similarly also in the case of the F -test.

We have applied this method for evaluating the territorial variability of the soils of Szeged; the last six rows of Table 4.3 contain the results of statistical tests, called here "simultaneous tests". Also the example quoted by SCHULTZE (1975a) (Section 6.2.3), or

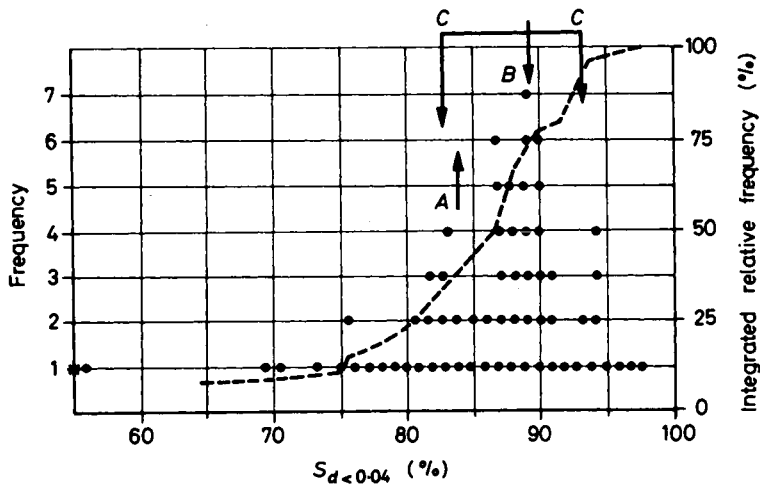


Fig. 6.10. Ring test for determining the mass percentage of $d < 0.04$ mm diameter grains; A — mean of the laboratories, B — mean of the three umpire laboratories and C — range of the standard deviations

the tests by three different CASAGRANDE-apparatuses (Section 6.2.4) could be evaluated similarly.

In literature we can find cases, where group means and standard deviations of results supplied by various laboratories (or by different test methods) are given (LUMB, 1971; BORUS and RÉV, 1970; SINGH, 1970; BORUS *et al.*, 1969). Though these data are unsuited for analyzing the "row and the column effects", they may offer interesting information for the comparison of various methods (laboratories, etc.). The results of a few similar test series are given in *Table 6.19*.

An interesting test program was organized by the American Council of Independent Laboratories (ACIL), with the participation of nearly 100 laboratories in a ring test (HARR, 1977). One task was the determination of the mass percentage belonging to $d < 0.04$ mm in a sample. 67 laboratories took part in this experiment and three additional "umpire laboratories" (these determined this characteristic very carefully by numerous iterations). *Fig. 6.10* showing the evaluation of these tests permit to establish that

a) nearly 50% of the S -values determined by the laboratories are beyond of the standard deviation range specified by the three umpire laboratories;

b) about 75% of all the measured values are lower than the mean specified by the umpire laboratories.

(The tested soil was a clay of medium plasticity, the requested mass percentage was determined by the hydrometer method.)

6.3.4. Compatibility of two measurement results

Let us assume that every x_{ij} value in *Table 6.18* represents the mean of n measurements. In this case the so-called *iterative standard deviation* (s_i) may be calculated as

$$s_i = \sqrt{s_m^2 + \frac{s_3^2}{n}}, \quad (6.33)$$

where s_m — standard deviation defined by Eq. (6.27),

s_3 — standard deviation defined by Eq. (6.23).

The iterative standard deviation is the index of dispersion caused by the differences between the measurement methods, the subjective and objective faults. If there is no column-effect ($s_m = 0$), i.e. all the laboratories are using an identical test method, then

$$s_i = s_{rp}, \quad (6.34)$$

where s_{rp} is the standard deviation defined in Section 6.1, characterizing the reproducibility.

The comparative standard deviation (s_c) reflects the differences existing between the laboratories:

$$s_c = \sqrt{s_k^2 + s_m^2 + \frac{s_3^2}{n}}, \quad (6.35)$$

where s_k is the standard deviation defined by expression (6.26), characterizing the row-effect.

On comparing the expressions (6.33) and (6.35) it is easily seen that

$$s_c > s_i. \quad (6.36)$$

This means that if the tests are performed by a *single* laboratory, the *effective situation is idealized* by choosing s_i as basis. In such cases s_i can be accepted as ruling only, if the state of the laboratory apparatuses and the skill of the laboratory assistants are better than average and they perform the tests very carefully.

The expressions (6.33) and (6.36) may be used for judging the limit of the purposeful increase of n . In this respect the problem is decided by the assumed values of s_3 relative to s_m , resp. s_k (LUMB, 1971).

Let us examine now the condition of the acceptable deviation between the repeated measurements x_1 and x_2 (regarding their means as the expected value of the tested characteristic).

If the physical characteristic has a normal distribution, we may write, with expression (2.18) taken into account,

$$\bar{x} - \lambda \frac{s_i}{\sqrt{2}} < \mu < \bar{x} + \lambda \frac{s_i}{\sqrt{2}}, \quad (6.37)$$

where $\bar{x}$ — mean of both measurements,

μ — expected value of the population (see also Section 7.4.1).

Accordingly—if for example we select a significance level $p = 0.05$ —the deviation between two measurements is acceptable if

$$|x_1 - x_2| < 2 \cdot 1.96 \frac{s_i}{\sqrt{2}}. \quad (6.38)$$

(We must proceed similarly, when the measurements are performed by two laboratories, only in this case we must perform the calculations with s_c instead of s_i .)

If condition (6.38) is satisfied, we say that the measurement results are *compatible*. In the contrary case the test must be repeated and from the three results the two, for which the inequality (6.38) is satisfied, are to be accepted.

Let us assume that $s_i = s_{rp}$, so we can utilize the values of the standard deviations described in Section 6.1.2. According to Table 6.1 the standard deviation characterizing the reproducibility of the water content of silt is $s_{rp} = s_i = 1.11$. According to expression (6.38) the result of the repeated measurements is acceptable, if

$$|w_1 - w_2| < 3.1.$$

(For the infusion loess only 1.2w-%.)

7. Evaluation of the test data before the planning

So far the determination of the statistical parameters of the soil physical characteristics, their variations in space and time, the relationships between them and the comparison of tests performed by various methods, or different apparatuses were discussed. The data set obtained as the result of the work phases described in Chapters 1–6 forms the basis for the design, but before solving a given planning task, the test data must be re-evaluated in most cases for

- screening out the suspicious, extremal values (Section 7.1),
- taking into account the results of earlier, or other parallel studies (Section 7.2),
- combining, resp. separating certain sample groups, or soil layers (Section 7.3),
- determining the confidence intervals (Section 7.4).

On surveying large areas the overall evaluation of the test material is also advisable (Section 7.5).

7.1. Screening out the outliers (extreme values)

7.1.1. General aspects of the evaluation of outliers

A frequent experience in analyzing data series of some soil physical characteristic is that some values are very far from the mean, which may be due to the following *reasons*:

- the layer contains foreign inclusions (e.g. organic lenses),
- some samples came from another layer (not the one to be tested) due to the inaccurate determination of the boundaries between the layers (see Section 7.3),
- the structure, or the state of the sample, giving the outlying result, changed during the sampling, the transport, or the storage (see Section 7.1.3),
- some samples were tested under different conditions (e.g. at a different temperature) relative to the others,
- the tested layer was subjected to some local effect (e.g. water infiltration from a damaged pipe into the soil),

- a part of the tests was performed by another laboratory assistant, or by a different method, or less carefully, though by the same assistant,
- the tests were evaluated with a gross error.

According to GRUBBS (1969) the general viewpoints in judging outliers values are as follows.

1. *If the physical cause is known*, one of the following solutions may be chosen:

- (1) rejection,
- (2) correction based on physical considerations, or
- (3) rejection and performance of additional tests.

2. *If the physical cause is unknown*, one of the following decisions must be accepted, based on the statistical analysis of the results:

- (1) rejection, or
- (2) correction by statistical methods,
- (3) new tests for replacing the obtained results,
- (4) treating the remaining values as a “truncated” group of samples.

In some cases the outlying values can be judged easily (e.g., if it turns out that the standard specifications were not kept, or the sample dried out, the error of the calculation may be corrected), but apart from these fortunate cases *great care* must be exercised. The erroneous decisions may have severe consequences, e.g., the possible presence of organic lenses, which settled in the layer, or dangerous local effects exerted on the soil might be excluded. Therefore, the guiding principle must be kept strictly that the exclusion of any data *must be justified by expert opinion*. At the same time the outliers may refer to the fact that the reality may not be reflected truly even by results, which were qualified as faultless.

If, e.g., a value exceeding 1 for the degree of saturation of a layer is obtained several times, it may be assumed by right that the determination of the density (i.e. the calculated value of the void ratio) was incorrect. This may be examined on the basis of the error limits of the measurement (see Section 6.1.1).

As every test represents an *economic value*, everything must be done for “saving” it, especially if it cannot be repeated, or only at great expenses. A possible way is the *correction* of the results, as mentioned above. This possibility is considered in Section 6.2.6 (in relation to small and large diameter boreholes), but other solutions are also conceivable, depending on the actual task. So, e.g., the rate of the capillary rise, or the water quantity flowing through the permeameter may be converted from one temperature to another one, whereby the result of a test performed under different conditions may be corrected unequivocally.

The extreme values may be examined generally by mathematical statistical methods (Section 7.1.2), but in fortunate cases even physical (deterministic) methods may be

used (Section 7.1.3). Whichever process is selected, the data series must be re-evaluated after the elimination of the outliers: a new mean and new central moments must be determined, the empirical distribution function must be assessed anew, etc. (Section 7.1.4).

7.1.2. Examination of outliers by statistical methods

This problem was first dealt with by SMIRNOV (1941) on the presently accepted basis. GRUBBS (1950), who summarized in one of his later studies the solutions formed till the end of the sixties, came also to a similar conclusion. The geotechnical importance of the problem was pointed out by numerous authors since then (HOLTZ and KRIZEK, 1971; FANG, 1975; VAN GANSE, 1975; LUMB, 1975b) and there exist even standards by now (e.g. GOST-20522, 1975), which prescribe the obligatory performance of the related tests.

The methods elaborated for the analysis of the outlying values are similar in many respects to the statistical tests discussed in Section 6.2.1:

- the decision depends on the chosen significance level, the number of the samples and on whether the expected value and the standard deviation of the population is known, or unknown,
- here too we may speak of first and second type errors,

As to the selection of the method, two important circumstances may imply a methodical difference, namely if there are one, two, or more suspicious values and if their deviation from the mean is of an identical, or a different sign.

It must be stated in advance that every method is based on the assumption that the distribution of the elements is *normal*. This must be controlled always, because in the contrary case we may exclude results which belong to the involved distribution, but are asymmetric due to the given physical circumstances,

Let us examine first the case, where *the standard deviation (σ) and the expected value (m) of the population is unknown*. As a first step, the measured data must be ordered in the form of a series:

$$x_1 \leq x_2 \leq \dots \leq x_n,$$

then the mean ($\bar{x}$) and the corrected standard deviation (s) defined by Eq. (1.8) must be calculated. This is followed by forming the quotient

$$T_n = \frac{x_n - \bar{x}}{s}, \quad \text{or} \quad T_1 = \frac{\bar{x} - x_1}{s}, \quad (7.1)$$

depending on whether *the first, or the last member* of the series is examined.

The quotient (7.1) must be compared with the critical T -value of *Table 7.1*. If

$$T_n > T, \quad \text{resp.} \quad T_1 > T,$$

the outlying value may be omitted, in the contrary case it must not be omitted.

Table 7.1. Critical T-values for unknown σ and m (GRUBBS, 1969; MSZ-COMECON-545, 1977)

Number of elements (n)	Value of T , when $\alpha =$		
	0.05	0.025	0.01
3	1.15	1.15	1.15
4	1.46	1.48	1.49
5	1.67	1.71	1.75
6	1.82	1.89	1.94
7	1.94	2.02	2.10
8	2.03	2.13	2.22
9	2.11	2.21	2.32
10	2.18	2.29	2.41
11	2.23	2.36	2.48
12	2.29	2.41	2.55
13	2.33	2.46	2.61
14	2.37	2.51	2.66
15	2.41	2.55	2.71
16	2.44	2.59	2.75
17	2.47	2.62	2.79
18	2.50	2.65	2.82
19	2.53	2.68	2.85
20	2.56	2.71	2.88
21	2.58	2.73	2.91
22	2.60	2.76	2.94
23	2.62	2.78	2.96
24	2.64	2.80	2.99
25	2.66	2.82	3.01
30	2.75	2.91	
35	2.82	2.98	
40	2.87	3.04	
45	2.92	3.09	
50	2.96	3.13	
60	3.03	3.20	
70	3.09	3.26	
80	3.14	3.31	
90	3.18	3.35	
100	3.21	3.38	

If it cannot be established a priori, if x_1 , or x_n is the more suspicious value, then *Table 7.1* must be used again, but in this case the significance levels appearing in it must be doubled.

According to GRUBBS (1969) the values given for $n > 25$ are only of an orientative character. The standard MSZ-COMECON-545 (1977) recommends to use the data of *Table 7.7* for this purpose.

Table 7.2. DIXON's criterion for examining extreme values

n	Criterion*	Level of significance		
		0.10	0.05	0.01
3	$r_{10} = \frac{x_2 - x_1}{x_n - x_1}$ and $\frac{x_n - x_{n-1}}{x_n - x_1}$	0.886	0.941	0.988
4		0.679	0.765	0.889
5		0.557	0.642	0.780
6		0.482	0.560	0.698
7		0.434	0.507	0.736
8	$r_{11} = \frac{x_2 - x_1}{x_{n-1} - x_1}$ and $\frac{x_n - x_{n-1}}{x_n - x_2}$	0.479	0.554	0.683
9		0.441	0.512	0.635
10		0.409	0.477	0.597
11	$r_{21} = \frac{x_2 - x_1}{x_{n-1} - x_1}$ and $\frac{x_n - x_{n-2}}{x_n - x_2}$	0.517	0.576	0.679
12		0.490	0.546	0.642
13		0.467	0.521	0.615
14	$r_{22} = \frac{x_3 - x_1}{x_{n-2} - x_1}$ and $\frac{x_n - x_{n-2}}{x_n - x_3}$	0.492	0.546	0.641
15		0.472	0.525	0.616
16		0.454	0.507	0.595
17		0.438	0.490	0.577
18		0.424	0.475	0.561
19		0.412	0.462	0.547
20		0.401	0.450	0.535
21		0.391	0.440	0.524
22		0.382	0.430	0.514
23		0.374	0.421	0.505
24		0.367	0.413	0.497
25		0.360	0.406	0.489

* The first criterion is to be used if the minimum value is suspected, and the second criterion if the maximum value is suspected.

The process proposed by DIXON (1953) is used in practice for a fast orientation, due to its simplicity. The basis of the decision is formed by the quotients r , calculated from the difference of the observed values (Table 7.2), otherwise the process is similar to the preceding one.

If the suspicion that both the lowest and the highest values must be omitted is well grounded, the method of DAVID *et al.* (1954) may be used: we calculate the quotient

$$\frac{w}{s} = \frac{x_n - x_1}{s} \quad (7.2)$$

and compare it with the corresponding value of Table 7.3.

If the latter is lower, we omit both extremal values. If the differences $x_n - \bar{x}$ and $\bar{x} - x_1$ deviate greatly from each other and the quotient w/s is close to the tabulated value, it is

Table 7.3. Critical values for w/s for the examination of both extreme values (DAVID *et al.*, 1954)

n	Level of significance			n	Level of significance		
	0.05	0.01	0.005		0.05	0.01	0.005
3	2.00	2.00	2.00	17	4.31	4.59	4.69
4	2.43	2.44	2.45	18	4.38	4.66	4.77
5	2.75	2.80	2.81	19	4.43	4.73	4.84
6	3.01	3.10	3.12	20	4.49	4.79	4.91
7	3.22	3.34	3.37	30	4.89	5.25	5.39
8	3.40	3.54	3.58	40	5.15	5.54	5.69
9	3.55	3.72	3.77	50	5.35	5.77	5.91
10	3.68	3.88	3.94	60	5.50	5.93	6.09
11	3.80	4.01	4.08	80	5.73	6.18	6.35
12	3.91	4.13	4.21	100	5.90	6.36	6.54
13	4.00	4.24	4.32	150	6.18	6.64	6.84
14	4.09	4.34	4.43	200	6.38	6.85	7.03
15	4.17	4.43	4.53	500	6.94	7.42	7.60
16	4.24	4.51	4.62				

advised to proceed as follows (GRUBBS 1969). The difference whose absolute value is higher (e.g. $x_n - \bar{x}$) is tested according to expression (7.1) and depending on the result

- the value x_n is rejected (if $T_n > T$), then for the remaining $n - 1$ elements the test $T_1 \leq T$ is performed with the new $\bar{x}$ and s values calculated,
- both the x_n and x_1 values are retained, if $T_n < T$.

In the next fundamental case *the two suspicious values are on the same side of $\bar{x}$* . GRUBBS (1950, 1969) recommends to calculate in this case first the standard deviation s (dividing the square sum of the deviations by n instead of by $n - 1$), then—if x_1 and x_2 are the suspicious values—to evaluate the square sum defined by the following expression:

$$s_{1,2}^2 = \frac{(n-2) \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2}{n-2}. \quad (7.3)$$

If the quotient $s_{1,2}^2/s^2$ is lower than the critical value (Table 7.4), then both values are omitted.

On using the DIXON-test we must proceed carefully, because the so-called *avalanche-effect* may also appear, causing ever more measurement data to drop out. Therefore, not more than 10% max. of the elements may be excluded by this method (NEMESDY, 1979).

Let us assume that σ and m are unknown, while having *more suspicious values*. FERGUSON (1961) gave a solution for this case. Accordingly, if the suspicious values are positioned on one side, we must calculate the quotient characterizing the skewness:

$$\sqrt{b_1} = \sqrt{n} \frac{\sum_{i=1}^n (x_i - \bar{x})^3}{s^3}, \quad (7.4)$$

Table 7.4. Critical values of the quotients $s_{1.2}^2/s^2$ and $s_{n-1,n}^2/s^2$ for examining two suspected values lying on the same side (GRUBBS, 1950)

n	Level of significance		
	0.10	0.05	0.01
4	0.0031	0.0008	0.0000
5	0.0376	0.0183	0.0035
6	0.0921	0.0565	0.0186
7	0.1479	0.1020	0.0440
8	0.1994	0.1478	0.0750
9	0.2454	0.1909	0.1082
10	0.2853	0.2305	0.1415
11	0.3226	0.2666	0.1736
12	0.3552	0.2996	0.2044
13	0.3843	0.3295	0.2333
14	0.4106	0.3568	0.2605
15	0.4345	0.3818	0.2859
16	0.4562	0.4048	0.3098
17	0.4761	0.4259	0.3321
18	0.4944	0.4455	0.3530
19	0.5113	0.4636	0.3725
20	0.5269	0.4804	0.3909

while if they are found on both sides, the quotient

$$b_2 = n \frac{\sum_{i=1}^n (x_i - \bar{x})^4}{(n-1)^2 \cdot s^4} \quad (7.5)$$

characterizing the kurtosis must be calculated. If they exceed the critical values given in Table 7.5, we omit the most remote result from $\bar{x}$ and continue this process until the zero-hypothesis is satisfied.

Let us return now to the case, where the standard deviation $s = \sigma$ is known (e.g. from earlier tests). The course of the test is similar to the first discussed method, with the difference that

- T_n and T_1 are replaced by the notations t_n and t_1
- the critical values of Table 7.6 are used.

Table 7.5. Examination of several suspected values on the basis of $\sqrt{b_1}$ and b_2 (FERGUSON, 1961)

α	n =											
	5	10	15	20	25	30	35	40	50	60	75	100
$\sqrt{b_1}$	0.01	1.34	1.31	1.20	1.11	1.06	0.98	0.92	0.87	0.79	0.72	
	0.05	1.05	0.92	0.84	0.79	0.71	0.66	0.62	0.59	0.53	0.49	
b_2	0.01	3.11	4.83	5.08	5.23	5.00			4.88		4.59	4.39
	0.05	2.89	3.85	4.07	4.15	4.00			3.99		3.87	3.77

Table 7.6. Critical values for t_n and t_1 in the case of known σ (MSZ-COMECON-545, 1977; GRUBBS, 1950)

n	Level of significance		
	0.05	0.01	0.005
3	1.738	2.215	2.396
4	1.941	2.431	2.618
5	2.080	2.574	2.764
6	2.184	2.679	2.870
7	2.267	2.761	2.952
8	2.334	2.828	3.019
9	2.392	2.884	3.074
10	2.441	2.931	3.122
11	2.484	2.973	3.163
12	2.523	3.010	3.199
13	2.557	3.043	3.232
14	2.589	3.072	3.261
15	2.617	3.099	3.287
16	2.644	3.124	3.312
17	2.668	3.147	3.334
18	2.691	3.168	3.355
19	2.712	3.188	3.375
20	2.732	3.207	3.393
21	2.750	3.224	3.409
22	2.768	3.240	3.425
23	2.784	3.255	3.439
24	2.800	3.269	3.453

The standard MSZ-COMECON-545 (1977) proposes the use of *Table 7.7* for the case $n \geq 25$, against the recommendation of GRUBBS (1969).

If both σ and m are known, the quotients

$$V_n = \frac{y_n - m}{\sigma}, \quad \text{resp.} \quad V_1 = \frac{m - y_1}{\sigma} \quad (7.6)$$

must be calculated and compared with the critical values of *Table 7.7*.

Let us assume that the following void ratio values were obtained for samples of the soil *S4* of Szeged:

0.70—0.75—0.76—0.79—0.80—0.82—0.84—0.87—0.88—0.89.

As the most remote value from the mean ($\bar{e} = 0.8055$) is the measured minimum void ratio ($\bar{e} = 0.70$), the statistical test must be performed with relation to this value. According to *Table 1.12* the distribution of the void ratio of the soil *S4* is approximately normal, or at least symmetric ($C_s = 0.36$), so the utilization of the methods elaborated for testing outliers is permissible. The standard deviation of the data set is $s = 0.0607$, so according to expression (7.1)

$$T_1 = \frac{0.8055 - 0.70}{0.0607} = 1.738.$$

Table 7.7. Critical values for V_n and V_1 in the case of known σ and m (MSZ-COMECON-545, 1977)

n	Level of significance			
	0.05	0.01	0.005	0.001
1	1.645	2.326	2.576	3.090
2	1.955	2.575	2.807	3.290
3	2.121	2.712	2.935	3.403
4	2.234	2.806	3.023	3.481
5	2.319	2.877	3.090	3.540
6	2.386	2.934	3.143	3.588
7	2.442	2.981	3.188	3.628
8	2.490	3.022	3.227	3.662
9	2.531	3.057	3.260	3.692
10	2.568	3.089	3.290	3.719
15	2.705	3.207	3.402	3.820
20	2.799	3.289	3.480	3.890
25	2.870	3.351	3.539	3.944
30	2.928	3.402	3.587	3.988
40	3.015	3.480	3.662	4.054
50	3.082	3.541	3.716	4.108
100	3.285	3.723	3.892	4.263
250	3.534	3.946	4.108	4.465
500	3.703	4.108	4.263	4.607

It is seen in Table 7.1 that $T=2.23$ belongs to $\alpha=0.05$ and $n=11$ and as $T_1 < T$, the value $e=0.70$ is retained.

Let us assume that there was also another measurement result: $e=0.59$. In this case

$$T_1 = \frac{0.7875 - 0.59}{0.0850} = 2.32 > T (=2.29),$$

i.e. this void ratio cannot be regarded to be a part of the set at the selected significance level.

The methods described above are suited for the evaluation of ring tests as well. In this case the aim is generally to find out, if there is a laboratory whose results are unacceptable in relation to the others. The method may be studied on the basis of the numerical example given in Section 6.3.2. Let us assume that the values of Table 6.18 were determined by $k=10$ laboratories, each testing $m=4$ samples. The variance within the laboratories is now, according to expression (6.30)

$$s_2^2 = \frac{56.397}{40-10} = 1.88,$$

so the estimated standard deviation of the means of each of the four measurements is

$$s_2 = \sqrt{\frac{1.88}{4}} = 0.6856.$$

Table 7.8. Critical T -values for analysis of ring tests (DAVID, 1956); the level of significance is $p=0.05$

$f \backslash k$	3	4	5	6	7	8	9	10	12
10	2.01	2.27	2.46	2.60	2.72	2.81	2.89	2.96	3.08
11	1.98	2.24	2.42	2.56	2.67	2.76	2.84	2.91	3.03
12	1.96	2.21	2.39	2.52	2.63	2.72	2.80	2.87	2.98
13	1.94	2.19	2.36	2.50	2.60	2.69	2.76	2.83	2.94
14	1.93	2.17	2.34	2.47	2.57	2.66	2.74	2.80	2.91
15	1.91	2.15	2.32	2.45	2.55	2.64	2.71	2.77	2.88
16	1.90	2.14	2.31	2.43	2.53	2.62	2.69	2.75	2.86
17	1.89	2.13	2.29	2.42	2.52	2.60	2.67	2.73	2.84
18	1.88	2.11	2.28	2.40	2.50	2.58	2.65	2.71	2.82
19	1.87	2.11	2.27	2.39	2.49	2.57	2.64	2.70	2.80
20	1.87	2.10	2.26	2.38	2.47	2.56	2.63	2.68	2.78
25	1.84	2.07	2.23	2.34	2.44	2.52	2.58	2.64	2.74
30	1.82	2.04	2.20	2.31	2.40	2.48	2.54	2.60	2.69
40	1.80	2.02	2.17	2.28	2.37	2.44	2.50	2.56	2.65
60	1.78	1.99	2.14	2.25	2.33	2.41	2.47	2.52	2.61
120	1.76	1.96	2.11	2.22	2.30	2.37	2.43	2.48	2.57
∞	1.74	1.94	2.08	2.18	2.27	2.33	2.39	2.44	2.52

The laboratory means ($\bar{x}_i$) may be evaluated on the basis of Table 7.8 as follows: in our case $k=10$, the number of the degrees of freedom is $f=30$, so the critical value is $T=2.60$. As the "great mean" $\bar{X}=95.09$, we obtain for the minimum mean ($x_1=93.85$) that

$$T' = \frac{95.09 - 93.85}{0.6856} = 1.81 < 2.60,$$

so the deviation is not significant. For the maximum mean we obtain

$$T' = \frac{97.73 - 95.09}{0.6856} = 3.85 > 2.60,$$

which means that *the results of No. 4 laboratory are unacceptable.*

7.1.3. Drying the samples before the test

The water content of the sample decreases after the sampling gradually. According to a test series performed in the Institute for Geodesy and Geotechnics (Budapest) the measure of the decrease is about 1.1 w-% in the case of core-samples and about 0.7 w-% in the case of partly disturbed samples (Table 7.9). In practice the situation is somewhat less favourable, because damages occur frequently during the transport (endangering mainly the paraffine-soaked insulation).

Table 7.9. Observation of the drying process on box and core samples insulated by various substances; S – insulation tape; C – chinoplast, P – paraffin (RÉTHÁTI und UNGÁR, 1978a)

Elapsed time (days)	Mazalane-box			Plastic	Small size metal box		
	S	C	F	C	S	C	F
0	29:50	30:20	30:21	29:50	29:50	27:80	27:80
2	—	—	—	29:39	—	—	—
4	29:23	29:81	30:02	29:32	29:24	27:65	—
6	29:07	—	—	29:12	28:94	—	—
8	28:70	—	—	—	28:87	27:46	27:74
10	28:62	29:81	29:98	29:08	28:84	27:47	27:74
12	—	29:77	29:94	—	—	27:35	27:66
14	28:50	29:77	29:90	29:03	28:83	27:31	27:67
16	—	29:49	29:23	28:99	—	27:22	27:66
18	28:42	29:41	29:19	—	28:64	27:22	27:66
20	28:38	29:37	29:15	28:95	28:65	—	—
22	—	—	—	28:90	28:54	27:21	27:63
24	27:97	29:21	29:15	—	28:51	26:98	27:62
26	27:97	29:05	29:11	28:90	28:44	26:98	27:63
28	27:97	29:05	29:07	28:86	28:43	—	—
$\Delta w =$	-1.53	-1.05	-1.14	-0.64	-1.07	-0.82	-0.17

The elaboration of a process for distinguishing between the really undisturbed and the more or less dried samples would be of a high importance for the practice. In some cases the fact of the drying can be recognized visually (by the lighter colour of the edges of the sample and by the “knocking” of the dry sample in the cylinder), but in the majority of the cases the evaluation may be made only in the possession of *laboratory* results.

On evaluating the drying process we must distinguish between samples originating from below and from above groundwater level respectively between samples of a *high* and a *low* plasticity. Below the groundwater level the degree of saturation is high (a low S_r value refers a priori to drying), while above the groundwater level S_r may assume a whole set of values, especially in the close vicinity of the surface. The volume and the void ratio of the high plasticity soils decreases gradually, while they discharge water until their water content reaches the shrinkage limit, but their degree of saturation remains practically unchanged, so S_r represents a safe basis only if $w < w_s$.

On summing up the above said we may state that the fact of the drying can be established (by S_r) with the highest certainty in the case of low plasticity samples originating from below the groundwater level, less reliably in the case of high plasticity samples from below the groundwater level and least certainly in the case of samples taken from above the groundwater level. The most difficult task is testing low plasticity samples taken from above the groundwater level.

Let us try now to determine and the critical value for the soil S4 (RÉTHÁTI und UNGÁR, 1978a). On the basis of the histogram of Fig. 7.1 we may be sure that the samples of $w < 21\%$ water content do not reflect their natural state: only 1.6% of the

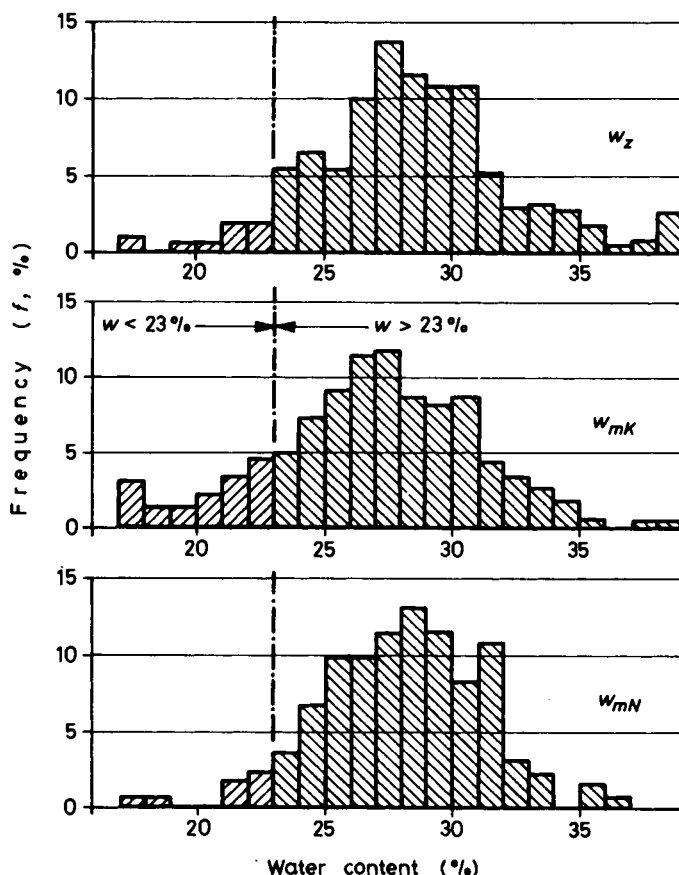


Fig. 7.1. Histograms of the water contents of the soil S4 by sample types (RÉTHÁTI and UNGÁR, 1978a)

water content of the large diameter samples, regarded as most reliable, drop to below this value. But it is more probable that we shall proceed correctly if we regard $w = 23\%$ as limit value (also for the sake of safety), on the basis of the following reasons:

- the related probability of the large diameter samples [$P(w < 23) = 5.4\%$] is more realistic than the value belonging to $w = 21\%$ ($P = 1.6\%$);
- in the histogram of the partly disturbed samples a step can be observed at $w = 23\%$;
- on assuming $S_r = 0.95$ the void ratio $e = 0.62$ belongs to $w = 21\%$, while the two lowest void ratio values, which were measured in large diameter samples were $e = 0.58$ and 0.63 .

In the following we examine the deviation of the degree of saturation and the void ratio of *small diameter samples* of $w < 23\%$ water content from the category-means.

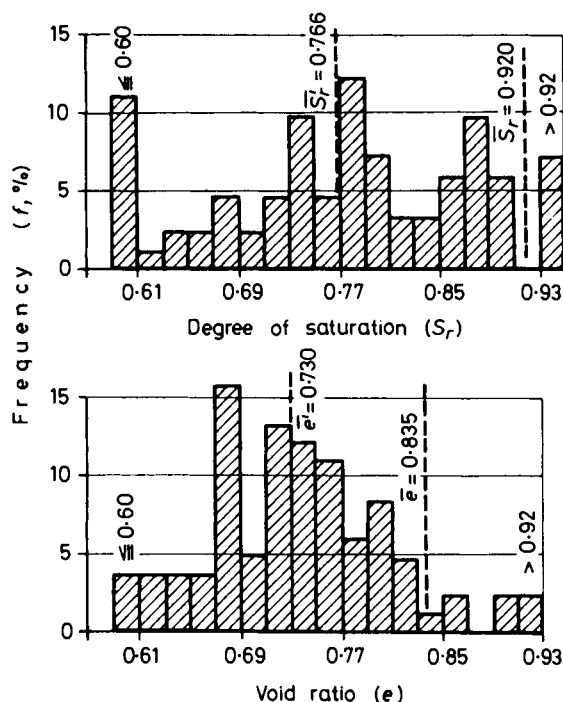


Fig. 7.2. Histograms of the degree of saturation and the void ratio of samples with $w < 23\%$ water content for samples of the soil S4 taken from a large diameter borehole (RÉTHÁTI and UNGÁR, 1978a)

First of all we may establish that a considerable number of the samples can be assigned here (82 pieces = 16%). The histogram of Fig. 7.2 shows the development of the above-mentioned characteristics. Accordingly the acceptance of $w = 23\%$ as critical water content seems to be realistic indeed: only 7% of the S_r and the e -values exceeded the means of samples which were qualified as satisfactory ($\bar{S}_r = 0.920$ and $\bar{e} = 0.835$).

The histogram proves also that in the case of small diameter samples both the compaction during sampling and drying must be taken into account. The extraordinarily low degree of saturation ($S_r < 0.80$ for the majority of the samples from below the groundwater level) points also to drying, while the compaction is proved by the fact that

a) there are also samples whose degree of saturation is $S_r > 0.85$,

b) the linear shortening calculated from the mean of the void ratios of the samples qualified as satisfactory and unsatisfactory

$$\varepsilon = \frac{\bar{e} - \bar{e}'}{\bar{e}} 100 = \frac{0.835 - 0.730}{0.730} 100 = 14.4\%$$

is greater than the linear shrinkage of the clays of a similar plasticity ($\bar{I}_p = 30\%$).

For investigating the problem more deeply, we have determined the empirical relationship $\bar{e} = f(w)$ for small diameter core samples. With regard to the very great

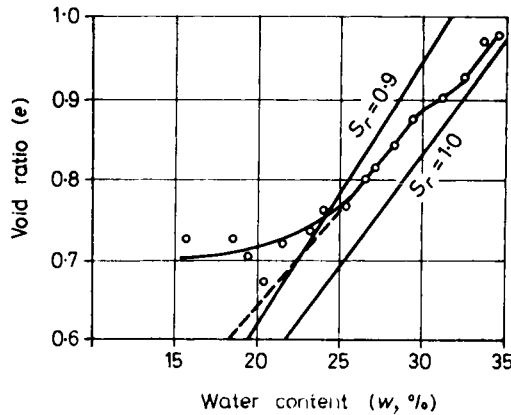


Fig. 7.3. The relationship $e = f(w)$ for core samples of the soil S4 taken from small diameter boreholes (RÉTHÁTI and UNGÁR, 1978a)

quantity of the available samples, the category means should lie on the straight line characterized by the equation

$$e = \frac{\rho_s}{S_r} w + A, \quad (7.7)$$

but the experience shows (Fig. 7.3) that the linearity stops at about $w = 25\%$, where the straight line described by the equation

$$e = 0.0241w + 0.158$$

goes over into an upwards concave curve. The degree of saturation increases with the void ratio; the relationship over the linear section is described by a straight line, whose form is

$$S_r = 0.247e + 0.733.$$

In the range of the low water contents S_r decreases rapidly (as demonstrated also by the straight lines belonging to $S_r = 1.0$ and 0.9), so here the decrease of e must have been caused by drying. The figure verifies also that the adoption of the “critical water content” was realistic.

On the basis of similar considerations the critical water content of the infusion loess variant found below the groundwater level (soil S3) is $w = 19\%$ and the distribution of S_r and e is as follows

	$w < 19\%$	$w \geq 19\%$
$\bar{S}_r$	0.731	0.923
$\bar{e}$	0.626	0.706

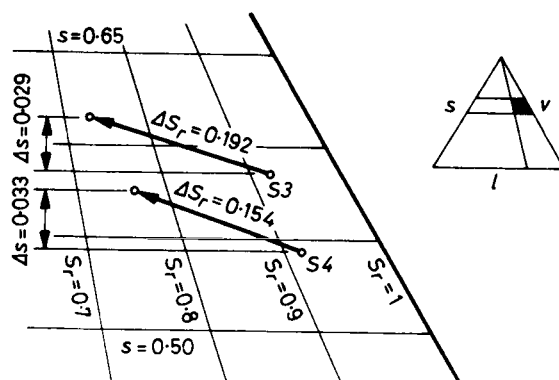


Fig. 7.4. Comparison of small diameter core samples which dried out and/or were compacted during the sampling with assumably reliable samples in the triangular-diagram of KÉZDI (RÉTHÁTI and UNGÁR, 1978a)

The linear shortening calculated with the above data is

$$\varepsilon = \frac{0.706 - 0.626}{0.626} 100 = 12.8\%,$$

which means that the *compaction* during the sampling is mainly involved, as here the mean plasticity index referring to the possibility of shrinkage amounts only to 11.5%.

The difference between the state changes of both soils may be studied by the triangular diagram of KÉZDI (Fig. 7.4), showing that the degree of saturation of the soil S4 of a higher plasticity displays a smaller and its void ratio a greater decrease than that of the soil S3.

It may be of interest to group the “dried” samples with $S_r < 0.85$ and/or $w < 23\%$ according to the *month of sampling*. As Table 7.10 shows, the highest frequency belongs

Table 7.10. Distribution by calendar months of small diameter core-samples of the soil S4 which dried and were compacted during the sampling (RÉTHÁTI and UNGÁR, 1978a)

Month	Number of samples	Samples of $S_r < 0.85$ and/or of $w < 23\%$	Proportion (%)
J	112	22	19.6
F	101	25	24.8
M	78	15	19.2
A	76	7	9.2
M	31	4	12.9
J	63	17	27.0
J	51	11	21.6
A	50	19	38.0
S	35	3	8.6
O	49	0	0.0
N	40	8	20.0
D	95	23	24.2

to the summer months (August and July), but the values of the months between November–March are not lagging by much. A possible explanation of this might be perhaps that the airtight closing of the samples in winter weather conditions (requiring the repeated melting of the paraffine, etc.) is often delayed and on the other hand, in cold weather the paraffine turns rigid, it breaks easily and is breaking off.

7.1.4. The re-evaluation of the data set

After the elimination of the outliers by statistical or by deterministic methods, the residual (truncated) data set must be re-evaluated.

Table 7.11 contains some statistical parameters of the unconfined compression strength of the soils S3 and S4, determined by samples taken from small diameter boreholes. The change is especially great in the case of the high plasticity soil, whose compression strength may increase even 2–3-fold under the effect of drying, with the consequence that – though there is no great difference between the $\bar{q}_u$ values— s , C_v , C_s and C_k decrease very considerably. (The significance of this is discussed in more detail in Chapter 8.)

Table 7.11. Statistical parameters of the unconfined compression strength of the soils S3 and S4 for the original data set and that truncated by the outliers

Soil	Examined data set	n	$\bar{q}_u$	s	C_v	C_s	C_k
S3	original	227	181.8	104.8	0.57	2.25	7.44
	truncated	190	168.4	88.2	0.52	2.44	11.84
S4	original	300	237.7	123.2	0.52	2.02	7.45
	truncated	257	215.6	85.7	0.40	1.00	2.29

7.2. Evaluation of the data using the Bayesian theory

The work of BAYES, published in 1763 (“An essay towards solving a problem in the doctrine of chances”) is utilized by numerous disciplines. FOLAYAN (1969) and TANG (1971) applied it first in soil mechanics and the subsequent studies (FOLAYAN *et al.*, 1970; SCHULTZE, 1975a; HARR, 1977; BOGÁRDI, 1977) contributed to the propagation of the method in geotechnics by some new ideas or examples.

The advantages of using the theorem of BAYES are

- the possibility of utilizing earlier data, whereby the number of the explorations and/or tests may be decreased;
- the better (more economic) estimation of statistical parameters;
- the determination of whether it is worthwhile to perform also (or only) indirect measurements,

- the decrease of the standard deviation (and thereby the factor of safety), which may have a great economic importance in certain cases;
- the combination of subjective judgment and objective information.

The Bayesian approximation shows also how far the probability of the successful technical analysis depends on the experiences of the engineers gained under similar circumstances (FOLAYAN *et al.*, 1970).

7.2.1. The theorem of BAYES

The original theorem of BAYES may be formulated as follows.

If the events $B_1, B_2 \dots B_n$ form a common system of events, $P(B_i) > 0$ and A is an event of an arbitrary positive probability, then

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^n P(A|B_i)P(B_i)} \quad (7.8)$$

where $|$ is the symbol of the conditional probability.

The conditional probability can be illustrated on the example of an experiment with a finite number of equiprobable outcomes (ELISHAKOFF 1983). Let n be the total number of the outcomes; $n(B)$ the number favourable to B ; $n(AB)$ the number favourable to both A and B . Then

$$P(B) = \frac{n(B)}{n}, \quad P(AB) = \frac{n(AB)}{n},$$

so that the conditional probability is

$$P(A|B) = \frac{n(AB)}{n(B)} = \frac{n(AB)/n}{n(B)/n} = \frac{P(AB)}{P(B)}. \quad (7.9)$$

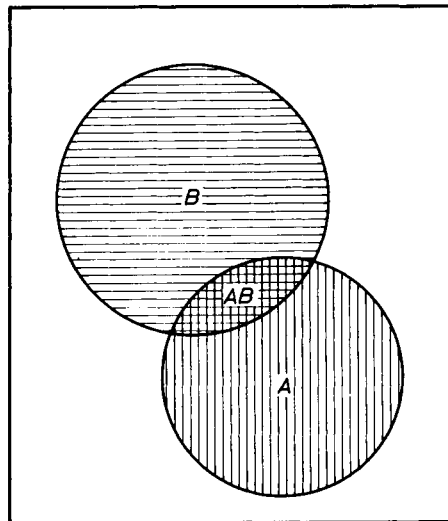


Fig. 7.5. Conditional probability $P(A|B)$

The concept may be illustrated graphically with the help of Fig. 7.5. It is easily admitted that if

$$P(A|B) = P(A) \quad (7.10)$$

then the events A and B are independent of each other.

The probabilities $P(B_i)$ are called *a priori* probabilities, as in using the theorem we start from given (earlier known) $P(B_i)$ probabilities, while the conditional $P(B_i|A)$ probabilities are called *a posteriori* probabilities. The concrete experimental information is the fulfilment of even A . Using this and relying on the probabilities $P(B_i)$ we determine the conditional probabilities $P(B_i|A)$. In practical cases we learn only after the performance of the experiment, which event A is involved, so the probabilities $P(B_i|A)$ can be calculated only then.

7.2.2. Determination of the “a posteriori” distribution

For dimensioning structures we must know the empirical distribution of the material characteristics. With this requirement in mind we may formulate the theorem of BAYES in the following way (TANG, 1971). If the *a priori* distribution of the soil physical parameter λ is $f_1(\lambda)$, then the *a posteriori* distribution $f_2(\lambda)$ may be determined by the following expression:

$$f_2(\lambda) = k L(\lambda) \cdot f_1(\lambda) \quad (7.11)$$

where $L(\lambda)$ is the so-called *likelihood function* (see Section 2.2.1) and k is a normalization constant. The relationship (7.11) is shown graphically in Fig. 7.6.

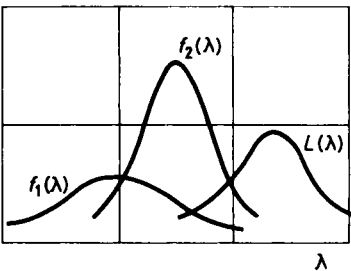


Fig. 7.6. Relationship between the a priori, a posteriori and likelihood functions

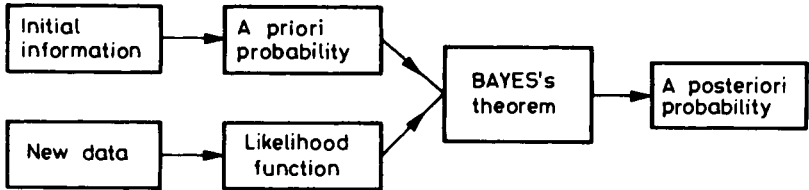


Fig. 7.7. The process of revising probabilities, given new data (ZELLNER, 1971)

The essence of the solution is that the a priori information is modified by a posteriori measurements (observations) using the theorem of BAYES. Figure 7.7 shows the course of the calculation after ZELLNER (1971).

Three characteristic cases may be distinguished according to the form and the position of the likelihood function (VIOLA, 1983). In the first case (Fig. 7.8a) the a priori distribution $f_1(\lambda)$ is concentrated around a value $\hat{x}$ of x , while the function $L(\lambda)$ is quite diffuse. In this case the a posteriori density $f_2(\lambda)$ is slightly different from the prior one. This means that the information contained in the observation may slightly modify the prior estimate. When the likelihood function $L(\lambda)$ and the a priori density have the same analytical form (Fig. 7.8, b), the a posteriori distribution takes the qualitative form shown in Fig. 7.8, b. If the distribution $f_1(\lambda)$ of the parameter x is at first diffuse and $L(\lambda)$ is concentrated in a small interval, the a posteriori density $f_2(\lambda)$ will conform to the likelihood function (Fig. 7.8, c). The influence of the a priori distribution on $f_2(\lambda)$ is practically negligible, which means that the posterior estimate is determined only by subsequent information.

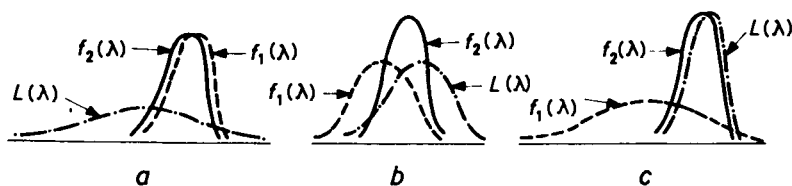


Fig. 7.8. Characteristics of a priori and a posteriori distributions (VIOLA, 1983, after BENJAMIN and CORNELL, 1974)

The concrete mode of the solution depends on our assumption concerning the *type of the distribution* of the physical characteristics. If the latter follow a *t-distribution*, then the *a posteriori* distribution may be written up in the following form (FOLAYAN *et al.*, 1970):

$$f''(\mu|m, k, v) = \frac{v^{0.5}}{\beta(0.5; 0.5v)} [v + k(\mu - m)^2]^{-0.5(v+1)} \sqrt{k}, \quad (7.12)$$

where $-\infty < \mu < \infty$, $k > 0$ and $v > 0$. The equation gives the distribution of the real (but unknown) mean μ , when m , k and v are given. From among the parameters m is the symbol of the empirical mean:

$$m = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}, \quad (7.13)$$

where n_1 and n_2 are the numbers of elements in both data series, m_1 and m_2 are the two empirical means. The factor k equals the inverse of the variance:

$$k = \frac{1}{s^2} = \frac{v_1 + \delta(n_1) + v_2 + \delta(n_2) - \delta(n)}{v_1 s_1^2 + n_1 m_1^2 + v_2 s_2^2 + n_2 m_2^2 - n m^2}, \quad (7.14)$$

where $n = n_1 + n_2$ and v is the degree of freedom:

$$v = v_1 + \delta(n_1) + v_2 + \delta(n_2) - \delta(n). \quad (7.15)$$

In the latter expression $\delta(n) = 0$, if $n = 0$ and $\delta(n) = 1$, if $n > 0$.

β in expression (7.12) is the symbol of the beta distribution, as interpreted in Section 2.1.5.

For the *normal distribution* several variants have been elaborated. TANG (1971) discusses the following sub-cases:

- A — the new data are obtained by direct measurements, resp.
- B — by (one or more) indirect measurements, or
- C — by direct and indirect measurements,
- D — the new data were determined by two different indirect measurements,
- E — as D, with the difference that a data series obtained by direct measurements is also available.

Be x the random variable to be determined and λ a parameter of the distribution, e.g., the mean related to the tested layer. If we have performed n independent measurements ($x_1, x_2, \dots, x_n$), then the likelihood function is:

$$L(\lambda) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}s} \exp \left[-\frac{1}{2} \left(\frac{x_i - \lambda}{s} \right)^2 \right], \quad (7.16)$$

where Π expresses that the n values following it must be multiplied. It may be proved that Eq. (7.16) corresponds to a normal distribution whose expected value is $\bar{x}$ and its standard deviation is $s/\sqrt{n}$.

If the *a priori* distribution is $N(m_1, s_1)$, the requested a posteriori distribution will be as follows:

$$N \left(\frac{m_1 \frac{s^2}{n} + \bar{x} s_1^2}{\frac{s^2}{n} + s_1^2}, \sqrt{\frac{\frac{s^2}{n} s_1^2}{\frac{s^2}{n} + s_1^2}} \right). \quad (7.17)$$

Let us assume that on surveying a given area only $n=8$ samples are available for determining the unconfined compression strength of the soil S4, whose test for q_u gave the following $x_1, x_2, \dots, x_n$ values:

274–265–209–215–307–255–228–217.

According to Table 1.12 the expected value of q_u is

$$m_1 = 219 \text{ kN/m}^2$$

and its standard deviation is

$$s_1 = 117 \text{ kN/m}^2.$$

Based on a great number of earlier tests for the eight samples $\bar{q}_u = \bar{x} = 246.25 \text{ kN/m}^2$, $s = 34.73 \text{ kN/m}^2$, so according to expression (7.17) (assuming normal distribution for the sake of the example)

$$m_2 = \frac{219 \frac{34.73^2}{8} + 246.25 \cdot 117^2}{\frac{34.73^2}{8} + 117^2} = 246.0 \text{ kN/m}^2,$$

$$s_2 = \sqrt{\frac{\frac{34.73^2}{8} 117^2}{\frac{34.73^2}{8} + 117^2}} = 12.21 \text{ kN/m}^2.$$

In the second case (B) we determine the variable X from an empirical relationship (or a calibration relationship):

$$X|y = a + by. \quad (7.18)$$

(In a given case y may be the number of blows required for the penetration and X the angle of friction of the soil.)

Let us assume in the following that the residual standard deviation s_f defined by Eq. (3.11) does not change with y .

If we perform a *single indirect measurement* (y_2) then the effective distribution of x is

$$N(m_{x|y_2}, s_f), \quad (7.19)$$

where $m_{x|y_2}$ is itself of a normal distribution, with the variance

$$s_0^2 = \frac{s_f^2}{n} \left(1 + \frac{(y_2 - \bar{y})^2}{s_y^2} \right), \quad (7.20)$$

where $\bar{y}$ is the mean of the y -values used for the regression and s_y is its standard deviation. The density function of the value X_2 determined by the measurement y_2 is

$$f(x_2) = \int_{-\infty}^{+\infty} N(m_{x|y}, s_f) \cdot N(a + by_2, s_0) dx, \quad (7.21)$$

which corresponds to a normal distribution of the following form:

$$N(a + by_2, \sqrt{s_f^2 + s_0^2}). \quad (7.22)$$

It may be proved that in this case the *a posteriori* distribution can be written as a normal distribution of the following form:

$$N\left(\frac{m_1 s_f^2 + (a + by_2) s_1^2}{s_f^2 + s_1^2}, \sqrt{\frac{s_f^2 \cdot s_1^2}{s_f^2 + s_1^2}}\right), \quad (7.23)$$

where

$$s_f^2 = s^2 + s_0^2, \quad (7.24)$$

m_1 and s_1 are parameters of the *a priori* distribution (the adoption of s is discussed in sections 7.2.3 and 10.6).

If we perform n_2 indirect measurements, then the parameters of the *a posteriori* distribution will be as follows:

$$N\left(\frac{m_1 s_{f''}^2 + m' s_1^2}{s_{f''}^2 + s_1^2}, \sqrt{\frac{s_{f''}^2 \cdot s_1^2}{s_{f''}^2 + s_1^2}}\right), \quad (7.25)$$

where

$$m' = \frac{\sum_1^{m_2} (a + b y_2)}{s_{f_i} \sum_1^{m_2} \frac{1}{s_{f_i}^2}}, \quad (7.26)$$

$$s_{f''} = \frac{1}{\sqrt{\sum_1^{m_2} (1/s_{f_i}^2)}} \quad (7.27)$$

and

$$s_{f_i}^2 = s^2 + s_f^2 + s_0^2. \quad (7.28)$$

In the third case (C), where n_2 indirect measurements and n_3 direct measurements are performed, the expression (7.25) may be regarded as an *a priori* distribution. The *a posteriori* distribution is obtained by substituting the parameters appearing in (7.25) into (7.17): the mean for m_1 , the standard deviation for s_1 , while $\bar{x}$ is identical with the mean of the n_3 measurements.

In the fourth case (D), where n_2 type I and n_3 type II indirect measurements are performed, the *a posteriori* distribution can be given again similarly to expression (7.25), with the following substitutions: n_1 and s_1 equal the mean and standard deviation, formed as in the case C and m' calculated according to (7.26), with the substitutions $m_2 = n_3$ and $y_2 = y_3$.

The fifth case (E), where n_2 type I and n_3 type II indirect measurements and n_4 direct measurements are performed, may be reduced to the case D (with the *a posteriori* distribution of case D adopted as *a priori* distribution).

On the basis of the formulae it may be verified that with the increasing quantity of information the standard deviation of the mean keeps decreasing.

7.2.3. Examples using the Bayesian theory

Example I (TANG, 1971). COZZOLINO (1961) proposed to use the following regression for determining the compression index in the Lowlands district of Santos in Brazil:

$$C_c = 0.0186(w_L - 30) \pm 0.41 \quad (7.29)$$

and

$$C_c = 1.21 + 1.055(e - 1.87) \pm 0.34. \quad (7.30)$$

Let us assume that $w_L = 75-80-60-72-65$ and $e = 1.64-1.80-1.72-1.50$ were measured in the given district. We know from earlier measurements that the standard deviation of C_c is about $s = 0.2$ in this area.

Our statistical analysis corresponds to the case *D*. As we have no *a priori* information on the means m_1 , we shall determine the *a priori* distribution by expression (7.29). The expected value is

$$\bar{x} = \frac{1}{5} \sum_1^5 0.0186 (w_L - 30) = 0.752,$$

and the standard deviation (with the assumption $s_0 = 0$, considering the great number of the samples used for the regression analysis) is

$$s_1 = \sqrt{0.2^2 + 0.41^2} = 0.4562$$

so the *a priori* distribution is

$$N\left(0.752; \frac{0.4562}{\sqrt{5}} = 0.204\right).$$

The *a posteriori* distribution is obtained from expression (7.25) with the following substitutions: $m_1 = 0.752$; $s_1 = 0.204$ and $m' = 0.9375$ (the latter is the mean of the compression indices calculated from the regression related to e). The value $s_{f''}$ appearing in expression (7.25) is

$$s_{f''} = \frac{1}{\sqrt{\frac{5}{0.2^2 + 0.34^2}}} = 0.1764$$

according to expressions (7.27) and (7.28). With the substitutions carried out, $N(0.859; 0.133)$ is obtained for the *a posteriori* distribution.

TANG (1971) examined the situation where w_L and e are determined by pairs from the same samples. For this case the tervariate correlation $C_d(w_L, e)$ must be used. The statistical calculations correspond now to case *B* and the result is: $N(0.896; 0.143)$. This means that though the sampling was less expensive, the reliability of C_c decreased as well.

Example 2 (TANG, 1971). Let us assume that one of the following alternatives must be selected:

- I* — we perform only indirect measurements,
- II* — we determine the calibration curve and additionally perform further indirect measurements.

Be c_1 and c_2 the cost of a direct, resp. an indirect measurement and c the sum, allowed for the measurements. The variance of the mean for the first alternative is

$$\sigma_I^2 = \frac{\sigma^2}{n_1} = \frac{c_1 \sigma^2}{c}, \quad (7.31)$$

where σ^2 is the variance of the variable.

In the case of the second alternative only the n_2 direct measurements contribute (as a part of the full information) to the determination of the variance of the mean, as all the n_2 measurement pairs of calibration relate to an identical sample, so the variance of the mean according to case C is

$$\sigma_{II}^2 = \frac{\frac{\sigma^2}{n_2} \cdot \frac{\sigma^2 + s_f^2}{n_3}}{\frac{\sigma^2}{n_2} + \frac{\sigma^2 + s_f^2}{n_3}}, \quad (7.32)$$

where the quantity n_3 of the further indirect measurements may be calculated as

$$n_3 = \frac{c - n_2(c_1 + c_2)}{c_2}. \quad (7.33)$$

If the principle of the selection from the alternatives is the reduction of the variances to the minimum, we adopt the second alternative if $\sigma_{II} < \sigma_I$. By combining expressions (7.31)–(7.33) we obtain for the requested criterion

$$\left(\frac{s_f}{\sigma}\right)^2 + \frac{c}{c - n_2 c_1} < \frac{c_1}{c_2}. \quad (7.34)$$

This means that it is advised to select the second alternative, if

- the calibration error s_f is small relative to the variability of the physical characteristic,
- the available finances are limited,
- the cost of the direct measurement is high, as compared to that of the indirect measurement.

Additional examples for applying the theorem of BAYES in geotechnics are found in Sections 9.3, 9.4 and 10.5.

7.3. Identification of sample groups and layers

Let us assume that during the soil investigation we determine p physical characteristics by testing all the n samples. In the knowledge of the results we form—on the basis of some index-property— k groups, each containing q elements. This operation means statistically that the limit values of the groups must be chosen in a way that the original $n \cdot p$ matrix should be replaced by the classifying matrix $k \cdot q$, as best as possible (LUMB, 1975b).

The task may be solved generally by a *multivariable analysis method* (by canonic variables, factor analysis, cluster analysis), based essentially on calculating pairwise correlation coefficients for the vectors formed from the data and assigning further vectors to the potential groups, starting from the maximum values of the latter. This process is continued until the average correlation coefficient attains a critical lower value. If every r -value is a priori lower than the critical value, we cannot form groups.

The situation is similar also if the physical characteristics are of a unimodal continuous distribution (LUMB, 1975b).

The cluster analysis may be used also in other fields of engineering practice, e.g., for determining the boundary lines of districts, within which the course of the groundwater is similar (BOGÁRDI *et al.*, 1978).

In soil mechanics the formation of sample groups means essentially the division of the explored soil block to *layers*. In this connection two basic questions arise:

- where should the layer boundaries be marked out within a borehole, and
- when can two or more layers be regarded as identical (or more exactly: what is the condition of neglecting smaller or greater deviations between them).

The task cannot always be solved unequivocally, mainly because the measure of the “affinity” of the layers may be different even *by physical characteristics*. It may be imagined, e.g., that the mean shear strength of silt found in two adjacent boreholes is closely identical, while the permeability of the two layers is very different.

The layer boundaries are established and the layers are identified in the soil-mechanical practice usually on the basis of *classifying characteristics*, but this attitude may lead to severe errors, as discussed in the preceding paragraph. This operation must be performed correctly with the *physical characteristics required for planning*. The same principle must be applied also for the calculation of the homogeneity, or some other statistical parameter of a layer (see in Section 1.3).

The error of the identification may imply severe consequences. For example, if not the proper PROCTOR-curve is assigned to a sample taken for controlling the density of earthworks, an error implying heavy costs may be incurred in the calculation of T_{ry} .

The tasks discussed in this chapter can hardly be solved by deterministic methods. As LUMB (1975b) stated, the processes used so far for the classification (identification) are similar to the principle “take to pieces and name the parts” of a drill sergeant.

7.3.1. Assessing layer boundaries

On testing samples taken from a borehole we can always find two or more subsequent samples over the depth with a certain similarity, i.e. which belong according to our judgment to the same layer. The layer boundaries are assessed in the present practice on the basis of classifying and qualifying characteristics (see Chapter 5), taking also the colour and the geological age of the soil into account.

The most important *statistical parameters* for the solution of a task, namely, the mean and the standard deviation can be determined for each layer and for each physical characteristic, so we must start always from both these parameters (see Chapter 8). The coefficient of variation C_v , defined in Section 1.3, by expression (1.13), i.e. the quotient of the two statistical parameters varies by physical characteristics, so *the upper limit, over which the layer must be regarded as heterogeneous*, can be determined with a sufficient accuracy. This is the basic idea of the standard

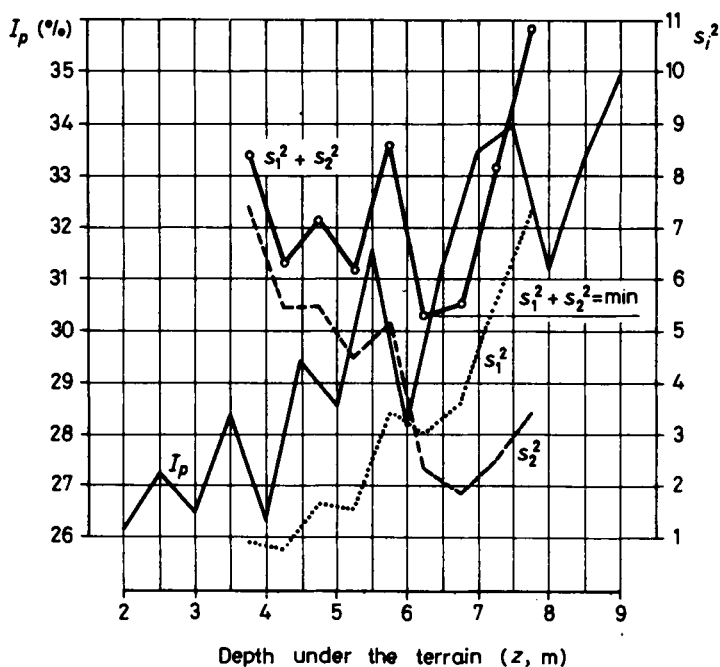


Fig. 7.9. Determination of the layer boundary on the basis of the sum of the variances

specification GOST-20522 (1975), where $C_v = 0.15$ is given for e and w and $C_v = 0.30$ for E_s , τ and q_u as upper limits for judging the homogeneity, i.e. for assessing the layer boundaries. A similar principle was adopted also by RECORDON and DEPEND in their papers read at the international conference in Tokyo (BIAREZ and FAVRE, 1977).

The task may be solved independently from the critical value of C_v , if we start from the variances of the physical characteristics. Figure 7.9 shows the series of the plasticity index values of samples taken at 0.5 m spacings. The mean of the complete series is $\bar{I}_p = 30.0\%$, its standard deviation is $s = 2.98$. Though C_v is relatively not high ($= 0.099$), it seems that the soil block represented by the last 5–7 samples may be treated as a separate layer. For this purpose let us split the data set into two parts ($n_1 + n_2 = n = 15$), then form the standard deviations (s_1 and s_2) of the obtained subsets and determine the sum of their squares (e.g. for $n_1 = 4$ and $n_2 = 11$ we have $s_1^2 + s_2^2 = 0.967 + 7.431 = 8.398$). By varying n_1 (and $n_2 = n - n_1$) we obtain a polygon, which has a minimum. The extremal value ($= 5.326$) belongs to $n_1 = 9$, resp. $n_2 = 6$, so the layer boundary is adopted at $z = 6.25$ m. The characteristics determined in this way are

$$\begin{aligned} \bar{I}_{p1} &= 28.6\%; & \bar{I}_{p2} &= 33.0\%; \\ s_1 &= 1.74\%; & s_2 &= 1.51\% \\ C_{v1} &= 0.061; & C_{v2} &= 0.046. \end{aligned}$$

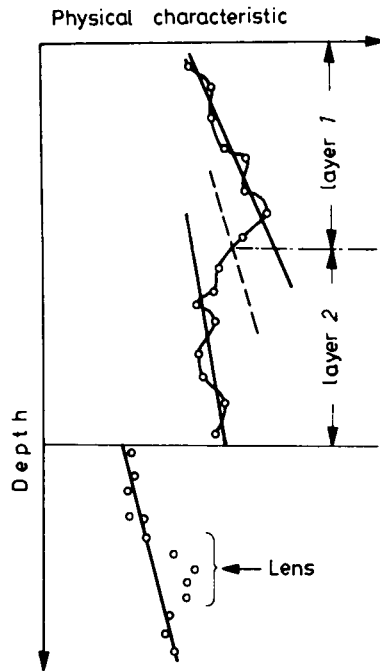


Fig. 7.10. Determination of the layer boundary and lenses in the case of physical characteristics varying with depth

If the tested physical characteristics *vary significantly with the depth* (see Section 4.2.2), we may proceed according to Fig. 7.10. We adopt the boundary between the layers 1 and 2, where the polygon representing the measurement results passes over the bisecting straight line of the regression straight lines. The *lens* appearing in the third layer may be regarded as an independent layer, if its deviation (d) from the regression straight line exceeds the critical value αs_f , determined by us.

The methods elaborated for determining the extremal values may also be used for solving the problem (see Section 7.1).

The question may be raised rightfully: why do not we increase the number of the layers beyond any limits. The answer is unequivocal:

- the multilayer systems *aggravate the geotechnical calculations* extraordinarily (e.g. the investigation of the stability of slopes, or the bearing capacity of foundations);
- with decreasing the number of samples the *reliability of the mean* decreases rapidly (see Section 7.4).

7.3.2. Comparison of two or more layers

In Section 6.2 we have dealt with the comparison of samples taken by different sampling technologies and tests performed by different apparatuses or by different persons. The tasks are not different from that referred to in the title of this section, meaning that also here we must resort to *statistical tests*, described in Section 6.2.1, so we must analyze here only their applicability and the examples quoted in literature.

FREDLUND and DAHLMAN (1971) determined the statistical parameters of the cohesive soils of Edmonton, Canada, first in four part areas of the town, then—after finding no significant differences between the four data sets—they treated the whole material as a single statistical sample. LADD *et al.* (1971) came to a similar conclusion with their groups of borings lowered in the soft clays of Bangkok. These authors used *t*-tests recommended also by the Soviet standard GOST-20522 (1975).

On analyzing the data of Table 7.12 MORSE (1971) came to the conclusion that the glacial silt (C) must be divided into further layers, because the standard deviation and the coefficient of variation of q_u were strongly outlying values, as compared to those of the other soils.

PAÁL (1974b) compared with the help of the empirical distribution functions the three typical Buda Clays, using the KOLMOGOROV-SMIRNOV-process. SZABÓ (1976) compared by the χ^2 -test the distribution functions of the liquid limits of various clays. On testing 108 samples from the boreholes of Avas and 237 samples from Mály he obtained

$$\chi^2 = 7.776.$$

As the critical value belonging to the probability level $p = 5\%$ is $\chi_p^2 = 12.59$, he determined that the greenish-grey clay of Avas is also of Pannonian age, as the clay of Mály (whose Pannonian age was proven). (More precisely: this assumption was not contradicted by the two empirical distributions.)

