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THE FOUNDATIONS OF MATHEMATICS

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Boole's Logic and Probability

SECOND EDITION

T. HAILPERIN

NORTH-HOLLAND

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BOOLE'S LOGIC AND PROBABILITY

A CRITICAL EXPOSITION FROM THE STANDPOINT OF CONTEMPORARY ALGEBRA, LOGIC AND PROBABILITY THEORY

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To my Family

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PREFACE TO THE FIRST EDITION

It was some 30 years ago that I first started to write an essay on Boole which I hoped would be ready by 1947, the 100-th anniversary year of the publication of his *Mathematical Analysis of Logic*. The many difficulties I ran into in trying to understand exactly what Boole had done took me by surprise and, not making the date I had set for myself, put the work aside. With the approach of 1954, the centenary of Boole's *Laws of Thought*, my interest was renewed and I tried again, but with no better result. However shortly thereafter I became acquainted with the newly developed subject of linear programming and recognizing in it, ready made, a tool which I had needed, used it to explain some of Boole's work in probability. The success encouraged me to persist and so, what started out 30 years ago to be an article, now ends up as a small monograph whose length, I hope, is more closely proportioned to the value of its subject.

Over the years I have had information and advice from various people. I am reluctant to mention names. Some of it was given so long ago that the donors probably have forgotten that they ever spoke to me, and some of it relates to blind-alley approaches which I have had to abandon. With regard to acknowledgements of such help, suffice it to say that we are all members of that confraternity of researchers and scholars who gladly assist each other like members of a close-knit family. I would, however, like to mention by name Professor A. E. Pitcher, my friend, colleague, and department chairman, whose enthusiastic encouragement, at times taking the material form of obtaining for me released time from teaching duties, is warmly appreciated.

I was delighted to find that I had a Boolean predecessor at Lehigh University. In 1879 Alexander Macfarlane, a Scottish mathematical physicist, published his *Algebra of Logic* which had as its aim "to correct Boole's principles, and place them on a clear rational basis". He later came to Lehigh as a lecturer in electrical engineering and mathematical physics, and was there from 1895 to 1908. Over a period of time he gave a series of addresses on 25 British mathematical scientists. The talks were given to an audience composed of students, instructors, and towns-people; and there was one, delivered April 19, 1901, devoted to George Boole. I wonder if any of that audience could have envisioned that three-quarters of a century later there would be in the catalogue an electrical engineering course offering:

> **241. Switching Theory and Logic Design (3)**
> *Boolean algebra and its application to networks with bivalued signals. Function simplification* [... *etc.*].

Bethlehem, Pennsylvania

Th. Hailperin

PREFACE TO THE SECOND EDITION

Since the first edition there has been a notable increase of interest in the development of logic—witness, for example, the several conferences on the history of logic which have taken place and the founding, in 1980, of a journal devoted to the history and philosophy of logic. This increased activity, and the accumulation of new results—a chief one being that Boole's work in probability is best viewed not as a new foundational approach but rather as a probability logic—were among circumstances conducive to a new edition.

Chapter 1 has been considerably enlarged to better render Boole's ideas on a mathematical treatment of logic, beginning with their emergence in his early 1847 work on through to his immediate successors. Chapter 2 includes additional discussion of the "uninterpretable" notion. Chapter 3 now includes a revival of Boole's abandoned propositional logic; and, also, discussion of his hitherto unnoticed brush with ancient formal logic. In Chapter 5 we have a revamped explanation of why Boole's probability method works. Chapter 6 is entirely new. All in all, the changes have brought about a three-fold increase in the Bibliography.

Special appreciation is due John Corcoran, historian of logic, who made available to me his list of corrections, comments, queries and criticisms written while reading the first edition. An extensive correspondence with him on these matters resulted in insights and ideas, as well as help on historical items.

Theodore Hailperin

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INTRODUCTION

A self-taught mathematician of uncommon power and originality, honored in his lifetime for various mathematical contributions, Boole is today largely remembered for his successful creation of an algebra of logic. Although, unknown to Boole, Leibniz had much earlier conceived of the idea of a mathematical treatment of logic and although, some decades later than Boole but with no awareness of Boole's work, Gottlob Frege had constructed a much deeper and more extensive mathematical logic, it is Boole's algebra of class terms in the revised and simplified Jevons-Peirce-Schröder form that is popularly associated with mathematical logic. This particular accomplishment of Boole's, the application of algebra to logic, though limited to the simple class calculus and propositional aspects, had a marked stimulating effect on the development of modern logic.

Also of importance, but not as well-known or as easy to evaluate, was the effect that Boole's introduction of a non-quantitative type of algebra had on the inception of modern abstract algebra. Boole himself was hardly aware that he was creating a new algebra, as the idea of *an* algebra had not yet emerged. In keeping with the notions of Symbolical Algebra then current he thought he was giving another interpretation to the general symbols of algebra, which symbols already had various interpretations—numerical, geometrical, or physical. But this new interpretation, Boole emphasized, was different in that it was the first not having to do with "magnitude". Accordingly, Boole allowed himself to use all the usual operations of ordinary algebra adding, however, the special requirement that symbols which are to stand for class terms must satisfy the law $x^2 = x$. This new condition, satisfied as an arithmetical law only by the numbers 0 and 1, enabled him to single out with remarkable ingenuity, if not complete clarity, those features of the full numerical algebra useable for logic. Boole's methods also included the use of expressions, such as $\frac{0}{0}$ and $\frac{1}{0}$, to which no meaning can be given in ordinary algebra let alone in logic. The elimination by later logicians of what was not fully understandable or not strictly relevant to the logic of class terms resulted in a calculus whose abstract formulation is now known as the theory of Boolean algebras or, in an equivalent form, the theory of Boolean rings.

Boole was a thorough and careful worker and the mathematical system which he elaborated for doing logic was not shown to be wrong by the historical simplification to Boolean algebra but merely replaced by it. Of logicians of note subsequent to Boole only Venn was an adherent of the strict Boolean system for doing logic (see § 1.11). In modern times (that is, after the ascendency of Boolean algebra) a number of expository studies of the original Boolean system have been published. The earliest of these is the extensive review in *Mind* by C. D. Broad occasioned by the appearance of the Open Court edition of Boole's Collected Works, Volume II (BROAD 1917). In C. I. Lewis' *Survey of Symbolic Logic* (LEWIS 1918) we have a clear and sympathetic account of Boole's system in the context of a history of the development of logic, and a recognition of the desirability of a rigorous justification for it. Using ideas of M. H. Stone, E. Hoff-Hansen, and Th. Skolem, BETH 1959, § 25, has a brief explanation for the success of Boole's methods, and in HOOLEY 1966 there is a proof that Boole's method for solving Boolean equations does give correct results, but no explanation why. There is also an essay at an explanation of the "secret" of Boole's method in STYAZHKIN 1969 (original Russian, 1964) which devotes a half-dozen pages to it.

In addition to the logico-algebraic phase of Boole's work there is another which we shall be interested in, that concerned with the logical foundations of probability. However, in contrast to the logic, this aspect of Boole's work has left no historical residue. Outside of perhaps a simple inequality that bears his name there is no mention of Boole in current probability literature, and the latest work that considers Boole's ideas on the subject is KEYNES 1921 that is, one published more than a half-century ago. In his treatise Keynes devotes a substantial amount of space to solving various logico-probability problems of the kind propounded by Boole, contrasting the simplicity of his own methods with those of Boole's whose methods, moreover, he declares to be constantly erroneous. According to Keynes, Boole's central error is his use of two incompatible conceptions of independence of events. Mentioning that Boole's writings are full of originality and genius, Keynes nevertheless gives no detailed discussion of this work.

While the object of our scrutiny—Boole's work dating from the 1850's—is historical, our investigation is not historical in the more conventional sense. We include practically no biographical data, nor do we trace the origins, the development, and the subsequent influences of the ideas. What we shall be occupied with is an intensive and extended study of Boole's mathematical theories which he used for doing logic and probability, explicating these never clearly understood theories on the basis of new contemporary notions. A brief summary of our results now follows.

Ostensibly Boole's logical system is about classes. He uses the symbols $x, y, z, \ldots$ to stand for classes and combines them with the familiar algebraic symbols $\times$, $+$, $-$, $\div$, 0, 1 to form other expressions. But these expressions may not represent classes—a product of the form xy always does, but a sum $x + y$ does only if the classes represented by x and y have no members in common. Expressions which do not represent classes are called by Boole "uninterpretable", and are formally recognizable as those which do not satisfy the law $x^2 = x$. Characteristic of Boole's method is that while expressions may be uninterpretable, equations always are when suitably interpreted by his rules. Boole states his algebraic laws in terms of class variables—commutivity, for example, is written $xy = yx$—yet he uses this and other laws without regard to whether the expressions involved are interpretable or not. In § 1.4 we collect together all the laws regarding $+$, $-$, $\times$, 0, 1 which Boole actually uses and, viewing them as a set of axioms for a mathematical theory, find that correct interpretations or models are obtained if we consider, not classes, but multisets as the entities over which the variables range. By a multiset we mean a collection in which more than one example of an object can occur (indistinguishable balls of various kinds in an urn, roots of an equation with multiplicities counted, etc.). And, just as the desire to escape the limited subtraction of natural numbers leads to their extension to the signed integers, so to reproduce Boole's unrestricted subtraction we are led to introduce the idea of a signed multiset. The algebra of signed multisets (§2.2) gives us all the features of Boole's system for doing logic except for division. The set of elements x of an algebra of signed multisets which satisfy $x^2 = x$ constitute a Boolean algebra. Variables restricted to range over this Boolean algebra we call Boolean variables. We show that every algebraic multiset equation involving only Boolean variables is equivalent to a purely Boolean algebra equation in these variables—a result the formal substance of which Boole often used.

To solve a logical equation for an unknown Boole makes use of "division", expressing the unknown as a quotient of Boolean polynomials. The algebraic properties required by him for this operation are quite rudimentary as compared with, say, division in the algebra of rationals. Of greater moment is Boole's application of his process of development (or expansion) to the resulting quotient so as to convert it to interpretable form. Here the assignment of values 0 and 1 to the Boolean variables produces values $\frac{1}{1}$, $\frac{0}{1}$, $\frac{0}{0}$, $\frac{1}{0}$ for the quotient, which values Boole uses to separate the corresponding basic products on the variables into four types instead of the two as for an ordinary Boolean polynomial. From such a development Boole obtains his interpretation in ordinary class terms. Clever as they were, the reasons Boole gives for his rules of interpretation convinced no one. In §2.7 we show not only how to justify Boole's procedure here but to make sense of it in all repects. We do this by going over to rings of quotients of Boolean elements, elements not from the original Boolean algebra but from a certain factor algebra.

In Chapter 3 we present and discuss both of Boole's versions of propositional logic—the one from *Mathematical Analysis of Logic* based on "cases, or conjunctures of circumstances" and the other from *Laws of Thought* based on "instants of time for which a proposition is true". In both versions Boole considered that he had reduced the theory to an application of his algebra [of classes]. By clarifying the notions and developing appropriate mathematical background we show how to make both versions into acceptable, if overly complicated, ways of doing propositional logic. Two of the selections from philosphical literature which Boole uses to illustrate his propositional logic—one being from Cicero and the other from Plato—come in for discussion as having relevance to the history of ancient logic.

Boole's ideas on probability can perhaps be best outlined through a discussion of his general probability problem. This problem is: for any set of logical conditions involving events (propositions) whose respective probabilities are given, determine the probability of any other event in terms of these probabilities. On the basis of his unusual logical and probability theories Boole gives a solution to this problem, the gist of which is as follows.

Since he takes logical sum disjunctively Boole can use for the probability of a logical function whose arguments are events the exact same algebraic function of the probabilities of these events—provided these events are stochastically independent (so that the probability of a product is the product of the probabilities). And, by a principle which he enunciates, events must be treated as independent if nothing is known or can be inferred respecting their connection. To solve the general problem Boole takes the event whose probability is sought (the objective event) and the events whose probabilities are given, all of these being known logical functions of "simple" events, as themselves simple events. Call these new simple events, respectively, w (for the objective event) and $s, t, u, \ldots$ (for the given events). The system of equations relating these new simple events to their corresponding functions (of simple events) which they represent, together with any logical relations given in the data, are taken as a system of logical equations. Applying his logical methods Boole eliminates from this system all but $w, s, t, u, \ldots$ and expresses w solely in terms of $s, t, u, \ldots$, say by an equation $w = F(s, t, u, \ldots)$. Boole's methods also supply the information as to which combinations of the simple events $s, t, u, \ldots$ are possible on the data, that is, which combinations are not excluded from happening. Call the universe of the possible events V. Since the logical equations imply that the universe of possibilities is V Boole claims, by another principle which he enunciates, that the corresponding probability relationship holding between w and $s, t, u, \ldots$ holds not as ordinary probabilities but as conditional probabilities on condition V. The sought for probability is then the conditional probability of $F(s, t, u, \ldots)$ given V, which probability can be determined once the (unconditional) probabilities of $s, t, u, \ldots$ are known. These unconditional probabilities of $s, t, u, \ldots$ are determined from a system of algebraic equations which Boole obtains by equating the conditional probabilities of $s, t, u, \ldots$ on condition V to the probability values of the originally given events they correspond to. The general solution of this system of algebraic equations in the probabilities of $s, t, u, \ldots$ occasioned Boole considerable difficulty. He finally succeeded—so he believed—in showing that the system has a unique solution with values in the probability range 0 to 1 if and only if a set of consistency conditions, called by Boole "conditions of possible experience" is satisfied. With this existence theorem Boole considered that the solution of the general probability problem had been effected. We doubt that anyone has gone through his long and complicated proof and checked it out carefully. (For example, the edited reprint of his paper containing the proof, BOOLE 1952 XVII, has all the mathematical errors of a typographical nature occuring in the original printing.) We carry out this checking chore in § 5.6.

In explaining Boole's ideas on probability we first introduce the preliminary notion of a simple probability algebra that is, a finitely generated free Boolean algebra with a probability measure whose algebraically independent generators are stochastically independent. The generators are what correspond to Boole's simple events about which nothing is known. In terms of a simple probability algebra we then define a conditioned events probability space. In such a space we bring out the idea of a conditioned event—an idea which was immanent but not explicit in Boole's work. The probability of a conditioned event is taken as a conditional probability (of events in a simple probability algebra). Using this as a background theory we interpret Boole's method of solving the general probability problem as a kind of embedding of the problem into a conditioned events probability space. There are several stages to the process and there may not be a unique outcome. One obtains Boole's solution if special assumptions, corresponding to what he believed were new principles of probability, are made. Viewing the matter quite differently we give our solution using modern linear programming techniques.

In our concluding chapter we examine the applications which Boole made of his methods to various problems on causes (inverse probability), probability of judgements and combinations of testimonies. Finally, we present a probability logic—a logic justifying inferences from probable premises to probable conclusions—to which the study of Boole's work has led us.

In keeping with our intentions to make this a narrowly focused study we include no general appraisal of, or historical background for, Boole's contributions to logic. For writings of this nature which have appeared in the last couple of decades we may cite: BRODY 1967, VAN EVRA 1977 and 1984, LAITA 1977, 1979 and 1980, and DUDMAN 1976. There are also general histories of logic which contain substantial material on Boole: LEWIS 1918, JØRGENSEN 1931, KNEALE and KNEALE 1962, and STYAZHKIN 1969. For interesting sidelights on Boole's personal life there is an (obituary) article HARLEY 1866 (reprinted as Appendix A in BOOLE 1952), and one by the distinguished scientist and grandson of Boole, TAYLOR 1956. A full length biography of Boole, MACHALE 1985, has just appeared.

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CHAPTER 0

REQUISITES FROM ALGEBRA, LOGIC AND PROBABILITY

The mathematical and logical material gathered together here in this Chapter 0 includes background notions and results needed for the understanding of the main body of the monograph. As my readers may be from many diverse fields I have included some elementary definitions and topics, as well as the more specialized and recondite results which will be referred to in later chapters. Even for those to whom much of the material of this chapter is familiar, it can serve the mundane purpose of fixing notation and terminology. However, for all except the very hardy, the mental malaise which an almost unbroken sequence of definitions and theorems can induce is to be avoided for this chapter by following the physicians's medicinal recommendation: "Take as needed".

§ 0.1. Preliminaries

We proceed from the working mathematicians point of view that the notion of *set* is clear. Ordered n-tuples of entities are denoted by $\langle x_1, \ldots, x_n \rangle$, but ordered pairs normally by (x, y). A (*binary*) *relation* on (or over) a set A is a set of ordered pairs of elements of A. A *function* (or a *mapping*) f from a set A to a set B (notation, f: $A \rightarrow B$) is a set of ordered pairs (x, y) where $x \in A$, $y \in B$ and such that for each $x \in A$ there is a unique $y \in B$ such that $(x, y) \in f$. The set A is called the *domain* of the function and B is its *range*. If f is a function and $(x, y) \in f$ then, as is customary, one writes $y = f(x)$, (or, also, $x \rightarrow y$) and refers to y as the (f-)*value* of x, or the *image* of x under the mapping f. We shall occasionally lapse into the common practice of referring to $f(x)$, an ambiguous value of a function, as a function. A function is *injective* (or *one-to-one*) if distinct members of the domain have distinct images; it is *surjective* (or *onto*) if each member of the range is the image of some member of the domain; it is *bijective* (or a *one-to-one correspondence*) if it is both injective and surjective.

A (binary) *operation* on a set A is a function from the set of ordered pairs of elements of A to A. In general an *n-ary operation* is one for which the domain of the function consists of ordered n-tuples of members of A. On occasion one may wish to consider operations for which the range is not limited to the set A—in such cases we say that the operation is *not closed in* A, or that A is *not closed under the operation*.

For two sets R and S, we take the *cartesian product* $R \times S$ to be the set of all ordered pairs (r, s) where $r \in R$ and $s \in S$. This notion, with R and S being the same set, was just used in the definition of a binary operation. We wish to define the more general notion of the cartesian product of a family (= set) of sets. A function from a set I to a family of sets A is called an *indexed set* (A is *indexed by* I). A common notation is $\{A_i \mid i \in I\}$, where for each $i \in I$, $A_i \in A$. Now by the cartesian product of the family A, $\mathsf{X}\{A_i \mid i \in I\}$, is meant the set of functions f on I for which $f(i) \in A_i$, for all $i \in I$.

A relation R on a class A is said to be an *equivalence relation* on A if it is reflexive ($R(x, x)$ holds for all $x \in A$), symmetric ($R(x, y)$ implies $R(y, x)$) and transitive ($R(x, y)$ and $R(y, z)$ imply $R(x, z)$). By putting together all elements of A which are R-related to each other into one and the same class, one effects a separation of A into mutually exclusive subclasses—the *R-equivalence classes* of A.

§ 0.2. Algebraic structures

Modern abstract algebra—to whose origins Boole in no small measure contributed—has singled out and studied in great depth a varied host of algebraic structures. In this monograph we shall be concerned with a number of well-known structures, which are briefly mentioned in this section, as well as some new ones especially introduced by us in Chapter 2 to give meaning to, and to make rigorous, Boole's logical system.

0.2.1. An (*algebraic*) *structure*, or (*mathematical*) *system*, or an *algebra*, consists of a non-empty set A together with a finite number of operations $O_1, \ldots, O_m$ on A where for each i $(1 \leq i \leq m)$ O_i is an n_i-ary operation $(n_i \geq 0)$. Such a structure we denote by the ordered $(m + 1)$-tuple $\langle A, O_1, \ldots, O_m \rangle$. The set A is referred to as the *universe* of the structure. Any other $(m + 1)$-tuple $\langle B, O'_1, \ldots, O'_m \rangle$ in which O'_i is also an n_i-ary operation (on B) will be said to be *of the same type* as $\langle A, O_1, \ldots, O_m \rangle$. How the operations for a structure are to be given is not specified—it can be done explicitly, but mostly this will be done "axiomatically", i.e. by stipulating certain conditions that the operations are to satisfy. The relationship between structures and axioms will be discussed in § 0.3.

In this monograph we shall generally though not exclusively be dealing with structures that involve at most two binary operations, conventionally denoted by $+$ (addition) and $\cdot$ (multiplication), and two constants or nullary operations 0 (zero) and 1 (one, unit). The term *trivial* will be used for structures that have merely a singleton set for a universe.

EXAMPLE 1. A structure $\langle A, \cdot \rangle$ is a *semigroup* if the indicated binary operation is associative.

One can have an *additive* semigroup—this simply means using $+$ as the symbol for the binary operation.

EXAMPLE 2. A structure $\langle A, + \rangle$ is a *semimodule* if

(i) $\langle A, + \rangle$ is an additive semigroup,

(ii) $a + b = b + a$ for all $a, b \in A$, and

(iii) $a + x = b$ has, for any $a, b \in A$, at most one solution for x (in A).

A prime example of a semimodule is the structure of the natural numbers under addition.

EXAMPLE 3. A structure $\langle G, \cdot, 1 \rangle$ is a *group* if

(i) $\langle G, \cdot \rangle$ is a semigroup,

(ii) $1 \cdot a = a \cdot 1 = a$ for all $a \in G$, and

(iii) $a \cdot x = x \cdot a = 1$ has, for any $a \in G$, a solution for x (in G).

A structure with multiplication in which there is an element 1 such that

$1 \cdot a = a \cdot 1 = a$ for anv a of the structure, is said to *possess* or *have a one* (*or unit*). It is easily shown that a structure can only have one unit. A group is referred to as being an *additive group* if the operations are written $+$ and 0 instead of $\cdot$ and 1.

EXAMPLE 4. A structure $\langle M, +, 0\rangle$ is a *module* if it is an additive group and $+$ is commutative over M.

A prime instance of a module is the structure of the integers under addition. There is also the more general notion of an R-module—a module whose elements can be suitably multiplied by elements from some ring R.

EXAMPLE 5. A structure $\langle S, +, \cdot\rangle$ is a *semiring* if
(i) $\langle S, +\rangle$ is a semimodule,
(ii) $\langle S, \cdot\rangle$ is a semigroup, and
(iii) multiplication is distributive over addition.

EXAMPLE 6. A structure $\langle R, +, \cdot, 0\rangle$ is a *ring* if
(i) $\langle R, +, 0\rangle$ is a module,
(ii) $\langle R, \cdot\rangle$ is a semigroup, and
(iii) multiplication is distributive over addition.

REMARK 1. In all the structures to be considered in this monograph both $+$ and $\cdot$ will be commutative and associative; hence we shall tacitly assume that sums and products may be arbitrarily parenthesized and rearranged. Moreover, as is customary, a^n shall mean $a \cdot a \cdot \ldots \cdot a$ for n factors and na shall mean $a + a + \cdots + a$ for n terms. For positive integers m and n we clearly have

$$a^m \cdot a^n = a^{m+n}, \qquad (a^m)^n = a^{mn},$$

$$ma + na = (m + n)\, a, \qquad n(ma) = (nm)a.$$

0.2.2. Let $\mathfrak{A} = \langle A, O_1, \ldots, O_m\rangle$ and $\mathfrak{B} = \langle B, O_1', \ldots, O_m'\rangle$ be structures of the same type. As is customary in abstract algebra we shall use the same symbol, say f_i, for both O_i and O_i'. (Confusion is usually avoided by noting, in context, whether the arguments of f_i come from A or from

B.) A mapping $\varphi : A \to B$ having the property that for any i ($1 \leq i \leq m$) and all $a_1, \ldots, a_{n_i} \in A$

$$\varphi f_i(a_1, \ldots, a_{n_i}) = f_i(\varphi a_1, \ldots, \varphi a_{n_i}),$$

is said to be a *homomorphism* of $\mathfrak{A}$ *into* $\mathfrak{B}$. If additionally the mapping is one-to-one it is called a *monomorphism* or an *embedding* of $\mathfrak{A}$ *into* $\mathfrak{B}$ (or, $\mathfrak{A}$ *is isomorphically embeddable* in $\mathfrak{B}$); if the mapping is one-to-one and onto, we say that it is an *isomorphism*, or $\mathfrak{A}$ and $\mathfrak{B}$ *are isomorphic*.

A structure $\mathfrak{A} = \langle A, f_1, \ldots, f_m \rangle$ is a *substructure* of $\mathfrak{B} = \langle B, f_1, \ldots, f_m \rangle$ if A is a (nonempty) subset of B and the operations f_i (for $\mathfrak{A}$), with arguments restricted to A, are closed in A.

Operations of a structure $\mathfrak{A}$ not explicitly listed in a structure $\mathfrak{B}$ may, if convenient, be thought of as being present "vacuously", i.e. as being operations with an empty set of values.

EXAMPLE 7. If $\mathfrak{N} = \langle N, +, \cdot \rangle$ is the semiring of the natural numbers and $\mathfrak{I} = \langle I, +, \cdot, 0 \rangle$ the ring of the integers, then the mapping $\varphi : N \to I$, where $\varphi(n) = +n$, is an embedding of $\mathfrak{N}$ into $\mathfrak{I}$; that is, the substructure of $\mathfrak{I}$ whose universe is the positive integers is an isomorphic image of the semiring of the natural numbers. Other examples: the integers are (isomorphically) embeddable in the rationals via $n \to n/1$, and the reals in the complex numbers via $a \to a + 0i$.

Given an indexed set $\{\mathfrak{A}_i \mid i \in I\}$ of algebraic structures of the same type we can define a new structure $\mathfrak{P} = \prod(\mathfrak{A}_i \mid i \in I)$ of the same type, called their *direct product* (or in some cases, as for modules and rings, their *direct sum*.) The universe of $\mathfrak{P}$ is the cartesian product $P = \mathsf{X}(A_i \mid i \in I)$ of the universes of the $\mathfrak{A}_i$. If $a \in P$ we denote its i-th component by $(a)_i$. The ν-th operation f_ν for P is defined by

$$(f_\nu(a_1, \ldots, a_n))_i = f_\nu((a_1)_i, \ldots, (a_n)_i),$$

that is, the n-ary operation f_ν for P is defined "componentwise".

EXAMPLE 8. If $\mathfrak{A}$ and $\mathfrak{B}$ are two structures of the same type as that for rings, then their direct sum would have operation $+$ and $\cdot$ defined by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2),$$
$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2).$$

The zero for the direct sum would be (0, 0).

0.2.3. We shall later be referring to certain substructures of direct sums called "subdirect sums". Let $\mathfrak{S}_1, \ldots, \mathfrak{S}_i, \ldots$ be an indexed set of structures of the same type (e.g. rings) and $\mathfrak{S}$ their direct sum. A substructure $\mathfrak{H}$ of $\mathfrak{S}$ is a *subdirect sum* of $\mathfrak{S}_1, \ldots, \mathfrak{S}_i, \ldots$ if its universe H is such that, for any i, the set of i-th components of the elements of H include S_i, i.e. each element of the universe of $\mathfrak{S}_i$ occurs at least once as the i-th component of some element of H.

0.2.4. A function $\theta : A \to A$ is a *congruence relation* on an algebraic structure $\mathfrak{A} = \langle A, f_1, \ldots, f_m \rangle$ if it is an equivalence relation on A which is preserved under the operations of A; that is, using $a \equiv b(\theta)$ to denote that the relation of equivalence holds between a and b, if

$$f_i(a_1, \ldots, a_{n_i}) \equiv f_i(b_1, \ldots, b_{n_i})\ (\theta),$$

whenever $a_j \equiv b_j(\theta)$, $j = 1, \ldots, n_i$. Such a congruence relation induces a new algebra, the *quotient* (or *factor*) *algebra* $\mathfrak{A}/\equiv_\theta$, whose universe is the set of equivalence classes, $A/\equiv_\theta$, and whose operations f_i (we use the same symbols as for $\mathfrak{A}$) are given by

$$f_i([a_1]_\theta, \ldots, [a_{n_i}]_\theta) = [f_i(a_1, \ldots, a_{n_i})]_\theta,$$

with $[a]_\theta$ denoting the equivalence class determined by a. It can be readily shown that these f_i are well-defined, i.e. do not depend on the particular representatives of the equivalence classes shown. Also one can show that the mapping

$$h_\theta: A \to A/\equiv_\theta, \qquad h_\theta(a) = [a]_\theta$$

is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\equiv_\theta$.

A congruence relation on $\mathfrak{A}$ is naturally induced by any homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ from $\mathfrak{A}$ onto a structure $\mathfrak{B}$, namely by having elements a_1 and a_2 equivalent if they have the same image under φ. If we denote this equivalence relation by θ_φ then we have the well-known and basic result:

THEOREM 0.21 (Homomorphism theorem). *If* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *is a homomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$ *and* θ_φ *is the induced congruence relation on* $\mathfrak{A}$, *then* $\mathfrak{A}/{\equiv_{\theta_\varphi}}$ *is isomorphic to* $\mathfrak{B}$ *under the mapping* $[a]_{\theta_\varphi} \to \varphi(a)$.

§ 0.3. First order theories, models, extensions

0.3.1. As mentioned in the preceding section, a structure, or kinds of structures, can be singled out by stating conditions ("axioms") which the operations of the structure are to satisfy. Independently of any structure, a set of such conditions in a precisely formulated language, in which only the logical symbols have specific meaning, is called a *formal axiomatic theory*. The symbols of the language are of two kinds:

A. Logical symbols
  (i) connectives: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\to$ (if ..., then), $\leftrightarrow$ (if and only if);
  (ii) variables: $x, y, z, \ldots$;
  (iii) quantifiers: $\forall x$ (for all x), $\exists x$ (for some x);
  (iv) $=$ (identity, equality).

B. Non-logical symbols
  (i) n-ary predicate symbols;
  (ii) n-ary function symbols.

The logical symbols of the language have a fixed interpretation—as indicated above in A. by the parenthetical English phrases; the non-logical symbols are open to interpretation: the variables are thought of as ranging over some non-empty set called the *domain* (*of individuals*), the predicate symbols are interpreted as relations over the domain, and the function symbols as operations on the domain. In what follows predicate symbols will not be mentioned, except perhaps incidentally, since the theories we are concerned with in this monograph can be stated using only function symbols.

A (*formal*) *term* is defined by generalized induction as follows:

(i) a variable is a term,

(ii) if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term. (Constants are here included, under $n = 0$.)

Similarly we define *formula* by:

(i) If P is an n-ary predicate symbol (or the 2-ary symbol $=$) and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ (or $t_1 = t_2$) is a formula.

(ii) If A and B are formulas, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.

(iii) If A is a formula and v a variable, then $\forall v\, A$ and $\exists v\, A$ are formulas.

The kind of formulas specified in (i) are referred to as *atomic*; formulas of the form (ii) are called, respectively, a *negation*, *conjunction*, *disjunction*, *implication* (or *conditional*), and *equivalence* (or, *biconditional*).

Formal axiomatic theories expressible in the language just described are called *first order* theories (also *elementary*); the epithet "first order" is used to indicate that the quantifiers apply only to individual variables and not to function or predicate variables. We have made no mention of logical axioms or of deducibility, implicitly taking such matters for granted as is customary in all branches of mathematics except foundations studies. However we shall have occasion to refer to a simple form of the "deduction theorem". Let $\vdash_{\mathbf{T}} \varphi$ denote that φ is deducible in $\mathbf{T}$ (i.e. from the axioms of $\mathbf{T}$), and $\psi \vdash_{\mathbf{T}} \varphi$ that φ is deducible in $\mathbf{T}$ from the premise ψ (holding free variables fixed). Then:

DEDUCTION THEOREM. *If* $\psi \vdash_{\mathbf{T}} \varphi$, *then* $\vdash_{\mathbf{T}} \psi \rightarrow \varphi$.

In Chapter 2 we are going to present certain axiomatically formulated theories whose models, we believe, are what Boole's "system" is about. The notion of a model has a precise meaning: a given structure is a *model* for a given first order theory if, when the formal function symbols of the theory are interpreted as operations on the universe of the structure, the axioms become true sentences about the structure. Thus what are traditionally called "groups" are structures which are models of the group axioms. A set of such axioms is given in Example 3 of § 0.2, taking the $\cdot$ and 1 as formal symbols.

0.3.2. A fundamental result about first order axiomatic theories is the following.

GÖDEL COMPLETENESS THEOREM. *A formula of a first order theory is provable from the axioms* (*i.e. is a theorem*) *if and only if it is true of* (*or in*) *each model of the theory.*

An equivalent statement is the following: a first order theory is consistent (i.e. no contradiction can be derived from the axioms) if and only if the theory has a model.

0.3.3. Model theory investigates the relationship between axioms and models. This has been done in great depth and there are many significant results. The following happens to be a somewhat minor result, but one which we shall have occasion to refer to later.

A formula of a first order language is a *McKinsey formula*[1] if it is a disjunction of formulas which are either atomic or the negation of an atomic formula and at most one of which is atomic. A first order theory is a *McKinsey theory* if it is equivalent to a theory all of whose axioms are McKinsey formulas. Relating to such theories there is the following result (SHOENFIELD 1967, p. 94, Problem 7(f)):

THEOREM 0.31. *A theory* **T** *is a McKinsey theory if and only if every substructure of a model of* **T** *is a model of* **T** *and every direct product of models of* **T** *is a model of* **T**.

0.3.4. We shall later be referring to certain theories as being decidable or undecidable. An axiomatic theory is *decidable* if there is a general mechanical procedure for determining whether or not any arbitrarily given formula of the theory is a theorem. In the contrary case the theory is *undecidable*. Before stating the next result of interest to us a brief explanation is needed.

[1] So named in SHOENFIELD 1967, p. 94. Despite Horn's explicit mention (*Journal of Symbolic Logic*, Vol. 16 (1951), p. 14) of their first use by McKinsey, they are now called "Horn" formulas.

Although our work is exclusively with axiomatic theories, it is possible to specify theories by other means than by giving a set of axioms. For example, if $\mathfrak{A} = \langle A, +, \cdot \rangle$ is a given structure with operations $+$ and $\cdot$ over A, then we may define the *theory of* $\mathfrak{A}$ as the set of formulas in two binary function symbols (corresponding to $+$ and $\cdot$) which are true of $\mathfrak{A}$. Thus we may define the theory of integers, **J**, as the set of formulas (with two binary function symbols corresponding to $+$ and $\cdot$) which are true of the structure $\mathfrak{I} = \langle I, +, \cdot \rangle$ where I is the set of integers and $+$ and $\cdot$ have their usual significance as operations on I. A theory $\mathbf{T}_1$ is a *subtheory* of a theory $\mathbf{T}_2$ if each sentence which is valid in $\mathbf{T}_1$ (true of all models of $\mathbf{T}_1$) is also valid in $\mathbf{T}_2$. Concerning the theory **J** there is the following result (TARSKI-MOSTOWSKI-ROBINSON 1953, p. 68):

THEOREM 0.32. *The theory of integers* **J**, *and any of its subtheories with the same symbols, is undecidable.*

0.3.5. We wish to make a few observations about solving algebraic problems by going over to an extension structure.

Suppose we are faced with the task of finding an integer n_0 such that $\varphi(n_0)$ is true. A commonly used tactic is to view it as a problem in some structure which is an "extension" of the integers, for example the rationals, use theorems true about all fields (the rationals being one) and, with the wider techniques such as unrestricted division available, derive consequences. If in this manner we should find a rational integer $n_0/1$ satisfying $\varphi(x)$ then we claim $\varphi(n_0)$ is the case. To what extent is this procedure justifiable?

Before answering this question let us make it precise. Consider two mathematical structures $\mathfrak{Z} = \langle \mathbb{Z}, 0, 1, +, \cdot \rangle$ and $\mathfrak{Q} = \langle \mathbb{Q}, 0/1, 1/1, +, \cdot \rangle$ where $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ the set of rationals and the other symbols have their usual significance in such contexts. By virtue of the customary mapping of the integers into the rationals, $h : \mathbb{Z} \to \mathbb{Q}$, where

$$h(n) = \frac{n}{1}$$

$$h(n + m) = h(n) + h(m),$$

$$h(nm) = h(n) \cdot h(m),$$

we have that h is an isomorphism between $\mathfrak{Z}$ and the substructure of $\mathfrak{Q}$ whose universe is $h(\mathbb{Z})$, the set of images of $\mathbb{Z}$ under h (i.e. $\mathfrak{Z}$ is embedded in $\mathfrak{Q}$ by h). Since the two structures are of the same type we can use the same first-order language for both—which language would have as its non-logical symbols 0, 1, $+$, $\cdot$. The condition $\varphi(x)$ we now take to be a *formula in this language* (and for simplicity let us suppose that $\varphi(x)$ has only the one free variable x present). Now we rephrase our question as follows:

Under what conditions is the formula $\varphi(x)$, when interpreted as a formula about $\mathbb{Z}$, true of the integer n if $\varphi(x)$, interpreted as a formula about Q, is true of the rational integer $n/1$?

We assume it is clear to the reader what is meant by the phrase "$\varphi(x)$, interpreted as about the structure $\mathfrak{A}$, is true of a (an element of the universe of $\mathfrak{A}$)"; we abbreviate this phrase by

$$\mathfrak{A} \models \varphi[a].$$

(Tarski's formal semantics gives this notion a precise meaning.) With this notation our question can now be succinctly and precisely stated:

$$\text{Does} \quad \mathfrak{Q} \models \varphi\left[\frac{n}{1}\right] \quad \text{imply} \quad \mathfrak{Z} \models \varphi[n]?$$

It can be quite readily seen, by virtue of the isomorphism properties of h, that the answer to the question is "yes" if $\varphi(x)$ is a purely algebraic equation built up from the non-logical symbols 0, 1, $+$, $\cdot$ and the variable x, and is still "yes" if $\varphi(x)$ is any propositional compound of such equations. However, if quantifiers are present in $\varphi(x)$ the answer can be "no"—for example let $\varphi(x)$ be

$$\exists y\,(xy = 1),$$

a formula expressing the idea that x has a multiplicative inverse. Here, since there is no multiplicative inverse for the integer $1 + 1$,

$$\mathfrak{Q} \models \varphi\left[\frac{1 + 1}{1}\right]$$

is true, but

$$\mathfrak{Z} \models \varphi[1 + 1]$$

is false. On the basis of the foregoing discussion of a specific case we can omit the readily supplied details and simply state the corresponding general result:

Let $\mathfrak{A}$ be a structure and $\mathfrak{B}$ a structure of the same type as $\mathfrak{A}$ in which $\mathfrak{A}$ is embedded by a mapping h. If $\varphi(x)$ is any quantifier-free formula in the language for $\mathfrak{A}$, then

$$\mathfrak{A} \models \varphi[a] \quad \text{if and only if} \quad \mathfrak{B} \models \varphi[h(a)].$$

We want to point out that what we have here is a (quite modest) model-theoretic result and as such is not dependent in how, or whether, we formalize the theory of the structure $\mathfrak{A}$, or $\mathfrak{B}$. Of course there is no objection to using an axiomatized theory that has $\mathfrak{B}$ as a realization (e.g. the *theory of fields* in our example) to derive consequences about $\mathfrak{B}$—such a theory can be much stronger than the theory for $\mathfrak{A}$ (in our example, the *theory of integral domains*). Still, no matter how we establish $\mathfrak{B} \models \varphi[h(a)]$, we can assert $\mathfrak{A} \models \varphi([a])$ if the conditions of the above-stated result apply.

§ 0.4. Semirings. Commutative rings with unit

0.4.1. An additive structure, that is a structure with a binary operation written '+' and called 'addition', *has a zero* if there is an element, 0, such that for all a in the universe, $0 + a = a + 0 = a$. For a semiring one can show that such an element is uniquely defined by this property and, moreover, is such that $0 \cdot a = a \cdot 0 = 0$.

An element a of a multiplicative structure with zero is a *nilpotent* if, for some n, $a^n = 0$. Thus a structure has no nonzero nilpotents if, for all a and all n, $a^n = 0 \to a = 0$. For this property (i.e. no nonzero nilpotents) the following theorem asserts that in a semiring one need only examine the case $n = 2$:

THEOREM 0.41. *In a semiring with* 0,

$$\textit{for all} \quad a, \quad a^2 = 0 \to a = 0$$

implies

$$\textit{for all} \quad a, \quad a^n = 0 \to a = 0.$$

The proof is by induction on n, separating out the even and odd cases.

One can also speak of *additive nilpotents*, that is elements such that for some n, $na = 0$. A structure having additive nilpotents is to be distinguished from one having characteristic n—in this latter case one has, for a fixed n, $na = 0$ for *every* a in the universe of the structure. If there is no such n, the structure is said to be of *characteristic* 0. A structure with no nonzero additive nilpotents must be of characteristic 0, unless it is a trivial structure and has only the element 0.

0.4.2. The following theorem summarizes well-known elementary results about rings. We shall tacitly use standard conventions, e.g. $ab = a \cdot b$, $a - b = a + (-b)$, $-ab = -(ab)$, etc.

THEOREM 0.42. *In a non-trivial commutative ring with unit, the following hold*:

$$1 \neq 0, \qquad 1a = a1 = a,$$

$$a + b = b + a, \qquad a + (b + c) = (a + b) + c,$$

$$a + 0 = a, \qquad a + x = 0 \textit{ has a unique solution } x = -a,$$

$$ab = ba, \qquad a(bc) = (ab)\,c,$$

$$a \cdot 0 = 0 \cdot a = 0, \qquad a(-b) = (-a)\,b = -(ab) = -ab,$$

$$a(b + c) = ab + ac, \qquad a(b - c) = ab - ac.$$

$$a^m \cdot a^n = a^{m+n}, \qquad (a^m)^n = a^{mn},$$

$$ma + na = (m + n)a, \qquad n(ma) = (nm)a,$$

for all integers m and n.

0.4.3. Certain substructures of rings called "ideals" are of considerable interest. If $\mathfrak{R} = \langle R, +, \cdot, 0\rangle$ is a ring and A a nonempty subset of R, we say $\langle A, +, \cdot, 0\rangle$ is an *ideal of* $\mathfrak{R}$ if

(i) $\langle A, +, 0\rangle$ is a submodule of $\mathfrak{R}$, and

(ii) A is closed under multiplication by elements of R ($ar \in A$ if $a \in A$ and $r \in R$).

It is easy to show that an ideal of a ring is a subring (which, however, may be trivial and contain only the 0). Equivalently, one can say $\langle A, +, \cdot, 0\rangle$ is an ideal of $\mathfrak{R}$ if A is closed under subtraction ($a - b \in A$ if $a, b \in A$) and under multiplication by elements of R.

In what follows we shall adopt the customary informal practice of referring to an ideal as being a set, metonymously using the universe of the structure for the structure itself.

As we shall later see, principal ideals will play an important role in our work:—if $a \in R$, $\mathfrak{R}$ a ring, then the set of elements

$$\{at \mid t \in \mathfrak{R}\}$$

is an ideal of $\mathfrak{R}$ called the *principal ideal* (*generated by* a); it is denoted by (a). As $\mathfrak{R}$ has a unity we know that $a \in (a)$. Also, it is readily shown that (a) is included in any ideal of $\mathfrak{R}$ having a as a member. Hence (a) is the smallest ideal of $\mathfrak{R}$ having a as a member.

With respect to an ideal N of a ring $\mathfrak{R}$, one can divide the elements of R into mutually exclusive classes, the *residue classes* (modulo N): any two elements of R are in the same class if their difference is an element of N. A given residue class is representable in the form $a + N$, where $a + N = \{a + n \mid n \in N\}$, in place of a one can have any member of the given class since $a + N = a' + N$ if a and a' are in the same residue class. Using the residue classes as elements, defining operations $+$ and $\cdot$ on these classes by

$$(a + N) + (b + N) = (a + b) + N,$$

$$(a + N) \cdot (b + N) = (ab) + N,$$

and with zero as $0 + N = N$, one readily shows that the structure so obtained is a ring, the *residue class* ring (also *quotient* or *factor* ring), denoted by $\mathfrak{R}/N$. It can be readily established that the correspondence $a \to a + N$ is structure-preserving:

THEOREM 0.425. *The mapping $a \to a + N$, from a ring $\mathfrak{R}$ to its residue class ring $\mathfrak{R}/N$ is a homomorphism of rings.*

0.4.4. If in a multiplicative structure with zero there is an element $b \neq 0$ such that $ab = 0$ for $a \neq 0$, then a is said to be a (nonzero) *divisor of zero*. A commutative ring without divisors of zero is an *integral domain*, the preeminent example of such a structure being the integers under the usual multiplication and addition. A non-trivial commutative ring in which the nonzero elements form a multiplicative group is a *field*; with the customary operations, the set of rationals, the set of reals, and the set of complex numbers are all fields.

In abstract algebra a so-called "structure theorem" is a theorem establishing a significant *structural* relationship between a type of structure and other, generally simpler, types of structures. Of interest to us later will be the following structure theorem of McCoy 1948, p. 123:

THEOREM 0.43. *In a commutative ring* $\mathfrak{R}$ *with unity the following conditions are equivalent*:

(i) $\mathfrak{R}$ *is a subdirect sum of integral domains*;
(ii) $\mathfrak{R}$ *has no nonzero nilpotents*;
(iii) $\mathfrak{R}$ *is a subring of a direct sum of fields.*

REMARK 0.44. A ring $\mathfrak{R}$ which is a direct sum of rings is without additive nilpotents if and only if each component ring likewise is without additive nilpotents.

By way of justification for this remark we note that if any component ring $\mathfrak{R}_p$ has an additive nilpotent a, say, then $a \neq 0$ and $na = 0$. But then the element of R having a as its p-th component and 0 elsewhere is an additive nilpotent of R. Conversely, if $A \neq 0$ and $nA = 0$ for some $A \in R$, then $na_i = 0$ for each component a_i of A. But since not all components of A can be 0, at least one $a_i \neq 0$.

§ 0.5. Boolean algebras and Boolean rings. Propositional logic

The algebraic structures to be described in this section are named after the man whose work in logic and probability is the subject of this monograph. The theory of these structures is universally regarded as *the* outcome of a systematization and simplification of Boole's unclear ideas. It is our contention, however, that the path leading from Boole's ideas to Boolean algebra is not the only way to systematize Boole's conceptions and, in Chapter 2, we shall show how this can be done using other structures which are essentially different from Boolean algebras. Not unexpectedly, however, Boolean algebras and Boolean rings will play a large role in our work and we here summarize various results which we shall be referring to later on.

0.5.1. A *Boolean algebra* $\mathfrak{B} = \langle B, +, \cdot, ', 0, 1 \rangle$ is a structure with two binary operations $+$ and $\cdot$, a unary operation $'$, and two constants (nullary operations) 0 and 1, satisfying the following axioms:

(i)	$ab = ba,$	$a + b = b + a,$
(ii)	$(ab)\,c = a(bc),$	$(a + b) + c = a + (b + c),$
(iii)	$1 \cdot a = a,$	$0 + a = a,$
(iv)	$aa' = 0,$	$a + a' = 1,$
(v)	$a(b + c) = ab + ac,$	$a + bc = (a + b)\,(a + c),$
(vi)	$aa = a,$	$a + a = a.$

(This is not an independent set of axioms—for example, the second equation in (v) is derivable from the others.)

The properties of Boolean algebras we have just written down are expressed by means of the symbols $+$, $\cdot$, $'$, 0, 1, $=$, and variables. If we consider these as formal symbols of a first-order language (§ 0.3), we then have a first-order axiomatic formulation of the *theory of Boolean algebras*, **BA**. Any Boolean algebra, by definition, satisfies these axioms, i.e. the axioms are true of any Boolean algebra, and any statement deducible from the axioms is true of (valid in) all Boolean algebras. Conversely, by virtue of the Completeness Theorem (§ 0.3), any formula expressed in the first-order language which is true of all Boolean algebras is deducible from the axioms. However, as the following theorem indicates, special preeminence should be accorded to validity in a two-element Boolean algebra.

THEOREM 0.51. *Any Boolean polynomial equation of the form* $f(a_1, \ldots, a_n) = 1$, *if true in a two element Boolean algebra (whose universe would then be the set* $\{0, 1\}$*), is true in all Boolean algebras, and hence is deducible from the axioms.*

Many assertions in the language of Boolean algebra can be reduced to the particular form $f(a_1, \ldots, a_n) = 1$ just mentioned in Theorem 0.51.

We have the following theorem, in which the notion of *inclusion* occurs and is defined by

$$a \subseteq b \quad \text{if and only if} \quad ab = a.$$

THEOREM 0.52. *Any Boolean polynomial equation* $f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$, *any Boolean polynomial inclusion* $f(a_1, \ldots, a_n) \subseteq g(a_1, \ldots, a_n)$, *and any conjunction of either or both of such relations, is equivalent to an equation of the form* $h(a_1, \ldots, a_n) = 1$.

THEOREM 0.53. *If the polynomial equation* $f(a_1, \ldots, a_n) = 1$ *is added as an axiom to the axioms for* **BA** *and if* $g(a_1, \ldots, a_n) = 1$ *is deducible from this enlarged set, then the inclusion* $f(a_1, \ldots, a_n) \subseteq g(a_1, \ldots a_n)$ *is deducible from the axioms of* **BA** *alone.*

This result (Theorem 0.53) follows readily from the fact that "If $f = 1$ then $g = 1$" and "$f \subseteq g$" are equivalent in the two-element Boolean algebra $\{0, 1\}$. For if $g = 1$ follows from $f = 1$ in **BA** then "$f = 1$ implies $g = 1$" is a theorem, and thus valid in a two-element Boolean algebra. But by the just mentioned result $f \subseteq g$ is also valid there and so, by Theorems 0.51 and 0.52, provable in **BA**.

As a simple consequence of Theorems 0.51 and 0.52 we have the well-known

THEOREM 0.535 (Law of Development).

$$f(a_1, a_2, \ldots, a_n) = f(1, a_2, \ldots, a_n)\, a_1 + f(0, a_2, \ldots, a_n)\, a_1'.$$

Theorem 0.51 provides the basis for connecting Boolean algebra with propositional calculus. The connection of Boolean algebra with the algebra of sets is provided for by Stone's representation theorem. In the statement of this theorem a *field of sets* is a collection of subsets of a set which contains this set and the empty set, and is closed with respect to union, intersection, and complementation relative to the set.

THEOREM 0.539 (Representation Theorem). *Every Boolean algebra is isomorphic to a field of sets.*

0.5.2. A *Boolean ring* is a commutative ring with unit which, in addition, satisfies the idempotency condition: $a^2 = aa = a$.

Although the definitions of Boolean algebra and Boolean ring look different it is well-known and easily shown that the theories of Boolean algebra, **BA**, and Boolean rings, **BR**, are equivalent in that there is a translation of any sentence of the one theory into a sentence of the other, such that the one sentence holds for Boolean algebras if and only if the other holds for Boolean rings. To state this properly we need to distinguish between the operations of **BA** and **BR**. We do this by using a subscript "B" on the operations of **BA** (except for $'$) and a subscript "Δ" on the operations of **BR**. The translation from **BA** to **BR** is effected via

$$a +_B b = a +_\Delta b +_\Delta a \cdot_\Delta b,$$
$$a \cdot_B b = a \cdot_\Delta b,$$
$$a' = 1 +_\Delta a,$$

and the translation from **BR** to **BA** via

$$a +_\Delta b = a \cdot_B b' +_B a' \cdot_B b,$$
$$a \cdot_\Delta b = a \cdot_B b,$$

with $0_B = 0_\Delta$ in both cases.

It can be verified that if one deletes from the twelve axioms (i)–(vi) the (redundant) $a + bc = (a+b)(a+c)$ then except for the last, namely $a + a = a$, the equations hold also for Boolean rings when the operations are interpreted as those of a Boolean ring (with $a' = 1 + a$); and if the exception $a + a = a$ is replaced by $a + a = 0$, then the resulting set of equations becomes a set of axioms for Boolean rings. One can formulate the theory of Boolean algebra in many ways—the particular set of axioms we have chosen brings to the fore the numerical analogy (which can be further enhanced by thinking of a' as $1 - a$).

One can readily show (or check by a Venn diagram) the following:

THEOREM 0.54. *In the theory* **BA** *the statements*

(i) $\exists v(w = a + vc)$,

(ii) $a \subseteq w \subseteq a + c$,

(iii) $w = a + wc$,

are all equivalent. The same theorem is true of **BR** *if the* $+$ *is that of* **BR**, *provided that one adds the hypothesis* $ac = 0$.

THEOREM 0.55. *In* **BA** *or* **BR**,

$$a'b = 0 \leftrightarrow \exists x(ab = b)$$

and

$$f(1)f(0) = 0 \leftrightarrow \exists x(f(x) = 0).$$

In both **BA** and **BR** a set of elements is *mutually exclusive* if the product of any two of them is 0, and is *exhaustive* if the sum of all is 1.

THEOREM 0.56. *In* **BA** *or* **BR**, *if a, b, c, and d are mutually exclusive and exhaustive then*

$$(a + b)\, w = a + d$$

if and only if

$$w = a + wc \quad \textit{and} \quad d = 0.$$

0.5.3. Since a Boolean algebra is also a ring, our § 0.4 discussion of ideals in rings carries over to Boolean algebras with $+_\Delta$ as the addition. We can also express the definition of an ideal in a Boolean algebra using $+_B$ as follows: a subset I of a Boolean algebra $\mathfrak{B}$ is an ideal of $\mathfrak{B}$ if its elements satisfy the condition

$$a +_B b \in I \quad \text{if and only if} \quad a \in I \quad \text{and} \quad b \in I.$$

Principal ideals will be of special interest to us. It is easy to show that the principal ideal generated by an element a consists of all elements b such that $b \subseteq a$, i.e. all the subelements of a. Since this set is also $\{ab \mid b \in B\}$ it is convenient to introduce the notation $a\mathfrak{B}$ for this ideal. The elements of the ideal $a\mathfrak{B}$ form a Boolean algebra under the $+$ and $\cdot$ of $\mathfrak{B}$ restricted to $a\mathfrak{B}$, with a and 0 as the 1 and 0 of the algebra, and with relative complementation, i.e. ae', as the complement of an element e. We denote this Boolean algebra by $\mathfrak{B} \mid a$. We have the following simple result.

THEOREM 0.57. *The mapping* $h : B \to B \mid a$, *defined by*

$$h(b) = ab,$$

is a homomorphism whose kernel (set of elements going into 0) *is* $a'\mathfrak{B}$ *and which is injective (one-to-one) over the subset* $a\mathfrak{B}$ *of* B.

An ideal I in a Boolean algebra, as in a ring, separates the elements of the algebra into mutually exclusive equivalence classes, the residue classes, via the condition: two elements $a, b \in B$ are in the same residue class if their symmetric difference $a +_{\Delta} b$ is in the ideal. We use $[b]$ $(= b +_{\Delta} I)$ to denote the residue class which is determined by the element b. The residue classes form a Boolean algebra, $\mathfrak{B}/I$, with operations specified by

$$[a] + [b] = [a +_{\Delta} b],$$
$$[a]\,[b] = [ab],$$
$$[a]' = [a']$$

and with [0] and [1] as the 0 and 1 of the algebra. In the present context Theorem 0.425 says that the mapping $h : B \to B/I$, $h(b) = [b]$, is a homomorphism. The following related result is peculiar to Boolean algebras (SIKORSKI 1964, p. 31):

THEOREM 0.58. *Under the mapping* $h(b) = [b]$ *the algebras* $\mathfrak{B} \mid a'$ *and* $\mathfrak{B}/(a)$ *are isomorphic.*

What Theorem 0.58 tells us, in other words, is that the Boolean algebra of subelements of a' is the same as the algebra obtained by identifying elements of B whose symmetric difference is contained in a.

Dual to the notion of an ideal of a Boolean algebra is that of a filter: a non-empty subset F of a Boolean algebra is a *filter* if the elements of F satisfy the condition

$$ab \in F \quad \text{if and only if} \quad a \in F \quad \text{and} \quad b \in F.$$

The *principal filter* generated by an element a of a Boolean algebra $\mathfrak{B}$ consists of all the superelements of a, i.e. the set of elements b such that $a \subseteq b$ or, equivalently, the set $\{a +_B b \mid b \in B\}$. Hence we shall use the notation $a +_B \mathfrak{B}$) (also $a +_B B$) for the principal filter generated by a.

0.5.4. In what follows we are contemplating Boolean algebras with finite universes since that is all we shall be concerned with in our discussion of Boole's probability theory. However most of what is said here is readily extended to arbitrary Boolean algebras.

A *basic product* (or *constituent*) on n Boolean elements $a_1, \ldots, a_n$ is a product of the form $A_1A_2 \cdots A_n$ in which each A_i is either a_i or a_i'. The set of elements $\{a_1, \ldots, a_n\}$ is said to be *(algebraically) independent* if each of the 2^n products of the form $A_1A_2 \cdots A_n$ is different from 0. A set of elements $\{g_1, \ldots, g_n\}$ of a Boolean algebra $\mathfrak{B}$ is said to be a set of *generators* for $\mathfrak{B}$ if each element of B, other than 0, is equal to a sum of basic products on $g_1, \ldots, g_n$. A set of generators G for a Boolean algebra $\mathfrak{B}$ is said to be *free* if every mapping φ of elements of G into the universe B' of an arbitrary Boolean algebra $\mathfrak{B}'$ can be extended to a homomorphism φ' of B into B'. Intuitively, the idea is that the elements of G are so free of entangling relationships (other than those required of any set of Boolean elements) that they can be "identified" (via the homomorphism φ') with elements of any Boolean algebra without engendering a contradiction. A Boolean algebra that has a set of n free generators is said to be *free (with n free generators)*. We state (in somewhat weaker form) a well-known result (SIKORSKI 1964, p. 43):

THEOREM 0.59. *A Boolean algebra is free (with n free generators) if and only if it is generated by an independent set of n free generators.*

One can show that a Boolean algebra with n free generators has 2^{2^n} elements, where $n \geq 0$. In particular then a free Boolean algebra has to have at least two elements.