In the following SZABÓ (1976) compared the cohesive soils covering the gravel terrace of the Szinva valley and the Quaternary clays covering the mezozoic formations. He obtained a significant agreement only when he completed the latter with the test results of the Helvetian clays, so it is probable that the covering of the gravel is a mixture of two clays of different ages.

PAÁL (1974b) remarks correctly that the significant agreement of the distribution functions does not mean an identical geological origin, resp. age.

Table 7.12. Statistical parameters of the unconfined compression strength of glacial clays in the state of Illinois (USA) (MORSE, 1971)

	A	B	C	D	A + B + C	A - D
n	22	21	18	25	61	86
$\bar{q}_u$ (kN/m ²)	383	286	447	215	369	324
s (kN/m ²)	56.7	42.0	138.0	42.6	107	117
C_v	0.15	0.15	0.31	0.20	0.29	0.36

The comparative analysis of the soils *S4* and *S5* of Szeged brought interesting results (RÉTHÁTI and UNGÁR, 1978c). It could be established still by *Table 1.12* that there were no great differences between the physical characteristics of these layers, lying on each other. This could be verified easily also by statistical tests. The only exception was the unconfined compression strength of the large diameter samples, whose statistical parameters were as follows:

$$\text{for } S4 (\equiv 1) n_1 = 108, \quad \bar{q}_u = 166, \quad s = 74;$$

$$\text{for } S5 (\equiv 2) n_2 = 85, \quad \bar{q}_u = 184, \quad s = 89,$$

so according to expression (6.11)

$$F = \frac{s_1^2}{s_2^2} = \frac{89^2}{74^2} = 1.446,$$

i.e. though there is a significant difference between the standard deviations according to *Table F8* ($f_1 = 84$, $f_2 = 107$, $F_p = 1.42$), yet F and F_p are close to each other. According to expression (6.10)

$$t' = \sqrt{\frac{108 \cdot 85 \cdot 191}{193}} \frac{184 - 166}{\sqrt{108 \cdot 74^2 + 85 \cdot 89^2}} = 1.526,$$

is lower than the STUDENT-value of $108 + 85 + 2 = 191$ degrees of freedom (according to *Table F4* this is 1.98 on the significance level of $p = 0.05$). This means that both $\bar{q}_u$ values may be regarded as practically identical. It follows from all the above that—in contrary to the present practice—the *foundation plane* must be in the deeper *S5* soil only if this is structurally necessary, or if we wish to increase the overburden stress, acting on the foundation plane. This is an *important economic consequence of the statistical tests*.

7.4. Determination of the confidence intervals

If the estimation a^* of the parameter a is unbiased (see Section 1.1) and the difference $a - a^*$ contains no unknown parameter, then in the case of an arbitrarily low ε -value selected by us we can find a d -value so that

$$P(a^* - d \leq a \leq a^* + d) = 1 - \alpha. \quad (7.35)$$

The interval $(a^* - d, a^* + d)$ is called the confidence interval, $100(1 - \alpha)\%$ is the confidence level, α is the significance level, the initial and the end points of the interval are called confidence limits (PRÉKOPA, 1962).

The expression (7.35) must be interpreted as follows. If we take n samples many times, independently of each other and determine each time the confidence interval of the parameter a , then in about $100(1 - \alpha)\%$ of the cases a will be within the confidence interval. This may be formulated also in the following way (RÉNYI, 1954; HARR, 1977):

the parameter a falls between the confidence limits calculated from n samples with a probability of $100(1 - \alpha)$.

On solving practical tasks first of all the confidence interval of the expected value and more rarely that of the standard deviation must be known.

7.4.1. The confidence interval of the expected value

If we have a *population of the distribution* $N(m, \sigma)$ and σ is *known*, we may start from expression (4.16) giving $\sigma/\sqrt{n}$ as the standard deviation of the expected value and the random variable

$$\frac{\sqrt{n}}{\sigma}(\bar{x} - m)$$

is of $N(0, 1)$ distribution. So the confidence interval may be given by the following expression:

$$\bar{x} - \lambda \frac{\sigma}{\sqrt{n}} \leq m \leq \bar{x} + \lambda \frac{\sigma}{\sqrt{n}}, \quad (7.36)$$

where $\lambda (=u_\alpha)$ is the argument of the normal distribution, which may be calculated from expression (2.18). If, e.g., $1 - \alpha = \Phi(x) = 1 - 0.05 = 0.95$,

$$\Phi(\lambda) = \frac{1 + 0.95}{2} = 0.975,$$

to which $\lambda = 1.96$ belongs according to *Table F2*. In this case we may state on the confidence level of 95% that the expected value m of the population falls between the limits

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}.$$

This is the case of the *bilateral test*, as against the *unilateral test*, for finding the limit value which is not surpassed (nor underpassed) by m . In this case we must calculate instead of α with 2α , so in the former example $\Phi(x) = 1 - 2\alpha = 0.90$, $\Phi(\lambda) = 0.95$, to which $\lambda = 1.64$ belongs (the confidence level is 95% also here!).

The λ -values belonging to the practically used α -values are as follows:

$\alpha =$	0.10	0.05	0.01
For bilateral tests	1.64	1.96	2.58
For unilateral tests	1.28	1.64	2.33

If the *standard deviation* σ of the population of $N(m, \sigma)$ distribution is *unknown*, then we must calculate with the STUDENT-distribution instead of the normal one. The

confidence interval in this case is

$$\bar{x} - t_{\alpha} \frac{s}{\sqrt{n}} \leq m \leq \bar{x} + t_{\alpha} \frac{s}{\sqrt{n}}, \quad (7.37)$$

where s is the unbiased (corrected) standard deviation of the sample, consisting of n elements. The value t_{α} versus the degree of freedom ($f = n - 1$) may be taken from *Table F4*, again with consideration to the unilateral, or the bilateral test.

On the basis of expressions (7.36) and (7.37) we may state that

- the reliability of the mean may be enhanced by increasing n ,
- if the standard deviation is unknown, the width of the confidence interval increases (because $t_{\alpha} > \lambda$), whereby the reliability of the mean is decreased.

According to the theorem of the central limit distribution the means may be regarded to be of a normal distribution, if the number of the elements is sufficiently great, so expression (7.36) may be used instead of (7.37), but this problem is of no great practical importance (HARR, 1977), because with the increase of n the STUDENT-distribution is converging anyway to the normal distribution.

We have applied the basic principles discussed so far also before, so

- we have determined the required number of the samples on the basis of expression (7.37) (see Section 4.4);
- the majority of the statistical tests (see Section 6.2.1) is based also on the determination of the confidence intervals, e.g., expression (6.10) gives a confidence interval for the deviation of the expected values of two populations of normal distribution (with an equal, but unknown standard deviation), resp. it calculates t' -values from it.

Equation (4.3) gave a solution for the special case of calculating the confidence limits of the autocorrelation coefficients r_k .

If the population is of an *unknown distribution*, the reliability of the expected value may be estimated on the basis of CHEBISHEV's inequality, according to which the probability P (%) that the expected value m of the population falls between the limits

$$\bar{x} \pm h \frac{s}{\sqrt{n}}$$

is

$$P \leq 1 - \frac{1}{h^2}. \quad (7.38)$$

According to CAMP and MEIDEL h^2 on the right side of the inequality may be substituted by $2.25h^2$, if we know for sure that the population is of an *unimodal distribution* and C_s is relatively low. Another refinement of CHEBISHEV's inequality is connected with the name of BERNSTEIN (RÉNYI, 1954).

Let us assume that from a field test consisting of $n=18$ measurements on the site $\bar{E}_s=10.2 \text{ MN/m}^2$ was obtained for the mean of the modulus of compression and $s=2.9 \text{ MN/m}^2$ for the standard deviation. According to expression (7.36) the confidence limits on the significance level $\alpha=0.05$ are

$$10.2 \pm 1.96 \frac{2.9}{\sqrt{18}} = 10.2 \pm 1.34$$

and

$$10.2 \pm 1.64 \frac{2.9}{\sqrt{18}} = 10.2 \pm 1.12,$$

depending on whether a bilateral, or a unilateral test is involved. On finding the value t_α of STUDENT in Table F4 (the number of the degrees of freedom is 17) the confidence limits are according to expression (7.37)

$$10.2 \pm 2.11 \frac{2.9}{\sqrt{18}} = 10.2 \pm 1.44$$

and

$$10.2 \pm 1.74 \frac{2.9}{\sqrt{18}} = 10.2 \pm 1.19.$$

Substituting $P=0.05$ into expression (7.38) we obtain $h=4.472$ and for the unilateral test ($P=0.10$) $h=3.162$. So the confidence interval calculated from CHEBISHEV's inequality is

$$10.2 \pm 3.06, \quad \text{respectively} \quad 10.2 \pm 2.16,$$

which is considerably higher than the value obtained in the knowledge of the distribution of the population.

7.4.2. The confidence interval of the standard deviation

If a sample of n elements is taken from a population $N(m, \sigma)$ distribution, the random variable ns^2/σ^2 will follow a χ^2 distribution of $n-1$ degrees of freedom. As this distribution is asymmetric, it is advised to maintain the following limit conditions:

$$P\left(\frac{ns^2}{\sigma^2} > \chi_{1-\alpha/2}^2\right) = 1 - \frac{\alpha}{2}$$

and

$$P\left(\frac{ns^2}{\sigma^2} > \chi_{\alpha/2}^2\right) = \frac{\alpha}{2}.$$

Accordingly, the confidence limits will be:

$$\frac{s\sqrt{n}}{\chi_{\alpha/2}} \leq \sigma \leq \frac{s\sqrt{n}}{\chi_{1-\alpha/2}}. \quad (7.39)$$

Let us determine the confidence limits of the standard deviation appearing in the numerical example of Section 7.4.1, with a significance level of $\alpha = 0.05$. As $n = 18$, the number of the degrees of freedom is 17, so we obtain from Table F3 by interpolation:

$$\chi^2_{2.5} = 30.427 \quad \text{and} \quad \chi^2_{97.5} = 7.491.$$

According to expression (7.39) the lower confidence limit of the standard deviation ($s = 2.9 \text{ MN/m}^2$) is

$$\frac{2.9\sqrt{18}}{\sqrt{30.427}} = 2.23$$

and the upper confidence limit is

$$\frac{2.9\sqrt{18}}{\sqrt{7.491}} = 4.50.$$

7.5. The comprehensive statistical evaluation of the surveyed area

It is advised to summarize the results of the soil investigation made with a statistical view in the expert opinion and/or the attached drawings for the planning based on similar principles in the following way.

The mean, the standard deviation and the range of every soil physical characteristic must be reported and if sufficient data are available, also the coefficients of skewness and kurtosis, resp. the type (the parameters) of the empirical distribution, the test methods (statistical, deterministic) used for determining the outliers (extremal values) and the results of the same must be given.

It is advised to supply a statistical model bore-profile for the characteristic layers of the area (Fig. 7.11), indicating the standard deviations and also the confidence intervals.

A convenient way of presenting the statistical properties graphically is in the form of a triangle, such as shown in Fig. 7.12, a. The 95 percent confidence limits are plotted to form the base of the triangle. The mode becomes the x-coordinate for the apex. The y-coordinate of the apex is established by setting zero kurtosis to some fixed value and plotting a plus or minus kurtosis to some predetermined scale. The median is plotted on the base of the triangle and the slope of the line passing through the median and the mode gives an indication of the skewness. The mean is plotted as a dashed vertical line. This allows a rapid examination of the statistical parameters and the comparison of the soil types, or their changes with depth.

Figure 7.12, b shows a variant of the preceding solution, indicating instead of the confidence limits $\bar{x} \pm 2s$ values and the kurtosis by a vertical line (the length of the line is zero, if $C_k = -2$ and it agrees with the height of the triangle, if $C_k = 0$).

It is advised to qualify the *cohesive soils* with the help of e , I_p and q_u , I_p point pairs by statistical methods (see Figs 5.8, 5.11 and 5.12), giving the distribution of I_p for every

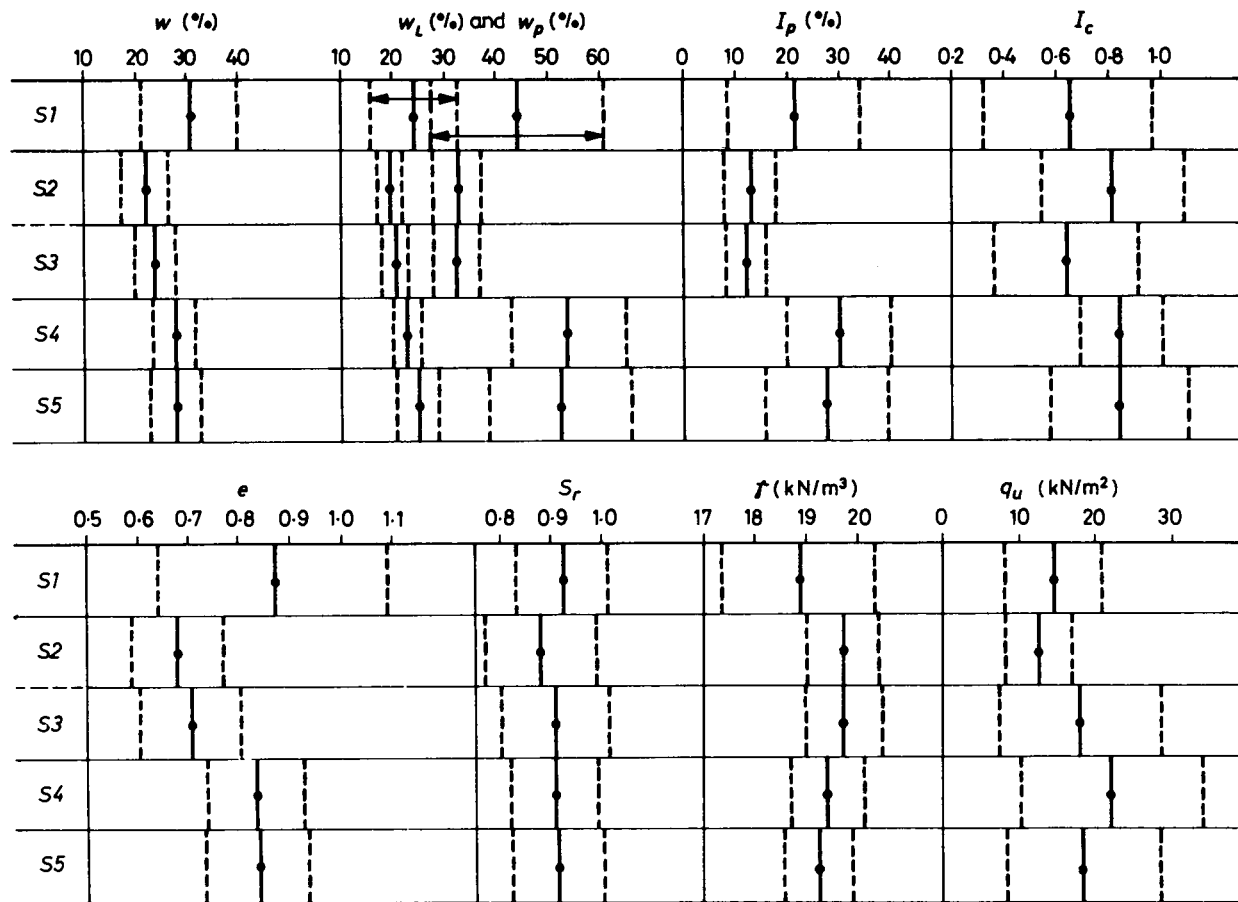


Fig. 7.11. Statistical model bore profile for Szeged (RÉTHÁTI and UNGÁR, 1978c)

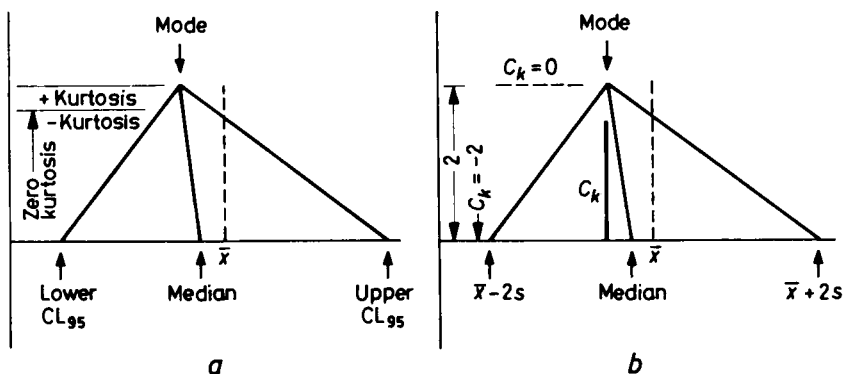


Fig. 7.12. Diagrammatic representation of geotechnical statistical properties; *a* — after FREDLUND and DAHLMAN (1971), *b* — a modification of this

layer (Fig. 1.10) and also its standard deviation in the system of coordinates of CASAGRANDE (Fig. 5.6). The statistical evaluation of *granular soils* may be given on the basis of grain size distribution curves and their characteristics.

The results of the *regression analyses* made for the laboratory and the field tests and possibly the correlation matrixes of the physical characteristics must also be reported.

The *trends* describing the horizontal and vertical variations of the physical characteristics and the statistical data on the territorial variability (see Section 4.2) may be very useful for the designer.

The possibilities are not yet exhausted by far. As a new tendency is involved the methodics of the preparatory planning operations reflecting numerous ideas and experiences of the experts will develop only in the decades to come.

8. Basic principles of the planning based on the theory of probability

A new attitude started to develop in the 1940s with the observation of engineering objects, the analysis of damages and the enhanced efforts towards economic rentability. The recognition that the loads and effects, as well as the strength and other characteristics of the materials are random variables affected basically the deterministic dimensioning process, which developed during centuries. This tendency was promoted also by the fact that mathematical-statistical methods were used rather extensively for the quality control of the materials (and products) by that time.

In this period new, or formerly rarely mentioned conceptions appeared, as the “technical-economic optimum”, the “economic risk”, the “degree of exploitation of the structure”, the “time factor” (the planned life span, the ageing of the materials, the variation of safety in time), and ever more people professed that the solution of these problems was impossible without the theses and methods of the theory of probability. With the perfection of the computation models and the material test methods, as well as with the introduction of new materials and technologies, the *factor of safety* came into the foreground, as a planning “parameter” which is a common numerical characteristic of the technical and economical demands.

The new attitude started to gain ground first in the field of the *structural design* (FREUDENTHAL, 1947, 1954, 1961; BROWN, 1960; JOHNSON, 1953) and only 8–10 years later in geotechnics (DAVIS, 1953; LÉVI, 1958; LAZARD, 1961; LANGEJAN, 1965).

This chapter of the book discusses the most important basic principles of the design. The subsequent chapters deal with the methods serving for the solution of various geotechnical tasks, based on these principles.

8.1. Uncertainties in the solution of geotechnical tasks

According to the wording of FREUDENTHAL (1947) the factor of safety is rather a measure of *incertitude* and less than of *ignorance*, intending to express that even the rapid scientific and technical progress attained by leaps and bounds could not and will never permit the *exact* dimensioning of the structures.

The main sources of the uncertainties appearing during the design of engineering objects are as follows:

- the more or less approximative nature of the *calculation model*, its connection to conditions which cannot be satisfied (in a desirable measure);
- the uncertainties in the calculation of the *loads and effects* (unforeseen effects, elemental phenomena, etc.);
- the deviation of the actual from the assumed *characteristics of the materials* (faulty production, damages during transport, etc.);
- errors and omissions incurred during the *production* and the *operation*, possible deviations from the plan.

In the field of geotechnics most difficulties are caused—in addition to the above—by the necessity of *learning to know* the properties of the soil and—apart from a few exceptions—the *impossibility of prescribing them* (as e.g. in the case of steel). In addition following sources of error must be taken into account.

1. During sampling the original stress state of the soil and a part of its physical characteristics varies.
2. Due to the relatively small number of borings
 - the number of the available *samples* is insufficient with regard to the heterogeneity of the soil;
 - the unfavourable *lenses* and discontinuities (preformed sliding surfaces, cavities, etc.) may remain hidden;
 - the soil profile is subject to interpolation faults.
3. The heterogeneity of the soil block affected by the object compels us to think “in layers”, whose separation and identification is difficult (see Section 7.3.1).
4. The soil is a *three-phase system*, which aggravates to forecast its behaviour (variation of the neutral stresses in the function of the load and time, the volume-change under the effect of water and heat, etc.).

As a result of the greater uncertainty as compared to the structural design, the possibilities of solutions in geotechnics *range over a wider scale*, so there is generally a greater relative difference between the costs of two extreme solutions.

8.2. The conventional safety factor

The uncertainties mentioned in Section 8.1 are taken into account in the conventional deterministic planning by making the quotient of the effects ruining the structure or structural element (Q) by those preventing this (R) lower than a prescribed $1/\nu$ value. The quotient

$$\nu = \frac{R}{Q} \quad (8.1)$$

is called safety factor, independently of the interpretation of the load and the resistance by the expected values, or otherwise (see Section 8.5.3).

Table 8.1. Conventional factors of safety proposed by TERZAGHI and PECK (1967)

Type of damage	Object	γ
Shear	Earthworks	1.3–1.5
	Retaining structures	1.5–2.0
	Foundation	2–3
Seepage	Buoyant force	1.5–2.5
	Piping	3–5

The order of magnitude of the safety factors used in the practice of *soil mechanics* is reflected well by the values proposed by TERZAGHI and PECK (1948, 1967) (Table 8.1). The upper limits refer to the usual operating conditions, the lower values are related to

- structures dimensioned to maximum loads,
 - temporary objects and
- to cases, where
- extensive in situ tests are performed, or
 - the dimensioning is based on the analysis of damages, which occurred with identical subsoil conditions.

If also extremal conditions (earthquake, the complete softening of the soil, etc.) are taken into account by the designer, or if the object is planned for a very short life, the safety factor may be reduced even to 1.1 (MEYERHOF, 1970).

The expression (8.1) is usually also called *total safety factor*, as against the *partial safety* proposed by BRINCH HANSEN (1956). According to his recommendation—taken over also by the Danish standards—the structures must be dimensioned for an equilibrium in the so-called nominal breaking condition. This may be expressed in the following way:

$$\frac{f_1 P_1 + f_2 P_2 + \dots + f_n P_n}{A} \leq \sigma \left(\frac{\tan \varphi}{f'}; \frac{c}{f''}; \dots \right), \quad (8.2)$$

where P_i is the load (effect) and A is the cross section area of the structure (e.g. the load-transferring surface area of the foundation body). The factor f_i is 1–1.05 for the dead load, 1–1.2 for water pressure, 1.2–1.5 for moving loads, 1.5 for wind pressure and 1.3 for static pressure (BRINCH HANSEN and LUNDGREN, 1960). The new idea is not the differentiation of the factors of the loads P in this way, but treating also the components of the *resistance* by proposing

$$f' = 2.0 \quad \text{and} \quad f'' = 1.2$$

for the bearing capacity of the foundations and

$$f' = 1.5 \quad \text{and} \quad f'' = 1.2$$

for the earth pressure and the stability of slopes. These values reflect the *different variability* of both components of the shear strength (see Section 1.3.6), so the process may be regarded as a forerunner of the *statistically based dimensioning*.

During the last decades numerous remarks were raised against the conventional safety factor. According to FREUDENTHAL (1947) the propagation of the efficient designing methods reflecting the equilibrium of safety and economy is prevented by the contradiction between the modern refined design and the rather arbitrarily adopted safety factor. The *true significance* and the real properties of the safety factor remain hidden. This opinion is shared also by LANGEJAN (1965) and by WU and CRAFT (1970). LÉVI (1958) demonstrates numerically how *uneconomic* solutions may be adopted; for example the safety factors ($\nu = 3 - 5$) accepted for the pylons of electric lines protect them against events which may occur only once in $10^6 - 10^9$ years. FREUDENTHAL (1961) points out that the designers impose sometimes so *irrealistic* conditions, as infinite life, or zero probability of failure. According to FELD (1965) many people believe in the "law" of FINNIGAN saying that "if misfortune is given a chance, it happens" also in the field of structural construction.

Most authors see the disadvantage of the conventional safety factor in its disregarding

- the *reliability* of the applied data (or taking them incorrectly into account) and
- the *risk* reflecting the economical background.

The role of both these factors is shown by the following example. The safety factor to be assigned to the maximum groundwater level observed till the time of the investigation was uniformly 1.0 m in the Hungarian practice of the early fifties. This was modified in the new regulations (NAGY and RÉTHÁTI, 1956) by making the additive safety dependent on the reliability of the investigation (the number of observation wells, the length of their time series, etc.) and on the magnitude of the damage caused by the inflow of water (dryness requirements of the basement).

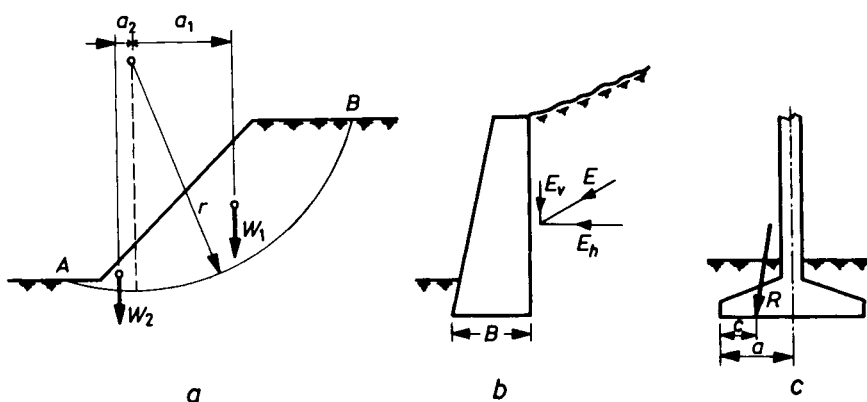


Fig. 8.1. Uncertainties in the interpretation of the conventional factor of safety (a and b: HÖEG and MURAKA, 1974, c: LESKE and KUSURELIS, 1965)

In some cases the conventional safety factor *cannot even be defined unequivocally* (HÖEG and MURARKA, 1974; LESKE and KUSURELIS, 1965). In case *a* of Fig. 8.1 the safety factor may be written in two forms:

$$v = \frac{s(AB)r}{W_1a_1 - W_2a_2} \quad (8.3)$$

or

$$v = \frac{s(AB) + W_2a_2}{W_1a_1}, \quad (8.4)$$

where

- s — shear strength of the soil,
- AB — length of the arc lying between points A and B ,
- r — radius of the failure surface.

It is characteristic for the uncertainty that TERZAGHI and PECK proposed to calculate, resp. to adopt the safety factor given by expression (8.4) in the first edition of their book (1948) and by expression (8.3) in its second edition (1967).

In case *b* we are faced with similar difficulties: on calculating the safety against overturning the moment BE_v is either added to the moment of the resistance, or subtracted from the overturning moment. Naturally, the two v -values may differ significantly from each other also here. The same task for the case of foundations is solved by LESKE and KUSURELIS (1965) by defining the safety as the quotient

$$v = \frac{a}{a - c} \quad (8.5)$$

(Fig. 8.1, detail *c*). In this way the circumferential conditions are satisfied (if $c=0$, $v=1$ and if $c=a$, $v \rightarrow \infty$), but the conventional interpretation of the safety factor does not change thereby.

8.3. Analysis of the damaging process

If the safety of a structure depends on the behaviour of N elements, two cases of the probability of failure may be distinguished.

1. If the drop-out of a *single* (any) element is sufficient for the failure, we speak of a *series connection* (system) between the elements. If the elements are independent and the probability of their failure is p_i , then the probability of failure of the whole system is

$$p_f = 1 - \prod_{i=1}^N (1 - p_i). \quad (8.6)$$

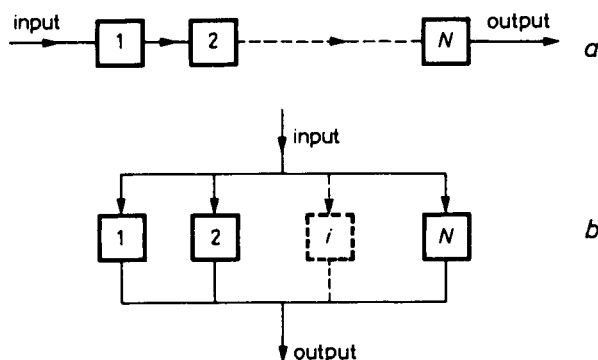


Fig. 8.2. Serial (a) and parallel (b) connection of the elements contributing to the failure

2. If the system reaches the unstable state only after the failure of *all* the elements (*parallel connection*), the probability of failure of the whole system is

$$p_f = \prod_{i=1}^N p_i. \quad (8.7)$$

The block diagrams of both cases are shown in Fig. 8.2.

There is a significant difference between the series- and the parallel systems. If, e.g., $N=4$ and the probability of failure of the individual elements is uniformly $p_1=p_2=p_3=p_4=0.015$, then in the former case $p_f=0.059$, in the latter case $p_f=5.1 \cdot 10^{-8}$. Consequently

a) for sensitive structures (e.g. in airplane production) the parallel system should be realized with built-in “redundant elements”;

b) for series systems a higher central safety (see in Section 8.5.3) must be selected (FREUDENTHAL, 1961).

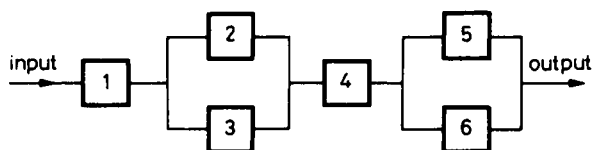


Fig. 8.3. Example for the combination of the serial and the parallel connection

In the majority of the practical cases the series and the parallel connections are *mixed*. In this case the probability of failure must be calculated by expressions (8.6) and (8.7) in common. For example, in the case of the block diagram shown in Fig. 8.3.

$$p_f = 1 - (1 - p_1)(1 - p_2 p_3)(1 - p_4)(1 - p_5 p_6). \quad (8.8)$$

Some research workers have studied the failure process by lattices (FRISCH *et al.*, 1961; DEAN, 1963). A method which could be used also in practice, is not yet available; one of the greatest difficulties is that the behaviour of the grids (e.g. in the case of the sliding of an embankment) is not independent of each other.

If the condition for the operation of the system is that a *certain number* of the elements must operate, we must calculate with the model called “*r* from *N*”. This case may occur, e.g., when the condition of the satisfactory foundation system is that the load should be supported at least by *r* piles out of *N* piles (HARR, 1977). In this case the probability of failure is according to the binomial theorem (see in Section 2.1.2)

$$p_f = 1 - \sum_{k=r}^N \binom{N}{k} (1-\bar{p})^k (\bar{p})^{N-k}, \quad (8.9)$$

where $\bar{p}$ is the mutually uniform probability of failure of the individual elements. If, e.g., $N = 30$, $r = 26$ and $\bar{p} = 0.01$, then

$$\begin{aligned} p_f = 1 - & \binom{30}{26} 0.96^{26} \cdot 0.01^4 - \binom{30}{27} 0.99^{27} \cdot 0.01^3 - \\ & - \binom{30}{28} \cdot 0.99^{28} \cdot 0.01^2 - \binom{30}{29} 0.99^{29} \cdot 0.01 - \\ & - \binom{30}{30} \cdot 0.99^{30} \cdot 1 = 1 - 0.999988 = 1.2 \cdot 10^{-5}. \end{aligned}$$

8.4. Uncertainties hidden in the calculation model

In geotechnics numerous processes were elaborated during the last decades for solving actual tasks. Though much advance was made both in the theoretical and the empirical fields (leading to the exclusion of erroneous theories), still several solutions, which are valid at identical initial and peripheral conditions, are available in every field for the designer. (This is the reason why the Hungarian standards refer to calculations “by some accepted theory”.)

This unfavourable circumstance acts towards *increasing the safety factor*. Its numerical effect may be assessed by determining the studied characteristic (e.g. the bearing capacity of the soil) with fixed physical properties, then by calculating their standard deviation, resp. their coefficient of variation.

In the following a few examples on the mode of determining the uncertainty and its numerical value originating from the model are given.

The calculation of the *stability of slopes* contains an uncertainty of 5%, resp. 7–10% according to SINGH (1970, 1971); WRIGHT (1969) came to a similar conclusion as well. CORNELL (1971) proposed (on the basis of the examination of failed embankments) to calculate the uncertainty with $C_v = 0.1$ for the variousness of the calculation models. HUDER (1977) plotted in connection with a numerical example the safety factor versus the aperture angle (2θ) of a circular sliding surface, comparing his own process—

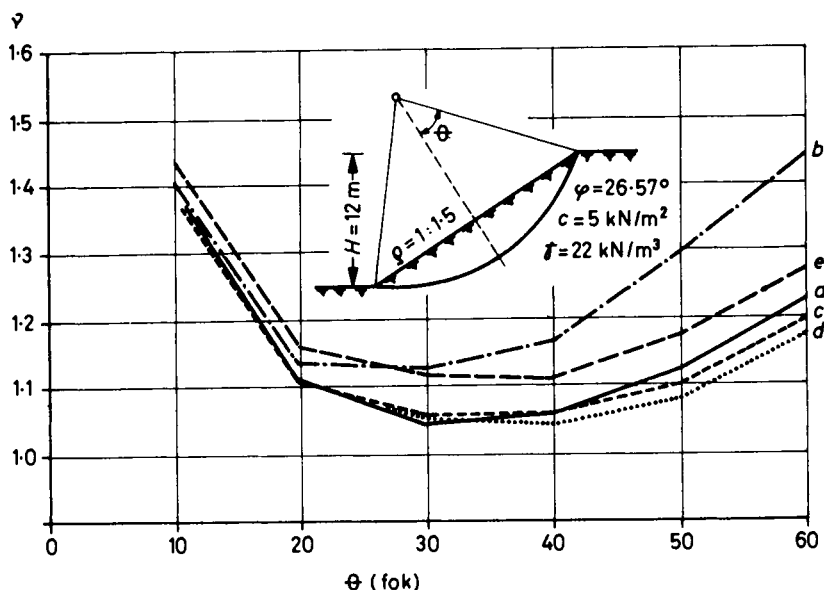


Fig. 8.4. Variation of the factor of safety v with the angle θ according to various authors, on the basis of HUDER's numerical example; *a* — HUDER's analytical method, *b* — BISHOP's method, *c* — FELLENIOUS's method, *d* — the original and *e* — the corrected method of JANBU

based on numerical solution by the method of FELLENIOUS—with those of BISHOP and FELLENIOUS and two processes of JANBU. The minimal safety factors (for judging the stability) are according to Fig. 8.4.

$$1.047-1.122-1.056-1.10-1.038,$$

so $\bar{v} = 1.073$ and $C_{vv} = 0.034$.

The ultimate bearing capacity (q) is subject to much greater calculation faults, resp. uncertainties. Let us consider a $B = 1.5$ m wide strip foundation standing on a soil of $c = 0$ cohesion, $\gamma = 19$ kN/m³ bulk density, and loaded by a vertical central force. Table 8.2 shows the mean, the standard deviation and the coefficient of variation of the q values calculated by six different theories versus φ , with the assumption of $t = 1.0$ m covering. On carrying out the calculations also for $t = 1.5$ m and 2.0 m we obtain for the relationship $C_v(\varphi)$ the curves shown in Fig. 8.5. Accordingly, the coefficient of variation expressing the uncertainty of the model decreases with the increase of t and φ , its mean is 0.08 in the ranges $t = 1-2$ m and $\varphi = 20-40^\circ$. This value is considerably higher in the case of foundations of a closed plan view, or lying on a cohesive soil and loaded by eccentric and/or oblique forces. So, e.g., the coefficient of variation calculated from the data of a comparative table of KÉZDI (1970) is 0.25-0.43.

SCHULTZE (1975a) described an interesting test method. Certain geotechnical tasks were solved by every member of a committee of 14 experts with fixed soil-physical and other characteristics. The coefficients of variation characterizing the scatter of the results were as follows:

Table 8.2. Magnitude of the ultimate bearing capacity (q kN/m²) according to various authors ($B=1.5$ m, $\gamma=19$ kN/m³, $t=1$ m)

Authors	$\varphi^0=10$	15	20	25	30	35	40
SOKOLOVSKY	61.8	94.5	162.9	304.5	572.9	1150.0	2304.7
TERZAGHI	55.6	96.9	171.0	318.3	560.5	1097.2	2465.3
MIZUNO	63.2	112.6	192.4	340.1	650.8	1211.3	2432.0
MEYERHOF	45.6	84.6	168.2	300.7	536.8	1021.3	2056.8
DIN	50.4	84.1	147.3	267.4	470.3	912.0	1900.0
MSZ-15004	57.5	96.9	170.1	311.6	555.8	1111.5	2384.5
$\bar{q} =$	55.7	94.9	168.7	307.1	557.9	1083.9	2257.2
$s =$	6.7	10.4	14.6	23.9	58.3	104.9	228.1
$C_v =$	0.12	0.11	0.09	0.08	0.10	0.10	0.10

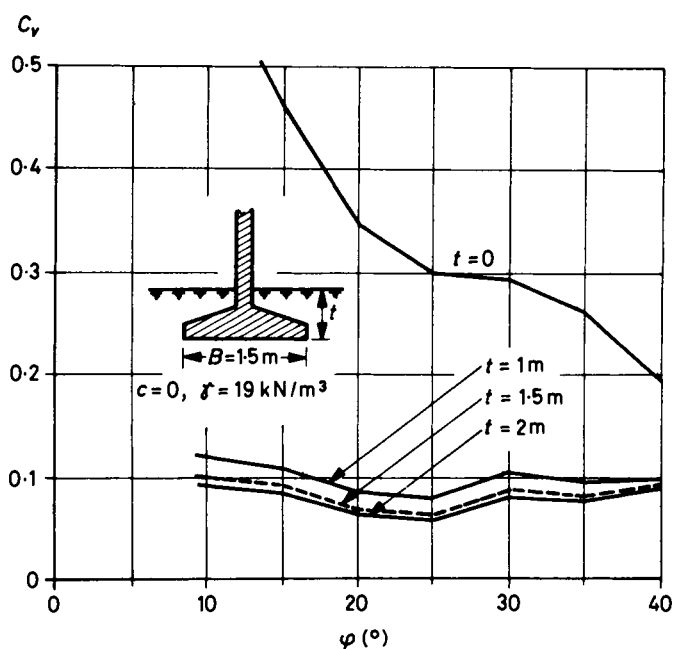


Fig. 8.5. Coefficient of variation of the ultimate bearing capacity calculated by six different theories versus φ (see Table 8.2)

$C_v=0.44$ for the calculated settlement,

$C_v=0.23$ for the safety of the retaining wall against overturning,

$C_v=0.28$ for the ultimate bearing capacity, and

$C_v=0.20$ for the stability of slopes.

Some calculations were repeated after the publication of the related *standards*, whereby the coefficient of variation for settlement decreased to 0.27, that for the stability of slopes to 0.05.

(It is to be noted here that the favourable effect of the standard may not be apparent unless recommended methods supported by observations, which may be adapted to the actual conditions, are given.)

8.5. Safety calculation based on the theory of probability

8.5.1. Classification of the calculation processes

Different methods of “measuring” safety are known (BIAREZ *et al.*, 1981) under the names

- deterministic (Level 0),
- semi-probabilistic (Level I),
- approximate probabilistic (Level II) and
- fully probabilistic (Level III).

In a Level I method the safety is estimated by a coefficient calculated on the basis of the fractiles of the soil and action parameters. By applying a Level II method a safety coefficient is calculated with the help of means and variances in the design parameters; the mean and the variance of this coefficient, together with a hypothesis of the probability distribution, allow the computation of the probability that it will be less than a given value. The complete statistical knowledge of all the random factors involved in the safety of structures is necessary for a Level III method to be applied. The estimation of the safety is then given through the probability of the occurrence of such and such a state of the structure.

With the propagation of the theory of probability ever more new dimensioning methods were developed. As a necessary implication of “path-finding” these methods were based on different principles. Nevertheless, certain aspects could be found by which the processes used so far were assigned to the following groups.

a) Mode of failure

FREUDENTHAL (1961) distinguishes between two fundamental cases: failure by the *load* and by *fatigue*. As the latter occurs rarely in geotechnics (e.g. the settlement of road pavements by repeated loading), we will not deal with this case.

b) The interpretation of Q and R

The modern dimensioning methods regard the effects bringing about the failure (Q) and those preventing it (R) as random variables (*Fig. 8.6*). In this respect we may distinguish between two cases: 1. only one of the above is a random variable (usually R , so $Q = \text{const.}$), 2. both effects are treated as random variables. *Fig. 8.6* shows clearly the great difference between these hypotheses (see Sections 8.5.2 and 8.5.5).

c) Relationship between Q and R

In the general case Q and R are independent of each other. In the contrary case the probabilistic solution of the task meets with extraordinary difficulties (e.g. when the distribution of the loads is changed by the settlement of one of the foundations, whereby also the distribution of the settlements is changed).

d) Relationship between the elements of the resistance (or of the load)

In this case again the question of *dependence* arises. E.g., the calculation is different, if both components of the shear strength (φ and c) are independent and if they are not so (see Sections 8.6.1 and 11.3.3).

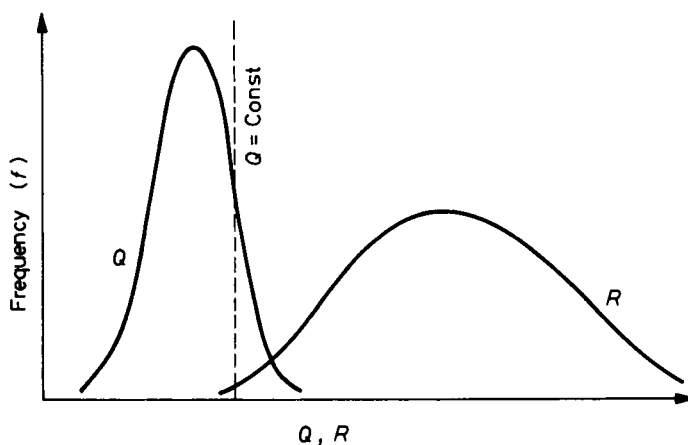


Fig. 8.6. The load and the resistance as random variables

e) The time dependence of Q and R

In the general case both the load and the resistance vary with time. Two great groups of solutions, depending on whether this process is followed up by the theory of probability or by other methods (see Section 8.5.5) are distinguished.

f) The interpretation of safety

The new methods differ greatly also in this respect: this problem is discussed in more detail in Sections 8.5.2–8.5.4.

g) The measure of the approximation

Today we cannot define accurately the criteria of the “exact dimensioning” as yet, but we may state that *up-to-date* is the method, which

- considers the load and the resistance, so also the measure of safety as random variables and

- permits to calculate the probability of failure, or more generally to determine the technical-economic optimum.

The significance of the *approximative processes* must not be underrated either, especially those which are based on the simultaneous evaluation of the “reliability” and the “risk”.

8.5.2. The separate examination of side R or side Q

There are cases—as mentioned already in Section 8.5.1—where one of the relations Q , R —mostly the load—is taken into account with its constant value (see Fig. 8.6). Tasks of this type may be solved by one of the following two methods:

- we determine the probability p of the effective resistance being lower than the load $Q = \text{const.}$;

- we examine, if the value $Q = \text{const.}$ is lying inside or outside of the confidence interval of R .

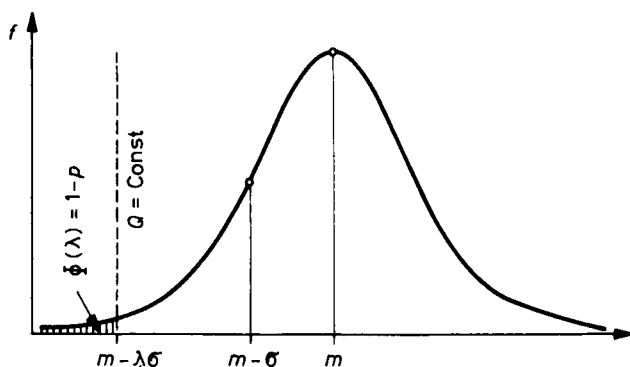


Fig. 8.7. Calculation of the probability of failure in the case of the normal distribution of R and $Q = \text{const}$.

In both cases the empirical distribution function of R must be known.

On applying the *first* method we are faced often with one of the two following fundamental cases.

a) If the studied physical characteristic of the soil has been determined by a *great number of tests* and its distribution is normal, we must determine the hatched area in Fig. 8.7. According to Section 2.1.3 we find as a first step λ , belonging to the value

$$\Phi(\lambda) = 1 - p$$

in Table F2. The requested lower threshold value x_a is

$$x_a = m - \lambda\sigma, \quad (8.10)$$

where m — expected value of the physical characteristic,
 σ — its standard deviation.

b) If the number of the tests is limited ($n < 30$), the lower threshold value is determined from the STUDENT-distribution:

$$x_a = \bar{x} - t_p s, \quad (8.11)$$

where $\bar{x}$ — mean, and

s — standard deviation of the measured values,

t_p — value belonging to the degree of freedom $n - 1$ and the probability $2p$ in Table F4.

The task may be solved similarly—with proper interpretation—also if the distribution of the physical characteristics is different (e.g. gamma-distribution).

According to the second method the lower threshold value must be determined from expression (7.36) or (7.37). As in practice the number of tests is relatively small, the latter

expression is used, so the lower threshold value x_a is

$$x_a = \bar{x} - t_p \frac{s}{\sqrt{n}}. \quad (8.12)$$

SCHULTZE (1971) wrote up the expression (8.12) for the lower threshold value of the resistance, the so-called base value (R_a) in the following form:

$$R_a = \bar{x} \left(1 - C_v \frac{t_p}{\sqrt{n}} \right) = \frac{\bar{x}}{\frac{1}{1 - C_v \tau}}, \quad (8.13)$$

where the quotient

$$\eta = \frac{1}{1 - C_v \tau} \quad (8.14)$$

is the so-called *statistical safety* and the value of τ may be found in Table 4.4. (As we have seen in Section 4.4, the *required number of the samples* may be connected with the safety of the dimensioning in this way.)

The so-called calculated value of the studied physical characteristic must be determined from expression (8.12) also according to the Soviet standard for foundations (SNIP—II—15, 1974).

The processes described above are applicable also if the “load”, or the “resistance” *cannot be defined*. Let us consider for example the case of a vertical slope. With the surface cracks caused by tension taken also into account the critical height is (KÉZDI, 1970):

$$m_0 = \frac{2.67c}{\gamma} \tan(45^\circ + \varphi/2). \quad (8.15)$$

On utilizing the relationship between the cohesion and the unconfined compression strength, Eq. (8.15) may be written also in the following form:

$$m_0 = \frac{2.67q_u}{2\gamma} = \frac{1.33}{\gamma} q_u. \quad (8.16)$$

Let us assume that the height of the slope is 2.8 m, the bulk density of the soil is $\gamma = 19.2 \text{ kN/m}^3$, the mean of the unconfined compression strength determined by $n = 15$ tests is $\bar{q}_u = 80 \text{ kN/m}^2$ and its standard deviation is $s = 28 \text{ kN/m}^2$ (the relative small standard deviation of γ may be neglected). The question is: what can we say about the safety of the slope?

According to Eq. (8.16) the unconfined compression strength belonging to the stable-unstable situation is

$$q_u = \frac{2.8 \cdot 19.2}{1.33} = 40.4 \text{ kN/m}^2,$$

so the conventional factor of safety is $v = 80/40.4 = 1.98$. The probability p_f that the

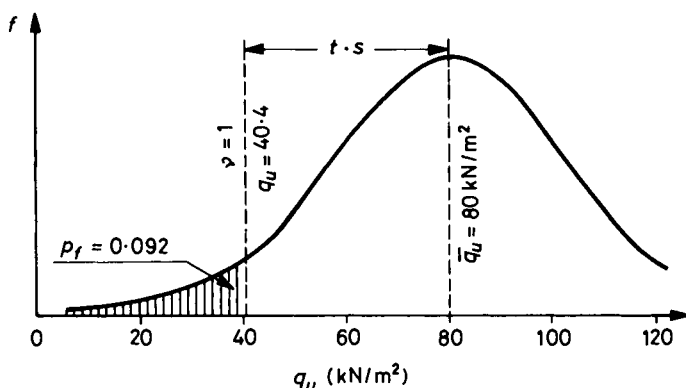


Fig. 8.8. Numerical example for calculating the probability of failure of a slope with vertical wall

slope fails, is given by the hatched area in Fig. 8.8. On determining the quotient

$$t = \frac{80 - 40.4}{28} = 1.414$$

we obtain for $n - 1 = 14$ a probability $p = 0.1834$ from Table F4, so

$$p_f = p/2 = 0.092 (= 9.2\%).$$

The p_f -value defined in this way is called *probability of failure* in literature.

According to the second method we obtain from expression (8.12) the following value for the lower confidence limit of $\bar{q}_u$ at the probability level $p = 0.001$:

$$q_{ua} = 80 - 4.14 \frac{28}{\sqrt{15}} = 50.1.$$

So the probability that the average compression strength exceeds the critical value $q_u = 40.4 \text{ kN/m}^2$ is 99.9%. This calculation method corresponds to the—disputable—assumption that the average cohesion is asserted on the failure surface.

UNGÁR (1974, 1975b) calculated on the basis of similar considerations the safety of a foundation pit, bordered by sheet piling, against *hydraulic soil fracture* by adopting the value of the coefficient of permeability whose superior value may occur merely with a probability of 2%.

MISTÉTH (1963, 1966) defined by using these basic principles the conception of the *limit stress* (Y_H):

$$Y_H = \bar{Y} - m \cdot s_Y, \quad (8.17)$$

which is the breaking stress appearing possibly during the planned life of the object in the case of the least favourable grouping of all the factors which influence the stress and may lead to the failure of the structure with a determined probability. In the formula $\bar{Y}$

is the expected value of the breaking stress, s is its standard deviation and m is a factor depending on the assumed risk, which may be determined on the basis of the distribution function of the load (for a normal distribution $m = \lambda$).

The ruling stress, resp. load (Y_M) may also be interpreted in a similar way:

$$Y_M = \bar{Y} + m \cdot s_Y, \quad (8.18)$$

where $\bar{Y}$ and s_Y are now statistical parameters of the load.

Examples for the unilateral test may be found also in other branches. For example, the limit compression strength of concrete is according to Hungarian specifications

$$\sigma_{bH} = \frac{0.9 K_{\min}}{k},$$

where $k = 1.25$ for new reinforced concrete structures, 1.8 for concrete structures and $K_{\min}$ may be calculated from the mean $\bar{K}$ and the standard deviation (s) of the cube strength—in the case of normal distribution—as follows:

$$K_{\min} = \bar{K} - 2s.$$

In actual cases the small number of the samples and the skewness of the distribution must also be taken into account (KORDA and SZALAI, 1973).

8.5.3. Common allowance for R and Q

Figure 8.9 shows various possibilities of the common allowance for R and Q . According to the conventional process the safety may be interpreted in two ways. The *central safety factor* is the quotient of the expected values of the resistance and the load:

$$v_c = \frac{\bar{R}}{\bar{Q}}. \quad (8.19)$$

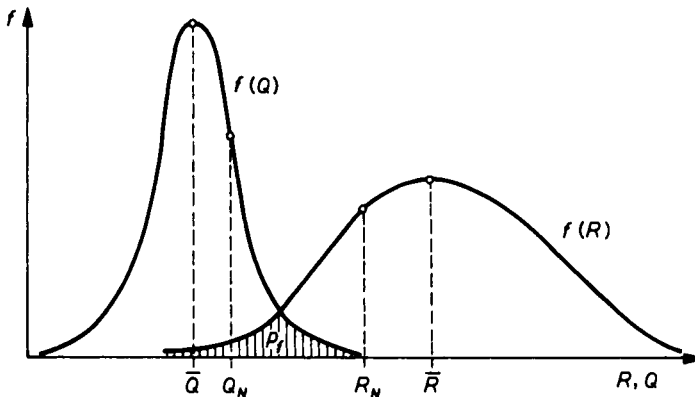


Fig. 8.9. Possibilities of the interpretation of the safety

The *conventional safety factor* differs from it by taking the resistance into account by a value $R_N < \bar{R}$ and the load by $Q_N < \bar{Q}$:

$$v_N = \frac{R_N}{Q_N} < v_c. \quad (8.20)$$

A variation of this is the solution proposed by BRINCH HANSEN (1956), who divides (resp. multiplies) both sides by partial safeties for rendering also $v_N = 1$ permissible.

The probabilistic interpretation of expression (8.20) is also possible if we determine the numerator and the denominator with the help of the distribution functions. Both these values are selected in a way that the probability of the appearance of superior, or inferior values should be lower than a threshold value defined by us. A possibility of this is discussed by SCHULTZE (1975b) in the following way. He determines the lower confidence limit (R_a) defined by Eq. (8.13), then obtains the standard deviation of the expected values of the population from the expression

$$\sigma_0 = s \sqrt{\frac{n-1}{\chi_{\alpha, n-1}^2}} \quad (8.21)$$

(see Section 7.4.2). So the requested ruling resistance will be:

$$R_N = R_a - \lambda \sigma_0, \quad (8.22)$$

where λ is the standardized variant of the normal distribution assigned to the probability level p .

In the numerical example of Section 8.5.2 the standard deviation σ_0 belonging to $\alpha = 90\%$ and $n - 1 = 14$ (see Table F.3) is

$$\sigma_0 = 28 \sqrt{\frac{14}{7.79}} = 37.5.$$

The lower confidence limit belonging to the probability level of 90% is:

$$q_{ua} = 80 - 2.977 \frac{28}{\sqrt{15}} = 58.5 \text{ kN/m}^2$$

so according to expression (8.22) the ruling unconfined compression strength belonging to the probability $p = \Phi(\lambda) = 0.90$ is

$$q_{uN} = 58.5 - 1.282 \cdot 37.5 = 10.4 \text{ kN/m}^2.$$

This means that only the slope of a height of $m_0 \leq 0.72 \text{ m}$ can be regarded to be stable, so the double reduction of SCHULTZE results in an *irrealistically high safety*.

The factor of safety defined by expression (8.20) has three disadvantages:

- it gives no information about the probability of failure,
- the (identical, or different) probability levels assigned to the numerator are selected rather arbitrarily and

— their simultaneous effect may lead to the assumption of an event, whose probability is unrealistically low, i.e. to uneconomic dimensioning.

These disadvantages may be eliminated by introducing the concept of the *safety margin* (SM), which is the difference between the “resistance” and the “load”:

$$SM = R - Q, \quad (8.23)$$

so it is itself a random variable, whose correlation with the probability of failure may be defined very simply as

$$p_f = P[(R - Q) \leq 0]. \quad (8.24)$$

The expected value of the safety margin is

$$\overline{SM} = \bar{R} - \bar{Q} \quad (8.25)$$

and its standard deviation (if Q and R are independent) is

$$s_{SM} = \sqrt{s_Q^2 + s_R^2}, \quad (8.26)$$

where s_R — standard deviation of the resistance,

s_Q — standard deviation of the load.

It may be easily verified that by this interpretation of the safety the uncertainties contained in expressions (8.3) and (8.4) may be eliminated unequivocally.

The determination of the statistical parameters and other details of the calculation are dealt with in Sections 8.6 and 8.7.

8.5.4. The safety as a random variable

A basic feature of the modern dimensioning processes is that they regard the safety factor (or the safety margin) as a random variable.

An interesting aspect of this view is that it permits to judge the risk (the probability of failure) not alone by v , or SM , but also on the basis of the distribution function of the involved *physical characteristics*, or of some geometrical dimension of the object. As an example *Fig. 8.8* shows the critical q_u value, which must not be underpassed, because in this case the slope with the given height would not be stable any more. We could have proceeded also by plotting the distribution function for defined m_0 heights from the measured q_u -values with the help of expression (8.16) and marking on it the *actual height* of the slope, with the advantage that in this way also the variation of the bulk density can be taken into account.

There are cases, where the *simultaneous values of two variables* must be examined. The process to be followed is illustrated by two examples.

The measure of the risk of *ice lens formation* depends on the cold quantity amount and the depth of the groundwater level (the water quantity supplied by capillary from here). Let us assume that results observed during 50 years are available for both

characteristics and the critical value of the rise of frost is 3 cm. In our case *the rise of the frost will be the random variable*, which may be plotted in the form of a distribution function, as determined from the actually observed yearly value pairs. On establishing the type of this distribution function we may calculate the probability of the rise exceeding 3 cm. This calculation method is more accurate and contains more information by far than, e.g., the calculation of a “ruling” value from the combination of the maximum water level and the cold quantity.

The second example refers to the investigation of the *stability of slopes*. If the height (H) and the inclination (β) of the slope are given and the bulk density—whose importance for the stability is lower—is taken into account with its mean value, the safety may be judged on the basis of Fig. 8.10, *a*. On connecting the centre of gravity of the coherent value pairs c , $\tan \varphi$ with the origin, we may calculate the quotient

$$v = \overline{OB} / \overline{OA},$$

called “conventional safety factor of FELLENIUS” by LAZARD (1961).

The position of the set of points relative to the curve $v = 1$ suggests in itself perceptibly the *distribution of the safety factor*, which may be obtained exactly only, if we determine the curves of identical safeties, as it was done by SINGH (1970). With the help of the graphs seen in Fig. 8.10, *b* the value of v for each value pair c , $\tan \varphi$ may be read off, then the distribution of v and p_f , the probability of the occurrence of the value $v = 1$ may be determined.

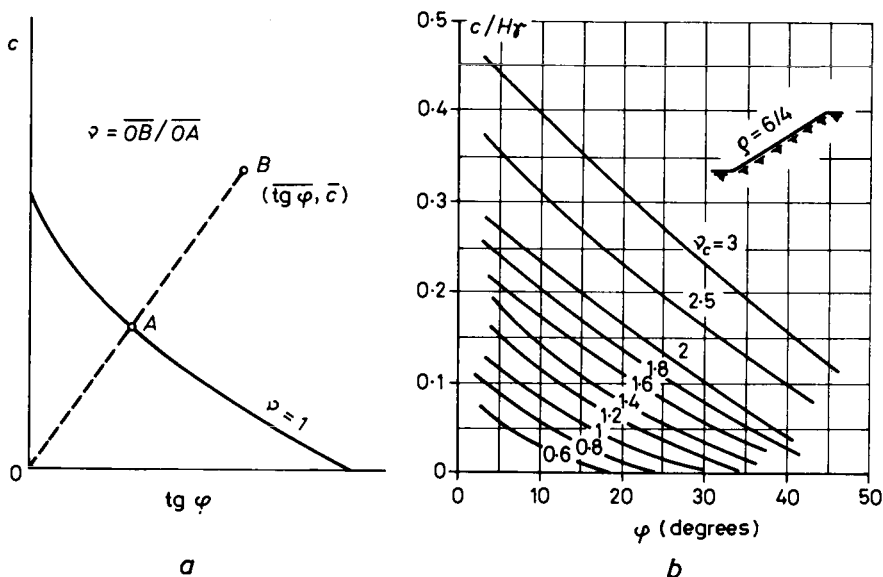


Fig. 8.10. Interpretation of the safety factors of slopes versus φ and c

8.5.5. Taking the time into account

The time deserves attention in the course of the dimensioning for two reasons:

- R and Q vary with the time and
- the probability of the occurrence of their extremal values is a function of the lifetime of the object.

The variation of the *soil physical characteristics* in time has been dealt with in Section 4.2.3, mainly with a view to the mode and the accuracy of the determination of their statistical characteristics, but other phenomena must be taken into account in the design, e.g. the following.

1. The deformation of the soil under the effect of load is a process protracting in time, with numerous consequences, one of which, deserving to be mentioned, is that by the *different consolidation processes* of the soil under the individual foundations it may occur that at a given time moment the differential settlement will be higher than the final value (RÉTHÁTI, 1955).

2. The subsoil of roads is getting compacted gradually by vehicular traffic, resulting in the appearance of a *trend component* in the variation of certain physical characteristics (e.g. the coefficient of bedding).

3. As the load acting on the soil is carried first by the porewater, the shear strength of the soil is very low in the first period of the loading, which means a serious risk, especially in the case of *sudden loads* (e.g. on charging silos, or at a sudden variation of the external water level). This example serves also as a warning to the effect that in given cases *the measure and the rate of the load-changing* must also be taken into account.

These examples demonstrate that *safety is a function of time as well*. In usual cases the safety diminishes gradually. For example, according to BURY (1975) the probability of failure after a time T has elapsed is

$$p_f = 1 - \exp \left[- \left(\frac{T}{\Theta} \right)^\lambda \right], \quad (8.27)$$

where

Θ — characteristic lifespan of the structure

λ — empirical constant (in the model of RAILEIGH $\lambda = 2$).

The probability of the occurrence of extreme values, resp. its *variation with the lifespan* may be evaluated with the help of the binomial theorem. The probability that the studied event will occur x -times in a period of n years is

$$f(x) = {}_nC_x p^x (1-p)^{n-x}, \quad (8.28)$$

where ${}_nC_x$ is the combination of n according to x , i.e.

$${}_nC_x = \binom{n}{x} = \frac{n!}{(n-x)!x!}, \quad (8.29)$$

and p is the probability of the yearly occurrence. Let us assume that $p = 0.001$ (i.e. the event occurs once in 1000 years), then the following probabilities of a single occurrence $f(x = 1)$ belong to the lifespans of various n years:

to $n = 10$ years	$f(x) = 0.010$
$n = 25$ years	$f(x) = 0.024$
$n = 50$ years	$f(x) = 0.048$
$n = 100$ years	$f(x) = 0.091$

Table 8.3. Classification of objects on the basis of their planned lifespan (t_0 years), the maximum of the damage ratio (C/D) and the accepted risk (MISTÉTH, 1981).

Class	Designation	t_0	$\left(\frac{C}{D}\right)_{\max}$	$\frac{1}{k}$
I.	Monumental buildings	150	200	10^{-4}
II.	Permanent objects (a)	50	200	10^{-4}
III.	Permanent objects (b)	50	20	10^{-3}
IV.	Semipermanent objects	15	20	10^{-3}
V.	Temporary objects	5	20	10^{-3}

In the design also the fact must be taken into account that not only the probability $f(x)$ of some undesirable event increases with n , but in the majority of cases also the *magnitude of the damage* caused by the possible failure. MISTÉTH (1981) recommends the categorization of the objects by their lifespan on the basis of such considerations (Table 8.3); further implications are discussed in Section 8.8.

8.6. Determination of the statistical parameters of the safety margin

8.6.1. Elemental statistical parameters of the R - and Q -functions

The standard deviation of the safety margin may be calculated in the case of the independence of Q and R by expression (8.26). If both these variables are not independent, the standard deviation may be determined by the following formula:

$$s_{SM} = \sqrt{s_Q^2 + s_R^2 - 2rs_Rs_Q}, \quad (8.30)$$

where r is the correlation coefficient between R and Q .

In Section 8.5.1 we have stated that the interdependence of the load and the resistance cannot be taken into account as yet in a reassuring way. Expression (8.30) corresponds to the simple case, where the

correlation existing between R and Q is indifferent with respect to the dimensioning model (see the numerical example given later).

The question is now how the standard deviation of the R - and the Q -functions can be determined.

Let us assume that the "resistance" can be expressed by a function $R(x_1, x_2, \dots, x_n)$ and the statistical parameters of the physical (geometrical, etc.) variables x_i are known. The *expected value* of the function $R(x)$ may be calculated on the basis of the TAYLOR-series of the function by the following expression (HAHN and SHAPIRO, 1967):

$$\bar{R} = R(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 R}{\partial x_i^2} s_i^2 + \frac{1}{6} \sum_{i=1}^n \frac{\partial^3 R}{\partial x_i^3} C_{si} s_i^3. \quad (8.31)$$

(On calculating the second and the third terms on the right side the variables must be replaced by their expected values.)

According to Eq. (8.31) $\bar{R}$ equals the value $R(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$, obtained by substituting the expected values only if the second and the third differential quotient of every variable is zero!

The standard deviation s_R may be calculated—with the neglect of the fourth-order terms—by the following expression:

$$s_R^2 = \sum_{i=1}^n \left(\frac{\partial R}{\partial x_i} \right)^2 s_i^2 + \sum_{i=1}^n \frac{\partial R}{\partial x_i} \frac{\partial^2 R}{\partial x_i^2} C_{si} s_i^2. \quad (8.32)$$

If $C_{si} = 0$ for every variable (so the distribution is symmetric, e.g. normal) the formula gives the statistical characteristic known in geodesy by the name "mean error of the function". KAZINCZY proposed with reference to this (already in 1942!) to characterize the safety of the engineering objects by the mean error.

The standard deviations of the individual factors participate with different weights in the resultant standard deviation, as pointed out, e.g., by LÉVI (1958), therefore, taking zero standard deviation into account for some factors (e.g. the bulk density) is in numerous cases permissible.

The coefficients of *skewness* and the *kurtosis* must be calculated by much more complex formulae. Their approximate value may be determined by the following expressions:

$$C_{sR} = \frac{1}{s_R^3} \sum_{i=1}^n \left(\frac{\partial R}{\partial x_i} \right)^3 C_{si} \cdot s_i^3, \text{ resp.} \quad (8.33)$$

$$C_{kR} = \frac{1}{s_R^4} \sum_{i=1}^n \left(\frac{\partial R}{\partial x_i} \right)^4 (C_{ki} + 3) s_i^4 + 6 \sum_{i=1}^n \left(\frac{\partial R}{\partial x_i} \right)^2 \left(\frac{\partial R}{\partial x_j} \right)^2 s_i^2 s_j^2. \quad (8.34)$$

For an example, let us consider again the case of the vertical slope. On neglecting the tensile cracks developing on the surface, the condition of the equilibrium is:

$$\frac{m_0^2 \gamma}{2} \tan^2 (45^\circ - \varphi/2) = 2cm_0 \tan (45^\circ - \varphi/2),$$

i.e.

$$\frac{m_0 \gamma}{2} \tan (45^\circ - \varphi/2) = 2c,$$

where the left side may be regarded as “load” (Q), the right side as “resistance” (R).

Let us assume an $m_0 = 4$ m high slope and the following measured values of the shear strength components of the soil:

φ°	21.0	23.5	24.6	19.5	19.0	22.4	23.5	24.2	20.6	20.9	22.8
$c(\text{kN/m}^2)$	28.0	23.5	19.4	33.7	35.2	35.2	28.0	29.2	27.8	18.5	26.4

Accordingly the expected values are $\bar{\varphi} = 22.0^\circ$ and $\bar{c} = 27.72 \text{ kN/m}^2$, their standard deviation is $s_\varphi = 1.91^\circ$, resp. $s_c = 5.69 \text{ kN/m}^2$, the mean of the $t = \tan (45^\circ - \varphi/2)$ values is $\bar{t} = 0.6748$ and its standard deviation is $s_t = 0.02426$.

On neglecting the standard deviation of the bulk density ($\gamma = 18.8 \text{ kN/m}^3$) the variance of the load is—if the distribution of t is assumed to be symmetric—according to Eq. (8.32)

$$s_Q^2 = \left[\left(\frac{m_0 \gamma}{2} t \right)' \right]^2 \cdot s_t^2 = 0.83$$

and the variance of the resistance is

$$s_R^2 = [(2\bar{c})']^2 \cdot s_c^2 = 4 \cdot 5.69^2 = 129.5.$$

The expected value of the resistance is (as both second differential quotients equal zero)

$$\bar{R} = 2 \cdot 27.72 = 55.48 \text{ kN/m}^2,$$

that of the load is

$$\bar{Q} = 25.37 \text{ kN/m}^2.$$

So the expected value of the safety margin is according to expression (8.25)

$$\overline{SM} = 55.48 - 25.37 = 30.11 \text{ kN/m}^2.$$

As between R and Q —i.e. between $\tan (45^\circ - \varphi/2)$ and c —an intermediately close correlation exists ($r = -0.46$), the standard deviation of the safety margin is according to Eq. (8.30):

$$s_{SM} = \sqrt{0.83 + 129.5 + 2 \cdot 0.46 \sqrt{0.83} \sqrt{129.5}} = 11.83 \text{ kN/m}^2.$$

The probability of failure is

$$p_f = P \left[2c - \frac{m_0 \gamma}{2} \tan 45^\circ - \varphi/2 < 0 \right].$$

On assuming a normal distribution for SM (see Section 8.6.2), p_f may be calculated as follows. The standardized variable is

$$\lambda = \frac{\overline{SM} - 0}{s_{SM}} = \frac{30.11}{11.83} = 2.545,$$

to which $\Phi(\lambda) = 0.9946$ belongs according to *Table F2*, so

$$p_f = 0.0054 = 0.54\%.$$

8.6.2. Determination of the distribution functions of the load and the resistance

For calculating the probability of failure we must know the distribution functions of R and Q (see Section 8.7), which may be determined by one of the following methods.

a) If R , or Q is a *linear function of a single variable*, the type of their distribution is identical with that of this variable.

b) If the relationship is *nonlinear*, or if *several* variables are involved, our first task is the determination of the statistical parameters according to Section 8.6.1. In the knowledge of the same we can usually assess, which type of distribution approximates the effective distribution best (e.g. with the help of PEARSON's coordinate system described in Section 2.1.1). If more than one type may come up, the most suitable one must be found by the fitting test (see Section 2.2). If $C_s \sim 0$ and $C_k \sim 0$, the normal distribution can be accepted, eventually at the cost of certain concessions.

c) If the initial conditions described under b) prevail, the Monte Carlo method may also be used successfully (see Section 8.6.3).

d) In the case of a single variable the distribution function may be determined also by CHEBISHEV's approximation. Accordingly, the density function is

$$f(z) = \varphi(z) - \frac{C_s}{3!} \varphi'''(z) + \frac{C_k}{4!} \varphi^{IV}(z) - \frac{C_s - 10C_s}{5!} \varphi^V(z) + \frac{C_6 - 15C_k}{6!} \varphi^{VI}(z). \quad (8.35)$$

$\varphi(x)$ in the formula is the standardized density function, where

$$z = \frac{\bar{x} - x}{s},$$

$\varphi''' \dots \varphi^{VI}$ are the 3rd ... 6th derivatives of it, further

$$C_s = \frac{1}{ns^5} \sum_{i=1}^n (x_i - \bar{x})^5 \quad (8.36)$$

and

$$C_6 = \frac{1}{ns^6} \sum_{i=1}^n (x_i - \bar{x})^6 - 15. \quad (8.37)$$

A form modified by POINCARÉ is also used, where the 4th and 5th terms on the right side are replaced by

$$\frac{10C_s^2}{6!} - \varphi^{IV}(z), \quad (8.38)$$

so there is no need to determine the fifth and the sixth central moments, which are always uncertain due to the small number of the elements.

If the distribution is close to normal, it is sufficient to take into account only the first two terms of Eq. (8.35); in this case

$$f(z) = \varphi(z) \left[1 - \frac{C_k}{6} (3z - z^3) \right], \quad (8.39)$$

whose correctness may be verified easily by the triple derivation of the function

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

8.6.3. The Monte Carlo method

According to the preceding section the statistical parameters of the R - and Q -functions can be determined also with the help of the Monte Carlo method. The essence of the process may be summed up—after SREJDER (1965)—as follows.

The Monte Carlo method is a stochastic modelling (numerical simulation); the designation itself first appeared in an article of METROPOLIS and ULAM published in 1949. Its name is derived from the fact that the roulette (reminding most people of the Casino of Monte Carlo) is one of the simplest means by which random numbers can be generated.

The solution of computation tasks by this process is closer, in its character, to physical experimentation than to the classic numerical processes. It may be used most successfully in fields where the original mathematical task itself is the examination of some stochastic process. But there exist also problems which demand extraordinarily much computation work and whose formulation is not connected with the theory of probability (e.g. the parameter problems of elliptic differential equations), yet it is advised to resort to this method for their solution.

The essence of the Monte Carlo method may be demonstrated by the solution of the definite integral

$$S = \int_0^1 f(x) dx. \quad (8.40)$$

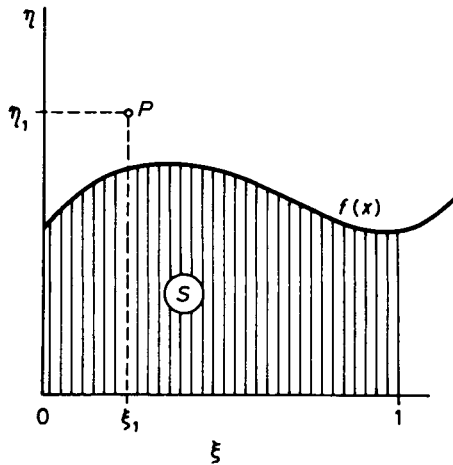


Fig. 8.11. Calculation of a definite integral by Monte Carlo method

The probability that a point, whose coordinates are ξ_1, η_1 falls in the hatched area shown in Fig. 8.11 is just S . So if we have a method for producing independent random variables ξ and η of a uniform distribution and if by sampling a single-element sample (ξ, η) we examine the assumption $f(\xi) > \eta$, then the quotient L/N (where N is the number of the samples and L is the number of cases where the inequality is satisfied) will equal just S .

For improving the accuracy it is advised to determine the "main part" of the result by the conventional process. If we find, e.g., a function $g(x) \sim f(x)$ whose integral is known, then the estimation Θ over the main part is

$$\Theta = \frac{1-0}{N} \sum [f(\xi_i) - g(\xi)] + I_1, \quad (8.41)$$

where I_1 is the integral of the function $g(x)$.

The earliest known application of the Monte Carlo method is BUFFON's classic "needle problem". In this connection the following question may be raised: Given a board ruled in parallel lines with a uniform spacing a , on which a needle of the length l is dropped at random; what is the probability of the needle falling across one of the lines (ELISHAKOFF, 1983). Let X be the random variable representing the distance between the midpoint M of the needle and the nearest line and Θ the random variable representing the acute angle between the needle (or its extension) and the line. x and ϑ denote any particulate values of X and Θ , it being apparent that the former can take on any value between 0 and $a/2$ and the latter any value between 0 and $\pi/2$. The probability we seek is given by

$$P(A) = \frac{4}{a\pi} \int_0^{\pi/2} \int_0^{(l/2)\sin\vartheta} d\vartheta dx = \frac{2l}{a\pi}. \quad (8.42)$$

A remarkable feature of this result is that it contains the number π , thereby providing an experimental means for its estimation. Dropping the needle numerous times and recording the hits, the relative frequency of these, $p(a)$, stabilizes near the real probability, so that equating them we obtain:

$$\pi = \frac{2l}{P(A)a}. \quad (8.43)$$

As ELISHAKOFF (1983) writes, such a series of experiments was carried out between 1849 and 1853 by the Swiss astronomer WOLF, using a ruled board with $a=45$ mm and a needle 36 mm long. The needle was dropped 5000 times and 2532 hits were scored, yielding a relative frequency of $P(a)=0.5064$. The corresponding approximation for π equals

$$\pi = \frac{2 \cdot 36}{0.5064 \cdot 45} = 3.1596,$$

deviating from the real value (3.1416...) by less than 0.02 (0.57%).

The accuracy of the method depends on

- the number of tests (according to CHEBISHEV's inequality, mentioned in Section 7.4.1, N must be increased 100-fold for a 10-fold improvement of the accuracy) and
- the mathematical characteristics of the set of the used "random numbers".

The table of the random numbers must meet *three basic requirements*: the uniformity of the distribution, the random nature of the numbers and their uncorrelation. The uniformity is tested usually by dividing the interval $[0, 1]$ to equal partial intervals and by performing the χ^2 -test. For testing the random nature of the numbers a process has been elaborated by KENDALL, BABINGTON and SMITH, consisting of the following four tests: a) frequency test and b) digit frequency test, c) interspace test and d) the test of the combinations. For series, produced with computers the length L of the aperiodic section may be given (in this respect the requirement $N < L$ must be observed).

Random numbers may be produced also by the computer, with the help of recurrent correlations, but these are more or less of a periodic character. For important calculations series obtained by recording the radiation of radioactive materials or the noise appearing in electron tubes are used.

Table F9 of the Appendix shows random numbers of a uniform distribution. With the help of these (or other similar) numbers we may produce random numbers of *another distribution type* as well. E.g., those of an exponential distribution may be obtained with the help of the following relationship:

$$x_i = -\frac{1}{\lambda} \ln(1 - R_i), \quad (8.44)$$

where R_i — random number taken from the uniform distribution,
 λ — parameter of the distribution (See Section 2.1.4).

Table F10 of the Appendix contains random numbers of a normal distribution.

The random numbers may be used successfully also for *marking out the samples*.

The Monte Carlo method is used today for solving numerous tasks. E.g., in the case of linear equation systems we can determine only a single unknown (so the number of the arithmetic operations will be proportional to the number of the equations and not to their third power). The Monte Carlo method is widely used for solving elliptic integrals, in mass service theories, in communication, as well as in tasks of heat conduction and numerous other problems. One of its important probabilistic utilizations is found in the field of random walk tasks (see Section 10.1.1).

The Monte Carlo method is gaining ground increasingly also in the field of geotechnics (SCHIMMING and GARVEY, 1967; SINGH, 1971; HÖEG and MURARKA, 1974; LUMB, 1975b; HARR, 1977). Examples of typical applications are to be found in Sections 9.1, 11.3.2 and 11.7.1.

8.7. The safety margin and the probability of failure

If we plot the density functions of R and Q in the same coordinate system, the probability of failure corresponds to the hatched area found in Fig. 8.12, *a*. This may be determined in the knowledge of the density functions $f(Q)$ and $f(R)$, resp. the distribution functions $F(Q)$ and $F(R)$ in the following way (HARR, 1977).

The probability of failure at some Q_1 load value is

$$P\left[Q_1 - \frac{dQ}{2} < Q < Q_1 + \frac{dQ}{2}\right] = f_Q(Q_1)dQ. \quad (8.45)$$

The probability of $R < Q_1$ corresponds to the hatched area in Fig. 8.12, *b*, i.e.

$$P[R < Q_1] = \int_{-\infty}^{Q_1} f_R(R)dR. \quad (8.46)$$

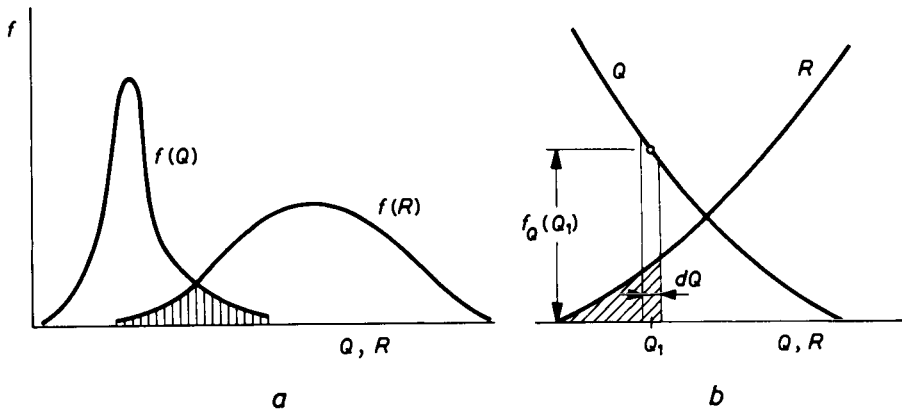


Fig. 8.12. Calculation of the probability of failure from the distributions of R and Q

The failure is the product of the probabilities defined by expressions (8.45) and (8.46):

$$dp_f = f_Q(Q_1)dQ \int_{-\infty}^{Q_1} f_R(R)dR, \quad (8.47)$$

so for all possible values of Q

$$p_f = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^Q f_R(R)dR \right] f_Q(Q)dQ. \quad (8.48)$$

This expression is identical to the following formula derived by FREUDENTHAL *et al.* (1966):

$$p_f = \int_{-\infty}^{+\infty} F_R(Q)f_Q(Q)dQ, \quad (8.49)$$

where F_R is the distribution function of the resistance.

The probability of failure may be brought into correlation also with the difference of both variables, i.e. the *safety margin*.

If both variables are of a *normal* distribution, then the density function of SM may be written in the following form:

$$f(SM) = \frac{1}{s_{SM}\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{SM - \overline{SM}}{s_{SM}} \right)^2 \right], \quad (8.50)$$

and the probability of failure is

$$p_f = 1 - \Phi \left(\frac{\bar{R} - \bar{Q}}{s_{SM}} \right), \quad (8.51)$$

where s_{SM} may be calculated from Eq. (8.30) and the values of the function Φ may be obtained from *Table F2*. The argument of Φ is called *safety index* in literature.

The probability of the failure may be expressed also by the central safety factor v_c . In the case of normal distribution

$$p_f = 1 - \Phi \left[\frac{v_c - 1}{\sqrt{v_c^2 C_{vR}^2 + C_{vQ}^2}} \right], \quad (8.52)$$

where $v_c = \bar{R}/\bar{Q}$, while C_{vR} and C_{vQ} are the coefficients of variation of the resistance and the load respectively.

The corresponding relationship for the lognormal distribution is as follows:

$$p_f = 1 - \Phi \left[\frac{\ln v_c \sqrt{\frac{1 + C_{vQ}^2}{1 + C_{vR}^2}}}{\sqrt{\ln(1 + C_{vQ}^2)(1 + C_{vR}^2)}} \right]. \quad (8.53)$$

In the numerical example of Section 8.6.2 we have calculated p_f from expression (8.51), with the assumption of a normal distribution. The central safety factor here is $v_c = 55.48/25.37 = 2.19$; $C_{vR} = 5.69/27.72 = 0.205$ and $C_{vQ} = 0.02426/0.67475 = 0.036$. So the argument of the function Φ is 2.642, which agrees with the value given there.

If the distribution does not follow any of both mentioned types, the area shown in Fig. 8.12, *a* may be determined by numerical, or graphical integration. If p_f is relatively high (0.005–0.01), then the type of the distribution function is of no great importance according to HÖEG and MURARKA (1974).

In the domain of the low probabilities it occurs that the usual function tables (e.g. F2) do not contain the requested value. In such cases the “tail” of the normal distribution may be determined from the following inequality (FELLER, 1957):

$$[z^{-1} - z^{-3}] \varphi(z) < 1 - \Phi(z) < z^{-1} \varphi(z). \quad (8.54)$$

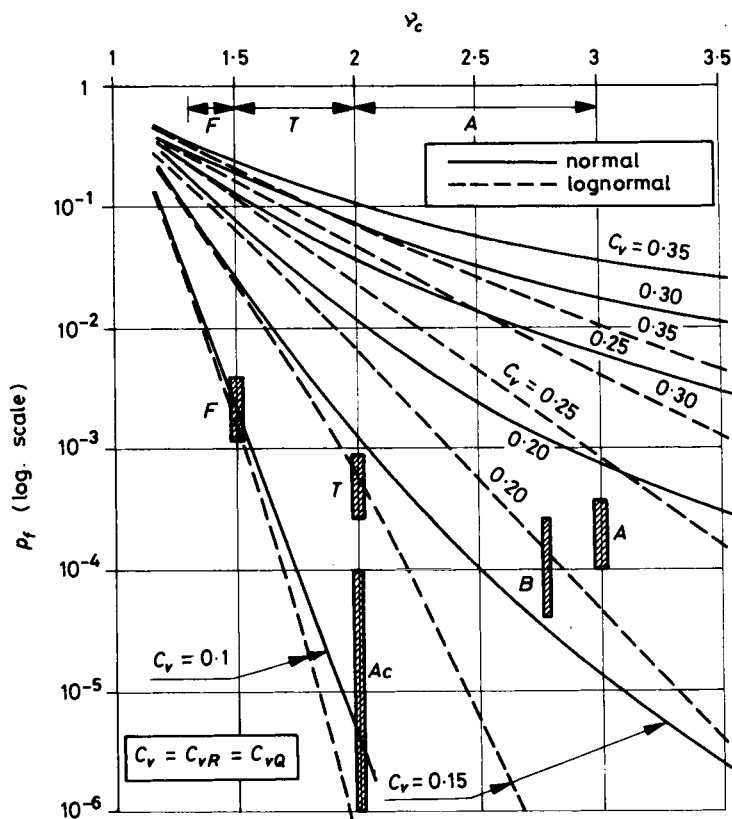


Fig. 8.13. Relationship between the central factor of safety (v_c) and the probability of failure (p_f), with the indication of the usual values after MEYERHOF (1970); *F* — earthworks, *T* — retaining walls, *A* — foundations; *Ac* — steel structure, *B* — concrete structures

If, e.g., $z = \lambda = 3.2$, then according to expression (8.54) and Table F1

$$\left[\frac{1}{3.2} - \frac{1}{3.2^3} \right] 0.00024 < 1 - \Phi(3.2) < \frac{1}{3.2} 0.0024,$$

i.e.

$$0.00075 < 1 - \Phi(3.2) < 0.00068,$$

which is indeed true according to Table F2, as $1 - \Phi(3.2) = 0.0007$. The calculation may be made more accurate by determining the value $\varphi(z)$ from the formula of the density function.

Let us return now to the correlations $p_f(v_c)$, which are of great interest, because they compare the concepts of the conventional and modern safety with each other. A very illustrative image (Fig. 8.13) may be obtained by the corresponding graphs of the expressions (8.52) and (8.53). In the coordinate system also the values used so far in the course of the solution of characteristic geotechnical tasks may be indicated, permitting to draw the interesting conclusion that a higher probability of failure belongs to the conventional geotechnical safety than the p_f -values of the concrete and reinforced concrete structure are (MEYERHOF, 1970). Most striking is the high probability assigned to the earthworks, especially in the case, where the coefficient of variation of the physical characteristics is about 0.3.

8.8. Decision strategy

The decision in the engineering sense takes always into account the *technical and the economical* considerations simultaneously, *in their interaction*. On applying the conventional designing methodology this principle is formulated usually as "selecting the cheapest of the technically equivalent solutions". But looking at it from the viewpoint of the theory of probability, this attitude is basically erroneous, as there exist no equivalent solutions, namely with the safety being a finite value, the probability of failure is different for each solution, with the consequence that the *expected probable cost* varies with the modes of the solution (geometrical dimensions, etc.), so our task is the minimization of this cost.

On felling the decision we must start from the fact that we have only an approximate knowledge of the true conditions. One of the possibilities of selecting from among the *variants* may be summed up as follows (HARR, 1977).

Let us assume that on designing a foundation the following three possibilities may be counted upon:

- unexplored soft clay lenses are present under the foundations (Θ_1),
- the surface of the rock of a high bearing capacity is unknown (Θ_2),
- the groundwater level varies considerably (Θ_3).

According to further assumptions the foundation may be solved by four methods:

- every foundation is standing on rock, on piles (a_1),
- solitair (block) foundations are formed (a_2),
- the structure is placed on a plate foundation (a_3),
- the highly loaded columns are placed on piles, the slighter loaded ones on block foundations (a_4).

Each Θ_i, a_i combination represents a *cost*, which may be a gain (+), or a loss (−); the latter case prevails, e.g., if there is no compressible lens, yet piles were used for the foundation. These combinations may be summarized in a *cost-matrix*. Let us assume that in our case this matrix is—in any arbitrary currency—the following:

	Θ_1	Θ_2	Θ_3
a_1	0	600	1000
a_2	200	400	200
a_3	−400	400	1000
a_4	1000	−1000	1400

The question is now, which solution should be chosen. According to the literature there is no unequivocal answer, as namely one of the following *four different strategies* may be accepted.

a) Selecting the best from among the worst possibilities according to the *maximin criterion* of WALD (1950), by writing out the lowest costs from each row of the matrix and accepting the highest of them as ruling. Accordingly,

$$a_{1\min} = 0, \quad a_{2\min} = 200, \quad a_{3\min} = -400, \quad a_{4\min} = -1000,$$

so we decide for the solution a_2 .

b) The *minimax criterion* of SAVAGE wishes to minimize the “regrets”, which appear after the effective conditions are recognized. If, e.g., the case Θ_1 applies, then at the solution a_2 the “regret” interpreted as above amounts to 800 units (the maximum in column Θ_1 is 1000 and 200 units belong to solution a_2). On the basis of such considerations we obtain the so-called “regret-matrix”:

	Θ_1	Θ_2	Θ_3	Max.
a_1	1000	0	400	1000
a_2	800	200	1200	1200
a_3	1400	200	400	1400
a_4	0	1600	0	1600

As from the maxima of the individual rows the minimum value belongs to solution a_1 , this foundation mode is accepted.

c) According to the principle of LAPLACE the probability of the occurrence of the events may be assumed to be identical. So, e.g., in the case of solution a_1 the expected value of the cost is

$$u(a_1) = 0 \cdot \frac{1}{3} + 600 \cdot \frac{1}{3} + 1000 \cdot \frac{1}{3} = 533,$$

while in the case of the other solutions it is 267; 333 and 467, in this sequence. So it seems most advised to accept solution a_1 a priori.

d) If the probability p_i of the events Θ_i can be estimated on the basis of preliminary explorations and tests, a better approximation may be obtained for the expected $u(a_i)$ values. If, e.g., $p(\Theta_1) = 0.60$, $p(\Theta_2) = 0.10$ and $p(\Theta_3) = 0.30$, then $u(a_1) = 0 \cdot 0.60 + 600 \cdot 0.10 + 1000 \cdot 0.30 = 360$, $u(a_2) = 220$, $u(a_3) = 100$ and $u(a_4) = 920$. So we proceed correctly, if we accept solution a_4 .

This principle may be applied also for the preliminary estimation of the dewatering cost (RÉTHÁTI, 1968a, 1975). Its value belonging to the given foundation plane is α -times the total dewatering cost K_v , where α is the

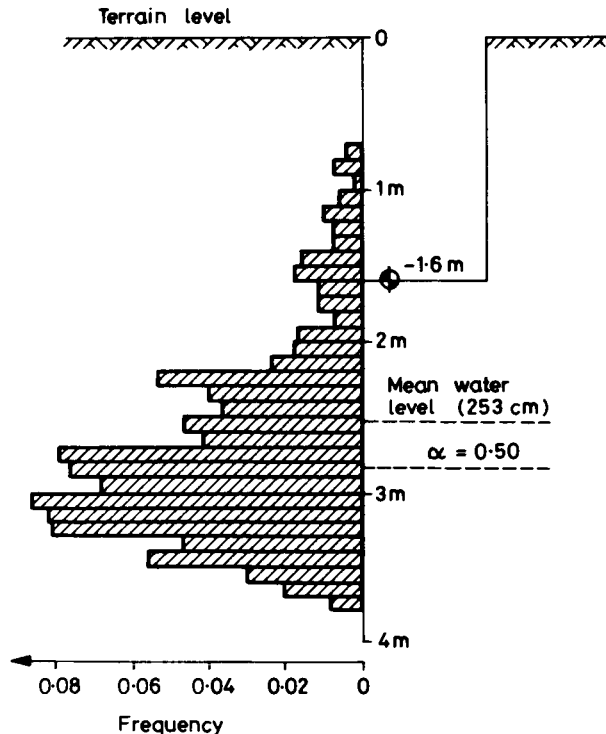


Fig. 8.14. Determination of the expectable draining costs for different foundation variants

probability that the effective water level shall exceed the foundation plane. If K_v varies with the foundation plane, resp. with the height of the water column (e.g. up to a certain water column height an open drain may be applied, beyond it the groundwater level must be lowered), then we may speak of K'_v and K''_v total costs and $\alpha'K'_v + \alpha''K''_v$ expected value. The α -values may be obtained from the empirical density, or distribution functions of the water levels; e.g., in the case shown in Fig. 8.14 $\alpha = 0.073$ belongs to $D = -1.6$ m.

The decision is facilitated by presenting the alternatives in a *tree-diagram*. The essence of the method may be studied by the following example. Let us assume that the geotechnical conditions of a residential quarter to be planned are as follows: the subsoil of the area consists of 70% sand and 30% refilling of a thickness of $H = 0 - 6$ m, and the mean groundwater level is higher on 20–30% of the area than the structurally required minimum foundation level ($D = -2$ m). Accordingly five foundation modes may be foreseen: shallow foundation at -2 m without draining (*A*); shallow foundation at -2 m, or at -3 m with draining (*B*, resp. *C*); foundation on exchanged soil (*D*) and pile foundation (*E*). Figure 8.15 shows all the possibilities with the corresponding probabilities. So, e.g., the probability that the soil must be exchanged is, according to the rule of multiplication, $P = 0.3 \cdot 0.3 \cdot 0.7 = 0.063$, which means in the given case that this probability appears only at 6% of the buildings.

With the help of the tree diagrams more justified decisions may be attained also concerning the type (and the amount) of the required *explorations and tests*. The initial basis is supplied by the data of earlier or preliminary tests, which may be refined with the help of the Bayesian theory (see Section 7.2). Methodical processes and examples for decisions based on tree diagrams are presented by RAIFFA and SCHLAIFER (1961), JAYNES (1963) and WU (1974).

Another useful tool of the dimensioning is the *loss function* (called also “utility function”). The essence of it is that the costs are plotted versus some ruling parameter (e.g. the slope angle), then the value of the studied parameter, to which the minimum cost belongs, is determined. Figure 8.16 illustrates the linear and the exponential

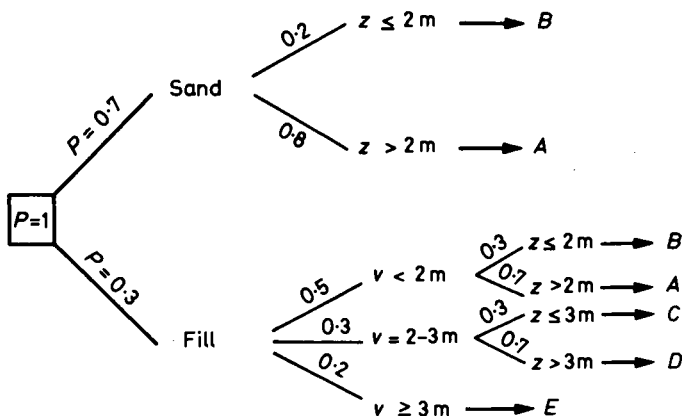


Fig. 8.15. Tree diagram for surveying the foundation possibilities

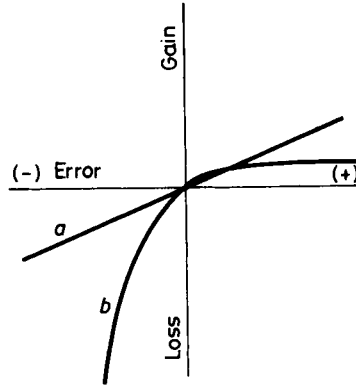


Fig. 8.16. Linear (a) and exponential (b) loss functions

variants of the loss function. In the first case the optimum value is identical with the expected value of the parameter, in the second case a detailed analysis is necessary (e.g. according to the curve seen in the figure the loss increases rapidly with a negative deviation, while a positive deviation brings hardly any economic gain).

An example for the loss function is given in Section 4.5, where the minimum cost of frost protection is determined versus the number of the boreholes, with the help of expression (4.24), resp. Fig. 4.17. Further examples are given in Sections 10.5 and 11.6.

With respect to economy another highly important question is the threshold value to be accepted for the *probability of failure*. According to MISTÉTH (1969) the following process is recommended. If $C(k)$ is the cost of rebuilding the object at a risk of $1/k$ assumed for the planned lifetime (t_0) and q is the coefficient of interest of the depreciation, then the yearly ratio of the depreciation is

$$j = \frac{q^{t_0}(q-1)}{q^{t_0}-1} C(k). \quad (8.55)$$

If the building fails after the time $t < t_0$, the depreciation of the sum $j(t-t_0)$ from the building cost is not written off. This sum, capitalized at the time moment of the failure is

$$j = \frac{q^{(t_0-t)}-1}{q^{(t_0-t)}(q-1)}. \quad (8.56)$$

By substituting j from Eq. (8.55), the residual ratio of the depreciation from the prematurely failed object, capitalized at the time moment of the failure, will be as follows:

$$\frac{q^{t_0}-q^t}{q^{t_0}-1} C(k). \quad (8.57)$$

If we designate the damage caused by the failure of the building (including the missing gain until its reconstruction) by D , then the total cost may be calculated by the

following expression:

$$K(k) = C(k) + \frac{1}{k} \left[C(k) \left(1 + \frac{q^{t_0-t}}{q^{t_0}-1} \right) + D \right]. \quad (8.58)$$

The minimum cost is found, where the first differential quotient of the function $K(k)$ according to k is zero. $C(k)$ may be obtained according to the related calculations with an accuracy of about 2% from the following expression:

$$C(k) = C_0(1 + b_1 \lg k), \quad (8.59)$$

where

C_0 — cost of the object belonging to $k = 1$,

b_1 — constant depending on the statistical characteristics of the dimensioning (its mean value may be assumed as 0.052).

After the derivation of equation (8.58) and substituting $b_1 = 0.05$, resp. $k = 740$, we obtain

$$k_{\max} = \frac{2.6}{0.05} \left(\frac{D}{C} + T \right), \quad (8.60)$$

where $T = 2$, if $t = 0$ and $T = 1$, if $t = t_0$. So if no damage is caused by the failure of the structure ($D = 0$), $t_0 = 1$ year (so there is no interest repayment), $k_{\max} \cong 50$. If $D/C = 100$, i.e. great damages may be caused, then $k_{\max} = 5000$. Accordingly, it is advised to foresee an assumed risk of $2 \cdot 10^{-2} \dots 2 \cdot 10^{-4}$.

In connection with the safety margin the risk should be interpreted according to expression (8.24). This means that if the distribution of SM is normal, then the quotient $s_{SM}/\overline{SM}$ must equal the λ -value, for which $\Phi(\lambda) = 1 - 1/k_{\max}$.

The permissible value of the probability of failure may be assumed also on the basis of other considerations. *The observation of the realized objects* plays an important role here. It would be a great advance in the field of dimensioning, if *guiding principles* for assuming *differentiated p_f -values* for each type of the object could be elaborated.

9. Determination of the bearing capacity of the soil

On studying the bearing capacity of shallow foundations we may write for a strip foundation loaded by a central vertical force the following equation:

$$q = \frac{B}{2} \gamma_1 N_\gamma + D \gamma_2 N_q + c N_c, \quad (9.1)$$

where

- q — ultimate bearing capacity,
- B — width of the foundation,
- γ_1 — bulk density of the soil under the foundation plane,
- γ_2 — bulk density of the soil over the foundation plane,
- D — covering over the foundation plane,
- c — cohesion of the soil,
- N_i — load bearing factors.

According to the conventional interpretation the limit stress (σ_H) may be calculated from the ultimate bearing capacity by dividing the latter by a safety factor v . According to the modern dimensioning methods all the terms in Eq. (9.1), so also the ultimate bearing capacity itself are regarded as *random variables* and the “limit stress” is determined on the basis of probabilistic principles (see Chapter 8).

9.1. Statistical parameters of the load bearing factors

B and D on the right side of Eq. (9.1) may be assumed to be constants, while the standard deviation of γ_1 and γ_2 is negligible (their coefficient of variation is 0.03–0.05 according to Section 1.3.4). This cannot be said for the cohesion and the angle of friction, the argument of the load bearing factors. As the statistical parameters of the cohesion are generally given, we must deal first of all with the variability of the N_i -factors.

The standard deviation of the load bearing factors may be determined according to Section 8.6 by two methods: the Monte Carlo simulation and the law of error propagation according to Eq. (8.32).

Table 9.1. Coefficients of variation of the load bearing factors defined by expressions (9.2) and (9.3) versus φ and $C_{v\varphi}$ (SINGH, 1971)

Load bearing factor	$C_{v\varphi}$	$\varphi =$					
		15°		25°		35°	
		$\bar{N}_i$	C_{vN}	$\bar{N}_i$	C_{vN}	$\bar{N}_i$	C_{vN}
N_q	0.10	3.983	0.143	11.081	0.268	37.242	0.474
	0.20	4.037	0.216	11.645	0.419	43.439	0.820
	0.30	4.115	0.291	12.515	0.590	55.763	1.380
N_c	0.10	11.029	0.086	21.147	0.180	49.357	0.342
	0.20	11.096	0.130	21.712	0.278	54.197	0.575
	0.30	11.192	0.175	22.563	0.388	63.123	0.940

SINGH (1971) studied the variability of the load bearing factors:

$$N_q = \tan^2 (45^\circ + \varphi/2) e^{\pi \tan \varphi} \quad (9.2)$$

and

$$N_c = (N_q - 1) \cot \varphi \quad (9.3)$$

used generally in practice by the first method. His calculation results are contained in Table 9.1, permitting to state that

- a) the variance of N_q is greater than that of N_c ,
- b) with the increasing angle of friction the coefficients of variation of the load bearing factors increase gradually,

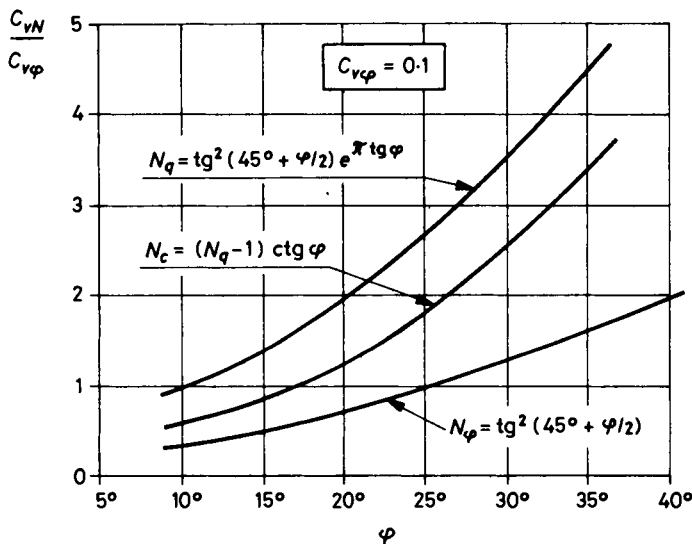


Fig. 9.1. Quotient of the coefficients of variation of the load bearing factors and of the angle of friction versus φ (SINGH, 1971)

c) the relationship between the coefficients of variation of the load bearing factors and that of the angle of friction is nonlinear, the quotient of these decreases with the increasing coefficient of variation of φ ($C_{v\varphi}$) (e.g. N_q decreases from 1.43 to 0.97 if $C_{v\varphi}$ increases from 0.10 to 0.30).

Figure 9.1 shows the variation of the mentioned quotient under c) versus φ for the case of $C_{v\varphi} = 0.1$. Accordingly, the variability of N_q and N_c in the case of angles of friction exceeding $\varphi = 10^\circ$, resp. 15° is greater than the variability of the angle of friction (at $\varphi = 35^\circ$ they are already 4.5–3.3-times as great). The figure supplies information also on the variability of COULOMB's hydrostatic factor of the passive earth pressure, drawing attention to its relatively low value.

According to SINGH (see above) the variability of the ultimate bearing capacity is considerable. Similar statements were made also by other authors, e.g. LUMB (1966) and SCHULTZE (1975a, 1977). The consequences of this fact are discussed in Section 9.3.

The variability of the load bearing factors may be calculated very simply also by Eq. (8.32), especially if we neglect the terms of a higher order. Let us assume, e.g., $\bar{\varphi} = 25^\circ$ for the mean value of the angle of friction, $s_\varphi = 2.8^\circ$ for its standard deviation, while we wish to determine the coefficient of variation of N_q defined by expression (9.2). The numerical value of the examined load bearing factor is

$$N_q = \tan^2 57.5^\circ \cdot e^{\pi \tan 25^\circ} = 10.66.$$

The differential equation appearing in Eq. (8.32) is

$$\left(\frac{\partial N_q}{\partial \varphi} \right)_{\bar{\varphi} = 25^\circ} = \frac{10.66\pi}{\cos^2 25^\circ} + \frac{\tan 57.5^\circ}{\cos^2 57.5^\circ} e^{\pi \tan 25^\circ} = 64.30.$$

As the value of s_φ expressed in radians is 0.04887, the standard deviation of N_q is

$$s_{N_q} = \frac{\partial N_q}{\partial \varphi} s_\varphi = 64.30 \cdot 0.04887 = 3.142,$$

and its coefficient of variation is $3.142/10.66 = 0.295$.

The *disadvantage* of the calculation based on error propagation, as compared to the Monte Carlo method, that it fails to supply any information on other statistical parameters of the load bearing factors, appears also here, so we must resort to assumptions concerning the type of the distribution function as well.

According to the above, the variance of the angle of friction influences the variance of the load bearing factors greatly, so also the value of the probability of failure. Consequently a preliminary condition of the economical design is the correct assumption of $C_{v\varphi}$. This may be ensured, in addition to the careful performance of the tests, also by other possible means. According to SCHULTZE and PATTHORST (1983) for the safety analysis of the bearing capacity not the $C_{v\varphi}$ value of φ 's measured at singular points, but the $C_{v\varphi}$ value of the mean angle $\bar{\varphi}$ of the entire shear area below the footing is determinative. On this basis they accepted, e.g., for the coefficient of variation of φ

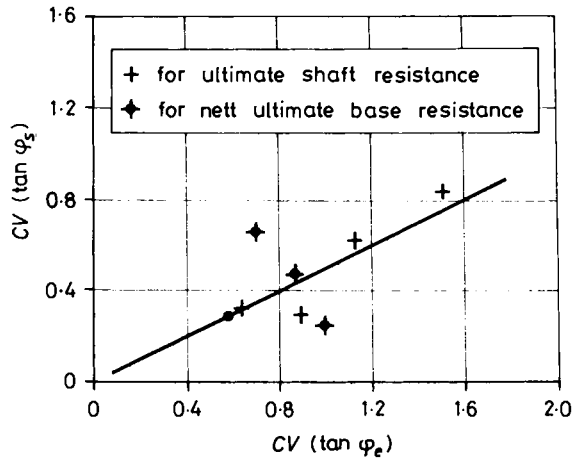


Fig. 9.2. $CV(\tan \varphi_s)$ vs. $CV(\tan \varphi_e)$ from pile load tests and investigation test (McANALLY, 1983)

examined by them in 300 samples $C_{v\varphi} = 0.075$ instead of 0.115, belonging to the values measured point by point.

$C_{v\varphi}$ may be determined by three different methods:

- a) from the standard deviation of the φ values determined point by point,
- b) from the standard deviation of the means measured in the individual verticals (drillings),
- c) from the confidence limits calculated for the mean of all the samples.

Although the foundation will be ever more economical on advancing from a) to c), yet solution c) is unacceptable; apparently solution b) is the most realistic.

On forming means for individual soil blocks the standard deviation of these is decreasing with the increasing volume of the block, as the equalization of ever more uncertain values may be expected. VANMARCKE (1977a) notes that the quotient s_w/s_i varies according to a reduction factor $\Gamma(V)$ (s_w is the standard deviation of the means belonging to the volumes V , s_i is the standard deviation of the whole set of the samples). McANALLY (1983) remarks, with reference to VANMARCKE's article that the variance σ_s^2 of the strength parameter $\tan \varphi_s$ of a net ultimate foundation response to footing loads has been observed to be less than the variance of $\tan \varphi_e$ estimated from test results. The relation between these variances can be expressed as $\sigma_s^2 = \sigma_e^2/k$ and the distribution of $\tan \varphi_s$ can also be taken to be lognormal. Figure 9.2 shows values of the coefficient of variation $C_s(\tan \varphi_s)$ estimated from the results of foundation load tests compared with values of the coefficient of variation $C_e(\tan \varphi_e)$ estimated from test results performed on the same sites. These values are based on relatively small numbers of test results and are approximate, but they indicate a ratio of the order of two. This indicates a typical k value of the order four.

9.2. Calculation of the load bearing capacity of shallow foundations

The limit stress is determined in most cases by analytical formulae, or from tables prepared for this purpose.

9.2.1. Determination of the bearing capacity on theoretical basis

As we mentioned in the introduction of Chapter 9, the limit stress is interpreted in the conventional dimensioning process as a ratio of the ultimate bearing capacity, while in the probabilistic dimensioning the *safety margin* interpreted in Section 8.5.3 is determined and correlated with the *probability of failure* and the *economic requirements* implied by it.

The safety margin required against the failure may be calculated from the statistical parameters of q . For this purpose we must know—according to Section 8.5.3—the expected value and the standard deviation of the ultimate bearing capacity and for solving more demanding tasks the type and the parameters of its distribution function. The variance of q may be calculated according to Eq. (8.32) as follows:

$$s_q^2 = \left(\frac{\partial q}{\partial B} s_B \right)^2 + \left(\frac{\partial q}{\partial \gamma} s_\gamma \right)^2 + \left(\frac{\partial q}{\partial \varphi} s_\varphi \right)^2 + \left(\frac{\partial q}{\partial D} s_D \right)^2 + \left(\frac{\partial q}{\partial c} s_c \right)^2. \quad (9.4)$$

As the variance of B , D and γ may be neglected generally in relation to that of φ and c , Eq. (9.4) may be written in the following form:

$$s_q^2 = \left(\frac{B}{2} \gamma_1 \frac{\partial N_\gamma}{\partial \varphi} s_\varphi \right)^2 + \left(D \gamma_2 \frac{\partial N_q}{\partial \varphi} s_\varphi \right)^2 + \left(c \frac{\partial N_c}{\partial \varphi} s_\varphi \right)^2 + (N_c s_c)^2 \quad (9.5)$$

provided that φ and c are independent random variables (see Section 8.6.1).

The derivatives of the load bearing factors defined by Eqs (9.2), (9.3) and

$$N_\gamma = 2(N_q + 1) \tan \varphi \quad (9.6)$$

are as follows (HARR, 1977).

For expression (9.2)

$$\frac{\partial N_q}{\partial \varphi} = \frac{N_q \pi}{\cos^2 \varphi} + \frac{\tan(45^\circ + \varphi/2)}{\cos^2(45^\circ + \varphi/2)} e^{\pi \tan \varphi}, \quad (9.7)$$

for expression (9.3)

$$\frac{\partial N_c}{\partial \varphi} = \frac{\partial N_q}{\partial \varphi} \cot \varphi - \frac{N_q - 1}{\sin^2 \varphi}, \quad (9.8)$$

Table 9.2. The differential quotients $\partial N_i / \partial \varphi$ defined by the expressions (9.7)–(9.9) and (9.11) versus φ

Examined expression	$\varphi =$									
	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°
(9.7)	5.1	8.1	13.0	21.4	36.4	64.3	119.6	237.2	511.3	1222.6
(9.8)	12.7	17.7	25.1	36.1	53.8	83.8	127.5	240.6	456.4	956.9
(9.9)	4.0	6.6	11.7	22.1	43.3	88.4	189.8	434.4	1080.3	2984.7
(9.11)	0.0	1.3	3.8	8.9	19.4	41.7	92.2	214.2	536.7	1488.3

for expression (9.6)

$$\frac{\partial N_y}{\partial \varphi} = \frac{2(N_q + 1)}{\cos^2 \varphi} + 2 \tan \varphi \frac{\partial N_q}{\partial \varphi}. \quad (9.9)$$

Table 9.2 shows values belonging to various angles of friction. Column 5 of the Table refers to the factor (proposed by MUHS and WEISS)

$$N_y = (N_q - 1) \tan \varphi \quad (9.10)$$

corresponding to the DIN specifications. The derivation in this case is

$$\frac{\partial N_y}{\partial \varphi} = \frac{N_q - 1}{\cos^2 \varphi} + \tan \varphi \frac{\partial N_q}{\partial \varphi}. \quad (9.11)$$

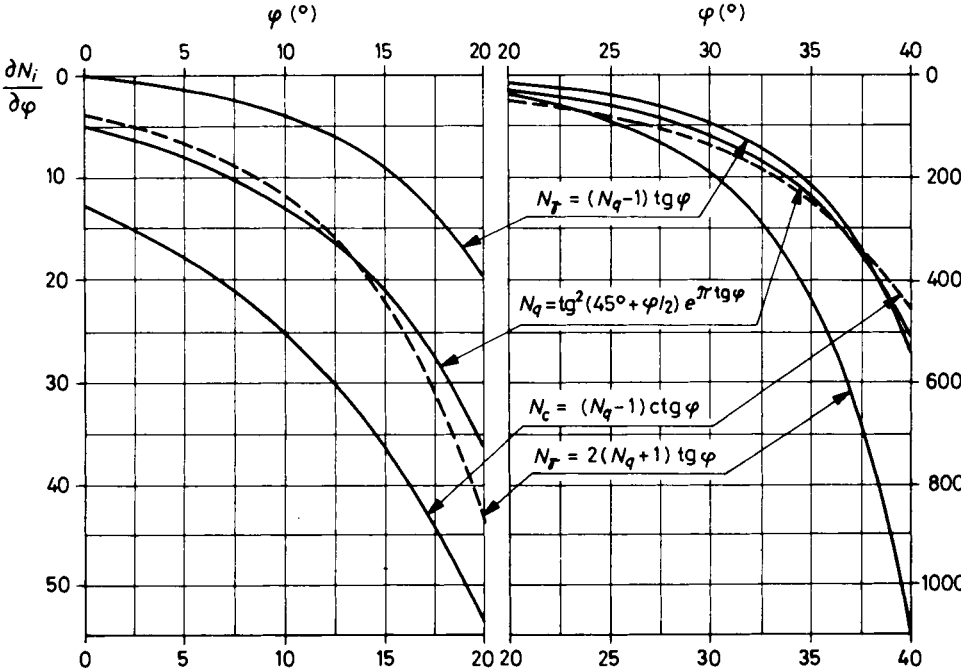


Fig. 9.3. Variation of the differential quotients according to φ of the load bearing factors

The values $\partial N_i / \partial \varphi$ belonging to angles of friction not appearing in the table may be taken from Fig. 9.3.

Numerical example. Let us calculate the permissible load Q_h kN/m which can be borne by a strip foundation of $B = 1.2$ m width, prepared with a covering of $D = 1.3$ m, for a building to which the addition of a storey is foreseen, with $p_f = 0.005$ failure probability. Let us assume the following soil physical characteristics: $\gamma_1 = \gamma_2 = 19$ kN/m³, $\bar{c} = 25$ kN/m², $s_c = 5.2$ kN/m², $\bar{\varphi} = 20^\circ$ and $s_\varphi = 2.2^\circ$ ($= 0.0384$).

The expected value of the ultimate bearing capacity is according to Eqs (9.1)–(9.3) and (9.6):

$$\bar{q} = 0.6 \cdot 19 \cdot 5.39 + 1.3 \cdot 19 \cdot 6.40 + 25 \cdot 14.83 = 590.3 \text{ kN/m}^2.$$

On substituting the values belonging to $\varphi = 20^\circ$ given in Table 9.2 and the basic data into Eq. (9.5) we obtain the following variance for the ultimate bearing capacity:

$$s_q^2 = (0.6 \cdot 19 \cdot 43.3 \cdot 0.0384)^2 + (1.3 \cdot 19 \cdot 36.4 \cdot 0.0384)^2 + (25 \cdot 53.8 \cdot 0.0384)^2 + \\ + (14.83 \cdot 5.2)^2 = 10,166,$$

so $s_q = 100.8$ kN/m².

According to Table F2 $\lambda = 2.57$ belongs to $p_f = 1 - \Phi(\lambda) = 0.005$, so the required limit stress is

$$\sigma_H = q - \lambda s_q = 590.3 - 2.57 \cdot 100.8 = 331.2 \text{ kN/m}^2$$

and the permissible load on the strip foundation is

$$Q_h = 1.2 \cdot 331.2 = 397 \text{ kN/m}.$$

The conventional factor of safety is

$$v = \frac{590.3}{331.2} = 1.78,$$

which is a realistic value, considering that the addition of a storey is involved.

(It may easily be verified that the coefficients of variation of N_γ , N_q and N_c are 0.309, 0.218 and 0.139 as compared to $C_\gamma = 0.208$ given for c and $C_\varphi = 0.11$ given for φ .)

On designing a new foundation which must bear a load Q kN/m safely, the following equation must be solved:

$$Q = B(\bar{q} - \lambda s_q). \quad (9.12)$$

As in this case $\bar{q}$ and s_q are functions of the still unknown width B , this variable must be separated from the others. If, for example, the dimensioning is performed with the help of the expressions (9.12) and (9.5), Eq. (9.12) may be written up in the following form:

$$\left(\frac{\gamma_1 N_\gamma}{2} B + E - \frac{Q}{B} \right)^2 = \lambda^2 \left[\left(\frac{\gamma_1}{2} \frac{\partial N_\gamma}{\partial \varphi} s_\varphi \right)^2 B^2 + F \right], \quad (9.13)$$

where E — sum of second and third terms on the right side of Eq. (9.1),

F — sum of the second, third and fourth terms on the right side of Eq. (9.5).

Equation (9.13) may be solved by gradual approximation, which is facilitated by the fact that the right side varies hardly at all with B . Using the data of the preceding numerical example, the width of the strip foundation loaded by $Q = Q_h = 350$ kN/m

may be determined from the following equation:

$$\left(\frac{19 \cdot 5 \cdot 39}{2} B + 528 \cdot 8 - \frac{350}{B}\right)^2 = 2 \cdot 57^2 \left[\left(\frac{19}{2} 43 \cdot 3 \cdot 0 \cdot 0384\right)^2 B^2 + 9807 \right].$$

The solution is as follows: $B = 1 \cdot 08$ m, the contact pressure, which is identical with the limit stress, is $350/1 \cdot 08 = 324$ kN/m² and the factor of safety is $\nu = 584/324 = 1 \cdot 80$.

If the variability (coefficient of variation) of the load is non-negligible, the standard deviation must be calculated according to expression (8.26).

The variability of loads may be taken into account in the following way (MCANALLY, 1983). There will be an uncertainty in the dead load Q_D due to approximation in its estimation and also to variations in the dimensions and the material density of the components of the structure. ELLINGWOOD *et al.* (1980, 1982) found that the *dead load* is approximately normally distributed, with a coefficient of variation of 0.10 for typical building structures. Also the code-determined design value was found to be approximately 0.95 of the expected value (Am. Nat. Std A58). Uncertainties in the *live load* Q_L will result from changes in the occupancy and the use of a structure during its life. ELLINGWOOD *et al.* also found that the 50 year maximum value of life load for typical building structures can be modelled as an Extreme Value of Type I distribution, with a coefficient of variation of 0.25. For USA design conditions (Am. Nat. Std A58) the design value selected will approximately equal the expected value. According to German prescriptions $C_{vD} = 0 \cdot 05$, $C_{vL} = 0 \cdot 63$; the nominal value is in the first case the expected value, in the second case the fractile of 99% (SCHULTZE and PATTHORST, 1983).

According to SCHULTZE and PATTHORST (1983) in the first step of the dimensioning q must be multiplied by a model factor K . The model factor considers variations between the calculatory ultimate bearing capacity and the test values, as reported by WEISS (1970) for 16 loading tests. Systematic variations are considered by the mean value m_K , random variations by the standard deviation σ_K . This leads to the statistical parameters of the model factor

$$m_K = 1 \cdot 37; \quad C_{vK} = 0 \cdot 165.$$

(As the bearing capacity of the test has been always greater than the calculated bearing capacity, a type of distribution was chosen permitting only values of ≥ 1 , i.e. a lognormal distribution limited up to 1.) Also MCANALLY (1983) considers the introduction of the model factor purposeful. With this method we naturally obtain a lower value for the probability of failure, as we have eliminated an uncertain factor of the dimensioning.

BIAREZ and FAVRE (1981) illustrate by a numerical example the difference between the dimensioning performed according to Level 0, I and II. Their initial data were as follows: $Q = 400$ kN/m, $B = 1 \cdot 5$ m, $D = 1$ m, $\bar{\gamma} = 20$ kN/m³, $\bar{c} = 10$ kN/m², $C_{vc} = 0 \cdot 2$; $\bar{\varphi} = 30^\circ$, $C_{v\varphi} = 0 \cdot 3$. (All the parameters were assumed to be normally distributed.) The results of their calculations for the individual levels were as follows.

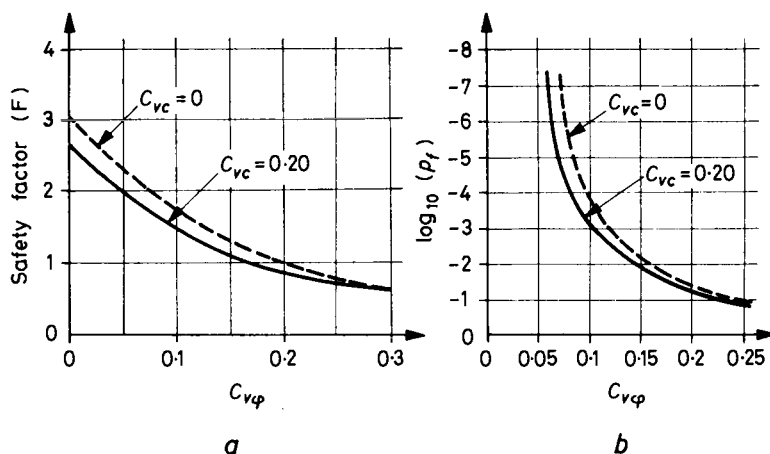


Fig. 9.4. Variation of FS (a) and $\log_{10}(p_f)$ (b) with $C_{v\varphi}$ and C_{vc} (BIAREZ and FAVRE, 1981)

Level 0. The safety factor calculated with the mean values was $F = 3.5$, i.e. greater than the prescribed $F_{cr} = 3$.

Level I. The fractile of 5% is applied for calculating φ and c and the fractile 95% for $q = Q/B$, then $q_{95} = 310$ kN/m² and the bearing capacity was

$$q(B, D, \gamma_{5\%}, \varphi_{5\%}, c_{5\%}) = 383 \text{ kN/m}^2,$$

so the safety factor was $F = 1.23 > 1$.

Level II. In this case (with the assumption of $C_{v\varphi} = 0.2$) the failure probability of was obtained as 0.02, resp. 0.04, on assuming normal distribution for the safety margin, resp. a lognormal distribution for the safety factor F .

The effect of the coefficients of variation C_{vc} and $C_{v\varphi}$ on F (Level I), resp. on p_f (Level II) is illustrated in Fig. 9.4.

9.2.2. Determination of the bearing capacity from tables

In this case two additional problems arise beside those discussed in Section 9.2.1:

- the magnitude of the contact pressure causing failure is unknown,
- the limit stress is not a continuous function of the physical characteristics.

The consequence of the former circumstance is that the *probability of failure* cannot be calculated with satisfactory accuracy. The only possibility is an approximate estimation of the ratio of the ultimate bearing capacity (q) by the limit stress (σ_H) given in the Table, i.e. an estimation of the mean safety factor (v_T) assumed during the preparation of the Table. The course of the calculation is as follows. We determine on the basis of the table a limit stress σ_{Hi} for each of the available n samples, then we

calculate the following quantities:

$$\bar{\sigma}_H = \frac{\sum_1^n \sigma_{Hi}}{n} \quad \text{and} \quad s_{\sigma_H} = \sqrt{\frac{\sum_1^n (\sigma_{Hi} - \bar{\sigma}_H)^2}{n-1}}.$$

Taking λ belonging to the value $p_f = 1 - \Phi(\lambda)$ foreseen by us from *Table F2* we obtain the limit stress σ_{Hm} forming the basis of the dimensioning as

$$\sigma_{Hm} = v_T \bar{\sigma}_H - v_T \lambda s_{\sigma_H}. \quad (9.14)$$

The first term on the right side of the formula is identical with the ultimate bearing capacity q , the second term with λ -times the standard deviation of q , so the course of the calculation is the same, as in the numerical example of Section 9.2.1.

Let us assume that we wish to determine the basic value of the limit stress (σ_a) of a cohesive layer ($I_p = 7 - 15\%$) in the function of e and I_c , in correspondence with the Hungarian standard (*Table 9.3*). Be the coherent value pairs e , I_c and the σ_a -values belonging to them as follows:

e	0.66	0.71	0.90	0.55	0.85	0.70	0.58	0.75	0.74
I_c	0.50	1.05	1.13	1.18	0.96	0.83	1.20	0.81	1.02
σ_a	220	310	288	396	243	266	392	240	293

Table 9.3. Basic limit stress value of cohesive soils (σ_a kN/m²) according to the Hungarian standard

$I_p(\%)$	e	$I_c =$		
		1.2	1.0	0.5
1 – 7	0.5	350	350	(300)
	0.7	300	250	180
7 – 15	0.4	500	400	(320)
	0.5	420	350	(300)
	0.7	350	300	200
	1.0	300	200	120
> 15	0.4	900	(750)	—
	0.6	720	(600)	—
	0.8	420	350	(200)
	1.0	350	250	120

The statistical parameters calculated from the above data are:

$$\bar{\sigma}_a = 294 \text{ kN/m}^2; \quad s_{\sigma_a} = 63.3 \text{ kN/m}^2; \quad C_v = 0.215.$$

If we assume that in *Table 9.3* the safety is $v_T = 2.5$ and we accept $p_f = 0.001$ as the probability of failure of the foundations, then according to Eq. (9.14)

$$\sigma_{am} = 2.5(294 - 3.09 \cdot 63.3) = 246 \text{ kN/m}^2.$$

The probable value of the safety against the failure is $v = 2.5 \cdot 294/246 = 2.99$.

Expression (9.14) refers also to the possibility of incurring grave errors by the use of the tables, because they do not permit to take into account the variability of the physical characteristics. So, e.g., in the case of our numerical example the assumption of $\bar{\sigma}_a = 294 \text{ kN/m}^2$ is *not equivalent* from the viewpoint of the safety, if its standard deviation is 36.3, or e.g. 95. It may be easily proved that in given cases a ruling contact pressure exceeding $\bar{\sigma}_H$ may also be assumed. The condition of this is

$$\sigma_{Hm} = v_T(\bar{\sigma}_H - \lambda s_{\sigma_H}) > \bar{\sigma}_H, \quad (9.15)$$

i.e.

$$s_{\sigma_H} < \frac{v_T - 1}{\lambda \cdot v_T} \bar{\sigma}_H.$$

If, for example, $p_f = 0.001$ (so at a normal distribution $\lambda = 3.09$) and $v_T = 2.5$, then the condition for satisfying the inequality (9.15) is

$$C_{v\sigma} < 0.19,$$

where $C_{v\sigma}$ is the coefficient of variation of the limit stress (or the basic value of the limit stress). (In the numerical example $\sigma_{v\sigma} = 0.215 > 0.19$, so $\sigma_{am} < \bar{\sigma}_a$.)

If σ_H is a function of the width B too, we must proceed similarly as described in Section 9.2.1. So, e.g., according to the Hungarian standard for cohesive soils and at $D = 0 \dots 2$

$$\sigma_H = \frac{1 + D + B}{4} \sigma_a.$$

If the load to be assumed by the foundation is Q , the required width B may be determined by the following expression:

$$\frac{Q}{B} = v_T \cdot 0.25(1 + D + B) [\bar{\sigma}_a - \lambda s_{\sigma_a}]. \quad (9.16)$$

Finally, we must examine the consequences of σ_H being a *non-continuous function* of the physical characteristics used for its determination.

(If, e.g., $I_p = 15\%$, $e = 0.9$ and $I_c = 1$, then two values may be read from Table 9.3: $\sigma_a = 233 \text{ kN/m}^2$ and $\sigma_a = 300 \text{ kN/m}^2$.) The effect of this forced "inconsistency" may be decreased by proceeding as described before, i.e. by interpreting, e.g., $\bar{\sigma}_a$ as the mean of the σ_{ai} -values.

9.3. Calculation of the bearing capacity of piles

As it is known, the soil failure under *shallow foundations* may occur in three different ways, depending on the density of the soil and the ratio D/B : a) by a general failure, b) by local failure and c) by penetration (pushing). The ultimate bearing capacity can be determined by theoretical methods only in the first case, which occurs but rarely, if piles are used. This is the reason why the conception of the bearing capacity is connected here with the settlement (penetration) of the pile, even in the case of endbearing piles.

9.3.1. Examples for calculation of the bearing capacity

If the bearing capacity of the piles is calculated from a failure theory, then the probabilistic dimensioning does not differ from the methodology elaborated for shallow foundations at all. In the following two examples, which are of interest from other viewpoints, the determination of the bearing capacity of piles is presented.

MISTÉTH (1963) studied the bearing capacity of a beam supported by $n = 14$ piles. According to the Hungarian standard the failure load is

$$Q_T = [(aH\gamma + bc)D^2 + 0.8 \cdot 4DfH]n, \quad (9.17)$$

where H — length of the pile,
 D — diameter of the pile,
 γ — bulk density of the soil,
 c — cohesion of the soil,

f, a and b — values obtained from the tables given in the standard (both a and b versus the angle of friction φ).

Let us assume that the expected values and the standard deviations of the quantities appearing in expression (9.17) are as follows:

$$\begin{aligned} \varphi &= 35^\circ & s_\varphi &= 3^\circ \\ H &= 6.0 \text{ m} & s_H &= 0.3 \text{ m} \\ \gamma &= 20 \text{ kN/m}^3 & s_\gamma &= 0.1 \text{ kN/m}^3 \\ c &= 30 \text{ kN/m}^2 & s_c &= 10 \text{ kN/m}^2 \\ f &= 28 \text{ kN/m}^2 & s_f &= 0.7 \text{ kN/m}^2 \\ D &= 0.3 \text{ m} & s_D &= 0.004 \text{ m} \end{aligned}$$

The constants are: $a = 14.8$ and $b = 20.8$.

Table 9.4. Numerical example for calculating the limit bearing capacity of a pile foundation (MISTÉTH, 1963)

x_i	$\frac{\partial Q_T}{\partial x_i}$		$\left(\frac{\partial Q_{Tsi}}{\partial x_i}\right)^2$	%-ratio
φ	$\left(H\gamma \frac{da}{d\varphi} + c \frac{db}{d\varphi}\right)D^2n$	$= 23.1$	4770	50.83
H	$(a\gamma D^2 + 3.2Df)n$	$= 74.7$	501	5.34
γ	aD^2n	$= 111.9$	125	1.33
c	bD^2n	$= 26.2$	684	7.29
f	$3.2DHn$	$= 80.6$	3183	33.92
D	$(aH\gamma + bc)2D + 3.2fH$	$= 2770$	122	1.20
Total:			9385	100.01

The components of the variance calculated from Eq. (8.32) are shown in *Table 9.4*. The last column in the Table gives the *weight* of the contribution of the quantities appearing in formula (9.17) to the standard deviation of P_T . (As it is seen, in this case the role of the angle of friction is predominant, in spite of the fact that the coefficient of variation of the cohesion is 0.33.) Taking also the terms of a higher order of Eq. (8.32) into account, the total standard deviation of the failure load is

$$s_p = \sqrt{9385 + 115.8} = 97.5 \text{ Mp} = 975 \text{ kN}.$$

On assuming $p_f = 0.001$ for the probability of failure, the limit load capacity of the pile group is

$$Q_H = 530 - 3.09 \cdot 97.5 = 229 \text{ Mp} = 2290 \text{ kN},$$

where the first term on the right side (530) is the expected value of the failure load.

HÁRS and PÖCZ (1977) examined the limit bearing capacity of *Franki piles* produced in Hungary on the basis of 162 loading tests, by which the failure loads of the piles were determined. The statistical characteristics of the failure load P_T calculated from these data are as follows:

$$\bar{P}_T = 2.892 \text{ MN}, s_p = 0.654 \text{ MN}, \quad C_v = 0.226.$$

Then they plot the density function of P_T which may be approximated by β -distribution and determine by graphical integration the load P_H belonging to the probabilities of failure $p_f = 0.0014$ and $p_f = 0.0228$. They obtained $P_H = 1.13$, resp. 1.44 MN, which would be—on assuming normal distribution— 0.93 , resp. 1.58 MN. They justified in this way the Hungarian practice of permitting a load of 1.3 MN max. on a *Franki pile*. They regard the empirical distribution function of the settlements belonging to $P = 1.44$ MN as a further proof, as namely 37% of the settlements exceeded 3 mm.

A single comment against their recommendation may be that they have examined the results of all the loading tests together, disregarding the differences existing between the physical characteristics of the soils, the lengths of the piles, etc.

9.3.2. Taking the loading tests into account using the Bayesian theory

KAY (1977) proposes to take the result of the loading tests into account with the help of the Bayesian theory.

On examining piles standing in London clay he found that the capacity values followed a lognormal distribution. So the distribution of the capacities could be characterized by an unknown mean u_{l_q} and a standard deviation σ_{l_q} . For the latter 0.08 may be accepted (but it must be noted that the number of the examined piles was only five).

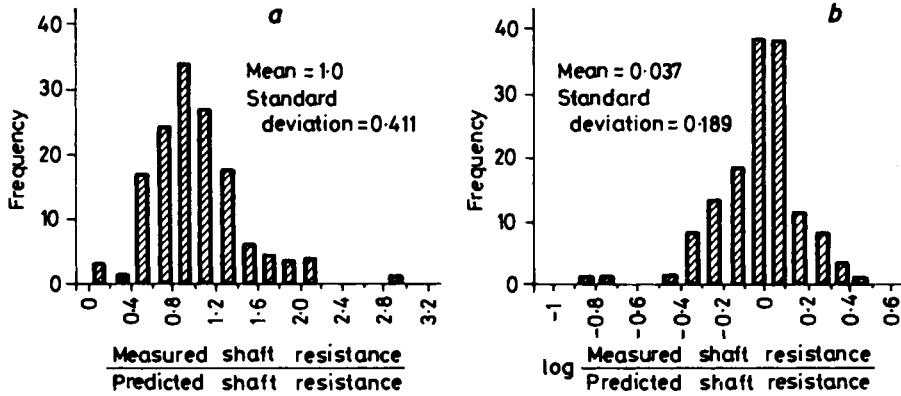


Fig. 9.5. Ratio of measured to predicted shaft shear resistance (KAY, 1977)

The bearing capacity of piles standing on clay is usually calculated from the undrained shear strength (s_u). VESIČ (1967), collecting literature data, established a relationship between s_u and the measured shaft shear resistance (C_A) of the piles. According to KAY (1977) the quotient of the measured and the calculated resistances shows a lognormal distribution (Fig. 9.5), with a standard deviation of $\bar{\sigma}_{l_q} = 0.189$.

The distribution obtained in this way is a Bayesian distribution with the mean $\bar{\mu}_{l_q}$ and the standard deviation $\bar{\sigma}_{l_q}$. The desired *a priori* distribution, which has a mean $m'_{\mu l_q}$ and a standard deviation $S'_{\mu l_q}$ can be determined by reducing the standard deviation to account for the site variability:

$$m'_{\mu l_q} = \bar{\mu}_{l_q}, \quad (9.18)$$

$$S'^2_{\mu l_q} = \bar{\sigma}_{l_q}^2 - \sigma_{l_q}^2. \quad (9.19)$$

The mean m''_q and the variance S''_q of the *a posteriori* distribution is given by

$$m''_{\mu l_q} = \frac{m'_{\mu l_q} \sigma_{l_q}^2 + S'^2_{\mu l_q} \log q_i}{\sigma_{l_q}^2 + S'^2_{\mu l_q}} \quad (9.20)$$

and

$$S''^2_{\mu l_q} = \frac{\sigma_{l_q}^2 S'^2_{\mu l_q}}{\sigma_{l_q}^2 + S'^2_{\mu l_q}}, \quad (9.21)$$

where q_i is an observation from the underlying distribution, the result of the pile load test. The estimated distribution of the pile capacity can then be represented as the normally distributed Bayesian distribution, whose parameters $\bar{\mu}_{l_q}$ and $\bar{\sigma}_{l_q}$ may be determined from Eqs (9.18) and (9.19), where $m'_{\mu l_q}$ and $S'_{\mu l_q}$ are replaced by $m''_{\mu l_q}$ and $S''_{\mu l_q}$.

A design load capacity q_d is defined in terms of the Bayesian distribution such that

$$\log q_d = \bar{\mu}_{l_q} - \beta \bar{\sigma}_{l_q}, \quad (9.22)$$

where β is the reliability index.