0.5.5. Propositional logic. Using *(propositional) variables*

$$X, Y, Z, X_1, X_2, \ldots$$

and *logical connectives*

$$\neg \text{ (not)}, \quad \wedge \text{ (and)}, \quad \vee \text{ (or)}, \quad \rightarrow \text{ (if ..., then)},$$

one constructs a formal language for use as a *propositional calculus* **PC**. A *formula* of **PC** is defined recursively by stipulating:

(1) any variable is a formula,

(2) if φ and ψ are formulas, so are

$$\neg\varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi), \quad \text{and} \quad (\varphi \rightarrow \psi).$$

In well-known fashion there is associated with any formula of **PC** a *truth-table*; those formulas whose associated truth-table comes out with all T's are the *valid* formulas of **PC**. The valid formulas of **PC** can be axiomatized (in many ways), i.e. one can give a recursive initial list of valid formulas of **PC** and a set of rules (of inference) by which, from the initial list, all valid formulas are derivable. If φ is derivable we say it is a *theorem* of **PC** and denote this by writing $\vdash \varphi$.

By a trivial modification of **PC** we can obtain a *quantifierfree monadic predicate calculus* (or, a *general term calculus*), namely by replacing the propositional variables by monadic predicates $X(t)$, $Y(t)$, $Z(t)$, ... representing propositional functions of one *individual variable* (the same variable throughout).

The connection of Boolean algebras with **PC** is established by constructing its *Lindenbaum algebra*. Two formulas φ and ψ of **PC** are *logically equivalent*, written $\varphi \equiv \psi$, if

$$\vdash ((\varphi \to \psi) \wedge (\psi \to \varphi)).$$

In terms of the truth-table notion, φ and ψ are logically equivalent if, when entered from their union set of variables (using vacuously occurring ones if necessary), their truth-tables are the same. It is straightforward to show that logical equivalence is an equivalence relation on the set F of formulas of **PC** and hence divides this set into mutually exclusive and exhaustive equivalence classes. The unique equivalence class determined by a formula φ is denoted by $|\varphi|$, and the set of equivalence classes by $F/\equiv$. This set has at least two members since there are two non-equivalent formulas, e.g. $X \wedge \neg X$ and $X \vee \neg X$. We make $F/\equiv$ into an algebra $\mathfrak{L} = \langle F/\equiv, +_B, \cdot, 0, 1\rangle$ by defining operations on it via

$$\begin{aligned} |\varphi|' &= |\neg\varphi|, \\ |\varphi|\,|\psi| &= |\varphi \wedge \psi|, \\ |\varphi| +_B |\psi| &= |\varphi \vee \psi|, \\ 0 &= |X \wedge \neg X|, \\ 1 &= |X \vee \neg X|. \end{aligned}$$

Concerning this Lindenbaum algebra $\mathfrak{L}$ we have:

THEOREM 0.595. *Under the indicated operations, $\mathfrak{L}$ is a Boolean algebra with* $0 \neq 1$.

From this theorem and Theorems 0.51, 0.52 one can obtain the important relationship between Boolean identities and theorems of **PC**:

THEOREM 0.560. *If $f(x_1, \ldots, x_n)$ is a Boolean polynomial then*

$$f(x_1, \ldots, x_n) = 1 \quad \textit{is true in all Boolean algebras}$$

if and only if

$$\vdash F(X_1, \ldots, X_n),$$

where $F(X_1, \ldots, X_n)$ is the formula of **PC** *obtained by replacing in $f(x_1, \ldots, x_n)$ each x_i by X_i, replacing the operations $+_B$, $\cdot$, $'$ respectively by $\vee$, $\wedge$, $\neg$, and replacing 0 by $X \wedge \neg X$ and 1 by $X \vee \neg X$. Similarly*

$$f(x_1, \ldots, x_n) = 1 \quad \textit{implies} \quad g(x_1, \ldots, x_n) = 1$$

if and only if

$$\vdash F(X_1, \ldots, X_n) \rightarrow G(X_1, \ldots, X_n).$$

0.5.6. There are Boolean polynomials which, although different, do not lead, as far as many kinds of problems are concerned, to essentially different solutions. Thus a Boolean identity remains one under application of the operations: (i) interchange of two variables, and (ii) replacement of a variable by its negation (complement). Two Boolean polynomials so related that one can be obtained from the other by a succession of applications of the operations (i) and (ii) will be said to be of the same *symmetry type*.

§ 0.6. Rings of quotients. Boolean quotients

0.6.1. Division of integers is, of course, not a closed operation in the set of integers—only if the dividend is a multiple of the divisor is the result an integer. However by suitably extending the structure of the integers to a larger structure, i.e. the rationals, a structure in which the integers are embeddable, one obtains closure of division. More precisely, and in greater detail, this is done as follows.

Effect a separation of all ordered pairs of integers of the form (m, n) with $n \neq 0$, into classes by stipulating that two such ordered pairs,

(m_1, n_1) and (m_2, n_2), are in the same class if and only if the relation $m_1n_2 = m_2n_1$ holds. One readily shows that this relation is an equivalence relation and that any ordered pair of a given class uniquely determines the class. Denote by m/n the equivalence class determined by (m, n), and take these equivalence classes as elements of a structure with operations $+$ and $\cdot$ defined by

$$\begin{aligned} m_1/n_1 + m_2/n_2 &= (m_1n_2 + n_1m_2)/(n_1n_2), \\ (m_1/n_1) \cdot (m_2/n_2) &= (m_1m_2)/(n_1n_2), \end{aligned} \tag{1}$$

and in which $0/1$ and $1/1$ are the zero and unity. One can then show that the resulting structure is a field in which the integers are embeddable by the injection $n \to n/1$, and that the inverse of m/n $(m \neq 0)$ is n/m.

The above method of producing an extension of the integers carries over with essentially no change from the particular structure of the integers to any ring which is an integral domain. Not so apparent is the generalization of this construction to a ring which is not an integral domain, i.e. one containing divisors of zero.

One way to accomplish a generalization is to exclude, or preclude, from being a denominator elements which are divisors of zero, for otherwise the relation $m_1n_2 = m_2n_1$ would lose its significance in that it would not be an equivalence relation: transitivity could fail—take for example the integers modulo 6. In this structure the "fractions" 3/4 and 3/3 are both "equal", via the proposed equivalence relation, to 0/2 but not to each other. Note that in this example 2, 3 and 4 are divisors of zero and that a product of non-zero denominators could result in zero. It turns out that if the set of denominator elements is a multiplicatively closed set not containing 0, then the resulting structure of quotients is indeed an extension ring (but not necessarily a field) and one in which the mapping $m \to m/1$ is an injection. However, practically all the elements in the rings we shall be dealing with in this monograph are divisors of zero, and to exclude them from being denominators would leave us with little of interest. Fortunately one can generalize the construction of a ring of quotients without having to exclude divisors of zero. This was shown initially by Chevalley for rings of a certain kind and then in the general case of any commutative ring with unit in Uzkov 1948. Uzkov's quite simple idea was to suitably modify the definition of the basic equivalence relation as follows.

Let $\mathfrak{R} = \langle R, +, \cdot, 0 \rangle$ be a commutative ring with unity and S a multiplicatively closed subset of R containing 1 (hence non-empty). For ordered pairs (r_1, s_1) and (r_2, s_2) in the set of all ordered couples $R \times S$, we define a relation to hold between (r_1, s_1) and (r_2, s_2) if and only if

(2) For some $s \in S$, $s(r_2 s_1 - r_1 s_2) = 0$.

One readily shows that for R and S as specified, this relation is an equivalence relation and hence divides the set $R \times S$ into mutually exclusive equivalence classes. As is customary we use r/s to denote the equivalence class determined by a couple (r, s). With the operations $+$ and $\cdot$ defined by

$$(r_1/s_1) + (r_2/s_2) = (r_1 s_2 + r_2 s_1)/(s_1 s_2),$$
$$(r_1/s_1) \cdot (r_2/s_2) = (r_1 r_2)/(s_1 s_2),$$

and with 0/1 and 1/1 as the zero and unit, one readily shows that the resulting structure is a commutative ring with unit which we call the *ring of quotients of* $\mathfrak{R}$ *by* S and denote it by $\mathfrak{R}S^{-1}$. The mapping from R to RS^{-1} determined by $r \to r/1$ is a homomorphism of rings but one which need not be an injection. Nevertheless, as we shall see, such rings of quotients will be of service to us.

Note that cancellation of a common factor is available if the factor is a member of S; for if $s \in S$ then by the definition $s/s = 1/1$ and

$$r_1 s/r_2 s = (r_1/r_2)(s/s) = r_1/r_2.$$

Oddly enough one need not exclude 0 from the denominator set—but at a price. If $0 \in S$ then all ordered pairs in $R \times S$ are equivalent and the ring $\mathfrak{R}S^{-1}$ is the trivial ring with one element.

0.6.2. Since a Boolean algebra is a ring with unit one can, provided a suitable denominator set is specified, define for it a ring of quotients—the elements of this ring we call *Boolean quotients*. Concerning Boolean quotients we have the following two theorems which are key results for us.

THEOREM 0.61. *For any element* e *of a Boolean algebra* $\mathfrak{B}$ *the filter* $e +_B B$ *is a denominator set for the ring of quotients* $\mathfrak{B}(e +_B B)^{-1}$. *The mapping* $\varphi : B \to B(e +_B B)^{-1}$, $\varphi(b) = b/1$, *whose kernel is* $e'B$ *and which establishes a homomorphism from* $\mathfrak{B}$ *to* $\mathfrak{B}(e +_B B)^{-1}$, *is injective over the set* eB.

PROOF. That $e +_B B$ is a denominator set is clear since any filter has 1 as an element and is closed under multiplication. To show that φ is injective over eB consider $p, q \in eB$, so that p and q are included in e. Hence, if

$$p/1 = q/1,$$

then by the fundamental equivalence relation

$$(e +_B b)\, p = (e +_B b)\, q \quad \text{for some} \quad b \in B,$$

which implies, since p and q are included in e, that

$$p = q.$$

The "canonical" mapping φ of the preceding theorem induces a natural equivalence relation on B by virtue of which two elements of B are equivalent if their images in $B(e +_B B)^{-1}$ are the same. It is easy to show that the equivalence classes determined by this equivalence relation are just the residue classes modulo the principal ideal (e'). For if $p, q \in B$ are such that

$$p/1 = q/1$$

which is to say, by (2), that for some $b \in B$

$$(e +_B b)\,(1p - 1q) = 0,$$

then $p - q$ is included in e' and hence p and q are in the same residue class modulo (e'). The converse is equally as easy to see, that is if p and q are in the same residue class then $p/1 = q/1$, by virtue of

$$p - q \subseteq e' \to (e +_B 0)\,(1p - 1q) = 0.$$

The mapping φ, moreover, is onto $B(e +_B B)^{-1}$ since, as is easy to check

$$\text{(3)} \qquad \frac{b}{e +_B a} = \frac{b}{e} = \frac{eb}{1} = \frac{b}{1} \quad \text{for any} \quad b \in B,$$

and hence each element of $B(e +_B B)^{-1}$ is the φ-image of some element of B. Thus by the Homomorphism Theorem 0.31 we can conclude

THEOREM 0.62. *For any element e of a Boolean algebra $\mathfrak{B}$ the quotient ring $\mathfrak{B}/(e')$ and the ring of quotients $\mathfrak{B}(e +_B B)^{-1}$ are isomorphic.*

It should be noted here that if $e = 0$ then all these isomorphic algebras are trivial, i.e. have a universe with a single element, namely 0.

To avoid a complicated notation we have used in the preceding discussion a deficient notation for Boolean quotients. A fully adequate notation should show that one has an equivalence class of fractions determined by the displayed numerator and denominator, and should also indicate the denominator set—in our case a principal filter. For a principal filter all one needs as an indication is the element generating it. A fully adequate notation for a Boolean quotient is then something like (using here $e + ae'$ in place of $e +_B a$)

$$\left[\frac{b}{e + ae'}\right]_e, \quad \text{or} \quad \left[\frac{b}{f}\right]_e \quad f \in e +_B B.$$

In this notation line (3) would read

$$\left[\frac{b}{e + ae'}\right]_e = \left[\frac{b}{e + 0e'}\right]_e = \left[\frac{eb}{e + e'}\right]_e = \left[\frac{b}{e + e'}\right]_e,$$

in which notation the quotient $[eb/1]_e$ is no longer deceptively symmetric in e and b, and the quotient $[b/1]_e$ explicitly displays its dependence on e. The mapping

$$h : B/(e') \to B(e +_B B)^{-1}$$

which establishes the isomorphism referred to in Theorem 0.62 is given by

$$h(b + (e')) = h(eb + (e')) = \left[\frac{eb}{e + e'}\right]_e = \left[\frac{b}{1}\right]_e.$$

The following theorem codifies, for future reference, some properties already noted during the course of discussion.

THEOREM 0.63. *If* $\mathfrak{B}$ *is a Boolean algebra,* $b, e \in B$, *and* $f \in e +_B B$, *then*

(i) $$\left[\frac{fb}{f}\right]_e = \left[\frac{b}{1}\right]_e$$

(ii) $$\left[\frac{b}{e}\right]_e = \left[\frac{eb}{1}\right]_e$$

(iii) $$\left[\frac{b}{f}\right]_e = \left[\frac{b}{e}\right]_e.$$

As the last item indicates, a Boolean quotient is independent of the particular member of the denominator set which is used as a denominator in the fraction. Accordingly, one would not expect as great a future for this notion as has been the case for numerical quotients, i.e. rationals—and Theorem 0.62 shows that residue classes, or elements of a cutdown algebra (Theorem 0.58), would do just as well.

§ 0.7. Fourier elimination. Solvability of linear systems

All the structures we have discussed so far have had operations, but no relations, defined over the universe of the structure. Here in this section we are considering *ordered fields*, that is fields (§ 0.4) over which a binary relation, $<$, is defined having the following properties:

$\neg(a < a)$	(irreflexive),
$(a < b \wedge b < c) \rightarrow a < c$	(transitive),
$a < b \vee a = b \vee b < a$	(connected),
$a < b \rightarrow (a + c < b + c)$	(additively monotone),
$(a < b \wedge 0 < c) \rightarrow ac < bc$	(multiplicatively monotone).

From these properties together with those for a field one can deduce all the usual rules of elementary algebra for addition, subtraction, multiplication, division, and the handling of inequalities in connection with these operations. Ordered fields also have the denseness property, that is between any two elements there is a third, i.e. if $x < y$, then there is a z, e.g. $z = (x + y)/(1 + 1)$, such that $x < z < y$. One can readily show that every ordered field has an ordered subfield isomorphic to the rationals.

The results which we wish to describe in this section require only that the elements belong to an ordered field. As the real numbers are an ordered field one can, if one chooses to, think of the elements we shall deal with as being real numbers. However, by pointing out that it is only ordered field properties which we are using we spare ourselves from being distracted by other additional properties of the reals which are not needed.

Fourier elimination is a generalization to systems of linear inequations of the well-known elimination method for solving systems of linear equations. To make this relationship apparent we shall first describe the usual elimination process for equations, but in a little different way.

Suppose we have a system (S) of m linear equations in n unknowns $x_1, \ldots, x_n$ which we may write in the form

$$\text{(S)}\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1, \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m. \end{aligned}$$

From this system we produce another system (S′) in $n - 1$ unknowns as follows. For each equation of (S) for which the coefficient of x_n is different from 0 solve that equation for x_n; e.g. if $a_{in} \neq 0$, then the i-th equation yields

$$\text{(i)}\qquad x_n = -\frac{1}{a_{in}}(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in-1}x_{n-1} - d_i).$$

By pairing the first of these m or fewer equations that are solved for x_n with each of the subsequent equations and equating the respective right hand sides, one obtains a set of linear equations in the unknowns $x_1, \ldots, x_{n-1}$. If now to these equations we adjoin the equations of (S) in which $a_{in} = 0$, one obtains a system (S′) which is the result of eliminating x_n from (S). It is readily seen that if an n-tuple of elements $(x_1, \ldots, x_n)$ satisfies (S) then the $(n - 1)$-tuple $(x_1, \ldots, x_{n-1})$ satisfies (S′), and for each $(n-1)$-tuple $(x_1, \ldots, x_{n-1})$ which satisfies (S′) there is a unique x_n (obtainable from $x_1, \ldots, x_{n-1}$ and any one of the equations (i)) such that $(x_1, \ldots, x_n)$ satisfies (S).

This elimination process can be made the basis for a method—not necessarily the most efficient—for solving systems of linear equations.

Turning now to systems of linear inequations of the type

$$(S) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq d_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq d_2, \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq d_m, \end{aligned}$$

we describe an elimination process for it. To each $a_{in} \neq 0$ we now have two possible types of solution for x_n:

$$\text{(i)} \qquad x_n \leq -\frac{1}{a_{in}}(a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} - d_i),$$

$$\text{(i')} \qquad -\frac{1}{a_{in}}(a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} - d_i) \leq x_n,$$

depending upon whether $a_{in} > 0$ or $a_{in} < 0$. If m_0 of the inequations in (S) have $a_{in} = 0$, there will be $m - m_0$ inequations of type (i) or (i′). Suppose the right hand sides of type (i) are $R_1, \ldots, R_r$ and the left hand sides of type (i′) are $L_1, \ldots, L_l$. Now write down the rl inequations asserting that each one of the L_i $(1 \leq i \leq l)$ is less or equal to every R_j $(1 \leq j \leq r)$; and to these inequations adjoin the m_0 inequations of (S) having $a_{in} = 0$. (If $rl = 0$, none are adjoined to the m_0 inequations.) Call this resulting system of linear inequations (S′). Again we see that if an n-tuple $(x_1,\ldots,x_n)$ satisfies (S), then the $(n-1)$-tuple $(x_1,\ldots,x_{n-1})$ satifies (S′), and for any $(n-1)$-tuple $(x_1,\ldots,x_{n-1})$ which satisfies (S′) there is an x_n such that, $(x_1,\ldots,x_n)$ satisfies (S)—for if (S′) is true with the given values of $x_1,\ldots,x_{n-1}$ then

$$\max(L_1,\ldots,L_l) \leq \min(R_1,\ldots,R_r),$$

and for any x_n for which

$$\max(L_1, \ldots, L_l) \leq x_n \leq \min(R_1, \ldots, R_r)$$

the n-tuple $(x_1, \ldots, x_n)$ satisfies (S).

As with systems of linear equations this *Fourier elimination* procedure can be made the basis for a method of solving systems of linear inequations of the type (S). The method is easily extended to the case of a system containing also linear equations as well as strict inequations. Here we omit strict inequations from consideration as they do not enter into our particular application to Boole's work. However equations are easily encompassed in the treatment of non-strict inequations, for any equation of the form $L = 0$ is equivalent to the pair of inequations $L \geq 0$, $-L \geq 0$.

Based on the preceding discussion we may now state a consistency theorem for linear (inequational) systems in a form sufficiently general for our purposes. A more general form may be found, e.g., in STOER-WITZGALL 1970, § 1.1.

THEOREM 0.71. *A linear system*

$$\text{(S)} \qquad \begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\geq d_1, \\ a_{21}x_1 + \cdots + a_{2n}x_n &\geq d_2, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\geq d_m \end{aligned}$$

is solvable if and only if the procedure of Fourier elimination applied successively to the variables $x_1, \ldots, x_n$ *results in a sequence of linear systems* (S_0) $[= (S)]$, $(S_1), \ldots, (S_n)$ *such that each relation in* (S_n) *is true.*

Observe that the necessary and sufficient condition of this theorem is *effective*, i.e. can be decided mechanically in finitely many steps. Also, that the values of the constants $d_1, \ldots, d_m$ enter linearly in the relations of (S_n).

§ 0.8. Linear programming

The problem of finding a solution of a linear system (of equations and/or inequations) which minimizes (or maximizes) a given linear function is called a *linear programming problem.* A particular type of linear program, of the kind we shall be interested in, is the following:

$$\begin{aligned} &\text{minimize} && \sum_{j=1}^{m} c_j x_j, \\ &\text{subject to} && \sum_{j=1}^{m} a_{ij} x_j = d_i, \quad i = 1, \ldots, n, \\ & && x_j \geq 0, \qquad\quad j = 1, \ldots, m, \end{aligned}$$

where the c_j, a_{ij}, d_i are given constants and the x_j are variables. Use of matrix notation enables are to express this succinctly as

$$\begin{aligned} &\text{minimize} && \boldsymbol{cx}, \\ &\text{subject to} && \boldsymbol{Ax} = \boldsymbol{d}, \\ & && \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{c} = [c_1 \cdots c_m]$ is a row vector ($1 \times m$ matrix), where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = [x_1 \cdots x_m]^{\mathrm{T}}$$

is a column vector, where $\boldsymbol{d} = [d_1 \cdots d_n]^{\mathrm{T}}$, and where $\boldsymbol{A}$ is the $n \times m$ matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1m} \\ a_{21} & a_{22} \cdots a_{2m} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \cdots a_{nm} \end{bmatrix}.$$

The superscript T indicates matrix transpose. The function $\boldsymbol{cx}$ is called the *objective function*. Vectors $\boldsymbol{x}$ satisfying the constraints, i.e. in this case

$$\boldsymbol{Ax} = \boldsymbol{d} \quad \text{and} \quad \boldsymbol{x} \geq \boldsymbol{0},$$

are called *feasible solutions*. Feasible solutions which minimize (or maximize) the objective function are called *optimal solutions*. In linear programs which are of interest it is generally the case that the number of unknowns exceeds the number of equations ($m > n$), and we assume that there are no redundant equations in the system $\boldsymbol{Ax} = \boldsymbol{d}$ (so that rank $\boldsymbol{A} = n$). A set of n independent column vectors of the matrix $\boldsymbol{A}$ is called a *basis* for the system (and the linear program); the variables x_j associated with these columns are called *basic variables* (the remaining variables are *non-basic*). A *basic feasible solution* is a feasible solution in which the values of the non-basic variables are 0, and an *optimal basic solution* is a basic feasible solution which optimizes $\boldsymbol{cx}$.

The understanding of linear programming is enhanced if relations and properties are visualized geometrically in m-dimensional vector space. Since common 3-dimensional geometrical notions (line, plane, convex set) generalize readily, we shall not trouble to state definitions of these. Inequations $\boldsymbol{ax} \leq d$ or $\boldsymbol{ax} \geq d$ specify *half-spaces*. The intersection of finitely many half-spaces is a *convex polytope* (polyhedron, if bounded). A supporting *half-plane* to a convex set S is a hyperplane containing a point of S and having all of S on one side of the hyperplane.

In terms of this geometric language some of the fundamental facts of linear programming can now be stated as follows.

The set of points corresponding to the feasible solutions of a linear program is a convex polytope, P. A basic feasible solution corresponds to an extreme (or "corner") point of P. If there are any optimal solutions then there are optimal extremal solutions. If non-empty, the set of optimal solutions (also a convex polytope) is the intersection of the polytope P with a supporting hyperplane whose equation is of the form $\boldsymbol{cx} = q_0$, where $\boldsymbol{cx}$ is the objective function (so that the supporting hyperplane is a member of the family of parallel planes $\boldsymbol{cx} = q$, q a parameter) and q_0 is the optimal value.

There is an important result in the theory which relates the so-called primal and dual forms of a linear program. For the types which will be of particular interest to us, namely

(i)	minimize	$\boldsymbol{cx}$,	(ii)	maximize	$\boldsymbol{cx}$,
	subject to	$A\boldsymbol{x} = \boldsymbol{d}$,		subject to	$A\boldsymbol{x} = \boldsymbol{d}$,
		$\boldsymbol{x} \geq \boldsymbol{0}$,			$\boldsymbol{x} \geq \boldsymbol{0}$,

we define, their respective duals to be

(i)	maximize	$\boldsymbol{d}^{\mathrm{T}}\boldsymbol{u}$,	(ii)	minimize	$\boldsymbol{d}^{\mathrm{T}}\boldsymbol{v}$,
	subject to	$A^{\mathrm{T}}\boldsymbol{u} \leq \boldsymbol{c}^{\mathrm{T}}$,		subject to	$A^{\mathrm{T}}\boldsymbol{v} \geq \boldsymbol{c}^{\mathrm{T}}$,
		$\boldsymbol{u}$ arbitrary,			$\boldsymbol{v}$ arbitrary.

We now state the result (DANTZIG 1963, p. 125, STOER-WITZGALL 1970, p. 28):

THEOREM 0.81 (Duality Theorem of Linear Programming). *For dual pairs of linear programs the following hold*:

(a) *The value of the objective function at a feasible solution in a minimization program is greater or equal to the value of the objective function for any of the feasible solutions of the maximization program.*

(b) *If both primal and dual forms have feasible solutions then both have optimal solutions, and the respective objective functions are equal for these optimal solutions*; *and*

(c) *If one of the programs has an optimal solution, then so does the other* (*and by* (b) *their objective functions are equal at these optimal solutions*).

Fourier (also called *Fourier-Motzkin*) *elimination* can be made the basis for a direct solution of a linear programming problem. But since at each stage the number of inequations added to the system is of the form rl (see preceding §), it is generally not a practical method for solving problems, as uncontrollably large numbers of inequations could arise. However we shall find it useful when parameters are involved in the coefficients of the constraint equations. The method is quite simple to describe:

Add the equation $z = \boldsymbol{cx}$, where $\boldsymbol{cx}$ is the objective function, to the system of constraints and eliminate all variables but z. If the resulting sets of inequations are

$$(1) \qquad \begin{array}{c} L_1 \leq z, L_2 \leq z, \ldots, L_l \leq z \\ z \leq R_1, z \leq R_2, \ldots, z \leq R_r \end{array}$$

then

$$(2) \qquad \max\{L_1, \ldots, L_l\} \leq z \leq \min\{R_1, \ldots, R_r\}$$

provides the least and greatest value for z subject to the constraints (DANTZIG 1963 § 4–4). Elimination of z produces

$$(3) \qquad \max\{L_1, \ldots, L_l\} \leq \min\{R_1, \ldots, R_r\},$$

which then provides the consistency conditions, i.e. the necessary conditions for the existence of a solution. If the L_i and R_j involve parameters, and hence have no fixed numerical values, then condition (3) converts to a set of conditions on these parameters, namely that each L_i $(i = 1, \ldots, l)$ shall be less than or equal to every R_j $(j = 1, \ldots, r)$.

Fractional linear programming. This is a generalization of linear programming in which the objective function is a linear fractional form

$$\frac{\mathbf{c}\mathbf{x} + d}{\mathbf{a}\mathbf{x} + b}, \tag{4}$$

which is to be optimized subject, as in the ordinary case, to linear constraints. CHARNES 1962 shows that such a problem can be reduced to solving a related ordinary linear programming problem with one more variable. In our particular case which we shall be looking at (4) reduces to

$$\frac{\mathbf{c}\mathbf{x}}{\mathbf{a}\mathbf{x}}$$

with $\mathbf{a}\mathbf{x}$ never negative. For this situation Charnes' result can be stated as follows.

THEOREM 0.82. *The linear fractional programming problem*:

$$\textit{optimize} \quad \frac{\mathbf{c}\mathbf{x}}{\mathbf{a}\mathbf{x}} \qquad (\mathbf{a}\mathbf{x} \geq \mathbf{0})$$

subject to the constraints

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0},$$

is equivalent to the linear programming problem:

$$\textit{optimize} \quad \mathbf{c}\mathbf{y}$$

subjects to the constraints

$$\mathbf{A}\mathbf{y} = t\mathbf{b}$$

$$\mathbf{a}\mathbf{y} = 1$$

$$t, \mathbf{y} \geq 0.$$

§ 0.9. Probability theory

For the purposes of this monograph we shall be needing only the simplest and most elementary aspects of the theory say, for example, that which would be required to handle finite stochastic situations. Accordingly our presentation is trimmed down to this level of sophistication.

Typically, contemporary expositions of probability take the basic notion of "event" to be a subset of a set Ω, the set of possible outcomes of an experiment or trial (the sample space). These subsets are assumed to form a field or algebra of subsets, i.e. a collection which includes Ω and is closed under the operations of union and of complement with respect to Ω. Probability is taken as a normed, additive (or, more specifically, σ-additive) measure on this algebra of sets. If Ω is finite then an assignment of probabilities to the singleton subsets of Ω suffices to endow every subset of Ω with a probability value.

Here we proceed from the point of view, somewhat more convenient for us, that probability is a numerical-valued function on a Boolean algebra, rather than on an algebra of sets (see e.g. KAPPOS 1960, 1969). Precisely put: an ordered pair $\langle \mathfrak{A}, P \rangle$ is a *probability algebra* if $\mathfrak{A}$ is a Boolean algebra and P a real-valued function (a *probability function*) defined on elements of the universe of $\mathfrak{A}$ which is

(i) *strictly positive*, i.e. for any $x \in A$

$$P(x) \geq 0,$$
$$P(x) = 0 \quad \text{if and only if} \quad x = 0.$$

(ii) *normed*, i.e.

$$P(1) = 1.$$

(iii) *additive*, i.e. for any $x, y \in A$

$$P(x \vee y) = P(x) + P(y), \quad \text{if } xy = 0.$$

From this definition one can readily show (using $\bar{x}$ for the complement of x):

THEOREM 0.91. *In any probability algebra*

(a) $$P(x) + P(\bar{x}) = 1,$$

(b) $$P(x) + P(y) = P(x \vee y) + P(xy).$$

(c) $$\textit{If} \quad x \subseteq y \quad \textit{then} \quad P(x) \leq P(y).$$

Two probability algebras $\langle \mathfrak{A}, P \rangle$ and $\langle \mathfrak{B}, Q \rangle$ are *isometric* if there is an isomorphism $\varphi: A \to B$ which preserves probability, i.e. such that $P(x) = Q(\varphi(x))$ for all $x \in A$. With respect to mathematical properties isometric probability algebras are indistinguishable. The elements of the Boolean algebra will be referred to as *events* (of the probability algebra).

Two events x, y of a probability algebra are *independent* if $P(xy) = P(x)\,P(y)$. A set of n events $x_1, \ldots, x_n$ in a probability algebra is a *mutually independent set* (or, the events are *mutually independent*) if the probability of any logical product of two or more events selected out of the n events is equal to the product of the respective probabilities. Since there are $2^n - \binom{n}{1} - \binom{n}{0}$ such selections this definition of mutual independence involves $2^n - n - 1$ equations. There is an equivalent condition in terms of the probabilities of the basic products (constituents) on the n events, which condition involves 2^n equations (see, e.g. RÉNYI 1970, p. 110):

THEOREM 0.915. *The system of 2^n equations, asserting the equality between the probability of each constituent on n events $x_1, \ldots, x_n$ with the product of the respective probabilities of the factors of the constituent, is a system equivalent to the $2^n - n - 1$ equations in the definition of the mutual independence of $x_1, \ldots, x_n$.*

By virtue of this theorem the probability of a Boolean function of mutually independent events can be given a particularly simple form, a form which is extensively used by Boole:

THEOREM 0.92. *Let V be a Boolean function of $x_1, \ldots, x_n$ expressed in disjunctive normal form, i.e. as a logical sum of constituents on $x_1, \ldots, x_n$. If the events $x_1, \ldots, x_n$ are mutually independent then $P(V) = [V]$, where $[V]$ stands for the result obtained from V be replacing x_i by $P(x_i)$, $\bar{x}_i$ by $P(\bar{x}_i)$ $(i = 1, \ldots, n)$ and taking logical sum and product as, respectively, numerical sum and product.*

We now present a result concerning circumstances under which two

Boolean functions of arbitrary mutually independent events are themselves independent. (For example, supposing x_1 and x_2 to be independent, the pair of functions $\bar{x}_1$ and $\bar{x}_2$ are independent, but the pair x_1 and $x_1 x_2$ are not.)

REMARK 1. The mutual independence of a set of events is unaffected by the addition or removal from the set of either the event 0 or the event 1.

REMARK 2. If $x_1, \ldots, x_n$ are mutually independent events and φ is a Boolean function of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ (i.e. of the variables $x_1, \ldots, x_n$ minus the variable x_i), then

$$P(\varphi x_i) = P(\varphi)\, P(x_i) \quad \text{and} \quad P(\varphi \bar{x}_i) = P(\varphi)\, P(\bar{x}_i).$$

REMARK 3. If for arbitrary mutually independent events $x_1, \ldots, x_n$ $P(\varphi(x_1, \ldots, x_n)) = P(\psi(x_1, \ldots, x_n))$ then $\varphi = \psi$.

Justification for these remarks is readily given: for Remark 1 one merely notes the definition of mutual independence (which is easily extended to refer also to singleton sets of events) and that $P(0) = 0$, $P(1) = 1$. The assertion of Remark 2 becomes evident if one replaces φ by its disjunctive normal form on the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, then distributes x_i over this logical sum, distributes P over the mutually exclusive terms of the sum, and then over the mutually independent events in the logical products (Theorem 0.915). Similarly, for Remark 3, one distributes P over the disjunctive normal forms of $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$; then the equality of the probabilities for any set of mutually independent events (which could be any one of the 2^n selection of 0's and 1's) implies the identity of the two polynomials in the products on $P(x_i)$ and $P(\bar{x}_i)$, and hence that the corresponding coefficients (either 0 or 1) of the disjunctive normal forms of $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are the same.

Let $\varphi(0/x_i)$ stand for the result of replacing x_i by 0 in φ. Similarly for $\varphi(1/x_i)$. We say that the Boolean function φ *depends inessentially on* x_i if $\varphi(0/x_i) = \varphi(1/x_i)$, and otherwise that it *depends essentially* on x_i. If φ depends inessentially on x_i then by the Law of Development (Theorem 0.535)

$$\begin{aligned}\varphi &= \varphi(1/x_i)\, x_i + \varphi(0/x_i)\, \bar{x}_i \\ &= \varphi(1/x_i)\,(x_i + \bar{x}_i) = \varphi(0/x_i)\,(x_i + \bar{x}_i) \\ &= \varphi(1/x_i) = \varphi(0/x_i).\end{aligned}$$

In particular, then, such a φ admits of a disjunctive normal form in which x_i is absent. Now for our theorem.

THEOREM 0.93. *Let φ and ψ be Boolean functions of n arguments. Then for any probability algebra the following two conditions are equivalent:*

(i) *For arbitrary mutually independent events, $x_1, \ldots, x_n, \varphi = \varphi(x_1, \ldots, x_n)$ and $\psi = \psi(x_1, \ldots, x_n)$ are independent events.*

(ii) *The sets of variables on which φ and ψ depend essentially are disjoint.*

PROOF. Assume (ii), i.e. that φ depends essentially on $x_{i_1}, \ldots, x_{i_k}$, that ψ depends essentially on $x_{j_1}, \ldots, x_{j_l}$ and that the sets of variables $\{x_{i_1}, \ldots, x_{i_k}\}$ and $\{x_{j_1}, \ldots, x_{j_l}\}$ are disjoint. Since $\varphi(x_1, \ldots, x_n)$ depends essentially only on $x_{i_1}, \ldots, x_{i_k}$, we have that

$$\varphi = \varphi(x_1, \ldots, x_n) = \sum\nolimits^{(\varphi)} K_r(x_{i_1}, \ldots, x_{i_k}), \tag{1}$$

when the index r ranges from 1 to 2^k and the superscript (φ) on the logical sum symbol indicates that in this sum not all basic products $K_r(x_{i_1}, \ldots, x_{i_k})$ appear but only those which "imply" φ—that is which are included in the event φ. Similarly,

$$\psi = \psi(x_1, \ldots, x_n) = \sum\nolimits^{(\psi)} L_s(x_{j_1}, \ldots, x_{j_l}). \tag{2}$$

Noting that the variables of any K_r and any L_s are disjoint, so that no product $K_r L_s$ vanishes, and that any two distinct products $K_r L_s$ are mutually exclusive, we have for mutually independent $x_1, \ldots, x_n$,

$$\begin{aligned}P(\varphi\psi) &= P(\Sigma^{(\varphi)} K_r \Sigma^{(\psi)} L_s) \\ &= P(\Sigma^{(\varphi)} \Sigma^{(\psi)} K_r L_s) \\ &= \Sigma^{(\varphi)} \Sigma^{(\psi)} P(K_r L_s) && \text{(the } \Sigma\text{'s are now numerical)} \\ &= \Sigma^{(\varphi)} \Sigma^{(\psi)} P(K_r)\, P(L_s) && \text{(by the mutual independence)} \\ &= \Sigma^{(\varphi)} P(K_r)\, \Sigma^{(\psi)} P(L_s) \\ &= P(\varphi)\, P(\psi).\end{aligned} \tag{3}$$

Now assume (i), i.e. for any mutually independent $x_1, \ldots, x_n$

$$P(\varphi(x_1, \ldots, x_n)\,\psi(x_1, \ldots, x_n)) = \\ = P(\varphi(x_1, \ldots, x_n))\,P(\psi(x_1, \ldots, x_n)). \tag{4}$$

If we take x_i to be 0, and 1, then (by Remark 1 the mutual independence still holding)

$$P(\varphi(0/x_i)\,\psi(0/x_i)) = P(\varphi(0/x_i))\,P(\psi(0/x_i)), \tag{5}$$

$$P(\varphi(1/x_i)\,\psi(1/x_i)) = P(\varphi(1/x_i))\,P(\psi(1/x_i)). \tag{6}$$

It will be convenient to introduce the abbreviations Φ_1 for $P(\varphi(1/x_i))$ and Φ_0 for $P(\varphi(0/x_i))$, and similarly Ψ_1 and Ψ_0 for ψ. Then by the Law of Development and Remark 2,

$$P(\varphi) = \Phi_1 P(x_i) + \Phi_0 P(\bar{x}_i), \\ P(\psi) = \Psi_1 P(x_i) + \Psi_0 P(\bar{x}_i), \tag{7}$$

and, by use of (5) and (6),

$$P(\varphi\psi) = \Phi_1 \Psi_1 P(x_i) + \Phi_0 \Psi_0 P(\bar{x}_i). \tag{8}$$

With a little bit of algebra we obtain from (7) and (8)

$$P(\varphi\psi) - P(\varphi)\,P(\psi) = (\Phi_1 - \Phi_0)\,(\Psi_1 - \Psi_0)\,P(x_i)\,P(\bar{x}_i).$$

By assumption the left hand side is 0, hence either $\Phi_1 = \Phi_0$ or $\Psi_1 = \Psi_0$, i.e., by virtue of Remark 3, either φ depends inessentially on x_i or ψ depends inessentially on x_i, in other words, no x_i is such that both φ and ψ depend essentially on it.

For later use we shall be needing the following result, for which we shall assume that for each (finite) m there is a Boolean algebra $\mathfrak{B}_m$ which is generated by m algebraically independent generators (see 6. in § 0.5).

THEOREM 0.94. *Let $\lambda_1, \ldots, \lambda_{2^m}$ be a set of non-negative real numbers whose sum is 1. Then there is a probability algebra $\langle \mathfrak{B}, P \rangle$ and events $E_1, \ldots, E_m$ (elements of B) such that $P(C_i) = \lambda_i$ $(i = 1, \ldots, 2^m)$, where the C_i are the constituents (basic products) on $E_1, \ldots, E_m$.*

PROOF. *Case* 1. No λ_i is 0. Let $\mathfrak{B}_m$ be a Boolean algebra generated by m independent generators $E_1, \ldots, E_m$. We define the probability function P over the elements of $\mathfrak{B}_m$ by setting

$$P(C_i) = \lambda_i \qquad (i = 1, \ldots, 2^m)$$

and then extending P to all other elements additively, i.e. since each element of $\mathfrak{B}_m$ is the logical sum of a set of constituents on $E_1, \ldots, E_m$, we take the P-value of the element to be the sum of the corresponding λ's. It is routine to check that this P is a probability function over $\mathfrak{B}_m$ so that $\langle \mathfrak{B}_m, P \rangle$ is the desired probability algebra.

Case 2. Some λ_i is 0. Again select a Boolean algebra $\mathfrak{B}_m$ with independent generators, say $G_1, \ldots, G_m$, and associate each constituent H_i on the $G_1, \ldots, G_m$ with a λ_i $(i = 1, \ldots, 2^m)$. Let D be logical sum of those H_i associated with λ's which are 0 and put $V = \bar{D}$. We go over to the reduced algebra $\mathfrak{B}_m \mid V$ which identifies elements of $\mathfrak{B}_m$ if they differ only outside of V, and let its elements be denoted by $X \mid V$, $X \in B_m$. By Theorem 0.57 the mapping $X \to X \mid V$ is a homomorphism of $\mathfrak{B}_m$ onto $\mathfrak{B}_m \mid V$; hence if $X = H_{i_1} + \cdots + H_{i_k}$ then

$$X \mid V = H_{i_1} \mid V + \cdots + H_{i_k} \mid V.$$

We define a P for $\mathfrak{B}_m \mid V$ by putting $P(H_i \mid V) = \lambda_i$ and extending P to the remaining elements of $\mathfrak{B}_m \mid V$ additively i.e. by putting

$$\begin{aligned} P(X \mid V) &= P(H_{i_1} \mid V) + \cdots + P(H_{i_k} \mid V) \\ &= \lambda_{i_1} + \cdots + \lambda_{i_k}. \end{aligned}$$

Again it is routine to check that this P is a probability function for $\mathfrak{B}_m \mid V$. (Note that $P(X \mid V) = 0 \to X \mid V = 0 \mid V$ since $X \mid V = 0 \mid V$ for any X included in D.) As for the elements E_i, we take these to be the $G_i \mid V$ and C_i we take to be $H_i \mid V$, where

$$\begin{aligned} H_i \mid V &= (g_1 g_2 \ldots g_m) \mid V \\ &= (g_1 \mid V)(g_2 \mid V) \ldots (g_m \mid V), \end{aligned}$$

with a g_j being either G_j or $\bar{G}_j$ depending on the particular H_i.

Based on the conventional definition of conditional probability,

$$P(x|y) = \frac{P(xy)}{P(y)}, \qquad (P(y) \neq 0)$$

the following theorem presents well-known properties.

THEOREM 0.95. *In any probability algebra (assuming that the conditional probabilities are defined)*:

(i) $P(xy) = P(x|y)P(y)$

(ii) $P(x+y|z) = P(x|z) + P(y|z), \quad \text{if } xy = 0$

(iii) $P(xy|z) = P(x|yz)P(y|z)$

$= P(x|y)P(y|z), \quad \text{if } yz = y$

(iv) $P(x|y) = \dfrac{P(y|x)P(x)}{P(y|x)P(x) + P(y|\bar{x})P(\bar{x})}$ (*Bayes' rule*)

Proof of the following is straightforward:

THEOREM 0.96. *Let $\langle \mathfrak{B}, P\rangle$ be a probability algebra and let b, $V \in B$ with $P(V) \neq 0$. If a function P^* is defined on elements $b|V$ of $B|V$ by setting*

$$P^*(b|V) = \frac{P(bV)}{P(V)},$$

then the pair $\langle \mathfrak{B}|V, P^*\rangle$ constitutes a probability algebra.

Since, as we shall later see, Boole's treatment of probability has propositions as events it will be useful to have a formulation in which this aspect is featured. We introduce the idea of a probability calculus.

Let S be a set of formulas of the propositional calculus closed with respect to formation of formulas using connectives $\neg$, $\wedge$, $\vee$ and propositional variables X_1, X_2, ... We say $\langle S, P\rangle$ is a *probability calculus* if P is real valued function on formulas from S into the interval $[0, 1]$ such that for arbitrary φ, $\psi \in S$.

P1. (i) $P(\varphi \wedge \neg\varphi) = 0$

(ii) $P(\varphi \wedge \psi) \leq P(\psi)$

P2. $$P(\neg\varphi) = 1 - P(\varphi)$$

P3. $$P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi).$$

From P1 (ii) it readily follows that the probabilities of logically equivalent formulas are equal. Reducing S modulo logical equivalence, i.e. "identifying" logically equivalent formulas, we obtain as set B of equivalence classes, each having associated with it a "natural" P-value, namely that of any of its members. Defining Boolean operations in the "natural" way we form the Lindenbaum algebra $\mathfrak{B}(S)$ of S (§ 0.5). The pair $\langle\mathfrak{B}(S), P\rangle$ constitutes a probability algebra.

A *probability calculus model* is an assignment of propositions (sentences of a formal language) $A_1, A_2, \ldots$ to the variables $X_1, X_2, \ldots$ such that all instances (and logical consequences) of P1–P3 hold with the A_i replacing the X_i. Note that what probability relations hold among the A_i in such a model is independent of any logical structure which the A_i may have. If we were to introduce the notion of an atomic proposition (of a fixed formal language L) then, considering P1–P3 to be collections of asserted probability relations (i.e. as φ, ψ range over the formulas of L), a probability calculus model is the same as a probability calculus with atomic propositions $A_1, A_2, \ldots$ replacing propositional variables.

Clearly, if $S = S(X_1, \ldots, X_n)$ is a set of formulas built up only using $X_1, \ldots, X_n$, then $\langle S(X_1, \ldots, X_n), P\rangle$ is defined as a probability calculus when the values of P are given for the 2^n constituents on $X_1, \ldots, X_n$.

§ 0.10. Miscellaneous

0.10.1. Boole's theorem on symmetric determinants of linear homogeneous forms. The theorem under consideration here, described by Muir in his monumental history of determinants (Muir 1920, Vol. 3, pp. 99) as "rather notable", plays an important role in proving a central result for Boole's probability method. The proof we give here is essentially Boole's (appearing in Boole 1862, pp. 235–238 = Boole 1952, pp. 400–406) but rephrased in terms of our present-day mathematical dialect. Although Muir complains "The proof is disappointingly lengthy, occupying very nearly three pages", in working the proof over we haven't succeeded in improving on Boole. Incidentally, Muir's description of this theorem omits a portion of the conclusion which Boole obtained—that in the end result the variables appear with power no higher than the first.

We consider linear homogeneous forms on variables $x_1, \ldots, x_p$ i.e. algebraic expressions

$$c_1x_1 + \cdots + c_kx_k + \cdots + c_px_p,$$

in which the coefficients c_k designate elements of an ordered field (e.g. the reals). Let $L(x_1, \ldots, x_p)$, or for short L, be the set of all such forms and let L^+ be the subset of L for which the coefficients are all non-negative. An $n \times n$ determinant whose entries are elements of L is said to have the *coefficient proportionality property* if for each k the following holds: let the coefficients of x_k taken in order in any row be $\alpha_1, \ldots, \alpha_n$ and let the corresponding coefficients of x_k in any other row be $\beta_1, \ldots, \beta_n$; then the vector of coefficients $(\alpha_1, \ldots, \alpha_n)$ is proportional to the vector of coefficients $(\beta_1, \ldots, \beta_n)$, i.e. there are constants a and b such that $a\beta_i = b\alpha_i$ for $i = 1, \ldots, n$. (Alternatively: either $\alpha_i = 0$ for all i or there is a λ such that $\beta_i = \lambda\alpha_i$ for all i.)

THEOREM 0.101 (Boole, 1862). *Let D be an $n \times n$ determinant whose entries are elements of L and such that*

(i) *it is symmetric,*

(ii) *all its elements on the principal diagonal are from L^+, and*

(iii) *it has the coefficient proportionality property.*

Then D is equal to a linear combination with positive coefficients of products of $x_1, \ldots, x_p$ each appearing, if at all, with exponent 1. *If the columns of D are independent then there is at least one term in the linear combination* (*which, otherwise, might be empty and D be identically* 0).

The last sentence in our statement of this theorem remedies a neglect in Boole's version. Before beginning the proof we state a few definitions and some observations.

The elements of an arbitrary determinant are denoted by a_{ij} $(i, j = 1, \ldots, n)$. A $(\lambda; i, j)$-row operation consists of subtracting from each element of the j-th row λ times the corresponding element of the i-th row. Similarly for a $(\lambda; i, j)$-column operation. The elements on the horizontal and vertical lines through the a_{ii} position will be referred to as an *i-system*. Define a *$(\lambda; i, j)$-system operation* as the application of a $(\lambda; i, j)$-row operation followed by a $(\lambda; i, j)$-column operation using the same λ, i and j. Since both the row and column operations leave the value of a determinant unchanged so does the system operation. Also, after a system operation the entries in the determinant are still in L if the original entries were.

PROOF. Assume the hypothesis of Theorem 0.101.

LEMMA 1. *The result of a $(\lambda; i, j)$-system operation on a determinant having properties* (i)–(iii) *is a determinant which still has these properties.*

That (i) still holds is clear since only elements of the j-th row and j-th column have been altered, and that symmetrically. To show that (ii) and (iii) still hold consider a variable x_k and let its coefficients in any two rows, excluding the j-th, be

$$\begin{matrix} \mu\alpha_1 \cdots \mu\alpha_i \cdots \mu\alpha_j \cdots \mu\alpha_n, \\ \nu\alpha_1 \cdots \nu\alpha_i \cdots \nu\alpha_j \cdots \nu\alpha_n. \end{matrix} \tag{1}$$

After the system operation these become

$$\begin{matrix} \mu\alpha_1 \cdots \mu\alpha_i \cdots \mu(\alpha_j - \lambda\alpha_i) \cdots \mu\alpha_n, \\ \nu\alpha_1 \cdots \nu\alpha_i \cdots \nu(\alpha_j - \lambda\alpha_i) \cdots \nu\alpha_n \end{matrix} \tag{2}$$

and thus all row coefficients, except possibly for the j-th, are proportional. Suppose now that (1) represents the coefficients of x_k in the i-th and j-th rows. After the $(\lambda; i, j)$-system operation these become

$$\begin{matrix} \mu\alpha_1 \cdots & \mu\alpha_i \cdots C_i \cdots & \mu\alpha_n, \\ (\nu - \lambda\mu)\,\alpha_1 \cdots & (\nu - \lambda\mu)\,\alpha_i \cdots C_j \cdots & (\nu - \lambda\mu)\,\alpha_n, \end{matrix}$$

where

$$C_i = \mu\alpha_j - \lambda\mu\alpha_i = \mu(\alpha_j - \lambda\alpha_i),$$

$$C_j = \nu\alpha_j - \lambda\mu\alpha_j - \lambda(\nu\alpha_i - \lambda\mu\alpha_i).$$

Using the fact that, by (i), $\mu\alpha_j = \nu\alpha_i$ we have

$$(3) \qquad \begin{aligned} C_j &= \nu\alpha_j - \lambda\nu\alpha_i - \lambda(\mu\alpha_j - \lambda\mu\alpha_i) \\ &= \nu(\alpha_j - \lambda\alpha_i) - \lambda\mu(\alpha_j - \lambda\alpha_i) \\ &= (\nu - \lambda\mu)(\alpha_j - \lambda\alpha_i). \end{aligned}$$

Thus C_i and C_j are in the same proportion (namely $\mu : \nu - \lambda\mu$) as the remaining pairs of coefficients and hence the vectors of coefficients are proportional. Since the i-th row coefficients are proportional to all the other rows, so is the j-th. To show that the coefficients in the a_{jj} position are non-negative, we note that if $\mu = 0$ then the $(\lambda; i, j)$-system operation doesn't alter the coefficient of x_k in a_{jj}. So suppose $\mu \neq 0$. Now by (i)

$$\mu(\alpha_j - \lambda\alpha_i) = \mu\alpha_j - \lambda\mu\alpha_i = \nu\alpha_i - \lambda\mu\alpha_i = (\nu - \lambda\mu)\,\alpha_i.$$

Whence substituting in (3) we have

$$C_j = (\nu - \lambda\mu)\frac{(\nu - \lambda\mu)\,\alpha_i}{\mu} = \left(\frac{\nu - \lambda\mu}{\mu}\right)^2 (\mu\alpha_i),$$

and since by hypothesis $\mu\alpha_i$ is non-negative so also is C_j.

LEMMA 2. *The variable x_k occurs in a_{ij} (i.e. has a non-zero coefficient) if and only if it occurs in a_{ii} and in a_{jj}.*

By (i), x_k occurs in a_{ij} if and only if it occurs in a_{ji}. Suppose m, l, n and m are the respective coefficients of x_k in a_{ij}, a_{ii}, a_{jj} and a_{ji}. If x_k occurs in a_{ij} then by (iii) and (i) there is a λ such that

$$m = \lambda n \quad \text{and} \quad l = \lambda n;$$

whence, since $m \neq 0$, we have that λ, n and l are $\neq 0$. In the other direction, if $l \neq 0$ then by (iii) and (i) there is a λ such that

$$n = \lambda m \quad \text{and} \quad m = \lambda l,$$

and hence, if in addition $n \neq 0$, then $m \neq 0$.

LEMMA 3. *If x_k occurs in a_{ij}, then it may be removed from the j-system by a suitable $(\lambda; i, j)$-system operation.*

Since by (iii) the vector of coefficients of x_k in the j-th row is proportional to the vector of coefficients of x_k in the i-th row, then by a suitable choice of λ the result of a $(\lambda; i, j)$-row operation produces a j-th row in which the coefficients of the x_k are all zero. By virtue of (iii) and (i) the subsequent $(\lambda; i, j)$-column operation removes all non-zero coefficients of x_k in the j-th column.

To complete the proof of Boole's theorem we observe that, by Lemmas 2 and 3, D can be converted by system operations to an equal determinant in which any particular variable, x_k say, occurs at most in one entry, and on the diagonal. If this is the a_{ii} position and α is its positive coefficient (Lemma 1) then on expanding the determinant by minors of the i-th row we have that x_k occurs only in terms coming from the part

$$(-1)^{i+i}\alpha x_k M_{ii},$$

where M_{ii} is the minor on element a_{ii}. By Lemma 1 M_{ii} has again properties (i)–(iii), but no occurrences of x_k. By an n-fold iteration of the argument until the minor becomes a single entry determinant, whose value is this entry, we see that x_k cannot occur with power higher than the first nor in a term with negative coefficient. Finally, if there are no non-zero terms in this expansion then D vanishes identically and hence has dependent columns.

The above theorem was developed by Boole for application to the following situation. Let V be a rational integral function of n variables $x_1, \ldots, x_n$ in which no variable appears with exponent greater than 1 and such that all coefficients are positive. If V_i designates the sum of the terms of V having x_i present and V_{ij} the sum of the terms of V having the product $x_i x_j$ present, then the determinant

$$(4) \qquad D(V) = \begin{vmatrix} V & V_1 & V_2 & \cdots & V_n \\ V_1 & V_1 & V_{12} & \cdots & V_{1n} \\ V_2 & V_{21} & V_2 & \cdots & V_{2n} \\ \vdots & & & & \\ V_n & V_{n1} & V_{n2} & \cdots & V_n \end{vmatrix}$$

will have in its expansion as a rational integral form in $x_1, \ldots, x_n$ only terms with positive coefficients—if there are any terms at all, i.e. if $D(V)$ is not identically 0. As an example of such a determinant let $V = ax_1x_2 + bx_1 + cx_2 + d$; then for this V, $D(V) =$

$$\begin{vmatrix} ax_1x_2 + bx_1 + cx_2 + d & ax_1x_2 + bx_1 & ax_1x_2 + cx_2 \\ ax_1x_2 + bx_1 & ax_1x_2 + bx_1 & ax_1x_2 \\ ax_1x_2 \qquad + cx_2 & ax_1x_2 & ax_1x_2 + cx_2 \end{vmatrix}$$

If we think of the products of the x_i's in such a V as the variables of the preceding theorem and if the coefficients of the terms are positive, then one can readily verify that the determinant in (4) satisfies the hypothesis of Theorem 0.101. This is immediate for items (i) (symmetry) and (ii) (positive terms on the diagonal); as for (iii) consider any particular term of V (product of x_i's) and let $\alpha_1, \ldots, \alpha_n$ be the list of coefficients for this product of x_i's in the first row of the determinant. Since the term occurs in V this list can't be all 0's. In any subsequent row the coefficients on this product will be $\lambda\alpha_1, \ldots, \lambda\alpha_n$ where $\lambda = 1$ or $\lambda = 0$—for the restriction represented by the subscripts on the V's will either leave this term throughout the given row (in the columns for which $\alpha_i \neq 0$) or else remove it entirely. Hence property (iii) holds and thus the expansion of such a determinant cannot have negative terms, though it could be identically 0.

There is one case however for which we are sure that $D(V)$ is not identically 0, namely when V (on n variables) has all 2^n terms present. For looking at the expansion of the determinant as a polynomial in the $n!$ products of elements one from each column and not in the same row, the term with product $a^n(x_1x_2 \ldots x_n)^n$ occurs only once, coming from the product of the principal diagonal elements. As it cannot occur in any other way it must remain in the final value.

Another simple observation concerning such determinants: if the value of $D(V)$ is $\neq 0$ for any particular set of non-negative values of $x_1, \ldots, x_n$, then $D(V) > 0$ for all positive values of $x_1, \ldots, x_n$. This is clear since the non-vanishing of $D(V)$ for a single set of values precludes it from being identically 0.

Finally, the product $x_1x_2 \cdots x_n$ is a factor of $D(V)$, for x_1 is a factor of the 2-nd column of the determinant, x_2 a factor of the 3-rd column, and so on.