Now the Bayesian estimate of q , $\bar{\mu}_q$ is given by

$$\bar{\mu}_q = 10^{\left(\bar{\mu}_{l_q} + \frac{\ln 10}{2} \sigma_{l_q}^2 \right)} \quad (9.23)$$

and if the safety factor FS is defined such that

$$FS = \frac{\bar{\mu}_q}{q_d} \quad (9.24)$$

then from Eq. (9.22)

$$FS = 10^{\left(\beta \bar{\sigma}_{l_q} + \frac{\ln 10}{2} \bar{\sigma}_{l_q}^2 \right)}. \quad (9.25)$$

This means that a general tabulation of FS may be made in terms of β . This is given in Table 9.5, based on the a priori standard deviation of 0.17, derived from Fig. 9.5, b and Eq. (9.19), and the underlying variable standard deviation of 0.08.

The course of the *design* is now as follows. C_A is calculated from s_u , then the expected pile load capacity from the formula

$$q_p = 9s_u A_t + C_A A_s. \quad (9.26)$$

Choosing a β -value from Table 9.5 the design load capacity is

$$q_d = \frac{q_p}{FS}, \quad (9.27)$$

where the factor of safety corresponds to the case where no load test is conducted.

For the design after the first pile load test it is necessary to determine the statistics of the logarithmic a-priori distribution. The standard deviation $S'_{\mu l_q}$ is 0.17 and the mean $m'_{\mu l_q}$ is given by

$$m'_{\mu l_q} = \log q_p - \frac{\ln 10}{2} S'_{\mu l_q} = \log q_p - 0.033. \quad (9.28)$$

Now the pile load test value q_i may be used in conjunction with Eq. (9.20) to obtain the posterior mean $m''_{\mu l_q}$, which is the same as the mean of the new Bayesian distribution $\bar{\mu}_{l_q}$. This may be converted to the mean of the arithmetic form $\bar{\mu}_k$ [using Eq. (9.23)] and

Table 9.5. Pile capacity safety factors in terms of the reliability index β ; $\sigma_{l_q}^2 = 0.064$ (KAY, 1977)

No. of pile load tests	$S_{\mu l_q}^2$	$\bar{\sigma}_{l_q}^2$	Factor of safety			
			$\beta = 2.0$	$\beta = 2.5$	$\beta = 3.0$	$\beta = 3.5$
0		0.0361	2.6	3.3	4.1	5.1
1	0.0289	0.0116	1.7	1.9	2.2	2.5
2	0.0052	0.0093	1.6	1.8	2.0	2.2
3	0.0029	0.0084	1.6	1.7	1.9	2.1
4	0.0020	0.0079	1.5	1.7	1.9	2.1
∞	0.0000	0.0064	1.5	1.6	1.8	1.9

$\bar{\mu}_q$ now represents the updated best estimate of the pile load capacity q_p . The design capacity q_d is then obtained from Eq. (9.27). Both $\bar{\sigma}_{lq}$ and FS are obtained from Table 9.5 for one pile load test.

The procedure may be repeated for additional pile load tests by using the a posteriori parameters $m''_{\mu lq}$ and $S''_{\mu lq}$ as the new a priori parameters $m'_{\mu lq}$ and $S'_{\mu lq}$.

Numerical example. On loading two timber piles standing in clay 660 kN, resp. 370 kN was obtained for the bearing capacity; the value calculated from the undrained shear strength of the clay was $q_p = 409$ kN. Table 9.6 illustrates the calculation of the design capacity with $\beta = 3$ as reliability index. An interesting result of the calculation was that after the second load test the design value had to be decreased, which is a consequence of the heterogeneity of the soil and the random selection of the piles.

The method may be used also for piles standing on sand. KAY (1976) proposed the distinction of three groups, depending on the method used for determining the bearing capacity. The three groups and the $\bar{\sigma}_{lq}$ -values belonging to them are as follows:

— Group I ($\bar{\sigma}_{lq} = 0.17$) applies to cone penetrometers, the JANBU-formula and the DANISH-formula;

— Group II ($\bar{\sigma}_{lq} = 0.22$) corresponds to the HILEY-formula and to MEYERHOF'S (1956) SPT-formula, after which

$$q = 4N A_p + \bar{N} A_s/50, \tag{9.29}$$

Table 9.6. Computations for example problem of KAY (1977); $\beta = 3.0$, $\sigma_{lq}^2 = 0.0064$, $0.5 \ln 10 = 0.151$, forces in kN

1	2	3	4	5	6	7	8	9	10	11
No. of load test	q_i (from load test)	$\log q_i$ (from Col. 2)	$\log q_p$ (from Col. 9)	$S'_{\mu lq}$ (from Table 9.5)	$m'_{\mu lq}$ (from Eq. 9.28)	$m''_{\mu lq}$ (from Eq. 9.20)	$\bar{\sigma}_{lq}^2$ (from Table 9.5)	q_p (from Eq. 9.26)	FS (from Table 9.5)	q_d (from Eq. 9.27)
0								409	4.1	100
1	660	2.820	2.612	0.0289	2.579	2.778	0.0116	618	2.2	281
2	370	2.568	2.791	0.0052	2.778	2.693	0.0093	493	2.0	247

Table 9.7. Pile capacity safety factors in term of β , for Group I and Group II (KAY, 1976)

Number of pile load tests	Group I						Group II				
	$S'_{\mu lq}$	$\bar{\sigma}_{lq}$	β			$S'_{\mu lq}$	$\bar{\sigma}_{lq}$	β			
			2.5	3.0	3.5			2.5	3.0	3.5	
0	0.120	0.170	2.9	3.5	4.2	0.184	0.220	4.0	5.2	6.7	
1	0.085	0.147	2.5	2.9	3.5	0.101	0.157	2.6	3.2	3.8	
2	0.069	0.138	2.3	2.7	3.2	0.077	0.143	2.4	2.8	3.3	
3	0.060	0.134	2.3	2.6	3.1	0.065	0.136	2.3	2.7	3.1	
4	0.054	0.132	2.2	2.6	3.0	0.057	0.133	2.3	2.6	3.1	
10	0.036	0.125	2.1	2.5	2.9	0.037	0.126	2.2	2.5	2.9	
∞	0.000	0.120	2.1	2.4	2.7	0.000	0.120	2.1	2.4	2.7	

or for small displacement piles:

$$q = 4N A_p + \bar{N} A_s / 100; \quad (9.30)$$

— Group III ($\bar{\sigma}_{l_q} = 0.34$) meets the requirements of the Engineering News-formula.

The relationship between FS and β for the first two groups and the values $S'_{\mu l_q}$ and $\bar{\sigma}_{l_q}$ are given in Table 9.7.

9.3.3. The use of loading tests without preliminary data

KAY (1976, 1977) based his method described in Section 9.3.2 on statistics compiled from literature, so he could use the Bayesian theory with the assumption of an initial distribution. JAEGER and BAKHT (1983) examine the case where the bearing capacity must be estimated *exclusively on the basis of loading tests*. They refer to several prescriptions ruling that the bearing capacity obtained from loading tests must be multiplied by a reduction factor. For example the Ontario Highway Bridge Design Code specifies that the factored axial capacity of a pile at the ultimate limit should be taken as 0.5-times the failure load obtained from a "routine" test and 0.6-times the failure load, if the test is a "high level" test. If several load tests are made, their mean must be used for the calculation, but the specifications disregard the fact that the reliability of this mean increases gradually with the number of the load tests. JAEGER and BAKHT (1983) propose to take this circumstance into account in the following way.

Let us assume first that the distribution of the bearing capacity of the individual piles is normal. Let the frequency distribution of the pile capacities be given by a mean μ and a standard deviation σ , as shown in Fig. 9.6, a. Then the average value of a group of N piles has a mean value μ and a standard deviation $\sigma/\sqrt{N}$ (Fig. 9.6, c).

Let k_1 be the fraction by which the capacity of a single pile is multiplied to obtain the design capacity. The mean value of this quantity is $k_1\mu$ and the standard deviation is $k_1\sigma$.

Let k_N be a fraction by which the average capacity of N single piles tested is multiplied in order to obtain the single pile design capacity. The mean of the latter is $k_N\mu$ and the standard deviation is $k_N\sigma/\sqrt{N}$ (Fig. 9.6, d).

If in a given design the number of piles in a particular group is G , then the frequency distribution of the actual mean capacity of these piles will be as shown in Fig. 9.6, e, with a mean value μ and a standard deviation $\sigma/\sqrt{G}$.

The design value x_d is

$$x_d = \mu - \frac{\lambda\sigma}{\sqrt{G}}, \quad (9.31)$$

where $\lambda \geq 3$. The location of x_d is shown schematically by the dashed line in Fig. 9.6, b–e.

A design value obtained from the test data should remain—with a high degree of reliability—smaller than x_d . Using this criterion and adopting λ -times the standard

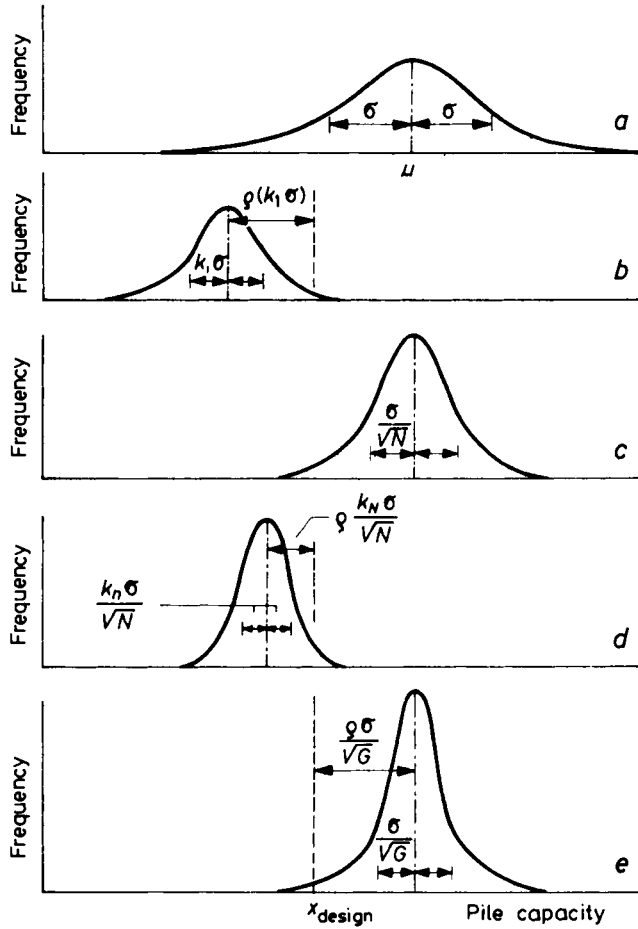


Fig. 9.6. Frequency distribution of pile capacity (a), of k_1x pile capacity (b), of the average capacity of N piles (c), of k_Nx average capacity of N piles (d), of the average capacity of G piles in a group (JAEGER and BAKHT, 1983)

deviation as appropriate level, the following equation is obtained from Fig. 9.6 for a test on *one* pile:

$$k_1\mu + \lambda(K_1\sigma) = \mu - \frac{\lambda\sigma}{\sqrt{G}}. \quad (9.32)$$

Proceeding similarly on the basis of the test of N piles, the following equation is obtained from Fig. 9.6:

$$k_N\mu + \lambda\left(\frac{k_N\sigma}{\sqrt{N}}\right) = \mu - \frac{\lambda\sigma}{\sqrt{G}}. \quad (9.33)$$

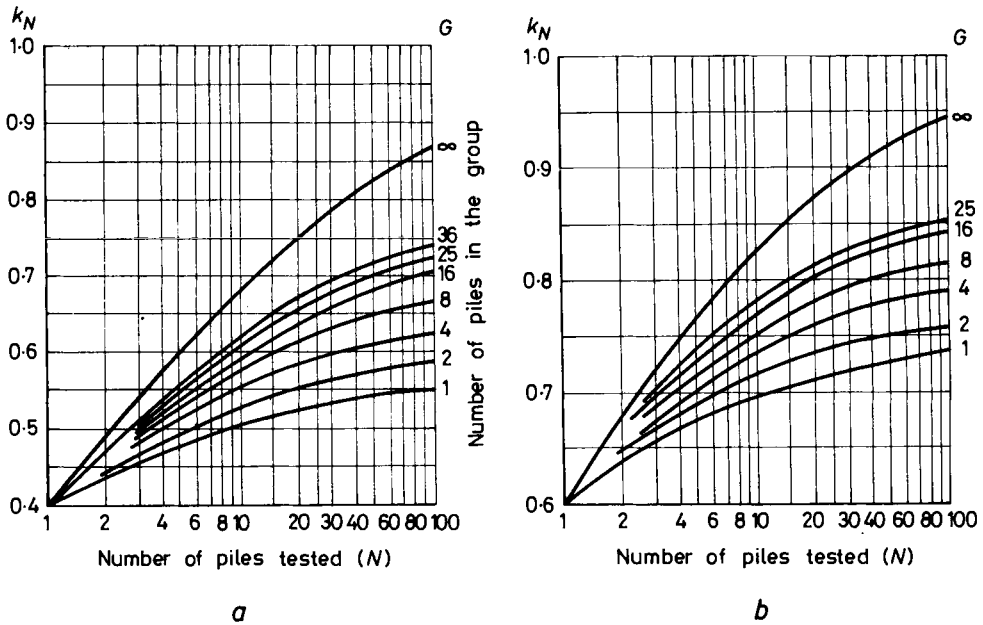


Fig. 9.7. Values of k_N corresponding to $k_1 = 0.4$ (a) and $k_1 = 0.6$ (b) (JAEGER and BAKHT, 1983)

From Eqs (9.32) and (9.33) the requested k_N -value is obtained as

$$k_N = k_1 \left[\frac{1 + \frac{1}{\sqrt{G}}}{\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{G}} - k_1 \left(1 - \frac{1}{\sqrt{N}} \right)} \right] \quad (9.34)$$

The value of k_N obtained from Eq. (9.34) is such that the probability of k_N -times the average capacity of N tests exceeding any predefined limit, x_d , is the same as the probability of k_1 times the value of a single test ($N = 1$) exceeding x_d .

Values of k_N obtained from Eq. (9.34) and corresponding k_1 values of 0.4 and 0.6 are given in Fig. 9.7.

If the pile capacities are assumably of a lognormal distribution (which is a sufficiently realistic assumption), k_N may be calculated by the following expression:

$$k_N = [k_1]^{[(1/\sqrt{N}) + (1/\sqrt{G})]/[1 + 1/\sqrt{G}]} \quad (9.35)$$

It may be easily verified that a higher k_N value is obtained from expression (9.35) than from (9.34).

9.3.4. Determination of the bearing capacity from archive data and loading tests

Loading tests are very costly (especially for piles of a high load capacity), therefore we must strive to utilize the results of earlier loading tests performed near to the actual site. Approximate values may be obtained by using piles made with an identical technology and standing in a similar soil. In some countries (so also in Hungary) the loading test results are centrally collected and stored for this purpose.

In this case we may use the Bayesian theory. The statistical parameters of the *a priori* distribution are given by the mean and the standard deviation of the earlier tests, the *a posteriori* distribution is determined with the help of loading tests performed in the examined area. The task may be solved as described in Section 7.2.2.

9.4. Economic aspects of dimensioning

The economic problems in designing foundations may be grouped around the following two questions:

- what type and how many preliminary tests were performed,
- what probabilities of failure should be assumed.

The first question is especially important, because the determination of the *failure load in situ* is a very expensive operation (e.g. think of the loading tests of shallow and pile foundations). On preparing the exploration and test plans numerous viewpoints must be taken into account: the physical characteristics of the subsoil and their variability, the importance of the object, the available financial resources, etc.

TANG (1971) presented an example of the *probabilistic decision*. Let us assume that $c = 150,000$ units—in any arbitrary currency—may be utilized for the preliminary tests of a pile foundation. The number of the point pairs required for the calibration of the blowing formula of the load capacity test is $n_2 = 10$, and the expected standard deviation in this case is $\sigma^* = 30$ t. The cost of driving a 21 m long steel pile into its place amounts to $c_2 = 500$ units, that of the loading test to $c_1 = 8000$ units. Let us assume that the distribution of the bearing capacity of the piles is normal, with an expected value of $Q_H = 120$ t and a standard deviation of $\sigma = 36$ t ($C_v = 0.3$). According to the inequality (7.34) based on the Bayesian theory (see Section 7.2)

$$\left(\frac{30}{36}\right)^2 + \frac{150,000}{150,000 - 10 \cdot 8000} < \frac{8000}{500},$$

i.e. $2.84 < 16$. This means that it is advised to decide in favour of the second alternative, performing 10 loading tests (at a cost of 85,000 units), then where

$$n_3 = \frac{150,000 - 85,000}{500} = 130,$$

indirect measurements (utilizing the calibration curve).

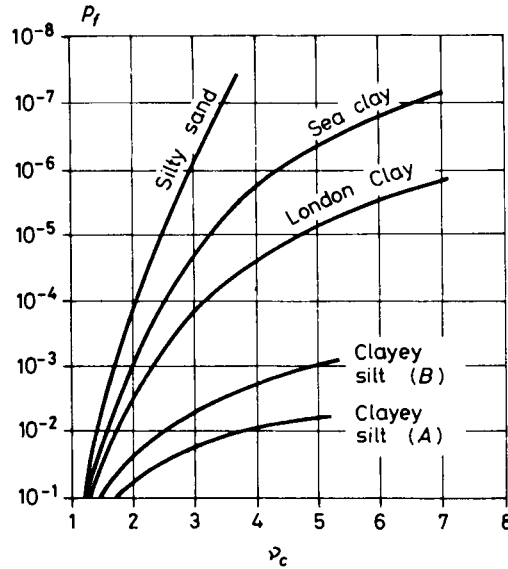


Fig. 9.8. The relationships p_f , v_c determined for the bearing capacity of soils found in the areas of Hong Kong and London (LUMB, 1966)

On selecting the *probability of failure* we must start from the fact that the soil rupture is a *very rare* phenomenon. FELD (1965) for example knew of only two such cases where the design was preceded by geotechnical tests. Later also MEYERHOF confirmed this (1970). A characteristic fact in this respect is that among about 800 damaged buildings in Hungary there was not a single one whose damage had been caused by soil rupture (RÉTHÁTI, 1968a, 1977a).

The variability of the load bearing factors is very great (see Section 9.1), due to the considerable scatter of the parameters of the shear strength (see Section 1.3.6), with the result that a relatively high probability of failure belongs to the usual values of the central safety factor ($v_c = 2 - 3$) (see Fig. 8.13). LUMB (1966) came also to similar conclusions; he determined the relationship v_c , p_f for known soils (Fig. 9.8) and established that on assuming $v_c = 3$ for the clayey silts of volcanic origin found in the Hong Kong area very high values for p_f are obtained (0.006–0.02).

From comparing the theoretical calculations with observations we may draw the conclusion that it is advised to choose $p_f \approx 0.001$, with 0.0001 and 0.01 as extreme values. Below the first value (0.0001) the design would be uneconomic with a high probability, while over the second value (0.01) the object would be exposed to an inadmissible risk. For deciding the question *numerous circumstances must be considered*, such as the importance of the object, the sensitivity of the structure, the degree of coworking between the structural elements (or the foundations), the reliability of the applied theory and, last but not least, the magnitude of the possibly occurring damage.

10. The forecasting of settlements

Although the probabilistic methods brought so far no essentially different practices as compared to the conventional methods concerning the forecast of settlement, yet they offer a multiple assistance to the solution of the related tasks. The calculation methods and the principal statements described in this chapter refer to the possibilities and the modes of applying modern basic principles in the design. This question is important, because in the last decades the deformations were regarded as determining the condition of the buildings (on the basis of conclusions drawn from their behaviour), instead of the bearing capacity.

10.1. The distribution of the vertical stresses

We are able to calculate by now rather accurately the stresses arising in the soil under the building, as proved by numerous instrumental measurements. Nevertheless it will be useful to learn the probabilistic aspects of the problem, the more so as they can be utilized also in other geotechnical fields.

10.1.1. The random walk

The random walk connected with the conception of the “random way” is observing the way covered by a “walking particle” in the function of time (or steps).

The so-called simple random walk may be interpreted on the basis of *Fig. 10.1*. Let us assume that the particle can move only along the diagonals of the square network and some random event (e.g. the tossing of a coin) determines if it gets from point x_i into $x_i + 1$, or $x_i - 1$. The basic question is the following: what is the probability $P[x | N]$ of the particle getting to point x after N steps?

This task may be studied by the model called the “*drunkard's walk*”. Let us assume that the drunkard is walking in a part of a town, consisting of mutually perpendicular streets (*Fig. 10.2*) and decides by tossing heads or tails at every street corner whether to go to the right, or to the left. On reaching the next street corner he turns south, covers the length of the street, then trusts himself again to fortune. The question is: what is the probability of his arriving from point a to his home, marked by point b ?

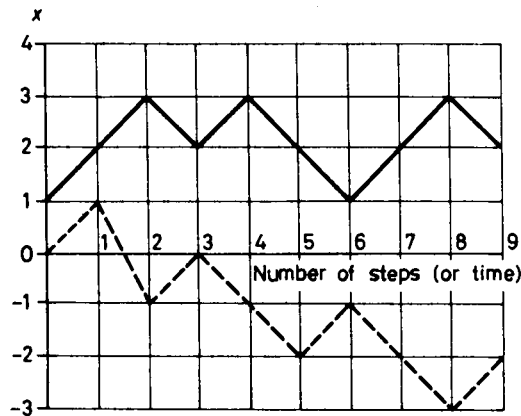


Fig. 10.1. Geometrical picture of the "simple random walk"

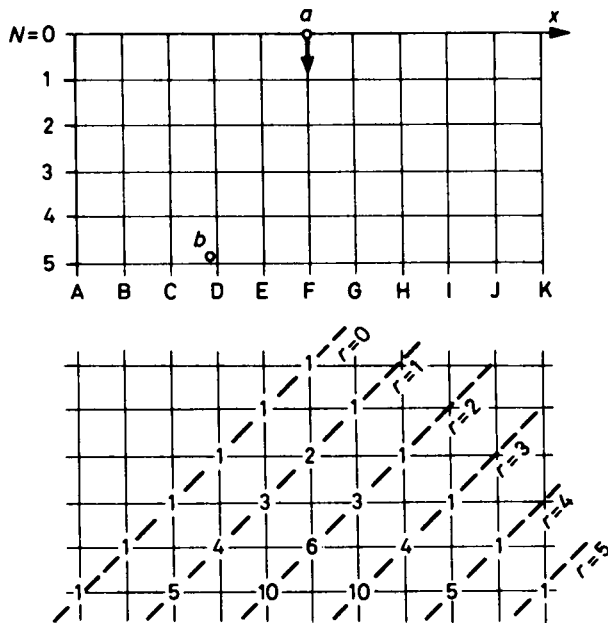


Fig. 10.2. The "drunkard's walk" model

The number of the possibilities of his arriving during his walk to a given point may be determined by PASCAL's triangle (Fig. 10.2), where a numerical value is obtained by adding the numbers standing to the right and to the left from it in the row found over it. In our case we enter the fifth row ($N = 5$) at the second position from the left ($r = 1$) so the number of combinations is 5. As the number of all the possible steps is $2^5 = 32$, the

probability of passing over the “favourable way” is

$$P[r | N] = \frac{\text{number of favourable cases}}{\text{number of all the cases}} = \frac{5}{32}.$$

In a general form the requested probability may be written as

$$P[r | N] = \frac{N!}{r!(N-r)!} \left(\frac{1}{2}\right)^N. \quad (10.1)$$

Expression (10.1) may be written also with the help of the abscissa x , if we take into account that $r = 0.5(N + x)$:

$$P[x | N] = \frac{N!}{[0.5(N+x)]![0.5(N-x)]!} \left(\frac{1}{2}\right)^N. \quad (10.2)$$

Figure 10.2 proves that the probability $P[x | N]$ can be interpreted only for the case, where $N + x$ is an even number.

Expressions (10.1) and (10.2) correspond to the special case of BERNOULLI's trial, where the probability of the occurrence of two events (i.e. turning right and left) is identical. If this is not so and there exist two probabilities: p and $q = 1 - p$, then Eq. (10.2) must be written in the following form:

$$P[x | N, p] = \binom{N}{0.5(N+x)} p^{0.5(N+x)} \cdot q^{0.5(N-x)}. \quad (10.3)$$

If the position of the points is given again by r instead of by x , the calculation may be performed instead of Eq. (10.3) by

$$P[r | N, p] = \binom{N}{r} p^r q^{N-r}, \quad (10.4)$$

which is the binomial distribution defined by Eq. (2.7), because

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}, \quad (10.5)$$

As mentioned in Section 5.1.2 of this book, the Brownian motion may be successfully studied with the help of the random walk theory.

10.1.2. Determining the stress distribution as a random walk problem

Let us examine the case of a linear unit load (e.g. 1 kN/m) acting on the surface and let us determine the distribution of the vertical stresses at the depth z . As the solution of the task based on the *theory of elasticity* is known, the question is essentially: how can we approach this task with the help of the *theory of probability* (HARR, 1977).

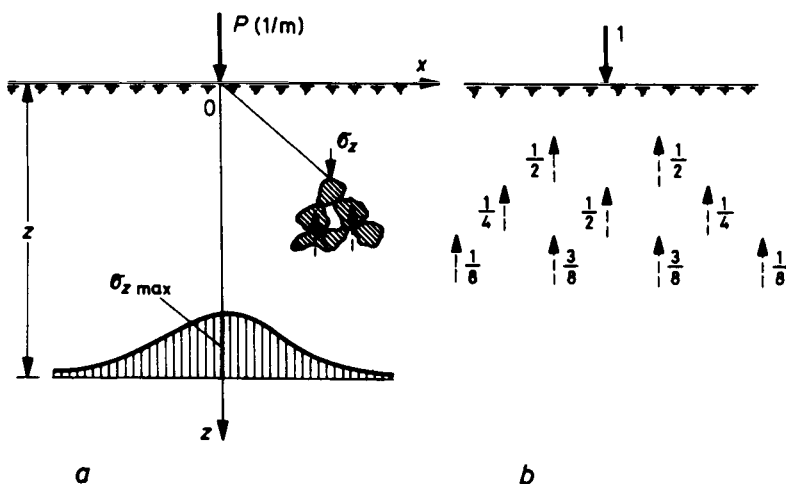


Fig. 10.3. Determination of vertical stresses developing under linear loads by the random walk theory

The stress acting on a grain of the subsoil characterized by the coordinates x, z , propagates into the positive or the negative direction of x (Fig. 10.3, detail a). If the soil is homogeneous, the propagation of the stresses may be assumed as shown in detail b of Fig. 10.3, so we have to do with the simple **BERNOULLI** trial ($p = q = 0.5$), to which the binomial distribution corresponds (see Section 10.1.1). The latter may be approximated (see Section 2.1.2) by the continuous *normal distribution* if the number of the grains is sufficiently great, i.e.

$$\sigma_z(x, z) = \frac{1}{s_z \sqrt{2\pi}} \exp \left[-\frac{x^2}{2s_z^2} \right]. \quad (10.6)$$

The standard deviation s_z is primarily a function of the depth (z), but it may reflect also the shape and the distribution of the grains, the spatial distribution and the form of the interspaces and the propagation of the forces along the grains or through them (HARR, 1977).

The standard deviation s_z appearing in Eq. (10.6) may be determined according to HARR (1977)—by simplifying its derivation—in the following way:

$$\sigma_z = \frac{2z^3}{\pi(x^2 + z^2)^2}, \quad (10.7)$$

and the maximum stress belonging to the abscissa $x = 0$ is

$$\sigma_{z \max} = \frac{2}{\pi z}. \quad (10.8)$$

Equating the maximum stress, calculated by Eq. (10.6) (belonging also to the abscissa $x = 0$)

$$\sigma_{z \max} = \frac{1}{s_z \sqrt{2\pi}}$$

with Eq. (10.8), we obtain

$$s_z = z \sqrt{\frac{\pi}{8}}. \tag{10.9}$$

So the final form of Eq. (10.6) is

$$\sigma_z(x, z) = \frac{2}{z\pi} \exp \left[-\frac{4x^2}{\pi z^2} \right]. \tag{10.10}$$

The difference between the products $z \cdot \sigma_z$ calculated from Eq. (10.10) resp. (10.7) is minimal, as proved also by the following table:

x/z	0	0.1	0.2	0.3	0.4	0.5	1	1.5	2
$z \cdot \sigma_z$ according to (10.10)	0.64	0.63	0.61	0.57	0.52	0.46	0.18	0.04	0.004
$z \cdot \sigma_z$ according to (10.7)	0.64	0.62	0.59	0.54	0.47	0.41	0.16	0.06	0.030

The task may be solved also for other peripheral conditions; the related details are given in the book of HARR (1977).

10.2. Statistical evaluation of the calculated settlements

For studying the possible applications of mathematical statistics let us start from the most frequently used formula:

$$s = \frac{Bqf}{E_s}, \tag{10.11}$$

where s — calculated settlement,
 B — width of the foundation,
 q — contact pressure,
 f — influence factor of settlement,
 E_s — modulus of compressibility of the soil.

Let us assume that we wish to forecast the settlement of foundation No. 1 seen in Fig. 10.4 and we have measured the values appearing in Table 10.1 for the modulus of compressibility. The statistical parameters calculated by the corresponding formulae of Section 1.2 are as follows:

$$\begin{aligned} \bar{E}_s &= 12.2 \cdot 10^3 \text{ kN/m}^2, & s_E &= 2.61 \cdot 10^3 \text{ kN/m}^2, & C_{vE} &= 0.214, \\ C_s &= -0.008 & \text{and} & & C_k &= -0.83. \end{aligned}$$

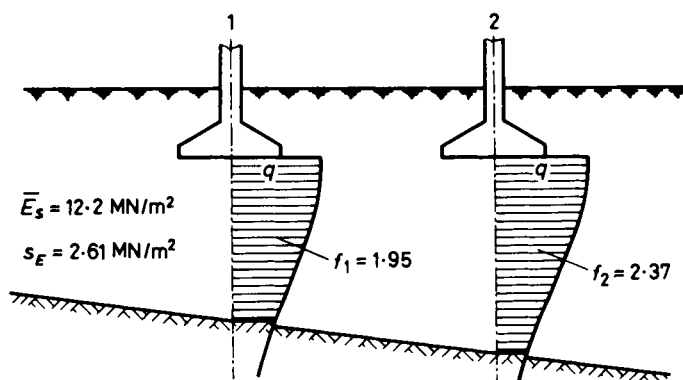


Fig. 10.4. Initial data for the numerical example of Section 10.2

Table 10.1. Initial data for the numerical example in Section 10.2

$E_s(\text{MN/m}^2)$	$s(\text{cm})$	$\sqrt{s}$	$\ln s$
14.2	3.845	1.961	1.3468
13.8	3.957	1.989	1.3755
16.3	3.350	1.830	1.2090
14.9	3.664	1.914	1.2986
9.5	5.747	2.397	1.7487
11.3	4.832	2.198	1.5753
12.4	4.403	2.098	1.4823
8.1	6.741	2.596	1.9082
9.9	5.515	2.348	1.7075
11.6	4.707	2.170	1.5491

(For distinguishing between the two notations of s here and in the following parts of Chapter 10 we shall always use a subscript with the symbol of the standard deviation.)

Column 2 of Table 10.1 shows the values of the settlement calculated from Eq. (10.11) for the 10 E_s -values. Their statistical parameters are as follows:

$$\bar{s} = 4.676 \text{ cm}, \quad s_s = 1.065 \text{ cm}, \quad C_{vs} = 0.228, \\ C_s = 0.725 \quad \text{and} \quad C_k = -0.31.$$

Accordingly the *coefficient of variation* changed hardly at all relative to C_{vE} , at the same time the s -series is *asymmetric*. (According to this example the authors who state that the distribution of E_s and s is "similar" are wrong; the same conclusion may be drawn also from Fig. 2.11.)

As according to expression (2.6)

$$\text{for } E_s \quad \beta_1 \cong 0, \quad \beta_2 = 2.17$$

and

$$\text{for } s \quad \beta_1 = 0.53, \quad \beta_2 = 2.69,$$

both characteristics have a typical reversed U-shaped *beta distribution* according to Fig. 2.2. As it is inconvenient to calculate with this, we shall carry out for s the α -transformation described in Section 2.1.6. Columns 3 and 4 of Table 10.1 contain the square roots and the natural logarithms of the s_i -values; the coefficient of skewness of the new series formed in this way is

$$C_{sg} = 0.541 \quad \text{and} \quad C_{sL} = 0.359,$$

so

$$A = \frac{C_{sg}^2}{C_{s1} \cdot C_{sL}} - 1 = \frac{0.541^2}{0.725 \cdot 0.359} - 1 = 0.1245$$

and according to Eq. (2.62)

$$\alpha_{cr} = 0.5 \pm \sqrt{\frac{1.1245}{0.4980}} = 2.003 \quad \text{and} \quad -1.003.$$

Of this two roots $\alpha = -1.003$ gives the required solution (to this belongs $C_{s\alpha} = 0$). This means that the *reciprocals* of the s_i -values give a symmetric distribution, differing from the distribution of E_s only by a constant factor according to expression (10.11) (and the coefficient of skewness is about zero indeed).

According to the above the *confidence interval* of the settlement can be calculated very simply. As the number of the measurements is relatively low ($n = 10$), we must use expression (7.37). Accordingly at the significance level of $\alpha = 0.05$

$$E_s = 12.2 \pm 1.833 \frac{2.61}{\sqrt{10}} = 12.2 \pm 1.5 \text{ MN/m}^2.$$

So we may state with a probability of 90% that the effective settlement is not less than

$$100 \frac{1.4 \cdot 200 \cdot 1.95}{13.7 \cdot 10^3} = 4.0 \text{ cm}$$

and not more than

$$100 \frac{1.4 \cdot 200 \cdot 1.95}{10.7 \cdot 10^3} = 5.1 \text{ cm}.$$

The confidence interval may be interpreted in cases, where E_s has *no horizontal trend* and- we wish to obtain numerical data for the *mean settlement* of the individual foundations, resp. its reliability. (When we wish e.g. to determine the dimensions of an elliptic aperture through a rigid foundation for transducing pipelines.)

The situation is different, if there is a definite *horizontal trend* in the modulus of compressibility. For an example let us assume that the first eight values in column 1 of Table 10.1 were measured in the axis of a rigid strip foundation at equal d distances from

each other (disregarding now that usually not even penetration tests are made so densely). The equation of the regression line determined for the individual s_i -values is

$$s = 0.346d + 3.01; \quad (r = 0.730).$$

This means that the expected settlement in point 1 is $s_1 = 3.36$ cm, in point 8 it is $s_8 = 5.78$ cm, so over the strip foundation a differential settlement of $\Delta s = 2.42$ cm will develop. We may examine with the help of Eq. (3.23) also the limits between which the regression coefficient ($a = 0.346$) may vary on a certain probability level. As in our case

$$s_f = s_s \sqrt{1 - r^2} = 1.159 \sqrt{1 - 0.73^2} = 0.792$$

and $s_x = s_d = 2.45$, the standard deviation of a is

$$s_a = \sqrt{\frac{0.792^2}{(10 - 1)2.45^2}} = 0.108.$$

So we may state with a probability of 90% that a does not exceed

$$0.346 + 1.28 \cdot 0.108 = 0.484$$

and Δs is not more than

$$(8 - 1)0.484 = 3.39 \text{ cm.}$$

Let us examine now the problems connected with the calculation of the *differential settlement of two independent foundations*.

Let us assume that in the case shown in Fig. 10.4 the 10 values of the modulus of compressibility given in Table 10.1 are at our disposal. On neglecting the third and fourth terms of Eq. (8.31), the expected value of the differential settlement is

$$\overline{\Delta s} = 100 \frac{qB}{E_s} (f_2 - f_1) = \frac{200 \cdot 1.4}{12.2 \cdot 10^{-3}} (2.37 - 1.95) = 0.96 \text{ cm.}$$

According to Eq. (8.32) the variance of the settlement of foundation No. 1—if we neglect the standard deviation of q and B in relation to that of E_s —is:

$$s_{s1}^2 = \left(-\frac{qBf_1}{E_s^2} \right)^2 s_E^2 + \left(-\frac{qBf_1}{E_s^2} \right) \frac{2qBf_1}{E_s^3} C_{sE} s_E^3. \quad (10.12)$$

Substituting the corresponding values we have:

$$\begin{aligned} s_{s1}^2 &= \left(-\frac{200 \cdot 1.14 \cdot 1.95}{12.2^2 \cdot 10^6} \right)^2 2.61^2 \cdot 10^6 - \\ &- \frac{2 \cdot 200^2 \cdot 1.4^2 \cdot 1.95^2}{12.2^5 \cdot 10^{15}} (-0.008) 2.61^3 \cdot 10^9 = \\ &= 91.67 \cdot 10^{-6} + 0.31 \cdot 10^{-6} = 91.98 \cdot 10^{-6}, \end{aligned}$$

so $s_1 = 0.00959$, i.e. 0.959 cm. On proceeding similarly we obtain for the standard deviation of the settlement of No. 2 foundation $s_{s2} = 1.164$ cm.

The differential settlement s , which may be exceeded with a probability of 10%, is

$$\Delta s = (0.0096 + 1.28 \sqrt{0.00959^2 + 0.01164^2}) 100 = 2.9 \text{ cm.}$$

Examining a living quarter UNGÁR (1974, 1975b) drew conclusions concerning the expected differential settlements with the help of *statistical tests*. There were buildings erected partly on brownish-yellow clay ($\bar{I}_p = 23\%$) and partly on dark brown clay ($\bar{I}_p = 25\%$). Though the latter was not organic ($I_{om} = 1.3-1.5\%$), the sensitivity of the structures to be built of great blocks commanded caution. As only few oedometer tests were available, he established the deformation properties of the soils on the basis of the consistency index and the void ratio, whose statistical characteristics were as follows:

- for the yellowish-brown clay $\bar{I}_c = 1.03$, $s = 0.60$, $\bar{e} = 0.68$ and $s = 0.047$,
- for the dark brown clay $\bar{I}_c = 0.99$, $s = 0.076$, $\bar{e} = 0.65$ and $s = 0.058$.

He compared the expected values and the standard deviations by F -, resp. t -tests (see Section 6.2.1) and found *no significant differences* between the consistency indexes and the void ratios of both layers on the probability level of 95%. The correctness of this conception was justified by the living houses, which had been erected since: the structures are perfectly sound in spite of their “mixed” foundations.

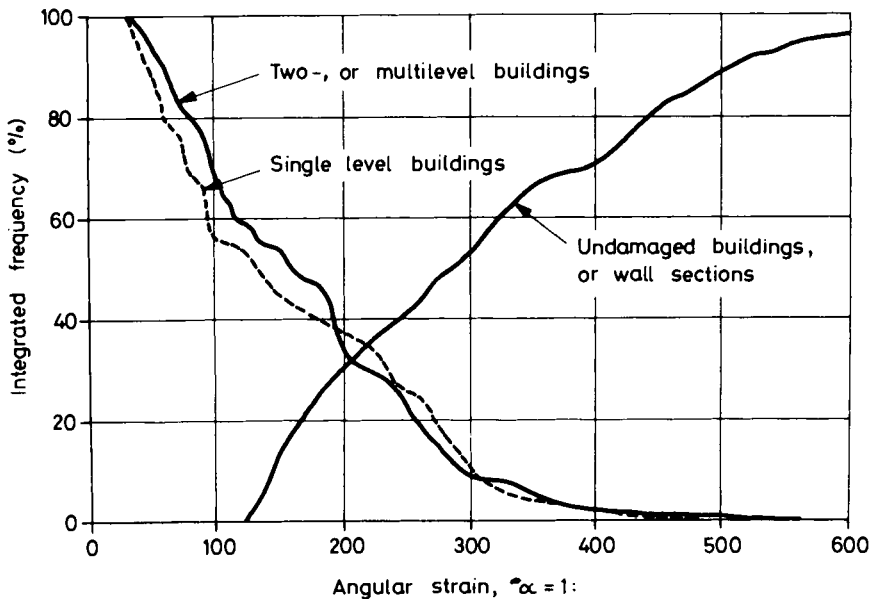


Fig. 10.5. Determination of the critical angular strain with the help of the empirical distribution functions of angular strains measured on sound and damaged buildings (walls) (RÉTHÁTI, 1977a)

In designing practice the calculated settlements are compared with the limit values of the *settlement criteria*. These are also results of statistical evaluations and supply simple methods supported by actual observations instead of the laborious and uncertain static calculations tracing the complex play of forces. The statistical evaluation of over 100 building damages in Hungary was a good occasion for assessing the limit values of one of these settlement criteria, namely the *angular strain* (RÉTHÁTI, 1977a). On plotting the empirical distribution functions of the angular strain measured in the walls of the damaged and the sound buildings, resp. walls (*Fig. 10.5*) the α -value read at the intersection of these may be regarded as the upper limit of the dangerous angular strain for the buildings.

10.3. Random motion components in homogeneous soils

The settlement of block foundations of identical surface areas, loaded identically and standing on a homogeneous soil is regarded in the conventional (deterministic) dimensioning process as *identical*.

But this assumption is very far from reality. The observations in situ prove that differential settlements develop also in such cases whose order of magnitude may be measured generally in centimeters (not in mms). The main reasons of it are as follows:

- the “homogeneity” is declared on the basis of relatively few samples, or field tests;
- the distribution of the live load, and often of the dead load too, is of a random nature (FREUDENTHAL, 1947);
- the conditions during the construction of the individual foundations are not identical (some foundation pits are soaked, or loosened, etc.);
- some foundations may be subjected to unforeseen local effects (e.g. the inflow of surface waters is not identical).

LEUSSINK (1955) studied—on the basis of literature data—buildings erected on—both horizontally and vertically—homogeneous subsoils standing on foundations of identical dimensions and of identical contact pressures. He determined for each building the ratio

$$\alpha = \frac{s_{\max} - s_{\min}}{\bar{s}} \quad (10.13)$$

and assigned them to two groups depending on the position of the compressible layer under or over a certain depth (half the strip width, or the smaller value from $B/2$ and $L/4$ in the case of block-foundations). He determined for both sets the empirical density function of the α -values and found that their mode was around $\alpha=0.3$, resp. $\alpha=0.9$. This means that the settlement of any foundation may deviate from the expected value even by $\pm 15\%$, resp. $\pm 45\%$, i.e. by a significant value.

The statistical data of LEUSSINK permit to draw the important conclusion that *if the foundation system is settling, also differential settlements may occur*. This means that the

magnitude of the average settlement must be limited even in the case of homogeneous soils, not alone for functional (e.g. pipeline connections), or other reasons, but also for protecting the buildings against undesirable uneven movements. The tolerable maximum value of $\bar{s}$ may be calculated as follows (RÉTHÁTI, 1968c).

According to the detailed statistical evaluation of SKEMPTON and MACDONALD (1956) the *permissible maximum differential settlement* on clay, resp. on sand (i.e. on slowly and rapidly consolidating soils) is

$$\Delta s_{\max} = 4.5 \text{ cm, resp. } 3.2 \text{ cm.}$$

On equating these values with those of LEUSSINK, the following equalities may be written up:

$$4.5 = 0.9\bar{s}, \text{ resp. } 3.2 = 0.9\bar{s}.$$

This means that the permissible average settlement is $\bar{s} = 5.0$ cm for clay and $\bar{s} = 3.5$ cm for sand. Similar results are obtained if we start from the other criterion of SKEMPTON and MACDONALD, namely from the maximum settlement, which may be measured within the building. Its permissible value is according to these authors 7.6 cm for clay and 5.1 cm for sand, so we obtain the following equalities:

$$7.6 = \bar{s} + 0.45\bar{s}, \text{ resp. } 5.1 = \bar{s} + 0.45\bar{s}.$$

The values obtained from these equalities for $\bar{s}$ (5.2 and 3.6 cm) are nearly identical with the previously determined critical values of $\bar{s}$.

RESENDIZ and HERRERA (1969) studied the motion components of a "solitaire" foundation of $2a \cdot 2b$ surface area, loaded by q contact pressure of a uniform distribution, in the case where the deformation characteristics of the soil have no horizontal trend. The expected displacement of the footing may be described by the following equation:

$$s(x, y) = s_0(x, y) + s_1 + \Theta_a x + \Theta_b y, \quad (10.14)$$

where $s_0(x, y)$ — deterministic part of the settlement,

s_1 — stochastic part of the settlement,

Θ_a — random tilting in direction x ,

Θ_b — random tilting in direction y ,

(the axis x is parallel with side a , y with side b).

The random components on the right side of the equation are non-zero, due to the standard deviation of the deformation characteristics, in spite of the "homogeneity" of the soil. The authors calculated with the compression ratio corresponding to expression (1.22)

$$C_r = \frac{C_c}{1 + e_0}, \quad (10.15)$$

not with E_s .

As a result of their rather extensive deduction they give the following formulae for calculating the expected value $E(\bar{s})$ of the mean settlement and its variance $\text{var}[s]$:

$$E(\bar{s}) = q \sum_{i=1}^N f_i; \quad (10.16)$$

$$\text{var}[s] = \frac{q^2}{16ab} \sum_{i=1}^N \frac{f_i K_i^2 v_i^2}{\alpha_i^2}. \quad (10.17)$$

The expected value of the *tilting* in the directions a and b and its variances are as follows:

$$E[\Theta_a] = E[\Theta_b] = 0; \quad (10.18)$$

$$\text{var}[\Theta_a] = \frac{9q^2}{16a^3b} \sum_{i=1}^N \frac{f_i^2 K_i r_{ai}^2 v_i^2}{\alpha_i^2} \quad (10.19)$$

$$\text{var}[\Theta_b] = \frac{9q^2}{16ab^3} \sum_{i=1}^N \frac{f_i^2 K_i r_{bi}^2 v_i^2}{\alpha_i^2}. \quad (10.20)$$

The factors f_i and v_i^2 may be calculated as follows:

$$f_i = \frac{\bar{C}_{ri} H_i \alpha_i}{2.3(p_i + Q)} \quad (10.21)$$

$$v_i^2 = \frac{A_0 \text{var}[C_{ri}]}{\bar{C}_{ri}^2}. \quad (10.22)$$

Notations used in the formulae:

- i — i -th lamella of the subsoil,
- H_i — thickness of the i -th layer,
- $\bar{C}_{ri}$ — mean value of the compression ratio,
- p_i — overburden pressure calculated from the foundation level,
- Q — q contact pressure + overburden pressure calculated up to the foundation level,
- A_0 — cross section area of the sample tested in the oedometer.

The values K_i , α_i , r_{ai}^2 and r_{bi}^2 may be taken from the graphs of *Fig. 10.6*.

The limit values belonging to the probability level P may be calculated as follows:

$$\left. \begin{aligned} \bar{s} &\leq q(f + \lambda\sqrt{F_0}) \\ |\Theta_a| &\leq q\lambda\sqrt{F_1} \\ |\Theta_b| &\leq q\lambda\sqrt{F_2} \end{aligned} \right\} \text{ and,} \quad (10.23)$$

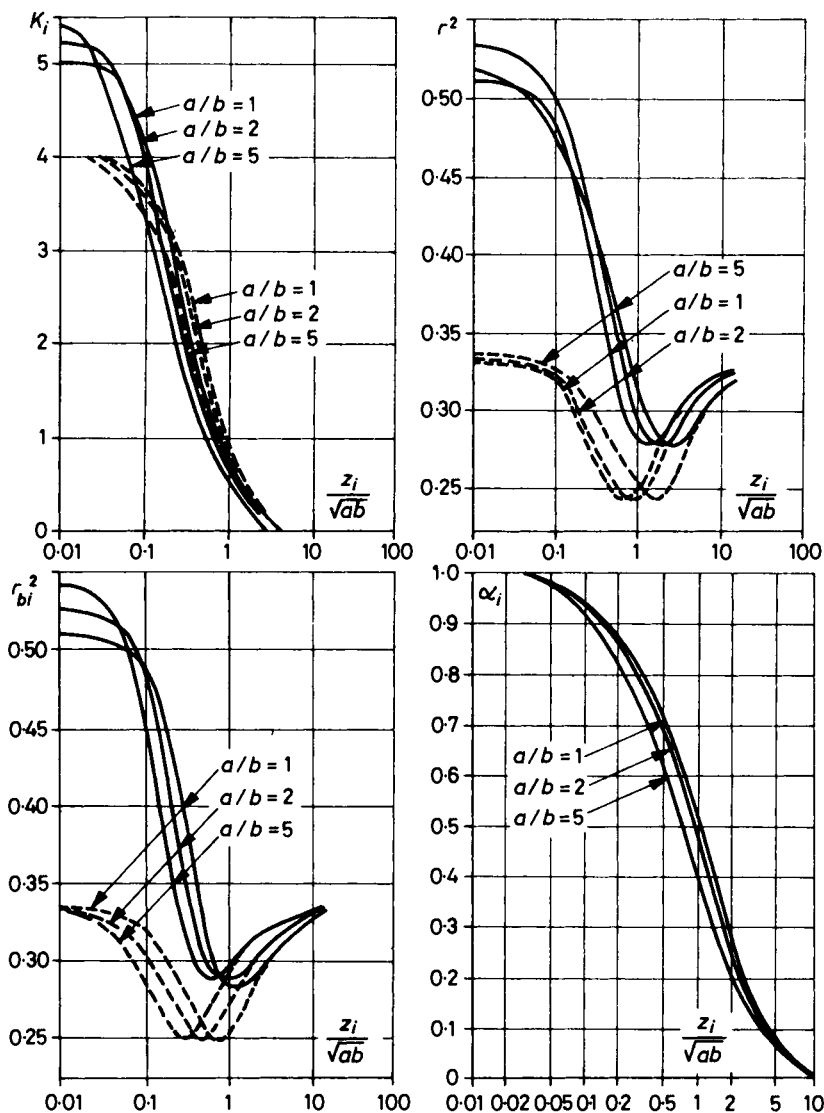


Fig. 10.6. Constants required for the calculation of the random tilting of foundations standing on a homogeneous subsoil (RESENDIZ and HERRERA, 1969)

where

$$F_0 = \frac{\text{var} [\bar{s}]}{q^2}$$

$$F_1 = \frac{9}{16a^3b} \sum_{i=1}^N \frac{f_i^2 K_i r_{ai}^2}{v_i^2} \quad \text{and}$$

$$F_2 = \frac{9}{16ab^3} \sum_{i=1}^N \frac{f_i^2 K_i r_{bi}^2}{v_i^2}.$$

The authors propose the following expression for calculating the permissible value Θ_p of the tilting:

$$\Theta_p = \sqrt{\Theta_a^2 + \Theta_b^2} = \frac{1}{100 + 3h}, \quad (10.24)$$

where h is the height of the building in m. So, e.g., for floors ($h=0$) the inclination 1 : 100 would be permissible.

It is not the analytical part of the process described above, which is of interest, as, e.g., the confidence interval of $\bar{s}$ could be calculated with other peripheral conditions as well (see Section 10.2). The principal significance of the problem is found in expressions (10.18) and (10.20) assigning a *non-zero probability* to the occurrence of an event, whose expected value is zero. The theory of probability in such cases is of a high importance, because these tasks cannot be solved by deterministic methods.

10.4. Statistical evaluation of measured settlements

The continuous instrumental measurement of the settlements (or more generally of the motion components) is of a great practical and theoretical importance. The following tasks can be solved with the help of measurements:

- controlling the calculations during the construction (for deciding on the eventually required technological or plan modifications);
- observing the development of safety in time, evaluating the movements on the basis of settlement criteria;
- collecting empirical data for designing buildings to be founded on the same soil in the future (e.g. in the case of residential and industrial quarters);
- detecting unforeseen local effects;
- demonstrating the effect of the construction on adjacent buildings;
- observing new structures, building technologies, or unusual foundation systems (e.g. for determining permissible motion components);
- determining the rate of the settlement (e.g. for deciding if the reconstruction can be started with safety, if the damaging factors are still active);
- controlling the methods of settlement forecasting.

Mathematical statistics may provide important assistance to solving most of the tasks listed above. This section deals with the extrapolation of the consolidation curve, the control of the calculation methods, the calculation of the modulus of compressibility from the measured settlements and the geometric evaluation of the tilting, from among the tasks referred to above.

10.4.1. Extrapolation of the measured consolidation curve

As it was mentioned in the introduction of Section 10.4, the development of the safety in relation to the critical settlement (differential settlement, angular strain, tilting, etc.) may be observed with the help of measurements. In given cases the rapid decrease of the safety makes necessary the knowledge of the *final value of the motion components*, for taking the required measures in the interest of the stability of the structure.

The first two of the three methods described below (RÉTHÁTI, 1971c, 1977b) may be used—strictly taken—only in the case of *one-dimensional consolidations*.

In the first case the advised process is to estimate the final settlement (s_∞) on the basis of measured results, then to calculate the supposed degree of consolidation of the settlements s_i belonging to the various t_i time moments

$$\kappa = \frac{s_i}{s_\infty} \quad (10.25)$$

The time factors T_i corresponding to the peripheral conditions belonging to the κ_i values are read from Fig. 10.7, then the quotients

$$\alpha_i = \frac{t_i}{T_i} \quad (10.26)$$

are formed.

The reliability of the settlement s_∞ may be characterized by the coefficient of variation of the α_i -values, by the quotient $C_v = s_\infty / \bar{\alpha}$. By varying the magnitude of s_∞ , its value, for which C_v is minimum, can be regarded as most probable.

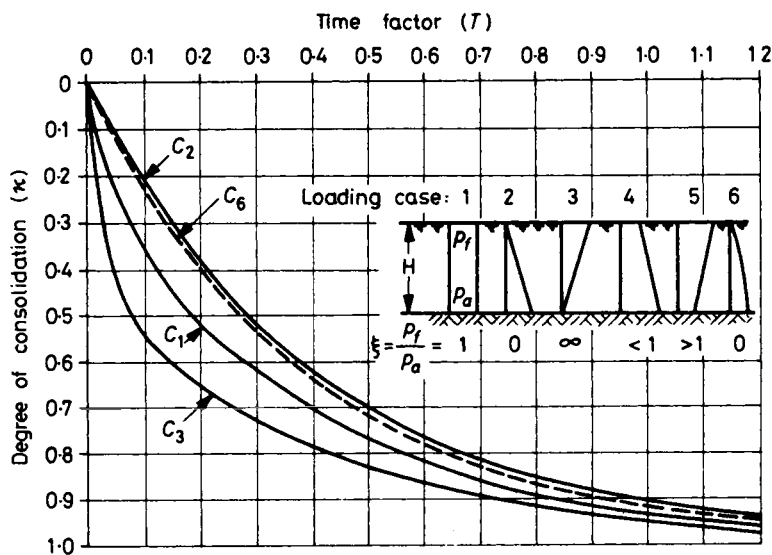


Fig. 10.7. Relationship between the time factor (T) and the degree of consolidation (κ) in the case of one-dimensional consolidation

Table 10.2. Determination of the final settlement to the numerical example in Section 10.4

s_{∞} (cm)	t_i (month): s_i (cm):	3 1.8	4.7 2.7	7.5 4.0	9.5 4.4	12 4.8	Standard deviation	Coefficient of variation
of α , resp. A								
6.0	$\alpha =$	24.0	21.4	17.6	18.6	19.0	2.58	0.128
6.6		27.3	25.4	22.1	22.4	23.5	2.20	0.091
6.8		27.3	26.1	22.1	23.8	25.5	2.05	0.082
7.0		30.0	27.6	24.2	24.7	26.7	2.34	0.088
7.2		31.6	27.7	25.0	26.8	28.2	2.42	0.087
6.0	$A =$	0.1189	0.1272	0.1465	0.1391	0.1341	0.0106	0.0799
6.8		0.1025	0.1076	0.1183	0.1096	0.1020	0.0066	0.0614
7.0		0.0991	0.1037	0.1130	0.1043	0.0965	0.0063	0.0611
7.2		0.0959	0.1000	0.1081	0.0994	0.0916	0.0061	0.0618

Let us assume that the distribution of the vertical stresses corresponds to case 5 of Fig. 10.7 and $\xi = p_f/p_a = 2$, i.e. the κ_i -values must be calculated from the following formula:

$$\kappa_i = \kappa_1 \frac{\xi - 1}{\xi + 1} (\kappa_1 - \kappa_2) = \frac{2}{3} \kappa_1 + \frac{1}{3} \kappa_2.$$

Let us assume further that the time series of the settlements s_i (cm), measured at t_i times (months) which had elapsed since the beginning of the construction corresponds to the values shown in Table 10.2. Rows 3–7 of this table contain the α_i -values belonging to $s_{\infty} = 6.6$ – 6.8 – 7.0 – 7.2 cm, their standard deviation and their coefficients of variation. Accordingly the estimated settlement belonging to the time moment $t \rightarrow \infty$ is

$$s = 6.8 \text{ cm.}$$

As in the one-dimensional case the second section of the consolidation curve may be approximated by an exponential curve, the settlements can be determined expectably also by a function of the following form:

$$s = s_{\infty}(1 - e^{-At}). \quad (10.27)$$

Here we can proceed again—since expression (10.27) cannot be linearized—by estimating s_{∞} , then calculating the A_i -values belonging to points t_i, s_i from the formula

$$A = \frac{\ln \left(1 - \frac{s}{s_{\infty}} \right)}{t}. \quad (10.28)$$

After determining the coefficients of variation belonging to the various A_i values we accept the estimated settlement, for which C_v is minimum.

According to the data given in rows 8–11 of Table 10.2 $s_{\infty} = 7.0$ cm may be accepted as the most probable solution which agrees well with the value obtained by the first method.

The time belonging to the degree of consolidation κ can be calculated very easily with the help of the factor A :

$$t_{\kappa} = - \frac{\ln(1 - \kappa)}{A}. \quad (10.29)$$

In the case of our example the mean of the A_i -values belonging to $s_\infty = 7.0$ cm is $\bar{A} = 0.1026$, so the consolidation of 95% occurs expectably in

$$t_{0.95} = - \frac{-2.996}{0.1026} = 29.2 \text{ months.}$$

The settlements belonging to $t_i = 3, 4, 7 \dots$ months, calculated in the numerical example with s_∞ - and with the A -values obtained by the second method, are compared below with the measured settlements:

s_m	1.8	2.7	4.0	4.4	4.8
s_c	1.86	2.68	3.76	4.36	4.96

(s_m = measured, s_c = calculated)

The correlation between the value pairs is very close: $r = 0.993$. This means that the expected value of the total settlement is a *very good approximation*.

It follows from the theory of consolidation that the factor A depends on the permeability of the soil, the thickness and the measure of closeness of the layer, as well as the distribution of the stresses over the depth.

SHERIF (1970) approximates the consolidation curve by a *hyperbola* and tries to trace also the variation of the load (contact pressure). The essence of his method is illustrated in Fig. 10.8. If the abscissa is the time t and the ordinate is the product

$$\frac{t}{s} \frac{p_t}{p} \quad (10.30)$$

the expected final settlement is

$$s_\infty = \cot \beta. \quad (10.31)$$

The advantage of this process is that the task can be solved by *linear regression*.

The reliability of the various prediction functions can be studied by verifying the effective measurements, by comparing the correlation indexes (I) and the residual standard deviations (s_f). From the evaluation of 30 measurement series carried out by the Institute for Geodesy and Geotechnics (Budapest), the following results were obtained,

for the exponential function

$$\bar{I} = 0.91 \pm 0.16; \quad s_f = 0.55 \pm 0.76,$$

for the hyperbolic function

$$\bar{I} = 0.93 \pm 0.14; \quad s_f = 0.52 \pm 0.59.$$

The difference is not great, but the predicted values are not identical: the quotient of the settlements s_∞ obtained by the hyperbolic and the exponential functions is

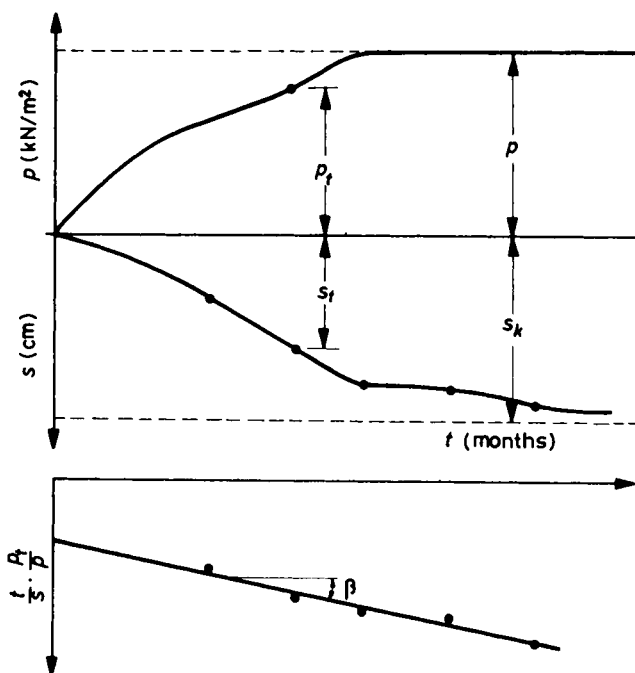


Fig. 10.8. Extrapolation of measured settlements by a hyperbolic function (SHERIF, 1970)

1.26 ± 0.14 . By using the former function the safety is increased, by using the latter function the second leg of the primary consolidation time curve may be approximated by an exponential function.

10.4.2. Statistical parameters of measured settlements

The statistical parameters of the motion components—by themselves and by their correlations—may reveal practically utilizable laws. In 1980 a large scale research, headed by the author of this book, was started with the aim of evaluating the measurements performed during 30 years at the Institute for Geodesy and Geotechnics (Budapest). The main results are summed up in the following (AL-WAKIL, 1983; RÉTHÁTI, 1981, 1985).

The settlement of the *various types* of the objects deviates from each other significantly. Table 10.3 shows the expected mean values of the settlement based on measurements. These data permit to establish the followings.

a) The settlement of silos bearing a great load is very considerable (over 700 mm was obtained for several silos). This category often caused difficulties in forecasting both the bearing capacity and the expected settlement.

Table 10.3. Statistical parameters of the settlement of various type objects

	n	$\bar{s}$ (mm)	$\bar{C}_v$
Reinforced concrete silos (> 1000 w)	22	212	0.15
Reinforced concrete silos (≤ 1000 w)	27	129	0.24
Metallic silos	25	26	0.31
Towers, stacks	17	27	0.38
Other objects	338	9	0.41

b) The average settlement of metallic silos, towers and chimney-stacks amounts to only 12% of the settlement of silos loaded by over 1000 w.

c) The residential and the industrial buildings showed an extraordinarily low average settlement. This agrees with the observation (RÉTHÁTI, 1977a) that under the Hungarian soil conditions damages may occur—assuming proper soil tests and careful construction—exclusively by exposure to external effects (water inflow to the foundations, etc.).

If we calculate the average settlement ($\bar{s}$) for all the objects and the *standard deviation* of the settlements measured at the individual pins we obtain value pairs whose regression may be determined. The equation of the regression line for reinforced concrete silos is

$$s_s = 0.07\bar{s} + 8.56 \text{ (mm)}, \quad (10.32)$$

for metallic silos it is

$$s_s = 0.29\bar{s} - 0.02 \text{ (mm)}. \quad (10.33)$$

(The correlation coefficient is $r = 0.61$, resp. $r = 0.95$.)

This means that if the absolute settlement increases, we must expect also the increase of the standard deviation, i.e. the *differential settlement*, too (compare with the statistics of LEUSSINK, Section 10.3).

From Eqs (10.32) and (10.33), resp. from the rather high value of r we may draw the conclusion that within the individual types of the objects the coefficient of variation (C_v) may be regarded as practically constant. (As every measured settlement and its standard deviation are taken into account in C_v , it is suitable to characterize the *uniformity* of the movement.) The last column of Table 10.3 contains the $\bar{C}_v$ -values calculated for the individual categories. These values are characteristic for the “average behaviour”. If the C_v -value of the examined building deviates essentially from the mean, the reason must be determined (superimposed stresses, unequal load, varying stratification, external effects, etc.).

The lowest C_p value was obtained for reinforced concrete silos. This points to the fact that their unequal movement (which means always tilting) is not great—as it was thought before—in relation to the absolute settlement. The higher value obtained for the metallic silos ($0.38 > 0.15$) proves that the superposition of the stresses plays an important role here, which is supported also by the fact that according to Eqs (10.32) and (10.33) there exists a proportionality between $\bar{s}$ and s_x ($b = -0.02$ mm merely) and not a simple linear relationship.

According to the measurements the heavily loaded *silos* show a significant absolute settlement and tilting. Their movement can be characterized purposefully by the following statistical parameters:

— average settlement ($\bar{s}$, mm), i.e. the mean value of the settlements measured on the pins;

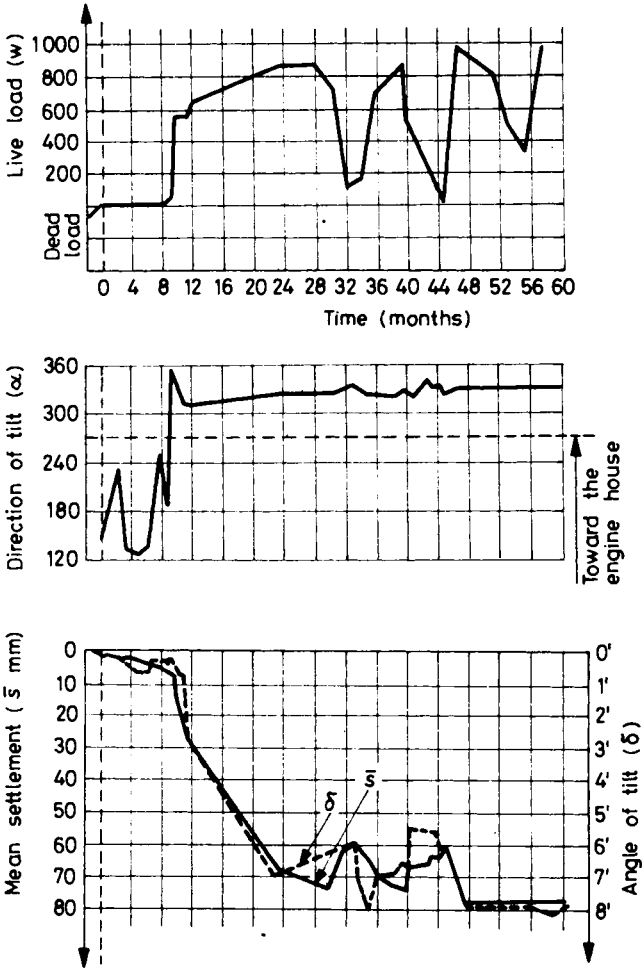


Fig. 10.9. Synoptic presentation of the movement component of a silo

- the angle α enclosed by the line of fall of the foundation and the longitudinal axis of the silo;
- the angle δ enclosed by the axis of the silo and the vertical, i.e. the value of the tilting expressed in degrees.

These statistical parameters vary constantly during the construction and the operation of the silo. As all three parameters have a significance, it is advised to draw them at identical t_i times, along with the variation of the load. Such a "synoptic" system of relationships is shown in Fig. 10.9, permitting to state the followings.

a) Under the effect of the dead load the silo tilts toward the *engine house*, because the foundation plate has an eccentricity in this direction. This tendency could be observed in the case of 78% of the silos; a consequently opposite component appeared only in 13% of the cases. After charging the silos the original direction was kept in 22% out of 78%, resp. in 4% out of 13%. In 9% of the direction oscillated both before and after the charging of the silos.

b) The line of the tilting follows relatively well the line of the *average settlement*. This tendency can be demonstrated also at the other silos (the closeness of the correlation may be characterized by $\bar{r}=0.7$).

For the three statistical parameters and the maximum tilting angle δ of reinforced concrete silos loaded by over 1000 w the following values were obtained, depending on the type of the soil:

Soil type	$\bar{s}$ (mm)	$\bar{s}_s$ (mm)	$\bar{C}_v$	δ
Granular	80.6 ± 61.8	94 ± 5.7	0.16 ± 0.09	$3'28'' \pm 2'10''$
Cohesive	231.5 ± 183.5	24.1 ± 17.4	0.16 ± 0.12	$7'25'' \pm 4'40''$

These values demonstrate the greatly deviating behaviour of both soil types and permit to predict—in the knowledge of the components of the silo's movement—if its behaviour will be more or less favourable than the average.

10.4.3. The control of the calculation models

The comparison of the calculated and measured settlements proves that the methodics of forecasting the settlement are not sufficiently developed as yet. Numerous research workers made suggestions for perfecting the models used in practice, by weighting the reliability of the individual elements of the model (stress distribution, limit depth, etc.).

A part of the investigations was connected with the settlement of buildings standing on *sand*. As the oedometer test gives no reliable results for granular soils and the

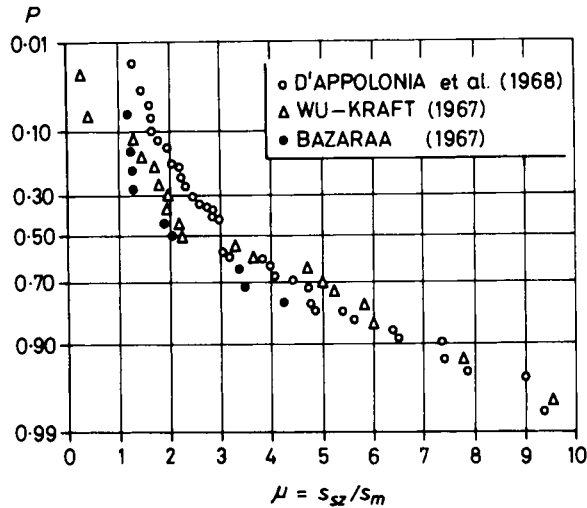


Fig. 10.10. Empirical distribution of the quotient μ of the settlements calculated on the basis of SPT-tests and those measured actually (WU, 1974)

loading test in situ is expensive, the most purposeful solution seems to be the statical or dynamical penetration test. TERZAGHI and PECK (1948) state that the contact pressure q_c which causes a settlement of 1 inch max. may be calculated on the basis of the standard penetration test (SPT) by the following formula:

$$q_c = 0.36(N-3) \left(\frac{B+1}{2B} \right)^2, \quad (10.34)$$

where N — number of blows required for a penetration of 30 cm,
 B — width of the foundation, expressed in feet.

WU (1974) collected numerous literature data for comparing the settlement calculated by formula (10.34) with the measured settlement. He calculated the former with the assumption that the effective contact pressure q_m is lower than the limit of proportionality, so $s_c = q_m(q_c)$. Figure 10.10 shows the empirical distribution functions of the quotients

$$\mu = \frac{s_c}{s_m}.$$

Accordingly the conventionally interpreted *mean* safety contained in the formula of TERZAGHI and PECK is about $\bar{\mu} = 3$. On determining the analytical form of the distribution function we may estimate the probability that the settlement will not exceed the threshold value determined by us.

D'APPOLONIA *et al.* (1968) mention that the coefficient of variation of the settlements calculated on the basis of penetration test was 0.3, that of the effective settlements was 0.4, so the main cause of the dispersion was the heterogeneity of the soil tested by them.

SCHULTZE and SHERIF (1973) went a step further and looked for a new relationship of a more general validity between N and s . For most of the 48 buildings whose data were processed by them, SPT-tests were performed and where static probes were used, they estimated N by SCHMERTMANN's formula. Four of the five constants appearing in their preliminary formula were determined by *multivariable correlation* and the fifth—which they treated as a *random number*—was taken to be identical with the value supplying the maximum correlation coefficient. They wrote up the final formula in the following form:

$$s(\text{cm}) = \frac{p \cdot f}{1.71 \cdot N^{0.87} \sqrt{B/B_1}} \frac{1}{(1 + 0.4t/B)}, \quad (10.35)$$

where p — $1/100$ of the contact pressure, expressed in kN/m^2 , not decreased by the overburden pressure,

f — influence factor of the settlement (e.g. according to KANY),

B — width of the foundation, cm ($B_1 = 1$ cm),

t — depth of the foundation, cm.

(The multiple correlation coefficient was 0.938!)

The formula is related to layer thicknesses of $d_s \geq 2B$. If $d_s < 2B$, the calculated settlements must be multiplied by the following reduction factors, depending on the L/B ratio (where L is the length of the foundation):

d_s/B	$L/B =$			
	1	2	5	100
1.5	0.91	0.89	0.87	0.85
1.0	0.75	0.73	0.69	0.65
0.5	0.52	0.48	0.43	0.39

According to expression (10.35) s varies linearly with p , which means that the bedding coefficient of the soil is not a function of the contact pressure.

On the basis of *Fig. 10.11* illustrating the relationship between the settlements calculated by expression (10.35) and the effective settlement values we may state that the error of the forecasts was less than 40% in the majority of the cases.

On studying the standard deviation of the ratios of the calculated and measured settlement values we must try to *differentiate* between them *according to physical or other characteristics* playing a role in calculating the settlement. A good example for such a differentiation is the evaluation of the settlement measured in the area of the Danube Iron Works (EGRI and RÉTHÁTI, 1959, 1960). The result $\bar{\mu} = 4.3$ obtained for 105 foundations standing on macroporous loess shows that a realistic prognosis of the settlement cannot be given for this soil on the basis of oedometer tests. At the same time it was striking that the highest μ -values belonged to foundations with extensive surface

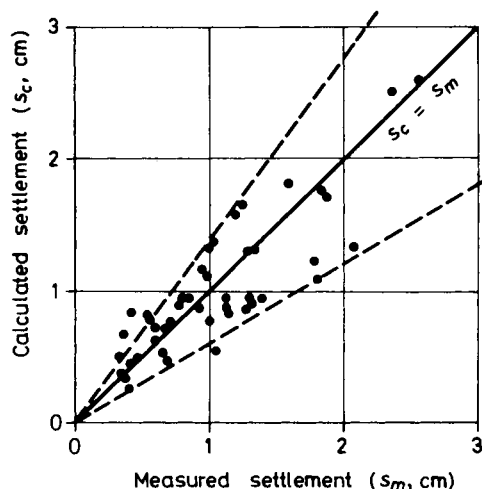


Fig. 10.11. Comparison of the actually measured settlements with those calculated from the SPT-tests with the help of Eq. (10.35) (SCHULTZE and SHERIF, 1973)

areas. On grouping the measured results according to the square root of the surface area of the foundations ($\sqrt{F}$), this observation could be demonstrated also in the form of a numerical correlation (Fig. 10.12).

The primary cause of this phenomenon is that the *limit depth* of the stresses arising under the foundations standing on a homogeneous soil is less than it was assumed by

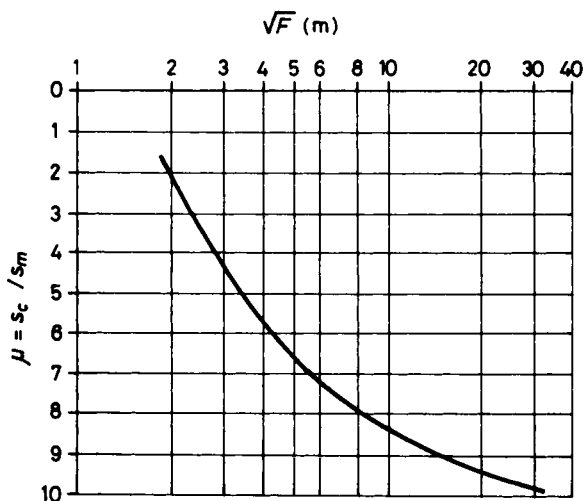


Fig. 10.12. Quotient of the settlement of foundations standing on macroporous loess, calculated on the basis of oedometer tests by the measured settlement versus the square root of the basic area of the footings (EGRI and RÉTHÁTI, 1959, 1960)

us. Most standards recommend to accept the depth $z = m_0$, where the vertical stress σ_z caused by the load of the foundations decreases to 10% or 20% of the overburden pressure. The measurements performed in situ during the last decades prove that with the increase of $\sqrt{F}$ the effective limit depth deviates—in the negative sense—ever more from this value, especially in the case of less compressible layers.

From among the analytical solutions based on measurements of the settlement the formula of ALTES (1976) deserves attention. This author describes the limit depth by the following empirical relationship:

$$m_0/B = f(\sigma_0, F, E_s, B/L), \tag{10.36}$$

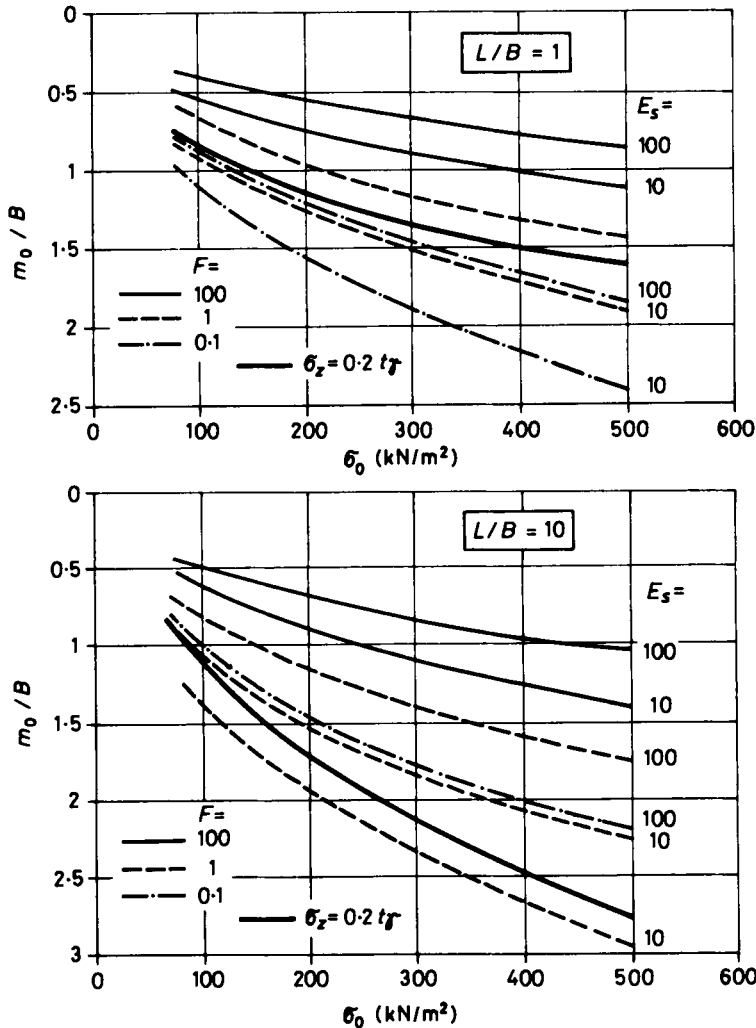


Fig. 10.13. Depth limit of the vertical stresses (m_0) in the function of the contact pressure (σ_0), the surface area of the foundation (F , m^2) and the ratio L/B of the sides (ALTES, 1976)

where B — shorter side of the foundation,
 L — longer side of the foundation,
 F — surface area of the foundation,
 σ_0 — mean contact pressure arising under the foundation,
 E_s — modulus of compressibility of the soil.

The author determined the formula (10.36) by *multivariate regression analysis* based on 132 measured settlement values found in literature and seven additional model tests. The multiple correlation coefficient (see Section 3.5.1) of the rather complex relationship was 0.82. (This value is sufficiently high for applying the formula also in practice.) *Figure 10.13* constructed on the basis of the formula for three different F and two different E_s -values each, permits to state that

- the limit depth decreases with the increase of E_s ,
- the quotient m_0/B decreases with the increase of B ,
- the limit depth defined by the condition $\sigma_z = 0.2 \text{ } t\gamma$ deviates from the value calculated by the formula the more, the smaller the B/L ratio and the greater the surface area F .

Table 10.4. m_0/B -values of footings with a side proportion of $A/B = 1$, according to ALTES (1976)

F (m ²)	E_s (MN/m ²)	σ_0 (KN/m ²)					
		90	100	200	300	400	500
0.1	2.5	0.99	1.38	1.92	2.32	2.66	2.96
	10.0	0.81	1.13	1.56	1.90	2.17	2.42
	100.0	0.61	0.85	1.18	1.43	1.64	1.82
	200.0	0.57	0.79	1.10	1.33	1.52	1.69
0.5	2.5	0.84	1.17	1.63	1.97	2.26	2.51
	10.0	0.69	0.96	1.33	1.61	1.85	2.05
	100.0	0.52	0.72	1.00	1.22	1.39	1.55
	200.0	0.48	0.67	0.93	1.12	1.29	1.44
1.0	2.5	0.79	1.09	1.51	1.84	2.10	2.34
	10.0	0.64	0.89	1.24	1.50	1.72	1.91
	100.0	0.48	0.67	0.93	1.13	1.30	1.44
	200.0	0.45	0.62	0.87	1.05	1.20	1.34
5.0	2.5	0.66	0.92	1.27	1.54	1.77	1.97
	10.0	0.54	0.75	1.04	1.26	1.45	1.61
	100.0	0.41	0.56	0.78	0.95	1.09	1.21
	200.0	0.38	0.52	0.73	0.88	1.01	1.12
10.0	2.5	0.61	0.85	1.18	1.43	1.64	1.82
	10.0	0.50	0.69	0.96	1.17	1.34	1.49
	100.0	0.38	0.52	0.73	0.88	1.01	1.12
	200.0	0.35	0.49	0.67	0.82	0.94	1.04
100.0	2.5	0.47	0.65	0.90	1.10	1.26	1.40
	10.0	0.38	0.53	0.74	0.90	1.03	1.14
	100.0	0.29	0.40	0.56	0.68	0.77	0.86
	200.0	0.27	0.37	0.52	0.63	0.72	0.80

Table 10.5. m_0/B -values of footings with a side proportion of $A/B=10$ according to ALTES (1976)

F (m^2)	E_s (MN/m^2)	$\sigma_0(\text{kN/m}^2)$					
		50	100	200	300	400	500
0.1	2.5	1.19	1.65	2.30	2.79	3.19	3.55
	10.0	0.97	1.35	1.88	2.28	2.61	2.90
	100.0	0.73	1.02	1.42	1.72	1.97	2.19
	200.0	0.68	0.95	1.32	1.59	1.83	2.03
0.5	2.5	1.01	1.41	1.95	2.37	2.71	3.02
	10.0	0.83	1.15	1.60	1.93	2.22	2.46
	100.0	0.62	0.87	1.20	1.46	1.67	1.86
	200.0	0.58	0.80	1.12	1.35	1.55	1.73
1.0	2.5	0.94	1.31	1.82	2.20	2.52	2.81
	10.0	0.77	1.07	1.48	1.80	2.06	2.29
	100.0	0.58	0.81	1.12	1.36	1.55	1.73
	200.0	0.54	0.75	1.04	1.26	1.44	1.61
5.0	2.5	0.79	1.10	1.53	1.85	2.12	2.36
	10.0	0.65	0.90	1.25	1.51	1.73	1.93
	100.0	0.49	0.68	0.94	1.14	1.31	1.45
	200.0	0.45	0.63	0.87	1.06	1.21	1.35
10.0	2.5	0.73	1.02	1.42	1.71	1.97	2.18
	10.0	0.60	0.83	1.16	1.40	1.61	1.78
	100.0	0.45	0.63	0.87	1.06	1.21	1.35
	200.0	0.42	0.58	0.81	0.98	1.12	1.25
100.0	2.5	0.56	0.78	1.09	1.32	1.51	1.68
	10.0	0.46	0.64	0.89	1.07	1.23	1.37
	100.0	0.35	0.48	0.67	0.81	0.93	1.03
	200.0	0.32	0.45	0.62	0.75	0.86	0.96

The numerical values of the quotient m_0/B are given for $L/B=1$ in Table 10.4 and for $L/B=10$ in Table 10.5.

SCHULTZE and SIEVERING (1977) compared various calculation methods by evaluating the measured settlement values of 148 buildings, 112 of which were standing on plate foundations, 29 on block foundations and 7 on strip foundations. The dimensions of the foundations varied between 1.1 m and 60.100 m, the contact pressure between 20–1263 kN/m². The modulus of compressibility of the soils was determined by dynamic (SPT) and static penetration tests, resp. by oedometer tests, as stated in the related publications. The limit depth was assumed to equal the width of the plate foundations and its double in the case of block foundations. The settlements were calculated by 10 different methods. In the first three cases $E_s=\text{const.}$ was assumed, accepting the influence factor of the settlement with the following combinations of FRÖHLICH's coefficient of concentration (ν) and POISSON's ratios (μ): $\mu=0$, $\nu=3$; $\mu=0$, $\nu=5$; $\mu=0.3$, $\nu=3$. In the other three cases they proceeded similarly, but assumed that

Table 10.6. Mean, standard deviation and coefficient of variation of the ratio μ of the effective (measured) and the calculated settlements (by various calculation methods) (SCHULTZE and SIEVERING, 1977)

Symbol of the calculation method	For all the foundations			For slab foundations			For block foundations		
	$\bar{\mu}$	s	C_v	$\bar{\mu}$	s	C_v	$\bar{\mu}$	s	C_v
1	1.82	1.43	0.79	1.99	1.54	0.78	1.51	0.79	0.52
2	2.16	1.66	0.77	2.28	1.78	0.78	2.07	1.05	0.51
3	1.49	1.15	0.77	1.60	1.24	0.78	1.31	0.69	0.52
4	1.41	1.12	0.79	1.56	1.21	0.77	1.05	0.52	0.50
5	1.63	1.27	0.78	1.78	1.37	0.77	1.36	0.66	0.49
6	1.13	0.89	0.78	1.24	0.96	0.77	0.89	0.44	0.50
7	3.71	3.37	0.91	4.30	3.64	0.85	2.05	1.05	0.51
8	1.70	1.45	0.85	1.32	1.58	0.82	1.06	0.49	0.46
9	0.55	0.49	0.90	0.64	0.53	0.82	0.35	0.16	0.46
10	2.02	1.75	0.87	2.31	1.90	0.82	1.27	0.59	0.46

the modulus of compressibility increased linearly with the depth. The calculations 7–10 were made on the basis of the following formulae:

$$s = 3.5 \frac{pR_H}{E_s}; \quad (10.37)$$

$$s = \frac{L \cdot B \cdot p}{E_s(L + B)}; \quad (10.38)$$

$$s = 2.3 \frac{N}{2E_s(L - B)} \log \frac{2B + z}{2E_s + z} \cdot \frac{L}{B}; \quad (10.39)$$

$$s = \frac{0.6pB}{E_s}. \quad (10.40)$$

The means of the μ -ratios of the calculated by the measured settlements determined by various calculation methods varied between 0.55 and 3.71. Within the individual groups the μ values were of POISSON-distribution, with means and standard deviations depending on the calculation method. On grouping the foundations by types they found that the POISSON-distribution was valid for the plate foundations and the normal distribution for the block foundations. An interesting result of the grouping was also that the coefficient of variation of the μ values decreased (Table 10.6).

The correlation coefficients of the relationships

$$s_m = a + bs_c$$

determined for the measured and calculated settlements varied between $r=0.81$ and $r=0.87$ and as $a \sim 0$, a correction factor can be assigned to each calculation method (see Table 10.6).

10.4.4. The posterior determination of the modulus of compressibility

A group of information gained from the measurement of the settlements supply numerically evaluable data on the *deformation characteristics of the subsoil*. The calculation method must be selected in correspondence with the subsoil conditions (*stratification*) and with the *measurement method of the settlement*. The latter may be

a) a continuous instrumental measurement started at the beginning of the building operations, or at the time of the supervision of the building,

b) levelling for determining the relative settlement line of the initially horizontal structural element (e.g. footings), carried out in the existing building.

The following examples present the selection of the *calculation model* depending on the soil conditions and the measurement method (RÉTHÁTI, 1980a, 1981).

Let us consider first the case of the *homogeneous stratification*.

If the *load* and the *surface area* of the foundations are identical, it is advised to select the process described in the example below. The settlement of one of the work halls at the Danube Iron Works (Hungary) was measured from the beginning of the construction. The mean of the settlements measured on 21 columns was $\bar{s} = 27.0$ mm, the standard deviation was $s_s = 4.22$ mm. The confidence intervals of the modulus of compressibility E_s may be determined with the help of the STUDENT-distribution. The confidence limits of the settlements belonging to $p = 0.05$ probability level and $n = 20$ degrees of freedom are according to Eq. (7.37) and Table F4:

$$s = 27 \pm 2.086 \frac{4.22}{\sqrt{20}} = 27 \pm 1.92 \text{ (mm)}.$$

If the area of the stress diagram of the congruent and identically loaded foundations is F , then the following inequality may be written up for the expected value of E_s :

$$\frac{F}{27 + 1.92} \leq E_s \leq \frac{F}{27 - 1.92}.$$

On summarizing the stresses up to the limit depth given by JÁKY, arising under the characteristic point of the foundations of $4 \cdot 2.5 \text{ m}^2$ surface area, transferring a load of 145 kN/m^2 (after subtracting the dead weight), $F = 298 \text{ N/mm}$ is obtained. So we may state with a probability of 95% that the modulus of compressibility of the loess, forming the subsoil is between $10.3 - 11.9 \text{ MN/m}^2$.

If the foundations are *differently loaded* and/or they have *different surface areas*, one of the following two methods may be applied:

a) E_s is calculated for every foundation to be measured and the reliability of their mean is characterized by the *standard deviation*.

b) A correlation is established between the area of the stress diagrams and the settlements (see the subsequent examples).

The possible contribution of the rigidity of the superstructure to the development (distribution) of the settlements must also be taken into account.

The second category consists of cases where the compressibility of one of the layers is exceedingly great. In this case the modulus of compressibility may be determined by correlating the settlement with the stresses falling to this layer and neglecting the deformation of the other layers. The calculation model must be selected so as to be able to write up one or more equations for the calculation of E_s .

The $B=0.45 \text{ m}^2$ wide strip foundations of the damaged building seen in Fig. 10.14, transferring a load of 146 kN/m^2 , are standing in the vaulted part of the building on sand of a high bearing capacity, in the other parts of the building on clay interwoven with turf veins ($\bar{w}=53\%$, $w_p=38\%$). As testified by the levelling of the footing the differential settlement was after 2.5 years since the construction $\Delta s=2.7 \text{ cm}$. According to the preceding paragraph

$$E_s = \frac{F}{s} \cong \frac{F}{\Delta s} = 3.14 \text{ MN/m}^2$$

where $F=(0.0848 \text{ MN/m})$ is the area of the stress diagram.

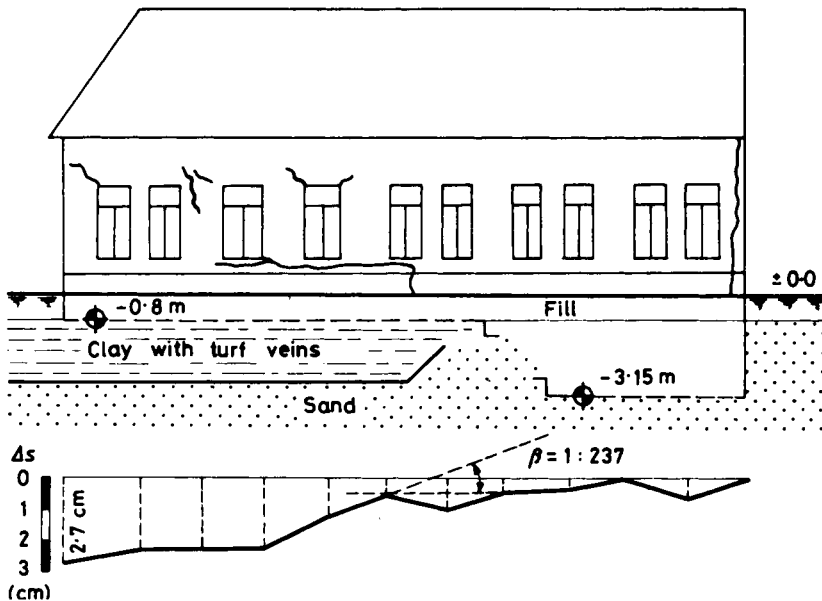


Fig. 10.14. Posterior determination of the modulus of compressibility from the deformation line of sockets in the case of a single compressible layer

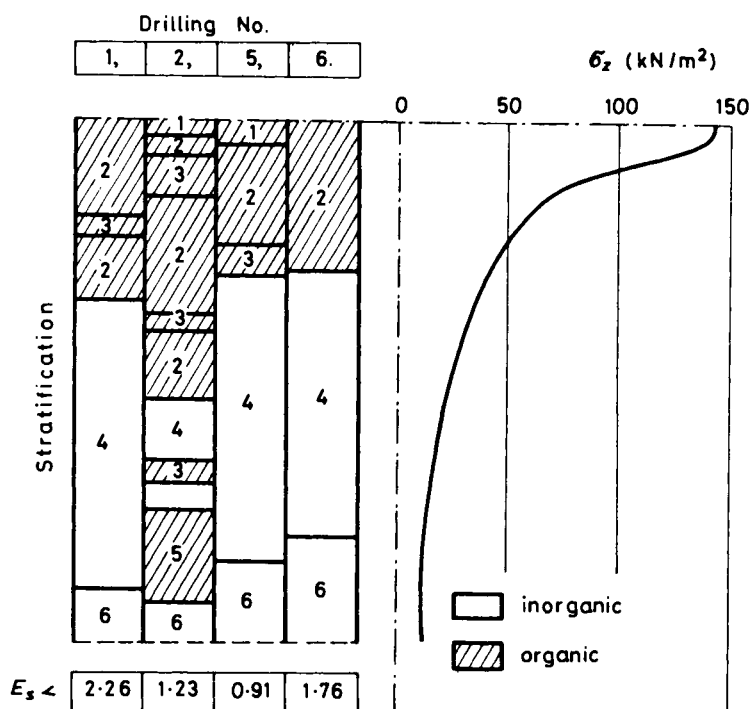


Fig. 10.15. Posterior determination of E_s from the relative settlements in the knowledge of four bore profiles and the Δs -values belonging to these

Figure 10.15 shows the schematic profile of four boreholes lowered around a considerably settled building ($\Delta s = 15$ cm). The ancient dead riverbed of the Danube has been filled up with granular soil (fine sand and silty sand) interwoven with 0.3–0.5 m thick turf veins. With the help of the levelling data a relative settlement Δs can be assigned to each borehole and an F -value to each Δs by summarizing the areas of the stress diagrams belonging to the organic layers:

Borehole	$F(\text{kN/mm})$	$\Delta s(\text{mm})$	$F/\Delta s$
1.	135.4	60	2.26
2.	184.1	150	1.23
3.	109.7	120	0.91
4.	105.6	60	1.76

The equation of the *regression line* expressing the relationship of both variables is as follows:

$$\Delta s = 0.792F - 8.4 \quad (r = 0.64).$$

The intercept ($b = -8.4$) is a very small value in relation to the measured differential settlements. According to Eq. (3.34)

$$t = \frac{0.64(-8.4)}{0.792} \sqrt{\frac{4-2}{(1-0.64^2) \cdot (36 \cdot 1^2 + 133 \cdot 7^2)}} = \pm 0.090,$$

so according to Table F4 the intercept *does not differ significantly from zero*, with a probability of 90%. Consequently, $\Delta s \sim s$ and the quotient $F/\Delta s$ may be regarded as the *modulus of compressibility* of the organic layers. The mean of the values obtained for the four boreholes is $\bar{E}_s = 1.54 \text{ MN/m}^2$, the standard deviation is 0.59 MN/m^2 . The reciprocal of the directional tangent of the regression line is 1.26, which corresponds to E_s determined by the method of the least squares.

The third category consists of cases, where *two (or more) layers with unfavourable properties* are present under the foundations. Now the following questions must be answered:

- a) which layer had a predominating role in the development of the settlements (damage);
- b) by what modulus of compressibility are these layers characterized.

Examples of establishing the calculation model are described below.

The hall structure of $60 \cdot 20 \text{ m}^2$ area of the Balatonfüred Shipyard was built in 1960; the roof covering half its width, the partition walls and strip foundations under each row of 12 columns were added in 1962/63. In 1964 it was found that the strip foundation and the floor beam of the side facing Lake Balaton were broken and wide cracks

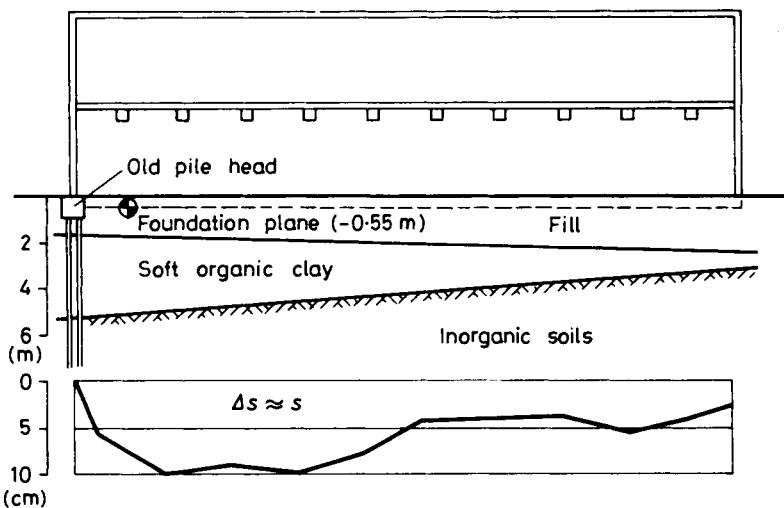


Fig. 10.16. Subsoil conditions of the hall of the Balatonfüred Shipyard and relative settlements measured at the pillars

Table 10.7. Relative settlement of the columns of the Balatonfüred shipyard (Δs , mm), thickness of the fill and the organic layer (v_1 and v_2 , m) and areas of the lamellae of the stress diagram (F_1 and F_2 , kN/mm)

Columns, No.	1	2	3	4	5	6	7	8	9	10
$\Delta s \approx s$	107	98	103	82	42	41	49	56	41	30
v_1	1.3	1.35	1.35	1.35	1.4	1.4	1.45	1.5	1.55	1.6
v_2	3.1	2.85	2.6	2.4	1.95	1.7	1.45	1.15	0.9	0.8
F_1	58.7	59.7	59.7	59.7	60.7	60.7	61.7	62.7	63.7	64.7
F_2	37.2	34.7	32.8	31.1	26.6	23.6	20.6	16.7	13.4	11.9

appeared in the partition walls. The measured maximum differential settlement was 107 mm.

The $B = 0.50$ m wide strip foundations transfer a contact pressure of 100 kN/m^2 to the subsoil. The lake side of one strip foundation was tied to an old, assumably *immobile* pilehead; under the rest of it there was a 1.2–1.8 m thick fill of brick fragments and sand, followed by a 0.8–3.8 m thick organic clay layer (Fig. 10.16), whose soil physical characteristics were as follows: $w = 76\%$, $w_p = 42\%$, $I_{om} = 11\%$.

Table 10.7 shows the relative settlement of the columns, the thickness of the fill and the organic layer found under them, as well as the area of the corresponding lamellae of the stress diagram. The equality $\Delta s \approx s$ expresses that due to the assumable immobility of the piles supporting column No. 1 the settlement may be regarded to be identical with the differential settlement.

The role of both compressible layers may be examined by *regression analysis*, where s is the dependent and F is the independent variable. The equation of the regression line obtained for the *fill* is

$$s = -12.1F_1 + 806 \quad (r = -0.80).$$

The negative sign of the correlation coefficient refers to the fact that *the differential settlement was not caused by the deformation of the fill*. The result is not surprising, as the thickness of the fill varies hardly at all and even this slight variation is just contrary to the trend of the deformation (as pointed out by the negative sign).

The equation of the regression line obtained for the organic clay is

$$s = 2.85F_2 - 5.9,$$

the correlation coefficient is $r = 0.88$, which refers to a very close correlation. The residual standard deviation is $s_f = 14.2$ mm, which means that the settlement is determined with this accuracy by the area of the stress diagram.

On considering the closeness of the correlation we must remember that the heterogeneity of the layer, the limited accuracy of the measurement, the interpolation of the layer boundaries between the drillings are all sources of error, so the value obtained for the correlation coefficient is indeed significant.

The intercept of the equation characterizing the organic clay ($b = -5.9$ mm) is a low value and the probability of its differing significantly from zero is merely 30% according to Eq. (3.34). This means that if $F_2 = 0$, $s \approx 0$, so $\Delta s \approx s$ is really true.

For the sake of completeness we may examine also the relationship between the thickness of the organic layer (v_2 , m) and the settlement. The equation of the regression line is

$$s = 32.3v_2 + 3.9$$

and the correlation coefficient is $r = 0.90$, i.e. similar as in the case of the relation F_2 , s .

The modulus of compressibility of the organic layer may be calculated by three different methods.

a) As the reciprocal of the directional tangent of the regression line:

$$E_{s2} = \frac{1}{2.85} = 0.35 \text{ MN/m}^2.$$

b) On accepting the assumption $\Delta s \sim s$ the modulus of compressibility interpreted as the quotient $F_2/\Delta s$ may be calculated in the position of each column (see Table 10.7). The result of the calculation is

$$\bar{E}_{s2} = 0.40 \pm 0.112 \text{ MN/m}^2.$$

The fact that the coefficient of variation of E_{s2} is merely 0.28, refers to the close correlation between the two variables (F and s).

c) The task may be solved also by variance analysis. On fixing the ratio $\alpha = E_{s1}/E_{s2}$ of the modulus of compressibility of both layers, an E_{s2} value may be calculated for each point of measurement. By determining in this way the mean ($\bar{E}_{s2}$) and the determined standard deviation (s_E) of 10 moduli of compressibility obtained, a coefficient of variation

$$C_v = \frac{s_E}{\bar{E}_{s2}}$$

is assigned to the given α . By varying the quotient α the following values are obtained for the studied parameters:

α	$\bar{E}_{s2}$	s_E	C_v	E_{s1}
2	0.98	0.30	0.309	1.97
3	0.79	0.23	0.294	2.37
4	0.69	0.19	0.280	2.77
5	0.63	0.17	0.273	3.17
10	0.52	0.13	0.258	5.20
∞	0.41	0.11	0.265	∞

We may assume rightfully that the true E_{s2} value is approximated best by the mean $\bar{E}_{s2}$, at which C_v is minimum. According to the table this extreme value exists indeed and falls between $\alpha = 10$ and $\alpha \rightarrow \infty$. This means that

$$0.41 < E_{s2} < 0.52 \text{ (MN/m}^2\text{)}.$$

The second example refers to a case where the building tilted as a rigid body.

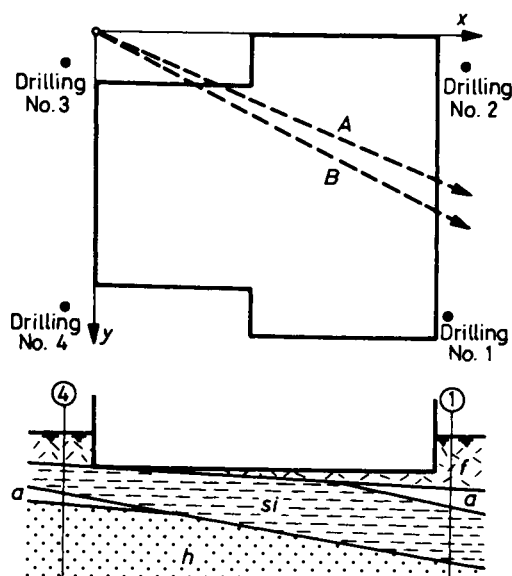


Fig. 10.17. Tilting direction of a panel-building measured instrumentally (A) and calculated from the thickness of the compressible layer (B) (notations in the soil profile: *f* — fill, *a* — clay, *si* — silt, *h* — sand)

In the second month after the panel mounting of a 10-storey building in Budapest it turned out that the lift-shaft tilted with an inclination of 12 cm. The boreholes lowered near the four corners of the building revealed the presence of a 0.15–1.35 m thick fill under the *plate-foundation*, followed by *silt*—with a thin clay lens wedged in at some places—, then by a 2.5–3.0 m thick layer of sand (Fig. 10.17). The fill, which consisted of silt, slag and building fragments was of a loose structure. The silt was rollable, its void ratio varied between $e=0.80-0.92$, referring also to a loose state.

From the determined *angles of inclination of the building edges* the relative settlement (Δs , mm) of the four corners could be calculated and on the basis of the bore profiles the thickness of the fill and the silt layers under the corners (v_1 and v_2 , m) and the partial areas corresponding to the diagram of the vertical stresses (F_1 and F_2 , kN/mm) could be established. The coherent values are compiled in the table below:

Borehole No.	Δs	v_1	v_2	F_1	F_2
3.	0	1.35	0	224.1	0
4.	24.3	0.15	1.75	24.9	290.5
2.	87.9	0.30	2.85	49.8	473.1
1.	124.8	0.90	3.20	149.4	531.2

The contribution of both unfavourable loose layers to the development of the tilting may be examined by *regression analysis*. The equation of the regression line calculated with the data of the table is

$$\Delta s = -0.115F_1 + 72.2,$$

and the correlation coefficient is $r = -0.19$. The negative sign cannot be interpreted physically, so the fill did not cause the tilting. The equation of the regression line obtained for the silt is

$$\Delta s = 0.223F_2 - 13.0$$

and the correlation coefficient is $r = 0.93$. This exceedingly high correlation proves unequivocally that the harmful tilting of the building was caused by the *uneven thickness of the silt*.

Now the question arises: how accurately do the deviations in the thickness of the layer determine the main direction of the tilting. This question can be examined with the help of *terviate correlation* (see Section 3.5.1). On interpreting the horizontal coordinates x and y according to *Fig. 10.17* we obtain for the regression plane the following equation:

$$v_2 = 0.1169x + 0.0611y + 0.348.$$

If $v_2 = 0$, then

$$y = -1.915x - 5.704,$$

which is the equation of the intersection line of the plane laid on the v_2 -values with the x, y -plane. The negative reciprocal of the directional tangent of the straight line is $m' = 0.5223$, so the angle of inclination of the v_2 -plane is

$$\alpha = \arctan 0.5223 = 27.6^\circ.$$

As according to the geodetical measurements the main direction of the tilting encloses an angle of $\alpha = 23.3^\circ$ with the x -axis, (see *Fig. 10.17*), we may state that *the variation of the layer thickness determine the angle of inclination very accurately*.

This example proves also that the expectable direction and magnitude of the tilting may be calculated with a good approximation still *in the designing phase*. The mathematical statistical methods supply not only the expectable values of both parameters, but also their standard deviation and thereby *the accuracy of the forecast* (see also Section 10.3).

10.4.5. Geometrical evaluation of the tilting

If the foundation system, or the superstructure is sufficiently rigid, the building behaves as a solid (rigid) body. If the load, acting on the foundations, is not central, or the stratification of the soil is not uniform, the deformation of the soil leads to the tilting of the object. (According to Section 10.3 it is sufficient, if the soil is "randomly" heterogeneous.)

This movement type—beside being aesthetically undesirable—may endanger the stability and the intended utilization of the object. For judging it we must know the geometrical data of the movement, both in its instantaneous and consolidated states.

The settlement s_i measured in individual points determines a *plane* whose equation may be obtained by a *terivariate regression analysis*. The *main direction of the tilting* is identical with the normal to the intersection line of this plane with the x, y -plane. (The course of the calculation is the same as in the numerical examples of sections 10.4.4 and 12.4.1.) The angle of inclination enclosed between the set of the straight lines running parallel with the main direction of the tilting and the horizontal is identical with the *angular displacement* δ of the vertical axis of the building object (see Section 10.4.2).

The regression analysis, as it controls also the measure of the linearity, supplies an information also on whether the object had tilted as a rigid body indeed. An answer to this question may be obtained by comparing the residual standard deviation with the accuracy of the measurement (mean error).

10.5. Economic considerations

Similarly to the problems connected with the bearing capacity (see Section 9.4) we must examine also in this case

- a) the required type and amount of the explorations (laboratory, in situ tests, etc.) and
- b) the economic consequences brought about by the deviation of the effective settlement from the forecast value.

An interesting example of considering the economic problems is presented by FOLAYAN *et al.* (1970). According to the plans the mud area around the San Francisco Gulf had to be made suited for building purposes by a 0.9–1.5 m thick fill. The subsoil consisted of a 9–15 m thick, very soft organic silt, which could be regarded as relatively homogeneous. For the height of the regulated surface after the consolidation a very strict requirement has been specified (with a permissible deviation of ± 3 cm).

For the preliminary estimation of the settlements five engineers were asked to give opinions concerning the realistic mean and the standard deviation of the compression ratio defined by expression (10.18). With these data the “*a priori*” distributions of C_r could be constructed (Fig. 10.18). Subsequently 37 boreholes were drilled and the compression ratio was determined on 39 samples. The test results were as follows:

$$\bar{C}_r = 0.34; \quad s^2 = 0.037.$$

So the standard deviation exceeded the expected value. In the knowledge of these parameters five “*a posteriori*” distributions (see Fig. 10.18) could be calculated with the help of Eq. (7.12).

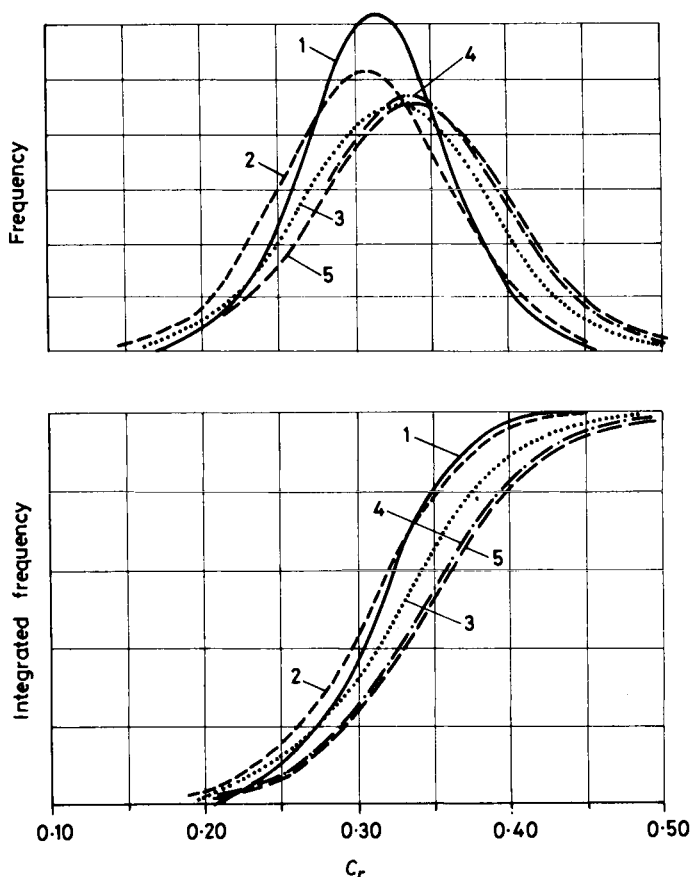


Fig. 10.18. A priori and a posteriori distributions of the compression ratio based on the opinions of five experts (FOLAYAN *et al.*, 1970)

The losses due to the inaccuracy of the forecasting were estimated in the following way. If $s_m < s_c$, the excess earth quantity must be subsequently removed, while if $s_m > s_c$, difficulties may appear in the operation of the public utility pipelines. From the approximate calculations—with these considerations taken into account—it was concluded that the losses were proportional with the square of the difference $|s_m - s_c|$, so the loss function (see Section 8.8) is a second-degree parabola.

The authors have determined also the required number of laboratory tests for reducing the relative error of C_r to below $\pm 20\%$, depending on the basic information of the consulting engineers.

The example described above is exceptional inasmuch as *independently of the sign* of the difference of the effective and the calculated settlement, financial losses must be counted upon. In the general case a *one-sided examination* must be performed to find

out the technical–economical consequences for the case where the effective settlement exceeds the forecast value.

According to ROSENBLUETH (1969) the loss functions should be determined for each type of movement on the basis of the following guiding principles.

a) The *average settlement* is determining for the condition of the limiting walls and the utilizability of the object. The cost may be written in the form of the following relationship:

$$D_a = A \left(\frac{s}{s_k} \right)^2,$$

where s_k is the absolute settlement which necessitates the complete repair of the building.

b) The tilting aggravates—in addition to the aspects listed in clause a)—the operation of the elevators, it decreases the resistance against horizontal forces (but it may have also other unpleasant consequences, e.g., aesthetic, or psychological effects). According to the author the loss may be written in the following form:

$$D_b = A \Theta^2,$$

where Θ is the tilting in the main direction, expressed in radians.

c) The *angular distortion* causes damages in the plaster work, the structure, the doors and windows, etc. The loss calculation is much more difficult in this case (according to the author it may be taken to be proportional with the square of the angle—expressed in radians—between two deformation lines).

11. The stability of free standing and supported slopes

Numerous articles discussing probabilistic dimensioning in geotechnics deal with the problems of the free standing slopes. The reasons are obviously that

- the stability can be defined well, even if not all the effects, e.g., the variation of the pore water pressure can be taken into account;
- the principle of modern dimensioning can be illustrated well by calculating the probability of failure and the optimum dimensions;
- generally important economic, life and property interests depend on the stability of the earthworks (valley dams, etc.).

Relatively few articles deal with the earth pressure acting on the retaining walls, though there exist numerous possibilities for the successful utilization of the theory of probability also in this field (see Section 11.7).

11.1 General methodological problems

11.1.1. Establishing the model

If we wish to examine the stability of slopes by probabilistic methods, distinctions must be made:

- between the assumption of flat or curved rupture surfaces, resp.
- between examining the stability of the slope on the basis of the forces acting on slices (lamellae) or treating the earth mass limited by the sliding surface as a *rigid body*.

These distinctions exist also in the deterministic methodology, but in our case they are of specific importance, as their role in calculating the probability of failure is determining.

The first group of the alternatives may be disposed of by the well-known fact that in homogeneous soils a *curved rupture surface* develops usually according to the observations, and this gives a lower factor of safety. So in the following the calculations

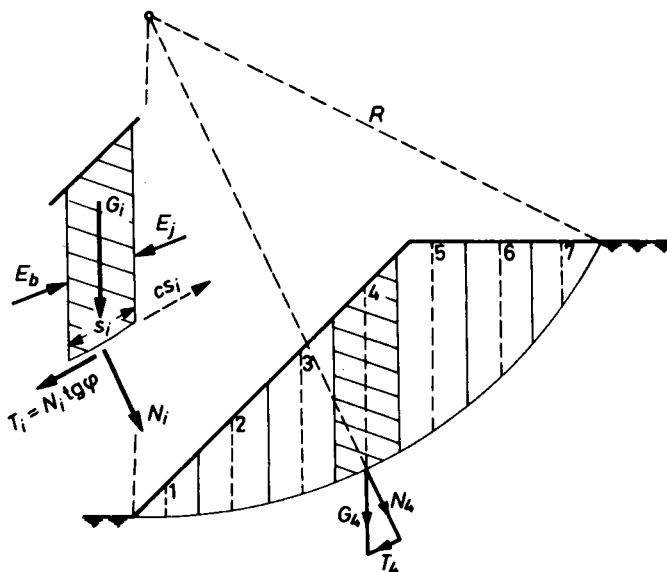


Fig. 11.1. Stability test of a slope based on the equilibrium of the slices

related to the planar rupture surface are presented, whose treatment is simpler, only for demonstrating the dimensioning method.

In the second group of the alternatives a distinction must be made depending on whether we wish to determine the probability of failure on the basis of the standard deviation of the *safety factor*, or on that of the *safety margin* (see Sections 11.3 and 11.4). The researchers prefer the methods based on the examination of the equilibrium of the *slices*, as in this case the safety margin can be interpreted very simply. Namely, the conventional factor of safety is—according to Fig. 11.1—the quotient of the resisting moment and the “driving” moment:

$$v_c = \frac{M_R}{M_Q} = \frac{R(cl_i + \tan \varphi \sum N_i)}{R \sum T_i} \quad (11.1)$$

and the safety margin is

$$SM = R(cl_i + \tan \varphi \sum N_i) - R \sum T_i. \quad (11.2)$$

The *Swedish method of moments* (see in Section 11.2.2), proposed by FELLENIUS and the process of BISHOP (1955) taking into account also the difference of the earth pressures E_b and E_j interpreted according to Fig. 11.1 may also be discussed similarly.

Two processes treating the sliding earthmass as a rigid body (TAYLOR, 1948 and JÁKY, 1944) determine the required cohesion with the help of *vector polygons* (Fig. 11.2). These processes suggest that the probability of failure should be examined

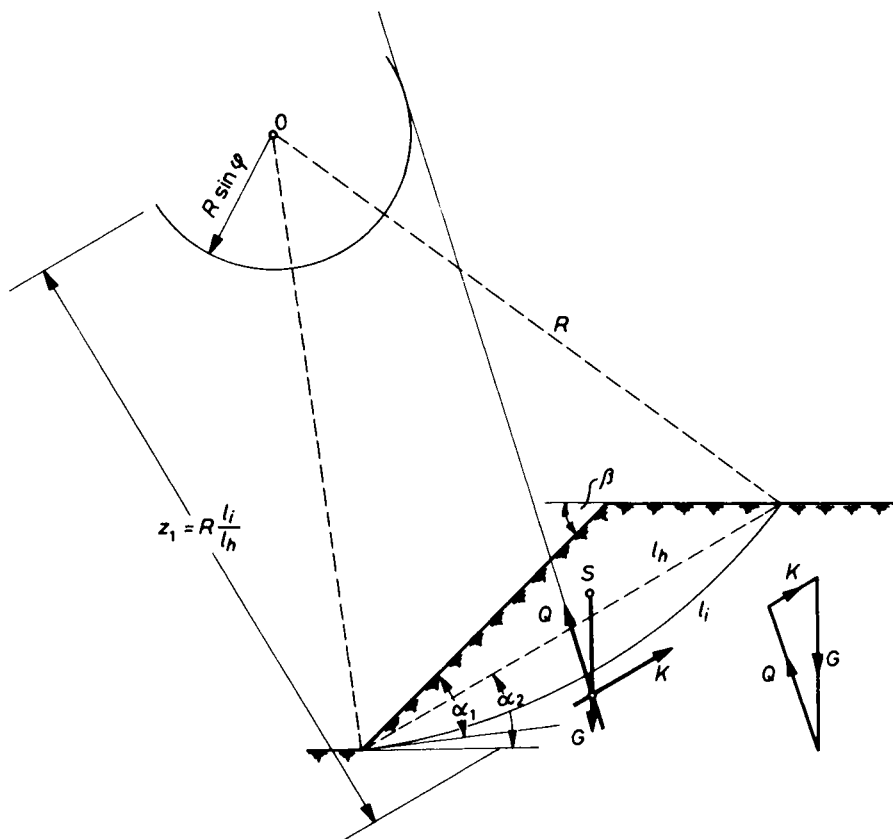


Fig. 11.2. Stability test of slopes according to TAYLOR (1948) and JÁKY (1944); in the latter case $\alpha_1 = 45^\circ - \varphi/2$ and $\alpha_2 = 0.5(\beta + \varphi)$

on the basis of the conventional factor of safety, which is identical in this case with the quotient

$$v_c = \frac{c \cdot l_h}{K}, \quad (11.3)$$

where c — effective value of the specific cohesion,
 l_h — length of the chord of the rupture surface.

A third method belonging here is that of FRÖHLICH (1950). Here we must determine from the resultant P of the forces acting on the sliding earth mass (dead weight, external load, seepage pressure, etc.) and from the cohesion force $K = c l_h$ an auxiliary force S , which is the resultant of the forces N and $T = N \tan \varphi$ too, acting on the rupture surface (Fig. 11.3). The points of intersection of the forces S interpreted on the basis of two different combinations, with the force K do not coincide generally; the factor of safety interpreted on this basis is

$$v_c = 1 + \frac{e}{a} \quad (11.4)$$

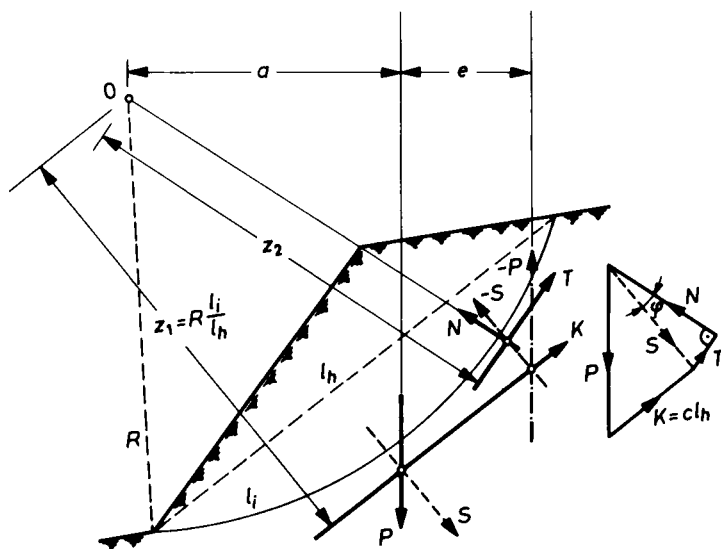


Fig. 11.3. Stability test of slopes according to FRÖHLICH (1950)

and the safety margin is

$$SM = K_{z1} + T_{z2} - Pa, \quad (11.5)$$

where $z_1 = Rl_i/l_h$ and z_2 may be calculated from the assumed distribution of the normal stresses acting on the rupture surface.

A common feature of these theories is that each assumes a *critical rupture surface*, to which the minimum safety belongs. On using a stochastic model we must find the rupture surface which gives the maximum probability of failure. The effective p_f probability is lower than the value calculated in this way, because the slope fails as a *system* (see Section 8.3). But we cannot take this difference into account, because the rupture surfaces which may be involved are lying close to each other, so there exists a very close correlation between the physical characteristics, the pore water pressures and also between the safety factors. Nevertheless according to CORNELL (1971) the value belonging to the critical rupture surface may be accepted for p_f and this surface is very near to that determined by deterministic methods (ALONSO, 1976), as it may be concluded from Fig. 11.4.

HARR (1977) made interesting statements concerning the expectable position of the rupture surface. He starts from the low probability of a *simultaneous failure at all points* of the rupture surface, even in homogeneous and isotropic soils. According to theoretical researches (BALADI *et al.*, 1967) the highest stress concentration appears at the *foot of the slope*; the failure is progressive and starts most probably in the vicinity of the foot of the slope (HARR *et al.*, 1972). Let us assume that the failure starts from point *B*, *D*, or *F* indicated in Fig. 11.5. The question is how can it "find" the foot of the slope. *Theoretically*—as there exists no preliminary information on the position of the foot of the slope—the rupture surface of an increasing length

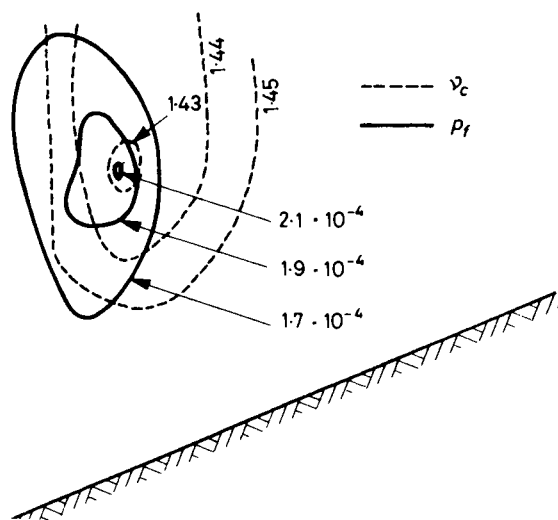


Fig. 11.4. Determination of the critical rupture surface on the basis of the minimum of the safety factor (v_c) and the maximum of the probability of failure (p_f) (ALONSO, 1976)

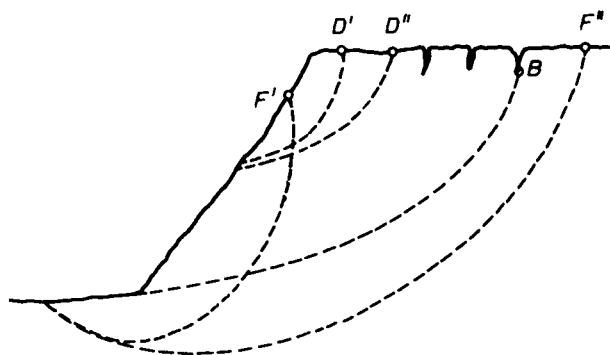


Fig. 11.5. The formation of sliding planes, when the sliding does not start from the base point (BALADI *et al.*, 1967; HARR *et al.*, 1972)

may reach *any point* of the slope with an *identical probability*, as it occurs indeed in the case of rupture surfaces starting from the points D' , D'' , F' and F'' . So if a *base point rupture surface* develops, the probability that the failure will start from the foot of the slope is the greatest by far. This is supported by the observation that such surfaces occur most frequently.

As to the selection of the model the question of the *scatter of the models* arises again. According to Section 8.4 the coefficient of variation of the factors of safety calculated for the same slope by various methods will certainly not exceed 0.1. This is a favourable circumstance, not only because of its absolute value, but also with regard to its relative magnitude *related to the cohesion* (see Section 1.3.6). By the same consideration also an

approach taking into account the inclination of the planar rupture surface with the value

$$\frac{\beta + \varphi}{2}$$

is permissible, the more so, as this assumption has been justified by examples (e.g. HARR, 1977).

The probability of the slope failure must be determined as described in Chapter 8. Numerous authors (e.g. YONG et al., 1977) emphasize that in the case of slopes where p_f takes on generally a value between 10^{-2} and 10^{-3} , the assumption accepted for the distribution of SM (or FS) is of no importance.

11.1.2. Determination of soil physical characteristics

The parameters of the shear strength show a considerable spatial dispersion, as mentioned already in Section 1.3.6, with the consequence that a relatively high probability of failure (see, e.g., Fig. 8.13) belongs to the factors of safety used in practice and which can hardly be increased any more because of economic reasons. The situation is still worse, if the *dispersion in time* is also taken into account, caused in the layers lying close to the surface by meteorological effects and in the deeper layers by the variation of the groundwater level and the pore water pressure (see Section 4.2.3).

The common dispersion of the physical characteristics in time and space may be estimated by deterministic considerations, or on the basis of soil tests performed at various time moments. This is how the parameters of the shear strength required for examining the stability of a hillside was determined by KRÁLIK (1980) taking the correlation w, τ into account.

As the coefficient of variation of the shear strength and the probability of failure are very closely correlated, the standard deviation of φ and c should be made as low as possible. As the *natural heterogeneity* cannot be diminished, this aim can be achieved only by improving the sampling and the test methods (see examples in Section 6.2).

Increasing the number of samples will generally not reduce the dispersion, only the reliability of the mean is increased by narrowing down the confidence interval according to Eqs (7.36) and (7.37). In connection with the numerical example of Section 8.5.2 we have established that the calculation of the probability of failure from the confidence intervals is justified only, if we are sure that *the mean shear strength* will act on the rupture surface. This assumption acts against the safety—and as SALEMBIER (1979) remarks—*the more so, the longer* the involved section of the slope is. We have namely no guarantee against the presence of a section between two boreholes in which τ is significantly lower than the mean. As the earth mass has no significant longitudinal “rigidity”, we would be exposed to the risk of failure of some sections of the slope.

Similar problems arise also if there is a significant spatial autocorrelation between the parameters of the shear strength. According to Section 4.2.1, dealing with this

Table 11.1. Data for the autocorrelation factor α of the soil physical characteristics (ALONSO, 1976)

Type of soil	Property	α, m^{-1}
Sand	Point pressure of the probe	0.9
Sand	Friction ratio (probe)	1.6
Clay	Point pressure of the probe	1.91
Soft silt	Water content	0.158
Soft silt	Sand-fraction	0.231
Soft silt	Silt-fraction	0.31
Soft silt	Bulk density	0.253
Soft silt	Void ratio	0.191
Soft silt	Liquid limit	0.23
Gravelly sand	Porosity	0.136

problem, the autocorrelation coefficient varies generally according to the expression

$$r_k = c \cdot e^{-\alpha k},$$

where k —step interval, m, α and $c \leq 1$ —constants characterizing the earth mass.

The example in Section 4.3 showed also how far the measured mean of the physical characteristic is changed by the *position* and the *number of the samplings*. So it is understandable that this problem is dealt with ever more in literature. The primary aim of the research is the determination of the coefficient α whose informatory numerical values are shown in Table 11.1.

SALEMBIER (1979) proposes to calculate the approximate value of the space δ between samplings according to Fig. 11.6, at whose surpassing the samples can be regarded to be statistically independent. According to his numerical example related to the examination of the stability of a dam, the dependence of the probability of failure (p_f) on δ is as follows:

$$\begin{aligned} \text{at } \delta = 10 \text{ m} \quad p_f &\cong 2 \cdot 10^{-3} \\ \delta = 5 \text{ m} \quad p_f &\cong 10^{-5} \\ \delta = 1 \text{ m} \quad p_f &< 10^{-6}. \end{aligned}$$

An example for taking the vertical *trend* of the shear strength into account is given in Section 11.3.1.

On calculating the probability of failure the degree of correlation between φ and c is not indifferent. This may be calculated by the methods described in Chapter 3, by determining the coefficient r which reflects the closeness of the correlation. Its sign is always negative, which means that within a layer c decreases with the increase of φ (and vice versa). Its absolute value is one of the statistical characteristics of the examined soil mass and it varies generally between 0.3 and 0.9. Some literature data are as follows:

HOLTZ and KRIZEK (1971): $-0.49(\varphi, c)$;

LUMB (1970): $-0.03, -0.72, +0.35, -0.37, -0.43, -0.70(\tan \varphi, c)$;

WEBER and GEHRISCH (1980): $-0.8 \div -0.9(\tan \varphi, c)$

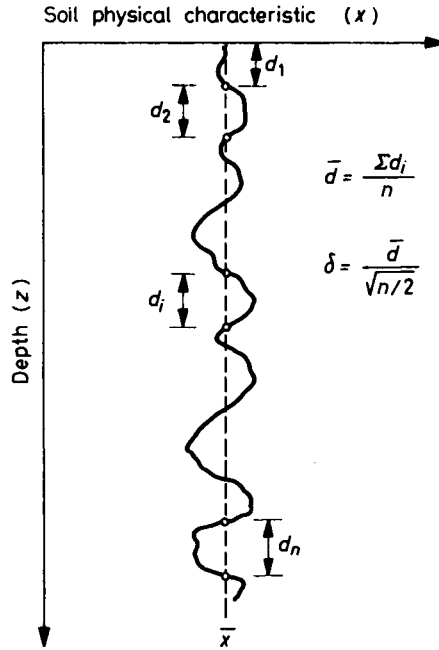


Fig. 11.6. Calculation of the autocorrelation distance δ according to the proposition of SALEMBIER (1979)

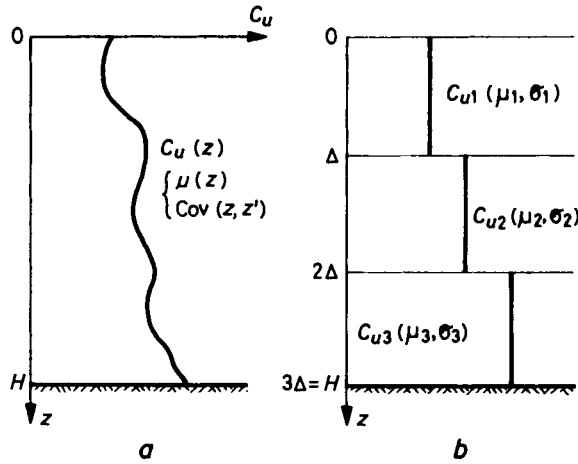


Fig. 11.7. The original (a) and the equivalent multi-layer system (b) (MATSUO and ASAOKA, 1983)

GRIVAS (1981): $-0.16(\varphi, c)$

CHERUBINI *et al.* (1983): $-0.60 \dots -0.85(\varphi, c)$.

The correlation must not be disregarded and the less so, the higher $|r|$ is (see Section 11.3.3).

If the layer is not homogeneous (e.g. c_u varies linearly with the depth), the calculation of the shearing resistance appearing along the individual rupture surfaces may be aggravated. MATSUO and ASAOKA (1983) suggest for this case the following process. The original situation shown in Fig. 11.7, *a* should be replaced by an “equivalent three-layer system” (detail *b* of the figure). In the figure c_{ui} ($i = 1, 2, 3$) are the mutually random variables with means μ_i and variances σ_i^2 . The line element dL in the equation

$$FS = \min_L \frac{R \int_L c_u(z) dL}{M_0} \quad (11.6)$$

is assumed to be well approximated by the quadratic weight function of z , i.e.

$$dL \cong (a_0 + a_1 z + a_2 z^2) dz. \quad (11.7)$$

When this is the case, since

$$E \int_L c_u(z) dz \cong a_0 \int_0^H \mu(z) dz + a_1 \int_0^H \mu(z) z dz + a_2 \int_0^H \mu(z) z^2 dz,$$

if μ_i ($i = 1, 2, 3$) satisfy

$$\left. \begin{aligned} \mu_1 \Delta + \mu_2 \Delta + \mu_3 \Delta &= \int_0^H \mu(z) dz \\ \mu_1 \frac{\Delta^2}{2} + \mu_2 \frac{3\Delta^2}{2} + \mu_3 \frac{5\Delta^2}{2} &= \int_0^H \mu(z) z dz \\ \mu_1 \frac{\Delta^3}{3} + \mu_2 \frac{7\Delta^3}{3} + \mu_3 \frac{19\Delta^3}{3} &= \int_0^H \mu(z) z^2 dz \end{aligned} \right\} \quad (11.8)$$

the multilayer system can give the same value for $E \int_L c_u(z) dz$, whatever values a_0, a_1 and a_2 may take (i.e. for any slip surface L). In Eq. (11.8) $\mu(z)$ on the right side is the mean value function of the original system $c_u(z)$.

11.2 The case of $c=0$ and $\varphi=0$

11.2.1. Examination of granular soils ($c=0$)

As it is known, the safety of the stability of an earth mass consisting of granular soil, limited by a slope angle β is:

$$v_c = \frac{\overline{\tan \varphi}}{\tan \beta}. \quad (11.9)$$

This means that the probability of failure may be determined with the help of the standard deviation, resp. of the coefficient of variation of $\tan \varphi$. As the standard

deviation of β (the inaccuracy of the construction work) is negligible, the coefficient of variation C_{vQ} of the “load side” is zero. So Eq. (8.52) corresponding to the normal distribution may be written in the form

$$p_f = 1 - \Phi \left[\frac{v_c - 1}{v_c \cdot C_{vR}} \right], \quad (11.10)$$

and Eq. (8.53) corresponding to the lognormal distribution in the form

$$p_f = 1 - \Phi \left[\frac{\ln \left(v_c \sqrt{\frac{1}{1 + C_{vR}^2}} \right)}{\sqrt{\ln(1 + C_{vR}^2)}} \right], \quad (11.11)$$

where C_{vR} is the coefficient of variation of $\tan \varphi$.