0.10.2. Implicit function theorem. For convenience of reference we state here a basic theorem in analysis concerning the circumstances under which a set of equations

$$\begin{aligned} F_1(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0, \\ F_2(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0, \\ \vdots \qquad & \quad \vdots \\ F_m(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0 \end{aligned}$$

has a solution for the y's as functions of the x's. We go over to vector notation, writing the system of equations as

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0},$$

where

$$\boldsymbol{F} = (F_1, \ldots, F_m),$$

$$\boldsymbol{x} = (x_1, \ldots, x_n), \qquad \boldsymbol{y} = (y_1, \ldots, y_m).$$

We shall need the Jacobian

$$J(\boldsymbol{F}, \boldsymbol{y}; \boldsymbol{x}), \quad \text{or} \quad \frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)},$$

defined as the determinant

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \cdots & \dfrac{\partial F_1}{\partial y_m} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} \cdots & \dfrac{\partial F_2}{\partial y_m} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial F_m}{\partial y_1} & \dfrac{\partial F_m}{\partial y_2} \cdots & \dfrac{\partial F_m}{\partial y_m} \end{vmatrix},$$

and we use R_n for the set of all n-tuples of reals.

IMPLICIT FUNCTION THEOREM. *Let $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y})$ be a function from $R_n \times R_m$ to R_m defined on some domain D and continuously differentiable there (i.e. of class C'). Suppose that $\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0}$ for some point $(\boldsymbol{a}, \boldsymbol{b}) \in D$ and that $J(\boldsymbol{F}, \boldsymbol{y}; \boldsymbol{x}) \neq 0$ at $(\boldsymbol{a}, \boldsymbol{b})$. Then there is a neighborhood of $\boldsymbol{a}$ and a unique vector function $\boldsymbol{f}(\boldsymbol{x})$ from R_n to R_m defined over the neighborhood and of class C' having the properties that*

$$\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{b} \quad \textit{and} \quad F(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = 0.$$

We shall have occasion later to refer to matters of functional dependence (in § 5.5). Neat and simple necessary and sufficient conditions are apparently unavailable. The following two theorems taken from OSTROWSKI 1951, pp. 262–263, give sufficient conditions, one for independence and one for dependence, which together will be sufficient for our purposes.

Sufficient Condition for Functional Independence. Let $F_1, \ldots, F_n$ be a set of n functions each from R_n to R, defined and with continuous first partial derivatives over some region D in R_n. Let $F(D)$ be the range of $(F_1, \ldots, F_n)$, i.e. the set of points in R_n over which $(F_1, \ldots, F_n)$ ranges as $(x_1, \ldots, x_n)$ ranges over D. Suppose further that the Jacobian of the functions with respect to $x_1, \ldots, x_n$,

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}$$

is not identically 0 over D. Then excepting functions that are identically 0 in a subregion of a region including $F(D)$, there is no continuous function φ such that $\varphi(F_1, \ldots, F_n) = 0$ throughout D.

Sufficient Condition for Functional Dependence. Suppose that the n functions

$$F_1(x_1, \ldots, x_m), F_2(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)$$

together with their first partials are continuous throughout some region D and suppose that the matrices

$$\left|\begin{matrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_1} \cdots & \frac{\partial F_n}{\partial x_1} \\ \frac{\partial F_1}{\partial x_2} & \frac{\partial F_2}{\partial x_2} \cdots & \frac{\partial F_n}{\partial x_2} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_1}{\partial x_m} & \frac{\partial F_2}{\partial x_m} \cdots & \frac{\partial F_n}{\partial x_m} \end{matrix}\right|$$

and

$$\left|\begin{matrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_1} \cdots & \frac{\partial F_r}{\partial x_1} \\ \frac{\partial F_1}{\partial x_2} & \frac{\partial F_2}{\partial x_2} \cdots & \frac{\partial F_r}{\partial x_2} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_1}{\partial x_m} & \frac{\partial F_2}{\partial x_m} \cdots & \frac{\partial F_r}{\partial x_m} \end{matrix}\right|$$

are, at each point of D, of the same rank r. Then each of the functions $F_{r+1}, \ldots, F_n$ are expressible in terms of $F_1, \ldots, F_r$, i.e. there are continuous and continuously differentiable functions $\psi_1, \ldots, \psi_{n-r}$ such that

$$F_{r+i} = \psi_i(F_1, \ldots, F_r) \qquad (i = 1, \ldots, n - r)$$

throughout D.

We point out that if there is even one such relation as expressed in the conclusion of this theorem, say

$$F_{r+1} = \psi_1(F_1, \ldots, F_r),$$

then the function

$$\varphi(F_1, \ldots, F_n) = \psi_1(F_1, \ldots, F_r) - F_{r+1}$$

is continuous and vanishes identically over D. Hence by the preceding theorem (since $\psi_1(y_1, \ldots, y_r) - y_{r+1}$ is not identically zero) the Jacobian

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}$$

is identically 0 over D.

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PART I. LOGIC

... he shrouded the simplest logical processes in the mysterious operations of a mathematical calculus. The intricate trains of symbolic transformations, by which many of the examples in the Laws of Thought are solved, can be followed only by highly accomplished mathematical minds; and even a mathematician would fail to find any demonstrative force in a calculus which fearlessly employs unmeaning and incomprehensible symbols, and attributes a signification to them by a subsequent process of interpretation.

W. Stanley Jevons, 1870

It would be a pretty piece of research to take Boole's algebra, find independent postulates for it (his laws are entirely insufficient as a basis for the operations he uses), complete it, and systematically investigate its interpretations.

C. I. Lewis, 1918

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CHAPTER 1

BOOLE'S LOGIC OF CLASS TERMS

§ 1.0. Symbolical Algebra

Boole wrote his *Laws of Thought* before the notion of an abstract formal system, expressed within a precise language, was fully developed. And, at that time, still far in the future was the contemporary view which makes a clear distinction between a formal system and its realization or models. Boole's work contributed to bringing these ideas to fruition. He was—in the 1840's along with W. R. Hamilton and H. Grassmann—one of the very first to originate a variant algebra, and among the earliest to use a formal algebraic system (as best as it was understood then) with more than one interpretation. But the change, from Boole's time to the present, in the conception of what constitutes Algebra has been so marked that a proper understanding of Boole's central contribution—the successful application of algebraic methods to logic—requires a recreation of the setting within which he worked. Accordingly we present a brief description of the "science of algebra", as it was then referred to, and in particular the so-called symbolical algebra. As representative works of the period on which to base our account we choose PEACOCK 1833 and GREGORY 1840. Although Boole seems never to have mentioned him, Peacock wrote extensively on the topic. Gregory was an early mathematical mentor of Boole and editor of the journal which published Boole's fledgeling works.

According to Peacock there are two sciences of algebra, *arithmetical* and *symbolical*. In the former the general symbols and the signs of operations refer to the numbers and operations of "common arithmetic", whose meaning required in many cases restrictions on the performability of the operations. For example, in arithmetical algebra one couldn't subtract a larger from a smaller number and hence the *form* $a-b$ involving the *general* symbols a, b could be meaningless if so interpreted. Symbolical algebra, although "suggested by" arithmetical algebra, needs to be based on "independent and *ultimate*" principles and "thus becomes essentially a science of symbols and their combinations, constructed upon its own rules, which may be applied to arithmetic and all other sciences by interpretation: by this means interpretation will *follow*, and not *precede*, the operations of algebra and their results; "Symbolical algebra arises from arithmetical by supposing that the "symbols are perfectly general and unlimited both in value and in representation, and that the operations to which they are subject are equally general likewise." The "laws" of symbolical algebra are obtained from those of arithmetical algebra by principles Peacock laid down:

(i) whatever forms in general symbols are equivalent in arithmetical algebra, are also equivalent in symbolical algebra;

(ii) whatever forms are equivalent in arithmetical algebra where the symbols are general in form, though specific in their value, will continue to be equivalent when the symbols are general in their nature as well as in their form.

Thus although $a(b-c) = ab - ac$ is true in arithmetical algebra only if b is not less than c, this restriction is removed in symbolical algebra and the equation is considered to be true. Likewise, from the arithmetical result that $a^m \times a^n = a^{m+n}$ for m and n whole numbers, it is inferred by (ii) that the equation is true in symbolic algebra when m and n are "any quantities". In addition Peacock included a restrictive principle to the effect that

(iii) the laws of combinations of symbols are to be such as to reduce to those of arithmetical algebra "when the symbols are arithmetical quantities and the [homonymous] operation symbols are taken to be those of arithmetical algebra." As we shall see, this principle was dropped by Gregory.

Considering himself to be dealing with the laws of combinations of symbols Peacock had no need to prove the existence of entities with prescribed properties but assumed the existence of appropriate "signs" (akin to + and − for positive and negative). E.g. "... the operation of extracting the square root of such a quantity as $a - b$ is *impossible*, unless a is greater than b. To remove the limitation in such cases, (an essential condition in symbolical algebra) we assume the existence of such a sign as $\sqrt{-1}$;..." And from this assumption Peacock then establishes that $\sqrt{a-b} = \sqrt{-1}\sqrt{b-a}$.

One should contrast this point of view of the nature of complex numbers with that contained in HAMILTON 1837 which appeared four years after the report of Peacock's under discussion. Hamilton defines complex numbers as ordered pairs of numbers and, on the basis of definitions of the operations of addition and multiplication for such pairs, derives their algebraic properties, i.e. the laws they satisfy. Thus Hamilton presents a specific mathematical structure. Peacock's conception, obscure and ill-formed as it is, bears more of a resemblance to our present day idea of an abstract formal system admitting of interpretations; as he says "... interpretation will *follow*, and not *precede*, the operations of algebra and their results, ..."

In Gregory's paper [1840] we find an advance in generality over Peacock. He defines symbolical algebra as "the science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject." However he considers operations in general and not simply those arising in arithmetical algebra—without comment Peacock's restriction, that common arithmetic be necessarily one of the interpretations, is dropped. Also, numerical quantities are treated as operations; e.g. a letter a being prefixed to a subject means a times it. Not only operations which are commutative and distributive but also those with other general properties are studied and interpretations sought for them.

Gregory believed that the general principles which he enunciates in his paper justified the practice of "separation of symbols", in which the symbols of operation are treated algebraically. For example a linear differential equation with constant coefficients

$$\frac{d^n y}{dx^n} + c_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + c_{n-1} \frac{dy}{dx} + c_n y = F(x)$$

would be written

$$f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)y = \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} + c_1\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + c_{n-1}\frac{\mathrm{d}}{\mathrm{d}x} + c_n\right)y = F(x),$$

and the differential operator working on y "factored", yielding

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - a_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - a_2\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x} - a_n\right)y = F(x),$$

from which the solution for y is obtained by successively operating on both sides of the equation with inverse operators $(\mathrm{d}/\mathrm{d}x - a_i)^{-1}$, $i = 1,\ldots,n$. An early paper of Boole's [1841] presents an innovation to this technique. He "divides" by the operator $f(\mathrm{d}/\mathrm{d}x)$ and resolves the symbolic fraction $1/f(\mathrm{d}/\mathrm{d}x)$ by the method of partial fractions into a sum of reciprocals of linear differential operators with numerical coefficients, and thence obtains the solution by having the sum work on $F(x)$. We bring this mathematical item up for attention as we believe it bears a resemblance to Boole's technique which he devised for solving a logical equation for an unknown by division, development, and interpretation (§§ 1.6, 1.7).

§ 1.1. Boole's first essay

One might contrast the title "The Mathematical Analysis of Logic" of Boole's small pamphlet of 1847 with that of his major work, "The Laws of Thought", appearing 7 years later. While it is the judgement of history that the earlier title more accurately portrays what Boole had accomplished, it is in the *Laws of Thought* that we have Boole's mature and systematically worked out ideas and which, accordingly, we shall be using as the basis of our detailed exposition of Boole's work. Nevertheless it is worthwhile to first have a brief summary of his initial ideas, for when we later on come to examine certain topics our understanding of these will be furthered by a comparison with the corresponding earlier treatment. Moreover we shall be able to see the suggestive role the calculus of operations, which Boole used in his mathematical work, had in the genesis of his calculus of logic, a role which came to be submerged in the *Laws of Thought*.

The Introduction to *The Mathematical Analysis of Logic* opens with a statement of the tenet of Symbolical Algebra: "... that the validity of [mathematical] analysis does not depend upon the interpretation of the symbols which are employed, but solely on the laws of their combination." Heretofore all uses of this "true principle of the Algebra of Symbols" have been in situations in which the elements represents magnitudes and the operations are on magnitudes. Now Boole wishes to present one, a Calculus of Logic, which is not of this type. By a *calculus* he means "a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation."

The symbol 1 is used to represent "the universe", which is to comprehend "every conceivable class of objects, whether existing or not, ...". Capital letters X, Y, Z are to represent (also be names for) members of classes, and corresponding symbols x, y, z are used as operators; e.g. x standing before "a subject [operand] comprehending individuals or classes" selects from the subject all the X's which it contains. When no subject is indicated he will understand it to be 1, that is

$$x = x(1)$$

"the meaning of either term being the selection from the Universe of all the X's which it contains, and the result of the operation being, in common language, the class X, i.e. the class of which each member is an X."

Note that Boole's 1 here is an absolute universe of discourse, a divergence from De Morgan who in 1846 had introduced the changeable, limited universe into logic [De Morgan 1966, p. 2]. (In *Laws of Thought* Boole adopts the De Morgan conception of a universe.) Note also that the result of the operator x on 1 is the class X, using this symbol also as a name ("applying to each member of a class") as well as for the class itself. We note also the absence of a symbol for a universal selector—comparable to the use of a lower case letter x for selecting X's—which would select everything from the universe.

Since Boole is using his "elective" symbols x, y, z as operators he takes it for granted that an indicated product or juxtaposition, as xy, has a meaning, namely as successive selection, first of the Y's and then of the X's, so that the result is a class whose members are both X's and Y's. Declaring it unnecessary to enter into analysis of "that mental operation which we have represented by the elective symbol", he enumerates the "laws of combination and succession" which govern it. The first of these is that "the result of an act of election is independent of the grouping or classification of the subject," which he writes mathematically as

$$x(u+v) = xu+xv,$$

in which $u+v$ represents "the undivided subject, and u and v the component parts of it." The second law is that the order of succession of the act of election is immaterial, and the third is that the repeated election of the same class results in nothing new. Concerning these laws,

$$x(u+v) = xu+xv \tag{1}$$

$$xy = yx \tag{2}$$

$$x^n = x \tag{3}$$

Boole asserts that they

> ... are sufficient for the basis of a Calculus. From the first of these, it appears that elective symbols are *distributive*, from the second that they are *commutative*; properties which they posess in common with symbols of quantity, and in virtue of which all the direct processes of common algebra are applicable to the present system. The one and sufficient axiom in this application is that equivalent operations performed on equivalent subjects produce equivalent results. [BOOLE 1847, p. 18].

It is an interesting commentary on the lack of sophistication in algebraic axiomatics, as evidenced in this quotation, that Boole believed that these laws plus the "axiom" cited were sufficient justification for the application of the "processes of algebra". In addition to commutativity and distributivity it was also usual to include the exponential law

$a^m a^n = a^{m+n}$, but clearly this would have no relevance here in view of Boole's index law $x^n = x$. Incidentally, in *Laws of Thought* Boole always writes this law as $x^2 = x$ and never with the general n.

The symbol '$1 - x$' is introduced in its unanalyzed entirety as an operation which selects from a subject all the not-X's and it is assumed, without justification, that the minus sign between the [operator?] symbols 1 and x acts as an algebraic inverse to the + previously introduced between the *results* of an operation, i.e. between classes. (It is surprising that Boole doesn't mention the logical law of double negation in the form $1-(1-x) = x$.)

The symbol 0 first comes in (unannounced) in connection with discussion of "All X's are Y's" which is symbolized

$$xy = x$$

and then, without explanation, converted to

$$x(1-y) = 0,$$

presumably by "common algebra" which he believes applies to his symbols.

After presenting this algebraic apparatus—in view of its fragmentary nature one can hardly call it a system—Boole turns to its use in doing term logic, which is conceived by him pretty much in the traditional postscholastic form. However we shall not continue with an exposition of this early version as the material receives fuller and more cogent treatment in *Laws of Thought* to which, as the main business of this Chapter, we are about to turn.

We conclude this section with Boole's own estimate of his earlier work as contrasted with a very general result which he announces he has obtained and which he claims can be used to solve general problems in the theory of probability. The paper from which it comes, Boole 1851a, was written four years after *Mathematical Analysis of Logic* and three years before the appearance of *Laws of Thought*:

> 1. In a hasty and (for this reason) regretted publication entitled "The Mathematical Analysis of Logic", an in a paper published in the Cambridge Mathematical Journal, entitled, "The Calculus of

> Logic", I have stated certain general laws of thought, mathematical in their expression, and constituting, as I believe, the true basis of formal logic. The actual development of those laws in the works referred to is far to imperfect to meet the requirements of the case now under our consideration. But that imperfection does not apply to the laws themselves. The results of subsequent investigations authorise me to say that there exists a general method, enabling us not only to educe any of the consequences of a system of propositions, but also to express in a scientific form and order the connexion which any proposed proposition bears to any other proposition, or system of propositions. [BOOLE 1952, VIII, p. 252]

§ 1.2. The basic principle

As the title "Laws of Thought" indicates Boole's conception of logic was not, as now viewed, namely as a discipline based on semantic ideas and concerned with truth-preserving transformations of sentences but, rather, as one having a psychological basis, concerned with mental operations and their normative expression in mathematical form. As our chief interest is in the formal and mathematical aspects of Boole's ideas we can, once our exposition is underway, ignore his psychological framework since, as we shall see, it plays no essential role. Boole of course thought otherwise and so we devote some space to his views.

The unquestioned opinion, that logic concerned thinking and operations of the mind, pervaded the contemporary English writings on logic which Boole was familiar with. Typical are the two works recommended to the reader by him in his Preface to *Laws of Thought* as sources for the technical terms of (traditional) logic: Whately's *Elements of Logic* [1852] and Thomson's *Outline of the Laws of Thought* [1853]. Both of these works appeared in many editions—those cited here are editions Boole could have had before him when writing his *Laws of Thought*. Both of these recommended books include discussions of the "higher mental faculties".

Whately has a chapter entitled "Of Operations of the Mind and

Terms" dealing with *apprehension*, *judgement*, and *reasoning*; similarly Thomson has among his main divisions "Part I. Conceptions," "Part II. Judgements," and "Part III. Syllogism. Reasoning." However with regard to the role of Language in connection with thought the two authors took opposite sides—Whately declaring that language is essential, Thomson that it is not. Boole, in opening his development (Chapter II. Signs and their Laws) declares that the question as to whether language is essential for reasoning to be "beside the design of this treatise", since "whether we regard signs as the representative of the conceptions and operations of the human intellect, in studying the laws of signs, we are in effect studying the manifest laws of reasoning." (p. 24)[1].

And now Boole takes the step, unprecedented as far as was generally known, of viewing the various combination of concepts as having an algebraic character. He claims (p. 27):

PROPOSITION I

> All the operations of Language, as an instrument of reasoning may be conducted by a system of signs composed of the following elements, viz:
>
> 1-st. Literal symbols, as x, y, etc., representing things as subjects of our conceptions.
>
> 2-nd. Signs of operations, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.
>
> 3-rd. The sign of identity, $=$.
>
> And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

[1] Unless the context indicates otherwise, page number references without an accompanying source will be to Boole's *Laws of Thought*, i.e. to the item BOOLE 1854 (or the Dover reprint BOOLE 1951) in our Bibliography. The pagination of the Open Court edition, BOOLE 1916, is different, but does include in the text the original page numbers.

Let us first note, for future reference, (i) the sweeping generality of the claim that *all* uses of language as an instrument of reasoning can be carried out by his system of signs, (ii) the absence of the sign of division, and (iii) the only sign of relation referred to is that of identity.

Although in the 1st item Boole states that the literal symbols x, y, etc. are to represent *things* as subjects of our conceptions subsequent discussion shows him to be using these to stand either for *classes* or for *general terms* and, moreover, not maintaining a distinction between the two. Thus on p. 28 we find the statements:

> Let us agree to represent the class of individuals to which a particular name or description is applicable, by a single letter, as x.
>
> If the name is "men," for instance, let x represent "all men," or the class "men."
>
> Let it be further agreed, that by the combination xy shall be represented that class of things to which the names x and y are simultaneously applicable.

In the first of these statements x stands for a class determined by a name or description (i.e. by a general term); in the second he refers to "men" both as a name and as a class. In the third, x and y are names, but the combination xy is a class. Compounding the confusion is his referring, in the 2nd item of Proposition I, to the signs of operation $+$, $-$, $\times$ as standing for operations of the mind, not as operators on names or classes.

But Boole's lack of clarity on these matters was no hindrance to his developing a workable calculus since (a) he believed that the laws of signs are the laws of reasoning, whether one takes the signs introduced as representing [classes of] things and their relations, or as representing conceptions and operations of the mind, and (b) it is the case that the logic of general terms is formally indistinguishable from a calculus of classes. (See CARNAP 1958, § 28c, or QUINE 1972, § 20.) By virtue of (a) Boole can replace subjective psychological explanation by objective talk about classes or terms, and by (b) he can use linguistic practice in the use of general terms to guide him to logical laws. We thus find him, on the basis of a variety of arguments involving conceptions of the mind, linguistic practice with nouns and adjectives, and the extensions of terms (i.e. classes), concluding:

> We are permitted, therefore, to employ the symbols x, y, z, etc., in the place of substantives, adjectives, and descriptive phrases subject to the rule of interpretation, that any expression in which several of these symbols are written together shall represent all the objects or individuals to which their several meanings are together applicable, and to the law that the order in which the symbols succeed each other is indifferent. [pp. 29, 30]

Boole expresses the formal property of this juxtaposition operation by writing an algebraic equation

(1) $$xy = yx,$$

and stating that the expressions

(2) $$zxy,\ zyx,\ xyz, \text{ etc.}$$

all represent the same class. He says the law expressed by (1) "may be characterized by saying that the literal symbols x, y, z are *commutative, like the symbols of Algebra*," (note the attribution of the property to the operands rather than the operation) but makes no mention of associativity, apparently not realizing its formal necessity to connect the binary operation of (1) with the ternary operation of (2). By considering x and y as representing the same name or quality he concludes that xy and x have the same signification. Hence $xx = x$, or

(3) $$x^2 = x,$$

which is "the expression of a second law of those symbols by which names, qualities, or descriptions are symbolically represented". By referring to (3) as the *second* law ((1) being the first) Boole confirms our surmise that he is not cognizant of the need for associativity.

Next Boole considers "Signs of those mental operations whereby we collect parts into a whole, or separate a whole into its parts." The collecting of parts into a whole is indicated in language by the use of "and" or "or" between general terms ("men and women," "barren mountains, or fertile vales"). Though later on admitting that common usage favors the non-exclusive meaning he maintains: "In strictness, the words "and," "or," interposed between the terms descriptive of two or more classes of objects, imply that these classes are quite distinct, so that no member of one is found in another. In this and in all other respects the words "and" and "or" are analogous with the sign + in algebra, and their laws are identical." Symbolizing this aggregation of classes by the algebraic symbol he then has the properties

$$x + y = y + x \tag{4}$$

$$z(x + y) = zx + zy, \tag{5}$$

which, he says, "govern the use of the sign +." Again, as for multiplication, there is no mention of associativity.

It is not clear why Boole considers his restricted notion of aggregating classes—restricted to have meaning only for disjoint classes—and that of arithmetical addition, to behave analogously in all respects and to have idential laws since performability of arithmetical addition is unrestricted. It is of course true that the number of elements in the union of two (finite) classes is the sum of the number in the respective classes if and only if the two classes are disjoint, but this is a property of the union of classes, not of addition. Note moreover that (4) and (5), as statements of arithmetic laws, hold without restriction, i.e. x, y, and z can be the same or different numbers. Likewise, as we shall see, Boole will also be using them unrestrictedly in his system, i.e. without first establishing that the summands are disjoint.

As an operation in logic inverse to his $x + y$ Boole has a substraction, $x - y$, which is expressed in "common language" by the use of the word *except* (e.g. "All men, except Asiatics"). Here, as with his addition, subtraction is restricted, its use implying that the "things excepted form a part of the things from which they are excepted." On the basis of linguistic example ("excepting Asiatics, men") Boole concludes that

$$x - y = -y + x \tag{6}$$

without explaining what meaning '$-y$' could have so as to be capable of being added to the class x, nor of providing justification for interpreting the comma in his example (men, excepting Asiatics) as logical addition. Also, on the basis of linguistic example, he concludes that in logic

$$z(x - y) = zx - zy. \tag{7}$$

Boole next turns to the question of formation of propositions and argues that the only relation sign needed is the sign $=$. By way of illustration he converts the proposition "Caesar conquered the Gauls" to "Caesar is one who conquered the Gauls", that is an equation between an individual term and a definite description—a somewhat inappropriate example, as the latter form is not a relation between *classes* unless one identifies an individual with its unit class. Boole, of course, is not fully aware of the many meanings of the word "is" which have subsequently been distinguished and clarified in logic: identity between individuals, equality of classes, logical equivalence of general terms, inclusion of classes, subsumption of terms, and membership of an individual in a class.

Viewing the sign $=$ as standing for 'equals' Boole affirms the axiom that equals may be added or subtracted from equals yielding results that are equal. He believes that this suffices to justify transposition of terms in an equation. For a contemporary mathematician what is additionally needed is existence of an additive inverse and of a zero with the laws

$$A + (-A) = 0, \tag{8}$$

$$B + 0 = B, \tag{9}$$

as well as associativity of $+$. (Our use of capital letters is explained in § 1.4 below.)

Boole observes that in logic both sides of an equation can be multiplied by the same factor, but that from $zx = zy$ we cannot deduce $x = y$. There he says the analogy with "common algebra" breaks down, but contents himself with the fact that this law, i.e.

$$zx = zy \quad \text{implies} \quad x = y,$$

does not have the full generality of the laws heretofore discussed in that it does not hold unless "it is known that z is not equal to 0". But, it should be noted, even the correctly stated algebraic law for numbers,

$$z \neq 0 \quad \text{and} \quad zx = zy \quad \text{imply} \quad x = y,$$

still fails for class logic, i.e. is not a general truth. Ignoring or not realizing that there are such exceptions he observes that all the laws examined hold for numerical algebra, and that the special law $x^2 = x$ does if x is confined to the values 0 and 1, and hence concludes that it is not with "Number generally" that the "symbols of Logic" should be compared but with "symbols of quantity" admitting only the values 0 and 1. A basic principle of the work is then stated:

> Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes of such an algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established. [pp. 37–38].

Of interest here is Boole's first use, without comment, of the terms "an algebra" and "an Algebra of Logic" where heretofore he had used "a calculus" and "a Calculus of Logic". It signifies, we believe, a realization that what he had created was not simply another (mathematical) calculus as with operators (see § 1.0) but a formal structure, similar to but different from common algebra, with two distinct interpretations. That these interpretations are distinct is (or should have been) clear since the one interpretation has only two entities 0 and 1, and an adequate Algebra of Logic can't do with only two classes. He speaks of the two interpretations as having the same (formal) laws, axioms, and processes, but neither in Chapter II (Signs and Their Laws) nor in Chapter III (Derivation of the Laws) do we find these laws spelled out. However in Chapter I (Nature and Design of this Work) he asserts they are "... identical in form with the laws of the general symbols of Algebra, with this single addition, viz., that the symbols of Logic are further subject to a special law [i.e. $x^2 = x$] (Chap. II), to which the symbols of quantity, as such, are not subject." (p. 6). Exactly what these laws of the general symbols of Algebra are we are not sure of since neither Boole nor any of the Symbolical Algebraists have enumerated or described them in full. As we have already observed, some which should have been listed, e.g. associativity of $+$ and $\times$, and existence of an additive inverse, were not. Moreover, it isn't clear whether something like

(10) If $xy = xz$, then either $x = 0$ or $y = z$

(implying non-existence of divisors of zero) should, or should not, count as one of the laws of general symbols of Algebra—it is not a statement of "equivalent forms" in Peacock's sense, yet it is a truth of arithmetic stated in general symbols. Is it a truth of Symbolical Algebra? (Property (10) is, of course, not a law for classes.)

We should also note that although Boole's statement of general algebraic laws (e.g. communativity, distributivity) is in terms of his literal symbols x, y, z, etc., it is clear from what follows that he intends these laws ($x^2 = x$ excepted), as general laws of symbols, to be taken in full generality, i.e. allowing substitution of any algebraic expressions for these letters, whether or not the expressions are restricted to those having logical meaning. We resume discussion of this topic in § 1.4.

§ 1.3. Symbols of Logic and "operations of the mind". The fundamental law

In contrast to *Mathematical Analysis of Logic* where he decides not to enter into an analysis of the mental operations corresponding to his elective symbols, in *Laws ofThought* Boole does devote a portion of his Chapter III (Derivation of the Laws) to showing that the laws of the symbols of logic can be derived from those of the operations of the mind.

First he disavows any particular theory of the mind, arguing that if one succeeds in obtaining by observation the laws of thought then they have a real existence as laws of the mind independently of any "metaphysical" theory of the mind. However he will use the "language of common discourse" and talk about "ideas, or conceptions", which he assumes are communicated by words. He will also talk about the mind as having "powers or faculties" such as Attention, Conception or Imagination, etc., which he refers to as *Operations* of the mind, without thereby implying the "real existence" of such activity. An objective of his Chapter III is stated as Proposition I:

> To deduce the laws of the symbols of Logic from a consideration of those operations of the mind which are implied in the strict usage of language as an instrument of reasoning. [p. 42]

The argument for the Proposition is prefaced by a discussion of the notion of a universe of discourse, here taken not in an absolute sense as in *Mathematical Analysis of Logic*, but as one which may be limited in extent and may change with the context. This is the notion introduced by De Morgan in 1846 [De Morgan 1966, p. 2], though Boole makes no mention of it. Boole apparently finds it quite natural to take language usage as an indicator of what goes on in the mind. The connection between mind and language is established when he claims that the use of a word, e.g. *men*, "... directs us to select mentally from the Universe [in this example the "actual" one] all the beings to which the term 'men' is applicable; so the adjective 'good', in the combination 'good men' directs us still further to select mentally from the class *men* all those who posess the further quality 'good'. ..." He finds it "perfectly apparent" that if the mental selection is performed in either order the result is the same. Hence one has a law of the mind corresponding to the law of literal symbols $xy = yx$. The law $x^2 = x$ is discussed in similar fashion. We note here the reappearance from *Mathematical Analysis of Logic* of the selection operation idea, only now on the mental level and not symbolized—the x, y, z, are not "elective" symbols but names, or class symbols, which "direct the mind" to select the appropriate individuals from the universe.

The remainder of Chapter III is devoted to matters relating to the fundamental law $x^2 = x$, which for Boole characterizes the difference between the logical and quantitative use of symbols; the first of these concerns the meaning of symbols 0 and 1.

Since, as numbers, 0 and 1 satisfy the algebraic equation $x^2 = x$ it is natural to inquire whether these symbols can be given an interpretation in Logic. Boole explains that the grounds for the interpretation of particular symbols (i.e. symbols for constants) depends on the laws which they are "subject to" in a given system, that this interpretation is to be in conformity with the adopted interpretation of the given system, and that "there may be both propriety and advantage in employing the same symbols in different systems of thought, provided that such interpretations can be assigned to them as shall render their formal laws identical, and their use consistent." (p. 46) From the arithmetical formulas

(1) $$0y = 0$$

(2) $$1y = y$$

Boole infers that in logic 0 must represent Nothing and 1 the Universe, these being the only interpretations consistent with logical multiplication. Also, from his meaning assigned to subtraction, $1-x$ then represents the complementary class not-x. It is of interest to note that even before determining meanings for 0 and 1 in logic, Boole is assuming (p. 47, last line) that (1) and (2) are valid in logic. This accords with Peacock's Symbolical Algebra view that "interpretation will *follow*, and not *precede* the operations of [symbolical] algebra, and their results."

In connection with $1-x$ it is surprising that Boole never comments on the logical law of double negation which would be expressed by $1-(1-x) = x$. The first logical theorem he does derive is

$$x(1-x) = 0,$$

which comes from $x^2 = x$ by transposing x^2, replacing x by $x1$ and factoring. Boole is evidently quite pleased at having derived "that 'principle of contradiction' which Aristotle has described as the fundamental axiom of all philosophy" from his fundamental law of thought $x^2 = x$. Discussing this law leads Boole to an observation which, in view of the development of many-valued logic, some 70 years later, is surprisingly prophetic:

> ... I desire to direct attention also to the circumstance that the equation (1) [i.e. $x(1-x) = 0$] in which the fundamental law of

> thought is expressed is an equation of the second degree.* [Footnote below]. Without speculating at all in this Chapter upon the question, whether that the circumstance is necessary in its own nature, we may venture to assert that if it had not existed, the whole procedure of the understanding would have been different from what it is. Thus it is a consequence of the fact that the fundamental equation of thought is of the second degree, that we perform the operation of analysis and classification, by division into pairs of opposites, or, as it is technically said, by dichotomy. Now if the equation had been of the third degree, still admitting of interpretation as such, the mental division must have been threefold in character, and we must have proceeded by a species of trichotomy, the real nature of which it is impossible for us, with our existing faculties, adequately to conceive, but the laws of which we might still investigate as an object of intellectual speculation.

In a lengthy footnote to the remark that the fundamental law of logic is of the second degree we find Boole mentioning for the first time a feature of his approach to logic which gave rise to serious objections, namely the use of algebraic expressions having no meaning in logic. We state this footnote (omitting its last paragraph) and discuss it in detail.

> *[Boole's footnote] Should it here be said that the existence of the equation $x^2 = x$ necessitates also the existence of the equation $x^3 = x$, which is of the third degree, and then inquired whether the equation does not indicate a process of trichotomy; the answer is, that the equation $x^3 = x$ is not interpretable in the system of logic. For writing it in either of the forms
>
> $$x(1-x)(1+x) = 0, \tag{2}$$
>
> $$x(1-x)(-1-x) = 0, \tag{3}$$
>
> we see that its interpretation, if possible at all, must involve that of the factor $1+x$, or of the factor $-1-x$. The former is not interpretable, because we cannot conceive of the addition of any

class x to the universe 1; the latter is not interpretable, because the symbol -1 is not subject to the law $x(1-x) = 0$, to which all class symbols are subject. Hence the equation $x^3 = x$ admits of no interpretation analogous to that of the equation $x^2 = x$. Were the former equation, however, true independently of the latter, i.e. were that act of the mind which is denoted by the symbol x, such that its second repetition should reproduce the result of a single operation, but not its first or mere repetition, it is presumable that we should be able to interpret one of the forms (2), (3), which under the actual conditions of thought we cannot do. There exist operations, known to the mathematician, the law of which may be adequately expressed by the equation $x^3 = x$. But they are of a nature altogether foreign to the province of general reasoning.

As his factorizations of $x - x^3$ and $x^3 - x$ show, Boole's use of common algebra leads him to expressions, namely $1 + x$ and -1, which have no logical meaning. The reasons he gives for them not being interpretable is different in the two cases: in the case of $1 + x$ the reason given is interpretive ("we cannot conceive of the addition of any class x to the universe") and in the case of -1 the reason is formal ("-1 is not subject to the law $x(1-x) = 0$ [$x^2 = x$?], to which all class symbols are subject to") It has been pointed out by John Corcoran (letter to the author of 2 May 1981) that Boole's reason for rejecting $x^3 = x$ as having an interpretation in logic could equally well be applied to his $x^2 = x$. For by writing it in the form $x^2 - x = 0$ and factoring, it converts algebraically to

(3) $$x(x-1) = 0.$$

Since this form involves the factor $x - 1$, which for Boole would have no meaning in logic, we could then declare (3) and so $x^2 = x$ as not interpretable in logic. We don't know what Boole would have replied to this, but we do note that he had examined both of the factorizations, that of $x - x^3$ and $x^3 - x$, and in each case does obtain an objectionable factor. However, in the case of $x^2 = x$ there is at least one form in which all factors have logical meaning. It is possible that considerations such as those occurring in his footnote, and not the appealing philosophical notion of dichotomy, are what may have led Boole to consistently write his fundamental law as $x^2 = x$ or $x(1-x) = 0$ in *Laws of Thought*, shunning the algebraically equivalent $x^n = x$ which he used in *Mathematical Analysis of Logic*. We shall resume discussion of the topic of interpretability in § 1.7 and in § 2.4 below.

Whether or not Boole had explored the consequences of assuming that -1 and $1+x$ *did* satisfy the fundamental law we don't know, but if he had he would have found that

$$\begin{aligned}(-1)^2 = -1 \rightarrow 1 &= -1\\ &\rightarrow 1+1 = 0\\ &\rightarrow x+x = 0, \quad \text{for any } x\end{aligned}$$

and

$$\begin{aligned}(1+x)^2 = 1+x &\rightarrow 1+x+x+x^2 = 1+x\\ &\rightarrow 1+x+x+x = 1+x\\ &\rightarrow x+x = 0,\end{aligned}$$

with both sets of implications reversible. Thus having -1 and $1+x$ satisfying $x^2 = x$ is equivalent to having $x + x = 0$. With his fixed idea of $+$ having to stand for the aggregation of disjoint classes Boole would certainly have found having $x + x = 0$ equally as inconceivable as adding a class x to the universe. From our present day vantage point it is easy for us to see that if Boole had only thought of representing symmetric difference of classes by addition—for which it is true that $x + x = 0$ and for which all the usual elementary properties of addition, including having an inverse, hold—he would have had an algebra which is a Boolean ring with unit, a structure equivalent to Boolean algebra in which all problems of class logic can be handled without encountering "uninterpretables". But one can hardly fault him for this since it wasn't until the 20th century that the equivalence of the two kinds of algebras was noticed (For a proof see STONE 1936).

Although Boole did not explicitly consider the matter, it is clear that he would have rejected $1 + 1 = 0$ as it implies $(-1)^2 = -1$, which he did reject. Likewise he would have certainly rejected $1 + 1 = 1$ since by subtraction of 1 it yields $1 = 0$. Thus for Boole addition was not a closed operation in the set $\{0, 1\}$.

§ 1.4. Boole's algebra of $+$, $-$, $\times$, 0, 1

Before continuing with our exposition of Boole's system for doing logic we wish to formally codify the algebraic laws which he had so far committed himself to, some explicitly, others implicitly.

Boole uses the literal symbols x, y, z, etc. in a two-fold capacity which are not clearly distinguished by him, namely as symbols standing for (arbitrary) classes with their special properties and, also, as algebraic variables in terms of which general algebraic properties for all "symbols" (e.g. commutativity, distributivity) are expressed. With these literals, and the two constants 0 and 1, algebraic expressions are constructed using the operations $+$, $-$, $\times$. However, only some of these expressions (e.g. xy, $1 - x$) are taken to represent classes while others either do not (e.g. -1, $1 + 1$) or else might not (e.g. $x + y$, $x - y$). This distinction is of course important since only *classes* are subject to the law $x^2 = x$. When Boole says "Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone" he does not thereby imply that 0 and 1 are the only values which can result since, as we have seen, having $1 + 1$ as 0 or as 1 is excluded. Thus there are legitimately constructed expressions (e.g. $1 + 1$) which x, y, z do not admit as substitutable values. In order then to state the laws which do hold in general we need to introduce a new set of variables since x, y, z, ... are (informally) restricted in what may be substituted for them. For such general variables we use capital letters A, B, C, ..., adopting the customary algebraic principle that properly constructed expressions are freely substitutable for these general variables in algebraic theorems. The substitutable expressions here are, of course, those built up from Boole's literal symbols x, y, z, ..., the constants 0 and 1, and the operations $+$, $-$, $\times$.

In terms of these new variables we can state the general algebraic principles. From an examination of Boole's practice in *Laws of Thought* it is readily seen that they include, in the first instance, properties we nowadays ascribe to a commutative ring with unit (§ 0.4). These include $0 \neq 1$, which Boole never states but clearly intends. Also we find him cancelling "definite" numerical factors, as in going from $2xy = 0$ to $xy = 0$. To formalize this one needs to add the law $A + A = 0 \rightarrow A = 0$. We believe this is now a complete list. For convenience we spell the list out:

BOOLE'S ALGEBRA OF $+, -, \times, 0, 1$

$$A + B = B + A \qquad AB = BA$$

$$(A + B) + C = A + (B + C) \qquad (AB)C = A(BC)$$

$$A + 0 = A \qquad A \cdot 1 = A$$

$$A + (-A) = 0$$

$$A(B + C) = AB + AC$$

$$x^2 = x$$

$$0 \neq 1$$

$$A + A = 0 \rightarrow A = 0$$

In the case of the fundamental law $x^2 = x$ it is assumed that any other "literal" $y, z, \ldots$ may replace x. No special mention of 0 and 1 is needed here since from the ring properties one can derive $0^2 = 0$ and $1^2 = 1$. With the exception of the last two—which Boole never explicitly recognized—all the laws are in the form of equations, accounting perhaps for Boole's belief that the only relation he needed was $=$.

§ 1.5. Primary propositions and class terms

We now describe how Boole rendered propositions of ordinary language into his symbolism.

Chapter IV (Division of Propositions) begins by classifying all propositions as *primary* or *secondary*, primary propositions being those which "express a relation among things" while secondary are those which "express a relation among propositions". This division of Boole's corresponds to the traditional one into categorical and hypothetical but, in keeping with his broadened view of logic, is more generally conceived. From the modern point of view Boole's doctrine of primary and secondary propositions has serious conceptual flaws but these need not detain us as there were no deleterious consequences. (Propositional logic is discussed in Chapter 3.) In this section our interest is in the logical form in which he casts primary propositions.

For Boole primary propositions have the general form of an equation between terms whose construction, unlike the subject and predicate of traditional logic, is more fully prescribed. Though not explicitly saying so he seems to believe that classes are defined in only two ways, namely by giving either (i) "the names or qualities common to all individuals which it contains" or (ii) "the different portions ... defined by different properties, names or attributes...." In the first case the symbolic expression is a concatenation of the names or qualities, e.g. xyz, $xy(1-z)$, etc., and in the second case there is a + sign connecting such expressions (or, in the case of exclusion from a class, a subtraction sign). Boole refers to expressions of the first kind as "class terms", in evident analogy with the mathematical use of "algebraic terms". In symbolizing disjunctively given classes Boole requires that the expressions explicitly indicate that the portions are disjoint. Hence 'x's or y's is to be rendered either as '$x(1-y)+y(1-x)$' or as '$x+(1-x)y$' according as 'or' is taken in the exclusive or inclusive sense. He remarks that each of these expressions he describes (i.e. xyz, $xy(1-x)$, $x(1-y)+y(1-x)$, $x+y(1-x)$) satisfies the fundamental law $x^2 = x$, and that this is characteristic of expressions representing classes, though without giving a general proof. It is convenient to have a name for any such expression—the designation *(logical) class term* seems to us particularly appropriate and, since Boole doesn't make much use of his so-named narrower notion (i.e. for a monomial term), we shall henceforth adopt it in this wider sense, that is for any polynomial expression P in his symbolism satisfying the law $P^2 = P$. When necessary we shall refer to Boole's monomial term as an *algebraic class term.*

Boole's contention is that primary propositions are of three kinds, each of the form of *an equation between (logical) class terms.* He accepts and uses the traditional terminology of categorical propositions, but deftly modifies the analysis of such propositions to his ends. An affirmative proposition is always of the form subject-copula-predicate, with the subject and predicate terms "understood to be universal or particular, i.e. whether we speak of the whole of that collection of objects to which a term refers, or indefinitely of the whole or of a part of it, the usual signification of the prefix 'some'."

The first of the three kinds of primary propositions is that in which the subject and predicate are both universal. To render this type into symbols Boole's rule is: form the separate expressions for the subject and predicate and connect them by the sign =. As an illustration he uses the definition of Wealth (due to the economist N.W. Senior): "Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventative of pain." Setting

w = wealth

t = things transferable

s = limited in supply

p = productive of pleasure

r = preventative of pain

he then has the equation

$$w = st\{p + r(1 - p)\}$$

or

$$w = st\{p(1 - r) + r(1 - p)\}$$

depending on whether one intends the 'or' of the definition to be taken in the non-exclusive or exclusive sense. Boole asks the question what if [in disregard of the requirement that + connect disjoint terms] one writes the definition as

$$w = st(p + r).$$

His answer is that it would be equivalent to the second of the above equations for w with the added implication that stp and str are disjoint. Foreshadowing later development contained in his Chapter VI he says:

"How the full import of any equation may be determined will be explained hereafter". By an 'equation' he means to include even those constructed without the restrictions on $+$ (or on $-$) needed for interpretability.

The second type of categorical proposition is the affirmative one in which the predicate is particular, as in 'all men are mortal' which Boole expands to 'All men are some mortal beings'. To render this Boole introduces a special symbol v, "a class indefinite in every respect but this, viz., that some of its members are mortal beings, and let x stand for 'mortal beings', then will vx represent 'some mortal beings'. Hence if y represent men, the equation sought will be

$$y = vx."$$

Concerning the symbol v Boole says: "It is obvious that v is a symbol of the same kind as x, y, etc., and that it is subject to the general law

$$v^2 = v, \quad \text{or} \quad v(1 - v) = 0."$$

The symbol v is also used by him to express a proposition with a particular subject (the third type) such as 'Some men are not wise'. With y standing for 'men' and x for 'wise beings' he writes the proposition as

$$vy = v(1 - x)$$

"introducing v as the symbol of a class indefinite in all respects but this; that it contains some individuals of the class to whose expression it is prefixed,..."

This device of Boole's for expressing particularity (i.e. some) by use of his special symbol v was adversely criticized by later writers. We postpone a discussion of the topic until our § 1.10.

In sum, if X and Y symbolize the terms of a primary proposition, then such a proposition will be of one of the three forms

$$X = vY$$

$$X = Y$$

$$vX = vY,$$

with v as described above. Boole remarks that the terms X and Y "if founded upon a sufficiently careful analysis of the meanings of the 'terms' of the proposition will satisfy the fundamental law of duality which requires that we have

$$X^2 = X \quad \text{or} \quad X(1-X) = 0,$$

$$Y^2 = Y \quad \text{or} \quad Y(1-Y) = 0."$$

By a "sufficiently careful analysis" we take it that Boole means expressing terms so that $+$ occurs only between disjoint class terms. One can formally prove this on the basis of the algebraic properties listed in § 1.4:

THEOREM. *If T is a term (of a proposition) constructed from literals x, y, z, ..., the operation of subtraction from* 1, *multiplication, and addition, and such that the sign $+$ stands only between disjoint class terms then T is a class term.*

PROOF. By induction on the structure of T using as a basis for the induction that literals are class terms, and then the following three results about class terms X and Y:

(1) $$(1-X)^2 = 1 - X - X + X^2 = 1 - X - X + X = 1 - X.$$

(2) $$(XY)^2 = X^2Y^2 = XY.$$

(3) $$(X+Y)^2 = X + Y \leftrightarrow X^2 + XY + XY + Y^2 = X + Y$$
$$\leftrightarrow XY + XY = 0$$
$$\leftrightarrow XY = 0.$$

One readily sees that

$$XY = 0 \rightarrow X + Y = X(1-Y) + Y(1-X) = X + (1-X)Y,$$

showing that for disjoint operands Boole's addition is equivalent to either symmetric difference or union. Also easily established is the fact that $X - Y$ is a class term if and only if $XY = Y$. Thus the above theorem can be taken so as to include subtraction if the subtrahend class is contained in the minuend class.

§ 1.6. Principles of symbolical reasoning. Development

If his rules are followed Boole's class terms and the primary propositions containing them are always logically interpretable. Nevertheless believing that he has to have the same laws as, and the same algorithmic freedom of, numerical algebra in order to conduct inferences from such propositions, Boole is compelled to consider the problem of what to do about expressions that arise and procedures carried on to which no logical significance can be given. Chapters V and VI of *Laws of Thought* are largely devoted to this problem.

In the opening paragraphs of Chapter V he acknowledges that the laws for addition were determined from the study of examples in which the summands were mutually exclusive. Even so he maintains that one need not restrict oneself, in the application of these laws, to these circumstances under which the knowledge of the laws was obtained—"If such restriction is necessary, it is manifest that no such thing as a general method in Logic is possible." However the problem is not one peculiar to logic but "to every developed form of human reasoning which is based upon the employment of symbolical language." According to Boole, as long as the symbols have a fixed interpretation, the laws of which have been correctly determined, and if the formal processes of solution and demonstration have been carried out in "obedience" of these laws, and if the end results are interpretable, then there can be no question as to the validity of the conclusion, even though in some of the intermediate steps there may be uninterpretable expressions or processes. This principle, he contends, rests on a general law of the mind by which the general principle is clearly manifested in the particular instance. He cites the example of the use of the imaginary $\sqrt{-1}$ in the "intermediary processes" of trigonometry and firmly asserts that no explanation (i.e. of the use of $\sqrt{-1}$ in trigonometry) can be given which does not "correctly assume the very principle in question". Boole could also have cited the example of solving an algebraic problem for an unknown natural number, the condition for which involves only natural numbers, and to which one applies the full algorithmic power of algebra without regard to whether the operations are meaningful (in terms of natural numbers), and in the course of which proper fractions or roots may appear; if the end result gives the unknown as a natural number, then one is satisfied that the solution is correct.

To a modern mathematician, however, the correctness of the procedure cited by Boole is justified by an appeal, not to a principle of the mind, but to the embeddability of the natural numbers in the reals or, in the trigonometric example, of the reals in the complex numbers (see § 0.3). And, in all such cases, one deals with specific structures (natural numbers, reals, complex numbers) and not with abstract axiomatic formalizations. In Boole's case, however, he is starting out with what roughly corresponds to a formal system (though never explicitly formulated) and then comes out with a "partial" interpretation. We intend to show in our Chapter 2 that Boole's procedures can be justified by contemporary standards, namely by having, for a formal system which codifies his algebraic usage, suitable structures or models in which there are entities for all terms (hence no uninterpretable expressions) and in which ordinary class calculus is embedded. But for the moment we consider only his presentation.

Having argued that the laws governing his logical symbols x, y, z, ... and the operations on them are, when interpretation is possible, "subject to laws identical in form with the laws of a system of quantitative symbols, susceptible only of the values 0 and 1 ...", Boole concludes that one can ignore questions of interpretation and operate with algebraic freedom:

> We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols susceptible only of the values 0 and 1; perform upon them as such all the requisite processes of solution; and finally restore them to their logical interpretation. [p. 70]

When Boole says here "all the requisite processes of solution"he is including division, an operation not mentioned so far in *Laws of Thought*.

However, unlike for the mathematical examples we have cited in which, when there is a solution, the end result is in interpretable form, the algebraic processes which Boole uses do not always lead to interpretable results in the original sense but only after the introduction of a special transformation involving the notion of the "development" (or "expansion") of a function, and which cannot be fully justified by present day standards because of, among other things, the occurrence of fractions with 0 denominator.

Boole uses the word 'function' not in its contemporary sense but rather to denote an algebraic expression, and, from examples, we see that what he has in mind is a rational fractional form (indicated quotient of two polynomials) in which, since he will only be concerned with "logical functions", the variables will appear therein at most to the first power. In the examples he gives of functions, e.g. x, $1-x$, $(1+x)/(1-x)$ for $f(x)$ and $x+y$, $x-2y$, $(x+y)/(x-2y)$ for $f(x,y)$, we encounter algebraic fractions for the first time in *Laws of Thought*.

A function $f(x)$ in which x is a logical symbol is said by Boole to be *developed* "when it is reduced to the form $ax+b(1-x)$, a and b [independent of x and] determined so as to make the result equivalent to the function from which it was derived." Boole does not say what he means by a function being "equivalent to" another, but we shall understand it to imply that the functions are equal for all possible assignments of 0's and 1's to the logical symbols. To show that any function $f(x)$ can be developed he *assumes*

(1) $$f(x) = ax + b(1-x)$$

and determines, by substituting $x = 1$ and $x = 0$, that $a = f(1)$ and $b = f(0)$, where $f(1)$ and $f(0)$ are the expressions resulting from $f(x)$ when x is replaced by 1 and 0. But as the equation

(2) $$f(x) = f(1)x + f(0)(1-x)$$

is true for the only two values with x admits, Boole conclude that the right-hand member "adequately represents the function $f(x)$" and that the assumption of (1) is thereby justified.

Boole's argument here lacks cogency for a number of reasons. Since the notion of development is important to his method we discuss the argument in some detail.

Firstly, in concluding that the laws governing class symbols are identical in form with those of quantitative symbols limited to the values

0 and 1, he did so without examining any laws involving fractions. Possibly Boole thought this unnecessary on the basis that all [equational] statements about division could be expressed as statements about multiplication, division being inverse to multiplication. But before one can legitimately operate with division the existence of a multiplicative inverse has to be established, either by postulate or from other properties. But clearly if x is a class (other than the universe) there is no class y such that $xy = 1$. If multiplicative inverses were available one could prove

$$\text{if} \quad xy = xz, \quad \text{then} \quad x = 0 \quad \text{or} \quad y = z,$$

which, as observed earlier, is not true for classes.

Secondly, the expressions $f(1)$ and $f(0)$ could involve 0 denominators which, to a modern mathematician, renders the expression meaningless in "quantitative" algebra.

Finally, Boole's argument is faulty in that he has given no adequate justification for assumption (1). Thus all he has shown is that if (1), then (2).

One should contrast Boole's "proof" with that of the corresponding result for standard Boolean algebra (§ 0.5) where one first *proves* that any $f(x)$ can be converted by identities of the algebra to the form $ax + b(1 - x)$, and *then* determines a and b by substituting 1 and 0 for x.

As the expansion for $f(x, y)$ with respect to x and y Boole gives

$$(3) \qquad f(x, y) = f(1, 1)xy + f(1, 0)x(1 - y) + f(0, 1)(1 - x)y + f(0, 0)(1 - x)(1 - y),$$

and similarly for any number of logical symbols x, y, z, etc. That portion of the product in each term involving only the variables is called a constituent of the expansion (e.g. here xy, $(1 - x)y$, $(1 - y)x$, $(1 - x)(1 - y)$ in the example). Boole readily shows (Props. III and IV. pp. 78–79 that, for any given expansion, the sum of any number of [distinct] constituents satisfies the fundamental law of logic, that the product of any two distinct constituents is 0, and that the sum of all is 1. Among the examples of expansions which Boole give are these:

$$\frac{1+x}{1+2x} = \tfrac{2}{3}x + (1-x)$$

$$\frac{1-x}{1-y} = \tfrac{0}{0}xy + 0x(1-y) + \tfrac{1}{0}(1-x)y + (1-x)(1-y).$$

From the first one we have by "cross-multiplication"

$$1 + x = (1 + 2x)(\tfrac{2}{3}x + 1 - x),$$

and if the right-hand side is multiplied out, simplified by "common" algebra, and x^2 replaced by x, the equation becomes an identity. However an attempt at a similar treatment for the second one of the examples results in nothing sensible. In general the connection between multiplication and division of ordinary algebra doesn't carry over for expansions, and Boole carefully avoids using it. Likewise the coefficients $\frac{0}{0}$ and $\frac{1}{0}$ are never used by him in an arithmetic sense but only for his special purposes to be presently explained.

Nevertheless, Boole asserts that the expansion theorem (e.g. as in (3)) is "perfectly true and intelligible" when x and y are limited to being quantitative symbols 0 and 1 [and the operations taken arithmetically] and hence may be "intelligibly employed in any stage of the process of analysis intermediate between the change of interpretation of the symbols from the logical to the quantitative system referred to and the final restoration of the logical interpretation [not as yet explained]." On the other hand, he remarks, if $f(x\,y)$ is such as to represent a *class* or *collection of things* then the right-hand side of (3) is always logically interpretable. Hence if the theorem is used for non-interpretable functions "it must be understood that x and y are quantitative and of the particular species referred to, ..." Although expansions are not necessarily interpretable it will, he claims, lead us to interpretable results. His Chapter VI is devoted to showing this.

§ 1.7. Interpretation

The terms 'logical equation' and logical function' are used by Boole to denote any equation or function arising either in expressing premises by equations or in the process of (algebraically) deriving other equations, irrespective of whether the symbols x, y, z, etc. are taken as logical or as "quantitative". Boole devotes considerable attention to the task of logical interpretation of functions and equations. Since he never raises the question, he is apparently taking it for granted that the results of algebraic transformations or derivations preserve logical content.

Proposition I in Chapter VI (Of Interpretation) establishes that the constituents of any logical function are always interpretable and represent a mutually exclusive and exhaustive classification of the universe of discourse "formed by predication and denial in every possible way of the qualities denoted by x, y, z, etc." The problem of the interpretation of logical equations is then taken up. It is asserted that, while functions may not be interpretable, equations always are. A key feature of such interpretation is the manner in which coefficients in an expansion modify the interpretation of constituents which they affect.

First equations of the form $V = 0$ are considered, and for the special case of a V involving the symbols x, y, z, etc. "in combinations which are not fractional". If, for simplicity, it is assumed that x and y are the only symbols present, then the development of the equation $V = 0$ will be of the form

$$(1) \qquad axy + bx(1-y) + c(1-x)y + d(1-x)(1-y) = 0,$$

where a, b, c and d are "definite numerical constants". (Note the tacit inference from '$V = 0$' to 'expansion of $V = 0$'). Now if in (1) any coefficient, say a, is not 0 then by multiplying the equation through by xy and making use of the fact that the product of distinct constituents is 0, one obtains

$$axy = 0$$

so that, "as a does not vanish",

$$xy = 0.$$

And if "a does vanish, the term xy does not appear ... and, therefore, the equation $xy = 0$ cannot thence be deduced." Boole then states the general rule that [for V's of the special kind under consideration] the interpretation of $V = 0$ consists of the collective assertion of the equations $t_i = 0$ for all constitutents t_i in the development of V whose coefficient is not 0. Boole's conclusion, i.e. that the $t_i = 0$, can be justified by having the general property

$$nA = 0 \rightarrow A = 0, \quad n \text{ a natural number,}$$

or, simply

$$A + A = 0 \rightarrow A = 0,$$

since for V's under consideration (no fractional forms) only integer coefficients could arise.