On designing earthworks $\tan \varphi$ and C_{vR} are given and the slope angle β belonging to the probability of failure determined by us is requested. By transforming Eqs (11.10) and (11.11) we obtain the following formulae:

for the normal distribution

$$\beta = \arctan [\overline{\tan \varphi} + (1 - C_{vR} \cdot \lambda_p)], \quad (11.12)$$

for the lognormal distribution

$$\beta = \arctan [\exp(\ln \overline{\tan \varphi} + \ln A - \lambda_p B)], \quad (11.13)$$

where

$$A = \sqrt{\frac{1}{1 + C_{vR}^2}} \quad \text{and} \quad B = \sqrt{\ln(1 + C_{vR}^2)},$$

while λ_p is the argument of the normal distribution, which may be calculated from the expression

$$p_f = 1 - \Phi(\lambda_p). \quad (11.14)$$

Let us examine the stability of a slope of $\beta = 33.7^\circ$ ($\rho = 6/4$), where the mean tangent of the angle of friction is $\overline{\tan \varphi} = 0.8185$ and its coefficient of variation is 0.05. As the central factor of safety in this case is

$$v_c = \frac{0.8185}{0.6667} = 1.228,$$

so according to Eq. (11.10)

$$p_f = 1 - \Phi \left[\frac{0.228}{1.228 \cdot 0.05} \right],$$

to which $p_f = 0.0001$ belongs in Table F2. On assuming a lognormal distribution

$$p_f = 1 - \Phi(4.0853)$$

from Eq. (11.11). As Table F2 can be used only for values of $\lambda_p \leq 3.8$, we may approximate the value of the function with the help of the inequality (8.54):

$$\left(\frac{1}{4.0853} - \frac{1}{4.0853^3} \right) \varphi(4.0853) < 1 - \Phi(\lambda_p) < \frac{1}{4.0853} \varphi(4.0853).$$

As

$$\varphi(4.0853) = \frac{1}{\sqrt{2\pi}} e^{-\frac{4.0853^2}{2}} = 9.48 \cdot 10^{-5},$$

the requested inequality is

$$0.000\,0218 < 1 - \Phi(4.0853) < 0.000\,0232,$$

i.e. $p_f = 2.25 \cdot 10^{-5}$ may be accepted with a good approximation.

On varying the coefficient of variation we obtain the curves shown in Fig. 11.8 for the relationship C_{vR} , p_f . Accordingly, the probability of failure increases rapidly with the variance of $\tan \varphi$; if $C_{vR} = 0.15$ (which is not an unrealistic value), $p_f \cong 0.1$, which means that we must count on the failure of every tenth slope, prepared under similar conditions.

Detail *b* of Fig. 11.8 corresponds to the case of the design where $p_f = 0.01$ is prescribed. In the case of a normal distribution $\beta = 28.1^\circ - 35.9^\circ$ and in the case of a lognormal distribution $\beta = 29.8^\circ - 36.1^\circ$ corresponds to the examined interval $C_{vR} = 0.05 - 0.15$, so the difference between both extreme values is $6 - 8^\circ$.

In homogeneous granular soils the slip surfaces running out to the side of the slope (*conchoid formation*) are potentially the most dangerous. EVANGELISTA *et al.* (1975) suggested the statistical evaluation of this formation. On examining a 50 m high dam built of sandy gravel they established that the dry bulk density was of a normal

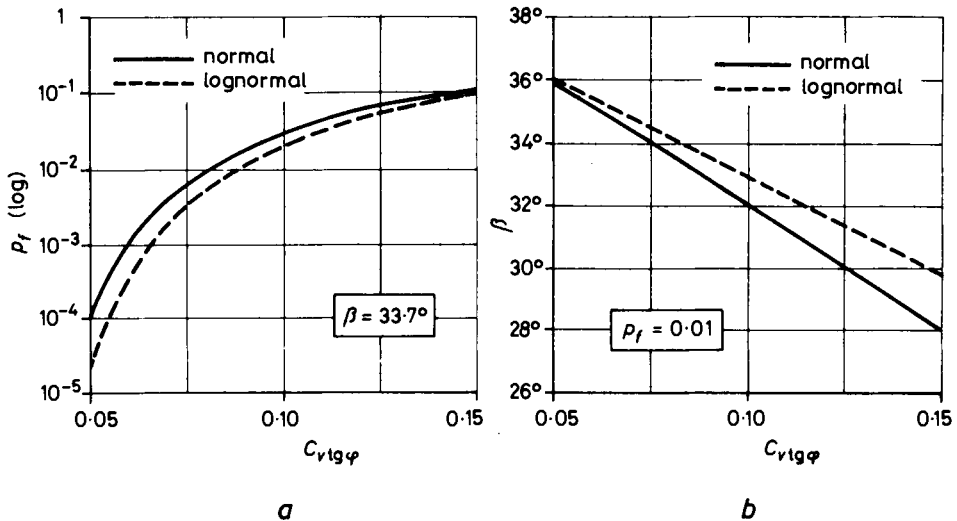


Fig. 11.8. Numerical example for calculating the probability of failure (p_f), or the slope angle (β) versus the coefficient of variation of $\tan \varphi$ in the case of a given value of β , or p_f

distribution, its spatial variation was of a random nature with 0.03–0.05 coefficient of variation and could be correlated closely with the angle of friction. They examined the stability of the dam in the following steps:

- a) they divided the earth mass into 1 m high horizontal slices and
- b) determined the characteristic dry bulk density (and thereby the angle of friction) of the individual layers with the help of random numbers, using a computer program, and generating 110 normal distributions,
- c) then they determined the factor of safety for the circular rupture surfaces of $a = 2, 4, 6$ and 8 m rise by the method of FELLENIUS.

On the basis of their examinations they made the following statements:

- the safety of the stability for a heterogeneous slope is lower than that of a homogeneous slope of an identical $\bar{\gamma}_d$ -value,
- the factor of safety increases gradually with the increase of a ,
- the effect of the rise (a) of the factor of safety is smaller than that of the standard deviation s of γ_d , especially in the case of higher s -values.

11.2.2. Examination of soft cohesive soils ($\varphi = 0$)

Let us assume that a planar rupture surface develops in the earth mass. The conditions of equilibrium for the general case ($\varphi \neq 0, c \neq 0$) are shown in *Fig. 11.9*; accordingly the so-called necessary cohesion c_n belonging to $v_c = 1$ for the case of $\varphi = 0$ (interpreting φ in the function of the total stress) is

$$c_n = H\gamma \frac{\sin^2 \frac{\beta}{2}}{2 \sin \beta}. \quad (11.15)$$

The conventional factor of safety is

$$v_c = \frac{\bar{c}}{c_n}$$

and the probability of failure (with the assumption of a normal distribution for c) may be calculated from Eq. (8.52) with the substitutions $C_{vR} = s_c/\bar{c}$ and $C_{vQ} = 0$.

The value λ appearing in Eq. (8.52) may be calculated also from the standard deviation of the factor of safety

$$s_v = \frac{s_c}{c_n} \quad (11.16)$$

by the following expression:

$$\lambda = \frac{v_c - 1}{s_v}. \quad (11.17)$$

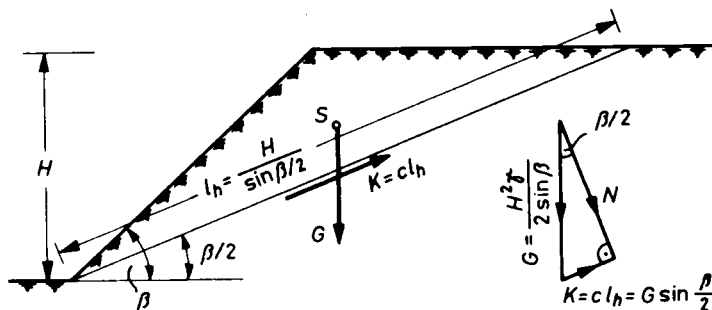


Fig. 11.9. Examination of the case $\varphi=0$ with the assumption of planar rupture surface

A similar result is obtained also if we start from the excess-cohesion:

$$\lambda = \frac{\bar{c} - c_n}{s_c}. \quad (11.18)$$

On dividing the right side by c_n we obtain expression (11.17) and from the latter λ appearing in expression (8.52) by substituting Eq. (11.16) for s_v and multiplying both the numerator and the denominator of the latter by $\bar{c}$:

$$\lambda = \frac{v_c - 1}{s_v} = \frac{v_c - 1}{\frac{\bar{c}}{c_n} \cdot \frac{s_c}{\bar{c}}} = \frac{v_c - 1}{v_c C_{vc}} \quad (\text{q.e.d.}).$$

The difference between Eqs (11.17) and (11.18) is that one starts from the “safety margin” of the factor of safety, the other one from that of the cohesion.

Let us assume that the height of the slope is $H = 6$ m, the bulk density of the soil is $\gamma = 18.4$ kN/m³, the cohesion is $\bar{c} = 26.2$ kN/m² with a standard deviation $s_c = 7.6$ kN/m². Let us calculate the probability of failure of the slope, if $\beta = 60^\circ$. According to Eq. (11.15), $c_n = 15.93$ kN/m², so

$$v_c = \frac{26.2}{15.93} = 1.645, \quad s_v = \frac{7.6}{15.93} = 0.477 \quad \text{and} \quad C_{vc} = \frac{7.6}{26.2} = 0.29.$$

The three different calculation methods gave for λ the following result:

$$\lambda = \frac{0.645}{1.645 \cdot 0.29} = \frac{1.645 - 1}{0.477} = \frac{26.2 - 15.93}{7.6} = 1.3513,$$

so, according to Table F2, $p_f = 0.088$. For the slope angle $\beta = 45^\circ$ we get $p_f = 0.026$, for the slope angle $\beta = 30^\circ$ $p_f = 0.007$.

The actual conditions must be approximated with the assumption of a *circular slip surface*. The stability must be determined according to Fig. 11.10. From the condition

$$v_c = \frac{Kz}{Ga} = -\frac{cl_h \cdot R \frac{l_i}{l_h}}{Ga} = \frac{cl_i R}{Ga} = 1$$

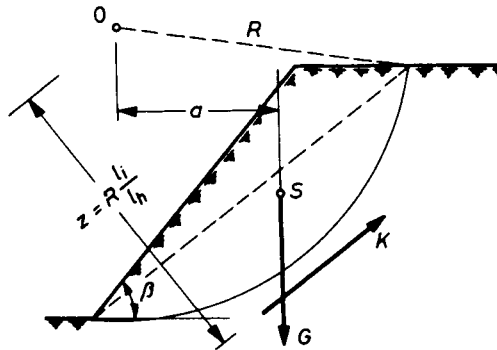


Fig. 11.10. Examination of the case $\varphi = 0$ with the assumption of a circular rupture surface (Swedish method of moments)

the necessary cohesion is

$$c_n = \frac{Ga}{Rl_i}. \quad (11.19)$$

The ruling rupture surface (belonging to the lowest factor of safety) is a function of the depth of the solid layer as well, beside the slope angle. The related research proved also that for slope angles of $\beta < 53^\circ$ (if the solid layer is lying deep) the *undercutting slip surface* becomes ruling. TAYLOR (1948) constructed, with all these factors taken into account, a set of graphs giving the stability factor

$$N_c = \frac{c}{H\gamma} \quad (11.20)$$

in the function of the slope angle and the so-called depth factor.

Accordingly the probability of failure may be calculated with the help of the factor of safety determined by the fraction

$$v_c = \frac{\bar{c}}{N_c H \gamma} \quad (11.21)$$

from the formulae used for the planar rupture surfaces.

From the preceding example the following N_c and c_n values are obtained, with the assumption of a circular slip surface:

$\beta =$	60°	45°	30°
$N_c =$	0.192	0.166	0.133
$c_n =$	21.20	18.33	14.68

Accordingly, $p_f = 0.25-0.15-0.065$, i.e. the values obtained for the planar slip surface are 2.8–9.3-fold multiples of the values calculated on a realistic basis.

If the material of the slope is not homogeneous, the method of MATSUO and ASAOKA (1983) elaborated for examining soil failures under dams may be used (compare with Fig. 11.7). The factor of safety (FS) can be considered as a function of three random variables (c_{u1}, c_{u2}, c_{u3}) i.e.

$$FS = FS(c_{u1}, c_{u2}, c_{u3} | G), \quad (11.22)$$

in which the symbol G denotes the condition of a design earthwork (slope geometry, groundwater table, etc.). The probability of failure is defined by

$$p_f(G) = \int \dots \int \xi(c_{u1}, c_{u2}, c_{u3}) dc_{u1} dc_{u2} dc_{u3}, \quad (11.23)$$

in which $\xi(\dots)$ denotes a joint probability function for c_{u1} , c_{u2} and c_{u3} . The approximate value of Eq. (11.23) can be obtained on the basis of ROSENBLUETH's (1975) "point estimates theory" as follows. Let m_G and s_G denote the mean and the standard deviation of the eight different factors of safety

$$FS(\mu_1 + \sigma_1, \mu_2 + \sigma_2, \mu_3 + \sigma_3 | G), \dots$$

$$FS(\mu_1 - \sigma_1, \mu_2 - \sigma_2, \mu_3 - \sigma_3 | G),$$

respectively (the signs of the σ_i values are: $+++$, $++-$, $+ - +$, $+ - -$, $- + +$, $- + -$, $- - +$, $- - -$). Then, $p_f(G)$ can be approximately evaluated by

$$p_f(G) = \Theta\left(\frac{1 - m_G}{s_G}\right), \quad (11.24)$$

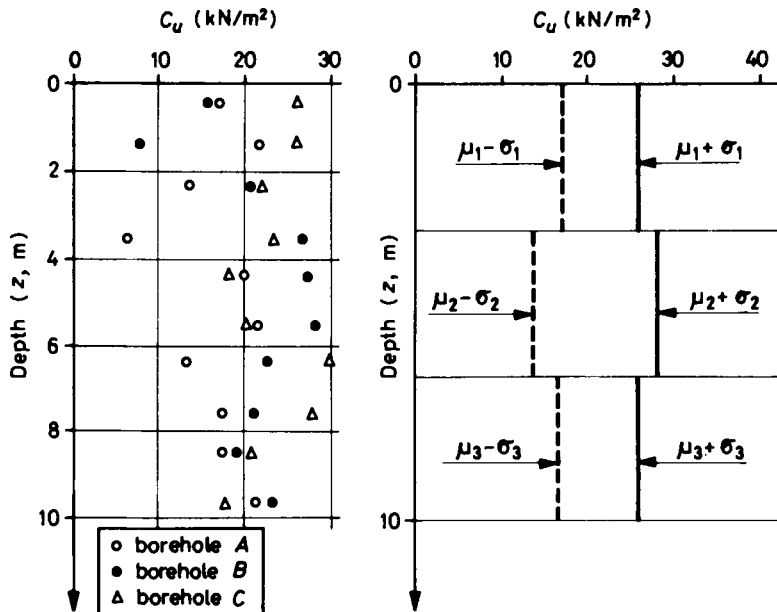


Fig. 11.11. Calculation of p_f by using the method of ROSENBLUETH (1975) (MATSUO and ASAOKA, 1983)

where Φ denotes the distribution function of a standardized normal random variable. The solution is illustrated in *Fig. 11.11*.

If in addition to the dead weight also other forces are acting (e.g. external loads), or if the slope is not limited by a plane, the probability of failure may be calculated with the help of the *safety margin evaluated from the moments* and its standard deviation (see in the following section).

11.3. The solution of the task in the general case ($\varphi \neq 0$, $c \neq 0$)

The circumstance that the shear strength of the soil is described by two parameters instead of one renders the calculation *more complicated* and necessitates the determination of the correlation between φ and c for the dimensioning based on the theory of probability. At the same time the fact that in the case of $\varphi > 5^\circ$ the base point slip surface is predominating is favourable.

The probabilistic dimensioning may be performed by two processes: by analyzing the standard deviation of the load and the resistance (Section 11.3.1), resp. by the Monte Carlo method (Section 11.3.2). Within these processes we may distinguish between the method of investigating whether the condition $SM = 0$, or $v_c = 1$ is fulfilled, but still more important is from the aspect of methodology, whether the correlation between the angle of friction and the cohesion is negligible or not (Section 11.3.3).

11.3.1. Examination of the stability with standard deviation of the load and the resistance

This process starts from the *expected value* ($\bar{\varphi}$, $\bar{c}$) of the shear strength of the soil and determines the probability p_f of the decrease of SM to zero, resp. of v_c to 1. For establishing the latter

- a) we may examine the equilibrium of the forces on the basis of the moments bringing about, respectively resisting the sliding, or
- b) we may compare the effective cohesion and that belonging to $v_c = 1$, determined by a theoretical method, or from a graph.

Let us assume that the stability is examined on the basis of the equilibrium of slices. According to Eq. (11.2), the expected value of the safety margin in this case is

$$\overline{SM} = R(\bar{c}l_i + \overline{\tan \varphi} \Sigma N_i - \Sigma T_i), \quad (11.25)$$

where N_i and T_i belong to the earth mass limited by the critical rupture surface, at which $v_c = \min.$ with the given $\bar{\varphi}$ - and $\bar{c}$ -values.

The course of the calculations is illustrated by the following example. Be the length of the critical slip surface of a slope $l_i = 17.6$ m, the sum of the normal and the tangential forces

$$\Sigma N_i = 124 \text{ kN}, \quad \text{resp. } \Sigma T_i = 68 \text{ kN}$$

and the parameters of the shear strength:

$$\begin{aligned} \overline{\tan \varphi} &= 0.404, & s_{\tan \varphi} &= 0.046, \\ \bar{c} &= 3.8 \text{ kN/m}^2, & s_c &= 1.32 \text{ kN/m}^2. \end{aligned}$$

With these data the central factor of safety is

$$v_c = \frac{\bar{c} l_i + \overline{\tan \varphi} \Sigma N_i}{\Sigma T_i} = 1.72$$

and the expected value of the safety margin divided by R is

$$\overline{SM'} = \frac{\overline{SM}}{R} = 3.8 \cdot 17.6 + 0.404 \cdot 124 - 68 = 48.98 \text{ kN/m}.$$

The standard deviation of the safety margin divided by R is

$$s'_{SM} = \frac{s_{SM}}{R} = \sqrt{l_i^2 s_c^2 + (\Sigma N_i)^2 s_{\tan \varphi}^2} = 23.92 \text{ kN/m}^2.$$

As the conditions $SM = 0$ (as also $v_c = 1$) and $SM' = 0$ lead to the same result, in the following we shall count with the latter (so the value R is indifferent now). As the standard deviation of v_c is

$$s_v = \frac{s_R}{\Sigma T_i} = \frac{s_{SM}}{\Sigma T_i} = \frac{23.92}{68} = 0.3518,$$

we obtain for λ —on assuming a normal distribution—the following three coinciding values:

$$\lambda = \frac{116.98 - 68}{23.92} = \frac{1.72 - 1}{0.3518} = \frac{1.72 - 1}{1.72 \frac{0.3518}{1.72}} \cong 2.047,$$

i.e. the probability of the failure is $p_f = 0.020$.

The values obtained for λ correspond to the expressions (11.16)–(11.18), i.e. p_f may be calculated from the standard deviation of both the safety factor and the safety margin.

If the stability is calculated by a method giving the *required cohesion*, then the probability of failure must be calculated as described in Section 11.2.2. Let us assume, e.g., a slope of $H = 5$ m height and the slope angle $\beta = 50^\circ$, with the following physical

characteristics of the soil:

$$\gamma = 18 \text{ kN/m}^3, \quad \bar{c} = 14.5 \text{ kN/m}^2.$$

$$\bar{\phi} = 15^\circ, \quad s_c = 4.6 \text{ kN/m}^2.$$

According to TAYLOR's graphs the stability factor N_c is 0.094, so the required cohesion is

$$c_n = 5.18 \cdot 0.094 = 8.46 \text{ kN/m}^2.$$

As according to Eq. (11.18)

$$\lambda = \frac{14.5 - 8.46}{4.6} = 1.313,$$

the probability of failure is $p_f = 0.095$.

So far only the variability of the shear strength parameters was considered. In the general case also other effects must be taken into account (e.g. the pore water pressure) and it may occur also that the variance of the driving moment is not negligible (if e.g. the surface level is loaded). On analysing the "sensitivity" in a concrete case ALONSO (1976) determined the variances shown in *Fig. 11.12* for the individual factors. (In this case the variance of M_Q is negligible and the variability of the pore water pressure is very significant.) The figure surveys thoroughly the factors influencing the stability of the examined slope and corresponds essentially to the percentage resolutions given in *Table 9.4*.

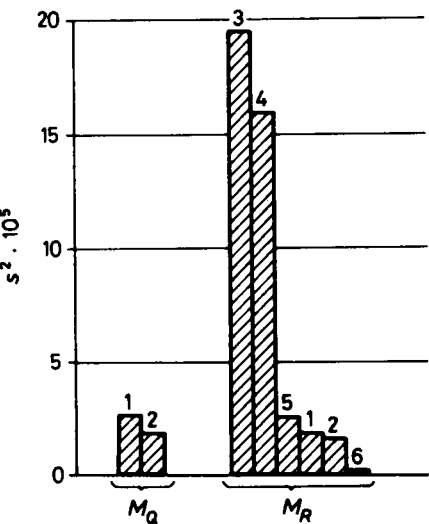


Fig. 11.12. Numerical example showing the components of the variance of the driving (M_Q) and resisting (M_R) moments

In given cases the *variation of the shear strength with the depth* may also be taken into account. For example, CORNELL (1971) established that the unconfined compression strength of the clay, forming the slope examined by him, varied with the depth z (m) according to

$$q_u(\text{kN/m}^2) = 41 + 6.2z.$$

He calculated the cohesion force arising on the rupture surface by assigning a $c = q_u/2$ value calculated from the relationship q_u, z to the centres of every section of the rupture surface.

The *model* uncertainty appears in the variance $s_{v,M}^2$ of the factor of safety in the following way:

$$s_{v,M}^2 = v_c^2 C_{vM} + s_v^2, \quad (11.26)$$

where C_{vM} — coefficient of variation of the safety factor due to the standard deviation of the model,

s_v^2 — variance of v_c calculated with the actually used model.

Let us assume in connection with the numerical example of Section 11.3.1. that $C_{vM} = 0.05$ (compare with Section 8.4). In this case

$$s_{v,M} = \sqrt{1.72^2 \cdot 0.05^2 + 0.3518^2} = 0.3622,$$

i.e.

$$\lambda = \frac{1.72 - 1}{0.3622} = 1.988,$$

to which $p_f = 0.023$ corresponds (in the preceding numerical example it was 0.020). If $C_{vM} = 0.1$, then $p_f = 0.033$; the deviation is not significant even so.

If M_Q and M_R are correlated and we wish also to take into account the uncertainty originating from the assumption of the model, the expected value of the safety factor μ_{FS} and its variance $\text{Var}(FS)$ may be calculated by the following expressions:

$$\mu_{FS} = \frac{\mu_R}{\mu_Q} \mu_\Delta - \frac{\mu_\Delta}{(\mu_Q)^2} \text{Cov}(M_R, M_Q) + \mu_\Delta \frac{\mu_R}{(\mu_Q)^3} \text{Var}(M_Q); \quad (11.27)$$

and

$$\begin{aligned} \text{Var}(FS) = & \left(\frac{\mu_\Delta}{\mu_Q} \right)^2 \text{Var}(M_R) + \frac{\mu_R \mu_\Delta}{\mu_Q^2} \text{Var}(M_\Delta) + \left(\frac{\mu_R}{\mu_Q} \right)^2 \text{Var}(\Delta) - \\ & - 2 \frac{\mu_R \mu_\Delta^2}{\mu_R^2} \text{Cov}(M_R, M_\Delta), \end{aligned} \quad (11.28)$$

where μ_R and μ_Q are the expected values of the moments M_R , resp. M_Q , Cov is the symbol of the covariance, Δ is a random dimensionless parameter representing the prediction error of the model. Both equations may be derived from the expression

$$FS = \frac{M_R}{M_Q} \Delta$$

with the help of the explanation described in Section 8.6.1.

On the basis of *Fig. 8.10* information may be obtained on the probability of failure also by another approximation. For each point pair c , $\tan \varphi$ we may determine—by interpolation between the curves $v_c = \text{const.}$ —various *safety factors*, then the mean and the standard deviation of these. The probability of failure may be calculated with the help of the last two parameters, e.g., by Eq. (11.17), if the distribution of v_c is normal.

11.3.2. Examination of the stability by the Monte Carlo method

An example of using the Monte Carlo method was presented by SINGH (1971), who examined the case of $\varphi \neq 0$, $c \neq 0$, assuming the normal distribution of both these parameters of the shear strength. He selected the angle of friction φ as one variable and the quotient $c/H\gamma$ as the other variable, assigning 200 random numbers to each. (With the tacit assumption that the variables are independent from each other.)

The initial data of one of his numerical examples were as follows:

	Mean	Coefficient of variation
$c/H\gamma$	0.13	0.30
φ	25°	0.10

Figure 11.13 shows the density and the distribution functions of the factor of safety v_c . The statistical parameters of the factors of safety were

$$v_c = 1.928 \quad \text{and} \quad s_v = 0.334,$$

so according to Eq. (11.17), $\lambda = 2.778$, to which $p_f = 0.003$ corresponds.

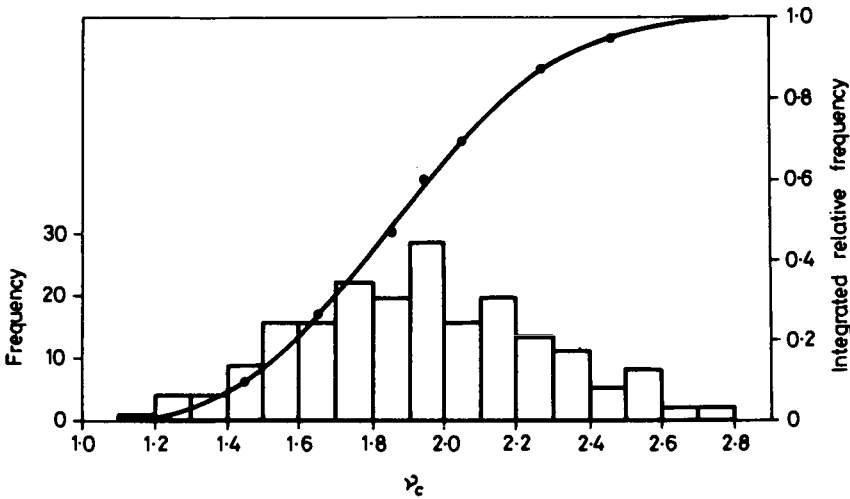


Fig. 11.13. Distribution of the factor of safety (v_c) of the slope stability determined by the Monte Carlo method (numerical example of SINGH, 1971)

SINGH (1971) examined several combinations by varying the statistical characteristics of the parameters of the shear strength. He established on the basis of the calculation results that

- the distribution of v_c may be regarded to be normal,
- the coefficient of variation of v_c is between those of φ and c ,
- the value $\bar{v}_c$ determined by the Monte Carlo method is nearly identical with the value calculated with the help of $\bar{\varphi}$ and $\bar{c}$.

The distribution of the v_{ci} -values is naturally not always normal "by necessity". In such cases we must proceed as described in Chapter 2.

The Monte Carlo method requires lengthy calculations (aided possibly by a high capacity computer), as compared to the process described in Section 11.3.1, but it has the following advantages against the latter.

a) Each value pair φ_i, c_i obtained from the random numbers determines a critical rupture surface (at which v_c is minimum), while the process described in Section 11.3.1 gives a single rupture surface (belonging to $\bar{\varphi}$ and $\bar{c}$), disregarding the fact that with the variation of φ also its position varies.

b) The distribution of the v_{ci} (or SM_i) values obtained from the random numbers can be controlled by fitting tests well, so we are not compelled to rely on assumptions.

CHERUBINI *et al.* (1983) used the Monte Carlo method for testing the simplified process of BISHOP. Their statements of interest are discussed in Section 11.3.3.

11.3.3. Taking the correlation between the angle of friction and the cohesion into account

In Section 11.1.2 some literature data were quoted, showing the closeness of the correlation between φ and c .

According to Eq. (8.30) the resultant standard deviation of two variables depends also on the correlation coefficient. This is true also, if both variables appear *only* in the expression of the load, or only in the expression of the resistance. Let us assume, e.g., that we examine the stability of a slope based on the equilibrium of the slices. In this case the expected value of the safety margin is according to formula (11.25)

$$\overline{SM} = R(\bar{c}l_i + \tan \varphi \Sigma N_i) - R \Sigma T_i,$$

i.e. both c and $\tan \varphi$ appear in the expression of the resisting moment. As the standard deviation of the "load" ($R \Sigma T_i$) is disregarded, the standard deviation of SM is

$$s_{SM} = \sqrt{l_i^2 s_c^2 + (\Sigma N_i)^2 s_{\tan \varphi}^2 - 2rl_i \Sigma N_i s_c s_{\tan \varphi}}. \quad (11.29)$$

Let us examine the role of the degree of correlation by the numerical example of Section 11.3.1. The standard deviation s_{SM} , the value λ calculated as the quotient

$\overline{SM}/s_{SM}$ and $p_f = 1 - \Phi(\lambda)$ taken from Table F2 vary with the correlation coefficient r as follows:

$r =$	0	-0.25	-0.5	-0.7	-0.9
s_{SM}	23.92	25.27	26.55	27.53	28.47
λ	2.047	1.938	1.845	1.779	1.720
p_f	0.020	0.026	0.033	0.038	0.043

As it is seen, p_f varies between the two extreme cases by more than 100%.

If the stability is calculated by the *Monte Carlo method*, it is advised to proceed by treating the angle of friction as a *random number* and assigning to it the cohesion calculated from the regression line (or from the curve determined by the fitting test) describing the relationship φ, c . In this case we may assume rightfully that the deviations caused by the residual standard deviation—due to the alternating sign of Δc —will be equalized if n is sufficiently great (which is a basic condition for the use of the Monte Carlo method).

NÉMETH (1978) divided the regression curve φ, c into six domains of an identical probability and selected six pairs of φ, c -values by throwing dice. The disadvantage of this method is that the number of the combinations is very small, especially as only 50 v_c -values were determined.

CHERUBINI *et al.* (1983) studied the factor of safety of a 8.0 m high slope, resp. the probability $FS < 1$ by the Monte Carlo method. Table 11.2 shows the means of φ and c and their standard deviations (the latter with two values each), as well as the p_f -values belonging to the various slope angles β versus $r_{\varphi, c}$. On this basis the authors established that

Table 11.2. Probability of failure (%) as a function of the slope angle β and of $r_{\varphi, c}$ (CHERUBINI *et al.*, 1983); $\bar{\varphi} = 25^\circ$, $\bar{c} = 30$ kN/m², $s_\varphi = 5^\circ$ (A) and 2.5° (B), $s_c = 5$ kN/m² (A) and 2.5 kN/m² (B)

$r_{\varphi, c}$	$\beta =$			
	20°	23°	25°	28°
-1.0 A	~0	~0	0.3	15.3
B	~0	~0	0	2.4
-0.5 A	~0	1.3	8.0	28.1
B	~0	~0	1.1	16.6
0.0 A	0.8	5.5	12.6	29.3
B	~0	0	1.3	21.7
0.5 A	1.4	7.9	14.2	30.0
B	~0	0.5	3.6	23.9
1.0 A	2.9	9.0	14.1	29.9
B	~0	1.5	5.1	24.5
$FS =$	1.46	1.28	1.18	1.06

- a) for correlation coefficients varying between 0 and -1.0 there are noteworthy variations in the probability of failure;
- b) the positive values of the correlation coefficients indicate minor variations in p_f ;
- c) as the mean factor of safety approaches 1, the incidence of the variation of r becomes ever lower (the higher the standard deviation, the more this statement is applicable).

The correlation between φ and c can be taken into account—in the possession of a sufficiently large test material—also by calculating a factor of safety for each laboratory φ, c -value pair (e.g. from Fig. 8.10) and determining their mean and their standard deviation. The situation is still more fortunate if we examine the stability of vertical walls; in this case we may carry out the calculation with a *single* variable (q_u) instead of two variables (φ and c), as described in Section 8.5.2.

11.4. Preliminary and posterior allowance of uncertain factors

In practice numerous designing tasks are met with, where the variability of certain factors (or the limited accuracy of their determination) can be allowed for in advance. The following example shows the possibility of estimating the *effective shear strength of the soil* (τ_e) on the basis of laboratory values (τ_l) (TANG *et al.*, 1976). The correlation between these two values is expressed as

$$\bar{\tau}_e = (\bar{N}_1 \cdot \bar{N}_2 \dots \bar{N}_n) \bar{\tau}_l,$$

where each N_i corresponds to a correction factor. The authors propose numerical values for the same and for their coefficients of variation on the basis of literature data in the following way.

The first correction factor must be applied, because the stress state of the soil varies during the sampling ($\bar{N}_1 = 1.03$, $C_v = 0.02$). Another consequence of the imperfect sampling is that the structure and certain physical characteristics of the soil vary ($\bar{N}_2 = 1.2$, $C_v = 0.12$). In addition, the geometrical dimensions of the sample play also an important role ($\bar{N}_3 = 0.85-1.00$), especially in the case of rigid soils, which incline to crack ($\bar{N}_3 = 0.55-0.85$). Also the rate of shearing in the laboratory is very fast in relation to its effective value ($\bar{N}_4 = 0.58-0.95$, $C_v = 0.03-0.22$, depending on the relative sensitivity of the actual soil). In the case of anisotropic soils the difference between the direction of the normal stresses attained in the laboratory and those acting effectively along the slip surface is not negligible either ($\bar{N}_5 = 0.80-1.08$). The field conditions deviate from the laboratory conditions also by the predominance of a plane stress state there ($\bar{N}_6 = 1.05$, $C_v = 0.03$). And finally also the fact must be remembered that the real situation is characterized generally by a progressive soil failure ($\bar{N}_7 = 0.95$, $C_v = 0.03$).

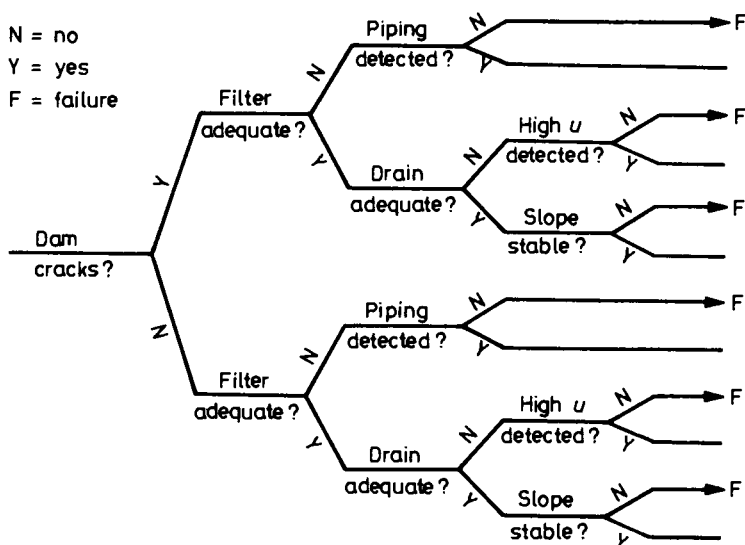


Fig. 11.14. Event tree for evaluation of geotechnical aspects of dam safety (WHITMAN, 1984)

The product of the correction factors is less than 1 in the majority of the cases (the authors demonstrated, e.g., 0.65 for one of the soils), which means that the *shear strength* determined by *laboratory tests* exceeds its effective value.

By filtering out the uncertainties (or faults) in advance we may allow a higher value for the *probability of failure*. We must proceed similarly if we take the occurrence of an event *for certain*, independent of when it will occur during the lifetime of the object. (This principle must be applied also in solving tasks of another type, e.g., simulating the effect of the rise of the groundwater level by examining artificially flooded samples.) The conditions of the simultaneous occurrence of several such events must be taken into account in all these cases.

WHITMAN (1984) presents an example of judging the behaviour of an object (in this case a dam). (The process is suitable also for examining designs.) Fig. 11.14 shows a simple *event tree*, in which each branch point represents some characteristic of the dam that has a major effect on the safety.

The phrase *dam cracks* includes any event that might cause a sufficiently strong water flow to start an internal erosion within the embankment, the foundation, or the abutments (e.g., cracking of the core, incomplete foundation cutoff). The *filter* includes all the provisions for preventing the growth of an internal erosion. The word *drain* refers to all the details intended to relieve pore pressures within the downstream portions, i.e. both internal and foundation drains. The provisions for detecting the onset of piping, or high pore pressures include not only physical measuring systems, but also procedures for making and interpreting observations, and acting upon them, concerning the physical state of the dam. The “no” branch at “*high μ detected*” means that no pore pressures are observed, which could be sufficiently high to cause failure, or they were counteracted in time to prevent the stability failure; the “yes” branch means that steps are taken to reduce these pore pressures. The phrase “*slope stable*” refers to the possibility of a slope failure when the pore pressures are essentially equal to those assumed for the design.

The example was intended to prove that a simple event tree may be used to identify the important features that must be evaluated in order to characterize the *overall* safety. It provides a systematic approach to accounting for multiple lines of defense against failure.

11.5. Examination of special cases

11.5.1. The examination of the three-dimensional case

Though several authors have dealt with slopes of a finite length (e.g. ANAGNOSTI, 1969), it was only a few years ago that they were discussed on the basis of the theory of probability (VANMARCKE, 1977b).

The task and the notations are shown in *Fig. 11.15*. The classic circular arc stability analysis defines the factor of safety for a finite length b as follows:

$$F_b = \frac{M_r}{M_o} = \frac{s_b L r b + R_e}{W a b}, \quad (11.30)$$

in which s_b is the average shear strength on the cylindrical failure surface of the length b and R_e is the contribution of the end-sections to the resisting moment.

For reducing the number of the data related to the shear strength we must calculate first the strength for each segment of unit length (which may be, e.g., the distance between the drillings) perpendicular to the cross section. These spatial average strengths will tend to differ from segment to segment. If n spatial averages $s_{i,j}$ are available, we estimate

$$\bar{s}_1 = \frac{\sum_i s_{1,i}}{n} \quad (11.31)$$

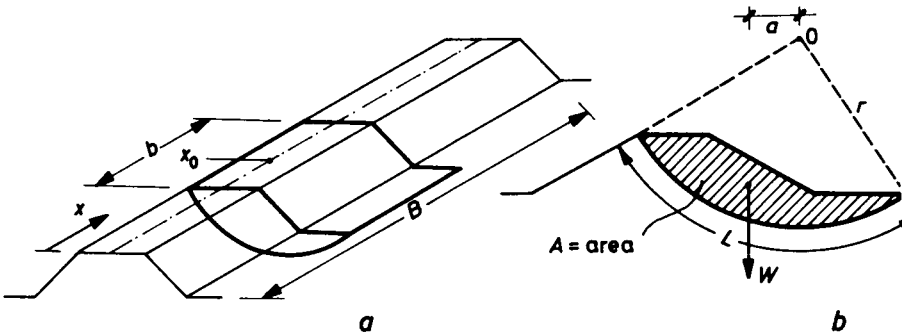


Fig. 11.15. (a) View of embankment and failure area and (b) cross section within failure area (VANMARCKE, 1977b)

and

$$\tilde{s}_1 = \sqrt{\frac{\sum_i (s_{1,i} - \bar{s}_1)^2}{n-1}}, \quad (11.32)$$

where $\tilde{s}_1$ denotes the standard deviation.

Next, the spatial averages of the strength on all sections of a width $\Delta x = 2$ are computed. The mean is again $\bar{s}_1$ and the standard deviation is denoted by $\tilde{s}_2$. The sections of an increasing width $\Delta x = b$ are considered similarly and their strength is averaged over the cylindrical failure surface Lb . While the mean of all the strength averages is bound to remain constant, $\tilde{s}_b$ will tend to decrease, as b increases and it may be related to the standard deviation $\tilde{s}_1$ of “unit width” as follows:

$$\tilde{s}_b = \Gamma(b) \cdot \tilde{s}_1, \quad (11.33)$$

in which $\Gamma(b)$ is a dimensionless factor. According to VANMARCKE (1977b), $\Gamma(b)$ must be assumed as follows:

$$\begin{aligned} \Gamma(b) &= 1; & b &\leq \delta \\ \Gamma(b) &= \sqrt{\frac{\delta}{b}}; & b &\geq \delta \end{aligned} \quad , \quad (11.34)$$

where δ is the characteristic distance and indicates the typical width of the sections of the embankment with a relatively low, or high *average* strength (see Sections 4.2 and 11.1.2).

The average factor of safety decreases with b tending to a minimum value, which corresponds to the case of an infinitely long ($b \rightarrow \infty$) shear surface, i.e. the plane strain case. The average of F_b is

$$\bar{F}_b = \frac{\bar{s}_1 Lrb + R_e}{Wab}. \quad (11.35)$$

On assuming that the uncertainty in the contribution of the end-section to the resisting moment is negligible, we have

$$R_e \approx \bar{R}_e = 2Ar'\bar{s}',$$

where $\bar{s}$ is the standard deviation of all “point” shear strength values, r' is the effective rotation arm for the end sections. (The ratio r'/r may be calculated by integrating the contributions to the resisting moment from all the points of the cross section.)

As b increases, $\bar{F}_b$ approaches the limit

$$\bar{F}_\infty = \lim_{b \rightarrow \infty} \bar{F}_b = \frac{\bar{s}_1 Lr}{Wa} = \bar{F}. \quad (11.36)$$

A useful way to express $\bar{F}_b$ in the context of this analysis is as follows:

$$\bar{F}_b = \bar{F} \left(1 + \frac{2Ar'}{Lrb} \right) = \bar{F} \left(1 + \frac{d_0}{b} \right), \quad (11.37)$$

where the length $d_0 \approx 2A/L$ depends only on the characteristics of the cross section and the shear surface.

The standard deviation of F_b is

$$\tilde{F}_b = \frac{\tilde{s}_b L r b}{W a b} = \frac{\tilde{s}_b}{s_1} \frac{\tilde{s}_1 L r}{W a} = \frac{\tilde{s}_b}{\tilde{s}_1} \bar{F} = \Gamma(b) C_{vs1} \bar{F}, \quad (11.38)$$

where $C_{vs1} = \tilde{s}_1/\bar{s}$ is the coefficient of variation of s_1 .

A useful parameter of the reliability is the ratio $\beta_b = (\bar{F}_b - 1)/\tilde{F}_b$ (see in Chapter 8); the failure risk $p_f(b)$ decreases as β_b increases. We may write:

$$\begin{aligned} p_f(b) &= P(F_b \leq 1) = P\left(\frac{F_b - \bar{F}_b}{\tilde{F}_b} \leq -\beta_b\right) = \\ &= P(U \leq -\beta_b) = F_u(-\beta_b), \end{aligned} \quad (11.39)$$

where F_u denotes the cumulative distribution of the standardized variable $U = (F_b - \bar{F}_b)/\tilde{F}_b$.

Inserting the value of $\bar{F}_b$ and $\tilde{F}_b$ into the expression defining β_b we obtain:

$$\beta_b = \frac{\bar{F}_b - 1}{\tilde{F}_b} = \left(1 + \frac{d_0}{b} - \frac{1}{F}\right) [\Gamma(b) C_{vs1}]^{-1}. \quad (11.39a)$$

At values of $b > \delta$ we have $\Gamma(b) = (\delta/b)^{1/2}$. Inserting this expression into Eq. (11.39a) and setting the derivative of β_b with respect to $b = 0$, we obtain the value of the width b , at which β_b is minimum:

$$b_c = \frac{\bar{F}}{\bar{F} - 1} d_0. \quad (11.40)$$

This *critical value* b_c locates the true minimum of β , if $b_c \geq \delta$. In the contrary case (if $b_c \leq \delta$) the minimum of β_b is theoretically at $b = \delta$.

VANMARCKE (1977b) demonstrated the correctness of his derivation by the example of a slope, a *finite section* of which slipped.

11.5.2. Examining the possibility of successive failure

Observations of natural slopes revealed that failures are often successive in character (Fig. 11.16, a). A single specific failure may involve only a limited area of the slope. This failure may then be followed by others. Thus it is of interest to consider not only the reliability of a long natural slope, but also the probability of failure propagation to the next stage, should the first failure of a limited extent take place (CHOWDHURY, 1983).

The stability of artificial earthworks is examined for the "critical failure surface" (in the case of a probabilistic model this is the surface belonging to $p_f = \max$). If a failure occurs along the assumed critical surface (or near to it), it may be followed by another failure (Fig. 11.16, b, c).

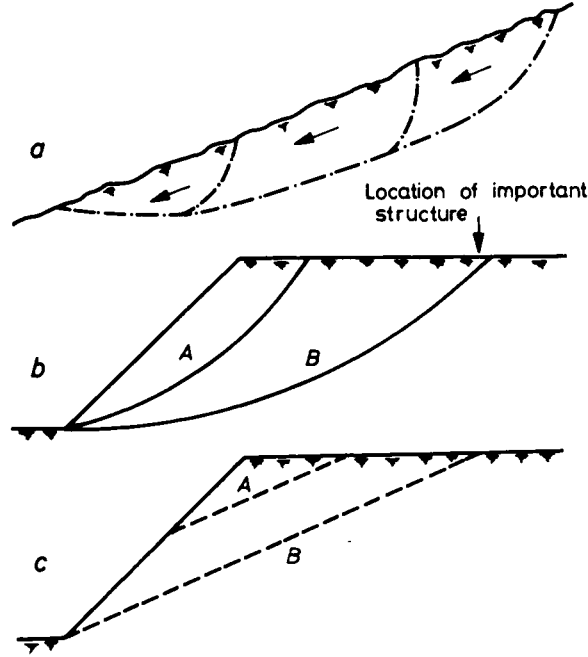


Fig. 11.16. (a) Potential successive failure in a natural slope; (b) two possible slip surfaces in a cut slope; (c) successive joints in rock slope-potential slip planes (CHOWDHURY, 1983)

CHOWDHURY (1983) discusses the case of the successive failure in the following way.

Considering a soil (rock) mass bounded by an assumed slip surface A , the failure of this mass may be defined in terms of the safety margin denoted by SM_A , as

$$(\text{Failure})_A = SM_A \leq 0. \quad (11.41)$$

SM_A may be expressed by the capacity R and the demand Q as:

$$SM_A = \sum_{i=1}^n (R_A - Q_A), \quad (11.42)$$

where i is any slice and n is the total number of slices.

The probability of failure is according to the usual definition

$$p_{f(A)} = P(SM_A \leq 0). \quad (11.43)$$

Now let us consider another slip surface B (Fig. 11.16, b). The probability of failure of the mass above B may be defined as

$$p_{f(B)} = P(SM_B \leq 0). \quad (11.44)$$

Failure along B following a failure along A is denoted as the following condition:

$$\begin{aligned} \text{Failure}_{AB} &= (\text{Failure } B | \text{Failure } A) \\ SM_B &\leq 0 | SM_A \leq 0. \end{aligned} \quad (11.45)$$

The probability of failure along A being followed by a larger failure along B (denoted by P_{AB}) may be defined as the following *conditional probability* according to Eq. (7.9):

$$P_{AB} = \frac{P[(SM_B \leq 0) \text{ and } (SM_A \leq 0)]}{P[SM_A \leq 0]}. \quad (11.46)$$

The joint distribution of SM_A and SM_B is assumed to be bivariate normal.

The safety margins of adjacent soil zones within a given mass may not be (and in general are not) independent of each other. (This is the reason for the assumption of a joint distribution.) Such a distribution requires five parameters for its description: the two mean values, $\overline{SM}_A$ and $\overline{SM}_B$, the two variances $s_{SM_A}^2$ and $s_{SM_B}^2$ and the correlation coefficient $r_{SM_A SM_B}$. After evaluating these parameters the joint probability (the numerator of Eq. 11.46) is obtained by numerical integration of the bivariate probability density function. The evaluation of the means and the variances of the safety margins follows established numerical procedures. For example, the first order moment approximation based on the TAYLOR-series expansion, or the point estimate method of ROSENBLUETH (see in Section 11.2.2) may be used. In both cases the only data required would be the geometry of the slip surfaces and the values $\bar{c}$, $\bar{\varphi}$, s_c , s_φ and $\text{Cov}(c, \varphi)$. The covariances of SM_A and SM_B may be also evaluated without any additional data, provided that the assumption of statistical homogeneity is made.

Example (CHOWDHURY, 1983). Let the height of a simple earth slope be 7.62 m. The slip surface A is conventionally a critical slip surface with a radius of 13.45 m; the slip surface B enclosing a larger and deeper soil mass has a larger radius than A and its lower sections are relatively flat. Obviously such a surface is, on its

Table 11.3. Calculated probabilities of failure for the example problem of CHOWDHURY (1983)

Probability of failure along A considered separately	p_{fA}	$8.4 \cdot 10^{-3}$
Probability of failure along B considered separately	p_{fB}	$9.5 \cdot 10^{-7}$
Correlation coefficient of safety margins SM_A and SM_B	r_{AB}	0.9588
Probability of failure along B if failure along A has occurred	p_{AB}	$1.14 \cdot 10^{-4}$

Table 11.4. Influence of $C_{vu}(u = \tan \varphi)$ on failure probabilities (CHOWDHURY, 1983)

C_{vu}	p_{fA}	p_{fB}	p_{AB}
0.10	$0.4 \cdot 10^{-3}$	$9.5 \cdot 10^{-7}$	$1.14 \cdot 10^{-4}$
0.15	$8.7 \cdot 10^{-3}$	$6.65 \cdot 10^{-6}$	$7.6 \cdot 10^{-4}$
0.20	$9.6 \cdot 10^{-3}$	$6.26 \cdot 10^{-5}$	$5.4 \cdot 10^{-3}$
0.25	$11.3 \cdot 10^{-3}$	$3.01 \cdot 10^{-4}$	$2.65 \cdot 10^{-2}$
0.30	$13.7 \cdot 10^{-3}$	$1.24 \cdot 10^{-3}$	$8.11 \cdot 10^{-2}$
0.35	$17.0 \cdot 10^{-3}$	$3.61 \cdot 10^{-3}$	0.17
0.40	$21.2 \cdot 10^{-3}$	$8.07 \cdot 10^{-3}$	0.28
0.45	$25.5 \cdot 10^{-3}$	$14.9 \cdot 10^{-3}$	0.39
0.50	$32.9 \cdot 10^{-3}$	$24.2 \cdot 10^{-3}$	0.49

own, far less critical than the slip surface A . The following soil parameters were assumed: $\beta = 45^\circ$, $\bar{c} = 23.94 \text{ kN/m}^2$, $C_{vc} = 0.50$, $\bar{u} = \tan \varphi = 0.79$, $C_{vu} = 0.10$, $\gamma = 18.84 \text{ kN/m}^3$, $r_{cq} = -0.20$.

The influence of the failure along surface A on the stresses along B was taken into account adequately. It was assumed that the soil mass found above A slips down the slope, if failure along A occurred. The results are given in Table 11.3.

The influence of C_{vu} on the progression of the failure was also studied. Keeping all other parameters unchanged, C_{vu} was increased from a low value of 0.1 to a relatively high value of 0.50. The probability P_{AB} of the failure progression increases dramatically, as shown in Table 11.4; the increase above $C_{vu} = 0.3$ is especially significant.

11.5.3. The examination of the progressive failure process

The progressive character of the earth slope failures has been recognized for many decades (TERZAGHI, 1936; CASAGRANDE, 1950). External observations and research studies have confirmed that the failure starts always at one location and then spreads progressively to other regions within a soil mass. Even when there are no external signs of failure, certain regions with a sloping ground may have been already overstressed and the propagation of the failure may have begun (CHOWDHURY and GRIVAS, 1982). In cuttings the failure spreads from the bottom upwards, because there is a great stress concentration at the bottom (BJERRUM, 1967; PALMER and RICE, 1973; CHOWDHURY, 1978), while in the case of natural slopes the situation is the reverse (PECK, 1967).

With the exception of the Swedish method all methods are based on the assumption that the "overall factor of safety" is the same as the "local factor of safety" at any point along a potential slip surface. Limit equilibrium methods (and generally deterministic methods) cannot be used to simulate the history of the construction, or to study the influence of the initial stresses on the safety of the slope.

The probabilistic study of the progressive failure has been taken up only recently (CHOWDHURY and GRIVAS, 1982; OBONI and BOURDEAU, 1983).

CHOWDHURY and GRIVAS (1982) discuss in detail the case where the failure spreads from the bottom upwards. On dividing the slipping mass to n slices $i = 1$ denotes the lowest slice. The failure is defined as the event whereby the safety margin assumes values less than or equal to zero, i.e.

$$(\text{Failure})_i = (SM_i \leq 0) = [(R_i - Q_i) \leq 0], \quad (11.47)$$

in which

$$R_i = cL_i + N_i \tan \varphi, \quad Q_i = W_i \sin \alpha_i. \quad (11.48)$$

(α_i is the inclination of the base to the horizontal.)

The safety margin of the different segments is a function of the same random variables, c and φ . Therefore, the safety margins of the adjacent slices are not independent of each other and their joint distribution must be considered. The assumption of a bivariate normal distribution for a pair of adjacent segments accords with the Central Limit Theorem. In this case the following five parameters must be

determined: $\overline{SM}_i$, $\overline{SM}_{i+1}$, $s_{SM_i}^2$, $s_{SM_{i+1}}^2$ and

$$r_{SM_i, SM_{i+1}} = \frac{\text{Cov}(SM_i, SM_{i+1})}{s_{SM_i} s_{SM_{i+1}}}$$

These parameters should be determined as described in Section 11.5.2 on the basis of the TAYLOR-series, or by the method of ROSENBLUETH.

The state of the failure progression may be identified by the number of the slices along a slip surface, to which the failure has progressed. Having reached a certain state, transition of the failure may, or may not occur to the next state, as shown in the transition diagram (Fig. 11.17). An absorbing state is one from which there is no further transition; e.g., when the last slice has failed (stage $i=n$), there is no further transition and the probability of being in stage n is equal to one.

From the transition diagram it is seen that when the failure has reached the i -th state, all the slices up to the slice i have failed. At this state there are two possibilities: the failure progression terminates at the i -th slice with a probability of p_{ii} , or it continues to the $(i+1)$ -th state, with a probability of $p_{i,i+1}$. The transition probabilities p_{ii} and $p_{i,i+1}$ ($i=1, 2, \dots, n$) can be represented in the form of a matrix P .

An element $p_{i,j}$ ($i, j=1, \dots, n$) of the $[n \times n]$ transition matrix provides for the probability of the transition from state i to state j . The transition matrix has the following form:

$$P = \begin{bmatrix} & 1 & 2 & 3 & i & i+1 & n \\ \begin{matrix} 1 \\ 2 \\ 3 \\ i \\ i+1 \\ n \end{matrix} & \begin{matrix} p_{1,1} & p_{1,2} & 0 & 0 & 0 & 0 \\ 0 & p_{2,2} & p_{2,3} & 0 & 0 & 0 \\ 0 & & & p_{i,i} & p_{i,i+1} & 0 \\ 0 & 0 & 0 & 0 & p_{i+1,i+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix} \quad (11.49)$$

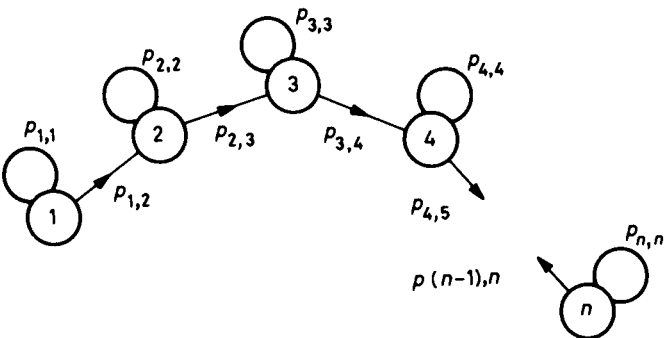


Fig. 11.17. Transition diagram on failure progression (CHOWDHURY and GRIVAS, 1982)

The conditional probability $p_{i,i+1}$ may be written as

$$p_{i,i+1} = \frac{P[(SM_i < 0) \text{ and } (SM_{i+1} < 0)]}{P[SM_i < 0]}. \quad (11.50)$$

The denominator of Eq. (11.50) is

$$P[SM_i \leq 0] = \int_{-\infty}^0 f_{SM_i}(x) dx = F_{SM_i} \leq 0 \quad (11.51)$$

and the numerator of Eq. (11.50) is

$$P[(SM_i \leq 0) \text{ and } (SM_{i+1} \leq 0)] = \int_{-\infty}^0 \int_{-\infty}^0 f_{SM_i, SM_{i+1}}(x, y) dx dy, \quad (11.52)$$

in which x and y are the dummy variables of integration (see, e.g., ELISHAKOFF, 1983).

The authors discuss as an example the case of the slope described in the example of Section 11.5.2. On dividing the sliding earth mass to nine slices and carrying out the calculations described before they obtained the following transition matrix:

$$\mathbf{P} = \begin{bmatrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} & \begin{matrix} 0.87 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0.13 & 0.81 & 0 & 0 & 0 & 0.26 & 0.19 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0.19 & 0.64 & 0.49 & 0 & 0 & 0 & 0.65 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0.36 & 0.51 & 0.31 & 0 & 0 & 0 & 0.31 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0.69 & 0.74 & 0.81 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}$$

As it is seen, at the beginning the probability is high that the slope remains in the initial state, as against the probability of the progression to the next stage. In the successive stages the probabilities of the failure progression increase steadily until the penultimate stage. The fact that the magnitude of the probability decreases again in this stage may be a reflection of the geometry of the particular slip surface (CHOWDHURY and GRIVAS, 1982).

OBONI and BOURDEAU (1983) determine the transition matrix also on the basis of similar considerations, with the following differences as compared to the preceding method:

- they regard the force P_{i+1} acting on the i -th slice as a random variable, i.e. the force transmitted by k elements ($k=i+1, \dots, n$) uphill to the i -th element;
- they assume for c , φ and p_{i+1} the beta distribution;
- they begin the computation with the evaluation of the transition probability of the n -th slice (*top* of the slope).

On examining a concrete slip the transition matrix was the following:

	1	2	3	4	5	6	7
1	0.178	0.822	0	0	0	0	0
2	0	0.018	0.982	0	0	0	0
3	0	0	0.002	0.998	0	0	0
4	0	0	0	0.001	0.099	0	0
5	0	0	0	0	1	0	0
6	0	0	0	0	0	1	0
7	0	0	0	0	0	0	1

As it is shown, the probability of the initiation of the sliding was found to be 0.2 and the transition probabilities increased very quickly uphill. It can be concluded that once the first element slipped, there was a high probability of a general movement. According to the authors the location of the existing *tension cracks* can be predicted by the model: such cracks appear, when the transition probabilities show a very rapid relative increase. These discontinuities in the transition diagram are due to the close proximity of highly unstable and less unstable zones.

11.6. The optimum geometrical dimensions of the slope

According to the data of FELD (1965) about 1800 earth dams were built in the United States before 1959 and 33 of them failed (mostly because their foundations were unsatisfactory). Combining these data with those of MIDDLEBROOKS (1953), MEYERHOF (1970) established that 1% of the earth dams built in 30 years failed on the basis of *soil-mechanical investigations* partly or fully in 1/3 part of the cases by sliding and in 2/3 part of them by having been washed out.

These data support the severally verified statement that with the usual values assumed for the central safety factor ($v_c = 1.3-1.6$) the probability of failure is 0.01-0.02. The designer may deviate from these values in both senses, depending on whether a temporary, or a permanent earthwork is involved, what consequences the failure might

have on the safety of life and property, at what cost it can be reconstructed, etc. As the probability of failure can be assigned to the geometrical dimensions of the slope, the latter may be determined on the basis of the *economic optimum*. Examples for this process are given below.

LANGEJAN (1965) determined the slope angle β for the cutting of $H = 2.5$ m depth, to which the cost minimum belonged. The author took the minimum cohesion required for the stability from TAYLOR's graph and carried out the subsequent calculations with the simplifying assumptions detailed below:

- a) the probability of failure is the function of only the shear strength acting on the slip surface;
- b) the construction cost is proportional with the mass of the shifted earth, i.e. to $\cot \beta$,
- c) the damage caused by the failure of the slope is proportional with the sliding earth mass.

The total cost consists of two components: the construction cost and the cost of the reconstruction multiplied by p_f . As the former cost decreases, the latter increases with the increase of β , the total cost is described generally by a curve, which has a *minimum*. The total cost calculated with the given physical characteristics of the soil (ϕ , c and their standard deviation, as well as γ) are shown in Fig. 11.18. Curves a and b correspond to the cases where p_f depends only on the distribution of ϕ , and only on that of c , respectively. Curves a' and b' were plotted on the basis of the assumption that the relative cost of the damage caused by failure is 100-times the specific construction cost. In the first case $p_f = 0.05$, in the case where the sliding causes heavy damages, $p_f = 0.0005$.

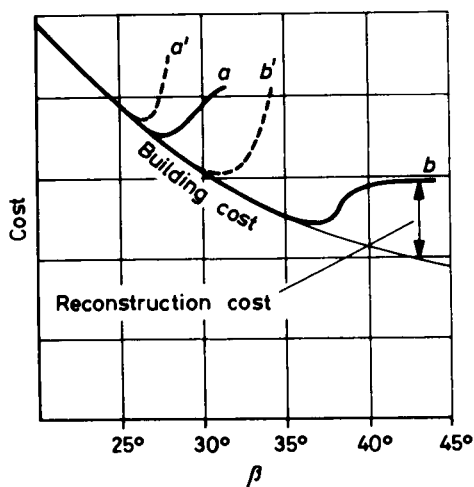


Fig. 11.18. Example for the determination of the optimum slope angle (LANGEJAN, 1965); a and b : relative costs of the construction and the damage are identical, a' and b' : the latter is 100-fold of the former

NÉMETH (1979, 1980) proposed to determine the optimum slope angle of the cutting as follows.

The construction cost P_e is composed of two parts: the cost of the excavation and the cost of the slope formation (grassing, etc.). If we assume that the depth H of the cutting is constant over its whole length L , then the construction cost will be as follows:

$$P_e = q_f V_f + q_r F = HL \left[q_f (k \times H \cot \beta) + 2 \frac{q_r}{\sin \beta} \right], \quad (11.53)$$

where

V_f — volume of earth to be excavated (m^3),

q_f — unit cost of the excavation of earth,

F — total surface area of the slope,

q_r — unit cost of the slope formation,

k — crown width of the cutting,

β — slope angle.

The cost K arising from the damage of the earthwork is composed also of two parts: the cost

$$P_k = l q_k$$

of the effective damage caused by sliding and the cost of the reconstruction:

$$P_h = V_h q_{hf} + F_h q_{hr},$$

where l — length of the sliding earth mass,

q_k — damage/1 m sliding over the length,

V_h — volume of the sliding earth mass,

F_h — damaged surface area of the slope,

q_{hr} — reconstruction cost of 1 m^2 surface area of the slope.

On introducing the relative numbers $\varepsilon = l/2L$ (denoted by λ in the quoted article) and $\alpha = a/A$ (where a = mean cross section area of the sliding earth mass, A = cross section area of the cutting), the total cost caused by the sliding is

$$K = \varepsilon 2L \left(q_k + \alpha A \cdot q_{hf} + \frac{H}{\sin \beta} q_{hr} \right). \quad (11.54)$$

The factor α (resp. the area a) may be obtained by constructing the ruling failure surface belonging to $\bar{\varphi}$ and $\bar{c}$. The factor ε may be taken, according to NÉMETH (1979, 1980), to be identical with p_f , which is the function of β . Their numerical values are obtained by calculating for each β the mean safety factor $\bar{v}_c$ and its standard deviation s_v , from which the probability $p_f = \varepsilon$ is calculated with the help of Eq. (11.17). (The author projects the standard deviation of the physical characteristics to the length of the slope.)

Figure 11.19 shows the optimum slope angle β , at which the sum $P_e + K$ is minimum.

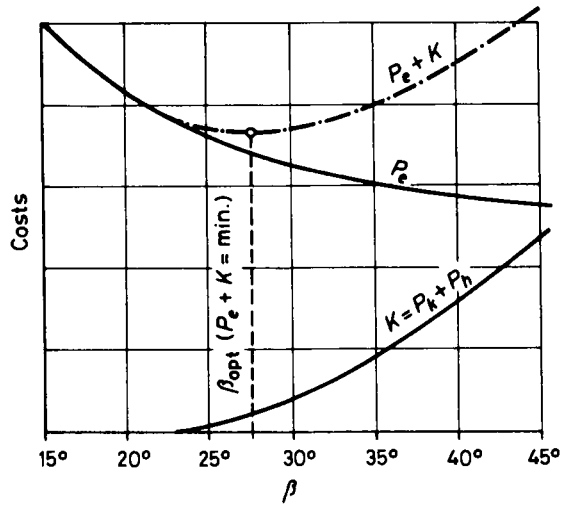


Fig. 11.19. Example for the determination of the optimum slope angle (NÉMETH, 1979, 1980)

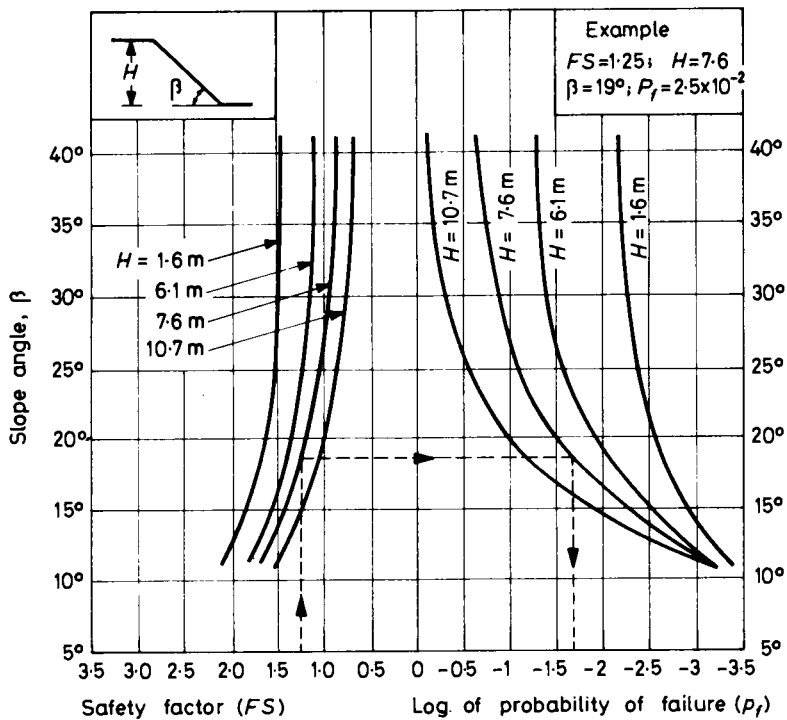


Fig. 11.20. Factor of safety vs. probability of failure for any slope angle β and height H (GRIVAS, 1981)

The economical dimensions may be determined in the first approximation with the help of graphs establishing the correlation between p_f , H and β , as shown in the example of *Fig. 11.20*, after GRIVAS (1981).

The method used by VAN DANZIG (1956) for determining the height of Dutch dikes may also be used (e.g. for flood dikes anywhere), though it is not related closely to the stability of slopes. Somewhat more than half the area of Holland lies under the level of the North Sea. The dikes protecting the land against the sea are exposed to very great forces, especially during the winter storms. As they cannot be dimensioned for a perfect safety, parts of the country are sometimes inundated (e.g. on 1 February 1953, 1800 people died, 150,000 ha were flooded, 9000 buildings were ruined, 38,000 damaged; the whole loss amounted to about 1.5–2 billion Dutch guilders).

The economic decision problem may be formulated as follows: "taking into account the cost of constructing the dike, the material losses, when a breach occurs and the distribution of different sea levels determine the optimum height of the dike". The solution of VAN DANZIG is described by ELISHAKOFF (1983) as follows.

Assume that the future dikes will all have the same height above a given standard level, so that the amount x by which the dikes must be elevated is

$$x = H - H_0. \quad (11.55)$$

The cost C of elevating the dikes from H_0 to H is a function of x .