After a brief discussion of the similar special case of $V = 1$, V again not containing x, y, z, etc. in fractional forms, Boole turns to the really important case of $w = V$, where w is a logical symbol and V is no longer restricted but may involve fractional forms. As a simple illustration of how such an equation and the need for interpretation arises, he takes the "definition of 'clean beasts' as laid down in the Jewish law, viz., 'clean beasts are those which both divide the hoof and chew the cud'", written in symbols as

(2) $$x = yz$$

(x = clean beasts, y = beast dividing the hoof, z = beasts chewing the cud) and the question is posed of determining "the relation in which 'beasts chewing the cud' stand to 'clean beasts' and 'beasts dividing the hoof'". This requires determining z as an *interpretable* function of x and y. Treating (2) as if it were an equation in ordinary algebra and solving for z yields

(3) $$z = \frac{x}{y},$$

an equation which is not in interpretable form. Boole claims: "If we càn reduce it to such a form it will furnish the relation required." He does this by replacing (3) by its "developed form"

(4) $$z = xy + \tfrac{1}{0}x(1-y) + 0(1-x)y + \tfrac{0}{0}(1-x)(1-y)$$

which, he will show, has the interpretation

Beasts which chew the cud [z] consists of all clean beasts (which

> also divide the hoof) [xy] together with an indefinite remainder (some, none, or all) [indicated by $\frac{0}{0}$] of unclean beasts which do not divide the hoof [$(1-x)(1-y)$].

It is also implied by (4) [presence of $\frac{1}{0}$] that clean beasts which do not divide the hoof [$x(1-y)$] do not exist.

This example illustrates a particular case of what Boole calls a "problem of the utmost generality in Logic." which he formulates as:

> Given any logical equation connecting the symbols x, y, z, [etc.], w, required an interpretable expression for the relation of the class represented by w to the classes represented by the other symbols x, y, z, etc. [p. 87].

In outline his solution runs as follows.

Since the logical equation under consideration is always of the first degree in w, it can be solved for w in terms of the other symbols; or, more generally, by development with respect to w the equation can be put in the form

$$Ew + E'(1-w) = 0,$$

where E and E' do not involve w; whence, by solving for w,

$$w = \frac{E'}{E'-E}.$$

Boole cautions against cancelling common factors of the numerator and denominator "unless they are mere numerical constants [$\neq 0$]". The next step in his procedure is to replace $E'/(E'-E)$ by its development with respect to x, y, z, etc. and to give the resulting equation an interpretation. Boole does this dividing the coefficients appearing in the development into the four categories (i) 1, (ii) 0, (iii) $\frac{0}{0}$, and (iv) all others (including $\frac{1}{0}$). The appearance of the coefficient 1 in the development means to take all of the class of which it is the coefficient, 0 take none of it, $\frac{0}{0}$ take an "indefinite portion of the class, i.e. some, none, or all", and, for those of the fourth class, the constituents having such coefficients are separately set equal to 0. Boole's reasons for so interpreting the developed equation, though ingenious, are far from adequate. For example, to justify the treatment for the fourth class he proves the theorem (p. 90) that if a coefficient, a, does not satisfy the law $a^2 = a$, then its corresponding constituent is to be set equal to 0. The proof comes from considering the equation in the form

$$w = a_1 t_1 + a_2 t_2 + \cdots + a_i t_i, \ldots,$$

where the a_i's are the coefficients and the t_i's the constituents, then squaring to obtain

$$w^2 = a_1^2 t_1 + a_2^2 t_2 + \cdots,$$

so that by subtraction

$$0 = (a_1^2 - a_1)t_1 + (a_2^2 - a_2)t_2 + \cdots.$$

If now a coefficient, a_j say, does not satisfy the fundamental law, then by multiplying the last of these equations through by t_j one obtains

$$(a_j^2 - a_j)t_j = 0; \quad \text{whence} \quad t_j = 0.$$

This argument of Boole's assumes (i) that functions are equal to their developments, (ii) that various algebraic processes (e.g. squaring) can be validly applied to developments, and (iii) that $a_j^2 - a_j$ is not a divisor of 0. Do the coefficients $\frac{0}{0}$ and $\frac{1}{0}$ which arise and play an important role satisfy the fundamental law? Boole circumspectly avoids stating the question arithmetically (i.e. are the equations $(\frac{0}{0})^2 = \frac{0}{0}$ and $(\frac{1}{0})^2 = \frac{1}{0}$ true?) In the case of $\frac{1}{0}$ he gives an oblique answer: "This is the algebraic symbol of infinity. Now the nearer any number approaches to infinity (allowing such an expression), the more it does depart from the condition of satisfying the fundamental law above referred to." And this is all he says—apparently leaving it up to the reader to make the inference that $\frac{1}{0}$ doesn't satisfy the law.

He also has trouble with $\frac{0}{0}$:

> The symbol $\frac{0}{0}$, whose interpretation was previously discussed, does not necessarily disobey the law we are here considering, for it admits of the numerical values 0 and 1 indifferently. Its actual interpretation, however, as an indefinite class symbol, cannot, I

> conceive, except upon the ground of analogy, be deduced from its arithmetical properties, but must be established experimentally [pp. 91–92].

The experimental grounds referred to consists of the examination of specific instances such as the equation

$$yx = y \tag{5}$$

(y = men, x = mortal beings) from which information about x *in* relation to y is desired. Boole says division cannot be *performed* in the case of logical symbols—he means that the y in (5) cannot be cancelled, or divided out. "Our resource, then, is to *express* the operation and develop the result..." Thus

$$\begin{aligned} x &= \frac{y}{y} \\ &= y + \tfrac{0}{0}(1-y). \end{aligned}$$

Hence, in terms of the meanings for x and y, asking what needs to be added to "men" (y) to produce "mortal beings" (x) leads him to conclude that the prefixed $\frac{0}{0}$ indicates taking an indefinite remainder (some, none, or all) of "not-men".

Resuming the discussion of the general case we see that Boole's method leads from the given equation to a solution for w which is of the form (using $\frac{1}{0}$ as typical of the fourth kind of coefficient)

$$w = 1A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D, \tag{6}$$

where A, B, C and D are the sums of the constituents having the indicated coefficient, and this equation is then given the interpretation

$$w = A + vC, \tag{7}$$

$$D = 0, \tag{8}$$

where v is an "indefinite class symbol". Of this Boole says:

> The interpretation of (7) [i.e., $w = A + vC$] shows what elements enter, or may enter, into the composition of w, the class

of things whose definition is required; and the interpretation of (8) [i.e. $D = 0$] shows what relations exist among the elements of the original problem in perfect independent of w.

We shall be discussing this result in our Chapter 2.

As a corollary to this result Boole establishes that if a logical function V is "independently interpretable" then it will satisfy $V(1 - V) = 0$.

> By an independently interpretable logical function, I mean one which is interpretable, without presupposing any relation among the things represented by the symbols which it involves. Thus $x(1 - y)$ is independently interpretable, but $x - y$ is not so. The latter function presupposes, as a condition of its interpretation, that the class represented by y is wholly contained in the class represented by x; the former function does not imply any such requirement.

The corollary then follows by noting that in the expansion of V there will be no constituents in the fourth category, so that all of its coefficients, and hence the entire sum, will satisfy the fundamental law. The theorem and the remarks at the end of § 1.5 show that the corollary can be directly established without appeal to the result of interpreting an expansion.

It should be noted that Boole gives logical interpretation to equations when algebraically transformable to the specific forms $V = 0$ and $V = 1$, V not containing division, and to $V = w$, V a quotient of polynomials and w a class symbol. Since equations of the latter kind, i.e. $P/Q = w$, can be written $P - wQ = 0$, the question of the consistency of the two kinds of interpretation arises. Boole never considered this nor, for that matter, does he show that the interpreting equations (7) and (8) imply (algebraically) the original equation from which they were obtained.

§ 1.8. Elimination. Reduction

In order to complete his method so as to have an effective calculus for reasoning on classes (and propositions) Boole requires two additional results. The first of these is a technique for eliminating unwanted logical symbols, that is symbols that are not to appear in the conclusion—for example the middle term of a syllogism, assuming that the traditional argument is expressed in class calculus terms. Boole will also be using this technique to eliminate his indefinite class symbol v. The second of the needed results is a way of reducing a system of equations to a single equation, which can then be "interpreted" as described in our preceeding section.

Boole considers it a remarkable fact that in logical algebra, in contrast to the "common" algebra, any number of class symbols can be eliminated independently of the number of equations. He shows (in three different ways!) that the results of eliminating x from $f(x) = 0$ is given by $f(1)f(0) = 0$.

Of the three proofs, all of which begin by replacing $f(x) = 0$ by

$$f(1)x + f(0)(1-x) = 0, \tag{1}$$

the first and third involve division. The second does not, but appeals to an elimination result from ordinary algebra. A fourth, a simpler and more natural derivation, can be obtained by multiplying (1) through first by $f(0)x$, then by $f(1)(1-x)$, deleting terms having $x(1-x)$ as a factor, replacing x^2 by x, $(1-x)^2$ by $(1-x)$ and adding the two equations to obtain

$$f(1)f(0)(x + (1-x)) = 0,$$

and thence $f(1)f(0) = 0$.

Boole refers to $f(1)f(0) = 0$ as "the complete result of the elimination of x" from the equation $f(x) = 0$. It isn't clear what he means by "complete". He has only shown that $f(1)f(0) = 0$ is an algebraic consequence of $f(x) = 0$, x being a logical symbol. The comment is made that just as in common algebra on carrying out an elimination one might obtain the identity $0 = 0$, indicating no independent relation connecting the remaining symbols. But nothing is said about the possibility of arriving at $1 = 0$ (or $a = 0$ for some a not 0) which would indicate that there is no x such that $f(x) = 0$. In standard Boolean algebra $f(1)f(0) = 0$ is the necessary and sufficient condition that there be an x such that $f(x) = 0$. (Theorem 0.55). We see no way that Boole could prove the sufficiency of the condition $f(1)f(0) = 0$ without assuming that $f(x)$ satisfied the fundamental law.

Boole gives many examples illustrating the use of elimination, the first being of special interest as it involves eliminating the indefinite symbol v, used by him to express inclusion and particularity ("some"). As we have seen Boole writes "All men are (some) mortal beings" as

$$y = vx \tag{2}$$

where v is "a class indefinite in every respect but this, viz., that some of its members are mortal beings,..." (p. 61). We read this as implying that Boole is taking the universal affirmative proposition as having existential import; for if the class x ($=$ mortal beings) is non-empty then so necessarily is vx, i.e. y, since it is being supposed that *some* of v's members are mortal beings. Rewriting (2) as $y - vx = 0$, and eliminating v by rule, gives

$$y(y-x) = 0$$

which, on multiplying out and replacing y^2 by y, yields

$$y(1-x) = 0. \tag{3}$$

Are (2) and (3) equivalent? Boole apparently thinks so: "It will generally be the most convenient course, in treatment of propositions, to eliminate first the indefinite class symbol v, wherever it occurs in the corresponding equations. This would only modify their form, without impairing their significance" (p. 105). We shall return to this topic in § 1.10.

The remaining examples in Boole's Chapter VII, which are devoted to illustrating elimination, are based on the equation $w = st\{p + r(1-p)\}$ (Senior's definition of Wealth) and are more or less routine. However in connection of one of these examples Boole remarks:

> 17. In the last example of elimination, we have eliminated the compound symbol *st* from the given equation, by treating it as a single symbol. The same method is applicable to any combination of symbols which satisfies the fundamental law of individual

> symbols. Thus the expression $p+r-pr$ will, on being multiplied by itself, reproduce itself, so that if we represent $p+r-pr$ by a single symbol as y, we shall have the fundamental law obeyed, the equation
>
> $$y = y^2, \quad \text{or} \quad y(1-y) = 0,$$
>
> being satisfied. For the rule of elimination ... [p. 112].

What is noteworthy here is Boole's explicit recognition that the compound $p+r-pr$ (expressing the non-exclusive "p or r") satisfied his fundamental law and hence is independently interpretable. Unlike Jevons and Peirce (see § 1.11), Boole apparently never realized that this notion could be used to develop a calculus of logic.

We turn now to the second of the two techniques which Boole needs to complete his method, namely that of reducing a system of logical equations to one equation so that the methods developed for a single equation can then be applied.

First he shows that if the equations are $V_1 = 0$, $V_2 = 0$, etc., and if in the developments of V_1, V_2, etc., there are only positive coefficients, then the single equation

$$V_1 + V_2 + \cdots = 0$$

has the same logical content as the system of equations, since they both furnish the same set of constituents with positive coefficients—this argument appeals to his earlier result that the logical interpretation of such an equation is given by the collective assertion that each such constituent is 0. If an equation $V = 0$ has an expansion with both positive and negative coefficients then its interpretation also consists of setting each constituent having such a coefficient equal to 0. However adding the left-hand side of expansions in a system $V_1 = 0$, $V_2 = 0$, etc., may cause constituents to drop out through cancellation of like positive and negative terms. To prevent this Boole goes over to the square of the V's and shows (Prop. III, p. 121) that the single equation

$$V_1^2 + V_2^2 + \cdots = 0$$

has the same interpretation as the system $V_1 = 0$, $V_2 = 0$, etc. The proof he gives depends on comparing the expansion of a $V = 0$, namely

$$a_1t_1 + a_2t_2 + \cdots = 0,$$

where the t_i's are the constituents and the a_i's are "numerical" coefficients, with that of $V^2 = 0$, namely

$$a_1^2t_1^2 + a_2^2t_2^2 + \cdots = 0,$$

and observing that the non-vanishing coefficients are the same for both. The argument assumes that $(at)^2 = a^2t^2$ for "numerical" a, and that $a^2 = 0 \leftrightarrow a = 0$.

Equations reduced to the form $V = 0$ in which $V^2 = V$ are clearly advantageous since no preliminary squaring is needed. Moreover equations derived from them by elimination of symbols likewise have the property and can be combined by addition (Prop. IV, p. 122). Accordingly Boole shows how to reduce each of his three general types of propositions to such a form. In the case of $X = vY$ (recall that such X and Y are assumed to satisfy the fundamental law) he eliminates v to obtain

$$X(1 - Y) = 0.$$

For $X = Y$ he transposes and squares obtaining

$$X - 2XY + Y = 0$$

i.e.

$$X(1 - Y) + Y(1 - X) = 0.$$

In the case of $vX = vY$ he says "but v is not quite arbitrary, and therefore must not be eliminated. For v is the representative of *some*, which, though it may include in its meaning *all*, does not include *none*. We must therefore transpose the second member to the first side, and square the resulting equation according to the rule". This gives

$$vX(1 - Y) + vY(1 - X) = 0.$$

Note that Boole is treating the v occurring in $vX = vY$ differently from the v in $X = vY$ where he does allow its elimination, thus implying that the universal affirmative is then taken without existential import, contrary to what he seems to be saying when he first describes it (§ 1.5 above).

A brief comment is made that if one doesn't attend to its "real meaning" a proposition could be represented by an equation in which terms may not satisfy the fundamental law—for example rendering Senior's definition of wealth by

$$w = st\{p + r\},$$

in which + is used for 'or'. "Such equations, however, as it has been seen, have a meaning. Should it, for curiosity, or any other motive, be determined to employ them, it will be best to reduce them by the Rule (VI.5) [reduce to the form $V = 0$ and equate each constituent having a non-zero coefficient to 0]." Thus, significantly, though expressions may have no logical interpretation, equations always do. With regard to interpretations for equations Boole neglects to consider the special case of equations for which, when reduced to the form $V = 0$, there are no non-zero coefficients in the expansion. This happens, e.g., in the case of the fundamental law $x = x^2$, which becomes $x - x^2 = 0$ and has the expansion

$$0 \cdot x + 0 \cdot (1 - x) = 0.$$

In such cases it would be appropriate to say that the equation has an *empty* interpretation, rather than *no* interpretation.

§ 1.9. Abbreviation. Perfection of method

In view of the complexity of Boole's machinery it is not surprising that, when solving class calculus problems is the sole object, considerable simplification or, as Boole terms it, abbreviation can be introduced. Limiting himself to algebraic operations on polynomials with (algebraic) class terms having positive coefficients he shows that an equation $V = 0$ has the same logical interpretation as one obtained from it by deleting any term having another as a factor – i.e. by replacing a part $x + xE$ by x—and that in such equations any positive coeffient can be replaced by 1; also in simplifying algebraic operations the product $(x + P)(x + Q)$

can be replaced by $x + PQ$ and $(1 + P)Q$ can be replaced by Q. Hence in such contexts Boole has in effect the simplifying equations

$$x + xP = x$$

$$(x + P)(x + Q) = x + PQ$$

$$(1 + \mathrm{P})\mathrm{Q} = Q$$

just as if his + were class union.

Boole also has a theorem whose use can simplify the calculation of resultants, i.e. equations obtained as a result of eliminating one or more class symbols. His statement and proof (Prop. III, p. 133) are somewhat obscure and we shall paraphrase the proposition as follows:

Consider an equation

$$(1) \qquad \phi(x, y, z, \ldots, s, t) = 0$$

whose expansion with respect to t is

$$\phi(x, y, z, \ldots, s, 1)t + \phi(x, y, z, \ldots, s, 0)(1 - t) = 0.$$

Let

$$\phi(x, y, z, \ldots, s, 1) = 0 \text{ have the resultant } E = 0$$

and

$$\phi(x, y, z, \ldots, s, 0) = 0 \text{ have the resultant } E' = 0,$$

both with respect to the same subset of the variables $x, y, z, \ldots, s$. Then the resultant of (1) with respect to the same subset of $x, y, z, \ldots, s$ is equivalent to the equation

$$Et + E'(1 - t) = 0.$$

By this theorem the calculation of a resultant function (to be equated to 0) can be replaced by that of the calculation of two resultant functions for equations obtained from the original one by replacing therein any one of the variables not being eliminated first by 1, and then by 0. Judicious choice of the variable can reduce computation.

The example in which Boole uses this result (Ex. 2, p. 138) is of interest for another, and more important, idea: Given Senior's definition of wealth, i.e.

(2) $$w = st\{p+r(1-p)\},$$

Boole asks what is the relation of things transferable and productive of pleasure (i.e. tp) to the remaining elements w, s, r of the definition? Boole considers this to be a problem "which cannot be fully answered, except in connection with the theory of systems of equations". His procedure is to introduce a new symbol y, write in place of (2) the system of equations

(3) $$\begin{aligned} w &= st\{p+r(1-p)\} \\ y &= tp \end{aligned}$$

eliminate from these t and p, and solve the resulting equation for y in terms of w, s, and r. Development and interpretation then complete the solution.

By considering not (2) but any equation $\phi(x, y, \ldots, z, w, \ldots) = 0$ and any class expression $\psi(x, y, \ldots)$ involving some of the symbols of the equation Boole then formulates (p. 140) the

General Problem

> Given any equation $[\phi(x, y, \ldots, z, w, \ldots) = 0]$ connecting the symbols $x, y, \ldots, w, z, \ldots$
>
> Required to determine the logical expression of any class $[\psi(x, y, \ldots)]$ expressed in any way by the symbols $x, y, \ldots$ in terms of the remaining symbols w, z, etc.

We shall not bother stating Boole's Rule (p. 142) for obtaining the answer to the general problem, as the idea is easily apprehended from the special example just discussed. Nor shall we present the several examples, all rather contrived, by which the Rule is illustrated. However we point out that being able to solve this type of problem is an important component in Boole's treatment of probability, which we shall be discussing in our Chapter 4.

Concerning the method of solution, Boole says (p. 146): "In none of the above examples has it been my object to exhibit in any special manner the power of the method. That, I conceive, can only be fully displayed in connection with the mathematical theory of probabilities."

However he doesn't forego displaying some of the capability of his system and, as final example of his chapter on Methods of Abbreviation, poses a problem whose statement involves 5 class symbols and three premises, each of which involves all 5 of them, and produces various conclusions from these premises (Ex. 5, pp. 146–149).

While evidently pleased with the power and generality of his methods one gets the impression that Boole would be happier of they were more direct. He devotes Chapter X (Of the Conditions of a Perfect Method) to describing what he considers "perfection of method", and how to attain it from equations (premises) not meeting the criteria, by reformulation of such premises. Foremost for Boole is the requisite that expressions should satisfy the fundamental law $x(1-x) = 0$. His Proposition I of Chapter X provides a rule for "reducing" any equation between polynomials on class terms to one of the form $V = 0$ with V satisfying the fundamental law: *transpose all terms to the left-hand side of the equation, develop the expression so obtained with respect to the class symbols present, and replace all non-zero coefficients by* 1. The result will be a sum of constituents equated to 0 and hence of the requisite form.

Boole remarks that fully developing expressions leads to expressions of "great length" and offers practical advice on circumventing the need for doing so. Thus for the "three great forms",

$$X = vY$$

$$X = Y$$

$$vX = vY,$$

he has already noted that if X and Y satisfy the fundamental law then so do the left-hand sides of his equivalent forms,

$$X(1-Y) = 0$$

$$X(1-Y) + Y(1-X) = 0$$

$$v\{X(1-Y) + Y(1-X)\} = 0,$$

and hence there would be not need for "reducing" such equations. However adding equations to obtain a simple equivalent one can lose the property; e.g. the equations $x = 0$, $y = 0$ when added give $x + y = 0$.

In general if $v = 0$, $v' = 0$, $v'' = 0, \ldots$ are the equations with $v, v', v'', \ldots$ satisfying the fundamental law then Boole recommends not

$$v + v' + v'' + \cdots = 0 \tag{4}$$

but rather

$$v + (1 - v)v' + (1 - v)(1 - v')v'' + \cdots = 0, \tag{5}$$

which, in effect, is forming the class union of the expressions $v, v', v'', \ldots$ rather than aggregating them with his $+$. Boole's trouble here is, of course, the possible occurrence of a common constituent in the different equations, but which occasions no difficulty when class union is the combining operation.

Equations $V = 0$ with $V(1 - V) = 0$ have a special property. Boole notes (Prop. II, p. 155) that if from such an equation a logical symbol is solved and the expression developed, then the only coefficients that can appear are 1, 0, $\frac{0}{0}$, and $\frac{1}{0}$; i.e. the only fourth type of coefficient being $\frac{1}{0}$. The result is evident since in the solution for the symbol the indicated quotient is, effectively, a quotient of constituents. Hence when 0 and 1 are substituted for the logical symbols only 0 and 1 can results as values. As a consequence we find Boole almost always writing a fourth type coefficient as $\frac{1}{0}$ even if the actual algebra results in some other value.

§ 1.10. Treatment of "some". Aristotelian logic

In keeping with his algebraic-equational conceptions of logic Boole believed that all primary propositions, that is propositions expressing relationships among things, are representable in the form of equations between class terms. Included among the primary propositions are the four categorical forms of the traditional logic. Here Boole is confronted with the notion of particularly, i.e. the use of "some", and his treatment of this notion was justifiably criticized by later logicians. (E.g. PEIRCE 1870 = 1933 III, pp. 90–91, and VENN 1894, Chapter VII)

No doubt influenced by the then current interest in quantification of the predicate Boole renders the universal affirmative, e.g. "All men are mortal," as "All men are some mortal beings" and symbolizes it as

$$(1) \qquad y = vx,$$

with 'vx' representing 'some mortal beings' hence implying non-emptyness of the subject, i.e. taking the proposition as having existential import. However, as we have seen in § 1.8, he considers the result of eliminating the indefinite class symbol v, namely

$$(2) \qquad y(1-x) = 0,$$

to have the same significance as that of the form with v. If we try to recover (1) from (2) (using Boole's techniques) we find we don't quite make it. Solving equation (2) for y and developing the result gives

$$y = \frac{0}{1-x}$$
$$= 0 \cdot (1-x) + \tfrac{0}{0}x,$$

which by Boole's rule is written

$$y = vx.$$

But this is not the same as (1) since here v is replacing $\frac{0}{0}$, which means "take some, none, or all" and hence this occurrence of v no longer carries existential import. Although *stating* universal affirmatives in the form (1) Boole's practice is to always eliminate the v and to work with form (2). Thus he effectively treats "all y's are x's" in contemporary fashion as being without existential import.

As one might imagine, Boole's ideas on existential import are somewhat off the track. As we have seen in § 1.7, for Boole the logical interpretation of an equation $V = 0$ consists in the system of equations $t_i = 0$, where the t_i are those constituents of V's expansion having nonvanishing coefficients. Hence (p. 84): "Every primary proposition can be resolved into a series of denials of the existence of certain defined classes of things, and may from that system of denials, be reconstructed." Continuing, he says,

> It might here be asked, how it is possible to make an assertative proposition out of a series of denials or negations? From what source is the positive element derived? I answer, that the mind

> assumes the existence of a universe not *a priori* as a fact independent of experience, but either *a posteriori* as a deduction from experience, or *hypothetically* as a foundation of the possibility of assertive reasoning. Thus from the Proposition, "There are no men who are not fallible," which is a negation or denial of the existence of "infallible men," it may be inferred either hypothetically, "All men (if men exist) are fallible," or absolutely, (experience having assured us of the existence of the race), "All men are fallible."

Thus by relegating the presence or absence of the premise "men exist" to the mind Boole can have his universal affirmative both with and without existential import. Boole is driven to this non-formal maneuver by the failure of his system to include any formal means of expressing existence (or denial of non-existence).

By, so to speak, backing into treating the universal categorical proposition as not having existential import Boole avoids trouble with the indefinite class symbol v. But in the case of the particular categorical there is no easy way out. As we have seen, Boole writes for "Some x's are y's"

$$vx = vy,$$

"introducing v as the symbol of a class indefinite in all respects but this, that it contains some individuals of the class to whose expression it is prefixed." Does this character attributed to the symbol v persist during algebraic manipulations? Is the following a correct derivation:

$$vx = vy$$

$$v - vx = v - vy$$

$$v(1-x) = v(1-y)?$$

If so we then have the invalid inference of "Some not-x's are not-y's" from "Some x's are y's". Although Boole allows—even requires—the elimination of v from $y = vx$, here in the case of $vx = vy$ he says: "...v is not quite arbitrary, and therefore must not be eliminated for v is the representative of *some*, which, though it may include in its meaning *all*, does not include *none*." To eliminate v from equation $vx = vy$ would result in the contentless conclusion $0 = 0$. So we find Boole treating the v in $y = vx$ differently from that in $vx = vy$, writing these respectively as

$$x(1-y) = 0$$

and

$$v\{x(1-y) + y(1-x)\} = 0,$$

with the *ad hoc* proviso that v not be eliminated in the latter.

It is only natural that Boole should want to compare his system with the traditional logic. This he does in Chapter XV, The Aristotelian Logic And Its Modern Extensions, Examined By The Method Of This Treatise. What the "modern extensions" amount to is essentially the introduction (due to De Morgan) of negative terms on an equal footing with positive terms in the categorical forms, resulting in an expansion of the four *A*, *E*, *I*, *O* forms to eight.

Boole expresses these as

1. All *Y*'s are *X*'s, $y = vx,$
1. No *Y*'s are *X*'s, $y = v(1-x)$
3. Some *Y*'s are *X*'s, $vy = vx,$
4. Some *Y*'s are not *X*'s $vy = v(1-x)$

with 5.–8. the same as 1.–4. but having $1 - y$ in place of y—all these coming under Boole's great primary forms. Boole shows how readily the traditional conversion rules follow. As would be expected, he is not consistent with regard to the existential content of v. Thus he goes from $y = vx$ to $y(1-x) = 0$ and then by this method to

$$1 - x = \frac{0}{y}$$

$$= \tfrac{0}{0}(1-y),$$

thus justifying "negative conversion" or contraposition. In this last equation the $\frac{0}{0}$ is allowed to include in its meaning "none." Hence the use of v in $y = vx$ should not entail existential import, otherwise $1 - x = \frac{0}{0}(1 - y)$ would not be a proper contrapositive. Yet from $y = vx$ he also infers, by multiplication with v, that $vx = vy$ (p. 229). So here the conclusion "Some x's are y's" does require the v to carry existential import.

Syllogistic inference is, for Boole, a special instance of his general method of reduction of systems of equations. Rather than treating cases individually Boole investigates, in characteristically mathematical fashion, the question of syllogistic rules by considering the several forms

$$(1) \qquad \begin{cases} vx = v'y \\ wz = w'y, \end{cases} \qquad \begin{cases} vx = v'y \\ wz = w'(1 - y), \end{cases}$$

with four parameters v, v', w, w' and, after eliminating the middle term y, solving by his method for x, for $1 - x$, and for vx. For each of these he obtains the general solution in terms of z, $1 - z$, and the parameters v, v', w, w'. Imposing the conditions that not both z and $1 - z$ should appear and that not both v and v' (or w and w') should be 1, he obtains the various syllogistic rules by appropriate specification of the parameters. Boole does not neglect to point out how much more general his methods are and that not all inference need be syllogistic.

It is of interest to note that from the systems (1) Boole only solves for x, $1 - x$, and vx, but not $v(1 - x)$. He says (p. 233): "The form $v(1 - x)$ is excluded, inasmuch as we cannot from the interpretation vx = Some X's, given in the premises, interpret $v(1 - x)$ as Some not-X's. The symbol v, when used in the sense of "some," applies to that term only with which it is connected in the premises." In other words the meaning of v, depending as it does on the context, has to be kept in mind. Thus in this regard Boole's system abrogates accepted canons for being a formal system.

Boole does not neglect to point out, in opposition to such weighty authorities as Archbishop Whately, J. S. Mill and Kant, the insufficiency of syllogistic in comparison with his general methods. Ironically, his system, lacking for one thing quantifiers, likewise falls short of sufficiency.

§ 1.11. De Morgan. Jevons, Peirce, Macfarlane, Venn

Appreciation of Boole's innovations in logic can be enhanced by a comparison with those of his most notable contemporary in the field, Augustus De Morgan. Like Boole De Morgan was a mathematician but, unlike Boole, especially interested in the foundations of algebra. His series of articles on this topic in the Transactions of the Cambridge Philosophical Society (1841–1847) counts as an important contribution to the beginnings of modern abstract algebra. He was in the forefront of those envisioning new algebras.

Clearly the idea of a mathematical treatment of logic was "in the air". We have but to note the title of Boole's pamphlet, *The Mathematical Analysis of Logic, Being an Essay towards a Calculus of Deductive Reasoning*, and that of De Morgan's book *Formal Logic: or, The Calculus of Inference, Necessary and Probable*, both appearing, according to De Morgan (see BOOLE 1952, p. 167n), on the same day in the fall of 1847. The two works are remarkably different. With Boole the algebraic character of logical combinations is primary, and the logic of the syllogism ancillary; with De Morgan the traditional logic, while considerably clarified, modified and generalized, is still the central core. From De Morgan's voluminous (and often discursive) writings on logic we select a number of items to describe, arranging these chronologically so as to see what interaction there was (if any) between him and Boole.

We begin with "On the Syllogism I" [DE MORGAN 1846 = 1966, pp. 1–21] which appeared a year before his *Formal Logic*. In it he uses letters X, Y, Z, etc. to stand for arbitrary general terms or names "which it is lawful to apply to any one of a collection of objects of thought: and in the language of Aristotle, that name may be *predicated* of each of these objects." Thus although the idea of a collection or class is in the background, De Morgan's X, Y, Z, etc. actually correspond to our present day conception of a (one-place) predicate rather than to the classes determined by them. He introduces an entirely new notion into logic, that of the "universe of a proposition, or of a name," which he says need not be taken as all possible conceptions, but "may be limited in any manner expressed or understood." Relative to the universe of a proposition, and differing with Aristotle, De Morgan allows formation of a "contrary" (= negative) of any name: if X is a name then x denotes its contrary, but neither is the positive or negative term except correlatively—as with a and $-a$ in algebra. For the traditional categorical forms A, O, E, I De Morgan uses the notation $X)Y$, $X:Y$, $X.Y$, XY. The symbols are borrowed from algebra but are neither analogous nor suggestive, except that the dot and juxtaposition are used for the convertible forms E and I, his reason being that the corresponding algebraic notions are convertible (i.e. commutative). Allowing the use of capital and small letters as terms in the four traditional fundamental forms plus the two converted forms $Y)X$ and $Y:X$, increases their number to twenty-four but, being equivalent in threes, only eight distinct forms. The various resulting syllogisms, their transformations, and their structural interrelations are investigated in detail and minutely tabulated. He uses surprisingly inappropriate algebraic notation to designated inferences; e.g. for the universal affirmative syllogism he writes

$$X)Y + Y)Z = X)Z \tag{1}$$

or, when the weaker particular affirmative conclusion is taken,

$$X)Y + Y)Z > XZ.$$

This notation is criticized by De Morgan himself in a later work [1850 = 1966, p. 87].

The question of existential import, though not named, comes up for discussion. He argues that for the syllogism (1) to be valid the middle term Y must exist. "The terms of the conclusion may be conditional: but inference requires that the middle term should be unconditional. Every X (if ever X existed) is Y; every Y is Z (if ever Z existed): therefore every X (if ever X existed) is Z (if ever Z existed). This is a good syllogism but Y here is absolute." In converting (1) to contrapositive form one needs the same absolute existence for y as for Y. The possibility of empty terms is mentioned in one sentence: "There is an extreme case; y may not exist, that is Y may contain the universe; but then [referring to (1)] Y and Z are identical, and the conclusion $X)Z$ is identical with $X)Y$ and $z)x$ contains nothing [has no content?]."

This first paper of De Morgan's is also of interest for its introduction of the idea of definite (numerical) quantity associated with terms of categorical propositions, generalizing an aspect of the quantifiers "some" and "all". An *addition* to the paper, dated 27 February 1847, states that he has found that "the whole theory of the syllogism might be deduced from the consideration of propositions in a form in which *definite quantity* of assertion is given both to the subject and the predicate of a proposition." In the *Addition* he also alludes to the possibility that he has been anticipated in this by Sir William Hamilton of Edinburgh though, as it later turned out, this was based on a minsunderstanding. Nevertheless, the subsequent acrimonious priority dispute which erupted stimulated Boole to resume his "almost-forgotten thread of former inquiries," resulting in his writing *Mathematical Analysis of Logic* within a few weeks. According to his wife Boole had thought of a mathematical treatment of logic when he was 17 (i.e. in 1832).

In *Mathematical Analysis of Logic*, although there is mention of De Morgan's paper, Boole attributes nothing to De Morgan except for a minor syllogistic matter (Boole 1847, p. 82). His use of 1 for a fixed universe of all conceivable objects contrasts with De Morgan's changeable, not necessarily unlimited, but unsymbolized, universe of a proposition. However by 1854, in *Laws of Thought*, Boole has switched over to a changeable, possibly limited, universe of discourse, again designated by the symbol 1, but with no mention of De Morgan. Having a symbol for the universe is, of course, a matter of no small import. For example with De Morgan that the contrary of the contrary of x is x has to be understood from the assigned meanings, whereas for Boole $1-(1-x)=x$ follows algebraically. Neither in *Mathematical Analysis of Logic* nor in *Laws of Thought* do we find Boole aware of problems associated with empty terms in the traditional categorical forms, even though an essential part of his method involves substituting 1 and 0 in for logical symbols.

De Morgan's *Formal Logic* [1847] includes the material of the paper we have just summarized, and considerably more. In it we find, for example, what turned into De Morgan's finest contribution to logic, i.e. logical relation theory, being foreshadowed by his abstract treatment of the copula, "conceived as obeying only these conditions necessary to inference", and by his contention that the inference "man is animal, therefore the head of a man is the head of an animal" is not justifiable syllogistically. A large part of the book is nonetheless devoted to a highly elaborate and intricate theory of the (more or less) traditional syllogism, of which we here consider but two aspects of interest to our study: the treatment of empty and universal names, and of compound names.

De Morgan stipulates that [in general] no name shall "fill the universe" or, by implication, be empty though "Nothing is more easy than to treat the supposition of a name being the universe as an extreme case." He asserts (p. 111) that if X exists and Y does not then 'All X is Y' is false and that 'All X is y' is true. But "If neither X nor Y exist, I will not so far imitate some of the questions of the schools as to attempt to settle what nonexisting things agree or disagree." Further, if Y exists but not X then 'all y is x' is true but not 'All X is Y', "for when x is, as here, the whole universe, the proof of $y)x = X)Y$ fails to present intelligible ideas, that is, fails to be a proof." Thus for De Morgan contraposition (or negative conversion) fails for "extreme" terms.

Turning to his discussion of compound names we find him using, where P, Q, and R are names, the juxtaposition 'PQR' as a name for "everything which is all three", and for what is either (one or more of them) the list of the three separated by commas: 'P, Q, R'. Thus De Morgan takes 'or' in the non-exclusive sense. Then to symbolize 'P and Q, or R' he writes 'PQ, R'. The contrary of PQR is p, q, r and that of P, Q, R is pqr (the so-called "De Morgan laws"). With these notions De Morgan deduces,e.g., that $X)P$ and $Y)Q$ give $XY)PQ$, as follows (p. 118):

$$XY)X + X)P = XY)P$$

$$XY)Y + Y)Q = XY)Q,$$

whence by the *conjunctive postulate* [i.e. $X)P + X)Q = X)PQ$] he has $XY)PQ$. We observe that the additional premises $XY)X$ and $XY)X$ are used by him without formal justification. By contrast Boole would consider the inference a case of reducing a system of equations

$$x = vp$$

$$y = v'q$$

(expressing the premises) to a single equation giving xy in terms of p and q. In this particular case the result comes easily by "multiplying" the equations so as to yield

$$xy = vv'pq.$$

Applying here the general rules of *Laws of Thought* to carry out the inference is quite tedious and involves (i) adjoining the additional equation $z = xy$, (ii) converting all three equations to the form $V = 0$, V satisfying the fundamental law, (iii) adding the equations, eliminating all but z, p, and q, and simplyfying, so as to obtain

$$z(1 - p + 1 - q) = 0.$$

Finally (iv) solving for z and expanding:

$$z = \frac{0}{1 - p + 1 - q} = 0\{p(1-q) + q(1-p)\} + \tfrac{0}{0}pq,$$

whence $z(= xy) = vpq$.

The idea of attaching a definite numerical quantity to terms of a proposition which De Morgan mentions in his addition to [1846] is elaborated in *Formal Logic* into a Chapter VIII, On the Numerically definite Syllogism. Briefly, the scheme assumes that there are fixed (finite) numbers of X's, Y's, and Z's in the universe as well as a total for the universe. The categorical forms may now have numerical quantities assigned to these terms; e.g. he writes mXY to denote that mX's (or more) are among the Y's, the quantifier 'm (or more)' generalizing 'some,' which is one or more. Similarly $mX:nY$ denotes that no one of the X's are among any of nY's. De Morgan makes a detailed case by case study of valid syllogistic forms, i.e. inferences involving three terms X, Y, Z having, respectively ξ, η, ζ individuals which they "name" in a universe of v individuals. Results of the following nature are obtained: from premises mXY and nYZ no inference is possible if $m + n \leq \eta$, but if $m + n > \eta$ then one can infer $(m + n - \eta)XZ$. As we shall see in § 1.12 Boole picks up this idea of definite numerical quantity attached to terms, frees it from De Morgan's syllogistic setting and elaborates a general treatment appropriate to his more general form of term logic.

In his *English Encyclopedia* article of 1860 on Logic De Morgan [1966, p. 256] praises Boole's "generalization of the forms of logic ... by far the boldest and most original of those which we have to treat." He illustrates Boole's ideas by showing how $(1 - A)(1 - B) = 0$ ("to be both not-A and not-B is impossible") is converted by common algebra to $A + B - AB = 1$ ("everything is either A or B or both"). Interestingly, he does not give Boole's partially defined meaning for $+$ but says "Let $A + B$ represent a class containing both A and B, with all the common parts, if any, counted twice," (This item was pointed out to me by John Corcoran.)

De Morgan was the only contemporary of Boole who, by virtue of common interests in symbolical algebra and logic, could have had substantial interaction with Boole. Despite their being correspondent friends there is little evidence of such interaction except, perhaps, for the paper which Boole wrote on propositions numerically definite—one which he never published and apparently had lost track of. (See § 1.12).

In the remainder of this section we present summaries of the work of several logicians who came after Boole but whose ideas had some immediate historical connection.

W. S. Jevons

Jevons, who had been a student of De Morgan's, published the first substantial criticism and revision of Boole's system. Appearing 10 years after the publication of the *Laws of Thought*, Jevon's *Pure Logic or the Logic of Quality apart from Quantity* [1864] contains a system of (term) logic which its author describes as "founded on that of Professor Boole", but, in marked contrast, uses only "processes of self-evident meaning and force". Explicit critical comments on Boole appear in a concluding chapter but the body of the work, in which it was shown that logical inferences of the kind Boole concerned himself with can be carried out without appeal to "dark and symbolic processes", in itself constituted a strong and eventually decisive argument against this aspect of Boole's system—by the end of the century, of all the leading logicians, only Venn was an adherent of the Boolean method.

Before explaining its principal features and summarizing its criticisms of Boole we should mention that Jevon's work was not without its share of misconceptions. For example, he believed that for simplicity and generality logic needed to be founded intensionally, i.e. on qualities or attributes rather than extensionally as classes. A proposition had to express sameness or equivalence of meaning, otherwise it was "imperfect". (Hence, as with Boole, his system is an equational one.) Negative propositions can only arise by changing a term, on one side only, to its contrary. (He follows De Morgan's practice of indicating a term and its contrary by using the same letter, capital and small.) The inference from $AB = AC$ to AB not $= Ac$, which is invalid for empty class A, is allowed since A is (informally) excluded from being contradictory. ("Any term not known to be contradictory must be taken as not contradictory.") The symbol 0 combined with (attached to ?) any term is to denote "that this term is contradictory, and thus excluded from thought". He then writes $Aa = Aa \cdot 0$, and abbreviates this to $Aa = 0$. Although 0 is not a term he writes expressions such as '$0 + 0$' and '$0 + A$'. All in all, compared with Boole there is a marked regression in standards of formality.

We turn to the positive aspects of Jevon's work. Like Boole he symbolizes the combination of terms by multiplication, this operation having its customary logical properties (i.e. $AB = BA$, $AA = A$, etc.). Negative or contrary terms, symbolized by the corresponding lower case letters, are however introduced *per se* and not, as with Boole, by subtraction from 1—neither subtraction nor 1 appearing in Jevon's system. But most conspicuous of all is the change from Boole's $x + y$ with its conditional meaning (i.e. having meaning only if x and y are disjoint) to Jevons' $A + B$ in which the '+' has the meaning of non-exclusive 'or', so that $A + B$ is meaningful under all circumstances. This results in laws like

$$A + A = A \quad \text{and} \quad A = A + AB,$$

which are not present in Boole's system. With these two changes, i.e. the introduction of negative, i.e. complementary, terms and the non-exclusive 'or', Jevons can express with his notation any "independently interpretable" class expression of Boole's system. Moreover Jevons shows that logical inference on terms can be carried out so that at no stage does any expression appear which transcends the bounds of his notation—hence all expressions remain interpretable. For example, to determine a class expression for a given term involved in a set of premises (= Boole's problem of solving a set of logical equations for a given variable) Jevons expands the given term (by the Law of Development) into a sum of all possible combinations on the set of terms and their contraries involved in the premisses (i.e. the sum of all possible constituents) and deletes from this sum those combinations implied to be contradictory by the premisses—e.g. if $A = B$ is a premiss then aB can be deleted since when $A = B$ is multiplied through by a the equation becomes $aA = aB$, with aA contradictory, and hence $aB = 0$. In place of Boole's algebraic-algorithmic solution of equations using subtraction, division and expansion into the form $1A+0B+\frac{0}{0}C+\frac{1}{0}D$, Jevons has a set of rules for operating with his equations. While the rules are not algebraic in the common sense they are evidently algorithmic, requiring no insight or understanding—a fact which is clearly borne out by Jevon's later construction of a mechanical contrivance for performing the operations (JEVONS 1870).

The objections to Boole's logical system were grouped under four headings. In the first Jevons contends that Boole's symbols were not the "names of symbols of common discourse". In particular Jevons disputed Boole's claim that 'or' could not be interposed between terms (classes) unless they were disjoint. (He cites examples from Aristotle, Shakespeare, Milton, Tennyson, Darwin, and Boole himself!) Jevon's second objection is that there are no such operations as addition and subtraction in logic—meaning operations with the same properties as in numerical algebra—these operations only being possible on sums of mutually exclusive alternatives. Thirdly, Boole's system is "inconsistent with the self-evident law of thought, the Law of Unity ($A+A = A$)". Finally, he argues "*that the symbols* $\frac{1}{1}$, $\frac{0}{0}$, $\frac{0}{1}$, $\frac{1}{0}$ *establish for themselves no logical meaning, and only have meaning derived from some method of reasoning not contained in the symbolic system*".

Although Boole died the year in which Jevons' [1864] appeared, we nevertheless have a pretty good idea of what he thought of Jevon's innovations from an exchange of letters between them in the latter half of 1863, extensive excerpts of which are contained in JOURDAIN 1913. A letter to Boole from Jevons with his criticism of Boole's system elicited a reply from Boole who said that with suitable added restrictions it was possible to work out logical problems such that all intermediate results would be interpretable: "I have somewhere laid by a paper on this subject which I wrote about two years ago. I cannot at this moment put my hand upon it, but I remember that it involved the application of such restrictions as should make the elementary operations always interpretable in ordinary language. Thus $x - vy = 0$ would become $x - xy = 0$, and so on. But I did this, not because I had any doubt of the validity of the processes of my work, which are unrestricted by any such conditions, but in order to determine for my then satisfaction and prospectively with a view to publication, how far my system could be made intelligible to those who knew nothing of mathematics."

Under direct questioning from Jevons Boole flatly asserted that $x + x = x$ was not true in logic, that Jevons was in error in trying to interpret expressions, rather than equations: "Thus the equation $x + x = 0$ is equivalent to the equation $x = 0$; but the expression $x + x$ is not equivalent to the expression x. Your principle of unity is not applicable to *expressions*. Apparently it didn't occur to either of them that they were talking about different notions symbolized by the same symbol. Both of them were certainly aware that the non-exclusive 'or' was expressible in Boole's calculus and, as we have pointed out in our § 1.1, Boole did know that $x + y - xy$ satisfied his fundamental law (i.e. was independently interpretable) but it never occurred to him that, considered as an operation on classes x and y it could be identified with Jevons' $+$.

C. S. Peirce

During the approximately half-century during which Peirce worked on logic his ideas underwent considerable evolvement, going far beyond Boole—and most other logicians—in extent and depth. We confine our attention here only to parts of his early writings that bear directly on our subject.

Appearing a few years after Jevons 1864, though independent of it, Peirce's "On an Improvement in Boole's Calculus of Logic" [1867], likewise replaces Boole's addition by non-exclusive logical addition. However, unlike Jevons, Peirce retains (in this article) many of the questionable features of Boole's calculus such as uninterpretables and indeterminates. His symbols include $+$, $\times$, $-$, $\div$, $=$, which are used with class symbols both in an arithmetic and a logical sense, the latter use being indicated by a comma under the symbol. Both meanings can appear in the same expression. He writes, for example,

$$1. \quad \text{If No } a \text{ is } b \qquad a \underset{,}{+} b \underset{,}{=} a + b$$

but with no prior explanation of the arithmetic $+$ (without the comma) occurring between *class* symbols. In contrast to Jevons, who says very little about the algebraic characteristics of logical operations, Peirce explicitly notes that logical addition and multiplication are "commutative and associative", and that these operations are "doubly [i.e. reciprocally] distributive". Logical subtraction is introduced simply by the stipulation

$$(10) \qquad \text{If} \quad b \underset{,}{+} x \underset{,}{=} a \qquad x \underset{,}{=} a \underset{,}{-} b$$

and it is noted that x is not completely determinate but may vary from a to a with b taken away. This latter minimum values is denoted by $a - b$. However, "if the sphere of b reaches at all beyond a, the expression $a \underset{,}{-} b$ is uninterpretable". Following Boole he uses v for a "wholly indeterminate class" but differs from him in not using a single symbol for uninterpretability but a special symbol in each case; e.g., in the case of subtraction, "if we allow $[0 \underset{,}{-} 1]$ to be a wholly uninterpretable symbol, we have

$$(11) \qquad a \underset{,}{-} b \underset{,}{=} v, a, b + a, \bar{b} + [0 \underset{,}{-} 1], \bar{a}, b\text{''}$$

(the bar over a symbol indicates the "contradictory negative") with no justification whatever for his use of the symbol $[0 \underset{,}{-} 1]$ in this algebraic context. The treatment of logical division, the inverse operation to logical multiplication, is in a similar vein. His treatment of the general expansion theorem for a logical functional $f(x)$ is no more cogent than Boole's, and less convincing intuitively. He cites three advantages of his logical addition and subtraction (as opposed to Boole's): that is gives unity to the system, abbreviates the labor of working with it, and enables him to express particular propositions, which Boole's system "cannot properly express". But this version of "some a", namely

$$\overline{a \mathbin{\bar{,}} (i, a)}$$

where i is "a class only determined to be such that only some one individual of the class a comes under it", is likewise faulty as Peirce tacitly recognized since he never again refers to it and replaces it by other treatments in later works.

This 1867 paper of Peirce's also contains a discussion of probability from Boole's point of view. We postpone a presentation of it to our § 4.8.

The short paper [1867a], containing a summary of the non-probability part of [1867] has additional comment on arithmetic addition and multiplication of classes. Concerning the arithmetic $a + b$, he says it is the same as $a \mathbin{+\!\!,} b$ if $a, b \mathbin{=\!\!,} 0$ but "if a and b are classes which have any extent in common, it is not a class" but no further specification as to what it is in this case. His notion of arithmetical multiplication is best discussed in context with his probability notions (§ 4.8).

The monograph length paper [1870], extraordinarily rich in new ideas, marks a distinct change in Peirce's logical conceptions. Here the notion of a relative term is basic and that of a class, or "absolute" term, subsidiary. He presents an algebra of relatives which still maintains the distinction between arithmetic and logical operations though the use of both $\mathbin{=\!\!,}$ and $=$ is no longer retained, $=$ now being used for the stronger notion of identity and, correspondingly he introduces an operation, $[t]$, giving the number of elements of t when t is a class, but when a relative term it is the average number of things so related to an individual. Uninterpretables are no longer mentioned. Having introduced the notion of inclusion in both non-strict ($-<$) and strict ($<$) forms he can express particularity (e.g. "Some a is b" is rendered "$a, b > 0$"). Peirce's very complicated treatment of the hypothetical proposition by use of relatives of a special kind is later (in his [1880]) replaced by a more usual treatment using $-<$ which, however, had for him a highly generalized sense including that of inference, inclusion, and implication. (See Dipert 1981)

Alexander Macfarlane

As with Jevons, Macfarlane's aim was to "correct Boole's principles and place them on a clear rational basis." However his eccentric *Principlies of the Algebra of Logic* [1879] in which he expounds his system seems not to have had any noticeable effect on other logicians. Yet it is worthy of some attention here in that it, like the early work of Peirce, attempts a generalized arithmetico-algebra system along Boole's line of thought so as to handle both logic and probability. Our sketch of his ideas ignores most of his misconceptions and some inchoate features.

The symbol U is used to signify "a definite collection of individuals of a given type" and, as in Boole's [1847], may be understood, i.e. left unexpressed. It has an *arithmetic value* which is an integer—presumably the number of such individuals—though Macfarlane doesn't explicitly say this; this value may be 0 as when U is "imaginary" and may also be infinite. Attributes, qualities, or characters (the latter being his preferred term) are denoted by symbols x, y, z, etc. which, like the operators in Boole's [1847] select from the universe U the individuals which posess the character or quality. But, unlike Boole who had no universal or empty selector, Macfarlane uses for this purpose the symbol 1, denoting 'all' or 'the whole' and 0, denoting 'none', and taken as operating symbols of the same kind as x, y, z, etc. He writes '$U\{x = y\}$', abbreviated to '$x = y$', to mean that the U's which have the character x are identical with the U's having the character y; also '$U\{x + y\}$' means 'U's which are x together with U's which are y' but with no restrictions such as Boole's mutually exclusive requirement for meaningfulness, and '$U\{x - y\}$' means 'U's which are x minus U's which are y' where "x and y destroy one another, so far as they coincide; and the result in general consists of a positive and a negative part." But no explanation is given as to what he means by 'positive' and 'negative' parts. '$U\{xy\}$' means 'U's which are both x and y' and, as with $+$ and $-$, it is assumed or, rather, taken for granted that there is an operation $\div$, inverse to $\times$, whose meaning he explores by converting equation containing $\div$ to one with $\times$. He also has a second kind of multiplication of characters used in connection with probability which we shall discuss in our § 4.8.

Of particular interest, and in contrast to Boole, Macfarlane gives meaning to $x+y$ when x and y are not mutually exclusive, namely as not necessarily denoting a "single" character but a "summation" of two characters. His definition of 'single' is: "A symbol is said to be single when it does not select any member of the universe more than once, and always with the same sign." We are not told explicitly, but presumably $x+x$ would select the x's from a U twice.

With regard to fractional expressions Macfarlane pretty much follows Boole. He examines the meaning of m/n when m and n are integral and finds that "m/n is impossible unless m divides n." Although "$\frac{1}{2}$ is impossible, ...$\frac{1}{2}x$ will be possible when $x = 2y$. For then we have $\frac{2}{2}y$; which is $= y$." In general he believes that his Algebra of Quality is a generalized form of the Algebra of Quantity (neither of which are spelled out in detail, only vaguely hinted at) and hence that "every theorem of the latter is true in the former, provided that any special conditions which have been introduced are removed."

A part of his book (Section XI) is devoted to examining conditions for a character to be single. He finds that, for x single and positive, $x^2 = x$ and, if x is single and negative, $x^2 = -x$. (In the proof of this latter he uses $-1^2 = 1$ without comment.) Thus Boole's fundamental law $x^2 = x$ is replaced by $x^2 = \pm x$.

Macfarlane introduces inequalities: $x > y$ denoting that $x-y$ is "positive and positive only". As a "corollary"(?) to this definition he has that if x and y are each single and positive then $x > y$ means "that the x includes the y". Marcfarlane apparently doesn't appreciate the logical importance of the notion of inclusion (as did Peirce) but uses it only in connection with finding upper and lower limits on the numerical value of a logical expression in terms of assigned numerical values to the argument. The treatment is quite different from De Morgan's or Boole's (see our next §) in that negative values are allowed as limits.

Among Macfarlane's logical eccentricities was his insistence that every general proposition referred to a definite universe—e.g. 'All men are mortal' and 'No men are perfect' both refer to a universe of men. With U standing for 'men' these he renders respectively as

$$U\{1 = \text{mortal}\}$$

$$U\{0 = \text{perfect}\}.$$

He argues against Boole's considering All men are mortal' as being the same judgement as 'All men are mortal beings' on the grounds that they apply to different universes of objects. This ability of Macfarlane's to be able to bring in the universe into formal statements will be referred to in connection with his treatment of probability discussed below in our § 4.8.

JOHN VENN

Unlike Boole and the other writers mentioned in this section Venn was well aware of the work in symbolic logic which preceded Boole, especially that of Leibniz and Lambert and their associates. He was also *au courant* with the latest in logical literature and was a correspondent of Peirce and Schröder. His treatise *Symbolic Logic* [1881, 1894] includes substantial discussions of the central logical problems of the time such as that of (existential) import and that of hypothetical propositions. Nevertheless, despite some departures from its main conceptions, he was an adherent—the last one of any logical stature—of Boole's general approach.

More or less in agreement with DE MORGAN 1860 (= 1966, p. 255, 256) PEIRCE 1867, and MACFARLANE 1879, Venn includes among the various notions of aggregating classes one which counts any common part twice. However, he considers it to be "Applied Logic" and, since he is treating only "Pure Logic", omits it from consideration. Differing from Jevons, who thought that there was only one correct way to render 'or' in logic, Venn stressed that a choice could be made between the exclusive and non-exclusive senses on the basis of "symbolic propriety or convenience". Among the changes which Venn made between the first (1881) and second (1894) editions of *Symbolic Logic* was a change in the use of the symbol + from Boole's sense to the non-exclusive sense of 'or', yielding in this change to the popular trend. But he makes no mention of any possible ensuing changes in the algebraic rules—indeed one looks in vain for an explicit statement of such rules—except in one place where he remarks that the solution of $z = x + y$ for y should be written $y = z - xy$, rather than $y = z - x$ if x and y overlap. He notes however: "The power of free subtraction without the need of any such correction is one of the conveniences of the Boolian plan of notation".

Venn followed Boole in using the indefinite class symbol v (or its equivalent $\frac{0}{0}$) to represent 'some' but in the extended sense which admits the possibility of 'none'. He emphasized the incapacity of the Boolean equational calculus to represent non-emptyness and introduced the formula $xy > 0$, replacing Boole's $vx = vy$, to express 'some x is y'. While removing one of Boole's confusions was a step forward, Venn did nothing further with the idea: he was content to use $>$ only in the context $A > 0$ (or $A < 1$) where it meant the denial of $A = 0$ (or $A = 1$). He cites no algebraic properties of $>$ nor does he try to give a meaning in logic to the more general $A > B$.