Let h denote the sea level along the dikes at any moment; then no loss is incurred so long as $h \leq H$; if $h > H$, one may disregard the possibility of partial losses and reckon with the total loss only, i.e. with the circumstance that all the houses, etc. are lost. Let V be the total value of the property and assume that the consequent losses (slump in the production, etc.) are included in it. We then have

$$S = \begin{cases} 0, & \text{if } h \leq H \\ V, & \text{if } h > H \end{cases} \quad (11.56)$$

VAN DANZIG assumed that a sum L would be reserved to cover all future losses. If L is invested at a certain rate of interest i (in percentage), it must cover the expected value of all the future losses, $P(H_m > H)V$ per annum, so we have

$$\begin{aligned} L &= P(H_m > H)V \sum_{t=0}^{\infty} (1 + 0.01i)^{-t} = P(H_m > H)V \int_0^{\infty} e^{-0.01it} dt = \\ &= \frac{100}{i} P(H_m > H)V. \end{aligned} \quad (11.57)$$

VEMELSFELDER found that the annual exceedence frequencies during high tides in the period 1888–1937 followed the exponential distribution

$$P(H_m > h) = \exp\left(-\frac{h - \alpha}{\beta}\right). \quad (11.58)$$

The cost of elevating the dikes is

$$I = I_0 + kx, \quad (11.59)$$

where I_0 is the initial cost and k the subsequent cost of elevation per meter.

Adding (11.57) and (11.59) we obtain

$$I + L = I_0 + kx + 100 \frac{V}{i} \exp [(-H - \alpha)/\beta]. \quad (11.60)$$

Now we determine x so that $I + L$ is minimal, i.e.

$$\frac{dI}{dx} + \frac{dL}{dx} = 0 \quad (11.61)$$

or

$$k = \frac{100 V \exp [(-H - \alpha)/\beta]}{\beta_i} = 0, \quad (11.62)$$

yielding the optimum dike elevation increment

$$x = \alpha + \beta \ln \frac{100 V}{k \beta_i} - H_0. \quad (11.63)$$

With the constants $\alpha = 1.96$ m, $\beta = 0.33$ m, $V = 24 \cdot 10^9$ Guilders, $k = 40 \cdot 10^6$ Guilders/m, $i = 1.5\%$, $H_0 = 4.25$ m. VAN DANZIG's analysis yielded the optimum increment of 1.57 m.

11.7. The stability of earth masses supported by retaining walls

If the earth mass is artificially delimited, we must examine the stability of the supporting structure instead of that of the earth mass. In the following the earth pressure acting on *retaining walls* (in the first place its *active* variant) in the case of $c = 0$ is discussed. (The methodological guiding principles are valid also for earth masses supported by bulkheads, or struts, as well as for the case of $c \neq 0$.)

A *direct* consequence of the variability of the angle of friction is that also the earth pressure varies around a mean value (Section 11.7.1) and its *indirect* effect—asserted through the earth pressure—is that the inclination and the eccentricity of the resultant, the safety against soil failure, overturning and sliding, so also *the geometrical dimensions of the retaining wall are random variables*.

From among the three criteria of the stability which must be examined together (see Section 11.7.3) it is worthwhile to deal separately with the overturning (Section 11.7.2), as a typical probabilistic task.

11.7.1. The variability of the earth pressure

The standard deviation of the earth pressure acting on the wall supporting granular soil is composed of the standard deviations of several components (angle of friction, bulk density, pore water pressure, etc.). In the following we disregard the effects of the precipitations and the groundwater and neglect the standard deviation of the bulk density (compare with Section 1.3.4), so we shall examine the variability of the earth pressure exclusively in the function of the standard deviation of the angle of friction.

SINGH (1971) studied the variability of the RANKINE factors (the "hydrostatic", or earth-pressure factors) defined by

$$K_A = \tan^2 (45^\circ - \varphi/2) \quad (11.64)$$

and

$$K_P = \tan^2 (45^\circ + \varphi/2). \quad (11.65)$$

Fixing the mean $\bar{\varphi}$ and the coefficient of variation $C_{v\varphi}$ of φ SINGH (1971) determined by the *Monte Carlo method*, on the basis of 200 random numbers, the means, the standard deviations and the coefficients of variation of K_A and K_P . He carried out the calculations for four different $\bar{\varphi}$ values (15° , 25° , 35° and 45°) and three $C_{v\varphi}$ values (0.10, 0.15 and 0.20) in each case. From his calculation results (Fig. 11.21) it may be established that

a) in the range of the greater angles of friction the coefficient of variation of K_P is greater than that of K_A ,

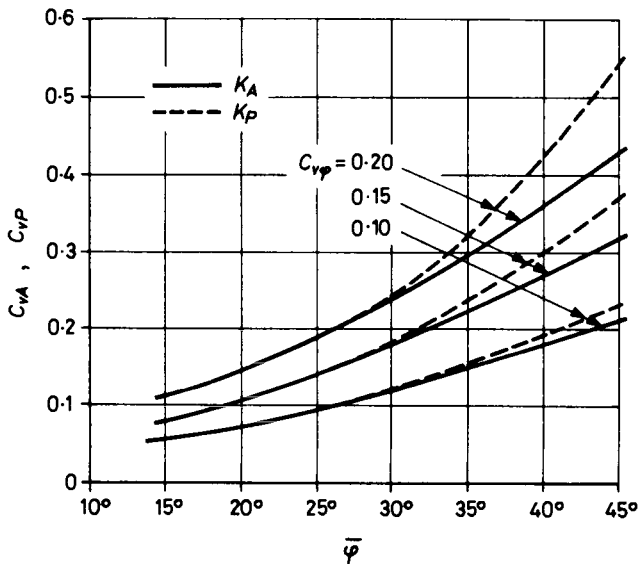


Fig. 11.21. Coefficients of variation of RANKINE's active and passive earth pressure factors (K_A , K_P) in the function of the expected value of the angle of friction ($\bar{\varphi}$) and its coefficient of variation ($C_{v\varphi}$) (SINGH, 1971)

b) under about $\varphi = 26^\circ$ the coefficients of variation of K_A and K_P are lower, over $\varphi = 26^\circ$ they are higher than the coefficient of variation of φ .

Statement b) may be verified also analytically. The equality $C_{vA} = C_{v\varphi}$ may be written also in the form

$$\frac{dK_A}{d\varphi} = \frac{K_A}{\varphi}. \quad (11.66)$$

The absolute value of the differential quotient of expression (11.64) is

$$\begin{aligned} \left| \frac{dK_A}{d\varphi} \right| &= 2 \tan(45^\circ - \varphi/2) \frac{1}{\cos^2(45^\circ - \varphi/2)} \left| -\frac{1}{2} \right| = \\ &= \frac{\tan(45^\circ - \varphi/2)}{\cos^2(45^\circ - \varphi/2)}. \end{aligned} \quad (11.67)$$

On equating this expression with the right side of Eq. (11.66) we obtain

$$\frac{\tan(45^\circ - \varphi/2)}{\cos^2(45^\circ - \varphi/2)} = \frac{\tan^2(45^\circ - \varphi/2)}{\varphi}, \quad (11.68)$$

i.e.

$$\varphi = \sin(45^\circ - \varphi/2) \cos(45^\circ - \varphi/2) = \frac{\sin(90^\circ - \varphi)}{2} = \frac{\cos \varphi}{2}, \quad (11.69)$$

whose solution is $\varphi \sim 26^\circ$.

The equation corresponding to expression (11.68) in the case of K_P is

$$\frac{\tan(45^\circ + \varphi/2)}{\cos^2(45^\circ + \varphi/2)} = \frac{\tan^2(45^\circ + \varphi/2)}{\varphi},$$

from which

$$\varphi = \sin(45^\circ + \varphi/2) \cos(45^\circ + \varphi/2) = \frac{\sin(90^\circ + \varphi)}{2} = \frac{\cos \varphi}{2}.$$

This is identical to expression (11.69), so the cases $C_{vP} \geq C_{v\varphi}$ are separated by $\varphi \cong 26^\circ$ also here.

The observation that the quotients $C_{vA}/C_{v\varphi}$ and $C_{vP}/C_{v\varphi}$ increase rapidly with the increasing φ means that the greater the angle of friction, the higher safety factor must be assumed. Let us assume, e.g., that $C_{v\varphi} = 0.1$ and the active earth pressure which will be exceeded only with a probability of $p = 0.05$ is requested. If $\varphi = 30^\circ$, then according to Fig. 11.21, $C_{vA} \cong 0.12$, so the standard deviation of the earth pressure is $s_A = 0.12 \bar{E}_a$. As according to the observation of SINGH (1971) the distribution of the hydrostatic factors may be regarded as normal and $\lambda = 1.645$ belongs to $p = 0.05$, the requested earth pressure is

$$E_{a,p} = \bar{E}_a + 1.645 \cdot 0.12 \cdot \bar{E}_a = 1.20 \bar{E}_a,$$

so the factor of safety is $v_c = 1.20$.

If $\varphi = 40^\circ$, then according to *Fig. 11.21*, $C_{vA} \cong 0.18$, so

$$E_{a,p} = \bar{E}_a + 1.645 \cdot 0.18 \cdot \bar{E}_a = 1.30 \bar{E}_a$$

and the factor of safety is $v_c = 1.30$.

The standard deviation of the earth pressure may be calculated also with the help of expression (8.32). If the distribution of φ is symmetric (e.g. normal), then the standard deviation of the hydrostatic factor defined by expression (11.64) is according to expression (11.67)

$$s_A = \frac{\tan(45^\circ - \varphi/2)}{\cos^2(45^\circ - \varphi/2)} s_\varphi. \quad (11.70)$$

If, e.g., $C_{v\varphi} = 0.2$ (so $s_\varphi = 0.2$) and $\varphi = 35^\circ$, then

$$s_A = \frac{\tan(45^\circ - 17.5^\circ)}{\cos^2(45^\circ - 17.5^\circ)} 0.2 \frac{35\pi}{180} = 0.0809$$

and

$$C_{vA} = \frac{s_A}{K_A} = \frac{\tan(45^\circ + \varphi/2)}{\cos^2(45^\circ - \varphi/2)} 0.2 \frac{35\pi}{180} = 0.298,$$

which agrees with the value read from *Fig. 11.21*.

If RANKINE's earth pressure is not calculated (as is the case mostly in practice), even then we can use the Monte Carlo method without any restriction, while the method based on error propagation can be used only if E can be expressed in an analytical form.

If the available test material is sufficiently large, we may proceed by determining the earth pressure for *each* φ -value; in the knowledge of these also the standard deviation of the required wall width, etc. can be calculated (SCHULTZE, 1975a).

11.7.2. The safety against overturning

We have stated in connection with *Fig. 8.1* that the overturning (tilting) of the retaining wall can be examined unequivocally only with the help of the safety margin SM of the moments. With the notations of *Fig. 11.22*, SM may be written in the following form:

$$SM = gG ABE_{av} - (bH + D)E_{ah}, \quad (11.71)$$

where $E_{av} = E_a \sin \delta$ — vertical component of the earth pressure,

$E_{ah} = E_a \cos \delta$ — horizontal component of the earth pressure,

a — factor expressing that the centre of rotation is not always identical with the corner point A of the retaining wall (specially in the case of soft soils).

The solution of the task is aggravated by the fact that E_a is not only a function of φ , but depends also on δ . If in addition we take into account that the earth pressure coefficient K_A can be given analytically by a very complicated expression only or not at

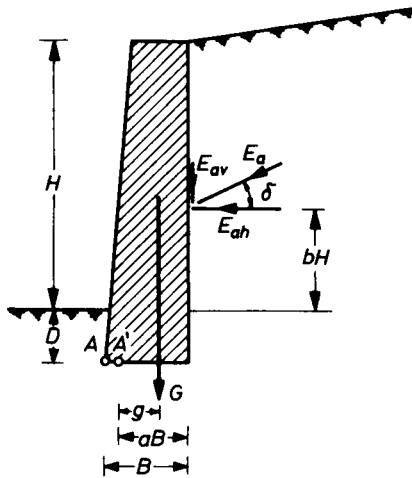


Fig. 11.22. Examination of the overturning of the retaining wall by probabilistic method (notations)

all, we must admit that the method based on the law of error propagation cannot be used here. Fortunately, another method, the so-called JÁKY-process permits to obtain for a given φ not alone E_a , but also the directional angle δ of the earth pressure. In this way the overturning can be examined in the following steps.

a) An earth pressure E_a and a directional angle δ is determined for each φ by using random numbers or test results.

b) The expected values of a and b and their standard deviations (e.g., $\bar{a}=0.95$, $s_a=0.02$; $\bar{b}=0.38$, $s_b=0.03$) are accepted on the basis of literature data.

c) For each φ we calculate the standard deviation of SM :

$$s_{SM} = \sqrt{G^2 s_a^2 + B^2 E_{av}^2 s_a^2 + H^2 E_{ah}^2 s_b^2}, \quad (11.72)$$

then the probability of the overturning with the help of the expected value

$$\overline{SM} = \bar{g}G + \bar{a}BE_{av} - (bH + D)E_{ah}. \quad (11.73)$$

d) The highest p_f -value obtained for the various angles of friction are accepted as ruling. (These will be expectably the value belonging to the lowest φ .) If the number of the tests is not sufficiently large, p_f may be calculated for a φ_p -value belonging to an arbitrary probability level p .

11.7.3. The dimensioning of retaining walls

The retaining wall as a structure must meet three requirements: it must be safe against overturning, soil failure and sliding.

This sequence corresponds also to the order of magnitude of the damage caused by the three modes of failure: in the case of overturning a great earth mass is crashing down

and even the retaining wall may become unusable, while the soil failure and the sliding imply usually less severe technical and economic consequences.

Accordingly, three different probabilities of failure should be assigned to the three criteria of stability, determining also their ratios, on the basis of economical considerations. Obviously "finding" the cross section area of the retaining wall which corresponds to this stipulation is no easy task.

The solution is somewhat facilitated by recent research results (KÉZDI, 1961, etc.), showing that the *sliding* should be examined according to the criterion

$$\nu_c \cdot \sigma \leq \sigma_t$$

and not only according to

$$T \leq N \tan \varphi$$

(σ is the effective contact pressure, σ_t the ultimate bearing capacity belonging to the eccentric and inclined forces). In this way the two failure modes which are less dangerous than overturning, can be combined (i.e. treated as a single criterion).

The designing of retaining walls is made more complex, as compared to the slopes, not alone by the need of ensuring a satisfactory protection against three (or two) failure modes, but also because *numerous forms of their cross section* can be applied (vertical, inclined or deflected backwalls, foundation with or without enlarging the wall, etc.). The task can be solved successfully only with the help of a computer, due to the great number of the possible combinations. Similar difficulties must be counted upon also on determining the *optimum dimensions*; the solution is by far not so unequivocal, as in the case of the slopes (HÖEG and MURARKA, 1974).

12. The quality control of earthworks

The earthwork is a *product* prepared of a natural building material: the soil. So we are not tied to given natural physical characteristics, as in the case of other geotechnical tasks (e.g. designing foundations), because we are planning them ourselves. In addition to the specifications of the construction technology and the technical requirements we have the possibility of selecting the appropriate *material*.

The designer considers, taking the intended aim of the object into account, *the physical characteristics to be specified* (e.g. the shear strength and the modulus of compressibility if the soil is exchanged, or the permeability in the case of earth dams). As every physical characteristic connected with the earthworks is a function of the *void ratio* which can be determined easily, the quality is controlled usually by measuring the *density*. The required density of the earthwork may be specified

- by taking it from the related standard, or
- by selecting it on the basis of its correlation with some other physical characteristic (whose importance is determining for the functioning of the earthwork).

An example for the latter solution is shown in *Fig. 12.1*. According to laboratory tests (RÉTHÁTI, 1957), supported also by field observations (RÉTHÁTI, 1977a), the loose

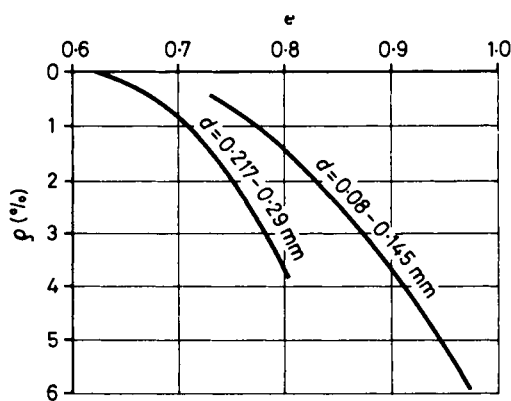


Fig. 12.1. Relative subsidence (ρ) of two sand fractions determined in the laboratory under the effect of capillary water absorption versus the void ratio (e)

granular soils are subject to a considerable subsidence due to absorption of capillary water. *Figure 12.1* presents the relationship ρ, e for two sand fractions (ρ is the relative subsidence caused by the absorption of water). According to the figure we must specify a void ratio (i.e. a degree of density T_{rr})—e.g. for internal fill under the floor—whereby the expectable relative subsidence will not attain a critical value prescribed by us.

The quality control based on the density measurement contains several error sources, connected with the methodics, in addition to those discussed in Section 12.2.1. So the correlation between T_{rr} and the involved physical characteristic is subject also to a residual standard deviation, which is added to the error, characterizing the determination of T_{rr} (see Section 12.2.1). A possible error may result also from disregarding the fact that the numerical value of the examined characteristic depends on the measured density belonging to the “dry” or the “wet” leg of the PROCTOR-curve (KÉZDI, 1970).

In the following we deal primarily with the mathematical statistical methods of the density control, examining the possibility of their generalization in each case (see e.g. Section 12.5).

12.1. Selection of the measuring sites and methods

As important technical and economical interests depend on the quality control of the earthworks, the guiding principles described in Section 4.1—in the first place the *representative* nature of the samples—must be observed especially strictly. This activity differs from the general case, as the earthwork is an artificial object, so it can be foreseen with a high probability that some of its sections will be denser (e.g. under the effect of a heavy vehicular traffic), others will be looser (e.g. those close to the slope). PETTITT (1967) remarks indeed that the sampling directed by the technical controller gives a “conservative” result (reflecting a less favourable image than the reality is), but there are also opinions, holding that the contractor himself is interested in revealing the faulty or weak parts, as if these emerge only later, during the operation, his loss will be greater than by repairing the faults in due time.

RÉNYI (1954) formulated the requirement of taking representative samples from the artificial product in the following way. If the examined data set consists of subsets, in which the involved characteristic is distributed differently, it must be ensured that the individual subsets are included in the sample with a *probability corresponding to the numerical ratio of their elements*.

The *measurement method* is determined by the type of the soil and the available instruments, but the transit time of the test may also be a determining factor. From the mathematical–statistical viewpoint, the measurement methods may be divided into two groups:

- methods based on sampling (stamping out by a metal cylinder, volume measurement by balloons, or sand throwing, etc.).
- methods without sampling (penetration test, isotopic density measurement, etc.).

The difference between the groups is that if methods belonging to the latter are selected, a *calibration* must also be performed, leading to a residual error of the correlation. (Another question is that, e.g., by the isotopic method the measured soil mass is greater, i.e. the relative number of the samples is quasi increased.)

12.2. Statistical parameters of the degree of density

According to the basic principles of modern quality control it is not enough to judge the satisfactory state of the earthwork (or a section of it) on the basis of its *average* degree of density, but also other statistical parameters must be known. Also the extreme values must be filtered out before the statistical processing of the data (see in Section 7.1).

12.2.1. Components of the standard deviation of the degree of density

The standard deviation of the degree of density

$$T_{r\gamma}(\%) = 100 \frac{\gamma_d}{\gamma_{d\max}} \quad (12.1)$$

is composed of two parts: the standard deviation of γ_d and of $\gamma_{d\max}$. According to expression (8.32)—if $C_s \sim 0$ for both characteristics—the variance s_T^2 of the degree of density is

$$s_T^2 = \left(\frac{100}{\bar{\gamma}_{d\max}} \right)^2 s_d^2 + \left(\frac{100\bar{\gamma}_d}{\bar{\gamma}_{d\max}^2} \right)^2 s_{dm}^2, \quad (12.2)$$

where s_d^2 — variance of the dry bulk density,
 s_{dm}^2 — variance of the maximum dry bulk density.

The variances appearing in Eq. (12.2) may be decomposed farther depending on whether they originate from measurements or from *local circumstances*.

The standard deviation of the *measured* γ_d -values is due mainly to the inaccuracy of the sampling (or the calibration, if no sampling was required for the measurement) and in the case of $\gamma_{d\max}$ to the uncertainties of the PROCTOR-test (e.g. the insufficient number of the partial samples). The reproducibility of both these physical characteristics may be controlled with the help of Table 6.3 and 6.4: accordingly the coefficient of variation characterizing γ_d is $C_v = 0.015 - 0.030$, depending on the type of the soil, and for $\gamma_{d\max}$ it is $C_v \approx 0.01$.

The standard deviation due to the *local circumstances* is—according to experiences—considerably greater. The following factors contribute to the dispersion of the *dry bulk density*:

- the heterogeneity of the soil (e.g. the smaller, or greater variation of its grain size distribution), whereby an identical compacting work does not give identical void ratios;
- the possible increase of the heterogeneity in the water content by meteorological effects (temperature and precipitations) during the period of the construction;
- the inequality of the compacting work/unit volume.

The role of the water content may be studied most simply by comparing the empirical density functions of w and γ_d . This was done, e.g., by TURNBULL, COMPTON and AHLVIN (1966), with the difference that instead of these characteristics they compared the histograms of the values

$$\Delta w = w - w_{opt} \quad \text{and}$$

$$\Delta \gamma_d = \gamma_d - \gamma_{de},$$

where γ_{de} is the prescribed value (Fig. 12.2). (At another workplace the standard deviation of $\Delta \gamma_d$ was only 0.30 kN/m^3 , which was explained by the authors by the more careful adjustment of the water content.) Also DAVIS (1953) called the attention to the importance of this problem, presenting examples of adjusting $\bar{w}$ to about w_{opt} and decreasing thereby also the *standard deviation* of the water content.

Another unfavourable effect of the heterogeneity of the soil is that the earth mass to be built into the object cannot be characterized by a single γ_{dmax} value. In practice the

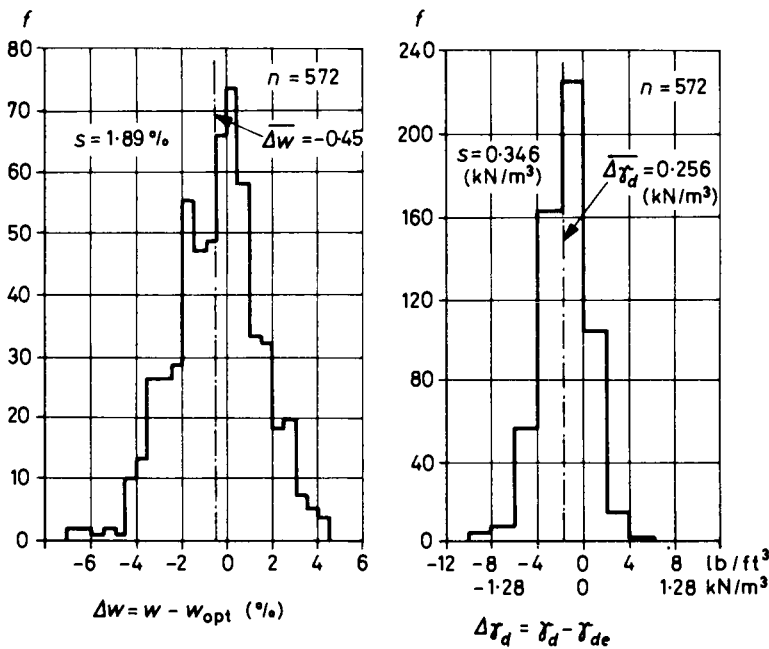


Fig. 12.2. Example of histograms of Δw and $\Delta \gamma_d$ differences (TURNBULL, COMPTON and AHLVIN, 1966)

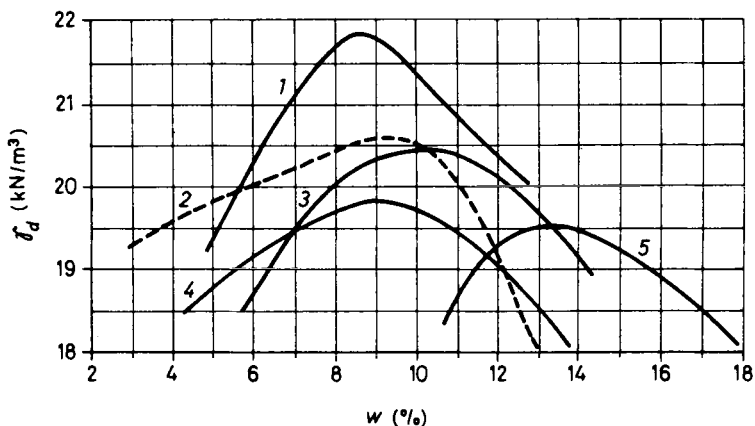


Fig. 12.3. PROCTOR-curves of five sands of a similar grain size distribution (VÁLÓCZI and GARAI, 1978)

ratio of the PROCTOR-tests and the sampling (measuring) places is 1 : 5 – 1 : 10 max., but often even less than 1 : 30. This means that we must find for each sample the PROCTOR-curve “belonging to it”, which is a very delicate operation, affecting the qualification of the earthwork.

Numerous trials were published in literature for calculating $\gamma_{d\max}$ from the identifying characteristics (e.g. KÉZDI, 1970; KOPPULA, 1983). But these relationships are not sufficiently close, as seen by Fig 12.3. The only feasible way is to establish this relationship for the local soils (for the actually used material). For example, correlation matrixes of the type determined by HOLTZ and KRIZEK (1971) may do good service (see Section 3.5.2 and Table 3.7).

Comparing the statements related to γ_d and $\gamma_{d\max}$ we come to the conclusion that the standard deviation of T_{ry} is the greater, the more different the physical characteristics of the built-in partial masses are. As a part of the standard deviation is due to the inaccurate approximation (a fault which cannot be eliminated practically and is loading the evaluation), therefore the dispersion of the density of an earthwork of heterogeneous composition must be judged less strictly.

The cause and the components of the standard deviation may be examined also by the relationship between the value pairs Δw , γ_d .

12.2.2. Empirical distribution function of the degree of density

Concerning the type of the distribution function only few literary data are available. Some authors (e.g. TURNBULL, COMPTON and AHLVIN, 1966) found that the distribution of T_{ry} was normal, but investigations which determined a strongly asymmetric distribution are also known. UNGÁR (1974, 1975b) calculating, e.g., the coefficient of skewness (of PEARSON) for an earthwork according to expression (1.18) obtained $S_k = 0.66$.

The degree of density has a lower and an upper *limit*, related to the given soil, its state and the compaction technology. The lower value is identical with T_{ry} of the uncompacted soil (in the case of certain soils this is 75–80%!), the upper limit may be formed as a result of the fortunate coincidence of the given circumstances (nearly optimum water content, maximum number of runs, etc.). Though these limits are not so sharp as, e.g., in the case of the degree of saturation (see Section 1.4), yet they may result in a positive asymmetry, if $\bar{T}_{ry}$ is closer to the lower limit, resp. a negative asymmetry, if $\bar{T}_{ry}$ is closer to the upper limit. This means also that we must grant a higher chance to the *beta distribution*, than to the distribution with an asymptote.

Table 12.1 shows the average degree of density and the statistical parameters calculated from individual measurements of 16 earthworks built in various places (n = number of measurements), permitting to make the following statements.

a) The mean value of the standard deviations is $s_T = 3.00$, the extremal values are 1.68–1.90 and 4.03–4.58, respectively. On the basis of these data the degree of homogeneity of the involved earthworks can be judged.

b) The coefficient of skewness is negative in 81% of the cases; its mean value is $\bar{C}_v = -0.55$.

c) The kurtosis is positive in 69% of the cases; its mean value is $\bar{C}_k = 0.81$.

d) If we accept the distribution, at which the conditions $|C_s| \leq 0.4$ and $|C_k| \leq 0.6$ prevail simultaneously, as practically normal distribution, then only the data set of the works Nos 6 and 7 can be regarded as normally distributed.

Table 12.1. Statistical parameters of the degrees of density measured at 16 different sites

No.	n	$\bar{T}_{ry}$	s_T	C_v	C_s	C_k
1	41	91.3	1.90	0.021	-1.87	3.95
2	51	86.7	3.88	0.045	-1.12	2.13
3	124	79.4	3.81	0.048	-0.68	0.33
4	183	91.0	2.74	0.030	-1.13	1.18
5	111	94.1	2.09	0.022	-1.63	2.83
6	85	83.2	4.03	0.048	-0.26	0.55
7	57	93.0	2.06	0.022	-0.38	0.58
8	33	92.7	1.68	0.018	0.54	-0.44
9	30	95.3	4.58	0.048	1.09	0.22
10	38	92.2	2.09	0.023	-0.74	0.11
11	25	86.4	3.07	0.036	-0.91	2.09
12	29	92.8	3.89	0.042	-0.19	-1.21
13	24	84.0	3.61	0.043	-0.55	-0.26
14	24	93.8	2.71	0.029	0.63	-0.17
15	40	95.7	3.36	0.035	-0.70	-0.43
16	44	95.2	2.54	0.027	-0.95	1.46
Mean:		90.4	3.00	0.034	-0.55	0.81

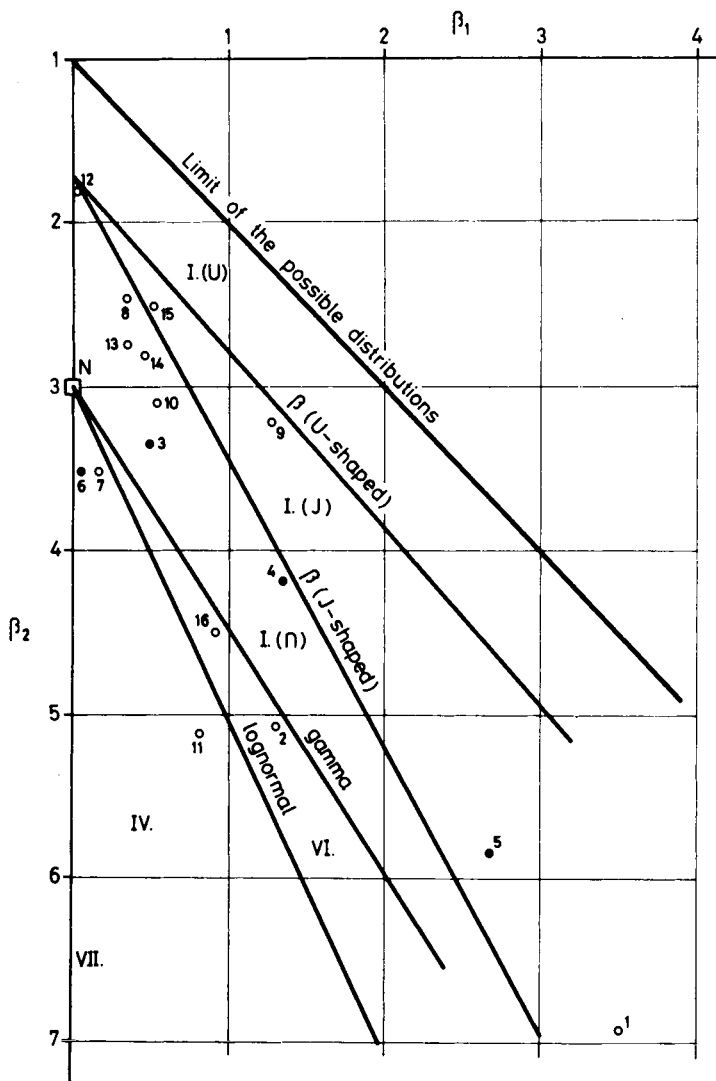


Fig. 12.4. Plotting the series of measurements seen in Table 12.1 in PEARSON's coordinate system

e) For the relationship $\bar{T}_{ry}$, C_s the following regression could be determined on the basis of samples of $n > 51$ elements:

$$C_s = -0.042\bar{T}_{ry} + 2.86$$

supporting the assumption that as the mean of the degree of density increases, the skewness becomes ever more negative. (The correlation is not very close, the correlation coefficient is $r = -0.47$.)

If we calculate β_1 and β_2 with the help of Eqs (2.6), the data series appearing in Table 12.1 may be evaluated also in PEARSON's coordinate system (Fig. 12.4).

12.3. Comparison of various compacting technologies and measurement methods

A generally known requirement is that the compacting means must be selected according to the type of the soil (e.g. sand can be compacted well by vibration, clay hardly at all). In the case of extensive earthworks *various compacting machines* may be used, partly for the above reason and partly for shortening the time of the construction. The result and the efficiency of the work of these machines may be compared by mathematical statistical methods, as proved by the following example.

PETTITT (1967) analyzed the measured density data of an earthwork prepared of a granular, volcanic material for road filling. During the work three different compacting tools were used: sheepfoot rollers (1), manually controlled vibrators (2) and sheepfoot rollers drawn by pneumatic-tyred vehicles (3). Columns 3–6 of *Table 12.2* show the means and the standard deviations of the attained degree of density, the mean deviation from the optimum water content and the standard deviation of the water content, permitting to state that

- a) the variants of the sheepfoot rollers gave a similar *average* density, but method (3) resulted in a significantly smaller standard deviation ($4.5 < 7.1$);
- b) the vibration gave a lower average density than the other methods, but its compacting effect was relatively uniform;
- c) the cause of the deviations between the results is not the difference in $\bar{w}$ (or in s_w);
- d) the maximum average density was attained by method (3), though Δw was here the greatest.

According to the conclusion of the author this method is the most efficient, but the assumption that w_{opt} is lower in the case of this compacting tool than the laboratory value, cannot be excluded either and it is also unknown, if the number of the runs was the same as in the case of method (1), or it was different.

Column 7 of *Table 12.2* shows the number of the samples which failed to attain the prescribed degree of density ($T_{ry} = 90\%$). The “counted values” agree well with those calculated from the normal distribution constructed for the data (*Fig. 12.5*).

Similar questions arise on evaluating the *field compaction tests*, whose purpose is to ensure the possible most efficient compaction of the earthwork. In this case we must determine the variation of $\bar{T}_{ry}$ and s_T with the thickness of the layer, the number of runs

Table 12.2. Density parameters attained by various means and $\bar{A}_w$ -values belonging to them (PETTITT, 1967)

Compacting means	n	$T_{ry}(\%)$		$w - w_{opt}(\%)$		$T_{ry} \leq 90\%$ (p%)
		$\bar{x}$	s	$\bar{x}$	s	
Sheepfoot roller	70	94.8	7.1	-2.0	2.8	22.8
Manual vibrator	40	93.1	5.3	-3.1	3.3	30.8
Vehicle with pneumatic tyre + sheepfoot roller	101	95.1	4.5	-5.2	2.9	13.9
With the measurements combined	211	94.9	5.7	-3.5	3.2	17.5

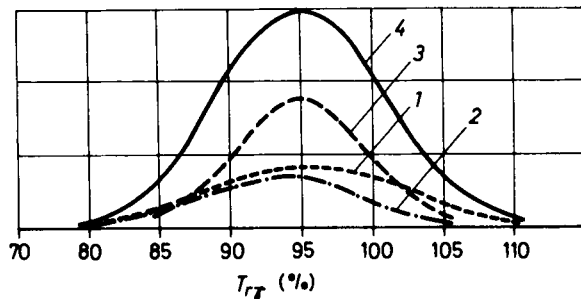


Fig. 12.5. Density functions of degrees of density attained by various means (PETTITT, 1967); 1 — sheepfoot roller, 2 — manually controlled vibrator, 3 — sheepfoot roller drawn by a pneumatic-tyred vehicle, 4 — total of all the measured data

and the natural or the “adjusted” water content. The various subcases may be compared with the help of *statistical tests*, assigning a *specific cost* to each.

The various PROCTOR-apparatuses may be regarded as representing different processes, though the test method is identical, only the construction of the apparatuses is different. The possibilities of error due to this difference are pointed out by the test series described in the following (BORUS and RÉV, 1968a). 20 laboratories performed a PROCTOR-test each of a homogenized sand and clay soil. The mean, the range and the standard deviation of the maximum dry bulk densities in kN/m^3 obtained by them were

$$\begin{aligned} \text{for sand } \bar{\gamma}_{d\max} &= 17.75, & R &= 1.4, & s_{dm} &= 0.46; \\ \text{for clay } \bar{\gamma}_{d\max} &= 18.09, & R &= 2.1, & s_{dm} &= 0.54. \end{aligned}$$

The authors concluded from the significant standard deviation that the missing control of the apparatuses may lead to a considerable dependence of the qualification of the earthwork on the laboratory, performing the test.

The *parallelly applied measurement methods* may be compared with the statistical tests described in Section 6.2. For example, according to the example of Section 6.2.3 the standard deviation of the isotopic measurement is smaller than that performed with the balloon, because—according to PETTITT (1967)—the former is a non-destructive test. Assumably the real situation is reflected best by the sample taken with a cylinder of great diameter, so this method may serve as the basis for the comparison, both with respect to the expected value and the standard deviation.

The various technologies, measurement methods and laboratories may be evaluated parallelly with each other also by the ring tests described in Section 6.3.

12.4. Qualification of earthworks on the basis of their density

For decades the only requirement, the earthworks had to meet was the *average* density prescribed by the designer. This view is basically erroneous, because it may lead to qualifying earthworks with both very dense and very loose sections as satisfactory. In this case we cannot speak of the equalization of both extremes, as the surplus density exceeding the prescribed value is *superfluous*, while the loose state is *dangerous*.

This recognition led to the inclusion of the *standard deviation* among the qualifying characteristics. With the use of this statistical parameter the earthworks may be qualified by one of the following two methods:

- a) determining the *confidence intervals* of the expected value on the basis of the standard deviation and the number of the samples (Section 12.4.2), or
- b) calculating the ratio of the samples (earthwork sections), whose density is less than a tolerable *lower threshold value* (Section 12.4.3); this latter method permits to fix also the risk of the supplier and the receiver (Section 12.4.4).

The processes a) and b) rely on the *distribution functions* of the measured T_{ry} -values, so naturally they cannot take into account the coordinates of the sampling (measuring) points. Therefore this—in itself correct hypothesis—disregards the possibility of damage which may be caused by the non-random *spatial distribution* of T_{ry} . We shall begin the discussion of the qualification of the earthworks with this question.

12.4.1. Spatial trend of the degree of density

According to expression (4.2) the spatial data set of the soil physical characteristics may contain not only random (stochastic), but also periodic and trend components. In the case of earthworks the autoregressive properties may be disregarded generally, but the trend is in many case not negligible, as it was proved severally.

An example of the *vertical trend* was presented by UNGÁR (1974, 1975b). A 3.5 m thick fill of various soils was prepared under the floor of an industrial building. The evaluation of the density by layers gave the distribution shown in *Fig. 12.6*. (The interpretation of the Q quantities see in Section 3.4.) The values of the statistical parameters of the density and the mean of the differences $\Delta w = |w - w_{opt}|$ by soil types are shown in *Table 12.3*. It is seen that the density of the sandy silt was the worst, due probably to the significant deviation of its water content from the optimum. As about 75% of the 50 cm thick top layer consisted of this material, the causality is unequivocal. As the behaviour of the floor was influenced most by this layer, it was removed and replaced by a well compacted layer of sandy gravel, trusting its favourable “crust effect”.

Figure 12.7 shows the *horizontal distribution* of the density of an exchanged soil layer under a plate foundation. The density values indicated at 42 measuring points are

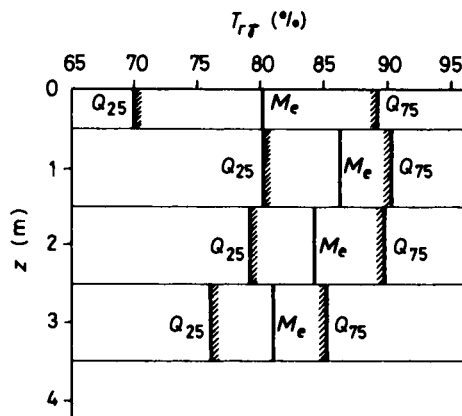


Fig. 12.6. Example for the evaluation of the degree of density by layers (UNGÁR, 1974, 1975b); M_e — median, Q_{25} and Q_{75} : quantiles belonging to $p=25\%$ and $p=75\%$, respectively

Table 12.3. Compaction parameters of the soils forming the inner fill in the example of UNGÁR (1974, 1975b); R — dispersion range (lower and upper limits of the measured values), M_e — median, Q_{25} and Q_{75} — quantiles belonging to $p=25\%$ and 75%

Soil physical characteristic	Statistical parameter	Type of soil		
		sandy silt	mixed	clay
$T_{rr}(\%)$	R	63–98	70–98	68–97
	M_e	74	84	85
	Q_{25}	71	82	80
	Q_{75}	79	90	90
$ w - w_{opt} $ mean		6.0	5.1	2.8

identical with the means of the measured values over the related vertical lines. (The trend over the depth is of no importance here, as the vertical stress—which is identical with the contact pressure—varies hardly at all in the exchanged soil.) The horizontal trend may be calculated with the help of the terivariate correlation described in Section 3.5.1. On introducing the notations

$$x_1 = T_{rr}, \quad x_2 = x \quad \text{and} \quad x_3 = y,$$

the total correlation coefficients will be as follows:

$$r_{12} = -0.785; \quad r_{13} = 0.296; \quad r_{23} = 0.$$

On writing up the determinant (3.38), the six subdeterminants may be calculated as

$$\begin{aligned} R_{12} &= 0.785; & R_{11} &= 1.000; \\ R_{13} &= -0.296; & R_{22} &= 0.912; \\ R_{23} &= 0.232; & R_{33} &= 0.384. \end{aligned}$$

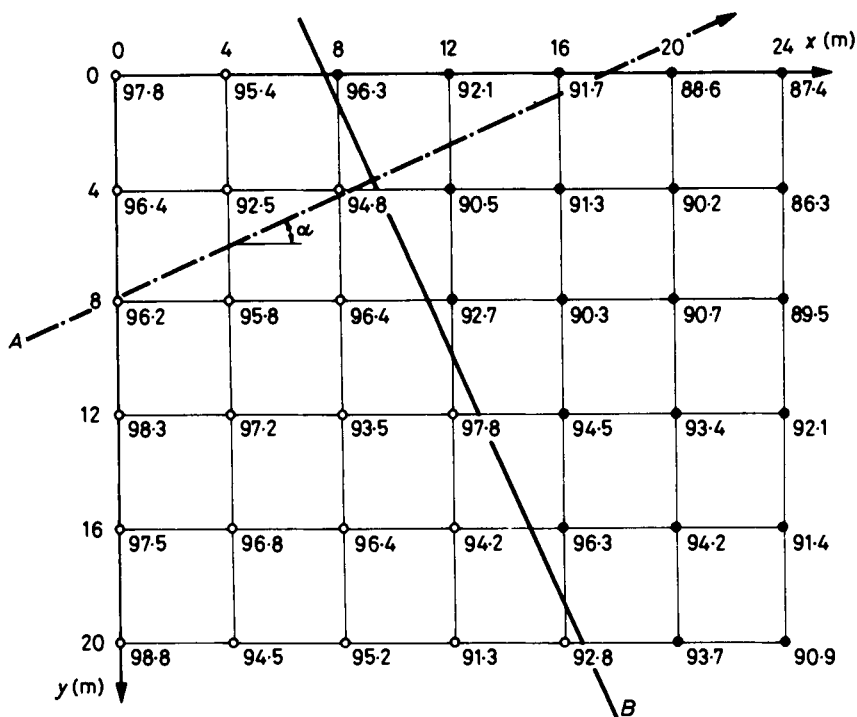


Fig. 12.7. Average degrees of density of the exchanged soil below a plate foundation measured in various verticals and horizontal trend of these means; A — main tilting direction of the regression plane, B — straight line drawn normally to the latter, dividing the measurement points into two equal parts

As $m_1 = 93.66$, $s_1 = 3.082$, $m_2 = 12$, $s_2 = 8.097$, $m_3 = 10$ and $s_3 = 6.914$, the constants of the regression plane are as follows:

$$a_{12} = -\frac{3.082}{8.097} \cdot \frac{0.785}{1} = -0.299;$$

$$a_{13} = -\frac{3.082}{6.914} \cdot \frac{-0.296}{1} = 0.132;$$

$$b_1 = 93.66 - (-0.299)12 - 0.132 \cdot 10 = 95.93.$$

So the equation of the regression plane is:

$$T_{ry} = 95.93 - 0.299x + 0.132y.$$

The intersection line of this plane with the plane x, y is (from the assumption of $T_{ry} = 0$):

$$0.132y = 0.299x - 95.93,$$

i.e.

$$y = 2.265x - 726.7.$$

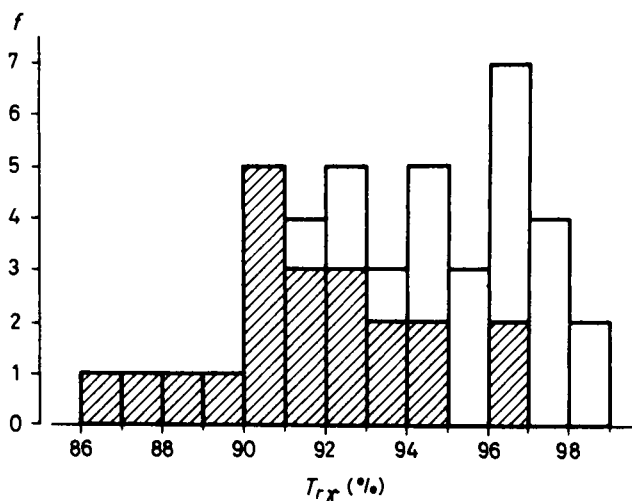


Fig. 12.8. Histogram of the T_{ry} values shown in Fig. 12.7 (the hatched areas refer to the values found to the right of the straight line B)

The angle of inclination α enclosed between the *line of fall* of the plane and the positive direction of the x-axis is:

$$\alpha = \arctan \left(-\frac{1}{2.265} \right) = -23.8^\circ.$$

The direction of the line of fall is drawn in Fig. 12.7 by a *dotted line*. The building would tilt in this direction, if, e.g., after the exchange of the soil and stopping the lowering of the groundwater level the rising groundwater caused a subsidence of the soil. The straight lines drawn normally to this direction correspond to the isometric lines of T_{ry} . The Figure shows from among these normals the one, which divides the measuring points to two equal parts, illustrating how significantly the distribution of the T_{ry} values found to the right and to the left of this line deviate from each other (Fig. 12.8).

As T_{ry} contains also stochastic elements, the probability of $\alpha=0$ is very little. The significant deviation of α from zero may be concluded from the magnitude of the *multiple correlation coefficient*. As in our case

$$|R| = 1 - 0.785 \cdot 0.785 + 0.296(-0.296) = 0.296,$$

we have according to expression (3.42)

$$\rho_1 = \sqrt{1 - \frac{0.296}{1}} = 0.84,$$

which is a considerable value.

12.4.2. Qualification based on the confidence interval

The reliability of the expected value may be judged with the help of the methods described in Section 7.4.1. In the numerical example of Section 12.4.1 the average degree of density calculated from $n=42$ measurements was 93.66% and the standard deviation was $s_T=3.082$. As the number of the samples is sufficiently great, the confidence intervals may be calculated—in a first approximation—from Eq. (7.36). Be, e.g., $1-\alpha=\Phi(x)=1-0.10=0.90$; in this case

$$\Phi(\lambda) = \frac{1+0.90}{2} = 0.95,$$

to which $\lambda=1.645$ belongs in *Table F2*. Accordingly, we may state with a reliability of 90% that $\bar{T}_{ry}$ falls between the limits expressed by the inequality

$$93.66 - 1.645 \frac{3.082}{\sqrt{42}} < \bar{T}_{ry} < 93.66 + 1.645 \frac{3.082}{\sqrt{42}},$$

i.e.

$$92.88 < \bar{T}_{ry} < 94.44,$$

so the requested range is rather narrow.

The justification of the approximation by the normal distribution may be judged in the knowledge of C_s . In our case $C_s = -0.36$, which is an acceptably low value, with regard also to the narrow confidence interval. In the contrary case we should have to determine the distribution best fitting the measured values, or to find the two limits (positioned asymmetrically around the mean) by the α -transformation described in Section 2.1.6.

The earthwork can be qualified on the basis of the confidence interval only, if

- the *mean* of T_{ry} —resp. the physical characteristic, which may be correlated with it—is determinant for soil failure (or for its inadmissible deformation), as e.g. under plate foundations,
- the *spatial trend* of the density does not exceed the critical value determined by the building object (see Section 12.4.1).

12.4.3. Qualification based on a lower threshold value

The lower the degree of density at a place, i.e. the farther it is from the mean, prescribed by us, the less favourable are the soil physical characteristics which determine the behaviour of the earthwork. So obviously the density of the earthwork should be qualified on the basis of a lower threshold value (T_A) by maximizing the probability p_a of the occurrence of lower T_{ry} values than this threshold value.

There exists an unequivocally definable relationship between the mean and the standard deviation of the degree of density on the one hand and T_A and p_a on the other

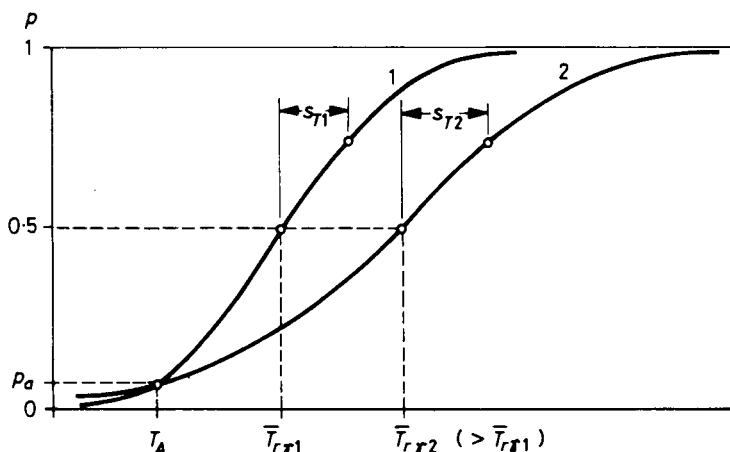


Fig. 12.9. Relationship between the lower threshold value of the degree of density (T_A), its standard deviation (s_T) and its mean ($\bar{T}_{r\gamma}$), fixing the probability p_a

hand. Let us start from the lower threshold value and let us assume that the number of the samples is sufficiently great and the distribution of $T_{r\gamma}$ is normal. In this case we can assign to the probability p_a prescribed by the equality

$$1 - p_a = \Phi(\lambda),$$

a λ -value permitting to calculate $\bar{T}_{r\gamma}$ from T_A as

$$\bar{T}_{r\gamma} = T_A + \lambda s_T. \quad (12.3)$$

The product λs_T gives the surplus, by which $\bar{T}_{r\gamma}$ must exceed T_A . As p_a (and thereby λ) is given, this surplus depends on the *standard deviation* alone. If the standard deviation is small, also the—indirectly—demanded $\bar{T}_{r\gamma}$ -value will be small and vice versa (Fig. 12.9). So the contractor is interested in lowering the standard deviation as much as possible by organizing the work better and by increasing the technological discipline (KORDA and SZALAI, 1973). Favourable results may be expected from the continuous measurement and control of the water content of the material to be built-in and the field compaction tests promoting the selection of the best solution.

The specification and the control of the quality requirements may be facilitated by the graphical presentation of the relationship between the quantities appearing in Eq. (12.3) (Fig. 12.10). If, e.g., the standard deviation of the density is $s_T = 5\%$ and the probability of the occurrence of lower densities than 90% is specified as $p_a = 0.10$, then the mean value must attain

$$\bar{T}_{r\gamma} = 90 + 6.4 = 96.4\%.$$

A prescription of general validity cannot be given for the value pair T_A, p_a . If T_A is fixed, p_a may be accepted on the basis of the following main considerations:

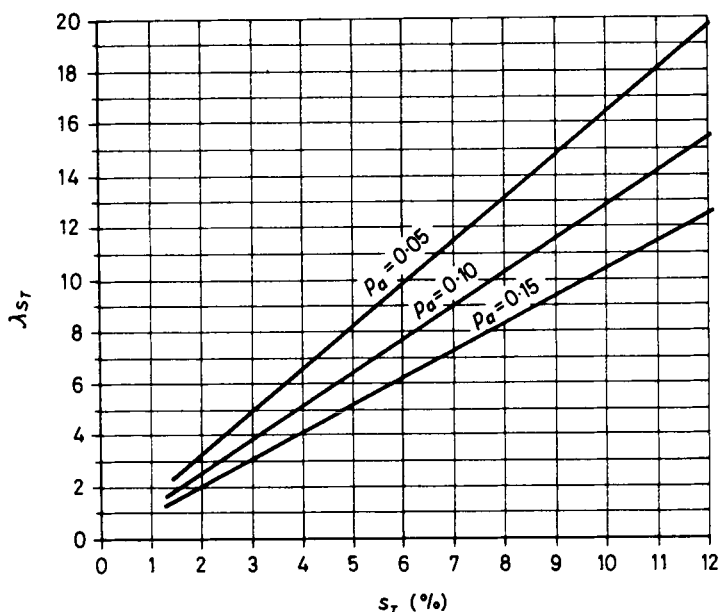


Fig. 12.10. "Lead" of the mean ($\lambda s_T = \bar{T}_{ry} - T_A$) versus the standard deviation of the degree of density (s_T) and the probability $p_a = P(T_{ry} \leq T_A)$

a) The greater the technical and economic interests, which are connected to the stability of the earthwork (or to its trouble-free function), the lower p_a must be selected.

b) The tolerated probability may be prescribed also in a differentiated way, e.g., with increasing values over the depth for the individual layers of the road embankment.

c) p_a may be higher for earthworks prepared of *heterogeneous*, than of *homogeneous* materials, mainly because the measurement errors are greater, as it was explained in Section 12.2.1.

Solutions combining the lower threshold value with the confidence interval can also be found in the literature (KÜHN, 1971). This author proposes two threshold values: an acceptance value T_a and a refusal value T_r , while the average degree of density can be lower than these with a probability of α_a , resp. α_r , max. Both criteria may be written up according to Eq. (7.37) as follows:

$$T_a = \bar{T}_{ry} - t_a \frac{s_T}{\sqrt{n}}, \quad (12.4)$$

respectively

$$T_r = \bar{T}_{ry} - t_r \frac{s_T}{\sqrt{n}}, \quad (12.5)$$

where t_a and t_r — values taken from STUDENT's table for α_a and α_r ,
 s_T — standard deviation of the degree of density,
 n — number of measurements.

According to the suggestion of KÜHN (1971) the designer must specify an upper threshold value also for the standard deviation s_T and check a posteriori with the help of Eq. (7.39), if the actual standard deviation does not exceed it on a given probability level.

12.4.4. Qualification based on the risk of the supplier and the receiver

This qualification process has been in use for decades by now, especially in the machine and the chemical industries, but also in other industrial branches (e.g. in pharmaceuticals). It spread gradually in the field of *road construction* as well, with the increasing volume and mechanization of the operations. It is used ever more for controlling the content of bitumen, filler material and crushings, the density, the water absorption, the strength of the concrete, etc., respectively, for regulating the conditions of the presentation and acceptance of finished works, until this assumably *most modern method* has been extended also to embankments, protective layers, etc.

The contractor and the receiver are partners of opposing interests, at least seemingly. Although the damage occurring later in the earthwork, or in the functioning of the superimposed structure affects also the contractor unfavourably, yet his momentary interest is that his product should be accepted. He expects it the more so, as a “perfect” work cannot be demanded by the receiver anyway, because of the serious economical consequences. So the “presentation–acceptation” process must be performed as a *rational compromise*, with due consideration to the *risk* of both parties.

On fixing a lower threshold value T_A for the degree of density, four numerical values must be specified in the contract:

- the *permitted* fault percentage p_1 , by which there is a probability of $1 - \alpha$ for the acceptance,
- the *tolerated* fault percentage p_2 , which is still accepted with a risk β by the receiver.

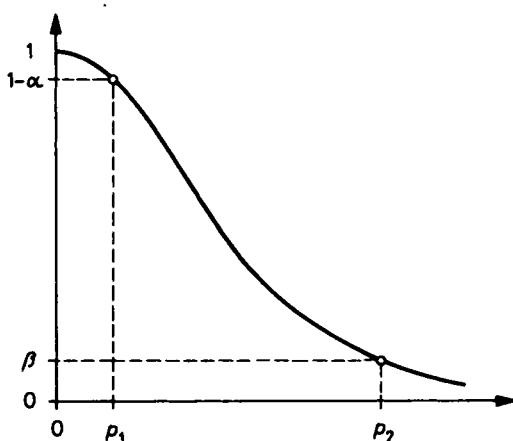


Fig. 12.11. Characteristic curve of the presentation-acceptation process: α — risk of the supplier, β — risk of the receiver, p_1 — permitted fault percentage, p_2 — tolerated fault percentage

In this case the probability of refusing a high quality work ($p \leq p_1$) is only α for the *contractor*. (For example, the probability of the acceptance with a risk of $\alpha = 0.05$ is $1 - \alpha = 0.95$, i.e. 95%.) For the *receiver* the probability of receiving an earthwork, whose quality is lower than the tolerated quality ($p \geq p_2$) is only β . (The certainty of the refusal, e.g., in the case of a risk of $\beta = 0.05$ is $1 - \beta = 0.95$, i.e. 95%.)

The relationship between the various fault percentages and risks may be described by the *characteristic curve of the presentation-acceptance process*, from which the variations of α and β with p_1 and p_2 can be read off (Fig. 12.11).

The quantification of the risks α and β will depend on the type and magnitude of the predictable damage, on the existence or not of penalty prices, and on the method and cost of the tests (LOPEZ CORAL and FONSECA GARCIA, 1983).

The process based on the risk of the supplier and the receiver may be classified according to whether

- the product is qualified by inspection, or by measurements,
- one (a lower, or an upper), or two (a lower and an upper) threshold values are fixed,
- samples are sampled once, or twice (or more times).

The qualification *by inspection* is the typical method in certain industries, which may be applied in cases, where the sample can be qualified as satisfactory, or *wasted* by inspection alone, or by some simple investigation (e.g. by transillumination). If a lot of N pieces contains M pieces of waste, the probability P_k that in the sample of n element we shall find k waste pieces is

$$P_k = \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}}, \quad (12.6)$$

i.e. k follows a so-called hypergeometric distribution. If the ratio $p = M/N$ is sufficiently small, k may be approximated by the binomial, resp. the POISSON-distribution (RÉNYI, 1954), so the characteristic curve of the presentation-acceptance process may be determined with the help of these functions. The method and the possibilities of its application are described in detail also by the product standards (e.g. MSZ-247/1, 1975) and in the literature.

Processes permitting to repeat the sampling, i.e. the so-called sequential analysis were also elaborated (WALD, 1947).

It is more advised to qualify the density by the other process, i.e. by *measurement*. In this case the decision is made on the basis of the total set of the T_{ry} data, i.e. their *empirical distribution*. This method permits a more differentiated analysis, making superfluous the rather subjective interpretation of the “waste”. The condition system of

the presentation-acceptance has been elaborated by a work team of the Columbia University (USA), headed by WALD: initially it was used for military purposes, so it was published only after the second world war. On assuming a lower (or an upper) threshold value x_A , the statistical parameters of the so-called *distribution straight line*

$$\xi = \bar{x} \pm ks \quad (12.7)$$

were examined as distribution function. According to the investigations of the above team the distribution of ξ may be taken to be normal in the case of $n \geq 5$, with an expected value of

$$\xi = \mu + k\sigma \quad (12.8)$$

and a standard deviation of

$$\sigma_\xi = \sigma \sqrt{\frac{1}{n} + \frac{k^2}{2(n-1)}} \quad (12.9)$$

On interpreting the probabilities by the expression

$$P = \Phi \left(\frac{\frac{\lambda - k}{\frac{1}{n} + \frac{k^2}{2(n-1)}}}{\frac{1}{n} + \frac{k^2}{2(n-1)}} \right) \quad (12.10)$$

we obtain from the equalities $P(p_1) = 1 - \alpha$ and $P(p_2) = \beta$:

$$n = \frac{2(\lambda_\alpha + \lambda_\beta)^2 + (\lambda_\alpha \lambda_\beta + \lambda_\beta \lambda_1)^2}{2(\lambda_1 - \lambda_2)^2} \quad (12.11)$$

and

$$k = \frac{\lambda_\alpha \lambda_2 + \lambda_\beta \lambda_1}{\lambda_\alpha + \lambda_\beta}, \quad (12.12)$$

where λ_α , λ_β , λ_1 and λ_2 are the arguments belonging to the probabilities α , β , p_1 and p_2 of the normal distribution function.

Be, e.g., the lower threshold value of the degree of density $T_A = 88\%$, the permitted and the tolerated fault percentages $p_1 = 0.06$, resp. $p_2 = 0.15$ and the risk of the supplier and the receiver $\alpha = \beta = 0.05$; in this case the required number of the samples (measurements) is

$$n = \frac{2(1.645 + 1.645)^2 + (1.645^2 + 1.645 \cdot 1.555)^2}{2(1.555 - 1.04)^2} = 93$$

and the directional tangent of the qualifying straight line

$$k = \frac{1.645(1.555 + 1.04)}{2 \cdot 1.645} = 1.298.$$

So, e.g., if the standard deviation of T_r is $s_T = 4.2\%$, then the required average degree of density is

$$\bar{T}_r = 88 + 1.298 \cdot 4.2 = 93.5\%.$$

Generally, the product standards specify the number of the samples depending on the lot size and permit a differentiation according to the various *degrees of strictness*.

FANG (1975) examined the case, where not only the lower (T_A), but also the upper threshold value (T_F) is fixed. For the solution he used the theory of LIEBERMAN and RESNIKOFF (1955). The method is correct, but specifying the upper threshold value of the density seems to be artificial and wasteful.

12.5. Strength and density control of stabilized soils

The stabilized soils are qualified on the basis of their *cube strength*. This does not mean that the efficiency of the compaction, or the *density* itself has no role at all: on assuming an identical quantity of binding material the strength increases with the decreasing

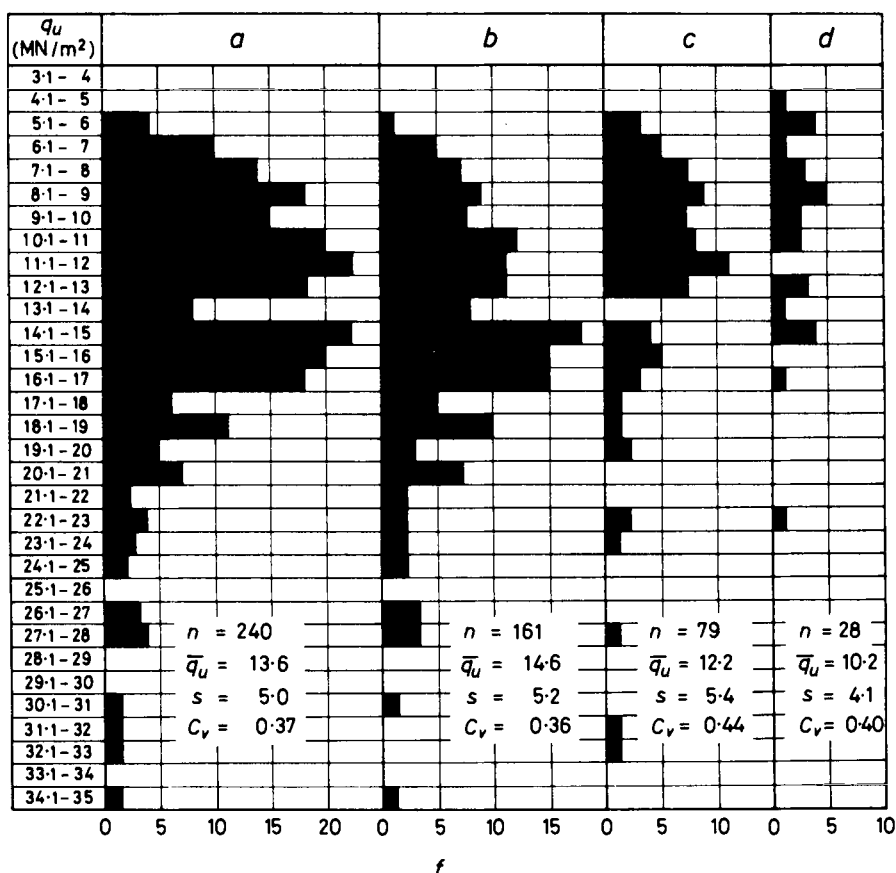


Fig. 12.12. Histograms of the cube strength of a soil stabilized by cement (BLAUT, 1975); a — for the total length of the road, b — for the section prepared with the prescribed cement dosage, c — for the section prepared with a cement dosage exceeding the prescribed amount, d — for the experimental section

void ratio. Consequently the *water content* of the applied soil is not indifferent either, as, e.g., the strength of the soil stabilized by cement is affected by it even twofold: partly by the water–cement ratio and partly by the attainable density.

The problems arising in the course of the construction work may be studied according to the example of BLAUT (1975). The designer prescribed the spreading of a 15 cm thick layer stabilized by cement under the pavement of a road. The water content of the available gravel was $w = 6\text{--}7\%$, so he specified $18.75 \text{ kg cement/m}^2$ for ensuring a satisfactory water–cement ratio (1.0–1.2).

During the construction there were heavy rains, whereby the water content of the gravel increased considerably. This was compensated for by the addition of more cement, whereby the contractor wished to attain the prescribed strength ($q_u = 7 \text{ MN/m}^2$). In this way he used an excess of 1000 ton cement over the planned quantity, whose necessity was disputed by the investor.

The dispute was settled on the basis of the histograms shown in *Fig 12.12*. The strength of cylindrical samples taken from the stabilized layer after 60 days elapsed since its preparation was determined (histogram *a*). The mean of the measured values was $\bar{q}_u = 13.6 \text{ MN/m}^2$, their standard deviation was $s = 5 \text{ MN/m}^2$. Then the samples were divided into two parts: one with 18.75 kg/m^2 cement addition (histogram *b*) and one with a greater cement addition (histogram *c*). According to *Fig 12.12* the average strength and the standard deviation of the two parts showed no significant difference, which means that *the overdosage was necessary*.

The correctness of the process was proved also by the evaluation of a 280 m long test section, prepared with the *prescribed* cement addition, inspite of the increased water content of the gravel. Here the average strength decreased to 10.2 MN/m^2 (histogram *d*) and the ratio of the samples, whose strength was $q_u \leq 7 \text{ MN/m}^2$ increased from 5.8% to 21.4%. The difference appeared also in the bulk density, which was $\bar{\gamma} = 21.85 \text{ kN/m}^3$ in the constructed length, but only $\bar{\gamma} = 21.60 \text{ kN/m}^3$ in the test section.

12.6. Continuous control by control cards

So far we have supposed tacitly that the density, or other characteristics of the earthwork are controlled directly before their acception. But admittedly this solution would imply very heavy expenses if the quality of the earthwork or a part of it proved to be unsatisfactory. In cases, where e.g. the mass (height) of the fill or the embankment is great, only a part of it could be repaired, as a compromise, whereby the probability of failure would increase and the safety of the operations decrease, inspite of the excess costs expended by the contractor.

This consideration led to the *continuous* control in the industry by taking—at least once a day—samples and recording the results of the quality control on so-called control cards, showing in this way also the variations in the quality of the product and permitting to take the necessary measures (by the plant, or the contractor) for

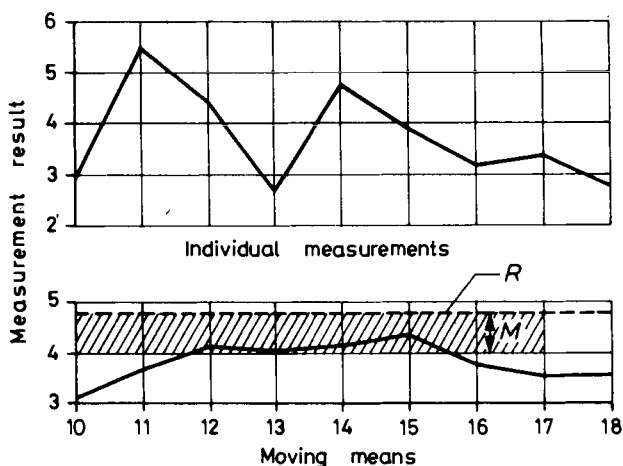


Fig. 12.13. Individual measurements and moving means on the control card (BEATON, 1968); R — threshold value for the refusal, M — range requiring the increase of the number of samples

improving it. This is generally *self-control* (the control cards are prepared by the plant itself), but the measurements may be performed also by the investor, or a third party. Where control cards are used, the production (or construction) is running—according to the statistical terminology—“under statistical control”.

From among the numerous variants of the control cards the following are the simplest.

DAVIS (1953) proposed to record the results of the daily (or rarer) measurements continuously on a sheet arranged by T_r categories (0.860–0.869; 0.870–0.879, etc.) and after collecting a sufficient amount of data, the statistical parameters, e.g., the ratio of the densities lying below the lower threshold value, can be determined.

Another possibility is the formation of *sample chains*, i.e. the *moving mean* of the measured characteristic continuously (BEATON, 1968). For example, in Fig. 12.13, every point of the graph indicates the mean of the last 4 days. We may mark in the coordinate system also the *refusal limit* and the *range*, where the number of the measurements must be increased.

The mean can be formed also *retrospectively* till the *first* measurement (PETTIT, 1967). In this case a gradually stabilizing polygon is obtained for the mean and also for the standard deviation (if calculated) (Fig. 12.14). (We have used this solution in the analysis of the tasks illustrated by Figs 1.2 and 1.3.)

More useful information is supplied by the so-called S- (SHEWART-) and C- (cumulative) control cards.

On the abscissa of the S-card the time, or some coordinate of the earthwork (e.g. the number of the road sections) and on the ordinate the *mean* of the measured characteristic is represented, indicating also the prescribed value and the various

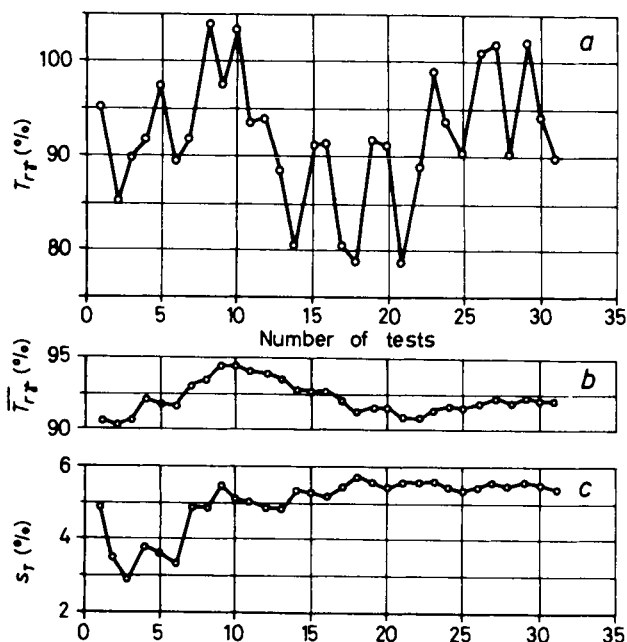


Fig. 12.14. Control card type indicating the individual measurements (a), the mean calculated continuously from the first measurement (b) and the standard deviation (c) (PETTITT, 1967)

standard deviation ranges of the mean. In the majority of the practical cases also another coordinate system is plotted below this for the continuous control of the *standard deviation* of the involved characteristic, i.e. of the degree of homogeneity of the product.

With this type of the control cards the prescription of a mean μ and a standard deviation σ for the degree of density of the earthwork is assumed. If n is the daily number of the samples and $\bar{x}$ is the daily mean of T_{rp} , then according to Section 7.4 the *confidence interval* of the latter value is proportional to the ratio

$$\sigma_k = \frac{\sigma}{\sqrt{n}}. \quad (12.13)$$

Accordingly, the probability of $\bar{x}$ differing significantly from μ can be judged with the help of a σ_k scale, containing in practice usually the values $1 \cdot \sigma_k$, $2 \cdot \sigma_k$ (warning limit) and $3 \cdot \sigma_k$ (control limit) (Fig. 12.15). An intervention (for finding and repairing the fault) is necessary in the following cases:

- if even a single point falls outside of the limit $\pm 3 \cdot \sigma_k$,
- if at least two subsequent points fall outside the straight lines $\pm 2 \cdot \sigma_k$,
- if at least three subsequent points fall outside the straight lines $\pm \sigma_k$,
- if 11, or more points lie on the same side of the line of the prescribed mean value μ .

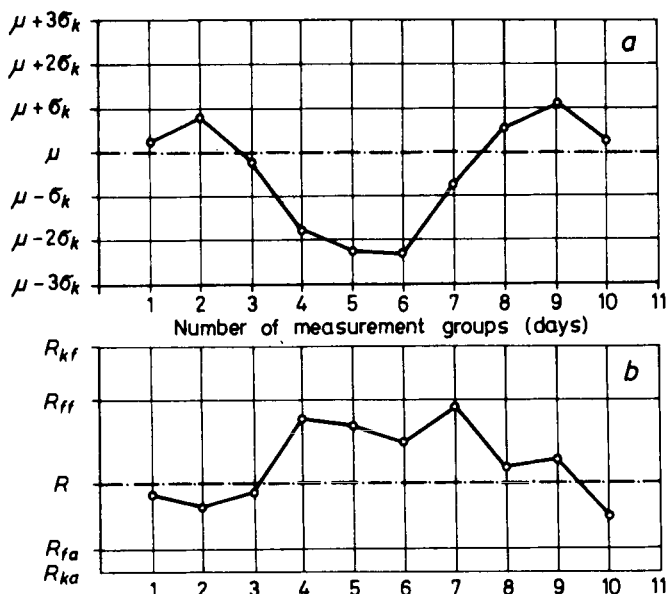


Fig. 12.15. S-type control card indicating (a) the mean of the measurement groups (of daily measurements) and (b) their range

The role of the densities *exceeding* the prescribed mean significantly must be judged individually. Such a value may be qualified as harmful, if it is the result of a considerable excess work (the control of w , an increased compaction work), or if it influences—alternating with weaker sections—the behaviour of the superimposed object unfavourably (see Section 12.4.1).

The second coordinate system on the S-card serves for controlling the *standard deviation*. As according to the theory of the ordered samples there exists a definite correlation between the standard deviation s and the *range* R (see Section 1.2.2), the latter is used in practice preferentially for its simplicity. The R value to be prescribed, the lower and the upper warning-, resp. control limits may be calculated as follows (NEMESDY, 1969b):

n	$d = R/\sigma$	$C_1 = R_{fa}/R$	$C_2 = R_{ff}/R$	$C_3 = R_{ka}/R$	$C_4 = R_{kf}/R$
2	1.128	0.04	2.87	0.00	4.12
3	1.693	0.18	2.17	0.04	2.98
4	2.059	0.29	1.93	0.10	2.57
5	2.326	0.37	1.82	0.16	2.34
6	2.534	0.42	1.72	0.21	2.21

The C-cards, drawn generally under the S-cards are used together with the latter and illustrate the continuously formed sum of the anomalies of the daily $\bar{x}$ values. For

example, the values of the first, second and third days are as follows:

$$(\bar{x}_1 - \mu);$$

$$(\bar{x}_1 - \mu) + (\bar{x}_2 - \mu);$$

$$(\bar{x}_1 - \mu) + (\bar{x}_2 - \mu) + (\bar{x}_3 - \mu).$$

The rising tendency of the polygon constructed in this way refers to the increase of the mean, its descending tendency to its decrease. (The instantaneous mean may also be calculated from the angle of inclination of the individual sections of the polygon.)

The *S*-card points rather to *sudden variations*, the *C*-card to *gradual variations*. Their introduction as control means would be desirable for every greater object.

Notations

a_{ik}	— regression coefficient (for more than two variables)
A	— SKEMPTON activity
	— cross section area of the sample
B	— symbol of the beta distribution
	— width of the foundation
c	— cohesion of the soil
c_n	— cohesion required for the stability of the slope
c_u	— undrained shear strength
c_r	— coefficient of consolidation
C_c	— compression index
C_k	— kurtosis
C_{kt}	— kurtosis of the transformed series
C_r	— recompression ratio
C_s	— coefficient of skewness
C_{st}	— coefficient of skewness of the transformed series
C_v	— coefficient of variation
CL	— confidence interval
Cov	— covariance
d	— diameter of the soil grain
D	— depth of the soil covering the foundation level (embedding depth)
$D(\xi)$	— standard deviation of the population
$D^2(\xi)$	— variance of the population
e	— void ratio
e_{rf}	— error function
E_a	— active earth pressure
E_{ah}	— horizontal component of the active earth pressure
E_{av}	— vertical component of the active earth pressure
E_s	— modulus of compressibility
f	— frequency
	— influence factor of the settlement
f or n	— number of degrees of freedom
$f(x)$	— density function
$f_1(\lambda)$	— a priori distribution
$f_2(\lambda)$	— a posteriori distribution
f_r	— relative frequency
F	— quotient of variances (F -test)
	— area of the figure of vertical stresses
F_p	— F -value belonging to the significance level p

FS	— safety factor
FV	— symbol of the in situ vane test
$F(x)$	— distribution function
h	— difference of water levels
H	— height of the slope (or the retaining wall)
$H(x)$	— entropy
$H_r(x)$	— relative entropy
i	— hydraulic gradient
I	— correlation index
I_c	— consistency index
I_{om}	— organic content of the soil
I_p	— plasticity index
k	— coefficient of permeability
	— directional tangent of the qualifying straight line
K	— PEARSON-criterion
K_A	— active earth pressure factor
K_p	— passive earth pressure factor
l	— length of the soil sample
l_k or L	— length of the slip surface
l_i	— length of the chord of the slip surface
L	— longer side of the foundation
LV	— symbol of the laboratory vane test
$L(\lambda)$	— likelihood function
m_d	— mode of the distribution
m_o	— height of the slope of vertical walls
	— limit depth of the vertical stresses
M	— symbol of the expected value
M_e	— median of the distribution
M_Q	— driving moment
M_R	— resisting moment
$M. D.$	— mean deviation
N	— normal force acting on the slip surface
N_c	— stability factor
$N(m, \sigma)$	— population of a normal distribution
N_c	— load bearing factors
N_q	
N_γ	
p_f	— probability of failure
p_1	— permitted fault percentage
p_2	— tolerated fault percentage
P or p	— probability
$P(x)$	— periodic component of the time- (data-) series
q	— ultimate bearing capacity (under footings)
$q_u(\alpha U)$	— unconfined compression strength of the soil
Q	— water quantity
	— load (causing the failure)
r	— correlation coefficient
r_e	— closeness of the relationship in the case of a sign-correlation
r_{ik}	— total correlation coefficient
r_B	— closeness of the regression in the case of a rank-correlation

R	— range of the sample — empirical correlation matrix — resistance (strength acting against the failure, etc.) — radius of the slip surface
s	— empirical standard deviation — settlement of the foundation (or its individual points)
s_a	— standard deviation of the regression coefficient
s_b	— standard deviation of the intercept
s_c	— calculated settlement — comparative standard deviation
s_{dm}	— standard deviation of the maximum dry bulk density
s_h	— standard deviation characterizing the natural heterogeneity
s_i	— reiterative standard deviation
s_k	— standard deviation characterizing the row-effect
s_m	— standard deviation characterizing the column-effect — measured settlement
s_r	— relative standard deviation — standard deviation of the correlation coefficient
s_{rp}	— standard deviation characterizing the reproducibility
s_T	— standard deviation of the degree of density
s_v	— standard deviation of the factor of safety
s_{∞}	— settlement at the end of the consolidation process
s^2	— empirical variance
S	— weight- (mass-) percentage
S_r	— degree of saturation
SM	— safety margin
$SI \dots S5$	— symbols of the soils of Szeged
t	— time
t_p or t_α	— value of the STUDENT-distribution function belonging to the probability p (resp. α)
T	— time factor (related to the consolidation) — tangential force acting on the slip surface
T_A	— prescribed lower threshold value of the degree of density
T_{ry}	— degree of density
$T(x)$	— periodic component of the time- (data-) series
u_p	— value of the normal distribution function belonging to the probability p
U	— uniformity coefficient — symbol of the unconfined compression test
UU	— symbol of the drained triaxial test
w	— water content
w_i	— coefficient of optimization
w_L	— liquid limit
w_m	— water content of the core sample
w_{mK}	— water content of a sample taken from a small diameter borehole
w_{mN}	— water content of a sample taken from a large diameter borehole
w_{opt}	— optimum water content obtained from the compaction test
w_p	— plastic limit
w_s	— shrinkage limit
w_z	— water content of the partly disturbed sample
z	— depth measured from the level of the terrain
α	— risk of the supplier

α_{cr}	— angular strain
	— power belonging to $C_s=0$
	— slope angle
	— risk of the receiver
β_1	— PEARSON-skewness
β_2	— PEARSON-kurtosis
Δs	— differential settlement
γ_d	— dry bulk density
γ_{dmax}	— max. dry bulk density (from the compaction test)
$\gamma(h)$	— variogram
γ_n or γ	— bulk density of the soil
γ_s	— specific weight of the soil grains
Γ	— symbol of the gamma function
δ	— directional angle of the earth pressure
η	— statistical safety
φ	— internal angle of friction of the soil
$\varphi(x)$	— density function of the normal distribution
$\varphi(\lambda)$	— standardized normal density function
$\Phi(x)$	— normal distribution function
$\Phi(\lambda)$	— standardized normal distribution function
κ	— degree of consolidation
λ	— argument of the normal distribution
μ	— quotient of the calculated settlement by the measured settlement
	— POISSON's ratio
	— expected value of the population
μ_k	— central moment
ν	— factor of safety
	— concentration factor of FRÖHLICH
ν_c	— central factor of safety
ρ	— relative subsidence due to the effect of water absorption
	— correlation coefficient of the population
ρ_i	— multiple correlation coefficient
ρ_{ik}	— partial correlation coefficient
τ	— shear strength of the soil
σ	— standard deviation of the population
σ_a	— basic limit stress value of the soil
σ_f	— residual standard deviation
σ_H	— limit bearing capacity of the soil (in connection with foundation)
σ^2	— variance of the population
ξ	— random variable forming the population

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Appendix

Table F1. Density function of the normal distribution

λ	$\varphi(\lambda)$	λ	$\varphi(\lambda)$	λ	$\varphi(\lambda)$	λ	$\varphi(\lambda)$	λ	$\varphi(\lambda)$	λ	$\varphi(\lambda)$
0.00	0.3989	0.53	0.3467	1.06	0.2275	1.59	0.1127	2.12	0.0422	2.65	0.0119
0.01	0.3989	0.54	0.3448	1.07	0.2251	1.60	0.1109	2.13	0.0413	2.66	0.0116
0.02	0.3989	0.55	0.3429	1.08	0.2227	1.61	0.1092	2.14	0.0404	2.67	0.0113
0.03	0.3988	0.56	0.3410	1.09	0.2203	1.62	0.1074	2.15	0.0396	2.68	0.0110
0.04	0.3986	0.57	0.3391	1.10	0.2179	1.63	0.1057	2.16	0.0387	2.69	0.0107
0.05	0.3984	0.58	0.3372	1.11	0.2155	1.64	0.1040	2.17	0.0379	2.70	0.0104
0.06	0.3982	0.59	0.3352	1.12	0.2131	1.65	0.1023	2.18	0.0371	2.71	0.0101
0.07	0.3980	0.60	0.3332	1.13	0.2107	1.66	0.1006	2.19	0.0363	2.72	0.0099
0.08	0.3977	0.61	0.3312	1.14	0.2083	1.67	0.0989	2.20	0.0355	2.73	0.0096
0.09	0.3973	0.62	0.3292	1.15	0.2059	1.68	0.0973	2.21	0.0347	2.74	0.0093
0.10	0.3970	0.63	0.3271	1.16	0.2036	1.69	0.0957	2.22	0.0339	2.75	0.0091
0.11	0.3965	0.64	0.3251	1.17	0.2012	1.70	0.0940	2.23	0.0332	2.76	0.0088
0.12	0.3961	0.65	0.3230	1.18	0.1989	1.71	0.0925	2.24	0.0325	2.77	0.0086
0.13	0.3956	0.66	0.3209	1.19	0.1965	1.72	0.0909	2.25	0.0317	2.78	0.0084
0.14	0.3951	0.67	0.3187	1.20	0.1942	1.73	0.0893	2.26	0.0310	2.79	0.0081
0.15	0.3945	0.68	0.3166	1.21	0.1919	1.74	0.0878	2.27	0.0303	2.80	0.0079
0.16	0.3939	0.69	0.3144	1.22	0.1895	1.75	0.0863	2.28	0.0297	2.81	0.0077
0.17	0.3932	0.70	0.3123	1.23	0.1872	1.76	0.0848	2.29	0.0290	2.82	0.0075
0.18	0.3925	0.71	0.3101	1.24	0.1849	1.77	0.0833	2.30	0.0283	2.83	0.0073
0.19	0.3918	0.72	0.3079	1.25	0.1826	1.78	0.0818	2.31	0.0277	2.84	0.0071
0.20	0.3910	0.73	0.3056	1.26	0.1804	1.79	0.0804	2.32	0.0270	2.85	0.0069
0.21	0.3902	0.74	0.3034	1.27	0.1781	1.80	0.0790	2.33	0.0264	2.86	0.0067
0.22	0.3894	0.75	0.3011	1.28	0.1758	1.81	0.0775	2.34	0.0258	2.87	0.0065
0.23	0.3885	0.76	0.2989	1.29	0.1736	1.82	0.0761	2.35	0.0252	2.88	0.0063
0.24	0.3876	0.77	0.2966	1.30	0.1714	1.83	0.0748	2.36	0.0246	2.89	0.0061
0.25	0.3867	0.78	0.2943	1.31	0.1691	1.84	0.0734	2.37	0.0241	2.90	0.0060
0.26	0.3857	0.79	0.2920	1.32	0.1669	1.85	0.0721	2.38	0.0235	2.91	0.0058
0.27	0.3847	0.80	0.2897	1.33	0.1647	1.86	0.0707	2.39	0.0229	2.92	0.0056
0.28	0.3836	0.81	0.2874	1.34	0.1626	1.87	0.0694	2.40	0.0224	2.93	0.0055
0.29	0.3825	0.82	0.2850	1.35	0.1604	1.88	0.0681	2.41	0.0219	2.94	0.0053
0.30	0.3814	0.83	0.2827	1.36	0.1582	1.89	0.0669	2.42	0.0213	2.95	0.0051
0.31	0.3802	0.84	0.2803	1.37	0.1561	1.90	0.0656	2.43	0.0208	2.96	0.0050
0.32	0.3790	0.85	0.2780	1.38	0.1539	1.91	0.0644	2.44	0.0203	2.97	0.0048
0.33	0.3778	0.86	0.2756	1.39	0.1518	1.92	0.0632	2.45	0.0198	2.98	0.0047
0.34	0.3765	0.87	0.2732	1.40	0.1497	1.93	0.0620	2.46	0.0194	2.99	0.0046
0.35	0.3752	0.88	0.2709	1.41	0.1476	1.94	0.0608	2.47	0.0189	3.00	0.0044
0.36	0.3739	0.89	0.2685	1.42	0.1456	1.95	0.0596	2.48	0.0184	3.10	0.0033
0.37	0.3725	0.90	0.2661	1.43	0.1435	1.96	0.0584	2.49	0.0180	3.20	0.0024
0.38	0.3712	0.91	0.2637	1.44	0.1415	1.97	0.0573	2.50	0.0175	3.30	0.0017
0.39	0.3697	0.92	0.2613	1.45	0.1394	1.98	0.0562	2.51	0.0171	3.40	0.0012
0.40	0.3683	0.93	0.2589	1.46	0.1374	1.99	0.0551	2.52	0.0167	3.50	0.0009
0.41	0.3668	0.94	0.2565	1.47	0.1354	2.00	0.0540	2.53	0.0163	3.60	0.0006
0.42	0.3653	0.95	0.2541	1.48	0.1334	2.01	0.0529	2.54	0.0158	3.70	0.0004
0.43	0.3637	0.96	0.2516	1.49	0.1315	2.02	0.0519	2.55	0.0154	3.80	0.0003
0.44	0.3621	0.97	0.2492	1.50	0.1295	2.03	0.0508	2.56	0.0151	3.90	0.0002
0.45	0.3605	0.98	0.2468	1.51	0.1276	2.04	0.0498	2.57	0.0147	4.00	0.0001
0.46	0.3589	0.99	0.2444	1.52	0.1257	2.05	0.0488	2.58	0.0143	4.10	0.0001
0.47	0.3572	1.00	0.2420	1.53	0.1238	2.06	0.0478	2.59	0.0139	4.20	0.0001
0.48	0.3555	1.01	0.2396	1.54	0.1219	2.07	0.0468	2.60	0.0136		
0.49	0.3538	1.02	0.2371	1.55	0.1200	2.08	0.0459	2.61	0.0132		
0.50	0.3521	1.03	0.2347	1.56	0.1182	2.09	0.0449	2.62	0.0129		
0.51	0.3503	1.04	0.2323	1.57	0.1163	2.10	0.0440	2.63	0.0126		
0.52	0.3485	1.05	0.2299	1.58	0.1145	2.11	0.0431	2.64	0.0122		

Table F2. The normal distribution

λ	$\gamma(\lambda)$	λ	$\gamma(\lambda)$	λ	$\gamma(\lambda)$	λ	$\gamma(\lambda)$	λ	$\gamma(\lambda)$
0.00	0.5000	0.51	0.6950	1.02	0.8461	1.53	0.9370	2.08	0.9812
0.01	0.5040	0.52	0.6985	1.03	0.8485	1.54	0.9382	2.10	0.9821
0.02	0.5080	0.53	0.7019	1.04	0.8508	1.55	0.9394	2.12	0.9830
0.03	0.5120	0.54	0.7054	1.05	0.8531	1.56	0.9406	2.14	0.9838
0.04	0.5160	0.55	0.7088	1.06	0.8554	1.57	0.9418	2.16	0.9846
0.05	0.5199	0.56	0.7123	1.07	0.8577	1.58	0.9429	2.18	0.9854
0.06	0.5239	0.57	0.7157	1.08	0.8599	1.59	0.9441	2.20	0.9861
0.07	0.5279	0.58	0.7190	1.09	0.8621	1.60	0.9452	2.22	0.9868
0.08	0.5319	0.59	0.7224	1.10	0.8643	1.61	0.9463	2.24	0.9875
0.09	0.5359	0.60	0.7257	1.11	0.8665	1.62	0.9474	2.26	0.9881
0.10	0.5398	0.61	0.7291	1.12	0.8686	1.63	0.9484	2.28	0.9887
0.11	0.5438	0.62	0.7324	1.13	0.8708	1.64	0.9495	2.30	0.9893
0.12	0.5478	0.63	0.7357	1.14	0.8729	1.65	0.9505	2.32	0.9898
0.13	0.5517	0.64	0.7389	1.15	0.8749	1.66	0.9515	2.34	0.9904
0.14	0.5557	0.65	0.7422	1.16	0.8770	1.67	0.9525	2.36	0.9909
0.15	0.5596	0.66	0.7454	1.17	0.8790	1.68	0.9535	2.38	0.9913
0.16	0.5636	0.67	0.7486	1.18	0.8810	1.69	0.9545	2.40	0.9918
0.17	0.5675	0.68	0.7517	1.19	0.8830	1.70	0.9554	2.42	0.9922
0.18	0.5714	0.69	0.7549	1.20	0.8849	1.71	0.9564	2.44	0.9927
0.19	0.5753	0.70	0.7580	1.21	0.8869	1.72	0.9572	2.46	0.9931
0.20	0.5793	0.71	0.7611	1.22	0.8888	1.73	0.9582	2.48	0.9934
0.21	0.5832	0.72	0.7642	1.23	0.8907	1.74	0.9591	2.50	0.9938
0.22	0.5871	0.73	0.7673	1.24	0.8925	1.75	0.9600	2.52	0.9941
0.23	0.5910	0.74	0.7703	1.25	0.8944	1.76	0.9608	2.54	0.9945
0.24	0.5948	0.75	0.7734	1.26	0.8962	1.77	0.9616	2.56	0.9948
0.25	0.5987	0.76	0.7764	1.27	0.8980	1.78	0.9625	2.58	0.9951
0.26	0.6026	0.77	0.7794	1.28	0.8997	1.79	0.9633	2.60	0.9953
0.27	0.6064	0.78	0.7823	1.29	0.9015	1.80	0.9641	2.62	0.9956
0.28	0.6103	0.79	0.7853	1.30	0.9032	1.81	0.9649	2.64	0.9959
0.29	0.6141	0.80	0.7881	1.31	0.9049	1.82	0.9656	2.66	0.9961
0.30	0.6179	0.81	0.7910	1.32	0.9066	1.83	0.9664	2.68	0.9963
0.31	0.6217	0.82	0.7939	1.33	0.9082	1.84	0.9671	2.70	0.9965
0.32	0.6255	0.83	0.7967	1.34	0.9099	1.85	0.9678	2.72	0.9967
0.33	0.6293	0.84	0.7995	1.35	0.9115	1.86	0.9686	2.74	0.9969
0.34	0.6331	0.85	0.8023	1.36	0.9131	1.87	0.9693	2.76	0.9971
0.35	0.6368	0.86	0.8051	1.37	0.9147	1.88	0.9699	2.78	0.9973
0.36	0.6406	0.87	0.8078	1.38	0.9162	1.89	0.9706	2.80	0.9974
0.37	0.6443	0.88	0.8106	1.39	0.9177	1.90	0.9713	2.82	0.9976
0.38	0.6480	0.89	0.8133	1.40	0.9192	1.91	0.9719	2.84	0.9977
0.39	0.6517	0.90	0.8159	1.41	0.9207	1.92	0.9726	2.86	0.9979
0.40	0.6554	0.91	0.8186	1.42	0.9222	1.93	0.9732	2.88	0.9980
0.41	0.6591	0.92	0.8212	1.43	0.9236	1.94	0.9738	2.90	0.9981
0.42	0.6628	0.93	0.8238	1.44	0.9251	1.95	0.9744	2.92	0.9982
0.43	0.6664	0.94	0.8264	1.45	0.9265	1.96	0.9750	2.94	0.9984
0.44	0.6700	0.95	0.8289	1.46	0.9279	1.97	0.9756	2.96	0.9985
0.45	0.6736	0.96	0.8315	1.47	0.9292	1.98	0.9761	2.98	0.9986
0.46	0.6772	0.97	0.8340	1.48	0.9306	1.99	0.9767	3.00	0.9986
0.47	0.6808	0.98	0.8365	1.49	0.9319	2.00	0.9772	3.20	0.9993
0.48	0.6844	0.99	0.8389	1.50	0.9332	2.02	0.9783	3.40	0.9996
0.49	0.6879	1.00	0.8413	1.51	0.9345	2.04	0.9793	3.60	0.9998
0.50	0.6915	1.01	0.8438	1.52	0.9357	2.06	0.9803	3.80	0.9999

Table F3. The χ^2 -distribution

<i>n</i>	<i>p</i> (%)													
	99	98	95	90	80	70	50	30	20	10	5	2	1	0.1
1	0.000	0.000	0.003	0.016	0.064	0.148	0.455	1.074	1.642	2.706	3.841	5.412	6.635	10.827
2	0.020	0.040	0.103	0.211	0.446	0.713	1.386	2.408	3.219	4.605	5.991	7.824	9.210	13.815
3	0.115	0.185	0.352	0.584	1.005	1.424	2.366	3.665	4.642	6.251	7.815	9.837	11.345	16.268
4	0.297	0.429	0.711	1.064	1.649	2.195	3.357	4.878	5.989	7.779	9.488	11.668	13.277	18.465
5	0.554	0.752	1.145	1.610	2.343	3.000	4.351	6.064	7.289	9.236	11.070	13.388	15.086	20.517
6	0.872	1.134	1.635	2.204	3.070	3.828	5.348	7.231	8.558	10.645	12.592	15.033	16.812	22.457
7	1.239	1.564	2.167	2.833	3.822	4.671	6.346	8.383	9.803	12.017	14.067	16.622	18.475	24.322
8	1.646	2.032	2.733	3.490	4.594	5.527	7.344	9.524	11.030	13.362	15.507	18.168	20.090	26.125
9	2.088	2.532	3.325	4.168	5.380	6.393	8.343	10.656	12.242	14.684	16.919	19.679	21.666	27.877
10	2.558	3.059	3.940	4.865	6.179	7.267	9.342	11.781	13.442	15.987	18.307	21.161	23.209	29.588
11	3.053	3.609	4.575	5.578	6.989	8.148	10.341	12.899	14.631	17.275	19.675	22.618	24.725	31.264
12	3.571	4.178	5.226	6.304	7.807	9.034	11.340	14.011	15.812	18.549	21.026	24.054	26.217	32.909
13	4.107	4.765	5.892	7.042	8.634	9.926	12.340	15.119	16.985	19.812	22.362	25.472	27.688	34.528
14	4.660	5.368	6.571	7.790	9.467	10.821	13.339	16.222	18.151	21.064	23.685	26.873	29.141	36.123
15	5.229	5.985	7.261	8.547	10.307	11.721	14.339	17.322	19.311	22.307	24.996	28.259	30.578	37.697
16	5.812	6.614	7.962	9.312	11.152	12.624	15.338	18.418	20.465	23.542	26.296	29.633	32.000	39.252
17	6.408	7.255	8.672	10.085	12.002	13.531	16.338	19.511	21.615	24.769	27.587	30.995	33.409	40.790
18	7.015	7.906	9.390	10.865	12.857	14.440	17.338	20.601	22.760	25.989	28.869	32.346	34.805	42.312
19	7.633	8.567	10.117	11.651	13.716	15.352	18.338	21.689	23.900	27.204	30.144	33.687	36.191	43.820
20	8.260	9.237	10.851	12.443	14.578	16.226	19.337	22.775	25.038	28.412	31.410	35.020	37.566	45.315
21	8.897	9.915	11.591	13.240	15.445	17.182	20.337	23.858	26.171	29.615	32.671	36.343	38.932	46.797
22	9.542	10.600	12.338	14.041	16.314	18.101	21.337	24.939	27.301	30.813	33.924	37.659	40.289	48.268
23	10.196	11.293	13.091	14.848	17.187	19.021	22.337	26.018	28.429	32.007	35.172	38.968	41.638	49.728
24	10.856	11.992	13.848	15.659	18.062	19.943	23.337	27.096	29.553	33.196	36.415	40.270	42.980	51.179
25	11.524	12.697	14.611	16.473	18.940	20.867	24.337	28.172	30.675	34.382	37.652	41.566	44.314	52.620
26	12.198	13.409	15.379	17.292	19.820	21.792	25.336	29.246	31.795	35.563	38.885	42.856	45.642	54.052
27	12.879	14.125	16.151	18.114	20.703	22.719	26.336	30.319	32.912	36.741	40.113	44.140	46.963	55.476
28	13.565	14.847	16.928	18.939	21.588	23.647	27.336	31.391	34.027	37.916	41.337	45.419	48.278	56.793
29	14.256	15.574	17.708	19.768	22.475	24.577	28.336	32.461	35.139	39.087	42.557	46.693	49.588	58.302
30	14.953	16.306	18.493	20.599	23.364	25.508	29.366	33.530	36.250	40.256	43.773	47.962	50.892	59.703

Table F4. The STUDENT-distribution

<i>n</i>	<i>p</i>												
	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.001
1	0.158	0.325	0.510	0.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	636.619
2	0.142	0.289	0.445	0.617	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	31.598
3	0.137	0.277	0.424	0.584	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	12.941
4	0.134	0.271	0.414	0.569	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	8.610
5	0.132	0.267	0.408	0.559	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	6.859
6	0.131	0.265	0.404	0.553	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.959
7	0.130	0.263	0.402	0.549	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	5.405
8	0.130	0.262	0.399	0.546	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	5.041
9	0.129	0.261	0.398	0.543	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.781
10	0.129	0.260	0.397	0.542	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.587
11	0.129	0.260	0.396	0.540	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.437
12	0.128	0.259	0.395	0.539	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	4.318
13	0.128	0.259	0.394	0.538	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	4.221
14	0.128	0.258	0.393	0.537	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	4.140
15	0.128	0.258	0.393	0.536	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	4.073
16	0.128	0.258	0.392	0.535	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	4.015
17	0.128	0.257	0.392	0.534	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.965
18	0.127	0.257	0.392	0.534	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.922
19	0.127	0.257	0.391	0.533	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.883
20	0.127	0.257	0.391	0.533	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.850
21	0.127	0.257	0.391	0.532	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.819
22	0.127	0.256	0.390	0.532	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.792
23	0.127	0.256	0.390	0.532	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.767
24	0.127	0.256	0.390	0.531	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.745
25	0.127	0.256	0.390	0.531	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.725
26	0.127	0.256	0.390	0.531	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.707
27	0.127	0.256	0.389	0.531	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.690
28	0.127	0.256	0.389	0.530	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.674
29	0.127	0.256	0.389	0.530	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.659
30	0.127	0.256	0.389	0.530	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.646
40	0.126	0.255	0.388	0.529	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.551
60	0.126	0.254	0.387	0.527	0.679	0.848	1.046	1.296	1.671	2.000	2.390	2.660	3.460
120	0.126	0.254	0.386	0.526	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	3.373
∞	0.126	0.253	0.385	0.524	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.291

Table F5. The gamma-distribution

$k \backslash p$	p																					
	0.05	0.1	0.5	1.0	2.5	5.0	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0	95.0	97.5	99.0	99.5	99.9	99.95	
0.5	0.039	0.016	0.039	0.016	0.038	0.039	0.016	0.064	0.148	0.275	0.46	0.71	1.07	1.64	2.71	3.84	5.02	6.63	7.9	10.8	12.1	
1.0	0.010	0.020	0.0100	0.0201	0.0506	0.103	0.211	0.446	0.713	1.02	1.39	1.83	2.41	3.22	4.61	5.99	7.38	9.21	10.6	13.8	15.2	
1.5	0.0153	0.0243	0.0717	0.115	0.216	0.352	0.584	1.00	1.42	1.87	2.37	2.95	3.67	4.64	6.25	7.81	9.35	11.3	12.8	16.3	17.7	
2.0	0.0639	0.0908	0.207	0.297	0.487	0.711	1.06	1.65	2.19	2.75	3.36	4.04	4.88	5.99	7.78	9.49	11.1	13.3	14.9	18.5	20.0	
2.5	0.158	0.210	0.412	0.554	0.831	1.15	1.61	2.34	3.00	3.66	4.35	5.13	6.06	7.29	9.24	11.1	12.8	15.1	16.7	20.5	22.1	
3.0	0.299	0.381	0.676	0.872	1.24	1.64	2.20	3.07	3.83	4.57	5.35	6.21	7.23	8.56	10.6	12.6	14.4	16.8	18.5	22.5	24.1	
3.5	0.485	0.598	0.989	1.24	1.69	2.17	2.83	3.82	4.67	5.49	6.35	7.28	8.38	9.80	12.0	14.1	16.0	18.5	20.3	24.3	26.0	
4.0	0.710	0.857	1.34	1.65	2.18	2.73	3.49	4.59	5.53	6.42	7.34	8.35	9.52	11.0	13.4	15.5	17.5	20.1	22.0	26.1	27.9	
4.5	0.972	1.15	1.73	2.09	2.70	3.33	4.17	5.38	6.39	7.36	8.34	9.41	10.7	12.2	14.7	16.9	19.0	21.7	23.6	27.9	29.7	
5.0	1.26	1.48	2.16	2.56	3.25	3.94	4.87	6.18	7.27	8.30	9.34	10.5	11.8	13.4	16.0	18.3	20.5	23.2	25.2	29.6	31.4	
5.5	1.59	1.83	2.60	3.05	3.82	4.57	5.58	6.99	8.15	9.24	10.3	11.5	12.9	14.6	17.3	19.7	21.9	24.7	26.8	31.3	33.1	
6.0	1.93	2.21	3.07	3.57	4.40	5.23	6.30	7.81	9.03	10.2	11.3	12.6	14.0	15.8	18.5	21.0	23.3	26.2	28.3	32.9	34.8	
6.5	2.31	2.62	3.57	4.11	5.01	5.89	7.04	8.63	9.93	11.1	12.3	13.6	15.1	17.0	19.8	22.4	24.7	27.7	29.8	34.5	36.5	
7.0	2.70	3.04	4.07	4.66	5.63	6.57	7.79	9.47	10.8	12.1	13.3	14.7	16.2	18.2	21.1	23.7	26.1	29.1	31.3	36.1	38.1	
7.5	3.11	3.48	4.60	5.23	6.26	7.26	8.55	10.3	11.7	13.0	14.3	15.7	17.3	19.3	22.3	25.0	27.5	30.6	32.8	37.7	39.7	
8.0	3.54	3.94	5.14	5.81	6.91	7.96	9.31	11.2	12.6	14.0	15.3	16.8	18.4	20.5	23.5	26.3	28.8	32.0	34.3	39.3	41.3	
8.5	3.98	4.42	5.70	6.41	7.56	8.67	10.1	12.0	13.5	14.9	16.3	17.8	19.5	21.6	24.8	27.6	30.2	33.4	35.7	40.8	42.9	
9.0	4.44	4.90	6.26	7.01	8.23	9.39	10.9	12.9	14.4	15.9	17.3	18.9	20.6	22.8	26.0	28.9	31.5	34.8	37.2	42.3	44.4	
9.5	4.91	5.41	6.84	7.63	8.91	10.1	11.7	13.7	15.4	16.9	18.3	19.9	21.7	23.9	27.2	30.1	32.9	36.2	38.6	43.8	46.0	
10.0	5.40	5.92	7.43	8.26	9.59	10.9	12.4	14.6	16.3	17.8	19.3	21.0	22.8	25.0	28.4	31.4	34.2	37.6	40.0	45.3	47.5	
10.5	5.90	6.45	8.03	8.90	10.3	11.6	13.2	15.4	17.2	18.8	20.3	22.0	23.9	26.2	29.6	32.8	35.5	38.9	41.4	46.8	49.0	
11.0	6.40	6.98	8.64	9.54	11.0	12.3	14.0	16.3	18.1	19.7	21.3	23.0	24.9	27.3	30.8	33.9	36.8	40.3	42.8	48.3	50.5	
11.5	6.92	7.53	9.26	10.2	11.7	13.1	14.8	17.2	19.0	20.7	22.3	24.1	26.0	28.4	32.0	35.2	38.1	41.6	44.2	49.7	52.0	
12.0	7.45	8.08	9.89	10.9	12.4	13.8	15.7	18.1	19.9	21.7	23.3	25.1	27.1	29.6	33.2	36.4	39.4	43.0	45.6	51.2	53.5	
12.5	7.99	8.65	10.5	11.5	13.1	14.6	16.5	18.9	20.9	22.6	24.3	26.1	28.2	30.7	34.4	37.7	40.6	44.3	46.9	52.6	54.9	
13.0	8.54	9.22	11.2	12.2	13.8	15.4	17.3	19.8	21.8	23.6	25.3	27.2	29.2	31.8	35.6	38.9	41.9	45.6	48.3	54.1	56.4	
13.5	9.09	9.80	11.8	12.9	14.6	16.2	18.1	20.7	22.7	24.5	26.3	28.2	30.3	32.9	36.7	40.1	43.2	47.0	49.6	55.5	57.9	
14.0	9.66	10.4	12.5	13.6	15.3	16.9	18.9	21.6	23.6	25.5	27.3	29.2	31.4	34.0	37.9	41.3	44.5	48.3	51.0	56.9	59.3	
14.5	10.2	11.0	13.1	14.3	16.0	17.7	19.8	22.5	24.6	26.5	28.3	30.3	32.5	35.1	39.1	42.6	45.7	49.6	52.3	58.3	60.7	
15.0	10.8	11.6	13.8	15.0	16.8	18.5	20.6	23.4	25.5	27.4	29.3	31.3	33.5	36.3	40.3	43.8	47.0	50.9	53.7	59.7	62.2	

Table F6. KOLMOGOROV's distribution function

z	$L(z)$	z	$L(z)$	z	$L(z)$	z	$L(z)$
0.28	0.000 001	0.71	0.305 471	1.14	0.851 394	1.57	0.985 544
0.29	0.000 004	0.72	0.322 265	1.15	0.858 038	1.58	0.986 426
0.30	0.000 009	0.73	0.339 113	1.16	0.864 442	1.59	0.987 260
0.31	0.000 021	0.74	0.355 981	1.17	0.870 612	1.60	0.988 048
0.32	0.000 046	0.75	0.372 833	1.18	0.876 548	1.61	0.988 791
0.33	0.000 091	0.76	0.389 640	1.19	0.882 258	1.62	0.989 492
0.34	0.000 171	0.77	0.406 372	1.20	0.887 750	1.63	0.990 154
0.35	0.000 303	0.78	0.423 002	1.21	0.893 030	1.64	0.990 777
0.36	0.000 511	0.79	0.439 505	1.22	0.898 104	1.65	0.991 364
0.37	0.000 826	0.80	0.455 857	1.23	0.902 972	1.66	0.991 917
0.38	0.001 285	0.81	0.472 041	1.24	0.907 648	1.67	0.992 438
0.39	0.001 929	0.82	0.488 030	1.25	0.912 132	1.68	0.992 928
0.40	0.002 808	0.83	0.503 808	1.26	0.916 432	1.69	0.993 389
0.41	0.003 972	0.84	0.519 366	1.27	0.920 556	1.70	0.993 823
0.42	0.005 476	0.85	0.534 682	1.28	0.924 505	1.71	0.994 230
0.43	0.007 377	0.86	0.549 744	1.29	0.928 288	1.72	0.994 612
0.44	0.009 730	0.87	0.564 546	1.30	0.931 908	1.73	0.994 972
0.45	0.012 590	0.88	0.579 070	1.31	0.935 370	1.74	0.995 309
0.46	0.016 005	0.89	0.593 316	1.32	0.938 682	1.75	0.995 625
0.47	0.020 022	0.90	0.607 270	1.33	0.941 848	1.76	0.995 922
0.48	0.024 682	0.91	0.620 928	1.34	0.944 872	1.77	0.996 200
0.49	0.030 017	0.92	0.634 286	1.35	0.947 756	1.78	0.996 460
0.50	0.036 055	0.93	0.647 338	1.36	0.950 512	1.79	0.996 704
0.51	0.042 814	0.94	0.660 082	1.37	0.953 142	1.80	0.996 932
0.52	0.050 306	0.95	0.672 516	1.38	0.955 650	1.81	0.997 146
0.53	0.058 534	0.96	0.684 636	1.39	0.958 040	1.82	0.997 346
0.54	0.067 497	0.97	0.696 444	1.40	0.960 318	1.83	0.997 533
0.55	0.077 183	0.98	0.707 940	1.41	0.962 486	1.84	0.997 707
0.56	0.087 577	0.99	0.719 126	1.42	0.964 552	1.85	0.997 870
0.57	0.098 656	1.00	0.730 000	1.43	0.966 516	1.86	0.998 023
0.58	0.110 395	1.01	0.740 566	1.44	0.968 382	1.87	0.998 145
0.59	0.122 760	1.02	0.750 826	1.45	0.970 158	1.88	0.998 297
0.60	0.135 718	1.03	0.760 780	1.46	0.971 846	1.89	0.998 421
0.61	0.149 229	1.04	0.770 434	1.47	0.973 448	1.90	0.998 536
0.62	0.163 225	1.05	0.779 794	1.48	0.974 970	1.91	0.998 644
0.63	0.177 753	1.06	0.788 860	1.49	0.976 412	1.92	0.998 744
0.64	0.192 677	1.07	0.797 636	1.50	0.977 782	1.93	0.998 837
0.65	0.207 987	1.08	0.806 128	1.51	0.979 080	1.94	0.998 924
0.66	0.223 637	1.09	0.814 342	1.52	0.980 310	1.95	0.999 004
0.67	0.239 582	1.10	0.822 282	1.53	0.981 476	1.96	0.999 079
0.68	0.255 780	1.11	0.829 950	1.54	0.982 578	1.97	0.999 149
0.69	0.272 189	1.12	0.837 356	1.55	0.983 622	1.98	0.999 213
0.70	0.288 765	1.13	0.844 502	1.56	0.984 610	1.99	0.999 273

Table F7.

j	$10^4 \cdot a_{nj}$									
	10	11	12	13	14	15	16	17	18	19
1	5739	5601	5475	5359	5251	5150	5056	4963	4886	4808
2	3291	3315	3325	3325	3318	3306	3290	3273	3253	3232
3	2141	2260	2347	2412	2460	2495	2521	2540	2553	2561
4	1224	1429	1586	1707	1802	1878	1939	1988	2027	2059
5	399	695	922	1099	1240	1353	1447	1524	1587	1641
6		0	303	539	727	880	1005	1109	1197	1271
7				0	240	433	593	725	837	932
8						0	196	359	496	612
9								0	163	303
10										0
11										
12										
13										
14										
15										

j	$10^4 a$									
	30	31	32	33	34	35	36	37	38	39
1	4251	4220	4188	4156	4127	4096	4068	4040	4015	3989
2	2944	2921	2898	2876	2854	2834	2813	2794	2774	2755
3	2487	2475	2463	2451	2439	2427	2415	2403	2391	2380
4	2148	2145	2141	2137	2132	2127	2121	2116	2110	2104
5	1870	1874	1878	1880	1882	1883	1883	1883	1881	1880
6	1630	1641	1651	1660	1667	1673	1678	1683	1686	1689
7	1415	1433	1449	1463	1475	1487	1496	1505	1513	1520
8	1219	1243	1265	1284	1301	1317	1331	1344	1356	1366
9	1036	1066	1093	1118	1140	1160	1179	1196	1211	1225
10	862	899	931	961	988	1013	1036	1056	1075	1092
11	697	739	777	812	844	873	900	924	947	967
12	537	585	629	669	706	739	770	798	824	848
13	381	435	485	530	572	610	645	677	706	733
14	227	289	344	395	441	484	523	559	592	622
15		144	206	262	314	361	404	444	481	515
16		0	68	131	187	239	287	331	372	409
17				0	62	119	172	220	264	305
18						0	57	110	158	203
19								0	53	101
20									0	
21										
22										
23										
24										
25										

The factors $a_{n,j}$ of the SHAPIRO–WILK test

if the number of the elements (n) is:

20	21	22	23	24	25	26	27	28	29
4734	4643	4590	4542	4493	4450	4407	4366	4328	4291
3211	3285	3156	3126	3098	3069	3043	3018	2992	2968
2565	2578	2571	2563	2554	2543	2533	2522	2510	2499
2085	2119	2131	2139	2145	2148	2151	2152	2151	2150
1686	1736	1764	1787	1807	1822	1836	1848	1857	1864
1334	1399	1443	1480	1512	1539	1563	1584	1601	1616
1013	1092	1150	1201	1245	1283	1316	1346	1372	1395
711	804	878	941	997	1046	1089	1128	1162	1192
422	530	618	696	764	823	876	923	965	1002
140	263	368	459	539	610	672	728	778	822
	0		228	321	403	476	540	598	650
			0	107	200	284	358	424	483
					0	94	178	253	320
							0	84	159
									0

if the number of the elements (n) is:

40	41	42	43	44	45	46	47	48	49	50
3964	3940	3917	3894	3872	3850	3830	3808	3789	3770	3751
2737	2719	2701	2684	2667	2651	2635	2620	2804	2589	2574
2368	2357	2345	2334	2323	2313	2302	2291	2281	2271	2260
2098	2091	2085	2078	2072	2065	2058	2052	2045	2038	2032
1878	1876	1874	1871	1868	1865	1862	1859	1855	1851	1847
1691	1693	1694	1695	1695	1695	1695	1695	1693	1692	1691
1526	1531	1535	1539	1542	1545	1548	1550	1551	1553	1554
1376	1384	1392	1398	1405	1410	1415	1420	1423	1427	1430
1237	1249	1259	1269	1278	1286	1293	1300	1306	1312	1317
1108	1123	1136	1149	1160	1170	1180	1189	1197	1205	1212
986	1004	1020	1035	1049	1052	1073	1085	1095	1105	1113
870	891	909	927	943	959	972	986	998	1010	1020
759	782	804	824	842	860	876	892	906	919	932
651	677	701	724	745	765	783	801	817	832	846
546	575	602	628	651	673	694	713	731	748	764
444	476	506	534	560	584	607	628	648	667	685
343	379	411	442	471	497	522	546	568	588	608
244	283	318	352	383	412	439	465	489	511	532
146	188	227	263	296	328	357	385	411	436	459
49	94	136	175	211	245	277	307	335	361	386
	0	45	87	126	163	197	229	259	288	314
			0	42	81	118	153	185	215	244
					0	39	76	111	143	174
							0	37	71	104
									0	35

Table F8. Critical values of the F-test ($p=0.05$); f_1 — degree of freedom of the numerator; f_2 — degree of freedom of the denominator

$f_2 \backslash f_1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	161	200	216	225	230	234	237	239	241	242	243	244	245	245	246	246	247
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.73	8.71	8.70	8.69	8.68
4	7.61	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.89	5.87	5.86	5.84	5.83
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.66	4.64	4.62	4.60	4.59
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.98	3.96	3.94	3.92	3.91
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.55	3.53	3.51	3.49	3.48
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.26	3.24	3.22	3.20	3.19
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.05	3.03	3.01	2.99	2.97
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.89	2.86	2.85	2.83	2.81
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.76	2.74	2.72	2.70	2.69
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.66	2.64	2.62	2.60	2.58
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.58	2.55	2.53	2.51	2.50
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.51	2.48	2.46	2.44	2.43
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.45	2.42	2.40	2.38	2.37
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.40	2.37	2.35	2.33	2.32
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.35	2.33	2.31	2.29	2.27
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.31	2.29	2.27	2.25	2.23
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.21	2.20
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.25	2.22	2.20	2.18	2.17
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.22	2.20	2.18	2.16	2.14
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.20	2.17	2.15	2.13	2.11
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.23	2.20	2.18	2.15	2.13	2.11	2.09
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18	2.15	2.13	2.11	2.09	2.07
25	2.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.14	2.11	2.09	2.07	2.05
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.12	2.09	2.07	2.05	2.03
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.17	2.13	2.10	2.08	2.06	2.04	2.02
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12	2.09	2.06	2.04	2.02	2.00
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.08	2.05	2.03	2.01	1.99
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.06	2.04	2.01	1.99	1.98

32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19	2.14	2.10	2.07	2.04	2.01	1.99	1.97	1.95
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17	2.12	2.08	2.05	2.02	1.99	1.97	1.95	1.93
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15	2.11	2.07	2.03	2.00	1.98	1.95	1.93	1.92
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14	2.09	2.05	2.02	1.99	1.96	1.94	1.92	1.90
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.97	1.95	1.92	1.90	1.89
42	4.07	3.22	2.83	2.59	2.44	2.32	2.24	2.17	2.11	2.06	2.03	1.99	1.96	1.93	1.91	1.89	1.87
44	4.06	3.21	2.82	2.58	2.43	2.31	2.23	2.16	2.10	2.05	2.01	1.98	1.95	1.92	1.90	1.88	1.86
46	4.05	3.20	2.81	2.57	2.42	2.30	2.22	2.15	2.09	2.04	2.00	1.97	1.94	1.91	1.89	1.87	1.85
48	4.04	3.19	2.80	2.57	2.41	2.29	2.21	2.14	2.08	2.03	1.99	1.96	1.93	1.90	1.88	1.86	1.84
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95	1.92	1.89	1.87	1.85	1.83
55	4.02	3.16	2.77	2.54	2.38	2.27	2.18	2.11	2.06	2.01	1.97	1.93	1.90	1.88	1.85	1.83	1.81
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.89	1.86	1.84	1.82	1.80
65	3.99	3.14	2.75	2.51	2.36	2.24	2.15	2.08	2.03	1.98	1.94	1.90	1.87	1.85	1.82	1.80	1.78
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93	1.89	1.86	1.84	1.81	1.79	1.76
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88	1.84	1.82	1.79	1.77	1.75
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.90	1.86	1.83	1.80	1.78	1.76	1.74
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85	1.82	1.79	1.77	1.75	1.73
125	3.92	3.07	2.68	2.44	2.29	2.17	2.08	2.01	1.96	1.91	1.87	1.83	1.80	1.77	1.75	1.72	1.70
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.85	1.82	1.79	1.76	1.73	1.71	1.69
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80	1.77	1.74	1.72	1.69	1.67
300	3.87	3.03	2.63	2.40	2.24	2.13	2.04	1.97	1.91	1.86	1.82	1.78	1.75	1.72	1.70	1.68	1.66
500	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.81	1.77	1.74	1.71	1.69	1.66	1.64
1000	3.85	3.00	2.62	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.80	1.76	1.73	1.70	1.68	1.65	1.63
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.72	1.69	1.67	1.64	1.62

Table F8. (continued)

$f_1 \backslash f_2$	18	19	20	22	24	26	28	30	35	40	45	50	60	80	100	200	500	∞
1	247	248	248	249	249	249	250	250	251	251	251	252	252	252	253	254	254	254
2	194	194	194	195	195	195	195	195	195	195	195	195	195	195	195	195	195	195
3	867	867	866	865	864	863	862	862	860	859	859	858	857	856	855	854	853	853
4	582	581	580	579	577	576	575	575	573	572	571	570	569	567	566	565	564	563
5	458	457	456	454	453	452	450	450	448	446	445	444	443	441	441	439	437	437
6	390	388	387	386	384	383	382	381	379	377	376	375	374	372	371	369	368	367
7	347	346	344	343	341	340	339	338	336	334	333	332	330	329	327	325	324	323
8	317	316	315	313	312	310	309	308	306	304	303	302	301	299	297	295	294	293
9	296	295	294	292	290	289	287	286	284	283	281	280	279	277	276	273	272	271
10	280	278	277	275	274	272	271	270	268	266	265	264	262	260	259	256	255	254
11	267	266	265	263	261	259	258	257	255	253	252	251	249	247	246	243	242	240
12	257	256	254	252	251	249	248	247	244	243	241	240	238	236	235	232	231	230
13	248	247	246	244	242	241	239	238	236	234	233	231	230	227	226	223	222	221
14	241	240	239	237	235	233	232	231	228	227	225	224	222	220	219	216	214	213
15	235	234	233	231	229	227	226	225	222	220	219	218	216	214	212	210	208	207
16	231	229	228	225	224	222	221	219	217	215	214	212	211	208	207	204	202	201
17	224	224	223	221	219	217	216	215	212	210	209	208	206	203	202	199	197	196
18	221	220	219	217	215	213	212	211	208	206	205	204	202	199	198	195	193	192
19	219	217	216	213	211	210	208	207	205	203	201	200	198	196	194	191	189	188
20	217	214	212	210	208	207	205	204	201	199	198	197	195	192	191	188	186	184
21	212	211	210	207	205	204	202	201	198	196	195	194	192	189	188	184	182	181
22	210	208	207	205	203	201	200	198	196	194	192	191	189	186	185	182	180	178
23	207	206	205	202	200	199	197	196	193	191	190	188	186	184	182	179	177	176
24	205	204	203	200	198	197	195	194	191	189	188	186	184	182	180	177	175	173
25	204	202	201	198	196	195	193	192	189	187	186	184	182	180	178	175	173	171
26	202	200	199	197	195	193	191	190	187	185	184	182	180	178	176	173	171	169
27	200	199	197	195	193	191	190	188	186	184	182	181	179	176	174	171	169	167
28	199	197	196	193	191	190	188	187	184	182	180	179	177	174	173	169	167	165
29	197	196	194	192	190	188	187	185	183	181	179	177	175	173	171	167	165	164
30	196	195	193	191	189	187	185	184	181	179	177	176	174	171	170	166	164	162

32	1.94	1.92	1.91	1.88	1.86	1.85	1.83	1.82	1.79	1.77	1.75	1.74	1.71	1.69	1.67	1.63	1.61	1.59
34	1.92	1.90	1.89	1.86	1.84	1.82	1.80	1.80	1.77	1.75	1.73	1.71	1.69	1.66	1.65	1.61	1.59	1.57
36	1.90	1.88	1.87	1.85	1.82	1.81	1.79	1.78	1.75	1.73	1.71	1.69	1.67	1.64	1.62	1.59	1.56	1.55
38	1.88	1.87	1.85	1.83	1.81	1.79	1.77	1.76	1.73	1.71	1.69	1.68	1.65	1.62	1.61	1.57	1.54	1.53
40	1.87	1.85	1.84	1.81	1.79	1.77	1.76	1.74	1.72	1.69	1.67	1.66	1.64	1.61	1.59	1.55	1.53	1.51
42	1.86	1.84	1.83	1.80	1.78	1.76	1.74	1.73	1.70	1.68	1.66	1.65	1.62	1.59	1.57	1.53	1.51	1.49
44	1.84	1.83	1.81	1.79	1.77	1.75	1.73	1.72	1.69	1.67	1.65	1.63	1.61	1.58	1.56	1.52	1.49	1.48
46	1.83	1.82	1.80	1.78	1.76	1.74	1.72	1.71	1.68	1.65	1.64	1.62	1.60	1.57	1.55	1.51	1.48	1.46
48	1.82	1.81	1.79	1.77	1.75	1.73	1.71	1.70	1.67	1.64	1.62	1.61	1.59	1.56	1.54	1.49	1.47	1.45
50	1.81	1.80	1.78	1.76	1.74	1.72	1.70	1.69	1.66	1.63	1.61	1.60	1.58	1.54	1.52	1.48	1.46	1.44
55	1.79	1.78	1.76	1.74	1.72	1.70	1.68	1.67	1.64	1.61	1.59	1.58	1.55	1.52	1.50	1.46	1.43	1.41
60	1.78	1.76	1.75	1.72	1.70	1.68	1.66	1.65	1.62	1.59	1.57	1.56	1.53	1.50	1.48	1.44	1.41	1.39
65	1.76	1.75	1.73	1.71	1.69	1.67	1.65	1.63	1.60	1.58	1.56	1.54	1.52	1.49	1.46	1.42	1.39	1.37
70	1.75	1.74	1.72	1.70	1.67	1.65	1.64	1.62	1.59	1.57	1.55	1.53	1.50	1.47	1.45	1.40	1.37	1.35
80	1.73	1.72	1.70	1.68	1.65	1.63	1.62	1.60	1.57	1.54	1.52	1.51	1.48	1.45	1.43	1.38	1.35	1.32
90	1.72	1.70	1.69	1.66	1.64	1.62	1.60	1.59	1.55	1.53	1.51	1.49	1.46	1.43	1.41	1.36	1.32	1.30
100	1.71	1.69	1.68	1.65	1.63	1.61	1.59	1.57	1.54	1.52	1.49	1.48	1.45	1.41	1.39	1.34	1.31	1.28
125	1.69	1.67	1.65	1.63	1.60	1.58	1.57	1.55	1.52	1.49	1.47	1.45	1.42	1.39	1.36	1.31	1.27	1.25
150	1.67	1.66	1.64	1.61	1.59	1.57	1.55	1.53	1.50	1.48	1.45	1.44	1.41	1.37	1.34	1.29	1.25	1.22
200	1.66	1.64	1.62	1.60	1.57	1.55	1.53	1.52	1.48	1.46	1.43	1.41	1.39	1.35	1.32	1.26	1.22	1.19
300	1.64	1.62	1.61	1.58	1.55	1.53	1.51	1.50	1.46	1.43	1.41	1.39	1.36	1.32	1.30	1.23	1.19	1.15
500	1.62	1.61	1.59	1.56	1.54	1.52	1.50	1.48	1.45	1.42	1.40	1.38	1.34	1.30	1.28	1.21	1.16	1.11
1000	1.61	1.60	1.58	1.55	1.53	1.51	1.49	1.47	1.44	1.41	1.38	1.36	1.33	1.29	1.26	1.19	1.13	1.08
∞	1.60	1.59	1.57	1.54	1.52	1.50	1.48	1.46	1.42	1.39	1.37	1.35	1.32	1.27	1.24	1.17	1.11	1.00

Table F9. Random numbers originating from the uniform distribution
 $R_U(0, 1)$ (10,000-multiples)

0854	3465	3734	6662	7157	6386	7489	4333	8450	5484
5854	7765	3634	5189	4137	1331	7022	3611	4101	2728
0794	8128	6239	5608	8639	0731	9576	2157	8428	0248
7053	9902	5574	3587	7610	9494	8279	1499	5555	4414
2464	9331	0731	8712	3824	4023	2794	7403	1785	7904
4619	3315	0545	3835	5268	3139	4803	4095	0530	7007
3366	7133	8522	8512	0605	7827	0819	9824	6762	6591
3540	2991	0805	9415	8417	2456	0619	8747	3675	6524
0824	8214	0234	4175	4279	6616	3682	3572	9883	0158
5236	6695	3515	5774	7305	8781	8246	5213	4613	1701
6597	0318	0393	1462	4643	1838	2670	7034	1151	7449
2767	7069	6748	0097	2399	2027	0473	7193	6404	3458
1886	6296	2884	5847	0386	8698	7964	7495	1743	4352
8602	6088	4497	0547	7646	3186	0896	0172	2142	9494
0022	7667	4637	2657	0907	3778	8440	7045	3099	4338
5492	8141	6804	7771	9761	5552	7326	4371	7348	4307
2479	0669	2314	2807	4525	8341	0704	7540	9253	3490
8854	9762	0103	2340	0447	8465	1127	4710	5188	8836
9678	7620	5834	6347	8749	8634	3329	6806	5569	2452
3662	0302	2185	7754	6688	1103	7618	6776	3810	0447
2476	4137	8978	5085	3562	2074	2146	5477	6591	2193
5093	1787	3694	9462	6507	1218	8196	0908	6586	1497
2355	3272	1426	1149	6316	0524	9134	1074	3650	4611
6944	5651	2676	3821	2665	9547	6358	6045	9251	5953
8926	7707	9333	6605	5900	8714	4013	5584	9532	7014
3965	3919	8175	9153	3111	3290	5060	8583	8829	7085
5247	1023	8063	6088	8866	2822	1627	7380	9932	9236
8265	1877	1555	4714	9047	4811	0430	1450	3270	7568
3858	4329	6176	0314	5475	9850	9933	1228	0454	8557
9192	0283	4605	9362	5719	3951	5750	5890	0937	8546
8746	1279	3778	5436	4819	7132	3094	7832	7711	8992
9195	8480	5395	8635	8066	8819	5087	4918	1768	3489
5664	8822	1001	6850	2975	6189	6823	1053	7423	4119
9177	1734	9046	9179	1741	7493	2326	5572	3445	1047
6169	7387	4257	8630	7934	2970	3739	7810	2120	2827
3227	0810	5540	0537	9049	2282	3233	9414	4844	6832
0481	7834	3173	6446	1480	7649	7502	3912	3030	3119
4648	9191	1054	7717	6690	3977	5368	1622	0679	3232
3518	3382	8515	2848	0524	6762	4166	4488	7967	4724
3992	6647	7245	0474	4586	6771	5847	3701	5504	4324
2841	5857	8411	5491	0542	5468	1345	3578	4990	9111
7372	3045	5583	7349	2054	4728	8106	8492	3647	7093
8178	0734	4123	7243	4404	3919	5511	0061	3078	5244
6825	2371	7324	6934	8240	8903	3637	8360	5574	1380
3227	9312	8721	4307	2832	2023	8810	1939	2762	3073
4254	9937	3515	3902	2252	5786	2770	4472	8834	0224
4178	7774	5590	5953	0530	4104	7067	9532	8525	3383
8525	0445	7698	3472	1827	8193	4488	4392	7066	1402
6624	5021	7376	7337	8001	8021	4825	2742	8572	9130
8436	8885	2891	9328	8526	2936	8212	4023	5084	0524

Table F10. Random numbers originating from the normal distribution $R_N(0, 1)$
(10,000-multiples)

-28000	4846	-8697	-17391	3389	13779	-9834	-5377	5765	-22865
-1069	0544	-29638	-0620	13503	3287	-6811	1090	18991	-2733
3567	-16327	1421	-1671	-7451	-13068	15195	-12492	-5017	-3993
0955	-2372	20840	6836	8059	4215	7130	2351	10643	16513
6519	-7241	1040	21431	7934	10684	11667	-18127	-15256	8121
17799	3473	4072	-5208	-17413	-6061	4112	8803	12471	-16207
-15620	3961	3196	13024	-4108	-10664	0283	-8678	6848	-3824
-23675	4716	-0975	-5680	-6321	3342	-1032	9123	24251	7807
-3010	-7238	-4404	-3057	8708	-15782	-7583	-13872	-0365	-4517
-4895	2417	-11176	-10027	17677	8066	0608	-5102	19826	8009
-23414	-7813	-11479	8131	15968	-18449	2499	-7918	-3217	11757
-4444	-1598	-1598	-4394	0116	-13065	-4520	-7521	-5740	-2702
-1490	24076	11794	11216	18042	-6263	18848	-9470	-2176	8735
-5069	3013	-7234	1336	11896	20449	-2169	-3776	-9960	4016
10218	10140	12050	10910	-7244	-5736	14018	9680	7409	-10589
3137	-7960	-16003	-22033	8362	-0122	5791	-2005	9180	5122
0723	-14041	0309	8989	-12117	-5577	-15971	-6280	5152	-13410
2795	-9247	-0787	12925	-18582	9391	-5774	-4990	1619	-6441
12356	-2348	-8152	9489	6752	15281	-11728	13287	4633	-1721
-6932	7463	-11091	28265	-11506	5221	-5228	22743	-2374	-10621
-5335	-8150	-8686	11239	1535	7967	-10669	-16381	7069	7552
1440	17980	1244	15034	-6904	-18467	6018	5075	20989	8012
-7186	-15999	-3027	-14969	3947	5323	15812	0698	18134	-7889
19942	-3999	21685	-5380	-14752	-0496	-6593	4825	-5051	-5161
-0320	9760	-7639	0309	-14218	-10786	8238	24043	24886	17816
-20127	-11749	5869	-6994	-3284	0681	4745	11213	-4034	-0954
-9311	-6913	-7953	9457	-13440	-7203	-7822	9208	-0839	-1501
-1662	12720	-8981	4781	13896	12469	-1183	0338	-11724	-13755
14308	-3675	1337	21628	-21439	1158	-6940	-8895	-1818	23281
8294	0435	-12695	-11628	-5107	6037	8932	-14813	-1177	9712
-12385	-13946	-8549	-1068	-11932	5931	6623	6824	6018	-2767
7271	4069	-11430	-1547	0794	-2493	5149	2860	-18792	-7018
-18765	5757	6772	-6812	-7661	-3924	-3783	-22620	-4390	0702
-8410	-10521	5889	17332	5197	-0209	10888	6923	-2206	-22490
-5003	3866	0340	6722	1139	-2739	-6002	9088	-12593	-0810
-12786	1175	-25405	-1840	4057	3968	-30579	-2627	-1024	-1608
8550	-21096	23872	-4412	6374	-1015	8875	5121	-11780	13890
0156	-6068	-14636	1812	13702	1221	-2947	-14254	17714	10230
-16151	7789	-1026	8805	-4580	9009	8882	2961	12986	2342
2410	-10739	-0008	-8099	1859	-5293	-11441	5920	0958	-21516
-0560	-12421	-11828	21449	-7205	-1172	-13924	-1224	6284	-19214
7732	6399	12771	20312	6795	1222	6484	1993	-4079	-19035
-11642	-10646	4675	1797	-9275	24975	-0893	-4194	13267	1785
-1212	0323	-0118	10026	-9024	-4759	3732	5979	11264	6887
6515	4006	11930	-17441	-12668	1520	7226	-12980	1335	-18173
-3300	-4967	-8904	-10189	1240	14463	-8828	4109	0695	14364
12189	1423	-4415	11889	1153	4457	-3566	-3137	9877	6052
-5913	-15095	11664	5705	9941	11554	25056	0810	3468	-4239
-0219	-0222	-22466	0087	-8899	3150	3824	-3606	-12296	20720
-16121	8447	12194	4251	-1161	9663	-2290	8129	-15263	18121

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