Very much like Jevons, Venn wished to give a meaningful logical explanation of the "Boolian" logic which would be "independent of the mathematical calculus". The results in the two cases were quite different. In Venn's opinion ([1894], p. xxviii):

> Jevon's individual reforms in the direction of our Logic seem to me to consist mainly in excising from Boole's procedure everything which he finds an "obscure form", "anomalous", "mysterious", or "dark and symbolic" (*Pure Logic*, pp. 74, 75, 86). This he has done most effectually, the result being to my thinking that nearly everything which is most characteristic and attractive in the system is thrown away. Thus every fractional form disappears, so does the important indeterminate factor $\frac{0}{0}$, and all the general functional expressions such as $f(x)$ and its derivatives.

In examining Venn's claim to have given an explanation of every logical expression in purely logical terms we find him, as with Boole, admitting partial or conditionally defined and non-single valued operations. There is a slight improvement over Boole in that Venn has addition as well as multiplication as a single-valued and closed operation on classes. However with regard to subtraction he says: "In using this symbol we must remember the condition necessarily implied in the performance of the operation which it represents. As we remarked, we cannot 'except' anything from that in which it was not included; so that $x - y$ certainly implies that y is a part of x."

This is no different from Boole. With regard to a meaning for logical division of classes, something which Boole shied away from, Venn has "... the expression x/y stands for a class, viz. for the most general class which will, on imposition of the restriction denoted by y, just curtail itself to x. But to this expression we must remember to attach the condition, that this presupposes that 'all x is y', as otherwise no such class as that which it is desired to determine could exist."

From its definition Venn finds this class to be $x + vxy$, v the indefinite class symbol, inasmuch as $(x + v\bar{x}\bar{y})y = xy = x$ with the presupposition that x is contained in y. With this meaning for division Venn can now explain why $\frac{0}{0}$ is the same as v; for a class which, on restriction by 0 yields 0, can be any class whatever.

With regard to the use of Boole's expansion theorem on expressions which have no direct interpretation as a class Venn argues [1894, pp. 266–267] that one needs it only for fractional forms for which he has given a logical meaning as a class. Hence

> We shall therefore apply our formula to x/y with no more hesitation than, for example, to $x + \bar{x}y$. When we do so, developing in accordance with the rule ... we obtain
>
> $$\frac{x}{y} = \tfrac{1}{1}xy + \tfrac{1}{0}x\bar{y} + \tfrac{0}{1}\bar{x}y + \tfrac{0}{0}xy. \tag{1}$$
>
> The result obtained by purely logical considerations in the third chapter, it will be remembered, was
>
> $$\frac{x}{y} = xy + v \cdot xy, \text{ with the attendant condition } x = xy \ldots. \tag{2}$$
>
> A comparison will show the complete identity of these two results....

Venn's argument for their "complete identity" involves identifying $\frac{1}{1}xy$ with xy, deleting $\frac{0}{1}\bar{x}y$, identifying the indefinite symbol v standing for 'a perfectly uncertain portion, some, all or none' with $\frac{0}{0}$, since "this is exactly the well-known meaning of $\frac{0}{0}$ in mathematics" and omitting the term $\frac{1}{0}x\bar{y}$. As a reason for the last he gives: "Now the meaning of $\frac{1}{0}$ in mathematics is *infinity*. What then is meant by offering us, in a simple class expression, a term multiplied by infinity? Surely that there is no such class in existence, for this is the only way of escaping the consequent absurdity."

Although aware of FREGE 1879 and PEIRCE 1880, Venn's ideas on propositional logic seems not to have been significantly influenced by them since his treatment bears close resemblance to Boole's, discussed below in Chapter 3.

§ 1.12. Propositions numerically definite

The topic to be discussed here was treated by Boole at two different times. One version appears in *Laws of Thought* Chapter XIX, Of Statistical Conditions, where it is made the basis for a method of obtaining bounds on probabilities. The second appeared as a paper, BOOLE 1868 (= BOOLE 1952 IV), published posthumously by De Morgan and described by the editor of BOOLE 1952 (on p. 5) as "probably written about 1850," i.e. about four years prior to *Laws of Thought*. This is probably the "lost" manuscript referred to by Boole in *Laws of Thought*, p. 310 footnote (See § 4.1 below). It is this earlier, though posthumously published, paper we describe in this section.

It is interesting to contrast De Morgan's and Boole's conception of the topic. For De Morgan numerically definite propositions were generalizations of the Aristotelian categorical forms, e.g. 'mXY' (m or more X's are Y's), replacing 'XY' (some X's are Y's); the principal objective was the determination of the valid numerically define syllogistic forms involving three terms X, Y, Z. Boole's quite different approach was to introduce an operator N on class terms (but then extended to other expressions) whose values are the numbers of individuals in the classes. For him the central problem was:

> Given any system of propositions, any of the terms of which, simple or compound, are made in numerically definite form, required the numerical limits within which the number of individuals contained in any proposed class will be; whether that class be defined by the presence or the absence of any single

> attribute, or by the presence of any collection of attributes and the absence of any other collection of attributes, or whether it consist of distinct groups and parcels of individuals each of which is thus defined. [BOOLE 1952, pp. 168–169]

Boole uses the notation Nx for the number of individuals in the class x, Nxy for the number in "the class whose members are X's and Y's," and $N(x+y)$ for "the number contained in the aggregate class whose divisions are x and y. This implies that the classes x and y are mutually exclusive." He then goes on to extend without any special comment the meaning of $N(x+y)$ to the case of x and y not being mutually exclusive. He does this in his Proposition I which asserts that

$$NP \pm NQ \pm NR \cdots = N(P \pm Q \pm R \cdots),$$

P, Q, R, ... being "class functions" and "provided that we develope $P \pm Q \pm R \ldots$ into constituents, apply N to each term of the result, and interpret any term Nat in which t is a constituent and a a numerical coefficient, by aNt." Thus, since the development of $x+y$ is $x\bar{y}+\bar{x}y+2xy$, $N(x+y)$ is to be interpreted as $Nx\bar{y}+N\bar{x}y+2Nxy$. Hence in this context it would seem natural to suppose that $x+y$ be considered as some kind of an aggregate in which individuals common to x and y are counted twice, and similarly $x-y$ as an entity in which individuals in $y\bar{x}$ are counted negatively. But throughout his writings Boole steadfastly resists making the inference and refers to such expressions as "uninterpretable".

After presenting the more general proposition,

$$aNP \pm bNQ \pm cNR \cdots = N(aP \pm bQ \pm cR \cdots),$$

a, b, c, ... being numerical quantities, Boole shows how readily his algebraico-logical methods lead to results about class terms by taking linear combinations. Thus

$$\begin{aligned} Nx + Ny - N(1) &= N(x+y-1) \\ &= N(xy-(1-x)(1-y)) \\ &= Nxy - N\bar{x}\bar{y} \end{aligned}$$

so that

$$(1) \qquad Nxy = Nx + Ny - N(1) + N\bar{x}\bar{y}.$$

The simple result (1) for Nxy is generalized by Boole to $Nx_1 \ldots x_n$. Starting with

$$x_1 \ldots x_n = (1-(1-x_1))(1-(1-x_2))\ldots(1-(1-x_n))$$
$$= x_1 + x_2 + \cdots + x_n - (n-1) + R$$

he finds, by developing $x_1 \ldots x_n - (x_1 + x_2 + \cdots + x_n - (n-1))$, an expression for R which is a linear combination (with numerical coefficients) of constituents on $x_1, \ldots, x_n$ having 2 or more of the x_i negated. Thus he has

$$(2) \qquad Nx_1 \ldots x_n = Nx_1 + \cdots + Nx_n - (n-1)N(1) + NR,$$

with R having a given form. The part

$$Nx_1 + \cdots + Nx_n - (n-1)N(1),$$

which equals $Nx_1 \ldots x_n$ when $NR = 0$, is called the *prime value* of $Nx_1 \ldots x_n$, and NR the *remainder*. From its structure, Boole notes, NR cannot be negative, but fails to mention that it could be larger than $N(1)$. The prime value, clearly a lower bound for $Nx_1 \ldots x_n$, is described as "the least number of individuals which can exist in the class $x_1 \ldots x_n$". Since prime values can be negative Boole should have qualified this with "if the prime value is positive, otherwise the least number is 0". While the assertion, that the prime value is the least number of individuals that could be in $x_1 \ldots x_n$ (if qualified), is correct (see e.g. HAILPERIN 1965), Boole's argument isn't fully cogent; for he hasn't shown that one can have $NR = 0$ by suitable arrangement of the classes $x_1, \ldots, x_n$ while maintaining $Nx_1, \ldots, Nx_n, N(1)$ fixed.

Boole proves a number of interesting theorems on prime values and shows how to obtain [a set of] prime values for P, P being a sum of constituents. These are lower bounds on NP which are of the same form as the prime value of a constituent of P's development but with possibly fewer variables; e.g. $Nx_1 + Nx_2 - 1 \cdot N(1)$ is a prime value for $P = x_1x_2x_3 + x_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3$. He takes it for granted (erroneously as we shall see in § 4.7) that "the highest of the prime values obtained will be the least number of individuals which can enter into the proposed class [i.e. P]".

Boole's paper goes on to present his solution of the general problem of numerical limits for a class. However we shall interrupt our exposition here as the substance of it is contained in his *Laws of Thought* version which we pick up in our § 4.7.

§ 1.13. Notes to Chapter 1

(for § 1.0)

For more details on Symbolical Algebra and the origins of abstract algebra see KOPPELMAN 1971, KNOBLOCH 1981, and PYCIOR 1981.

(for § 1.1.)

NOTE 1. There is a study of the influence of Boole's mathematical work on his creation of an algebra of logic, mainly with reference to BOOLE 1847, in LAITA 1977.

NOTE 2. See our § 3.1 for a modern formalization of Boole's operator calculus, which is then used in § 3.2 to develop a form of propositional logic.

NOTE 3. For a very different perspective from ours on BOOLE 1847 see CORCORAN-WOOD 1980.

NOTE 4. The manuscript notes by Boole in possession of the Royal Society referred to by the editor of BOOLE 1952, p. 119, are now available as:

Manuscript Additions to 'Mathematical Analysis of Logic' B. Boole. Edited with an introduction by G. C. Smith. History of Mathematics Paper #18, Department of Mathematics, Monash University, Clayton, Victoria, Australia.

(for § 1.2)

NOTE 1. Recent discussions of logical psychologism, with reference to Boole, are in MUSGRAVE 1972 and RICHARDS 1980.

NOTE 2. The surprising fact that there are no semantic notions (validity, invalidity, logical consequences, etc.) in Boole's work is brought out and discussed in the Corcoran-Wood paper mentioned above in Note 3 for § 1.1. Presumably Boole thought that by virtue of his basic principle ("The laws, axioms, and the processes of such an algebra will be identical in their whole extent with the laws, the axioms and the processes of an Algebra of Logic.") correct algebraic derivations in such an algebra would correspond to correct logical inferences.

(for § 1.3)

It is curious that Boole's discussion of the possibility of having a trichotomous division in logic makes no mention of De Morgan who, in a letter to Boole of 3 April 1849 said:

> I have considered a little the problem of—not name and contrary—X and x—but any number of names—a proposition which the alternatives are more than X and not-X. I looked at it enough to see the possibility of wider classes of numerically definite distributions and logical syllogisms arising therefrom—but I never had the curiosity to investigate more than some simple cases of three alternatives—I hope you will go on with it. [Excerpt from Letter 16 in SMITH 1982]

(for § 1.4)

The absence of any mention of associativity, of either multiplication or addition, in both BOOLE 1847 and BOOLE 1854 is very puzzling. Without parentheses 'xyz' is ambiguous as between a ternary operation on x, y, and z, and either of the two composite binary forms '$x(yz)$' and '$(xy)z$', none of which are formally identifiable with the others in the absence of a postulate. Associativity of a binary operation was implicitly recognized in HAMILTON 1837 and explicitly in HAMILTON 1844. It also occurs in DE MORGAN 1844 (as was pointed out to me by John Corcoran). Concerning this latter paper, in a letter written in 1845 to De Morgan, Boole says that he read De Morgan's memoir on Triple Algebra "with great interest". (SMITH 1982, p. 15) At the very beginning of this memoir

De Morgan discusses the question of associativity in connection with his algebraic triples which he devised in analogy with W. R. Hamilton's quaternions.

(for § 1.10)

According to Alonzo Church (The History of the Question of Existential Import of Categorical Propositions, *Proceedings of the* 1964 *International Congress for Logic, Methodology and Philosophy of Science*, North-Holland Publishing Co. 1968), the earliest (implicit) appearance of the modern doctrine of existential import (i.e. that universal propositions are true and particular are false if the subject term is empty) is in Cayley 1871.

Cayley describes his result as a "more concise and compendious form" of the theory of the Syllogism in Boole's "The Calculus of Logic," *Camb. and Dubl. Math. Jour.*, t. III (1848), pp. 183–198 [= Essay II in Boole 1952]. Cayley circumvents Boole's use of the indefinite v by introducing the relation "not = 0".

(for § 1.11)

The original, and the edited collections Peirce 1933 and Peirce 1982, all have "$a \mathbin{-\!\!,} i, a$" for the rendition of "some a". But neither $(a \mathbin{-\!\!,} i), a$ nor $a \mathbin{-\!\!,} (i, a)$ could be intended, the former being uninterpretable if i had some a's and the latter being all a's but the one in i. Thus a bar over the latter, indicating complementation, is called for in order to render Peirce's intention.

CHAPTER 2

FORMALIZATION OF BOOLE'S LOGIC

§ 2.1. The calculus of multisets. Axioms for multiset algebra

There can be no question that by modern standards Boole's logical system falls far short of being a satisfactory theory. To his contemporaries and immediate successors much of it was obscure and, in a comparatively short time, it was supplanted by the Boole-Jevons-Peirce-Schröder calculus. But Boole does have a formal system if one is indulgent enough in what one understands by this. The trouble is that we don't know of what it is a formalization. Starting off with mostly intuitively clear ideas he ends up with algebraic notions without logical sense which then need special treatment to get back to understandable notions. One could follow the historic route and simplify it down to a theory of Boolean algebras (or Boolean rings) and all class calculus problems could be satisfactorily solved. Yet by so doing this we would lose the distinctive character of Boole's system and, of course, the possibility of understanding what is going on behind it all. We intend to show in this chapter that, other than the simplification to a Boolean algebra or Boolean ring, there is another way to make sense of Boole's system.

Customarily in theory building one has an intuitive meaningful idea of a mathematical structure and then proceeds to formalize it as an abstract system. So from the idea of a group we go to the theory of groups, axiomatized in some fashion, or from the algebra of subsets of a set to Boolean algebra. Here, however, our problem is the reverse of this. We have a formal system, admittedly badly formulated, and we wish to find out what kind of mathematico-logical structure it formalizes. Once we do, and have a clearly described (type of) structure in mind, we can then in the reverse direction use it to straighten out Boole's poorly formulated system. Inklings of how this may be accomplished can be found in the remark of De Morgan's about Boole's '+' implying double counting and in Peirce's and Macfarlane's attempts at revising Boole's system with classes having both logical and numerical meaning attached to them (see § 1.11).

Our basic contention is: *To obtain a meaningful interpretation of Boole's system we have to use not the notion of a class (class = set) but that of a multiset.*

By a multiset we mean a collection of objects as a whole in which more than one example of an object may occur—for example, the collection of 5 indistinguishable red balls and 3 indistinguishable black balls in an urn, or the set of roots of a polynomial equation in which multiplicities are counted. The objects will be referred to as *members* or *elements* of the multiset. In the example of the urn just mentioned there are 8 elements. If we wish to disregard multiplicities we shall refer to *distinct kinds of elements*. As a generalization of the "roster" method of representing sets, i.e. $\{a_1, \ldots, a_n\}$, for the set whose elements are the objects $a_1, \ldots, a_n$, we write $\{h_1)a_1,\ (h_2)a_2, \ldots, (h_n)a_n\}$ to represent the multiset having h_1 of the objects a_1, h_2 of the objects $a_2, \ldots, h_n$ of the objects a_n. One can, if one chooses, define (mathematically) multisets in terms of ordinary sets, and in a number of different ways. For example, as functions on sets into the natural numbers, i.e. as collections of ordered pairs (a_i, h_i); or, as the set of all permutations of the ordered m-tuple

$$\langle \overbrace{a_1, \ldots, a_1}^{h_1}, \ldots, \overbrace{a_i, \ldots, a_i}^{h_i}, \ldots, \overbrace{a_n, \ldots, a_n}^{h_n} \rangle,$$

where $m = h_1 + h_2 + \cdots + h_n$. Since an m-tuple is expressible as a set, this set of sets adequately represents the notion we are representing by $\{(h_1)a_1, (h_2)a_2, .., (h_n)a_n\}$; for, by taking the set of all permutations we abstract from the order and, from any one of the m-tuples, we can recover the distinct kinds of elements. But equally well the other way around, the notion of set is obtainable from that of multiset by specializing to those multisets for which the multiplicities h_i are 0 or 1. For our purpose here we wish to think of the notion of multiset as primary.

Two multisets

$$A = \{(h_1)a_1, \ldots, (h_i)a_i, \ldots\}, \qquad B = \{(k_1)b_1, \ldots, (k_i)b_i, \ldots\}$$

are *equal* if and only if, for each i, a_i and b_i are the same and $h_i = k_i$.

As in the calculus of classes we shall have a "universe" which we designate by 1. This is to be some fixed non-empty set of elements *without* repetitions; from these elements, replication permitted, all other multisets are constructed. Absence of members of 1 from a multiset can be indicated by allowing the notion of zero multiplicity; the totally empty multiset $\{(0)a_1, \ldots, (0)a_i, \ldots\}$ we designate by 0.

We turn now to defining operations on multisets. The intuitive idea of dumping the contents of two urns together will be represented by the algebraic operation of *adding* two multisets. Thus if $A = \{(h_1)a_1, \ldots, (h_i)a_i, \ldots\}$ and $B = \{(k_1)a_1, \ldots, (k_i)a_i, \ldots\}$, then

$$A + B = \{(l_1)a_1, \ldots, (l_i)a_i, \ldots\}, \quad \text{where } l_i = h_i + k_i.$$

From this we readily see that, for any multisets A, B, C,

$$A + B = B + A, \quad (A + B) + C = A + (B + C), \quad A + 0 = A.$$

(The danger of confusing multiset operations and constants with the homonymous arithmetic operations and constants is so slight that we shall not trouble to use different notation.) Clearly 1 is not the "largest" multiset in the same way as the universal class is in the calculus of classes, for $1 + A$ is "larger" than 1 for any $A \neq 0$.

Next we consider the operation of multiset *multiplication*. For arbitrary multisets $A = \{(h_1)a_1, \ldots, (h_i)a_i, \ldots\}$ and $B = \{(k_1)a_1, \ldots, (k_i)a_i, \ldots\}$ we write

$$AB = \{(l_1)a_1, \ldots, (l_i)a_i, \ldots\}, \quad \text{where } l_i = h_i k_i.$$

so that, in the product AB, the multiplicities of corresponding elements are multiplied. Clearly we have $AB = BA$, $A(BC) = (AB)C$, $A1 = A$. But note, as in a Boolean algebra, we can have divisors of zero, i.e. one can have $AB = 0$ even if A and B are non-zero, this occurring when A and B, although non-empty, have no element in common. Since

$h_i(k_i + l_i) = h_ik_i + h_il_i$ we see that distributively holds for multisets. Moreover, since for $n > 0$, $nh_i = 0$ if and only if $h_i = 0$ and likewise $h_i^n = 0$ if and only if $h_i = 0$, we see that $nA = 0$ if and only if $A = 0$ and $A^n = 0$ if and only if $A = 0$. (By nA we of course mean $A + \cdots + A$ for n terms, likewise, A^n is $A \cdot A \cdot \ldots \cdot A$ for n factors.)

We formalize these observations by stating that a calculus of multisets is an example of an algebraic system with the following properties: there are two binary operations $+$ and $\cdot$ and two distinct elements 1 and 0 such that with respect to these the system is a commutative semi-ring with unit and zero; and the system has no nilpotents, either additive or multiplicative. It is convenient to have all this spelled out:

Axioms for Multiset Algebras

$$A + B = B + A, \qquad AB = BA,$$

$$A + (B + C) = (A + B) + C, \qquad A(BC) = (AB)C,$$

$$A + 0 = A, \qquad A \cdot 1 = A,$$

$$A + 1 \neq A, \qquad A \cdot 0 = 0,$$

$$A(B + C) = AB + AC,$$

$$1 \neq 0,$$

$$nA = 0 \quad \text{only if} \quad A = 0, \qquad A^n = 0 \quad \text{only if} \quad A = 0.$$

By virtue of Theorem 0.41 the axiom

$$A^n = 0 \quad \text{only if} \quad A = 0$$

can be replaced by

$$A^2 = 0 \quad \text{only if} \quad A = 0;$$

however we leave the redundancy in for the sake of symmetry. Similarly, for aesthetic reasons, we have included both $A + 1 \neq A$ and $1 \neq 0$ as axioms although in the presence of the remaining axioms each is derivable from the other. The axiom

$$A + 1 \neq A$$

can be written

$$A + 1 = A + 0 \rightarrow 1 = 0,$$

in which form it represents a remnant of cancellation that is left to the semi-ring (as opposed to full additive cancellation for a ring).

These axioms have many models—the natural numbers for instance, as well as any algbra of multisets for any choice of the "universe" 1. As the axioms are all McKinsey formulas we know, by Theorem 0.31, that any substructure of a model of the axioms is also a model, and the direct product of models is a model. Thus as a fairly general model to keep in mind for the axioms one can think of a system which is a direct product of an arbitrary number of copies of the natural number system.

§ 2.2. Boole's Algebra (SM algebras)

An algebraic structure of the kind we have just discussed in § 2.1, while capable of giving meaning to Boole's "uninterpretable" $x + x$, still does not adequately represent Boole's system, for the notion of subtraction, forming the inverse to addition, is absent. We can, disregarding meanings, formally obtain this operation by adding to the axioms the assumption that each element of the system has an additive inverse, i.e. by converting the semi-ring to a ring. But then the structure of multisets we have had in mind is no longer a model for these axioms. However, a consideration of the relationship of the semi-ring of natural numbers to that of the ring of integers readily suggests how to obtain a suitable model, namely by introducing the notion of a *signed* multiset. If one goes back to the discussion on multisets and allows the coefficients h_i in $\{(h_1)a_1, \ldots, (h_i)a_i \ldots\}$ to be any integers—positive, negative, or zero—one readily sees that the resulting structure is a model for the so extended axiom system. While the notion of a signed multiset is not as intuitively simple as that of an unsigned multiset, a brief reflection on the history of the difficulties which were experienced until negative numbers were in good standing, should help one overcome resistance to the acceptance of signed multisets as a meaningful notion. Accordingly, as a codification of the algebraic properties actually required by Boole we present:

AXIOMS FOR BOOLE'S ALGEBRA (SM ALGEBRAS)

B1. $A + B = B + A$,

B.2. $A + (B + C) = (A + B) + C$,

B3. $A + 0 = A$,

B4. $A + X = 0$ has a (unique) solution for X,

B5. $AB = BA$,

B6. $A(BC) = (AB)C$,

B7. $A1 = A$,

B8. $A(B + C) = AB + AC$,

B9. $1 \neq 0$,

B10. $A^2 = 0$ only if $A = 0$,

$\mathrm{B11}_n$. $nA = 0$ only if $A = 0$ $(n = 1, 2, 3, \ldots)$.

'Boole's Algebra' is, of course, to be distinguished from 'Boolean Algebra'.

The above axiom set has been obtained from that for (unsigned) Multiset Algebras given in the preceding section, by adding the postulate B4 guaranteeing an additive inverse, and then deleting $A + 1 \neq A$ and $A \cdot 0 = 0$, both of which become derivable. In the language of modern algebra these are axioms for the theory of *commutative rings with unit and with no non-zero nilpotents, either multiplicative or additive*. For brevity's sake we refer to this as the *theory* **SM**, or the *theory of* SM *algebras*, not thereby implying that algebras of signed multisets are the only type of model (an algebra of signed multisets is a special kind of SM algebra). Rings which satisfy these axioms must be of characteristic 0 since from B9 and $\mathrm{B11}_n$ we have $1 \neq 0$, $1 + 1 \neq 0, \ldots$, $1 + 1 + \cdots + 1 \neq 0, \ldots$. Here are a few general properties of the theory **SM**.

Unlike the (elementary) theory of Boolean algebras, the theory of SM algebras (as well as that for unsigned Multiset Algebras of § 2.1) is undecidable. This follows immediately from a result of Tarski, Mostowski and Robinson (Theorem 0.32) since **SM** is a subtheory of the theory **J** (of integers) having the same symbols as **J**. Interestingly enough, there is a decision procedure for a subclass of sentences of this theory, namely the subsystem consisting of the universal sentences. (SIMMONS 1970, THEOREM 1, p. 549). As a consequence of this *we need not, for formulas which are quantifier-free (e.g. equations), distinguish between 'true in all* SM *algebras' and 'provable from the axioms for* SM *algebras'*.

There is also an algebraic structure theorem which can be given for the theory **SM**. According to the result of McCoy (Theorem 0.43), any commutative ring without nonzero nilpotents is isomorphic to a subdirect product of integral domains. Moreover, if the ring has the property that $nA = 0$ only if $A = 0$, then (by Remark 0.44) so does each component in the subdirect product. Thus: *every* SM *algebra is isomorphic to a subdirect product of integral domains each of which is without additive nilpotents.* We note that each of these component integral domains has embedded within it a structure like the integers; for if 0 and 1 are the zero and unit of such an integral domain, then 0, ± 1, $\pm(1+1), \ldots$ are all distinct and present in the domain and have the algebraic properties of the integers. It is enlightening to compare this structure theorem with that for Boolean algebras (SIKORSKI 1969, § 16): *Every Boolean algebra is isomorphic to a subdirect product of two-element Boolean algebras.* .

§ 2.3. Idempotents. Boolean multiset terms

As our interests from now on will be exclusively with signed multisets (and their algebraic codification), we can shorten terminology by dropping the qualifying adjective 'signed'.

From our conception of multiplication in a multiset algebra we have that $A^2 = \{h_1^2)a_1, \ldots, (h_i^2)a_i, \ldots\}$ so that $A^2 = A$ if an only if for each i, $h_i^2 = h_i$. But the equation $h_i^2 = h_i$ holds if and only if $h_i = 0$ or $h_i = 1$. Thus for such an algebra the idempotency condition $A^2 = A$ is equivalent to the condition that A have no negative or repeated element. This idempotency condition was referred to by Boole as the "condition of interpretability" and terms not satisfying it were "not interpretable" or interpretable only under special conditions on the variables in the terms. We discuss this in detail in our next section. Here we explore the general algebraic properties of idempotents in SM algebras.

We consider an abstract SM algebra $\mathfrak{M} = \langle M, +, \cdot, 0, 1 \rangle$, i.e. an arbitrary model of the axioms for Boole's Algebra and investigate properties of the subset B of M containing all its idempotents. Clearly B includes 0 and 1 as members. Following Boole's example we shall use lower case Latin letters at the end of the alphabet as variables ranging over B, referring to them as *Boolean* variables. From simple ring properties (§0.4) and the idempotency of x and y one readily establishes:

THEOREM 2.31. *In* $\mathfrak{M}$ (*i.e. as theorems about the theory* SM) *the following hold*:

(a) $x^2 = x$ *and* $x(1-x) = 0$.

(b) $(1-x)^2 = 1-x$.

(c) $(xy)^2 = xy$.

(d) $(x+y)^2 = x+y$ *if and only if* $xy = 0$.

(e)
$$(x+y-xy)^2 = (x+(1-x)y)^2 = x+(1-x)y = x+y-xy.$$

(f)
$$(x+y-2xy)^2 = [x(1-y)+y(1-x)]^2 = x(1-y)+y(1-x) = x+y-2xy.$$

(f′)
$$[(x-y)^2]^2 = (x+y-2xy)^2 = x+y-2xy = (x-y)^2.$$

(g) $x = y$ *if and only if* $x(1-y)+y(1-x) = 0$.

These results show that the set B of idempotents is closed under the operations of subtraction-from-1 and multiplication, but not under addition (since $1 \cdot 1 \neq 0$, we have by (d), $(1+1)^2 \neq 1+1$). The sum is, however, idempotent if the summands are mutually exclusive (i.e. if their product is 0). Hence (as (e) and (f) show) the operations represented by $x+y-xy$ $[=x+y(1-x)]$ and $x+y-2xy$ $[=(x-y)^2 = x(1-y)+y(1-x)]$ are closed in B; (f′) is an alternative version of (f). Let us put

$$x +_B y \quad \text{for} \quad x+y-xy \qquad \text{(Boolean sum)},$$

$$x +_\Delta y \quad \text{for} \quad x+y-2xy \qquad \text{(Symmetric difference)},$$

and note that $x +_B y = x +_\Delta y = x + y$ if $xy = 0$.

One easily shows the following (§ 0.5):

THEOREM 2.32. *The structure* $\langle B, +_\Delta, \cdot, 0, 1\rangle$ *is a Boolean commutative ring with unit* ($+_\Delta$ *being the ring addition*). *The structure* $\langle B, +_B, \cdot, \bar{\ }, 0, 1\rangle$, *where* $\bar{A}$ *is* $1-A$, *is a Boolean algebra. These two structures are equivalent via the equations*

$$1 - x = 1 +_\Delta x,$$

$$x +_B y = x +_\Delta y +_\Delta xy,$$

$$x +_\Delta y = x(1-y) +_B y(1-x).$$

The Boolean algebra in Theorem 2.32 will be referred to as $\mathfrak{M}$'s Boolean algebra.

If we were to limit ourselves to idempotents and to the operations of either of the two systems in Theorem 2.32, then we would arrive at the historical simplification of Boole's ideas to modern Boolean algebra. But in so doing we would *not* be using Boole's method. While he did limit himself only to variables ranging over idempotents he also used, in effect, addition in M which, as we have noted, is not closed in B. We therefore, since we do wish to follow Boole's course, turn to obtaining results about SM algebras in which the explicit variables used range only over idempotents, but with no restriction on the operations.

By a *Boolean* (signed) *multiset term* (i.e. a Boolean SM term) we shall mean any finite composition of constants and/or Boolean variables with the operation of SM addition, multiplication, and subtraction. When not needed for clarity we shall omit the qualifying adjectives 'Boolean multiset'.

We first establish that Boole's Law of Development holds for any Boolean multiset term. Our proof is different from Boole's since we can't use his assumption that an equation is established if it holds for all assignments of 0 and 1 to the variables—in fact, contrariwise, we shall prove this is so by using the Law of Development (second paragraph following the PROOF).

THEOREM 2.33 (Law of Development). *If $f(x)$ is any term and x a Boolean variable, then*

$$f(x) = f(1)x + f(0)(1-x),$$

where $f(1)$ is the term obtained from $f(x)$ by replacing x throughout by 1 and, similarly, $f(0)$ is obtained by replacing x by 0.

PROOF: Clearly $f(1)$ and $f(0)$ are terms. From the idempotency of x and simple ring properties one sees that $f(x)$ (which may have other variables present) is equal to a linear form, i.e. $f(x) = Ax + B$ where A and B do not involve x, and thus $f(1) = A + B$ and $f(0) = B$. The result is now immediate since $Ax + B = (A+B)x + B(1-x)$.

On the basis of this theorem one readily establishes Boole's result that a term is expandable in constituents on any set of Boolean variables. It is clear that, on a given set of Boolean variables, the product of distinct constituents is 0, that the sum of distinct constituents is an idempotent (Boole's $+$ can be taken as either $+_B$ or $+_\Delta$), and that the sum of all constituents is 1.

One can also justify, as far as equations are concerned, Boole's conception of his algebra as an algebra of 0 and 1. It is easy to show that an equation between Boolean multiset terms, e.g. $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$—in which all the Boolean variables $x_1, \ldots, x_n$ need not be present on both sides—is provable if and only if it becomes an identity for each of the 2^n possible assignments of the values 0, 1 to the variables; this is readily seen by replacing $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ by their respective expansions in terms of the constituents on the variables $x_1, \ldots, x_n$ and noting that each assignment makes one and only one constituent have the value 1 and all the rest 0.

It is easy to formally prove that any Boolean multiset term is equal to a sum of constituents on its variables with coefficients that are integers, i.e. are of the form '0' or '± 1' or '$\pm(1 + 1 + \cdots + 1)$'. Accordingly, when its variables represent sets in a multiset algebra, a term represents a multiset built up in obvious fashion from sets represented by the constituents. Thus

$$x + x = 2 \cdot x + 0 \cdot \bar{x},$$

and hence, for a set x, $x + x$ represents a multiset which has two copies of each element of x (and none of $\bar{x}$); and from

$$x - y = 1 \cdot x\bar{y} + 0 \cdot (xy + \bar{x}\bar{y}) + (-1) \cdot \bar{x}y,$$

we see that, as a multiset, $x - y$ would have the elements of $x\bar{y}$ singly together with negative copies of $\bar{x}y$ (and none of xy or $\bar{x}\bar{y}$).

Boole's important elimination theorem (§ 1.8) can be proved for Boolean multiset terms. We include more detail in the proof than is our usual custom since we wish to use this type of proof as an example later on (§ 2.5). Our present-day conception of algebraic elimination as a species of existential quantifier elimination was, naturally, not clearly understood in Boole's time.

THEOREM 2.34. *If $f(x)$ is a Boolean multiset term then*

$$f(1)f(0) = 0 \leftrightarrow \exists x(f(x) = 0).$$

PROOF. (a) Suppose $\exists x(f(x) = 0)$.

Let x be such that $f(x) = 0$. Then by Theorem 2.33

$$f(1)x + f(0)\bar{x} = 0.$$

Multiplying through by $f(0)x$ and simplifying gives

$$f(1)f(0)x = 0.$$

Similarly using $f(1)\bar{x}$ gives

$$f(1)f(0)\bar{x} = 0.$$

Whence, by addition, $f(1)f(0) = 0$.

(b) Suppose $f(1)f(0) = 0$.

Let t_i $(i = 1,\ldots,2^n)$ be the set of constituents on a set of variables which includes those present in $f(1)f(0)$. Let $f(1) = \sum a_i t_i$ and $f(0) = \sum b_i t_i$, where the a_i and b_i are purely numerical, i.e. integers. By the hypothesis (b),

$$(\sum a_i t_i)(\sum b_i t_i) = \sum (a_i b_i) t_i = 0,$$

so that

$$a_i b_i = 0 \quad (i = 1,\ldots,2^n).$$

Define

$$c_i = \begin{cases} 1 & \text{if } a_i = 0 \\ 0 & \text{if } a_i \neq 0 \quad \text{(and hence} \quad b_i = 0) \end{cases}$$

and set $x_0 = \sum c_i t_i$. Then by Theorem 2.33,

$$\begin{aligned} f(x_0) &= [f(1) - f(0)]x_0 + f(0) \\ &= [\sum a_i t_i - \sum b_i t_i] \sum c_i t_i + \sum b_i t_i \\ &= \sum (a_i - b_i) c_i t_i + \sum b_i t_i \\ &= \sum [(a_i - b_i) c_i + b_i] t_i \\ &= 0. \end{aligned}$$

As was known to Boole (Chapter X, Prop. I, p. 151) any equation "among logical symbols" can be reduced "to the form $V = 0$, in which V satisfies the law of duality, $V(1 - V) = 0$." For us this becomes

THEOREM 2.35. *Every Boolean multiset equation (i.e. an equation between Boolean multiset terms) is equivalent to a uniquely determined Boolean equation $V = 0$ in the same variables.*

PROOF. By algebraic (ring) operations any Boolean multiset equation can be reduced to the form $U = 0$. Let $\pm a_i$ $(i = 1,\ldots,m)$ be the non-zero coefficients in U's complete expansion—if there is no such a_i we are done, merely take V as $x\bar{x}y\bar{y}\ldots$ for as many variables as are in U. Writing $U = 0$ equivalently as

$$\sum \pm a_i t_i = 0$$

and multiplying the equation through by $\mp t_i$ $(i = 1,\ldots,m)$ gives

$$a_i t_i = 0$$

so that, by $B11_n$, $t_i = 0$ $(i = 1,\ldots,m)$.

Then $V = \sum_{i=1}^{m} t_i = 0$ is the desired Boolean equation—Boolean because addition of distinct constituents (on the same set of variables) is the same as Boolean sum. Each of the described steps is reversible, so that the asserted equation is then established. As Boole notes, one gets V from U's complete expansion by replacing each non-zero coefficient by 1.

One should not conclude from this theorem that Boole's Algebra is without interest when only Boolean variables are involved. For example the fact that $(x+x)y = 0$ is equivalent to the purely Boolean equation $xy = 0$ results in significant information when applied to classes, namely that duplicating the elements of a class x does not affect its exclusiveness or non-exclusiveness with another class. However Boole never used his algebra other than as a means of getting from premises to conclusion, and expressions not satisfying his "law of duality" ($A^2 = A$) occurred only in intermediate stages (and were considered to be "uninterpretable").

When starting with a purely Boolean equation we can, by means of operations of Boole's Algebra, arrive at equations involving non-idempotents. A natural question arises: Is Boole's Algebra conservative with respect to deductions in Boolean algebra or could there be deductions carried out by the full algebra but which could not be carried out in Boolean algebra? That is, if ψ is derivable from θ with free variables held fixed (both formulas in purely Boolean terms) by use of Boole's Algebra is there a purely Boolean derivation of ψ from θ? Symbolically, does $\theta \vdash_{\mathrm{SM}} \psi$ imply $\theta \vdash_{\mathrm{BA}} \psi$? By use of the deduction theorem (0.3.1) this question becomes:

$$\text{Does } \vdash_{SM} \theta \to \psi \quad \text{imply} \quad \vdash_{BA} \theta \to \psi\,?$$

An affirmative response is immediate from Theorem 2.37, for which the following is a lemma.

THEOREM 2.36. *Any Boolean algebra* $\mathfrak{B}$ *is isomorphically embeddable in the (algebra of) idempotents of an SM algebra.*

PROOF. (Informal, ignoring foundational questions of set theory). Let $\mathfrak{B}^S$ be an algebra of sets isomorphic to $\mathfrak{B}$ (Theorem 0.539), and let $M = B^S$ be the union of all sets of B, i.e. the set consisting of elements which are in any set of B^S. Assuming the axiom of choice, let M be well-ordered and let a_ξ be its ξth element. We define the universe M^* of the SM algebra to be the set of all sets ordinally similar to M and whose ξth element is the ordered pair $\langle n, a_\xi \rangle$, where n is an integer (allowed to be different for different elements of M). For any two elements A and B of M^*, whose ξth entries are $\langle n, a_\xi \rangle$ and $\langle m, a_\xi \rangle$ respectively, define $A + B$ to have $\langle n + m, a_\xi \rangle$ as its ξth entry, and $A \cdot B$ to have $\langle nm, a_\xi \rangle$ as its ξth entry. With obvious definitions for its 0 and 1, the structure is seen to be an SM algebra. Any $b \in B$ corresponds uniquely to a set $b^2 \in B^S$ which in turn corresponds to an idempotent of M^* whose ξth entry is $\langle 0, a_\xi \rangle$ if $a_\xi \notin b^S$ and is $\langle 1, a_\xi \rangle$ if $a_\xi \in b^S$. This correspondence sets up the isomorphism.

THEOREM 2.37. *Let* φ *be a quantifier-free formula in the language of Boolean algebras. If* $\vdash_{SM} \varphi$, *then* $\vdash_{BA} \varphi$.

PROOF. Suppose $\vdash_{SM} \varphi$ but not $\vdash_{BA} \varphi$. Then by the Gödel completeness theorem (0.3.2) there is a Boolean algebra. $\mathfrak{B}$ say, in which φ is false, that is $\mathfrak{B} \models \neg\varphi[a, b, c, \ldots]$ where a, b, c, ... are elements of B replacing free variables of φ (see 0.3.5 for notation). Let $\mathfrak{M}^*$ be an SM algebra in whose set of idempotents $\mathfrak{B}$ is isomorphically embeddable (Theorem 2.36). Then by 0.3.5,

$$\mathfrak{M}^* \models \neg\varphi[a^*, b^*, c^*, \ldots],$$

a^*, b^*, c^*, ... being the isomorphs of a, b, c, ... But this contradicts $\vdash_{SM} \varphi$, since provability of φ implies its truth in all models.

§ 2.4. Boole's notion of "uninterpretable"

Considering that Boole, an adherent of the Symbolical Algebra school of thought, was content with an uninterpreted or partially interpreted calculus and that he had no idea of logical semantics, it is understandable that there would be unclarity in his use of such semantic terms as 'uninterpretable' and 'interpretation' which he uses in connection with both equations and terms. We show here that substantially all of it—with exception of his 'interpreting' the expression $w = A+0B+\frac{0}{0}C+\frac{1}{0}D$, which requires special treatment—can be made clear in the context of SM algebras.

We return to Boole's footnote discussion (see our § 1.3) as to whether $x^3 = x$ indicates a process of logical trichotomy. His answer is that $x^3 = x$ is "not interpretable in the system of logic" since either factored form of the equation,

$$x(1-x)(1+x) = 0$$

$$x(1-x)(-1-x) = 0,$$

involves "uninterpretable" factors: $1+x$ is not interpretable "because we cannot conceive of the addition of any [non-empty] class x to the universe 1", and $-1-x$ is uninterpretable "because the symbol -1 is not subject to the law $x(1-x) = 0$, to which all class symbols are subject."

The topic of interpreting equations comes up again in Boole's Chapter VI (Of Interpretation) and, in apparent conflict with the above remarks on $x^3 - x = 0$, he gives a method of interpreting in logic any equation, $U = 0$, U containing logical symbols x, y, z, etc. "in combinations not fractional". This interpretation does not rely on the interpretation of U's terms but consists of the conjoint assertion of equations $t_i = 0$ for those constituents t_i in U's complete expansion which have a non-zero coefficient. (Equivalently, of the assertion of the one equation $\sum t_i = 0$). Boole says nothing about the special case of U having no non-zero coefficients—precisely the case of $x^3 - x$ and $x^2 - x$, both of which have the expansion $0 \cdot x + 0 \cdot \bar{x}$. If we consider the theory of SM algebras as codifying Boole's system, then the basis for this Chapter VI method, in more precise form, is contained in Theorem 2.35 which establishes that the equivalence,

$$U = 0 \leftrightarrow V[= \textstyle\sum t_i] = 0,$$

is true in all SM algebras, and hence in any model the two equations would have interpretations (in our contemporary sense) which are equivalent. Thus when x, y, z, etc. represent classes of some universe the (purely Boolean) equation $V = 0$ has its usual class calculus meaning and $U = 0$, an assertion (in general) about multisets, is true of the classes x, y, z, etc. precisely for the same values as $V = 0$, even though U may involve terms not interpretable as classes. This is analogous to the situation in ordinary mathematics with regard to, e.g., the equations $3\sqrt{2}x - 9\sqrt{2} = 0$ and $3x - 9 = 0$ with x restricted to range over integers. The first of these equations contains terms not "interpretable" in integer arithmetic; nevertheless they both are satisfied by the same integer(s) (i.e. 3). If we use Theorem 2.35 to extend Boole's method of interpreting equations, $U = 0$, to the case which he does not mention (all 0 coefficients in U's expansion), this would give the same (Boolean) interpretation for $x^3 - x = 0$ and $x^2 - x = 0$, namely that of $x\bar{x} = 0$. Presumably it is not this kind of interpretation of equations which Boole has in mind in his trichotomy discussion but rather that with regard to terms. We turn to a discussion of this notion.

As noted earlier Boole gives two distinct kinds of reasons for a (constant) term being uninterpretable: a formal or syntactic one ("-1 is not subject to the law $x(1-x) = 0$") and an informal or semantic one ("we cannot conceive of the addition of a class x to the universe 1"). For terms containing variables he has the more general notion of "independently interpretable", which he defines as "interpretable without supposing any relationship among the classes represented by the class symbols". (E.g. $x(1 - y)$ is independently interpretable, but $x + y$ is not since it needs the supposition that x and y are disjoint; and similarly $x - y$ needs the supposition that x includes y.) As a corollary to his rule for writing the interpretation for an equation $w = V$, V fractional in form (see our § 1.7), Boole "shows" that if a function V is independently interpretable then it satisfies the law $V(1 - V) = 0$. Boole's ideas and arguments are quite informal. To make them precise we again resort to SM algebras.

Let $\mathbf{L} = \{0, 1, +, \cdot\}$ be a (first-order) language for the theory of SM algebras and $\mathfrak{M} = \langle M, O_M, 1_M, +_M, \cdot_M \rangle$ an SM algebra, i.e. a model for the theory. Let $\mathfrak{B}_M$ be the Boolean algebra of $\mathfrak{M}$. It will be convenient to think of formal terms of $\mathbf{L}$ as being built up using Boolean as well as unrestricted variables. We say a_M is a *valuation of terms* of $\mathbf{L}$ (we are here on purpose avoiding the customary model-theoretic word 'interpretation' for this notion) if it is an assignment of members of M to variables of $\mathbf{L}$, with members of B_M assigned to Boolean variables, such that

$$a_M(0) = 0_M, \qquad a_M(1) = 1_M$$

$$a_M(T) = a_M(X), \quad \text{if } T \text{ is a variable } X$$

$$= a_M(x), \quad \text{if } T \text{ is a variable } x$$

$$a_M(T_1 \cdot T_2) = a_M(T_1) \cdot_M a_M(T_2)$$

$$a_M(T_1 + T_2) = a_M(T_1) +_M a_M(T_2).$$

DEFINITION. A Boolean multiset term T is *interpretable with respect to a valuation* a_M if $a_M(T) \in B_M$.

This corresponds to Boole's idea of a term representing a class, i.e. being interpretable when the class symbols represent classes.

We now give a semantic definition of 'independently interpretable' and show its equivalence with Boole's syntactic conception, i.e. idempotency.

DEFINITION. A Boolean multiset term T is *independently interpetable* if, for each SM algebra $\mathfrak{M}$ and any evaluation of terms a_M, T is interpretable with respect to a_M.

THEOREM 2.41. *For any Boolean multiset term T, the necessary and sufficient condition that T be independently interpretable is that $T^2 = T$ be a theorem, i.e. be provable from the axioms of SM algebras.*

PROOF (sketch). For arbitrary $\mathfrak{M}$ and a_M,

$$(1) \qquad a_M(T) \in B_M \leftrightarrow a_M(T) \cdot a_M(T) = a_M(T)$$
$$\leftrightarrow a_M(T^2) = a_M(T)$$

Hence if T is independently interpretable then $T^2 = T$ is true in all SM algebras and so provable (see the italicized remark near end of § 2.2). And if $T^2 = T$ is provable then it is true in all SM algebras and hence, by (1), T is independently interpretable. In what follows we shall shorten 'independently interpretable' to 'interpretable', and also use it interchangeably with '(provably) idempotent'.

To sum up, we agree with Boole that the equation $x^3 = x$ does not represent a process of trichotomy, though for a somewhat different reason. Speaking in terms of SM algebras—which is what we are using to give meaning to Boole's + symbol—we agree that

$$(2) \qquad x(1-x)(1+x)$$

is not interpretable in [the] logic [of classes] since (assuming $x \neq 0$) $1+x$ does not represent a class, i.e. is not idempotent. However (2) does have meaning in an algebra of multisets and does represent a threefold separation, though not a mutually exclusive one since $1+x$ overlaps with (in fact includes) x and $1-x$. (We have not given formal definitions of 'overlaps' or 'includes', but the intuitive meaning for multisets is clear.) Finally, all three equations

$$(3) \qquad x - x^3 = 0, \qquad x(1-x)(1+x) = 0, \qquad x - x^2 = 0,$$

are equivalent—being theorems of SM algebra—despite the fact that the second one has a term which is not 'interpretable' (in our technical sense). They are all satisfied by the same values of x, namely all, x being restricted to idempotents.

§ 2.5. The indefinite class symbol v

As we noted in § 1.10, Boole's use of the indefinite class symbol v does entail some confusion with respect to existential import and yet, for the most part, intelligible results do come out. Exactly what is going on?

Our explanation is that Boole is using, without being fully cognizant of it, one of the techniques of "natural deduction", in which one drops quantifiers so as to carry out deduction on the propositional level.

Consider a sentence of the form $\exists v F(v)$. If, from the assumption $F(v)$, one can deduce a result P which does not involve v, then one validly concludes that P is a logical consequence of $\exists v F(v)$. An example of this occurs in our proof (the (a) part) of Theorem 2.34. Arguments of this type are often embodied in so-called natural deduction formulations of the predicate calculus or sometimes, in other formulations, they may be established as derived rules. Clearly, when the quantifier $\exists v$ is dropped, the v in $F(v)$ is not a free variable subject to generalization by a universal quantifier but a "quasi-constant" subject to special safeguarding regulations. We shall not bother to spell out these regulations—the interested reader may consult Mendelson 1979, Chapter 2, Section 7, or Quine 1972, discussion of *EI*, p. 162. Boole's actual use of this technique is quite rudimentary and involves no other quantifiers, which is where the difficulties come in.

Let us look at his treatment of the universal affirmative, "All y's are x's", taken without existential import. Introducing quantification over class variables, this is expressible as $\exists v(y = vx)$, so that when Boole writes instead $y = vx$ he is in effect dropping the existential quantifier. However, as we have remarked, if one deduces, subject to appropriate safeguard conditions, a result not containing the variable v, then this result is indeed a logical consequence of the existentially quantified sentence. We have seen that the first thing Boole usually does with a sentence involving v is to eliminate it—e.g. he goes from $y = vx$ immediately to $y(1-x) = 0$; anything following from this involves no v and hence correctly follows from $\exists v(y = vx)$. Thus Boole's treatment of the universal affirmative (without existential import) is correct and justifiable provided that the special nature of the indefinite symbol v is recognized, which Boole informally does. However not having any means of expressing 'some' or 'non-empty' in his formalism the treatment of the particular affirmative doesn't fare as well.

As we know (§ 1.10) Boole thought that $vx = vy$, v an indefinite class symbol, expressed "Some x's are y's". With this symbolic rendition some traditional logical principles do come out correctly, e.g. simple conversion ("Some x's are y's" implies "Some y's are x's") and conversion by limitation ("All y's are x's" implies "Some x's are y's"), which Boole establishes by the derivation (p. 229)

$$y = vx$$
$$vy = vvx = vx$$
$$vx = vy.$$

But as Peirce has pointed out [1870, 1933 III, p. 91], one could argue similarly from "Some x's are not y's" to the invalid conclusion "Some y's are not x's" in the following manner:

$$vx = v(1 - y)$$
$$v - vx = v - v(1 - y) = vy$$
$$vy = v(1 - x).$$

To see where, or how, Boole was misled we express "Some x's are y's" not by the customary $xy \neq 0$ but by the equivalent

$$\exists v(vx = vy \text{ and } vx \neq 0)$$

in which now the clause $vx \neq 0$ carries the necessary existential import. But when Boole writes instead,

$$vx = vy,$$

not only is he, in effect, dropping the existential quantifier but also the side condition $vx \neq 0$, which however is not dispensable. If in the Peirce example the side condition $vx \neq 0$ were made explicit and carried through one would see that the conclusion is not the purported 'Some y's are not x's'.

In summary, then, if the Aristotelian particular sentence forms are ignored, Boole's use of the indefinite class symbol v is justifiable and sound, merely amounting to an understood existentially quantified variable. For example, when applying his general method so as to obtain a solution which he writes as

$$\begin{cases} w = A + vC, \\ D = 0, \end{cases}$$

we shall here understand by $w = A + vC$ the sentence $\exists v(w = A + vC)$. That the v here could be taken to be interpretable was taken for granted by Boole. He could have shown that, in all cases of interest, it indeed is idempotent:

THEOREM 2.50. *If $AC = 0$ and $w = A + QC$, with A and C idempotent, then there is an idempotent v such that $w = A + vC$.*

PROOF. If $w = A + QC$, C idempotent, then $w = A + (QC)C$. The proof is completed by showing that QC is idempotent:

$$\begin{aligned} 0 = w^2 - w &= (A + QC)^2 - (A + QC) \\ &= A + (QC)^2 - (A + QC) \\ &= (QC)^2 - (QC). \end{aligned}$$

§ 2.6. The solution of Boolean multiset equations for an unknown

In contrast to contemporary Boolean algebra a prominent feature of Boole's logical sytem is the occurrence of division in the process of solution of Boolean multiset equations. But a close inspection of Boole's technique indicates that division is rather incidental and serves mainly to separate constituents into four types according as one or the other of the "numerical" coefficients 1, 0, $\frac{0}{0}$, or $\frac{1}{0}$ is obtained in the development of a fractional expression. Indeed, we shall show in this section that all the results of Boole's method of solving class calculus problems can be obtained within his system without recourse to division. Our result here will also show that, for class calculus problems expressible in the form of a Boolean equation for an unknown, Boole's method always gives the correct solution, as the large number of worked examples in the *Laws of Thought* bear witness.

We recall that Boole's method is to first reduce the equation (or system of equations) in an unknown Boolean w to the form

(1) $$Ew = F,$$

where E and F are multiset terms not involving w, then solve for w to obtain $w = F/E$; next develop F/E with respect to the Boolean variables appearing in E or F, so as to obtain

$$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D, \tag{2}$$

and then writing for this equation the *interpretation*

$$\begin{cases} w = A + vC \\ D = 0. \end{cases} \tag{3}$$

By virtue of our discussion in § 2.5 we take Boole's $w = A + vC$ to be $\exists v(w = A + vC)$ and, making use of Theorem 0.54, we have then that Boole's solution is equivalent to any one of the three forms

$$\text{(4)} \quad \begin{array}{ll} \text{(i)} & \begin{cases} \exists v(w = A + vC), \\ D = 0; \end{cases} \\ \text{(ii)} & \begin{cases} w = A + wC, \\ D = 0; \end{cases} \\ \text{(iii)} & \begin{cases} A \subseteq w \subseteq A + C, \\ D = 0, \end{cases} \end{array}$$

all these being expressed in purely Boolean terms (the $+$'s can be taken as either $+_B$ or $+_\Delta$). Boole believed that the argument via his interpretation of (2) correctly established the result that (4) expressed the solution of $Ew = F$. To prove the result directly in **SM** we use as an intermediary the equation,

$$(A + B)w = A + D, \tag{5}$$

which by Theorem 0.56 is equivalent to (4) in **BA**, and hence also in **SM**:

$$\vdash_{SM} (A + B)w = A + D \leftrightarrow (w = A + wC)(D = 0). \tag{6}$$

For Boole, (5) would obviously be equivalent to $Ew = F$ since solving it for w gives $(A + D)/(A + B)$, whose expansion is $A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D$. This is exactly the same as that for F/E except that all the coefficients of the fourth type are $\tfrac{1}{0}$, whereas for F/E there may be others of this type; but either expansion receives the same interpretation. The following furnishes a direct proof of their equivalence without resort to expansions of fractional forms.

THEOREM 2.61.

$$(7) \qquad \vdash_{SM} Ew = F \leftrightarrow (A+B)w = A+D,$$

where E and F are Boolean multiset terms not containing w and where A, B and D are, respectively, the sums of constituents which appear with coefficient 1, 0 and $\frac{1}{0}$ (or any other fourth type coefficient) in Boole's complete development of F/E. If E and F are Boolean then $A+B$ and $A+D$ are their respective expansions.

PROOF. Although in the statement of this theorem we mention F/E, the definition of A, B, D depends only on the ordered pair of terms E and F and there is no actual division. Let the equation $Ew = F$ involve, other than w, n Boolean variables and let t_i ($i = 1,\ldots,2^n$) be the constituents on these variables. Let $\sum e_i t_i$ and $\sum f_i t_i$ be the respective expansions of E and F, the $\sum$ understood to run from $i = 1$ to $i = 2^n$, and e_i and f_i are numerical coefficients (integers) which may be either positive, negative or zero. We then have

$$(8) \qquad \begin{aligned} Ew = F &\leftrightarrow \left(\sum e_i t_i\right) w = \sum f_i t_i \\ &\leftrightarrow \sum (e_i w - f_i) t_i = 0 \\ &\leftrightarrow \bigwedge \ (e_i w - f_i = 0), \end{aligned}$$

where the last expression denotes the conjunction of 2^n equations. Now define

$$a_i = \begin{cases} 1 & \text{if } e_i = f_i \text{ and } e_i, f_i \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$b_i = \begin{cases} 1 & \text{if } e_i \neq 0 \text{ and } f_i = 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$c_i = \begin{cases} 1 & \text{if } e_i = f_i = 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$d_i = \begin{cases} 1 & \text{if } f_i \neq 0 \text{ and } e_i \neq f_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is relatively easy to show that, for all i,

$$(9) \qquad e_i w - f_i = 0 \leftrightarrow (a_i + b_i)w - (a_i + d_i) = 0.$$

For instance, if $e_i = f_i = r, r \neq 0$, so that $a_i = 1$, $b_i = d_i = 0$, then

$$\begin{aligned} e_i w - f_i = 0 &\leftrightarrow rw - r = 0 \\ &\leftrightarrow r(w-1) = 0 \\ &\leftrightarrow w - 1 = 0, \quad \text{since } r \neq 0 \\ &\leftrightarrow (a_i + b_i)w - (a_i + d_i) = 0; \end{aligned}$$

or, if $e_i = r$, $f_i = s$, $s \neq 0$, $r \neq s$, so that $a_i = b_i = 0$, $d_i = 1$, then

$$\begin{aligned} e_i w - f_i = 0 &\rightarrow rw - s = 0 \\ &\rightarrow rw = s \\ &= (rw)w = s, \quad \text{since } w = w^2 \\ &\rightarrow (s)w = s \\ &\rightarrow w = 1, \quad \text{since } s \neq 0 \\ &\rightarrow r = s \\ &\rightarrow 0 \cdot w - 1 = 0, \quad \text{since } r \neq s. \end{aligned}$$

As the converse implication is trivially true and

$$0 \cdot w - 1 = 0 \leftrightarrow (a_i + b_i)w - (a_i + d_i) = 0,$$

we have (9) for this case.

From (8) and (9) we obtain

$$(10) \qquad Ew = F \leftrightarrow \left(\sum a_i t_i + \sum b_i t_i\right) w - \left(\sum a_i t_i + \sum d_i t_i\right) = 0$$

and since, as is easy to check, $\sum a_i t_i = A$, $\sum b_i t_i = B$ and $\sum d_i t_i = D$, we have our result. The last sentence in the statement of the theorem is also readily checked.

We now show that Boole's method always gives the correct result for Boolean algebra problems.

Let Π ("premises") be a conjunction of quantifier-free formulas in the language of **BA**, and suppose

$$\vdash_{\text{SM}} \Pi \rightarrow Ew = F$$

then by (6) and (7)

$$\vdash_{SM} \Pi \to (w = A + wC)(D = 0).$$

But as the displayed formula is quantifier-free and in the language of **BA** we have, by Theorem 2.37,

$$\vdash_{BA} \Pi \to (w = A + wC)(D = 0).$$

We close this section with a number of items connected with Boole's solution,

$$\text{(11)} \qquad \begin{cases} w = A + vC \\ D = 0, \end{cases}$$

for $Ew = F$.

The first of the equations in (11) is described by him (p. 92) as showing "what elements [i.e. classes x, y, z, etc.] enter, or may enter, into the composition of w, the class of things whose definition is required; and the second equation "shows what relations exist among the elements of the original problem in perfect independence of w." This last assertion of Boole's is equivalent to saying that $D = 0$ is the result of eliminating w from $Ew = F$. To show this we go over to the equivalent equation $(A + B)w = A + D$; by Theorem 2.34 the result of eliminating w is

$$[(A + B) - (A + D)][-(A + D)] = 0,$$

which reduces to $D = 0$.

Boole doesn't consider the particular case of D coming out to be 1 (the sum of all constituents). This can only happen if the original equation is of the form $0 \cdot w = c$, c a constant other than 0; in which case $1 = 0$ is, appropriately, the "solution".

Finally, we show that Boole's solution for $Ew = F$ gives (i) the best possible Boolean bounds on w, and (ii) the strongest condition not involving w which follows from $Ew = F$. The proof is as follows:

(i) Since the equation $Ew = F$ is equivalent to the combined conditions

$$\begin{cases} A \subseteq w \subseteq A + C \\ D = 0, \end{cases}$$

if $Ew = F$ has a solution then $D = 0$. In that case $Ew = F$ is equivalent to $(A + B)w = A$, which is satisfied by $w = A$ and by $w = A + C$. Hence if there were Boolean terms P and Q (not containing w) such that for every solution w

$$A \subseteq P \subseteq w \subseteq Q \subseteq A + C,$$

then by taking first $w = A$ we find that $P = A$, and by taking $w = A + C$ that $Q = A + C$. Hence A and $A + C$ are the narrowest Boolean bounds obtainable which include between them any w satisfying $Ew = F$.

(ii) We have

$$\exists w(Ew = F) \leftrightarrow \exists w(A \subseteq w \subseteq A + C \wedge D = 0)$$
$$\leftrightarrow D = 0.$$

If now for some D' not involving w,

$$(Ew = F \rightarrow D' = 0) \wedge (D' = 0 \rightarrow D = 0),$$

then

$$(\exists w(Ew = F) \rightarrow D' = 0) \wedge (D' = 0 \rightarrow D = 0).$$

But as $\exists w(Ew = F) \leftrightarrow D = 0$, we have

$$D' = 0 \leftrightarrow D = 0.$$

Thus $D = 0$ is the strongest condition not involving w which follows from $Ew = F$. We can then speak of $D = 0$ as being *the* necessary condition relating the elements other than w in $Ew = F$.

A similar argument shows that the result of eliminating $x, y, \ldots, z$ from $f(x, y, \ldots, z, u, v, \ldots, w) = 0$ gives the strongest conclusion implied by this equation relating $u, v, \ldots, w$.

§ 2.7. Boolean equations and division

In the preceding section we have shown how to obtain the solution of a Boolean multiset equation for a Boolean unknown without appealing to the notion of division (or of a quotient). We now wish to explicate Boole's quite different procedure which seemingly does involve division, though to a limited extent and with restrictions on the operation not clearly justified.

Recall that his procedure for solving an equation, e.g. $ax = b$, for x involves the following stages:

(1) $ax = b$

(2) $x = \frac{b}{a}$ (solving (1) for x by division)

(3) $\frac{b}{a} = 1 \cdot ab + 0 \cdot a\bar{b} + \frac{0}{0} \cdot \bar{a}\bar{b} + \frac{1}{0} \cdot \bar{a}b$ $\left(\text{expanding } \frac{b}{a}\right)$

(4) $x = 1 \cdot ab + 0 \cdot a\bar{b} + \frac{0}{0} \cdot \bar{a}\bar{b} + \frac{1}{0} \cdot \bar{a}b$ (combining (2) and (3))

(5) $\begin{cases} x = ab + v\bar{a}\bar{b} \\ \bar{a}b = 0 \end{cases}$ (interpreting (4))

We shall show how to get from (1) to (5) by steps which closely resemble Boole's *and* which are mathematically justifiable.

Assume that (1) is an equation in a Boolean algebra (or Boolean ring) $\mathfrak{B}$. There are two impediments in the way of making the solution of (1) look as if one were solving an equation in ordiary algebra so as to get (2). In the first place, whereas in ordinary algebra (1) always has a solution (save for $a = 0$, $b \neq 0$), in Boolean algebra this need not be the case: a necessary and sufficient condition for a solution being that b be contained in a, i.e. that $\bar{a}b = 0$ (Theorem 0.55). Secondly, even if there were a solution it need not be unique—any element of B differing from a solution of (1) by a Boolean multiple of (i.e. a part of) $\bar{a}\bar{b}$ is also a solution, since $ax = b \rightarrow a(x + v\bar{a}\bar{b}) = b$. The difficulty with regard to non-uniqueness can be obviated by going over to a homomorphic structure in which the differences are "factored out" (as one does, for example, in a problem concerning days of the week by going over to arithmetic modulo 7).

To say that two solutions of (1) differ by a (Boolean) multiple of $\bar{a}\bar{b}$ is to say that the solution set is a residue class in the factor ring $\mathfrak{B}/(\bar{a}\bar{b})$, where $(\bar{a}\bar{b})$ is the principal ideal generated by $\bar{a}\bar{b}$(= set of classes which are included in $\bar{a}\bar{b}$). (Note that we are considering $\mathfrak{B}$ to be a ring.) Let us introduce the abbreviated notation $[r]$ for the residue class

$r + (\bar{a}\bar{b})$ determined by the element r of B. (We are using '+' for the ring addition.) Then the mapping $r \to [r]$ is a homomorphism of rings (Theorem 0.425) and hence (1) implies that

$$(6) \qquad [a][x] = [b]$$

holds in the (Boolean) ring $\mathfrak{B}/(\bar{a}\bar{b})$. Conversely (6) implies (1); for by (6),

$$ax - b = v\bar{a}\bar{b}.$$

On multiplying this equation through by $a +_B b$ (i.e. by $ab + \bar{a}b + a\bar{b}$) and using simple Boolean identities one arrives at (1). Thus the problem of solving (1) for x is equivalent to solving (6) for $[x]$. Since any two solutions of (6) for $[x]$ differ by a multiple of $\overline{[a]}\overline{[b]}$, and $\overline{[a]}\overline{[b]} = [\bar{a}\bar{b}] = [0]$, i.e. the zero of the ring $\mathfrak{B}/(\bar{a}\bar{b})$, we see that the solution of (6), if any exist, is unique. To bring out the relation of what we are doing with Boole's procedure we alter the customary notation for a principal ideal in a ring and write $\frac{0}{0}\bar{a}\bar{b}$ in place of $(\bar{a}\bar{b})$. We then sum up the preceding discussion as follows:

THEOREM 2.70. *The equation $ax = b$ in a Boolean ring $\mathfrak{B}$ is equivalent to the equation $[a][x] = [b]$ in the factor ring $\mathfrak{B}/\frac{0}{0}\bar{a}\bar{b}$; the latter equation having at most one solution. If the solution for $[x]$ is given by*

$$[x] = [p] = p + \tfrac{0}{0}\bar{a}\bar{b}$$

then in $\mathfrak{B}$

$$x = p + v\bar{a}\bar{b} \qquad v \in B$$

and the set of all such x as v ranges over the elements of B is the solution set for the equation $ax = b$.

In order to reproduce the part of Boole's technique which uses division we employ the notion of a ring of quotients for a Boolean algebra (§ 0.6), and using not $\mathfrak{B}$ but $\mathfrak{B}^* = \mathfrak{B}/\frac{0}{0}\bar{a}\bar{b}$ so that we can use (6) rather than (1). Accordingly let

$$\mathfrak{Q} = \mathfrak{B}^*([a] +_B B^*)^{-1}$$

be the ring of quotients of B^* by $[a] +_B B^*$, and suppose we do have an

$[x]$ such that

(7) $$[a][x] = [b] \quad \text{in } B^*.$$

Then, since $[a]$ is in the denominator set for $\mathfrak{Q}$, we have

(8) $$\left(\frac{[a][x]}{[a]}\right)_{[a]} = \left(\frac{[b]}{[a]}\right)_{[a]} \quad \text{in } \mathfrak{Q}.$$

By cancellation (Theorem 0.63(i)),

(9) $$\left(\frac{[x]}{[1]}\right)_{[a]} = \left(\frac{[b]}{[a]}\right)_{[a]}.$$

Using Theorem 0.63 (ii), and the fact that $r \to [r]$ is a homomorphism to replace $[a][b]$ by $[ab]$, we obtain from (9)

(10) $$\left(\frac{[x]}{[1]}\right)_{[a]} = \left(\frac{[ab]}{[1]}\right)_{[a]}.$$

We now go over to the ring $\mathfrak{B}^*/(\overline{[a]})$ which, by Theorem 0.62 is isomorphic to $\mathfrak{Q}$. But first note that $\overline{[a]} = [\bar{a}]$, and that $[\bar{a}] = [\bar{a}b]$ since $\bar{a}$ and $\bar{a}b$ differ by $\bar{a}\bar{b}$ and thus determine the same residue class modulo $\bar{a}\bar{b}$. Hence the ideal $(\overline{[a]})$ is the same as $([\bar{a}b])$. We change the usual notation and represent this ideal by $\frac{1}{0}\bar{a}b$. (Note that $\frac{1}{0}\bar{a}b$ is an ideal in $\mathfrak{B}^*$, whereas $\frac{0}{0}\bar{a}\bar{b}$ is an ideal in $\mathfrak{B}$.) Using '$\approx$' to indicate the relation of isomorphism between elements of $\mathfrak{Q}$ and $\mathfrak{B}^*/(\overline{[a]})$ we have, from Theorem 0.62,

$$\left(\frac{[b]}{[a]}\right)_{[a]} \approx [ab] + \tfrac{1}{0}\bar{a}b$$

or, in fuller notation,

(11) $$\left(\frac{[b]}{[a]}\right)_{[a]} \approx 1 \cdot ab + 0 \cdot a\bar{b} + \tfrac{0}{0}\bar{a}\bar{b} + \tfrac{1}{0}\bar{a}b$$

which is what for us corresponds to Boole's equating b/a to its expansion. (Note that in (11) the three plus signs have different meanings.) Since Boole has $x = b/a$ he can then write

(12) $$x = 1 \cdot ab + 0 \cdot a\bar{b} + \tfrac{0}{0}\bar{a}\bar{b} + \tfrac{1}{0}\bar{a}b,$$

whereas we can only equate the isomorphic images of the two sides of (10) to obtain

$$(13)\qquad [x]+\tfrac{1}{0}\bar{a}b = 1\cdot ab+0\cdot a\bar{b}+\tfrac{0}{0}\bar{a}\bar{b}+\tfrac{1}{0}\bar{a}b.$$

(Recall that Boole never used the equality in (12) as such.) The next step in Boole's procedure was the (inadequately justified) interpretation of (12) in terms of classes. For us the corresponding thing is the restatement of (13) in terms of $\mathfrak{B}$. If $b = ax$ then $\bar{a}b = \bar{a}ax = 0$, so that

$$\tfrac{1}{0}\bar{a}b = ([ab]) = ([0])$$

is the zero ideal of $\mathfrak{B}^*$. Thus from (13)

$$[x] = [ab] = 1\cdot ab+0\cdot a\bar{b}+\tfrac{0}{0}\bar{a}\bar{b}$$

so that by Theorem 2.70,

$$x = ab+v\bar{a}\bar{b}$$

Consequently $ax = b$ implies

$$(14)\qquad \begin{cases} x \;= ab+v\bar{a}\bar{b} & v\in B\\ \bar{a}b = 0 \end{cases}$$

(One readily shows that (14) implies $ax = b$).

Thus Boole's (1)–(5) for us translates to

$$(1^*)\qquad [a][x] = [b]$$

$$(2^*)\qquad \left(\frac{[x]}{[1]}\right)_{[a]} = \left(\frac{[b]}{[a]}\right)_{[a]}$$

$$(3^*)\qquad \left(\frac{[b]}{[a]}\right)_{[a]} \approx 1\cdot ab+0\cdot a\bar{b}+\tfrac{0}{0}\bar{a}\bar{b}+\tfrac{1}{0}\bar{a}b$$

$$(4^*)\qquad [x]+\tfrac{1}{0}\bar{a}b = 1\cdot ab+0\cdot a\bar{b}+{}_0\bar{a}\bar{b}+\tfrac{1}{0}\bar{a}b$$

$$(5^*)\qquad \begin{cases} x \;= ab+v\bar{a}\bar{b} & v\in B\\ \bar{a}b = 0 \end{cases}$$

with each step meaningful and mathematically justifiable.

Boole's actual argument is conducted in terms of solving an equation

$Ew = F$ for the class term w, and E and F are (effectively) Boolean class terms. For this our (13) would give

$$[w] + \tfrac{1}{0}\bar{E}F = 1 \cdot EF + 0 \cdot E\bar{F} + \tfrac{0}{0}\bar{E}\bar{F} + \tfrac{1}{0}\bar{E}F$$
$$= 1 \cdot A + 0 \cdot B + \tfrac{0}{0}C + \tfrac{1}{0}D$$

using Boole's designations A, B, C, D for the sums of constituents in the respective expansions of EF, $E\bar{F}$, $\bar{E}\bar{F}$, $\bar{E}F$. It is clear that, when 0's and 1's are assigned to the class symbols present in E and F, then those assignments corresponding to constituents present in A are assignments that produce $\tfrac{1}{1}$ for F/E, those in B produce $\tfrac{0}{1}$, those in C, $\tfrac{0}{0}$ and those in D, $\tfrac{1}{0}$. Hence, as with Boole, we can write

$$\begin{cases} w = A + vC & v \text{ "indefinite"} \\ D = 0 \end{cases}$$

as the solution for $Ew = F$.

Of the various forms a Boolean quotient can take, such as

$$\left(\frac{[F]}{[1]}\right)_{[E]}, \quad \left(\frac{[EF]}{[1]}\right)_{[E]}, \quad \left(\frac{[F]}{[E]}\right)_{[E]},$$

all of which are equal, it is the last which is closest in appearance to Boole's fractional F/E from which he gets his expansion $A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D$. Accordingly we shall take this to be the "standard" form for a Boolean quotient. In the reverse direction, one gets this form from an expansion $A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D$ by constructing the fraction

$$\frac{A + D}{A + B}$$

or, in official language,

$$\left(\frac{[A + D]}{[A + B]}\right)_{[A+B]},$$

with the square brackets referring to the ideal $\tfrac{0}{0}C$.

§ 2.8. Additional remarks on $\frac{0}{0}$ and $\frac{1}{0}$

Throughout *Laws of Thought* Boole carefully refrains from using division or the symbols $\frac{0}{0}$ and $\frac{1}{0}$ in any way but in the prescribed manner we have discussed, namely in connection with solving an equation $Ew = F$ for w, expanding F/E into the form $1 \cdot A + 0 \cdot B + \frac{0}{0} \cdot C + \frac{1}{0} \cdot D$, where A, B, C, D are mutually exclusive sums of constituents on the Boolean variables in F and E, and then giving his interpretation. Any algebraic operations are done before using division, which is only used once; there are no algebraic operations involving either fractions as such or the constants $\frac{0}{0}$ and $\frac{1}{0}$. (But see our endnote to § 2.8.)

In VENN 1894 we have, however, a few instances of these notions being considered outside of Boole's specifically limited context. At the end of his Chapter XI (Logical Statements and Equations) Venn sums up the use of indefinite terms in the table [1894, p. 302]:

(i)	$x = z$	leads to	$x\bar{z} = 0, \quad \bar{x}z = 0$
(ii)	$x = z + \frac{0}{0}w$	leads to	$x\bar{z}\bar{w} = 0, \quad \bar{x}z = 0$
(iii)	$x + \frac{0}{0}y = z + \frac{0}{0}w$	leads to	$x\bar{z}\bar{w} = 0, \quad \bar{x}\bar{y}z = 0$

(It is not clear from the text whether he is claiming the conditions on the right to be necessary, or necessary and sufficient, for those on the left.) The first of these is of course unproblematic. As for the second, $x = z + \frac{0}{0}w$ is given no meaning in Boole's official procedure since $z + \frac{0}{0}w$ (with the variables 'z' and 'w') could never arise in an expansion. By interpreting $\frac{0}{0}$ as standing for some indefinite class between 0 and 1 (so that $z + \frac{0}{0}w$ stands for an indefinite class between z and $z + w$) Venn concludes that there can be no z outside of x (hence $\bar{x}z = 0$) and [there being no x outside of $z +_B w$] that $x\bar{z}w = 0$. By using the (quantifier elimination) Theorem 2.34 we can formally justify the equivalence of the conditions in (ii) (on the right) with

$$(1) \qquad \exists v(x = z + vw).$$

Thus to avoid the unformalized notion of an "indefinite" term we would take '$x = z + \frac{0}{0}w$' to be '$x \in z + \frac{0}{0}w$', that is treating $z + \frac{0}{0}w$ as a residue class, as in our preceding section. Interpreted in this manner Venn's (ii) is then correct.

In the case of (iii) the equation

$$x + \tfrac{0}{0}y = z + \tfrac{0}{0}w \tag{2}$$

likewise presents difficulties as to its meaning. Of it Venn says: "Adopting the same explanations as before [in connection with (ii)], all we can say now is that there can be no x outside of $z + w$, and no z outside of $x + y$". This leads to his "$xz\bar{w} = 0, \bar{x}\bar{y}z = 0$". We can come out with this conclusion if we take (2) to mean

$$\exists v_1 \exists v_2 (x + v_1 y = z + v_2 w) \tag{3}$$

and eliminate the quantifiers. However, one could also view (2) as an equality between residue classes, so that instead of (3) we have

$$\forall u(\exists v_1 (u = x + v_1 y) \leftrightarrow \exists v_2 (u = z + v_2 w)), \tag{3'}$$

which leads to the stronger conditions

$$\begin{cases} \bar{x}z = 0, \; y\bar{z}\bar{w} = 0 \\ x\bar{z} = 0, \; \bar{x}\bar{y}w = 0. \end{cases} \tag{4}$$

Without a more specific context one has no way of choosing between these two possible meanings for (2).

Similar difficulties occur in connection with the logical meaning of the equation $a/b = c/d$ which Venn [1894, p. 343] explores in a discussion of the question:

If $ad = bc$, is it correct to conclude that $\dfrac{a}{b} = \dfrac{c}{d}$?

an inference form Venn says was used by Lambert [1782] more than once. Venn answers the question in the negative by comparing the class denials involved in $ad = bc$ (i.e. those classes which must be set equal to 0) with the class denials coming from $a/b = c/d$. These he obtains by considering the "developed form" of $a/b = c/d$, which he writes

$$ad + \tfrac{0}{0}ad = bc + \tfrac{0}{0}bc \tag{5}$$

(for which there is no precedent in Boole) and obtaining the denials by use of his (iii). To these denials he adds $a\bar{b} = 0$, $c\bar{d} = 0$ (coming from the $\frac{1}{0}$ terms in the expansions of a/b and c/d). Since the latter set of denials is more than that for $ad = bc$ he concludes to the invalidity of the inference.

Schröder [1891, p. 533] also discusses this question and likewise comes to a negative conclusion, though for different reasons. For Schröder a/b and a/d are conditionally defined solutions of the respective equations $ax = b$ and $cx = d$, under the respective conditions $a\bar{b} = 0$ and $c\bar{d} = 0$. Thus since such solutions, when they exist, need not be unique there are manifold possibilities for interpreting $a/b = c/d$, whether as an equation between arbitrary solutions, or sets of solutions. In all cases Schröder comes out with a negative conclusion to the inference.

From our point of view a quotient is only defined with reference to a denominator set, and equality of quotients is defined only for quotients in the same ring of quotients. Thus $a/b = c/d$ has no meaning unless b and d are in a common denominator set. We can *give* the equation a meaning by assuming we are dealing with the ring of quotients determined by taking the filter $bd +_B B$ (where a, b, c, $d \in B$) as a denominator set. Then $ad = bc$ does imply

$$\left(\frac{a}{b}\right)_{bd} = \left(\frac{c}{d}\right)_{bd}, \tag{6}$$

inasmuch as from the hypothesis $bd(ad - bc) = 0$.

§ 2.9. The partial algebra of Boolean quotients

Corresponding to each element a of a Boolean algebra $\mathfrak{A}$ there is an algebra of Boolean quotients $\mathfrak{A}(a +_B A)^{-1}$. For differing a's these various algebras have an autonymous existence and are not interrelated. However, in connection with our discussion of Boole's ideas on probability we would like to consider the collection of all possible Boolean quotients (for a given $\mathfrak{A}$) as forming a kind of structure. The notion that does this for us is a "partial algebra".

An operation on a set which need not be defined for all argument values is a partial operation. By allowing the operations referred to in the definition of an algebraic structure (§ 0.2) to be partial operations, we obtain a *partial algebra*. (A treatment of partial algebras from the viewpoint of universal algebra is given in Chapter 2 of GRÄTZER 1968.)

Let $\mathfrak{B}$ be a Boolean algebra. We define BQ($\mathfrak{B}$), *the partial algebra of Boolean quotients* (for $\mathfrak{B}$) as follows. The universe of BQ($\mathfrak{B}$) is to be the union of the universes of the rings of quotients $\mathfrak{B}^*([F] +_B B^*)^{-1}$, where $\mathfrak{B}^* = \mathfrak{B}/\frac{0}{0}\bar{E}\bar{F}$, and E and F range over B. The operations of BQ($\mathfrak{B}$) are to be those of the rings of quotients. Thus there are as many operations (of each kind) as there rings of quotients.

Of particular interest to us in Chapter 5 will be elements $[E]/[F]$ of a BQ($\mathfrak{B}$) for which $\bar{E}\bar{F} = 0$. The development of such a quotient lacks the $\frac{0}{0}$ part. In such cases there is no point in maintaining a distinction between $G \in B$ and $[G] \in B^*$ since $[G] = G + \frac{0}{0}0 = G + (0)$. Consequently in such cases we drop the use of the square brackets; for example, since $(\overline{\bar{F} + EF})\bar{F} = 0$, we will write

$$\frac{\bar{F} + EF}{F} \quad \text{instead of} \quad \frac{[\bar{F} + EF]}{[F]}.$$

Elements of BQ($\mathfrak{B}$) of the kind just described fall naturally into (Boolean) subalgebras:

THEOREM 2.90. *If b and $\bar{V}$ are elements of a Boolean algebra $\mathfrak{B}$, with $V \neq 0$, then the set of fractions of the form*

$$\frac{\bar{V} + Vb}{V}\left[= \frac{V \to b}{V} = Vb + 0V\bar{b} + \tfrac{0}{0}0 + \tfrac{1}{0}\bar{V}\right]$$

constitutes, as b ranges over B, a Boolean algebra isomorphic to $\mathfrak{B}|V$.

PROOF. From the standard form of the quotient,

$$\left(\frac{[\bar{V} + Vb]}{[V]}\right)_{[V]},$$

we can drop the square brackets since here they refer to the zero ideal $\frac{0}{0}0$. As it is easily verified (§ 0.61) that

$$\left(\frac{\bar{V} + Vb}{V}\right)_V = \left(\frac{b}{V +_B b'}\right)_V \quad \text{for any } b' \in B$$

we see that as b ranges over B we have the set of all elements of

$$\mathfrak{B}(V+_B B)^{-1},$$

which by Theorem 0.62 is isomorphic to $\mathfrak{B}|V$.

§ 2.10. Notes to Chapter 2

(for § 2.1)

In the first edition of this book, not realizing that another name for the notion was current, I used in place of 'multiset' the term 'heap'. According to Donald E. Knuth (letter of August, 1977): "The word 'multiset' is due to N. G. deBruijn who suggested it to me in private correspondence some years ago. I used the term in my book *Seminumerical Algorithms* in 1969 (see especially p. 551) and more extensively in my book *Sorting and Searching* [Vol. 2 of *The Art of Computer Programming*, Addison-Wesly, Reading, Mass.] in 1973 (see the index to that book). The word is used by hundreds of mathematicians and is the most popular word for the concept."

(for § 2.2)

NOTE 1. We are, to be sure, not attributing the idea of a calculus of multisets to Boole, only using it to explain his partially interpreted system. So far as I know the first recognition that multisets admits of mathematical treatment occurs in WHITNEY 1933. In Part I of his paper Whitney shows how to reproduce operations with (ordinary) sets in a more typically algebraic form by using characteristic functions, i.e. numerical-valued functions c_A on elements of a "universe" with c_A having the value 1 when $x \in A$ and 0 if $x \notin A$. Being numerical-valued functions, ordinary algebraic operations of addition and multiplication then apply to characteristic functions. In Part II of the above paper Whitney considers the idea of a function associating to each element *any* integer, and not just 1 or 0. Such a function will not be the "characteristic" function of a "real" set "but we may consider it as the characteristic function of a *generalized set* where each element is counted any number [positive, negative, or zero] times." However, instead of operating with generalized sets *per se* Whitney deals only with their associated characteristic functions.

NOTE 2. Multisets (defined as functions from objects into the set of cardinals, and having an appropriate inverse) are introduced as a fundamental tool in an intensive study of families of sets in RADO 1975. This reference was called to our attention by I. Grattan-Guinness.

(for § 2.3)

NOTE 1. That the set of idempotents of a commutative ring with unit forms a Boolean algebra (under the operations of ring multiplication, Boolean sum and complement) was shown in FOSTER 1945.

NOTE 2. A *pseudo-Boolean* function is a mapping $F: B_2^n \to Z$ from n-tuples of elements of $B_2 = \{0, 1\}$ to the integers Z. (See, e.g., P. L. Hammer (Ivanescu) and S. Rudeanu, *Boolean Methods in Operations Research*. Springer-Verlag 1968). If the variables of a multiset term are restricted to range of the set $\{1, 0\}$ then the multiset term represents a pseudo-Boolean function, and every pseudo-Boolean function is representable by such a term. This is clear since for any term $\Phi(x_1, \ldots, x_n)$ we have

$$\Phi(x_1, \ldots, x_n) = \sum_{i=1}^{2^n} z_i C_i,$$

where the C_i are the constituents on $x_1, \ldots, x_n$ and the z_i are integers, and for any pseudo-Boolean function F we can construct such a term with the z_i being the F values for each of the 2^n possible assignments of 0 and 1 to its arguments and the C_i the constituents corresponding to such assignments.

(for § 2.4)

We are indebted to John Corcoran who, in the course of a correspondence with us in 1981, pointedly brought out the need for semantic clarification of the notions of 'interpretable' and 'uninterpretable' used by Boole in connection with equations and terms.

(for § 2.5)

Difficulties with regard to Boole's use of the indefinite term v (and division as well) were noted immediately on the publication of *Mathematical Analysis of Logic* by De Morgan and discussed by him in a draft of a letter to Boole which, however, was never sent. See SMITH 1982, p. 26.

(for § 2.8)

A rare exception to the assertion of the first paragraph of § 2.8 occurs on p. 322 of BOOLE 1854 where he goes from

$$w = \frac{-xy}{x\bar{y}}$$

to

$$1 - w \left[= 1 - \frac{-xy}{x\bar{y}} = \frac{x\bar{y} + xy}{x\bar{y}} \right] = \frac{x}{x\bar{y}}.$$

In general, if

$$w = \frac{E}{F} = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D$$

then

$$1 - w = \frac{F - E}{F} = B + 0A + \tfrac{0}{0}C + \tfrac{-1}{0}D,$$

leading to the Boole interpretations

$$\begin{matrix} A \subseteq w \subseteq A + C \\ D = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} B \subseteq 1 - w \subseteq B + C \\ D = 0 \end{matrix},$$

which are equivalent. Thus this particular algebraic operation on a fraction (i.e. subtraction from 1) is valid, that is, preserves logical sense.

CHAPTER 3

BOOLES'S PROPOSITIONAL LOGIC

§ 3.0. The two theories

The preceding chapters of Part I of this monograph dealt with the first of the two main divisions of logic which are described in *Laws of Thought* (*LT*)[1], namely the part having to do with *primary* propositions, i.e. with propositions which, as Boole phrased it, "relate to things". The other main division—is that of *secondary* propositions which "concern, or relate to, other propositions regarded as true or false". Not only is the topic of interest here for its obvious logical importance but also for the use which Boole makes of it in his theory of probability which he grounds not on classes but on propositions.

The subject of propositional logic was also treated earlier by Boole in his *Mathematical Analysis of Logic* (*MAL*)[1] under the heading "hypothetical propositions", but there it is based on quite a different theory from that which he adopted in *LT*. Concerning the change Boole merely says (*LT*, p. 176):

> In a former treatise on this subject (*Mathematical Analysis of Logic*, p. 49), following the theory of Wallis respecting the Reduction of Hypothetical Propositions, I was led to interpret the symbol 1 in secondary propositions as the universe of "cases" or "conjunctures of circumstances"; and it is certain, that whatever

[1] Since there will be frequent reference in this chapter to both of Boole's logic books, we shall when necessary use the acronyms "LT" and "MAL" to distinguish between them.

is involved in the term beyond the notion of time is alien to the objects, and restrictive of the processes, of formal Logic.

We intend to show, first of all, that Boole need not have given up the earlier approach, for when suitable clarifications and corrections are made, still remaining within the ambit of his ideas, a viable logic of propositions results. Indeed we shall see that, precisely contrary to what Boole says, it is the notion of time which is "alien to the objects, and restrictive of the processes, of formal logic" and that, correctly conceived "cases" have nothing to do with time. That Boole had a promising approach in *MAL* has already been noted in the literature, e.g. by KNEALE and KNEALE [1962, p. 414], and there is a fairly extensive appraisal in PRIOR [1948, pp. 176–182]. Many of the points we make were made by Prior, but here the ideas are firmed up by the construction, in accordance with modern standards, of a formal system for Boole's calculus of elective symbols. This formal system, presented in § 3.1, is implicit in what he did, namely applying general algebraic principles and processes to his particular type of operators but, as we have noted in § 1.1, these algebraic principles were not clearly or fully stated, and much that is needed for mathematical cogency is absent. Our presentation remedies this deficiency. We then go on to show, in § 3.2, how one could come to a propositional calculus along Boole's *MAL* lines of thought.

In § 3.3 we take up the *LT* approach, i.e. the one based on the notion of portions of time and, in § 3.4, explain why it "works" even though there is some semantic confusion. In § 3.5, discussion of Boole's illustrative material for his methods relating to propositional logic affords us the opportunity of bringing out a number of items we consider to be of some interest.

§ 3.1. The calculus of elective symbols (operators)

Boole habitually uses a linguistic mode of expressing himself, e.g. referring to elective "symbols" and elective "functions", the latter being for him expressions built up from elective symbols. These expressions denote operators (of a certain kind). Rather than following Boole we shall instead use the more usual material mode of speech, speaking directly of operators and not of the symbols which designate them.

We consider as given a class U which is to be simply any class and not, as with Boole, a "class of all conceivable objects". With this class there is to be associated a set of (elective) operators, among which is one designated by "**1**". The operators take as operands subclasses of U, the result of the operation being again a subclass (i.e. we have closure in $P(U)$, the power set of U). We shall use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$,... as free variables standing for arbitrary elective operators, and the letters u, v,... for arbitrary subclasses of U. With regard to these operators the following is assumed:

O1. $\mathbf{1}u = u$

O2. $\mathbf{x}(u+v) = \mathbf{x}u+\mathbf{x}v$

O3. $\mathbf{x}(uv) = u\mathbf{x}v$

O4. $\exists \mathbf{x}(\mathbf{x}U = u)$.

We have a few words to say about each of these assumptions. In virtue of O1, **1** is an "identity" operator. The plus sign in O2 we are taking to be a notion of class aggregation which is commutative and associative; it is also, by our general assumption concerning operators, closed in $P(U)$. (In this latter respect we are "modernizing" Boole—recall that for him $x+x$, need not be a class.) The property expressed by O2, i.e. distribution of an operator over logical sum, appears in *MAL* on on p. 17, but for u and v disjoint. In O3, which does not appear in *MAL*, juxtaposition of class expressions (i.e. of "u" and "v", and "u" and "$\mathbf{x}v$") stands for class intersection. From O3, by replacing v by U and then uU by u, one obtains

$$\mathbf{x}u = u\mathbf{x}U,$$

a result which accords with Boole's conception of the elective symbol x "operating upon any subject [here u]..., shall be supposed to select from that subject all the Xs [here $\mathbf{x}U$] which it contains". Finally, assumption O4 guarantees that to any subclass of U there corresponds a selector.

The following definitions D1–D3 introduce three binary combinations of operators—multiplication, addition, and subtraction. For these we shall use customary algebraic notation, as we are able to distinguish them from like-symbolized operations by virtue of the boldface letters.

D1. $(\mathbf{x}\mathbf{y})u = \mathbf{x}(\mathbf{y}u)$

D2. $(\mathbf{x}+\mathbf{y})u = \mathbf{x}u+\mathbf{y}u$

D3. $(\mathbf{x}-\mathbf{y})u = \mathbf{x}u-\mathbf{y}u$.

While not present in *MAL*, definition D1 is implicit in Boole's algebraic manipulating two juxtaposed operator symbols as if the combination were a single symbol for an operator, and assuming that it means the same as their successive application. In D3 we are assuming that the minus sign on the right is an operation inverse to the + of class aggregation (in O2), and hence that our class aggregation is the "symmetric difference" which (in contrast to class union) does have an inverse.

Next we introduce a relation, $=$, between elective operators by stipulating that, for any well-formed operator-algebra expressions ϕ and ψ,

"$\phi = \psi$" means "$\phi u = \psi u$, for arbitrary $u \in P(U)$"

From O1 and D1 one readily has

E1. $\mathbf{1x} = \mathbf{x1} = \mathbf{x}$.

Thus $\mathbf{1}$ is a unit for the multiplication of elective operators. By introducing the constant operator $\mathbf{1}$, and distinguishing it from the universe U, we eliminate a source of confusion in *MAL*, where Boole uses only one symbol for both.

From D2 and the properties of class aggregation it follows that:

E2. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$

E3. $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$

Now from D1 and O3 we have

$$xy(uv) = x(y(uv)) = x(uyv) = x((yv)u) = yvxu = xuyv,$$

and hence

O5. $(xy)(uv) = xuyv$.

It is now straightforward to show these additional algebraic properties of elective operator-algebra:

E4. $x(y+z) = xy+xz$

E5. $x^2 = xx = x$ (*MAL*, p. 17)

E6. $xy = yx$ (*MAL*, p. 17)

E7. $x(yz) = (xy)z$

The absence of associativity, i.e. E3 or E7, from *MAL* (and also *LT*) should be noted. While distributivity of operators, i.e. E4 (not to be confused with O2), likewise is not in *MAL* it does appear in BOOLE 1848 (see BOOLE 1952, p. 127) where (without comment) it replaces (what corresponds to) O2 in *MAL* in a listing of "laws" of elective symbols.

By virtue of

O6. $(\mathbf{x}-\mathbf{x})u = \mathbf{x}u - \mathbf{x}u = \emptyset = (\mathbf{y}-\mathbf{y})u$, ($\emptyset$ the empty class)

one can introduce a constant operator $\mathbf{0}$ and show that it has the properties:

E8. $\mathbf{0}+\mathbf{x} = \mathbf{x}$

E9. $\mathbf{0x} = \mathbf{0}$.

Formal and explicit statement of these properties is missing in *MAL*.

It is clear that, for θ, ϕ, and ψ being any expressions of the operator algebra,

$$\theta = \phi \quad \text{implies} \quad \theta+\psi = \phi+\psi$$
$$\theta\psi = \phi\psi$$

and, in general, if $\theta = \phi$ then θ and ϕ are intersubstitutable in any operator-algebra expression. Hence, as anticipated by our choice of symbol for it, the relation $=$ is an equality (or identity) relation for the algebra of elective operators. Now

$$\begin{aligned} \mathbf{x}U = \mathbf{y}U &\rightarrow u(\mathbf{x}U) = u(\mathbf{y}U) \\ &\rightarrow \mathbf{x}(uU) = \mathbf{y}(uU) \quad \text{by O3} \\ &\rightarrow \mathbf{x} = \mathbf{y}, \end{aligned}$$

so that

O7. $\mathbf{x}U = \mathbf{y}U \rightarrow \mathbf{x} = \mathbf{y}$.

As a consequence of O7 and O4 we have that for any $u \in P(U)$ there is a unique operator $\mathbf{x}$ such that $\mathbf{x}U = u$. Let $E(U)$ be the set of elective operators so associated with elements of $P(U)$. Then, as $\mathbf{0}$ and $\mathbf{1}$ are in $E(U)$, and multiplication and addition of elective operators is closed in $E(U)$, we have that

$$\langle E(U), +, \times, \mathbf{0}, \mathbf{1} \rangle$$

is a mathematical system (an algebra), and one of the same type as the Boolean algebra (more exactly, Boolean ring)

$$\langle P(U), +, \times, \emptyset, U \rangle.$$

Moreover, since

$$\begin{aligned} \mathbf{xy}U &= \mathbf{x}U\mathbf{y}U \\ (\mathbf{x} + \mathbf{y})U &= \mathbf{x}U + \mathbf{y}U \\ \mathbf{1}U &= U, \\ \mathbf{0}U &= \emptyset. \end{aligned}$$

we have that these two algebras are, under the mapping $\mathbf{x} \rightarrow \mathbf{x}U$, isomorphic, i.e. are mathematically identical.

What we have just shown, that an algebra of elective operators which selects classes from U can be constructed so as to be the same as an algebra of subclasses of U, is hardly surprising, if not obvious. But the reason why this is so is that we are familiar with Boolean algebra. When Boole was trying to find (or construct) an algebra for doing logic Boolean algebra was not yet in existence, so he used what was familiar to him, namely operator calculus and designed for it operators which would reproduce intuitive class operations. Subsequently, and before he came to write *LT*, he must have realized that the symbols and operations of his algebra could just as well directly represent classes and operations on classes. Yet, interestingly, we still find him in *LT* retaining the idea of selection, only now the selection operations are operations of the "human mind", and the laws of his algebra are then "laws of thought".

One could compare our elective operator algebra with what Boole has (and has not) in *MAL* to see how poorly the concept of an algebraic system was understood at the time. However, our primary interest here is in the use of such an algebra for developing a propositional logic, and to this we now turn.

§ 3.2. The logic of hypotheticals

Opening this discussion of hypotheticals in *MAL* with a quotation (from Whately's *Elements of Logic*) to the effect that a hypothetical proposition is "two or more categoricals united by a copula", Boole nevertheless recognizes that the validity of the hypothetical syllogism does not depend on the categorical structures of the component propositions involved in the syllogism but only on their truth or falsity. Accordingly he introduces symbols X, Y, Z, etc. "expressive" of the elementary (i.e. unanalyzed) propositions involved in the inference. He claims that one can use his calculus of elective symbols to (formally) carry out such inferences. This is how he begins:

> To the symbols X, Y, Z representative of Propositions, we may appropriate the elective symbols x, y, z, in the following sense.
>
> The hypothetical Universe, 1, shall comprehend all conceivable cases and conjunctures of circumstances.
>
> The elective symbol x attached to any subject expressive of such cases shall select those cases in which the Proposition X is true, and similarly for Y and Z.
>
> If we confine ourselves to the contemplation of a given proposi-

> tion X, and hold in abeyance every other consideration, then two cases only are conceivable, viz. first that the given Proposition is true, and secondly that it is false. As these cases together make up the Universe of the Proposition, and as the former is determined by the elective symbol x, the latter is determined by the symbol $1 - x$. [*MAL*, 49].

Boole doesn't explain what he means by "all conceivable cases and conjunctures of circumstances" and, as our earlier quotation indicates, he must have had trouble, or at least dissatisfaction, with the idea. As a first clarification of Boole's ideas we take "the hypothetical Universe" not as absolute but relative to a given context. To see why note that in his discussion just quoted he is allowing X to be true or false, in effect taking X to be a two-valued propositional variable, and the Universe then has two cases. Similarly when he has three propositions under discussion there are 8 cases in the Universe. In general, for $X_1,\ldots,X_n$ (taken as propositional variables) we define the hypothetical universe (relative to $X_1,\ldots,X_n$) to be the class consisting of 2^n cases, namely the 2^n assignments of t or f to the n letters (the 2^n "lines" in a conventional truth-table). We symbolize this notion by $U(X_1,\ldots,X_n)$, or by $U(X_1,\ldots)$, or by U if no misunderstanding would ensue. The elements of $U(X_1,\ldots,X_n)$ are ordered n-tuples of t's and/or f's and there can be fewer than 2^n such elements if any of the X_i's are constant, i.e. are propositions. (Boole's logical notation made no provision for distinguishing a letter used as a variable from one used as a constant.) Thus if X_1 and X_2 are propositional variables then the class $U(X_1, X_2)$, given by

$$U(X_1, X_2) = \left\{ \begin{array}{c} X_1\ X_2 \\ (t,\ t) \\ (t,\ f) \\ (f,\ t) \\ (f,\ f) \end{array} \right\},$$

has four elements, but if X_1 is a constant, say a true proposition, then $U(X_1, X_2)$ has only the two elements (t, t), (t, f). As with Boole we take the symbols $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3, \ldots$ to be elective symbols (operators) on U (but now our U is $U(X_1, X_2, X_3, \ldots)$), or any subclass of U, such that $\boldsymbol{x}_i$ (for example) selects those cases in which X_i has been assigned the value t, i.e. those cases whose ith component is t. Note that here $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3, \ldots$ are associated with particular subsets of $U(X_1, X_2, X_3, \ldots)$ and are not, as in the preceding section, free variables standing for arbitrary operators selecting subclasses of U. Nevertheless we still can apply to them the laws of our elective operator algebra, since these laws are general and apply to any U and any selection operators. By virtue of the properties of elective symbols $\mathbf{1} - \boldsymbol{x}$ then selects those cases for which X has the value f. If we confine our attention to just X (as Boole does in the quotation) then the "Universe of the Proposition" is $U(X)$ and indeed, when X is a variable, has two cases. Note that for us "$\mathbf{1}$" in "$\mathbf{1} - \boldsymbol{x}$" designates an elective operation and not the Universe. This is important since we are taking the universe to be $U(X_1, \ldots, X_n)$—which depends on the X_i—whereas $\mathbf{1}$ does not.

Continuing, Boole wants to express that a given proposition X is true and argues (*MAL*, 51):

> The symbol $1 - x$ selects those cases in which the Proposition X is false. But if the Proposition is true, there are no such cases in its hypothetical Universe, therefore
>
> $$1 - x = 0$$
>
> or
>
> $$x = 1.$$

Boole has no explanation of what he means by "its [i.e. X's] hypothetical universe" but clearly what it is depends on whether X is true or false, since he says "But if the Proposition is true, there are no such cases in its hypothetical universe,...". However, if the universe can change, one no longer can follow Boole's practice of understanding (i.e. omitting) it. The "understood" subject in the equations of the preceding quotation cannot be $U(X, \ldots)$ with *variable* X as defined above since there are then cases in which X is both t and f. We can make Boole's argument come out right if, when X is true, we take its hypothetical universe to be the subclass of $U(X, \ldots)$ for which X has only t values. Let us designate this by $U(t,\ldots)$ or, for short, by U_1. Then Boole's quoted equations become

$$(\mathbf{1}-\boldsymbol{x})U_1 = \mathbf{0}U_1$$

and

$$\boldsymbol{x}U_1 = \mathbf{1}U_1,$$

which are true statements. Likewise if U_0 is the subclass of $U(X,\ldots)$ for which X has only f values, then if X is false so is $\boldsymbol{x}U_0 = \mathbf{1}U_0$ ($\boldsymbol{x}$ selecting nothing from U_0, $\mathbf{1}$ selecting all of U_0). We then have that, for propositional variable X, the equation

$$\boldsymbol{x}U(X,\ldots) = \mathbf{1}U(X,\ldots) \tag{1}$$

does represent, or express, X; for when X is true (1) becomes $\boldsymbol{x}U_1 = \mathbf{1}U_1$ (which is true), and when X is false (1) becomes $\boldsymbol{x}U_0 = \mathbf{1}U_0$ (which is false). (We are assuming that "$\boldsymbol{x}$" in (1), as an operator, refers to the X-position of the cases in $U(X,\ldots)$, and that that doesn't change with X (the proposition) being true of false.)

Following Boole we now correlate to each compound proposition form an equation in the operator algebra:

$$X \text{ represented by } \boldsymbol{x} = \mathbf{1}$$

$$\neg X \text{ represented by } \mathbf{1} - \boldsymbol{x} = \mathbf{1}$$

$$X \wedge Y \text{ represented by } \boldsymbol{x}\boldsymbol{y} = \mathbf{1}$$

$$X \rightarrow Y \text{ represented by } \mathbf{1} - \boldsymbol{x}(\mathbf{1} - \boldsymbol{y}) = \mathbf{1}$$

etc.,

where the understood subject for the operators is $U(X, Y, \ldots)$. If we consider a propositional language using variables $X, Y,\ldots$, and connectives $\neg$ and $\wedge$, then it is easy to see that for any formula $\Theta(X, Y,\ldots)$ of the language there is a unique polynomial $\vartheta(x, y,\ldots)$ built up from $x, y,\ldots$ and the operations of subtraction from 1 and multiplication, such that $\Theta(X, Y,\ldots)$ is represented by $\vartheta(\boldsymbol{x}, \boldsymbol{y},\ldots) = \mathbf{1}$ in the sense that the equation, considered as an operator equation on $U(X, Y,\ldots)$, is true for those and only those values assigned to $X, Y,\ldots$ making $\Theta(X, Y,\ldots)$ true. Thus as a simple example let $\Theta(X, Y)$ be

$\neg X \wedge \neg Y$. Then the correlated operator equation is

$$(\mathbf{1} - \boldsymbol{x})(\mathbf{1} - \boldsymbol{y}) = \mathbf{1},$$

which, on supplying the omitted $U(X, Y)$, is

$$(2) \qquad (\mathbf{1} - \boldsymbol{x})(\mathbf{1} - \boldsymbol{y})U(X, Y) = \mathbf{1}U(X, Y).$$

Considering all possible truth-values for X and Y we have

$$(\mathbf{1} - \boldsymbol{x})(\mathbf{1} - \boldsymbol{y})\{(t, t)\} = \mathbf{1}\{(t, t)\}, \text{ or } \phi = \{(t, t)\}$$
$$(\mathbf{1} - \boldsymbol{x})(\mathbf{1} - \boldsymbol{y})\{(t, f)\} = \mathbf{1}\{(t, f)\}, \text{ or } \phi = \{(t, f)\}$$
$$(\mathbf{1} - \boldsymbol{x})(\mathbf{1} - \boldsymbol{y})\{(f, t)\} = \mathbf{1}\{(f, t)\}, \text{ or } \phi = \{(f, t)\}$$
$$(\mathbf{1} - \boldsymbol{x})(\mathbf{1} - \boldsymbol{y})\{(f, f)\} = \mathbf{1}\{(f, f)\}, \text{ or } \{(f, f)\} = \{(f, f)\},$$

showing that (2) is true if and only if $\neg X \wedge \neg Y$ is true.

One can show that $\Theta(X, Y, \ldots)$ is a propositional tautology if and only if $\vartheta(x, y, \ldots) = 1$ is an algebraic identity true for all $0, 1$ values assigned to $x, y, \ldots$; and that the propositional calculus rules of substitution and modus ponens take obviously valid forms in the algebraic symbolism. We shall not go through the exercise of showing that the algebraic equation format is an adequate surrogate for doing propositional logic.

Why did Boole change his mind and switch from the "universe of cases" idea to that of "time for which the proposition is true" on which to base his propositional logic? It could be, as the quote in our first section seems to indicate, that he thought the notion of time more fundamental. On the other hand, our work here shows that getting his "cases" idea to work would have required of him a considerable emendatory effort: we had to use, not an absolute universe U, but one which depended on the context, and had to clearly distinguish between its arguments being variables or constant and, moreover, distinguish it from the identify operator $\mathbf{1}$; we had to develop an algebra of operators; and, finally, a definition of "cases" was provided. It would hardly have been easy for Boole at that stage in the development of logic to steer his way through to a clear understanding of the matter.

§ 3.3. Secondary propositions and "time"

The semantic basis for propositional logic which Boole advocates in Chapter XI of *Laws of Thought* is quite different from the earlier one in *Mathematical Analysis of Logic.* Our present section is devoted to a description of this later, supposedly more carefully considered, theory. Here too, as in much else of his work, we shall find that Boole is close enough to being correct that he is able to develop useable techniques.

We have already noted his separation of propositions into primary and secondary. While the core of the idea is sound—amounting to classifying propositions with regard to whether they do (secondary), or do not (primary), involve an analysis in terms of propositional connectives—the actual description Boole gives entails some semantic confusion. For instance he believes that secondary propositions express judgements concerning the truth or falsity of propositions, citing as an example.

(i) It is true that the sun shines.

But he also considers as a secondary proposition.

(ii) If the sun shines the day will be fair,

on the grounds that it expresses the dependence of the truth of "The day will be fair" on that of "The sun will shine". Contemporary semantic theory however would relegate example (i) to a meta-language and indeed, with Boole, as an assertion about the truth of a proposition; on the other hand example (ii) would be in the language proper and not an assertion *about* propositions of the language. And yet while not entirely clear Boole is nevertheless aware of a distinction, for he qualifies his characterization of secondary propositions by saying that they may "relate to" propositions considered as true or false, as well as being outright assertions of truth or falsity.

Effectively, Boole doesn't make a "semantic ascent" to a meta-language but works with and analyzes the propositions themselves. But this analysis, unlike our present-day analysis in terms of truth-functions, is based on the idea of time:

> ...Let us take, as an instance for examination, the conditional proposition, "If the proposition X is true, the proposition Y is true". An undoubted meaning of this proposition is, that the *time*

> in which the proposition X is true, is *time* in which proposition Y is true. This indeed is only a relation of coexistence, and may or may not exhaust the meaning of the proposition, but it is a relation really involved in the statement of the proposition, and further, it suffices for all the purposes of logical inference. [p. 163].

(In this quote we see Boole expressing the form of a conditional metalinguistically using the predicate "is true", but in his actual examples it is always correctly of the form "If X, then Y.") Thus while not admitting it, Boole is interpreting the conditional as a categorical (i.e. primary) proposition whose subject is "time in which the proposition X is true" and whose predicate is "time in which the proposition Y is true"; he writes the sentence symbolically as

$$x = vy,$$

where "x denotes the time for which the proposition X is true" [and similarly for y]. This is Boole's *operative* meaning for the symbols $x, y, z, \ldots$, i.e. representing portions of time, although "officially" they represent mental constructs which he uses, as in the case of primary propositions to find the "laws" which they obey. We quote from *LT* (pp. 164–165):

Proposition II

> 7. *To establish a system of notation for the expression of Secondary Propositions, and to show that the symbols which it involves are subject to the same laws of combination as the corresponding symbols employed in the expression of Primary Propositions.*
>
> Let us employ the capital letters X, Y, Z to denote the elementary propositions concerning which we desire to make some assertion touching their truth or falsehood, or among which we seek to express some relation in the form of a secondary proposition. And let us employ the corresponding small letters x, y, z, considered as expressive of mental operations, in the following sense, viz.: Let x represent an act of

> the mind by which we fix our regard upon that portion of time for which the proposition X is true; and let this meaning be understood when it is asserted that x denotes the time for which the proposition X is true. Let us further employ the connecting signs $+$, $-$, $=$, etc., in the following sense, viz.: Let $x + y$ denote the aggregate of those portions of time for which the propositions X and Y are respectively true, those times being entirely separated from each other. Similarly let $x - y$ denote the remainder of time which is left when we take away from the portion of time for which X is true, that (by supposition) included portion for which Y is true. Also, let $x = y$ denote that the time for which the proposition X is true, is identical with the time for which the proposition Y is true. We shall term x the representative symbol of the proposition X, etc.

On the basis of this mentalistic conception Boole determines, very much as he did for primary propositions, the "laws of combinations" which these representative symbols obey, and finds these laws to be identical with those for primary propositions. (We omit Boole's extensive philosophical remarks on this, to him, quite remarkable circumstance.) While the laws and the "mathematical processes founded on them" are unchanged, new interpreting rules are introduced: the symbol 0 represents *nothing* "in respect to the element of time", and the symbol 1 the *universe*, i.e. the "whole of time, to which the discourse is supposed in any manner to relate." To express "X is true" (Boole puts it: "To express the Proposition 'The Proposition X is true'") he writes

$$x = 1$$

i.e. identifying the time for which X is true with the (whole) universe (of time under consideration.) Although Boole is clearly assuming that the truth-value of X may depend on time, nevertheless x, the portion of time for which X is true, is fixed. Thus "$x = 1$" expresses that X has the fixed truth-value "true" during the entire time under consideration; likewise "$x = 0$" expresses that X is true at no time or, in Boole's words "The Proposition X is false." Significantly, Boole doesn't consider expressing "X is sometimes true."

As we have already noted, the conditional is expressed by $y = vx$, where v is "regarded as a symbol of time indefinite" or, on elimination of v, also by the equation $y\bar{x} = 0$. As in the case of primary propositions, v is allowed to be empty.

The verbal form of the disjunctive proposition which Boole gives is: "Either the proposition X is true or the proposition Y is true", which by his rule should be "$x = 1$ or $y = 1$." However the symbolic form he does give is

$$x(1-y)+y(1-x) = 1,$$

in other words actually expression "Either X or Y is true" or, even better, "Either X or Y." (Boole is, of course, taking "or" in the exclusive sense.) A similar semantic fault is also corrected (unintentionally) by Boole when he symbolically expresses the form of a conditional in which the antecedent or consequent may be a disjunction. Thus "If either X is true or Y is true, then Z is true" is rendered symbolically as

$$x(1-y)+y(1-x) = vz$$

In the case of the class calculus interpretation for his formal system Boole only casually notes (*LT*, pp. 112–113) that an equation of the form $w = A+vC$ expresses (may be interpreted as) a two-sided inclusion of the class w. In the case of his propositional version he elevates this to the statement of a PRINCIPLE (p. 173), to the effect that $w = A+vC$ interprets as a pair of conditionals: (i) A implying w and (ii) w implying $A+C$.

Boole's introduction of the notion of time to develop propositional logic was later criticized, e.g. by Macfarlane [1879, pp. 9–10] and Venn [1894, Chapter XVIII]. Boole himself recognized that, once the calculus was developed, the notion of time played no role: "We may here call attention to the remark, that although the idea of time appears to be an essential element in the theory of the interpretation of secondary propositions, it may practically be neglected as soon as the laws of expression and of interpretation are definitely established."

In anticipation of its later use in connection with the theory of probabilities Boole observes (p. 167): "Instead of appropriating the symbols x, y, z, to the representation of the truths of propositions, we might with equal propriety apply them to represent the occurrence of events. In fact, the occurrence of an event both implies, and is implied by, the truth of a proposition, viz., of the proposition which asserts the occurrence of the event." It should be noted that Boole's term "event" is not that of contemporary probability theory where an event is a set of outcomes of an experiment (see, e.g., RENYI 1970, § 1.1) and hence corresponds to a class of Boole's events. In modern probability the calculus of events is indeed a calculus of classes, not a calculus of propositions.

Boole concludes his Chapter XI with some general philosophical remarks on the analogy between the theory of Primary and Secondary Propositions and their relation to Space and Time, contending that the notion of time is essential for secondary propositions but that space is not for the primary—his principal reason being:

> ...Dismissing, however, these speculations as possibly not altogether free from presumption, let it be affirmed that the real ground upon which the symbol 1 represents in primary propositions the universe of things, and not the space they occupy, is, that the sign of identity = connecting the members of the corresponding equations, implies that the things which they represent are identical, not simply that they are found in the same portion of space. Let it in like manner be affirmed, that the reason why the symbol 1 in secondary propositions represents, not the universe of events, but the eternity in whose successive moments and periods they are evolved, is, that the sign of identity connecting the logical members of the corresponding equations implies, not that the events which those members represent are identical, but that the times of their occurrence are the same. These reasons appear to me to be decisive of the immediate question of interpretation. [p. 176]

The argument here involves a number of confusions. In the first place even if, in the case of primary propositions, 1 were a universe of *things*, the symbols x, y, z, as Boole uses them stand for (nearly enough) subclasses of 1 and hence, as classes, are abstractions—discussion concerning the space they occupy is meaningless. Secondly, since in the case of primary propositions it is *classes* of things which are related by $=$, the parallel which Boole draws between identity of things and identity of events is faulty—the corresponding relationship in the case of secondary propositions, if 1 is the universe of events, would be identity of classes of events. Thus there is no objection to having 1 being the universe of events provided one uses an appropriate meaning for "$=$" to go with it, but then one doesn't obtain propositional logic but, rather, a calculus of events (in the contemporary sense of this word.) Boole mistakenly thought that $=$ had to be the relation of identity for *elements* of the universe of propositions and baulked at the idea of so strong a relation for propositional logic. In a (roughly) similar situation Frege did accept the consequences of identifying, for the purposes of propositional logic, all true propositions and all false propositions, but allowed for diversity by his distinction between sense (Sinn) and denotation (Bedeutung)—according to this view there are only two distinct denotations (or referents) for propositions, the truth-values True and False, whereas propositions with the same truth-values may have many distinct senses. Boole's difficulty with $=$, when taken as identity of propositions as being too strong a relation, was resolved by his going over to a universe of classes—his "portions of time" functioning exactly as classes, although Boole never brought himself to say that they actually were classes. Propositions, then, though different could be equivalent if the portions of time during which they were true are the same.

§ 3.4. Boole's illustrative examples

Following the presentation (in Chapter XI) of his theory of secondary propositions Boole has three additional chapters relating to this topic. In the first of these he codifies the methods to be used, illustrating these methods with two simple examples of arguments, one taken from

Cicero's *De Fato* and the other from Plato's *Republic*. The second of these chapters contains an analysis, by his symbolic methods, of various arguments from Samuel Clarke's *Demonstrations of the Being and Attributes of God*, and also discussion of a portion of Spinoza's *Ethica Ordine Geometrico Demonstrata*. In the last of these three chapters he illustrates the method of converting a set of equations to a single equation of the form $V = 0$ in which V satisfies the condition $V(1 - V) = 0$, using as an example a set of equations arising in connection with the symbolization of one of Clarke's arguments. While the strictly logical matters in these chapters are all rather minor, discussion of this material will enable us to bring out a number of items which are of some interest.

1. At the beginning of Chapter XII (Methods in Secondary Propositions) Boole states, in the form of a RULE, his procedures for carrying out inferences involving secondary propositions. These are pretty much as expected: (i) the indefinite symbol v is to be removed (by elimination, § 1.8) from any equation in which it appears (ii) next eliminate those (propositional) symbols which are not to appear in the "final solution", but first reducing (§ 1.8) to a single equation those equations containing any such symbols and (iii) combining all such resulting equations into a single equation of the form $V = 0$. In connection with propositional logic Boole's elimination is a form of (partial) truth-value analysis; for replacing

$$f(x, y, z, \ldots) = 0$$

by

$$f(1, y, z, \ldots)f(0, y, z, \ldots) = 0$$

and algebraically simplifying (Boole tacitly assumes this) amounts to replacing a propositional formula

$$f(p, q, r, \ldots)$$

by

$$(f(T, q, r, \ldots)f(F, q, r, \ldots))$$

(T and F truth-values) and "resolving" (QUINE 1972, § 5), the result being the strongest logical consequence obtainable which does not contain p. In connection with his RULES Boole's last prescription "to ascertain whether a particular elementary proposition x [involved in a set of premises] is true or false [i.e. whether X, or not-X, is a consequence]" contains an omission: he states what, on eliminating all symbols but x, each of the outcomes $x = 1$, $x = 0$, and $0 = 0$ would indicate, but makes no mention of the possible outcome $1 = 0$ which, of course, would indicate inconsistent premises.

2. We consider Boole' first illustrative example (p. 179):

> 4. Ex. 1.—The following prediction is made the subject of a curious discussion in Cicero's fragmentary treatise, De Fato:—"Si quis (Fabius) natus est oriente Canicula, is in mari non morietur." I shall apply to it the method of this chapter. Let y represent the proposition, "Fabius will die in the sea." In saying that x *represents* the proposition, "Fabius, and Co," it is only meant that x is a symbol so appropriated (XI. 7) to the above proposition, that the equation $x = 1$ declares, and the equation $x = 0$ denies, the truth of that proposition. The equation we have to discuss will be
>
> (1) $$y = v(1 - x)$$
>
> The symbolic version (1) of the conditional sentence about Fabius is converted to
>
> (2) $$yx = 0$$

Boole then says (pp. 179–180):

> The interpretation of this result is:—"It is not true that Fabius was born at the rising of the dogstar, and will die in the sea." Cicero terms this form of proposition, "Conjunctio ex repugnantibus;" and he remarks that Chrysippus thought in this way to evade the difficulty which he imagined to exist in contingent assertions respecting the future: "Hoc loco Chrysippus...".

Boole's quotation (from *De Fato*) which follows records Cicero's mocking scorn of Chrysippus' views, but Boole gives us nothing of Cicero's reasons or the background discussion. In this discussion, which involves ideas such as free-will, fate, divination, necessity and the like, Cicero uses as an example the rule ("percepta", Greek "theoremata") of astrologers: "If anyone is born at the rising of the dogstar, he will not die at sea." From this he proceeds to refute Chrysippus by drawing a conclusion agreeing with Diodorus. ("Careful, Chrysippus, that in the big dispute you have with the mighty logician Diodorus, you do not leave your position undefended.") We translate a portion of Cicero:

> ...If indeed what is so connected [i.e. the conditional]. "If anyone is born at the rising of the dogstar, he will not die in the sea" is true, so also is "If Fabius is born at the rising of the dogstar, Fabius will not die at sea." Therefore it is incompatible (*Pugnant igitur haec inter se,...*) for Fabius to be born at the rising of the dogstar and to die in the sea; and since of Fabius one is sure that he was born at the rising of the dogstar, it is also an incompatibility for Fabius to exist and to die in the sea. Hence the conjunction "Fabius exists and will die in the sea" is an incompatibility (*Ergo haec quoque coniunctis est ex repugnantibus,...*), which as a subject of discourse (*propositum*) is impossible (*est ne fieri quidam potest*). Therefore "Fabius will die in the sea" is of the kind that is impossible. Consequently every false statement about the future is impossible. [*De Fato*, VI 12]

We see here that Cicero's main premise is not the conditional statement about Fabius (as Boole assumes by his inserting "(Fabius)" after Cicero's "*quis*") but the generalized form ("if anyone...") from which Cicero then infers the instance about Fabius. This is not inconsequential for, in the first place, Cicero is especially interested in generalized (law-like) conditionals and, secondly, it brings out an inadequacy in Boole's logical theory, namely its inability to handle singular inference. So we find Boole tacitly avoiding the problem by adopting Cicero's second step as the initial premise. Further, we see that Boole has mistakenly referred to his rendition of $xy = 0$—"It is not true that Fabius was born at the rising of the dogstar, and will die in the sea"—as Cicero's "Conjunctio ex repugnantibus" whereas what Cicero so refers to is "Fabius exists [was born] and will die in the sea." Again this is not inconsequential, for Chrysippus' rebuttal (as given by Cicero) is that one should state [làw-like] conditionals in the form of a denial of a conjunction. We translate from Cicero:

> At this point Chrysippus in anger (*aestuans*) hopes that the Chaldeans and other diviners are wrong, and for pronouncement of their observations (percepta) will not use "If anyone was born at the rising of the dogstar, he will not die in the sea" but rather "There is no one who is both born at the rising of the dogstar and who will die in the sea." (*Non et natus est quis oriente Canicula et is in mari morietur*) O joking license! to avoid a run in with Diodorus he instructs the Chaldeans how they should present their observations. [*De Fato*, VIII 15]

Cicero goes on to show the ridiculousness (in his opinion) of Chrysippus' version of the conditional by translating examples from medicine and geometry into that form. These examples, and other scattered remarks, leave no doubt that for Cicero the conditional form implies some necessary connection in nature.

It is remarkable that in selecting material from *De Fato* as his first illustrative example in propositional logic Boole happened to choose an important early source of information on Stoic logic, of which so little is extant. The historical value could hardly have been appreciated by Boole since it wasn't until the present century that an adequate historical understanding of Stoic logic as a propositional logic was achieved (MATES, 1953, 1961). Nevertheless Boole clearly saw that what was involved was propositional logic, not term (or class) logic. On the other hand, it does appear that the significant point in the discussion from which Boole took his example, namely that the dispute described by Cicero hinged on rival versions of the conditional, was missed by Boole; for by writing "If Y, then not-X" as $y = v(1-x)$, with his meaning of this equation, he automatically precludes there being any possible logical difference with $xy = 0$.

3. The second of Boole's examples used to illustrate his methods in propositional logic is taken from Book II of Plato's *Republic* (380, 381). Boole praises it as a "very fine example of the careful induction from familiar instances, and of the *clear and connected logic* by which he deduces from them the particular inferences which it is object to establish." [p. 181. Our emphasis]. An analysis of the dialogue leads Boole to his list of Plato's premises (Plato, arguing demonstratively, takes these to be true):

> 1. If the Deity suffers change, He is changed either by himself or by another.
> 2. If He is in the best state, He is not changed by another.
> 3. The Deity is in the best state.
> 4. If the Deity is changed by himself, He is changed to a worse state.
> 5. If he acts willingly, He is not changed to a worse state.
> 6. The Deity acts willingly.

Here too, as in the preceding illustrative example, Boole circumvents inferences from a general to a singular proposition by adopting the singular proposition as the premise. Introducing the symbols

x for the Deity suffers change

y for He is changed by himself

z for He is changed by another

s for He is in the best state

t for He is changed to a worse

w for He acts willingly.

Boole then expresses the premises symbolically and, after eliminating the indefinite symbol v from the conditionals, has:

(1) $xyz + x(1-y)(1-z) = 0$

(2) $sz = 0$

(3) $s = 1$

(4) $y(1 - t) = 0$

(5) $wt = 0$

(6) $w = 1$

Boole then successively eliminates (see § 1.8) z, s, y and t and obtains Plato's result $x = 0$.

It is instructive to compare Boole's algebraic elimination procedure of arriving at the conclusion with the "clear and connected logic"—as Boole himself styles it—that Plato uses. We extract from the dialogue (using Boole's "translation" to facilitate comparison) the relevant steps and alongside a corresponding symbolic version in modern notation—in our symbolic version all general propositions have been instantiated so as to apply to "God". We use the symbol "$\underline{\vee}$" for exclusive "or".

> Must not that which departs from its proper form be changed either by itself or by another thing? Necessarily so

$$x \rightarrow (y \underline{\vee} z)$$

> In this way, then, God should least of all bear many forms [be changed by another]? Least indeed of all

$$\neg z$$

> And whatever is in a right state,..., admits of the smallest change [least liable to] from any other thing. So it seems

$$s \rightarrow \neg z$$

> But God and things divine are in every sense in the best state. Assuredly

$$s$$

> ...should He transform and change Himself? Manifestly He must do so, if He change at all

$$x \rightarrow y$$

> Changes He then Himself to what is more good and fair, or to that which is worse or baser? Necessarily to the worse, if He be changed
>
> $y \to t$
>
> ...seems it to you... that God [or man] willingly makes Himself in any sense worse? Impossible said he
>
> $\neg(wt)$
>
> [God acts willingly] (Premise supplied) w
>
> Impossible, then, it is, said I, that a god should wish to change himself: but ever being fairest and best, each of them ever remains absolutely in the same form
>
> $\neg x$

The logical structure of Plato's argument can be brought out by displaying it in tree form:

$$\dfrac{\dfrac{x \to (y \underline{\vee} z), \quad \dfrac{s \to \neg z,\ s}{\neg z}\,(a)}{x \to y,}\,(d) \qquad \dfrac{y \to t, \quad \dfrac{w,\ \neg(wt)}{\neg t}\,(b)}{\neg y}\,(c)}{\neg x}\,(c)$$

Here the formulas occuring without a line above them are premises, and each pair of formulas above a line has as immediate logical consequence the formula below it. There are four different kinds of such immediate inferences, (a)–(d). Apparently Plato's audience needed no convincing as to the validity of such immediate inferences. The contrast of this intuitively "clear and connected logic" with Boole's laborious algebraic-equational technique is marked. Yet, one should not hold it against Boole for not being able to see that logical deduction could be carried out formally in this fashion, i.e. by general inferential, rather than (mostly) transformational rules, since it wasn't until well into the 20th century that "natural deduction" formulations of logic were introduced [Jaskowski 1934, Gentzen 1934-5].

4. In addition to its introduction of algebraic symbols and techniques another novel feature of Boole's work in logic was the generality with which problems were conceived and methods, both theoretical and practical, were devised for their solution. A typical example is that of obtaining from any given set of premises those of its logical consequences which refer only to a selected set of components (symbolically, letters) of the premises. (We are using the term "logical consequences" uncritically—Boole had no semantic basis for his logic and simply took it for granted that the common algebraic operations on a set of equations representing logical statements led one correctly to its logical consequences.) The method given involves two steps: (i) consolidating the premise equations into a single equation and (ii) eliminating from this equation all letters but those which it is desired to retain. In specific cases Boole, of course, avails himself of shortcuts. E.g. if $x = 1$ is one of the equations then the elimination of x is accomplished before consolidation by dropping this equation and replacing x throughout by 1. Likewise elimination can be carried out first in any subset of the equations if the letters to be eliminated do not occur in the complementary set of equations. The techniques are illustrated by examples obtained by analyzing portions of Samuel Clarke's "Demonstration of the Being and Attributes of God". Analysis of Clarke's proof of "Something has existed from eternity" leads Boole to extract the premises

$$
(1) \qquad
\begin{array}{ll}
x = 1 & x \\
x = v\{y(1-z)+z(1-y)\} & x \to y \underline{\vee} z \\
x = v\{p(1-q)+q(1-p)\} & x \to p \underline{\vee} q \\
p = vy & p \to y \\
q = v(1-z) & q \to \neg z
\end{array}
$$

Boole then shows how to obtain all (independent) conclusions respecting one or more of the five symbols x, y, z, p, q which are implied by (1); these are, in addition to $x = 1$, $y = 1$, $z = 0$ and $p(1-q)+q(1-p) = 1$. Another of his examples (set of premises for "Some one unchangeable and independent Being has existed from eternity") is

$$
(2) \qquad
\begin{array}{ll}
1-x=0 & x \\
x = v\{y(1-z)+z(1-y)\} & x \rightarrow y \veebar z \\
z = v\{p(1-q)+q(1-p)\} & z \rightarrow p \veebar q \\
p = 0 & \neg p \\
q = 0 & \neg q
\end{array}
$$

A third example is of interest in two respects: firstly Boole resorts to the theory of primary propositions and, secondly, he uses a singular term as if it were a class term. The proposition is that the unchangeable and independent Being (whose existence was "established" by the preceding argument) must be self-existent (Proposition III, p. 207). One of Boole's extracted premises is

> 3. The unchangeable and independent Being has not been produced by an external cause.

which is rendered symbolically by

$$w = v\bar{y},$$

with w as the *singular* term "The unchangeable and independent Being", and y the *class* term "Beings which have been produced by an external cause". The simplest way of putting things right here is by taking w to be the class whose only member is the singular term.

Although Boole devotes a great many pages (all of Chapter XIII) to his analyses of the metaphysico-theological arguments of Clarke and Spinoza, we shall not follow suit—to us of the 20th century it is clear that any such logical analysis, using only at most propositional and class term logic, is almost sure to be superficial; and, of the two, even more so for Spinoza, whose arguments, as Boole concludes after an examination of them, are "largely a play on terms defined to be equivalent". As regards the demonstrativeness of the arguments, Boole approvingly cites, in the case of Clarke, Bishop Butler's objection to a central premise, and in the case of Spinoza, Boole refers to "...fallacies, dependent chiefly on the ambiguous use of words...". Boole concludes his chapter with a peroration on "the futility of all endeavors to establish, entirely *a priori*, the existence of an Infinite Being, His attributes, and His relation to the universe".

4. The final related set of examples in propositional logic which Boole discusses—constituting his Chapter XIV—illustrates various of his techniques, e.g. for converting a set of premises into a single equation of the form $V = 0$, with V satisfying the fundamental law $V(1 - V) = 0$; from this Boolean form he derives various conclusions respecting any symbol (proposition), or any logical combination of symbols in terms of any preselected set of them. In these examples we see how Boole circumvents the awkwardness associated with his + leading outside of class terms (or the analogue for propositions) while avoiding excessive build-up of additional terms.

The set of premises he uses as a basis for his example is

$$\begin{aligned} xyt + x\bar{y}\bar{t} &= 0 \\ tzw &= 0 \\ y\bar{v} &= 0 \\ vx &= 0 \\ x\bar{z} &= 0, \end{aligned} \tag{3}$$

which arises from an analysis of Clarke's argument for the proposition "Matter is not a necessary being", and deleting the premise "Motion exists". (In (3) the letter "v" is not the indefinite class symbol.) A simple and direct way of converting (3) to a single equation of the desired (Boolean) kind would be to replace each term by an equivalent sum of constituents on all six letters appearing in (3), adding the resulting equations and then deleting repeated constituents. But, in general, this would result in an equation with a large and unwieldy number of terms. Instead Boole proceeds by first adding the equations as they stand (no preliminary squaring being needed here since all terms are "positive"—see § 1.9) so as to obtain

$$xyt + x\bar{y}\bar{t} + tzw + y\bar{v} + vx + x\bar{z} = 0 \tag{4}$$

One could now apply his theorem (see § 1.9) that

$$v_1 + v_2 + v_3 + \cdots = 0 \tag{5}$$

is replaceable by ("reducible to")

$$v_1 + \bar{v}_1 v_2 + \bar{v}_1 \bar{v}_2 v_3 + \cdots = 0 \tag{6}$$

so as to obtain an equation with an equivalent logical content satisfying $V(1 - V) = 0$. (Note that the left-hand sides of (5) and (6) are not, in general, equal.) Instead of applying the reduction process to the full equation (4), Boole applies one of his "abbreviative" techniques. A subset of these letters is selected (in this case x, y) and (4) is developed with respect to them:

$$\begin{aligned}(t + \bar{v} + v + \bar{z} + tzw)xy + (\bar{t} + v + \bar{z} + tzw)x\bar{y}\\ + (\bar{v} + tzw)\bar{x}y + tzw\bar{x}\bar{y} = 0,\end{aligned} \tag{7}$$

and reduction (replacing $v_1 + v_2 + v_3 + \cdots$ by $v_1 + \bar{v}_1 v_2 + \bar{v}_1 \bar{v}_2 v_3 + \cdots$) is applied to each of the coefficients of the constituents of (7) so as to obtain

$$\begin{aligned}1 \cdot xy + (\bar{t} + tv + t\bar{v}\bar{z} + t\bar{v}zw)x\bar{y}\\ + (\bar{v} + tzwv)\bar{x}y + tzw\bar{x}\bar{y} = 0\end{aligned} \tag{8}$$

(For a justification of this technique see our endnote in § 3.7.) Although reduction is applied four times instead of once, there is a net gain in simplicity. Boole now uses this form of the premises (3) to illustrate how one obtains "the whole relation connecting any particular set of symbols", i.e. what necessary relation is implied by the equation independently of the remaining symbols. Since (8) is of the form $V = 0$ with $V(1 - V) = 0$, he can apply his Proposition III, Chapter X (see § 1.9), namely to develop the left-hand side of the equation with respect to the particular set of symbols (here x and y), and to equate to 0 the sum of the constituents on these symbols whose coefficients in the development are 1. Hence the result here $xy = 0$. The correctness of the procedure is intuitively clear, for only those terms whose coefficient in the development is 1 cannot be made to drop out by an appropriate choice of 0 or 1 values for the other symbols. (Note that this argument assumes that V is Boolean.)

Boole has a Rule enabling him to obtain such a "whole relation" without the necessity of first converting the equation to Boolean form as in the just discussed example. We recall (§ 1.9) that Boole's statement of the GENERAL PROBLEM of logic was that of determining, from any given equation involving symbols x, y,..., w, z, ... "the logical expression of any class expressed in any way by the symbols x, y, ... in terms of the remaining symbols w, z, ... &c." That is, given an equation, e.g. $V = 0$, and a class (or proposition) $t = \varphi(x, y, \ldots)$, to determine what is implied by $V = 0$ with respect only to t and $w, z, \ldots$. The practical rule (here stated for class terms) he gives is:

> Rule.—Expand the given equation [$V = 0$] with reference to the symbols x, y, [...]. Then form the equation
>
> $$Et + E'(1 - t) = 0$$
>
> in which E is the product of the coefficients of all those constituents in the above development, whose coefficients in the expression of the given class [$\varphi(x, y, \ldots)$, when developed with respect to $x, y, \ldots$] are 1, and E' the product of the coefficients in the expression of the given class are 0. The value of t deduced from the above equation by solution and interpretation will be the expression required. [p. 142]

Boole demonstrates this rule by adjoining to the given equation (assumed to have only positive terms) the equation $t = \varphi(x, y, \ldots)$, forming a single equation for the system, and eliminating x, y,.... We shall forgo discussion of the demonstration. As an illustration of the Rule here in Chapter XIV Boole uses the equation

$$xw + x\bar{w}y + x\bar{w}\bar{y}\bar{z} = 0, \tag{9}$$

which comes from (8) on elimination of t and v, and determines the relation connecting $u[= wx]$, x, and y. The Rule gives

$$xu + xy\bar{u} = 0$$

so that, by "solution and interpretation",

$$u = 0x\bar{y} + \frac{0}{0}\bar{x} + \frac{1}{0}xy$$

and hence

$$wz = \frac{0}{0}\bar{x}, \qquad xy = 0.$$

§ 3.5. Justification of the logic of secondary propositions

Our objective here is an explanation of why Boole's approach to propositional logic was, speaking generally, successful. Hence we need not adhere stringently to his system in all details; in particular we shall ignore the non-, or partially, defined Boolean notions.

As we have seen, Boole refers to "time in which the proposition X is true". While it could be that Boole had in mind something like our present-day notion of a propositional function—so that an actual expression for X would contain an occurrence of a variable t ranging over time values—it seems more likely that he thought of X as unchanging, i.e. not depending on a variable, but that with circumstances (in the world) changing there could be times at which X would be true and times at which it would be false. To render this conception we introduce a two-place (material) predicate T such that $T(X, t)$ expresses that X is true at time t. It is easy to see that having an order structure on the set of time values is irrelevant to the use which Boole is putting it to. Hence we take his "portion of time for which X is true" to be simply the set of t values $\{t: T(X,t)\}$ using "1" to designate the set of all t values (under consideration), we can indicate the essential features of Boole's symbolization of secondary propositions by a table:

Linguistic form	*Semantics*	*Algebraic form*
X	$\forall t\, T(X,t)$ or $\{t: T(X,t)\} = 1$	$x = 1$
not X	$\forall t\, \neg T(X,t)$ or $\{t: T(X,t)\} = 0$	$x = 0$

X and Y	$\forall t(T(X,t) \wedge T(Y,t))$ or $\{t: T(X,t)\} \cap \{t: T(Y,t)\} = 1$	$xy = 1$
X or Y	(similarly)	$x + \bar{x}y = 1$
If X, then Y	$\{t: T(X,t)\} \subseteq \{t: T(Y,t)\}$	$x\bar{y} = 0$
X iff Y	$\{t: T(X,t)\} \cup \{t: T(Y,t)\}$	$x = y$
If X or Y, then Z	$\{t: T(X,t)\} \cup \{t: T(Y,t)\}$ $\subseteq \{t: T(Z,t)\}$	$(x + \bar{x}y)\bar{z} = 0$

in which the last item embodies the assumption about T (tacit with Boole) that

$$\{t: T(X \text{ or } Y, t)\} = \{t: T(X,t)\} \cup \{t: T(Y,t)\}.$$

Equations of the form $f(x, y, \ldots) = 0$ are, of course, replaceable by $1 - f(x, y, \ldots) = 1$, and $f(x, y, \ldots) = g(x, y, \ldots)$ by $f(x, y, \ldots)g(x, y, \ldots) + \bar{f}(x, y, \ldots)\bar{g}(x, y, \ldots) = 1$. Thus when Boole analyzes a secondary proposition and expresses the result as an algebraic equation, we may take this equation to be of the form

(1) $$f(x_1, \ldots, x_n) = 1.$$

where $f(x_1, \ldots, x_n)$ is a Boolean algebra expression (in Boole's symbols), and to which one may uniquely associate a formula $F(X_1, \ldots, X_n)$ of **PC** (§0.5), obtainable from $f(x_1, \ldots, x_n)$ by replacing x_i by X_i and the symbols for addition (these occur only between exclusive terms), multiplication and subtraction from 1 by the symbols $\vee$, $\wedge$, $\neg$. Boole uses his class calculus methods to derive consequences from (1). Since the result is always an interpretable equation (or a pair of such equations, replaceable by an equivalent single equation—see § 1.8) we can take it to be of the form

(2) $$g(x_1, \ldots, x_n) = 1,$$

with + signs occurring only between exclusive terms, so that $g(x_1, \ldots x_n)$ is a Boolean algebra expression. Using the table in the reverse direction provides a **PC** formula $G(X_1, \ldots, X_n)$ associated with

$g(x_1,\ldots,x_n)$. (If 1 or 0 should occur in $g(x_1,\ldots,x_n)$ then, in obtaining $G(x_1,\ldots,x_n)$ we need to replace these, respectively, by $X_1 \vee \neg X_1$ and $X_1 \wedge \neg X_1$.) Since Boole's class calculus methods are sound (§ 2.6) we have that $f(x_1,\ldots x_n) = 1$ implies $g(x_1,\ldots,x_n) = 1$, i.e. that

(3) $$\vdash_{\mathrm{BA}} f(x_1,\ldots,x_n) = 1 \to g(x_1,\ldots,x_n) = 1.$$

Boole's method of doing propositional logic would then be sound and complete provided that

$$\vdash_{\mathrm{PC}} F(X_1,\ldots,X_n) \to G(X_1,\ldots,X_n)$$

is equivalent to (3)—but this is the assertion of Theorem 0.560.

Boole's awkward and indirect way of doing propositional logic was, of course, the result of his not realizing that one could construct, on the basis of truth-value semantics, a formal system for propositional logic independently of his algebraic-equational class logic. Even so, granted that one were to use Boole's idea, one can accomplish the needed correlation of propositional forms with class calculus equations with a far simpler meaning for $T(X,t)$ than that of "time for which a proposition is true". The same result can be accomplished by letting $\dot{1}$, the universe over which t ranges, be the singleton set whose only member is the truth-value "true", and taking $T(X,t)$ now to mean "the truth-value of X is t", so that $\{t: T(X,t)\} = \dot{1}$ says that X has the truth-value "true". Thus the equation $\{t: T(X,t)\} = \dot{1}$ or, in Boole's notation $x = 1$, does represent X, since it is true if X is true and false if X is false.

§ 3.6. A two- to four-valued connective

In the preceding section, to avoid unnecessary distraction, we deliberately omitted aspects of Boole's propositional logic which would involve explicit use of uninterpretable expressions. This was easy to do since we were only considering Boole's use of his calculus for conducting logical inference on the propositional level. We may note, with reference to our discussion there, that in going from $f(x_1,\ldots,x_n) = 1$ to $g(x_1,\ldots,x_n) = 1$ there was no need, as Boole strongly emphasized, to interpret intermediate steps; and if at each stage one were to replace the intermediate equations by equivalent ones having present only independently interpretable expressions (which we know can be done), then there would seem to be little interest in a propositional interpretation of Boole's non-interpretable forms. However Boole does make explicit use of such forms—specifically expansions of Boolean quotients—in connection with this theory of probability, and as we shall be presently looking at this theory it behooves us to discuss this aspect of Boole's propositional logic. (We are using the term "propositional logic" in a somewhat loose sense, our interest here being primarily in the introduction of certain new connectives and not with valid inference and its systematic organization.) Boole didn't have our modern concept of a semantic basis for logical notations and we shall have to ferret out, as best we can from what he does, what such a semantics might be.

In place of the complete disjunctive normal form of a formula on a given set of propositional variables we have Boole's development

$$1A + 0B, \tag{1}$$

or, as an asserted proposition,

$$1A + 0B = 1, \tag{2}$$

where between them the sums represented by A and B include all possible 2^n constituents on the n given variables. Thus Boole's development (1) corresponds to the usual truth-table representation of a propositional formula, since it depicts both those constituents corresponding to truth-possibilities which the formula agrees with (the A part) as well as those which it disagrees with (the B part). But Boole also uses developments of the form

$$1A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D, \tag{3}$$

in which there is a separation of the constituents into four, rather than two, categories. Can we give propositional significance to expressions, such as x/y for example, whose expansion is

$$1xy + 0\bar{x}y + \tfrac{0}{0}\bar{x}\bar{y} + \tfrac{1}{0}x\bar{y}? \tag{4}$$

As far as the truth-table idea is concerned one can formally extend the notion to correspond to expressions such as (3) by allowing assignments of values 1 and 0 (whatever they may mean) to the Boolean arguments and have values 1, 0, 0/0, 1/0 (whatever they may mean) for the (so-prefixed) compounds. Mathematically such tables are functions from the set $\{1,0\}$ into the set $\{1,0,0/0,\ 1/0\}$. Associated with (4) for example would be the table

(5)

x	y	x/y
1	1	1
1	0	1/0
0	1	0
0	0	0/0

The set of such 2- to 4-valued functions differs from the set of truth-functions of 2-valued logic in a major respect, namely in not being closed under functional composition—only those functional expressions whose values are limited to 1 or 0 can be substituted in for the (Boolean) argument variables. However, absence of the property of compositional closure (in full generality) need not be a hindrance to the use of such functions in logic if one is willing to accept a more complicated syntax, e.g. that "x/y" cannot replace a Boolean variable in a formula. (Another alternative—generalizing connectives so as to be representable by functions on $\{1,0,0/0,1/0\}$—we shall not pursue.)

Suggestive ideas as to how to give meaning to the values $1, 0, 0/0, 1/0$ can be obtained from Boole's use of Boolean quotient expansions in connection with his probability theory (in which context one talks about events happening or not happening rather than of propositions being true or false). Boole treats the question of a propositional interpretation for Boolean quotients, whose expansions have the full complement of four types of constituents, not in his chapters on secondary propositions but later on in the *Laws of Thought* in the second chapter on probability, and there it isn't really separated from the probability ideas. Only in a subsequent publication (BOOLE 1854d = BOOLE 1952 XV) do we find a clear propositional interpretation for the equation

$$(6) \qquad w = 1A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$

where A, B, C, D are sums of constituents on variables x, y, z,... Concerning (6) Boole says:

> 1. A represents those combinations of [constituents on] events x, y, z,... which must happen if w happen.
> 2. B represents those combinations which cannot happen if w happen, but may otherwise happen.
> 3. C those combinations which may or may not happen if w happen, and
> 4. D those combinations which cannot happen at all.

Note that Boole is here explicating the meaning of (6) as a whole and not giving separate meaning to the right-hand side. In order for us to do so recall that the proposition X is represented by the equation $x = 1$. If in (6) we replace w by 1 and assume that "1 happens" is true (and hence can be dropped as the antecedent of a conditional) then

$$(7) \qquad 1 = 1A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$

which corresponds to the assertion of the "proposition" $1A + 0B + \frac{0}{0}C + \frac{1}{0}D$ (with small letters turned into capitals) has the meaning

> 1. A represents those combinations of events, x, y, z,... [propositions X, Y, Z,...] which must happen [are "true"]
> 2. B represents those combinations which cannot happen [are "false"] but which may otherwise happen [i.e. under other circumstances]

3. C represents those combinations which may or may not happen [may be "true" or may be "false"], and
4. D represents those combinations which cannot happen at all [under any circumstance, i.e. are impossible].

What kind of events are these type 4. events which "cannot happen at all", and how do we recognize them in ordinary discourse? Such events, we believe, are involved whenever there is a restriction or limitation of the universe of events by some presupposition or condition. For example, consider the "loaded" question "Have you stopped beating your wife?" Here the interrogator is denying the possibility of there being nonwife-beating cases, i.e. the question presupposes that there are only wife-beating cases. As another example, in a stochastic setting, suppose we consider a universe of events consisting of the outcomes of two successive tosses of a coin (i.e. the elementary outcomes H_1H_2, $\bar{H}_1H_2$, $H_1\bar{H}_2$, $\bar{H}_1\bar{H}_2$) together with all possible (Boolean) logical combinations of them. If we know for a fact that Heads has come up on the first toss then any event inconsistent with this, e.g. $\bar{H}_1H_2$, is an event which cannot happen. Departing from factual matters, if we introduce the *condition* that H_1, then on this condition the event $\bar{H}_1H_2$ cannot happen, i.e. the "event"

$$\bar{H}_1H_2, \text{ on condition } H_1 \tag{8}$$

is an event of the 1/0 kind.

Appropriating language from conditional probability, where one refers to the condition determining the events that can happen as "given", we can express the newly envisioned connective X/Y, whose "truth"-table is

X	Y	X/Y
1	1	1
1	0	1/0
0	1	0
0	0	0/0

in words as:

$$XY \text{ at least, } XY + \bar{X}\bar{Y} \text{ at most, given } XY + \bar{X}Y + \bar{X}\bar{Y}$$

Note that $(Y \to X)/Y$, whose associated table is

X	Y	$(Y \to X)/Y$
1	1	1
1	0	1/0
0	1	0
0	0	1/0

has its A (and $A+C$) part equivalent to X and its $A+B+C$ part equivalent to Y, and thus has the reading "X, given Y." It will be convenient to denote this connective by "$X|Y$".

We shall recur to this topic in § 5.1.

§ 3.7. Notes to Chapter 3

The material in §§ 3.1–3.2 was presented to a conference "The Birth of Mathematical Logic: Nineteenth Century Logic from Boole to Schröder" held at the State University of New York, College at Fredonia, March 16–18, 1983. It subsequently appeared as an article "Boole's Abandoned Propositional Logic" in *History and Philosophy of Logic*, vol. 5 (1984), pp. 39–48.

(for § 3.1)

As far as I know the first explicit symbolization and discussion of an identity operator occurs in CAYLEY 1854: "...the symbol 1 will naturally denote an operation which (either generally or in regard to a particular operant) leaves the operand unaltered,...". Note that Cayley's paper appeared in the same year as *LT.*

(for § 3.3)

It is interesting to note that Schröder, who did so much by way of eliminating many of Boole's notions which are now considered as irrelevant to logic, nevertheless went along with him in retaining the notion of time as something needed to establish propositional logic. See SCHRÖDER 1891, §28. There is a fairly extensive historical discussion, in the context of a wider topic, in PRIOR 1957, pp. 108–111.

(for § 3.4)

Interpretive scholarly treatments of the *De Fato* material occuring in Boole's example can be found in:

Josiah B. Gould. *The Philosophy of Chrysippus.* State University of New York Press 1970.

Michael Frede. *Die Stoische Logic.* Van Den Hoeck & Ruprecht, 1974.

(for § 3.4)

Boole's technique of applying reduction to coefficients of an expansion can be justified as follows. For terms $v_1, v_2, v_3, \ldots$ let

$$\mathrm{Red}(v_1 + v_2 + v_3 + \cdots) = v_1 + \bar{v}_1 v_2 + \bar{v}_1 \bar{v}_2 v_3 + \cdots,$$

where a *term* is a product consisting only of letters and negated letters. The operator *Red* has the properties:

(1) for any terms t_1, t_2,

$$\mathrm{Red}\,(t_1 + t_2) = t_1 + \bar{t}_1 t_2 = t_1 \vee t_2$$

(2) for mutually exclusive terms t_1, t_2

$$\mathrm{Red}\,(t_1 + t_2) = t_1 + t_2$$

(3) if each term a_i of a sum $\sum a_i$ is mutually exclusive with every term b_j of a sum $\sum b_j$, then

$$\mathrm{Red}\,(\textstyle\sum a_i + \sum b_j) = \mathrm{Red}\,(\sum a_i) + \mathrm{Red}\,(\sum b_j)$$

(4) for terms of the form $a_i C$

$$\begin{aligned}\text{Red}\,(\textstyle\sum a_i C) &= a_1 C \vee a_2 C \vee a_3 C \cdots \\ &= (a_1 \vee a_2 \vee a_3 \cdots)C \\ &= (\text{Red}\,(\textstyle\sum a_i))C\end{aligned}$$

Thus when C_1, C_2, C_3,... are distinct constituents on a given set of letters and $\sum a_i$, $\sum b_i$, $\sum c_i$,... are sums of terms not containing any of these letters,

$$\begin{aligned}&\text{Red}\,((\textstyle\sum a_i)C_1 + (\textstyle\sum b_i)C_2 + (\textstyle\sum c_i)C_3 + \cdots) \\ &\quad = \text{Red}\,((\textstyle\sum a_i)C_1) + \text{Red}\,((\textstyle\sum b_i)C_2) + \cdots \\ &\quad = (\text{Red}\,(\textstyle\sum a_i))C_1 + (\text{Red}\,(\textstyle\sum b_i))C_2 + \cdots.\end{aligned}$$

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PART II. PROBABILITY

I regard this work of Boole's on probability as being of the utmost brilliance and importance. I am not aware that the general problem which he solves has been solved before or since. So far as I can judge Boole's solution is essentially sound, ...

C. D. Broad, 1917

... he takes a general indeterminate problem, applies to it particular assumptions not definitely stated in his book, ... and with these assumptions solves it; that is to say, he solves a particular determinate case of an indeterminate problem, while his book may mislead the reader by making him suppose it is the general problem which is being treated of.

Henry Wilbraham, 1854

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CHAPTER 4

PROBABILITY FROM BOOLE'S VIEWPOINT

The basic relationship between Boolean algebra and the calculus of events, now commonplace in treatises on probability, was fully understood and exploited by Boole. There is no mention of probability in his *The Mathematical Analysis of Logic* of 1847 but by 1854, in the *Laws of Thought*, we have an extensive development. Here Boole not only uses this basic relationship but also presents a distinctively new approach to conditional probability making essential use of his peculiar logical system. As an indication of the importance which Boole attached to this application of his logical system it may be noted that fully one-third of the *Laws of Thought* is devoted to probability and associated matters. Subsequent to the publication of the *Laws of Thought* he wrote a number of articles on probability which are of significance to our study as they contain fuller and more cogent statements of his ideas, and also an important technical result needed to justify his procedures. Consequently our exposition of Boole's work, which is the task of this chapter, while mainly based on the *Laws of Thought*, will take these later developments into consideration. Never clearly understood, and considered anyhow to be wrong, Boole's ideas on probability were simply by-passed by the history of the subject, which developed along other lines.

§ 4.1. Critique of the standard theory

In the main Boole accepts the prevailing views on the nature and principles of probability—Laplace and Poisson, for example, are cited as authorities. His contention isn't that the received theory is wrong but that the principles normally adopted are insufficient to develop a general theory. Associated with Boole's more general standpoint are a number of distinctive ideas of which, for the present, we mention just two.

Instead of speaking of the probability of an event one can, as Boole remarks, equivalently speak of the probability ["of the truth"] of the proposition asserting the occurence of the event. This possibility of replacing 'event' by 'proposition' had been noted earlier in the history of the subject by Ancillon in 1794 (see KEYNES 1921, p. 5 footnote 2) but only with Boole's construction of a logical calculus of propositions did such a replacement have any material consequence—as he puts it "its adoption ['proposition' for 'event'] will be attended with a practical advantage drawn from the circumstance that we have already discussed the theory of propositions, have defined their principle varieties, and established methods for determining, in every case, the amount and character of their mutual dependence" (p. 248). (Following Boole's practice we shall use 'event' interchangeably with 'proposition'.) This assertion of Boole's is stronger than it appears to be. For, as we shall presently see, he believed that a conditional probability was the same as the probability of a conditional proposition, so that all probabilities, whether unconditional or conditional, were probabilities of combinations that could be encompassed by his logical system. Additionally, probabilities of events obtained from observations of frequencies were, in his opinion, ultimately expressible as probabilities of combinations of, and hence logically expressible in terms of, simple events. Accordingly, he views the general object of a theory of probabilities as: "Given the probabilities of any events, of whatever kind, to find the probability of some other event connected with them."

To compare this projected general goal with what is attainable from the "received" theory Boole gives a list ("chiefly taken from Laplace") of the principles which have been applied to questions of probability (p. 249):

> PRINCIPLE 1st. If p be the probability of the occurence of any event, $1-p$ will be the probability of its non-occurence.
>
> 2nd. The probability of the concurrence of two independent events is the product of the probabilities of those events.

3rd. The probability of the concurrence of two dependent events is equal to the product of the probability of one of them by the probability that if that event occur, the other will happen also.

4th. The probability that if an event, E, take place, an event, F, will also take place, is equal to the probability of the occurrence of events E and F, divided by the probability of the occurence of E.

5th. The probability of the occurrence of one or the other of two events which cannot concur is equal to the sum of their separate probabilities.

6th. If an observed event can only result from some one of n different causes which are *a priori* equally probable, the probability of any one of the causes is a fraction whose numerator is the probability of the event, on the hypothesis of the existence of that cause, and whose denominator is the sum of the similar probabilities relative to all the causes.

7th. The probability of a future event is the sum of the products formed by multiplying the probability of each cause by the probability that if that cause exists, the said future event will take place.

Boole remarks that these principles [in fact only the 1st–5th] suffice to determine the probability of any "compound" event in terms of those of the "simple" events of which it is composed, provided that these latter are all independent events. This tacitly assumes that all such events are expressible as compounds of the independent simple events by means only of negation, conjunction and disjunction. Taking the 6th and 7th into consideration, Boole asserts that the most general problem formulable from all these principles is the following (p. 250):

DATA

1st. The probabilities of the n conditional propositions:

If the cause A_1 exist, the event E will follow;

" A_2 " E "

....

" A_n " E "

2nd. The condition that the antecedents of those propositions are mutually conflicting.

REQUIREMENTS

The probability of the truth of the proposition which declares the occurrence of the event E; also, when that proposition is known to be true, the probabilities of truth of the several propositions which affirm the respective occurrences of the causes $A_1, A_2 \ldots A_n$.

He then goes on to remark on its lack of generality:

Here it is seen, that the data are the probabilities of a series of compound events, expressed by *conditional* propositions. But the system is obviously a very limited and particular one. For the antecedents of the propositions are subject to the condition of being mutually exclusive, and there is but one consequent, the event E, in the whole system. It does not follow, from our ability to discuss such a system as the above, that we are able to resolve problems whose data are the probabilities of *any system of propositions whatever*. And, viewing the subject in its material rather than its formal aspect, it is evident, that the hypothesis of *exclusive* causation is one which is not often realized in the actual world, the phaenomena of which seem to be, usually, the products of complex causes, the amount and character of whose co-operation is unknown.

Some comment is in order.

The contemporary understanding is that the notion involved in the inverse probability principle (i.e. Boole's 6th) is not something involving "cause" but a two-argument notion of conditional probability usually defined as a quotient of probabilities, $P(AB)/P(B)$, referred to as the conditional probability of A, given B, and representing the chances of A also being the case if B is the case. In the preceding quotation we see Boole referring to this as the probability of a "conditional proposition". This would be incorrect if one takes this to be the ordinary conditional equivalent to "not both A and not-B". However, save in one instance (to be discussed below in §4.6), Boole always assigns to such "conditional propositions" the value of a quotient of probabilities. Boole never makes clear the logical status of these "conditional propositions".

As a further comment to the above quotation we point out that Boole gives no real justification for his assertion that the most general formulable problem by the known theory is as he states it, nor that it could not handle, in some fashion or other, the more general type of problem he envisions. Of course, this could be settled by an actual instance of a problem solvable by the one but not by the other. Indeed, in Boole 1851c, we find him proposing a problem "as a test of the sufficiency of the received methods". We discuss this problem below in §4.7 and again in §6.2.

§ 4.2. An additional principle

Although Boole gives the impression that he is merely adding a new probability principle to the accepted ones he is, in fact, additionally introducing new concepts, some of which involve a special understanding of those already in use, e.g. independence. We discuss these matters in some detail.

For the probability of the conjunction of two events Boole "deduces" from the definition of probability as a ratio of numbers of equally likely cases the principle (p. 235):

> II. The probability of the concurrence of two events is the product of either of these events by the probability that if that event occur, the other will occur also.

From this the usual product rule for the conjunction of independent events (§0.9) is derived on the basis of this definition:

> Definition.—Two events are said to be independent when the

> probability of the happening of either of them is unaffected by our expectation of the occurrence or failure of the other.

But contrast this DEFINITION with what would be a modern definition of independence if it would be stated in Boole's kind of language:

Two events are *independent* if the probability of the happening of either of these events is the same as the (conditional) probability that if the other occur, that one will occur also.

Note that Boole's Principle II refers to the "occurrence" of events whereas in his DEFINITION he speaks of "our expectation of the occurrence". Despite the discrepancy the two are combined to obtain the product rule for independent events. While Boole's product rule is the same as that of standard theory he does argue for the independence of certain kinds of events—"simple unconditional events"—which would not be accepted as independent on the basis of today's understanding of this notion. We first consider Boole's classification of events as simple or compound.

Although he doesn't give precise definitions it is clear from examples that compound events are those whose syntactic expression in a language is composed out of those for simple events by means of the logical connectives and thus, as Boole carefully points out, what are simple events depends upon the choice of language (p. 14):

> By a compound event is meant one of which the expression in language, or the conception in thought, depends upon the expression or the conception of other events which, in relation to it, may be regarded as simple events. To say "it rains", or to say "it thunders", is to express the occurrence of a simple event; but to say "it rains and thunders", or to say "it either rains or thunders", is to express that of a compound event. For the expression of that event depends upon the elementary expressions, "it rains", "it thunders". The criterion of simple events is not, therefore, any supposed simplicity in their nature. It is found solely on the mode of their expression in language or conception in thought.

Boole goes on to maintain that if the set of data of a problem does not imply a connection or dependence of events then these events are stochastically[1] independent (p. 256):

> 4. Now if this distinction of events, as simple or compound, is not founded in their real nature, but rests upon the accidents of language, it cannot affect the question of their mutual dependence or independence. If my knowledge of two simple events is confined to this particular fact, viz., that the probability of the occurrence of one of them is p, and that of the other is q: then I regard the events as independent, and thereupon affirm that the probability of their joint occurrence is pq. But the ground of this affirmation is not that the events are simple ones, but that the data afford no information whatever concerning any connexion or dependence between them. *When the probabilities of events are given, but all information respecting their dependence withheld, the mind regards them as independent.* [Italics supplied] And this mode of thought is equally correct whether the events, judged according to actual expression, are simple or compound, i.e., whether each of them is expressed by a single verb or by a combination of verbs.

In contrast to Boole, the contemporary view is that positive information, or an hypothesis to that effect, is necessary for the assertion of stochastic independence and, indeed, as we shall see in § 4.3, in the face of a specific problem with material content Boole appears to back down from his position. It is clear that one can't prove Boole wrong formally, for if a set of premises includes x and y as unanalyzed events, and if nowhere is there any compound of them present (to indicate connexion), then one can adjoin to the premises the equation $P(xy) = P(x)P(y)$, i.e. a statement of their stochastic independence, without engendering inconsistency.

We have earlier mentioned Boole's not distinguishing between the probability of a conditional proposition and conditional probability. Although neither the notion of conditional probability *per se* nor, of

[1] Since Boole uses "dependence" in more than one sense, we shall often insert this adjective for clarity or emphasis.

course, a symbol for it appears in Boole's writings, his treatment of situations requiring the notion are essentially correct. When he speaks of "the probability that if the event V occurs, an event V' will also occur", he refers to the event V' as being *conditioned by* the event V, and he takes this probability to be the quotient of the probability of VV' by the probability of V, i.e. as the conditional probability of V' given V. Simple unconditioned events are given a special status (p. 258):

> The simple events x, y, z will be said to be "conditioned" when they are not free to occur in every possible combination; in other words, when some compound event depending upon them is precluded from occurring. Thus the events denoted by the propositions, "It rains," "It thunders," are "conditioned" if the event denoted by the proposition, "It thunders, but does not rain," is excluded from happening, so that the range of possible is less than the range of conceivable combination. *Simple unconditioned events are by definition independent.* [Italics supplied]

Thus Boole believes that only by virtue of some conditioning can events fail to be independent. Further discussion of Boole's ideas on independence of simple events will be found in a chapter endnote to this section.

And now for the statement of Boole's new principle (pp. 250–257):

> VI. The events whose probabilities are given are to be regarded as independent of any connexion but such as is either expressed, or necessarily implied, in the data; and the mode in which our knowledge is to be employed is independent of the nature of the source from which such knowledge has been derived.

While seemingly innocuous, by virtue of Boole's taking, for simple events, absence of knowledge of any connexion to imply that events are unconditioned and hence stochastically independent, the consequences of using this principle are nontrivial. Exactly what Boole intends by this principle will emerge as we examine its use in establishing his general method.

§ 4.3. A general method

The demonstration of Boole's general method for solving "any" problem in probability is presented by means of a series of Propositions in Chapter XVII of *Laws of Thought*. It also appears with greater clarity, though more succinctly, in two later papers, Boole 1854e and 1857. We shall make the fuller *Laws of Thought* version the basis of our initial exposition.

First he takes up (Proposition I, p. 258) the uncomplicated case of simple unconditioned events (by "definition" independent) and the determination of the probability of a compound event depending on them. Noting that for such events x, y, with respective probabilities p, q, the conjunctive combinations xy, $x(1-y)$, $(1-x)(1-y)$, and the disjunctive combination $x(1-y)+y(1-x)$, have probabilities which are the "same functions of p and q as the former is of x and y" Boole concludes in general "If p, q, r are the respective probabilities of any unconditioned simple events, the probability of any compound event V will be $[V]$, this function $[V]$ being, formed by changing, in the function V, the symbols x, y, z [, etc.] into p, q, r, etc." By this result Boole then can immediately pass from the logical expression for a compound event to an algebraic expression giving its probability in terms of the unconditioned simple events of which it is composed. In this same proposition he also includes the following result on finding the conditional probability of V' given V: "... Under the same circumstance [p, q, r respective probabilities of unconditioned simple events x, y, z], the probability that if the event V occur any other event V' will also occur, will be $[VV']/[V]$, wherein $[VV']$ denotes the result obtained by multiplying together the logical functions V and V', and changing in the result x, y, z, etc. into p, q, r, etc." Clearly this result gives, correctly, the value of the conditional probability of V' given V and not for the conditional event "If V, then V'". But the fact that he includes it in his Proposition I along with genuine compound events, although in a separate category, reinforces our aforementioned belief that Boole does not realize that they are quite distinct notions. It should also be noted that although Boole states these results of Proposition I for "uncondition simple events x, y, z" they hold equally well if x, y, z are compound events so long as they are mutually independent (Theorem 0.92). None of the results of Proposition I depend on Boole's new principle.

Next Boole considers the case of simple events which are conditioned. We quote his proposition and the accompanying discussion in full (p. 261):

PROPOSITION II

10. It is known that the probabilities of certain simple events $x, y, z, \ldots$ are $p, q, r, \ldots$ respectively when a certain condition V is satisfied; V being in expression a function of $x, y, z, \ldots$ Required the absolute probabilities of the events $x, y, z, \ldots$, that is, the probabilities of their respective occurrence independently of the condition V.

Let p', q', r', etc., be the probabilities required, i.e. the probabilities of the events $x, y, z, \ldots$, regarded not only as simple, but as independent events. Then by Prop. I, the probabilities that these events will occur when the condition V, represented by the logical equation $V = 1$, is satisfied, are

$$\frac{[xV]}{[V]}, \quad \frac{[yV]}{[V]}, \quad \frac{[zV]}{[V]}, \text{ etc.},$$

in which $[xV]$ denotes the result obtained by multiplying V by x, according to the rules of the Calculus of Logic, and changing in the result x, y, z, into p', q', r', etc. But the above conditioned probabilities are by hypothesis equal to $p, q, r, \ldots$ respectively. Hence we have,

$$\frac{[xV]}{[V]} = p, \quad \frac{[yV]}{[V]} = q, \quad \frac{[zV]}{[V]} = r, \text{ etc.},$$

from which system of equations equal in number to the quantities $p', q', r', \ldots$, the values of those quantities may be determined.

Now xV consists simply of those constituents in V of which x is a factor. Let this sum be represented by V_x, and in like manner let yV be represented by V_y, etc. Our equations then assume the form

$$\frac{[V_x]}{[V]} = p, \quad \frac{[V_y]}{[V]} = q, \text{ etc.}; \tag{1}$$

where $[V_x]$ denotes the results obtained by changing in V_x the symbols x, y, z, etc., into p', q', r', etc.

Evidently Boole is here considering p, q, r to be *conditional* probabilities of x, y, z given V and then wishes to find in terms of these values the unconditional (or prior) probabilities of x, y, z. Although in the statement of the proposition he refers to x, y, z as "simple" events, by virtue of the fact that he uses the results of Proposition I to write the probability of V as $[V]$ (and similarly $[xV]$ for that of xV, etc.) indicates that the prior events are taken to be *simple unconditioned*, i.e., by "definition", independent events. We also point out that the system of equations,

$$\frac{[xV]}{[V]} = p, \quad \frac{[yV]}{[V]} = q, \quad \frac{[zV]}{[V]} = r, \text{ etc.},$$

which relate the given p, q, r, etc. with the unknowns p', q', r', etc. are indeed equal in number to them, but they are not necessarily linear equations and their treatment, as Boole later realized, entails considerable difficulties regarding the existence and uniqueness of solutions and, in addition, the existence of solutions in the range 0 to 1 necessary for probability values.

As an illustration of his Proposition II Boole gives the following example (p. 262):

> Suppose that in the drawing of balls from an urn attention had only been paid to those cases in which the balls drawn were either of a particular colour, "white", or of a particular composition, "marble", or were marked by both these characters, no record having been kept of those cases in which a ball that was neither white nor of marble had been drawn. Let it then have been found, that whenever the supposed condition was satisfied, there was a probability p that a white ball would be drawn, and a probability q that a marble ball would be drawn: and from these data alone let it be required to find the probability that in the next drawing, without reference at all to the condition above mentioned, a

white ball will be drawn; also the probability that a marble ball will be drawn.

Here if x represent the drawing of a white ball, y that of a marble ball, the condition V will be represented by the logical function

$$xy + x(1 - y) + (1 - x)y.$$

Hence we have

$$V_x = xy + x(1 - y) = x, \ V_y = xy + (1 - x)y = y;$$

whence

$$[V_x] = p', [V_y] = q';$$

and the final equations of the problem are

$$\frac{p'}{p'q' + p'(1 - q') + q'(1 - p')} = p,$$

$$\frac{q'}{p'q' + p'(1 - q') + q'(1 - p')} = q;$$

from which we find

$$p' = \frac{p + q - 1}{q}, \qquad q' = \frac{p + q - 1}{p}.$$

Boole is apparently uneasy about this solution for he then goes on to say:

> To meet a possible objection, I here remark, that the above reasoning does not require that the drawings of a white and a marble ball should be independent, in virtue of the physical constitution of the balls. The assumption of their independence is indeed involved in the solution, but it does not rest upon any prior assumption as to the nature of the balls, and their relations, or freedom from relations, of form, colour, structure, etc. It is founded upon our total ignorance of all these things. Probability always has reference to the state of our actual knowledge, and its numerical value varies with varying information.

What is overlooked here, however, is that although the drawings of the white and of the marble balls are not required to be independent "in virtue of the physical constitution of the balls", taking them to be independent, which is what Boole in effect does by writing $[V]$ for the probability of V, necessarily implies a physical restriction on the contents of the urn—for not every urn has the property that the drawing of a white and of a marble ball from it are stochastically independent events. Boole must have realized his error but too late for correction in the text, for we find a NOTE printed at the beginning of the *Laws of Thought*[1] following the table of contents which acknowledges the inappropriateness of this example as an illustration of Proposition II and stating that the correct solution of the urn problem as given should be

$$p' = cp, \qquad q' = cq,$$

"in which c is the arbitrary probability of the condition that the ball should be either white, or of marble, or both at once". Boole gives no further explanation other than referring the reader to Case 2 of Proposition IV, where one has events $x, y, z, \ldots$ which are "conditioned"—presumedly, in the example, by the unknown constitution of the urn; hence one cannot take the probability of V to be $[V]$ as in the case of x and y being unconditioned simple events. Thus, as Boole's error and his correction of it indicates it may not be always evident when, in a specific application, events are or are not to be taken as simple unconditioned events. But this is really a question of how the theory is to be correctly *applied*—a topic to which Boole devotes little attention—rather than one of whether the theory is correct.

Preparatory to giving his solution of what he considered to be the general problem in probability theory, Boole needs to show how to determine an arbitrary event (whose probability is sought) as a function of other events (whose probabilities are given). This preliminary logical problem is stated as Proposition III and we quote the argument which follows (pp. 263–264):

> Let S, T, etc., represent any compound events whose probabili-

[1] The Note appears in the 1951 Dover reprint but is omitted from the 1916 Open Court edition. The substance of the Note also appears in a letter from Boole to De Morgan of 15 February 1854. (SMITH 1982, pp. 63–64).

ties are given, S and T being in expression known functions of the symbols x, y, z, etc., representing simple events. Similarly let W represent any event whose probability is sought, W being also a known function of x, y, z, etc. As S, T, ... W must satisfy the fundamental law of duality, we are permitted to replace them by single logical symbols, s, t, ..., w. Assume then

$$s = S, t = T, w = W.$$

These, by the definition of S, T, ..., W will be a series of logical equations connecting the symbols s, t, ..., w, with the symbols $x, y, z, \ldots$

By the method of the Calculus of Logic we can eliminate from the above system any of the symbols $x, y, z, \ldots$, representing events whose probabilities are not given, and determine w as a developed function of s, t, etc., and of such of the symbols x, y, z, etc., if any such there be, as correspond to events whose probabilities are given. The result will be of the form

$$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$

where A, B, C, and D comprise among them all the possible *constituents* which can be formed from the symbols s, t, etc., i.e. from all the symbols representing events whose probabilities are given.

The above will evidently be the complete expression of the relation sought. For it fully determines the event W, represented by the single symbol w, as a function or combination of the events similarly denoted by the symbols s, t, etc., and it assigns by the laws of the Calculus of Logic the condition

$$D = 0,$$

as connecting the events s, t, etc., among themselves. We may, therefore, by Principle VI, regard s, t, etc., as *simple* events, of which the combination w, and the condition with which it is associated D, are definitely determined.

We consider first Boole's assertion that the event W is "fully determined" by the equation $w = A + 0B + \frac{0}{0}C + \frac{1}{0}D$ with w representing W, s representing S, etc. If this were so then we should be able to obtain, under the necessary condition expressed by $D = 0$, a Boolean expression for W in terms of $S, T, \ldots$. One could try using Boole's interpretation for the equation, namely

$$(1) \qquad \begin{aligned} w &= A(s, t, \ldots) + vC(s, t, \ldots) \\ D(s, t, \ldots) &= 0, \end{aligned}$$

and obtain such an expression by substituting in it W for w, S for s, T for t, etc. But what of v? It too depends on $s, t, \ldots$ and, moreover, in general in no explicitly (Boolean) determinable manner. However, if we use not (1) but

$$(2) \qquad \begin{aligned} &A(s, t, \ldots) \subseteq w \subseteq A(s, t, \ldots) + C(s, t, \ldots) \\ &D(s, t, \ldots) = 0 \end{aligned}$$

we have, on making the substitution W for w, S for s, T for $t, \ldots$ that

$$\begin{aligned} &A(S, T, \ldots) \subseteq W \subseteq A(S, T, \ldots) + C(S, T, \ldots) \\ &D(S, T, \ldots) = 0 \end{aligned}$$

thus obtaining only Boolean inclusion *bounds* on W in terms of $S, T, \ldots$—subject of course to the condition that $D(S, T, \ldots) = 0$. As explained in § 2.6, these Boolean bounds are "best possible". It is only in this sense that we can agree with Boole that $w = A + 0B + \frac{0}{0}C + \frac{1}{0}D$ determines W as a combination of $S, T, \ldots$.

We next consider the assertion in the last sentence of the above quotation. Having introduced $s, t, \ldots$ as "single logical symbols", Boole appeals to his Principle VI to justify considering w as a combination of the *simple* events $s, t, \ldots$, these events being conditioned by $D = 0$ so that only those combinations contained in $A + B + C$ are possible. It is hard to see how this is justified by Principle VI since it refers to "The events whose probabilities are given", whereas $s, t, \ldots$ are newly introduced. But more than this: he takes $s, t, \ldots$ as simple *unconditioned* events, i.e. as independent events, which, when "conditioned" by $D = 0$, have the same probabilities as the given events $S, T, \ldots$. To see this we turn to the statement of Boole's Proposition IV, embodying the nub of his probability method (p. 25):

PROPOSITION IV

14. *Given the probabilities of any system of events; to determine by a general method the consequent or derived probability of any other event.*

As in the last Proposition, let S, T, etc., be the events whose probabilities are given, W the event whose probability is sought, these being known functions of x, y, z, etc. Let us represent the data as follows:

$$\begin{aligned}&\text{Probability of } S = p;\\ &\text{Probability of } T = q:\end{aligned} \tag{1}$$

and so on, p, q, etc., being known numerical values. If we then represent the compound event S by s, T by t, and W by w, we find by the last proposition,

$$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D; \tag{2}$$

A, B, C, and D being functions of s, t, etc. Moreover the data (1) are transformed into

$$\text{Prob. } s = p, \quad \text{Prob. } t = q, \text{ etc.} \tag{3}$$

Now the equation (2) is resolvable into the system

$$\begin{aligned}&w = A + qC,\\ &D = 0,\end{aligned} \tag{4}$$

q being an indefinite class symbol (VI. 12). But since by the properties of constituents (V. Prop. III), we have

$$A + B + C + D = 1,$$

the second equation of the above system may be expressed in the form

$$A + B + C = 1.$$

If we represent the function $A + B + C$ by V, the system (4) becomes

$$w = A + qC, \tag{5}$$

$$V = 1. \tag{6}$$

Let us for a moment consider this result. Since V is the sum of a series of constituents of s, t, etc., it represents the compound event in which the simple events involved are those denoted by s, t, etc. Hence (6) shows that the events denoted by s, t, etc., and whose probabilities are p, q, etc., have such probabilities not as independent events, but as events subject to a certain condition V. Equation (5) expresses w as a similarly conditioned combination of the same events.

That p, q etc. are conditional probabilities of s, t, etc. on condition V makes some sense since s, t, etc. have these probabilities given that $s = S$, $t = T$, etc., and V is the logical content of these equations in respect to s, t, etc. We examine this later in § 5.4. To continue:

Now by Principle VI, the mode in which this knowledge of the connexion of events has been obtained does not influence the mode in which, when obtained, it is to be employed. We must reason upon it as if experience had presented to us the events s, t, etc., as simple events, free to enter into every combination, but possessing, when actually subject to the condition V, the probabilities p, q, etc., respectively.

Boole goes on now to apply his Propositions I and II which, as we have noted, requires the *a priori* $s, t, \ldots$ to be simple *unconditioned*, i.e. independent, events:

Let then p', q', ..., be the corresponding probabilities of such events, when the restriction V is removed. Then by Prop. II of the present chapter, these quantities will be determined by the system of equations,

$$\frac{[V_s]}{[V]} = p, \qquad \frac{[V_t]}{[V]} = q, \text{ etc.}; \tag{7}$$

> and by Prop. I. the probability of the event w under the same condition V will be
>
> $$\text{Prob. } w = \frac{[A + cC]}{[V]}; \tag{8}$$
>
> wherein V_s denotes the sum of those constituents in V of which s is a factor, and $[V_s]$ what that sum becomes when $s, t, \ldots$, are changed into $p', q', \ldots$, respectively. The constant c represents the probability of the indefinite event q; it is, therefore, arbitrary, and admits of any value from 0 to 1.

There are minor lapses in Boole's argument here. First of all (8) should be written:

$$\text{Prob. } w = \frac{[A + qC]}{[V]},$$

where 'q' is Boole's indefinite class symbol, since the square brackets are applied only to logical expressions and $A + cC$ is not one if c is a probability. Secondly, the application of Prop. I to get (8) requires that the events involved be simple events; while we know (granted Boole's argument) that this is the case for s, t, etc., we have no explicit justification for this in the case of the indefinite event q. Perhaps Boole thought it self-evident that q is a simple unconditioned event on a par with $s, t, \ldots$

Boole's sentence about the constant c should then be postponed and placed after his equation (10) (see our next quotation).

We continue with his discussion of Proposition IV:

> Now it will be observed, that the values of p', q', etc., are determined from (7) only in order that they may be substituted in (8), so as to render Prob. w a function of known quantities, p, q, etc. It is obvious, therefore, that instead of the letter p', q' etc., we might employ any others as s, t, etc., in the same quantitative acceptations. This particular step would simply involve a change of meaning of the symbols s, t, etc.—their ceasing to be logical, and becoming quantitative. The systems (7) and (8) would then become
>
> $$\frac{V_s}{V} = p, \qquad \frac{V_t}{V} = q, \text{ etc.}; \tag{9}$$

$$\text{Prob. } w = \frac{A + cC}{V}. \tag{10}$$

In employing these, it is only necessary to determine from (9) s, t, etc., regarded as quantitative symbols, in terms of p, q, etc., and substitute the resulting values in (10). It is evident, that s, t, etc., inasmuch as they represent probabilities, will be positive proper fractions.

15. It remains to interpret the constant c assumed to represent the probability of the indefinite event q. Now the logical equation

$$w = A + qC,$$

interpreted in the reverse order, implies that if either the event A takes place, or the event C in connexion with the event q, the event w will take place, and not otherwise. Hence q represents that condition under which, if the event C take place, the event w will take place. But the probability of q is c. Hence, therefore, c = probability that if the event C take place the event w will take place. Wherefore by Principle II.,

$$c = \frac{\text{Probability of concurrence of } C \text{ and } w}{\text{Probability of } C}.$$

Boole's argument isn't quite clear, but that what he gives for c is correct can be seen by noting that $w = A + qC$ gives

$$P(w) = P(A) + P(q|C)P(C)$$

and that $P(q|C) = P(w|C)$, since C implies that w and q are equivalent. Boole now summarizes the above discussion and states his method as a General Rule which he divides up into two cases—the first when the given probabilities $p, q, \ldots$ are probabilities of unconditioned events x, $y, \ldots$ and a second case as follows:

Case II.—When some of the events are conditioned.

If there be given the probability p that if the event X occur, the event Y will occur, and if the probability of the antecedent X

be not given, resolve the proposition into the two following, viz:

$$\text{Probability of } X = c,$$

$$\text{Probability of } XY = cp.$$

If the quaesitum be the probability that if the event W occur, the event Z will occur, determine separately, by the previous case, the terms of the fraction

$$\frac{\text{Prob. } WZ}{\text{Prob. } W},$$

and the fraction itself will express the probability sought.

It is understood in this case that X, Y, W, Z may be any compound events whatsoever. The expressions XY and WZ represent the products of the symbolical expressions of X and Y and W and Z, formed according to the rule of the Calculus of Logic.

It thus appears that when conditioned events are part of the data then new parameters, e.g. the c = Probability of X in Case II just quoted, can enter. But then Boole's general method isn't quite what one may have understood it to be! For when he says in Proposition IV "*Given the probabilities of any system of events; to determine by a general method the consequent or derived probability of any other event*" one would naturally suppose it to mean that the derived probability would be in terms of the given probabilities. Now we see that, when conditioned events are part of the data, we are to understand this as meaning "in terms of the given probabilities plus possibly parameters representing unknown probabilities."

After presenting his general method for the solution of problems in probability Boole brings his Chapter XVII to a close with three short sections of philosophical argument, concluding that one may either start with the "ordinary numerical definition of the measure of probability" (i.e. as a ratio of cases) and derive the formal identity between the logical expression of events and the algebraic expression of their values or, conversely, by starting with the assumption that the measure of probability has such values so as to bring about this formal identity, one can obtain the ordinary numerical definition.

In view of the extensive exegetical comments with which we have interspersed our presentation of Boole's method it will be helpful, before going on to the next section, to recapitulate its main features as given in his Propositions I–IV of Chapter XVII.

In Proposition I we have the result that the probability of a Boolean polynomial V of simple unconditioned events (with $+$ being used disjunctively) is the homonymous arithmetic function $[V]$ of the respective probabilities; and for the conditioned event V' given V, the probability is given by the quotient $[VV']/[V]$.

In Proposition II it is required to find the absolute probabilities of events $x, y, z, \ldots$ if one is given that, on condition V, their respective probabilities are $p, q, r, \ldots$ As Boole explains in the emendatory NOTE printed at the front of the *Laws of Thought*, "In Prop. II, p. 261, by the 'absolute probabilities' of the events $x, y, z, \ldots$ is meant simply what the probabilities of those events ought to be, in order that, regarding them as independent [sic], and their probabilities as our only data, the calculated probabilities of the same events under the condition V should be $p, q, r, \ldots$". By virtue of this meaning for "absolute probabilities" Boole is then justified in replacing the equations

$$(1) \qquad \frac{P(xV)}{P(V)} = p, \qquad \frac{P(yV)}{P(V)} = q, \text{ etc.,}$$

expressing that p, q etc., are conditional probabilities, by the equations

$$(2) \qquad \frac{[xV]}{[V]} = p, \qquad \frac{[yV]}{[V]} = q, \text{ etc.}$$

which involve only the absolute probabilities of $x, y, \ldots$

In Proposition III Boole determines "in any question of probabilities" the connection of the event $W(x, y, \ldots)$ whose probability is sought as a logical function of events $S(x, y, \ldots)$, $T(x, y, \ldots)$, ... whose probabilities are given. As we have shown above there is a sense in which one can give this connection explicitly in terms of $S, T, \ldots$, but Boole does this through the intermediary of *simple* events $s, t, \ldots, w$ and by means of the equation

(3) $$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$

obtained from the system of equations

(4) $$s = S, \quad t = T, \ldots, \quad w = W$$

by eliminating events whose probabilities are not given, and then solving by his Calculus of Logic for w. The interpreted form for (3) is given by the pair of equations

(5) $$w = A + qC,$$

(6) $$V = 1, \quad \text{where } V = A + B + C.$$

If the data should include logical relationships among the $x, y, z, \ldots$ then these relationships expressed in equational form, are to be included along with the system (4).

In Proposition IV Boole argues, on the basis of his new principle in probability (called Principle VI above), that the equation (6) shows that the events $s, t, \ldots$ whose probabilities are the given $p, q, \ldots$ have these probabilities not as independent events but as events conditioned by V, and that (5) similarly expresses w as a combination of these conditioned events. Accordingly, by use of Propositions II and I, Boole can write this result as

(7) $$\frac{V_s}{V} = p, \qquad \frac{V_t}{V} = q, \ldots,$$

(8) $$w = \frac{A + cC}{V}$$

where, employing Boole's convention, the letters 's', 't', ... in these equations now stand for their respective (absolute) probabilities. The [unique?] numerical values of $s, t, \ldots$ determined by [are they?] equations (7) are to be found [how?] and substituted in (8) to determine the probability sought. (It is to be regretted that Boole had no special symbol for conditional probability—nor for that matter for probability itself other than the rudimentary „Prob."—as his discussion would improve in clarity by a visual distinction between $P(s), P(t), \ldots$ and $P(s \mid V), P(t \mid V), \ldots$).

§ 4.4. The problem of "absolute" probabilities I

We here undertake a preliminary investigation of Boole's Proposition II (*Laws of Thought*, p. 261) and his assertion that the absolute (i.e. unconditioned) probabilities of $x, y, z, \ldots$ "may be determined" from the equations

$$\frac{[xV]}{[V]} = p, \quad \frac{[yV]}{[V]} = q, \quad \frac{[zV]}{[V]} = r, \text{ etc.},$$

which are "equal in number" to the number of unknowns. Since the solution for the probability of the sought for event in his general probability problem is expressed in terms of these absolute probabilities, the question as to whether there are such values is of no small moment to his method. Initially Boole says nothing concerning possible difficulties attendant upon the solution of such a system of equations—difficulties relating to necessary conditions on the parameters $p, q, r, \ldots$, to dependence or independence of the equations, and to the handling of what in general is a non-linear system of equations. In the *Laws of Thought* Boole treats these matters on an *ad hoc* basis for the particular illustrative problems he studies, and only in a series of papers published subsequently to the *Laws of Thought* does he tackle the general situation. In this section, in preparation for the later general treatment, we shall look at some very simple cases so as to gain an appreciation of these difficulties.

Before continuing let us rephrase Boole's Proposition II in contemporary terms.

Problem on Absolute Probabilities. Given real numbers $p, q, r, \ldots$ in the range 0 to 1, and a Boolean function V of mutually independent events $x, y, z, \ldots$ with $P(V) \neq 0$, determine whether there are values of $P(x), P(y), P(z), \ldots$ (necessarily in the range from 0 to 1) satisfying

$$(1) \qquad p = \frac{P(xV)}{P(V)}, \quad q = \frac{P(yV)}{P(V)}, \quad r = \frac{P(zV)}{(V)}, \ldots$$

and, if so, find all such values.

In our discussion of this problem we shall adopt Boole's convention of using letters $x, y, z, \ldots$ both for events and for their respective probabilities; although by this convention the logical and arithmetical symbols 0, 1, $+$, $^-$ are indistinguishable, one can readily tell by the context which is meant. For example, if p is a number then in $(x\bar{y} + \bar{x}y)\, p = x\bar{y}$, the numerical interpretation for $x, y, \bar{x}, \bar{y}$ is clearly intended. It will also be convenient to use V for $[V]$, i.e. the result of replacing events $x, y, z, \ldots$ by their respective probabilities, and V_x for $[xV]$; we avoid using 'xV' in the numerical sense since it would be ambiguous as to whether $[x]\,[V]$ or $[xV]$ is meant. The assumption of the mutual independence of the events $x, y, z, \ldots$ enables one to replace $P(V)$ by $[V]$, $P(xV)$ by $[xV]$, etc.

One readily sees that in investigating the problem at hand it suffices to consider only one function from a given symmetry type (see 0.5.6 in § 0.5); for example the solution of the problem for

$$V = xy + \bar{x}\bar{y}, \quad p = \frac{V_x}{V}, \quad q = \frac{V_y}{V}, \tag{2}$$

is equivalent to that for

$$V = x\bar{y} + \bar{x}y, \quad p = \frac{V_x}{V}, \quad q = \frac{V_y}{V}, \tag{3}$$

since replacing y by $\bar{y}$ in (2) gives (3) by virtue of

$$1 - \frac{V_{\bar{y}}}{V} = \frac{V_y}{V}. \tag{4}$$

Consider an arbitrary Boolean function V of two variables. Since we are requiring $P(V) \neq 0$, we can rule out $V = 0$. In addition, the case of $V = 1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$ is trivial, the solution being clearly $x = p$, $y = q$. We then restrict our attention to functions V having 1, 2 or 3 constituents and of these select the following as representatives of each of the four possible symmetry types:

(i) xy,
(ii) $x\bar{y} + \bar{x}y$,
(iii) $xy + x\bar{y}\ (= x)$,
(iv) $xy + \bar{x}y + x\bar{y}\ (= x \vee y)$.

Case i. $V = xy$.

In this case the system of equations (1) becomes

$$p = \frac{xy}{xy}, \qquad q = \frac{xy}{xy},$$

and we see that there is a solution only if $p = q = 1$; and, if this is the case, then two equations become identical and any pair of values (x, y) with x and y in the half-closed interval $(0, 1]$ constitutes a solution.

Case ii. $V = x\bar{y} + \bar{x}y$.

In this case equations (1) are

$$p = \frac{x\bar{y}}{x\bar{y} + \bar{x}y}, \qquad q = \frac{\bar{x}y}{x\bar{y} + \bar{x}y},$$

which upon addition gives $p + q = 1$ and hence there is a solution only if p and q satisfy this condition. If the condition is satisfied then there is only one independent equation which we may write, on introducing $x_1 = \bar{x}/x$, $x_2 = \bar{y}/y$, as

$$x_2 = \frac{p}{1-p} x_1 \qquad (x_1, x_2 \geq 0)$$

indicating a "half-line" of solutions in the (x_1, x_2)-plane.

Case iii. $V = x$.

Here equations (1) are

$$p = \frac{x}{x}, \qquad q = \frac{xy}{x},$$

and a solution exists only if $p = 1$. In this case x can have any value in the range $(0, 1]$ and y must be q.

Case iv. $V = xy + \bar{x}y + x\bar{y}$.

Here the equations (1) are

$$(7) \qquad p = \frac{xy + x\bar{y}}{xy + \bar{x}y + x\bar{y}}, \qquad q = \frac{xy + \bar{x}y}{xy + \bar{x}y + x\bar{y}}.$$

By addition of these equations we find

$$(8) \qquad p + q - 1 = \frac{xy}{V},$$

and thus a solution only if

$$(9) \qquad 0 \leq p + q - 1.$$

Subcase iv_1. $p + q - 1 = 0$.

By (8) this implies $xy = 0$, so that either $x = 0$ or $y = 0$. In the first case we have by (7) $p = 0$, $q = y/y$ so that the solutions are $x = 0$, y any value in (0, 1]; in the second case (7) tells us that $p = x/x$, $q = 0$ so that the solutions are x any value in (0, 1] and $y = 0$.

Subcase iv_2. $0 < p + q - 1$.

By (8) we have that $xy \neq 0$. Hence equations (7) can be written

$$(10) \qquad p = \frac{1 + x_2}{1 + x_1 + x_2}, \qquad q = \frac{1 + x_1}{1 + x_1 + x_2},$$

where as before $x_1 = \bar{x}/x$, $x_2 = \bar{y}/y$. On rearranging (10) into the form

$$(11) \qquad px_1 - px_2 = \bar{p}, \qquad -\bar{q}x_1 + qx_2 = \bar{q}$$

we see that we have a pair of simultaneous linear equations in x_1 and x_2. Such a system has a unique solution if and only if the determinant of the coefficients, namely

$$\begin{vmatrix} p & -\bar{p} \\ -\bar{q} & q \end{vmatrix} = pq - \bar{p}\bar{q} = p + q - 1,$$

is different from 0. Since in this case we do have $p + q - 1 \neq 0$, there is a unique solution. One readily finds

$$x_1 = \frac{\bar{p}}{p + q - 1}, \qquad x_2 = \frac{\bar{q}}{p + q - 1}$$

and then

$$x = \frac{p + q - 1}{q}, \qquad y = \frac{p + q - 1}{p},$$

and that the values for x and y are in the appropriate probability range since $p + q - 1$ which, by hypothesis, is greater than 0 is also less or equal to q or p.

On the basis of these simple cases we can make some observations which will serve to illustrate points coming up later in our general discussion of this topic (in § 5.6).

It is not sufficient, for the existence of solutions to (1), that p and q be merely in the probability range 0 to 1—equations (1) imply in general more stringent necessary conditions than this. If the implied necessary condition is an *equation*, then the system (1) is not independent, and we are prepared for families (subspaces) of solutions. In the one case for which there is a unique solution, namely for $V = x \vee y$ (or any of its fellow symmetry-type mates) the necessary condition relating p and q is a linear *inequation*. Although in this case the equations (1) come out to be linear (in the variables $\bar{x}/x$ and $\bar{y}/y$) it is easy to see that, for V's with more than two variables, such a circumstance cannot in general prevail.

§ 4.5. Boole's method without the mutual independence

Although the simple events introduced in his general method were, from Boole's standpoint, always mutually independent, there are particular problems of the kind treated by Boole which can be solved without the need to assume that these simple events are mutually independent. We devote a few paragraphs to this topic.

Recall that, in the uncomplicated case of Boole's general method, one is given

$$P(S(x, y, \ldots)) = p,$$
$$P(T(x, y, \ldots)) = q, \quad \text{etc.}$$

and is asked to find $P(W(x, y, \ldots))$ where $S, T, \ldots, W$ are known Boolean functions of $x, y, \ldots$ Boole puts $s = S(x, y, \ldots)$, $t = T(x, y, \ldots), \ldots,$ $w = W(x, y, \ldots)$ and expresses w in terms of $s, t, \ldots$ The result is then converted into developed form

$$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$

where A, B, C, D are sums of constituents on $s, t, \ldots$ Boole argues—a contention we examine in our next chapter—that $P(W)$ is the conditional probability

$$(1) \qquad \frac{[A + qC]}{[V]}$$

and its value is to be determined by solving the equations

(2) $$p = \frac{[sV]}{[V]}, \qquad q = \frac{[tV]}{[V]}, \ldots$$

for the probabilities of $s, t, \ldots$ appearing therein which are then substituted into (1). In using the square brackets Boole is in effect (see our preceding section) assuming that the events $s, t, \ldots$ are mutually independent, but without this assumption he still could have argued that (using present day notation)

(3) $$P(W) = \frac{P(A + qC)}{P(V)}$$

with, similarly,

(4) $$p = \frac{P(sV)}{P(V)}, \quad q = \frac{P(tV)}{P(V)}, \ldots, \quad 1 = \frac{P(V)}{P(V)}.$$

(In comparison with (2) we here have an additional equation $1 = P(V)/P(V)$, which, although trivially true, will be needed.) From (3) and (4) one can still determine $P(W)$ in terms of the given $p, q, \ldots, 1$ *if* $P(A + qC)$ could be expressed as a linear combination of $P(sV), P(tV), \ldots, P(V)$ for, since by (4) these quantities are respectively equal to $pP(V)$, $qP(V), \ldots, 1 \cdot P(V)$, by substituting this linear combination in (3) for $P(A + qC)$ the $P(V)$ in numerator and denominator cancels, leaving as a value for $P(W)$ this same linear combination of $p, q, \ldots, 1$. Conditions on V, A, and C so that $P(A + qC)$ is so expressible are determined as follows.

Note that any two distinct constituents (on the same variables) are mutually exclusive so that the probability operator distributes over sums of such constituents. Call any one such term a *constituent probability*. Now since A and C are mutually exclusive,

$$\begin{aligned} P(A + qC) &= P(A) + P(qC) \\ &= P(A) + P(q \mid C)\, P(C) \\ &= P(A) + cP(C), \end{aligned}$$

where, in the last equation, we have put c for $P(q \mid C)$. If possible we would like to have values for $a_1, \ldots, a_{n+1}$ so that (assuming that the variables $s, t, \ldots$ are n in number)

$$(5) \qquad P(A) + cP(C) = a_1P(sV) + a_2P(tV) + \cdots + a_{n+1}P(V)$$

is an identity. If now in (5) we replace $P(A), P(C), P(sV), \ldots, P(V)$ by the respective sums of constituent probabilities which are equal to them and equate the total coefficients of like constituent probabilities of the two sides of the equation, we would obtain m (linear) equations in the $n+1$ unknowns $a_1, \ldots, a_{n+1}$, where m $(1 \leq m \leq 2^n)$ is the number of constituents in V. It is evident that in these m equations the coefficients of the a_i can be only 0 or 1, and the constant term in each equation can be only 0, 1, or c. From the theory of linear equations we know that a system of linear equations is consistent (has a solution) if and only if the rank of the augmented matrix is equal to that of the matrix of coefficients. If $P(W)$ can be found in this manner, i.e. if the m linear equations in the a_i have a solution, we will say that we are applying the *Method of Linear Composition*. In the next section we shall see examples of this method. It should be pointed out that the "chances" of this method being applicable in an arbitrary situation are small since (in the case of a V on n variables) there can be up to 2^n equations involving only $n+1$ unknowns—and, statistically speaking, the more equation there are the "harder" is it for the system to be consistent. On the other hand, if the Method of Linear Composition is applicable then one is spared having to solve the system of equations (2) for the probabilities of the simple events (see § 4.2). The possibility of this was not mentioned in the *Laws of Thought* but occurs later in BOOLE 1854c (see BOOLE 1952, p. 285).

§ 4.6. Elementary illustrations of Boole's method

After the exposition of his "general method" for the solution of problems in probability Boole next has a chapter devoted to illustrations of the method, using simple examples for which, as he says, the results are "readily verified". As might be expected, these examples involve logical relationships to a much more significant extent then had heretofore been the case in published works on the subject of probability. It will be worthwhile to go through these in some detail so as to familiarize ourselves with the method and so as to be able to contrast Boole's method with current practice.

We quote his first example (pp. 276–277) in full:

> Ex. 1. The probability that it thunders upon a given day is p, the probability that it both thunders and hails is q, but of the connexion of the two phenomena of thunder and hail, nothing further is supposed to be known. Required the probability that it hails on the proposed day.
>
> Let x represent the event—It thunders.
> Let y represent the event—It hails.
>
> Then xy will represent the event—It thunders and hails; and the data of the problem are
>
> $$\text{Prob. } x = p, \qquad \text{Prob. } xy = q.$$
>
> There being here but one compound event xy involved, assume, according to the rule,
>
> $$xy = u. \tag{1}$$
>
> Our data then become
>
> $$\text{Prob. } x = p, \qquad \text{Prob. } u = q; \tag{2}$$
>
> and it is required to find Prob. y. Now (1) gives
>
> $$y = \frac{u}{x} = ux + \tfrac{1}{0}u(1 - x) + 0(1 - u)x + \tfrac{0}{0}(1 - u)(1 - x).$$
>
> Hence (XVII. 17) we find
>
> $$V = ux + (1 - u)x + (1 - u)(1 - x),$$
> $$V_x = ux + (1 - u)x = x, \qquad V_u = ux;$$
>
> and the equations of the General Rule, viz.,
>
> $$\frac{V_x}{p} = \frac{V_u}{q} = V,$$
> $$\text{Prob. } y = \frac{A + cC}{V}$$

become, on substitution, and observing that $A = ux$, $C = (1 - u)(1 - x)$, and that V reduces to $x + (1 - u)(1 - x)$,

$$\frac{x}{p} = \frac{ux}{q} = x + (1 - u)(1 - x), \tag{3}$$

$$\text{Prob. } y = \frac{ux + c(1 - u)(1 - x)}{x + (1 - u)(1 - x)}, \tag{4}$$

from which we readily deduce, by elimination of x and u,

$$\text{Prob. } y = q + c(1 - p). \tag{5}$$

In this result c represents the unknown probability that if the event $(1 - u)(1 - x)$ happen, the event y will happen. Now $(1 - u)(1 - x) = (1 - xy)(1 - x) = 1 - x$, on actual multiplication. Hence c is the unknown probability that if it does not thunder, it will hail.

The general solution (5) may therefore be interpreted as follows: The probability that it hails is equal to the probability that it thunders and hails, q, together with the probability that it does not thunder, $1 - p$, multiplied by the probability c, that if it does not thunder it will hail. And common reasoning verifies this result.

Boole does not say how he eliminated x and u from equations (3) and (4). The logical function V in this problem is $ux + \bar{u}x + \bar{u}\bar{x}$ (i.e. $x \vee \bar{u}$), which is of the symmetry type for which the Problem on Absolute Probabilities (§ 4.4) has a unique solution and equations (3) can be solved as a linear system so as to obtain x and u. But the Method of Linear Composition is also available since $A + cC\,(= ux + c\bar{u}\bar{x})$ is expressible linearly in terms of $xV(= ux), uV(= ux)$, and $V(= ux + \bar{u}x + \bar{u}\bar{x})$; for, using the constituents ux, $\bar{u}x$, $\bar{u}\bar{x}$ to stand also for their respective probabilities, the condition

$$\text{(8)} \qquad ux + c\bar{u}\bar{x} = a_1(ux + \bar{u}x) + a_2(ux) + a_3(ux + \bar{u}x + \bar{u}\bar{x})$$

leads, on equating coefficients of the constituent probabilities of the two sides, to the consistent system of equations,

$$
\begin{aligned}
a_1 + a_2 + a_3 &= 1, \\
a_1 \qquad + a_3 &= 0, \\
a_3 &= c,
\end{aligned} \tag{9}
$$

which has $(-c, 1, c)$ as its solution. Thus the conditional probability $(A + cC)/V$ is the linear combination of p, q, 1 gotten by using these values as coefficients, namely $-c \cdot p + 1 \cdot p + c \cdot 1$, which is $q + c(1 - p)$. (We point out that all this, being in terms of the probabilities of the constituents ux, $\bar{u}x$, $\bar{u}\bar{x}$, doesn't require the events u and x to be independent.)

A present-day solution of the problem would run as follows: From the logical identity

$$y = xy + \bar{x}y$$

it results that

$$
\begin{aligned}
P(y) &= P(xy) + P(\bar{x}y) && \text{(addition law)} \\
&= P(xy) + P(y \mid \bar{x})\, P(\bar{x}) && \text{(multiplication law)} \\
&= q + P(y \mid \bar{x})\, \bar{p},
\end{aligned}
$$

which coincides with Boole's answer.

While the solution just given is certainly much simpler than Boole's nevertheless it does require one to think of writing y in terms of xy and $x\bar{y}$. While how to do this here is quite evident, conceivably in other problems a corresponding needed relationship among the given elements may not be so readily ascertainable—a noteworthy claim of Boole's method is that it includes an algorithm for obtaining such a relationship in all circumstances.

Boole's next example (Ex. 2, p. 278) poses the problem of finding $P(x\bar{y} + \bar{x}y)$ if one is given that $P(x \vee y) = p$ and $P(\bar{x} \vee \bar{y}) = q$. (All this is stated verbally by Boole—he makes no use of either the probability symbol, except occasionally for the rudimentary "Prob.", or of the symbol $\vee$.) By his logical methods he expresses $w = x\bar{y} + \bar{x}y$ in terms of

$s = xy + \bar{x}y + x\bar{y}$ and $t = x\bar{y} + \bar{x}y + \bar{x}\bar{y}$ obtaining

$$w = st + 0 \cdot s\bar{t} + 0 \cdot \bar{s}t + \tfrac{1}{0} \cdot \bar{s}\bar{t}.$$

Here, as in the preceding example, the logical function $V(= st + \bar{s}t + s\bar{t})$ is of the type yielding a linear solution for the unknown absolute probabilities; but the Method of Linear Composition also works and requires no assumption of independence of s and t. It is easy to verify by standard techniques that the answer Boole gives, namely $p + q - 1$, is correct: from the logical identity

$$x\bar{y} + \bar{x}y = (x \vee y)(\bar{x} \vee \bar{y})$$

and the general addition law we have

$$\begin{aligned} P(x\bar{y} + \bar{x}y) &= P(x \vee y) + P(\bar{x} \vee \bar{y}) - P(x \vee y \vee \bar{x} \vee \bar{y}) \\ &= p + q - 1. \end{aligned}$$

In Ex. 3, p. 279, Boole has a problem on the combination of testimonies of witnesses—a type of problem much discussed by early writers in probability. Keynes [1921, p. 180] argues that all these writers (Boole included) fallaciously assume that the probability of two independent witnesses speaking the truth is the product of the probabilities that they separately speak the truth. Ignoring the question as to whether or not Boole has correctly expressed the material conditions of the problem he poses, we look at the one actually formulated, which may be stated as follows:

Given

$$P(x) = p, \quad P(y) = q, \quad P(x\bar{y} + \bar{x}y) = r,$$

find the conditional probability

$$\frac{P(xy)}{P(xy + \bar{x}\bar{y})};$$

or, since the denominator $P(xy + \bar{x}\bar{y}) = 1 - P(\bar{x}y + x\bar{y}) = 1 - r$, one merely needs to find $P(xy)$.

We will not trouble ourselves to go through the details of Boole's solution. Suffice it to say that the V encountered is

$$xy\bar{s} + x\bar{y}s + \bar{x}ys + \bar{x}\bar{y}\bar{s} \qquad (s = x\bar{y} + \bar{x}y)$$

and is not of the type which leads to a linear system of equations for the absolute probabilities. However, the Method of Linear Composition does work and one can easily obtain here the $A + cC\,(= xy\bar{s})$ as a combination of pV, qV and rV. Presumedly this is how Boole solves the algebraic part of the problem for he says of it only "we readily deduce". Again Boole's answer, here that $P(xy) = \frac{1}{2}(p + q - r)$, is easily obtained from standard principles if one starts out with the right connections—in this case the value of $P(xy)$ is immediately forthcoming from the pair of equations

$$\text{(i)} \qquad \begin{aligned} P(xy) &= P(x) + P(y) - P(x \vee y) \\ &= p + q - P(x \vee y), \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} P(x \vee y) &= P(xy) + P(x\bar{y} + \bar{x}y) \\ &= P(xy) + r \end{aligned}$$

by elimination of $P(x \vee y)$.

Boole's next illustration (Ex. 4, p. 281) is stated in story form as a kind of public health problem concerning the incidence of "fever", "cholera" and "defective sanitary condition" and, when symbolically formulated, results in the following:

Given

$$P(x) = p, \quad P(y) = q, \quad P(\bar{x}\bar{y}\bar{z}) = r,$$

find $P(z)$.

Boole puts $w = (1 - x)(1 - y)(1 - z)$, solves algebraically for z obtaining

$$\begin{aligned} z &= \frac{(1 - x)(1 - y) - w}{(1 - x)(1 - y)} \\ &= \bar{x}\bar{y}\bar{w} + 0 \cdot \bar{x}\bar{y}w + \tfrac{0}{0} \cdot (xy\bar{w} + x\bar{y}\bar{w} + \bar{x}y\bar{w}) \\ &\qquad + \tfrac{1}{0} \cdot (xyw + \bar{x}yw + xy\bar{w}). \end{aligned}$$

The V is thus $xy\bar{w} + x\bar{y}\bar{w} + \bar{x}y\bar{w} + \bar{x}\bar{y}w + \bar{x}\bar{y}\bar{w}$, i.e. $\bar{w} + \bar{x}\bar{y}w$. The algebraic equations for the absolute probabilities is then found to be

$$\text{(10)} \qquad \frac{x\bar{w}}{p} = \frac{y\bar{w}}{q} = \frac{\bar{x}\bar{y}w}{r} = \bar{w} + w\bar{x}\bar{y} \quad (= V).$$

Here for this problem the Method of Linear Composition fails—it can be verified that the $A + cC$ is not linearly expressible in terms of pV, qV, rV and V. However it is the case that equations (10) have a unique solution: using $\bar{x}\bar{y}w/r = V$ to replace $\bar{x}\bar{y}w$ in $\bar{w} + w\bar{x}\bar{y} = V$ on finds that $\bar{w} = \bar{r}V$; using this to substitute for $\bar{w}$ in the first two equations (i.e. the equations obtained by equating $x\bar{w}/p$ and $y\bar{w}/q$ respectively to V) one obtains $x = p/r$, $y = q/r$; substituting these values for x and y back into $\bar{w} = \bar{r}V = \bar{r}(\bar{w} + w\bar{x}\bar{y})$ gives w. Using the values of the absolute probabilities thus obtained one gets Boole's result

$$\text{(11)} \qquad \text{Prob. } z = \frac{(1-p-r)(1-q-r)}{1-r} + c\left(p + q - \frac{pq}{1-r}\right),$$

where c is interpreted to be the probability of z, given (x or y). A bit of algebra applied to the first term of the right-hand member enables one to write (11) as

$$\text{(12)} \qquad \text{Prob. } z = \left(\bar{r} - (p+q) + \frac{pq}{\bar{r}}\right) + c\left(p + q - \frac{pq}{\bar{r}}\right).$$

This solution Boole "verifies" only for the particular case of $c = 1$.

The application of standard techniques does not reproduce Boole's result: beginning with

$$\begin{aligned} 1 - r &= 1 - P(\bar{x}\bar{y}\bar{z}) \\ &= P(x \vee y \vee z) \\ &= P(x \vee y) + P(z) - P((x \vee y)\, z) \\ &= P(x \vee y) + P(z) - P(z \mid x \vee y)\, P(x \vee y) \end{aligned}$$

and the replacing $P(x \vee y)$ by $p + q - P(xy)$ gives

$$\text{(13)} \quad P(z) = (\bar{r} - (p+q) + P(xy)) + P(z \mid x \vee y)\,(p + q - P(xy)),$$

which is like Boole's solution except for having $P(xy)$ where his has $pq/\bar{r}$. Comparing (12) and (13) we see that when Boole verifies his solution in the special case of $c = 1$, he is taking precisely the value for which there is no difference in the two answers; and since $P(xy)$ may vary within the constraints of the given data while $pq/\bar{r}$ is fixed, it appears that Boole's result is in error. But not badly so! For as x and y vary, the value of Prob.

z in (12) could vary (since c is the probability of z, given x or y), but the set of such values is contained in the set of values of $P(z)$ in (13) since $pq/\bar{r}$ lies between the minimum and maximum values of $P(xy)$. (For a proof of this see our Note 3 for § 4.6.) A last remark in connection with this problem before going on. It is here, for the first time in the *Laws of Thought*, that Boole mentions that the given probabilities may not be just any numbers between 0 and 1 but are subject to restrictions. In this particular instance he says simply that the constants p, q, r are subject to the conditions

$$p + r \leq 1, \qquad q + r \leq 1,$$

but says nothing further. As we shall later see, the topic comes to play a significant role under the rubric "Conditions of Possible Experience".

Boole's next illustration, Ex. 5, p. 284, asks for the probabilities of the conclusion of a hypothetical syllogism given the probabilities of its premisses:

> Let the syllogism in its naked form be as follows:
> Major premiss: If the proposition Y is true X is true.
> Minor premiss: If the proposition Z is true Y is true.
>
> Conclusion: If the proposition Z is true, X is true. Suppose the probability of the major premiss to be p, that of the minor premiss q.
>
> The data then are as follows, representing the proposition X by x, etc., and assuming c and c' as arbitrary constants:
>
> $$\text{Prob. } y = c, \qquad \text{Prob. } xy = cp;$$
>
> $$\text{Prob. } z = c', \qquad \text{Prob. } yz = c'q;$$
>
> from which we are to determine,
>
> $$\frac{\text{Prob. } xz}{\text{Prob. } z} \quad \text{or} \quad \frac{\text{Prob. } xz}{c'}.$$

Note that Boole is here writing the probabilities of the premisses and the conclusion as conditional probabilities, thus confusing the probability of a conditional statement, If Y then X, with the conditional probability of X, given Y. Since we are primarily interested in how Boole's method functions let us look at the actual problem, in terms of the conditional probabilities, which he solves namely (in modern notation):

Given that

$$P(x \mid y) = p \quad \text{and} \quad P(y \mid z) = q,$$

find

$$P(x \mid z).$$

Boole's method gives the answer

$$P(x \mid z) = pq + a(1 - q),$$

where "the arbitrary constant a is the probability that if the proposition Z is true and Y false, X is true". Immediately after this Boole adds:

> This investigation might have been greatly simplified by assuming the proposition Z to be true, and then seeking the probability of X. The data would have been simply
>
> $$\text{Prob. } y = q, \qquad \text{Prob. } xy = pq;$$
>
> whence we should have found Prob. $x = pq + a(1 - q)$. It is evident that under the circumstances this mode of procedure would have been allowable, but I have preferred to deduce the solution by the direct and unconditioned application of the method. The result is one which ordinary reasoning verifies, and which it does not indeed require a calculus to obtain. General methods are apt to appear most cumbrous when applied to cases in which their aid is the least required.

That the argument here is sound is moot. When Boole writes that under the assumption of z the data of the problem is

$$\text{(14)} \qquad \text{Prob. } y = q \quad \text{and} \quad \text{Prob. } xy = pq,$$

he is in effect assuming that $P(y \mid z) = q$ and $P(xy \mid z) = pq$. But the original data gives $P(x \mid y)\, P(y \mid z) = pq$. Now by the general multiplication law $P(xy \mid z) = P(x \mid yz)\, P(y \mid z)$. Thus when Boole takes (14) to be the data of the problem he is really adding

$$P(x \mid y) = P(x \mid yz)$$

as an additional assumption. Granted this tacit assumption we can come out with Boole's answer by current rules:

$$\begin{aligned} P(x|z) &= P(xy + x\bar{y} \mid z) \\ & P(xy \mid z) + (x\bar{y} \mid z) \\ &= P(x \mid yz)\, P(y \mid z) + P(x \mid \bar{y}z) (\bar{y} \mid z) \\ &= P(x \mid yz)\, q + a\bar{q} \\ &= pq + a\bar{q} \quad \textit{if} \quad P(x \mid yz) = P(x \mid y). \end{aligned}$$

Without the additional assumption we would have to stop at the fourth line with $P(x|z) = bp + a\bar{q}$, where b is $P(x|yz)$ and, like a, is "arbitrary" (thought not independent of a).

It is a little surprising that Boole did not realize that there was a difference between the probability of a conditional and conditional probability, for there is a pre-*Laws of Thought* article on probability, 1851a (= 1952, VIII), in which he explains that one cannot "contrapose" a conditional probability—that the probability of x, given y, is not equal to the probability of not-y, given not-x. Boole handles the counter-intuitive situation resulting from his confusion by arguing that logically equivalent propositions need not have equal probabilities (p. 286):

> 8. One remarkable circumstance which presents itself in such applications deserves to be specially noticed. It is, that propositions which, when true, are equivalent, are not necessarily equivalent when regarded only as probable. This principle will be illustrated in the following example.
>
> Ex. 6. Given the probability p of the disjunctive proposition "Either the proposition Y is true, or both the propositions X and Y are false", required the probability of the conditional proposition, "If proposition X is true, Y is true".

Boole finds (by treating the desideratum as a conditional probability) that the probability of "If X, then Y" is

$$\frac{cp}{1 - p + cp}, \tag{15}$$

where c is the conditional probability of $y + \bar{y}\bar{x}$, given x. That this value is different from p (except, as he notes, when $p = 0$ or $p = 1$) is for Boole a "remarkable circumstance" rather than an indication that something is amiss. (One can readily check that (15) is correct for $P(y \mid x)$ if $P(y \vee \bar{x}) = p$.)

We now come to Ex. 7, the last of the elementary illustrations of Boole's Chapter XVIII. This is a three part problem with successively additional hypotheses. The problem, stated in symbols, is to find $P(x)$ given either (i), or (i) and (ii), or (i), (ii) and (iii), where these three conditions are:

(i) $P(x + \bar{x}yz) = p$,
(ii) $P(y + \bar{y}xz) = q$,
(iii) $P(z + \bar{z}xy) = r$.

In customary fashion Boole introduces $s = x + \bar{x}yz$, $t = y + \bar{y}xz$, $u = z + \bar{z}xy$ and, for each case, eliminates unwanted terms, solves for x, and writes the algebraic conditions which determines $P(x)$. We summarize the results:

First case

Here it is supposed that only (i) is given. Boole easily deduces $x\bar{s} = 0$ so that $x = \frac{0}{0}s + 0\bar{s}$, and thence

$$\text{Prob. } x = cp,$$

where c is $P(x \mid x + \bar{x}yz)$. This answer is clearly correct by virtue of the general multiplication law of probability.

Second Case

We are given (i) and (ii). Here Boole deduces $\bar{s}x + s\bar{t}\bar{x} = 0$ and so

(16) $$x = s\bar{t} + 0(\bar{s}t + \bar{s}\bar{t}) + \frac{0}{0}st,$$

from which Boole gets

(17) $$\text{Prob. } x = p\bar{q} + cpq,$$

where $c = P(x \mid st) = P(x \mid xy + x\bar{y}z + \bar{x}yz)$. To see if Boole's result can be obtained by ordinary rules we need a suitable logical relationship among x, y and z. For this relationship we don't need to fumble around but use Boole's (16), and the results of § 2.6, to write

$$x = s\bar{t} + xst, \tag{18}$$

where $s = x + \bar{x}yz$ and $t = y + \bar{y}xz$. From (18) we have

$$\begin{aligned} P(x) &= P(s\bar{t}) + P(xst) \\ &= P(s\bar{t}) + P(x \mid st)\, P(st) \\ &= p\bar{q}\,\frac{P(s\bar{t})}{P(s)\,P(\bar{t})} + cpq\,\frac{P(st)}{P(s)\,P(t)}, \end{aligned}$$

which differs from Boole's result given in (17) except when s and t are independent, for then $P(st) = P(s)\,P(t)$ and $P(s\bar{t}) = P(s)\,P(\bar{t})$. Could we perhaps, by assuming that x, y and z are mutually independent, conclude that the events $s = x + \bar{x}yz$ and $t = y + \bar{y}xz$ are independent and so have Boole's result? Theorem 0.93 tells us "no", so even that won't do it and we conclude that Boole's method does not correctly give $P(x)$—except for the special case of s and t being stochastically independent, i.e. only by adding an additional assumption.

Third case

We are supposing (i)–(iii). Boole's logical algorithm for expressing x in terms of s, t and u gives (with some effort)

$$\begin{aligned} x = {} & s\bar{t}\bar{u} + 0(\bar{s}\bar{t}u + \bar{s}t\bar{u} + \bar{s}\bar{t}\bar{u}) \\ & + \tfrac{0}{0}\, stu + \tfrac{1}{0}\,(s\bar{t}u + st\bar{u} + \bar{s}tu) \end{aligned}$$

from which one obtains the algebraic equations for the absolute probabilities of s, t, u needed to express the solution for $P(x)$. Here for the first time Boole has to face the full complexity of the system of equations for the absolute probabilities. He succeeds in expressing the absolute probabilities in terms of a single numerical parameter λ, equal to $V/\bar{s}\bar{t}\bar{u}$ which is determined by the condition that it is a root of the cubic equation

$$(a\lambda - 1)\,(b\lambda - 1)\,(c\lambda - 1) = 4(d\lambda + 1), \tag{19}$$

where a, b, c, d are certain linear combinations of p, q, r which we need not write down. Concerning this equation Boole says (p. 293):

> 10. Now a difficulty, the bringing of which prominently before the reader has been one object of this investigation, here arises. How shall it be determined, which root of the above equation ought to taken for the value of λ. To this difficulty some reference was made in the opening of the present chapter, and it was intimated that its fuller consideration was reserved for the next one; from which the following results are taken.
>
> In order that the data of the problem may be derived from a possible experience, the quantities p, q, and r must be subject to the following conditions:
>
> $$1 + p - q - r \geq 0,$$
> $$1 + q - p - r \geq 0,$$
> $$1 + r - p - q \geq 0.$$
>
> Moreover, the value of λ to be employed in the general solution must satisfy the following conditions:
>
> $$\lambda \geq \frac{1}{1 + p - q - r}, \qquad \lambda \geq \frac{1}{1 + q - p - r}, \qquad \lambda \geq \frac{1}{1 + r - p - q}. \tag{15}$$
>
> Now these two sets of conditions suffice for the limitation of the general solution. It may be shown, that the central equation (13) [our (19)] furnishes but one value of λ, which does satisfy these conditions, and that value of λ is the one required.

Since equation (19) contains parameters (i.e. p, q, r) Boole cannot give an explicit expression for the unique root which he proves to exist under the "conditions of possible experience", nor of course does he give one for the $P(x)$ expressed in terms of this root. We make no attempt to check Boole's result in this case as we shall, in our Chapter 5, be undertaking a general consideration of this method.

On the basis of our examination of these elementary illustrations we must admit that Boole's method cannot be entirely correct—in some of the illustrations he comes out with the correct answer, but in others there are either near misses or else one needs special additional assumptions to get his answer. Nevertheless we persist in our presentation of Boole's theory which, at this point, requires giving an account of his determination of the "conditions of possible experience".

§ 4.7. Conditions of possible experience. Bounds on the probability of events

In the preceding section we have noted that at the conclusion of his illustrative Ex. 4 Boole mentions that the literal quantities p, q, r entering into this problem are subject to conditions, namely that these quantities must satisfies the inequations

$$p + r \leq 1, \qquad q + r \leq 1.$$

That is to say, not all values of the parameters p, q, r (between 0 and 1) are consistent with the data relating them and so could not furnish a solution unless the conditions were satisfied. As a simple example, $P(A) = p$, $P(AB) = q$ would not be a possible set of data unless $q \leq p$. How does one determine, for an arbitrary problem, what these conditions are? Answering this question is one aspect of a general investigation to which Boole devotes his Chapter XIX, Of Statistical Conditions. Boole's use of the adjective "statistical" comes from his view that typically probability values are obtained from statistical observations and hence statistical conditions are "those conditions which must connect the numerical data of a problem in order that those data may be consistent with each other, and therefore such as statistical observations might actually have furnished". In later works he used the more appropriate term "conditions of possible experience" rather than "statistical conditions".

Another related aspect of this general investigation of Boole's is the developing of a method for finding bounds or, as he calls them, limits within which a probability value must lie. He specifically needs this result, as illustrated by him in his Ex. 7 Case III, (discussed in our § 4.6) so as to be able to determine which root of a higher degree equation to select as the answer to his probability problem. Apart from this particular application Boole is quite aware of the importance of having such a general method for finding bounds on probabilities.

Accordingly, to the matters under investigation in his Chapter XIX Boole devoted a good deal of attention. We find him occupied with it not only in the *Laws of Thought* but also, in connection with another matter, in an earlier paper "Of Propositions Numerically Definite", written about 1850, and published posthumously by De Morgan (see our § 1.12) and after the *Laws of Thought*, in BOOLE 1854c (= BOOLE 1952, XIII). In the first part of Chapter XIX he expounds a method for finding "... to an extent sufficient at least for the requirements of this work", the conditions of possible experience and, at the same time, determining bounds on the probability of the event sought. At the end of the presentation, admitting that the method may not always give the "narrowest" bounds, he states "a purely algebraic" method for finding the narrowest bounds. He gives no proof but refers to a lost manuscript (very likely BOOLE 1868) written some four years ago. However in BOOLE 1854c (= BOOLE 1952, XIII) an entirely new "easy and general method" is presented for finding the conditions of possible experience, and it is this method which is then used exclusively by him. The topic gains even more in significance for, in BOOLE 1854e (= BOOLE 1952, XV), it is asserted that the Problem of Absolute Probabilities (as we call it in § 4.4) has a unique solution if and only if the conditions of possible experience are satisfied. However complete justification for this assertion eluded Boole for many years and finally appeared in BOOLE 1862 (= BOOLE 1952, XVII).

We shall first present the topic in the less general form as given in the *Laws of Thought*. There, as well as in the posthumous paper referred to above, Boole's investigation is couched not in terms of probability but rather in terms of the numerical operator n, where $n(x)$ is the number of individuals in a class x selected from some universe 1, the number of whose individuals in $n(1)$. Transition is made to probability *via* the frequency interpretation as thus explained (pp. 296–297):

In like manner if, as will generally be supposed in this chapter, x represents an event of a particular kind observed, $n(x)$ will represent the number of occurrences of that event, $n(1)$ the number of observed events (equally probable) of all kinds, and $n(x)/n(1)$, or its limit, the probability of the occurrence of the event x.

Hence it is clear that any conclusions which may be deduced respecting the ratios of the quantities $n(x)$, $n(y)$, $n(1)$, etc. may be converted into conclusions respecting the probabilities of the events represented by x, y, etc. Thus, if we should find such a relation as the following, viz.,

$$n(x) + n(y) < n(1),$$

expressing that the number of times in which the event x occurs and the number of times in which the event y occurs, are together less than the number of possible occurrences $n(1)$, we might thence deduce the relation,

$$\frac{n(x)}{n(1)} + \frac{n(y)}{n(1)} < 1,$$

or

$$\text{Prob. } x + \text{Prob. } y < 1.$$

And generally any such statistical relations as the above will be converted into relations connecting the probabilities of the events concerned, by changing $n(1)$ into 1, and any other symbol $n(x)$ into Prob. x.

In justification of the laws of operation of the symbol 'n' within his calculus of logic Boole devotes only one short paragraph (p. 297):

It is evident that the symbol n is distributive in its operation. Thus we have

$$n\{xy + (1 - x)(1 - y)\} = nxy + n(1 - x)(1 - y),$$

$$nx(1 - y) = nx - nxy,$$

and so on. The number of things contained in any class resolvable into distinct groups or portions is equal to the sum of the numbers of things found in those separate portions. It is evident, further, that any expression formed of the logical symbols x, y, etc. may be developed or expanded in any way consistent with the laws of the symbols, and the symbol n applied to each term of the result, provided that any constant multiplier which may appear, be placed outside the symbol n; without affecting the value of the result. The expression $n(1)$, should it appear, will of course represent the number of individuals contained in the universe. Thus,

$$n(1 - x)(1 - y) = n(1 - x - y + xy)$$
$$= n(1) - n(x) - n(y) + n(xy).$$

Again,

$$n\{xy + (1 - x)(1 - y)\} = n(1 - x - y + 2xy)$$
$$= n(1) - nx - ny + 2nxy.$$

In the last member the term $2nxy$ indicates twice the number of individuals contained in the class xy.

If one were to use the operator n distributively over *any* expression then, as the example $n(x + x) = 2n(x)$ shows, one would have to think of $+$ as heap addition. However, it will be noted, Boole does not accept or, rather, ignores this full generality for n and restricts its application to any class "resolvable into distinct groups or portions". Throughout his Chapter the symbol n is applied only to sums of constituents, which are of course mutually exclusive and thus any distinction between heap addition and exclusive addition which might have been brought out, is thereby lost.

On the basis of these two principles:

1st. If all the members of a given class possess a certain property x, the total number of individuals in the class x will be a superior limit of the number of individuals contained in the given class.

2nd. A minor limit of the number of individuals in any class y will be found by subtracting a major numerical limit of the contrary class, $1 - y$, from the number of individuals contained in the universe.

Boole readily determines

> Major limit of $n(xy)$ = least of the values $n(x)$ and $n(y)$.
> Minor limit of $n(xy) = n(x) + n(y) - n(1)$.

For the minor limit Boole should really have: greatest of 0 and $n(x) + n(y) - n(1)$, since the latter value might be negative, but Boole chooses to omit, or understand, the 0 for "... 0 is necessarily a minor limit of any class ..."

This result on the limits of a logical product of two classes Boole immediately extends to any constituent:

> 1st. The major numerical limit of the class represented by any constituent will be found by prefixing n separately to each factor of the constituent, and taking the least of the resulting values.
>
> 2nd. The minor limit will be found by adding all the values above mentioned together, and subtracting from the result as many, less one, times the value of $n(1)$.

Boole now turns to the important problem of finding the major numerical limit of a class which is the sum of a series of constituents on x, y, z, etc. these limits to be expressed as a function of $n(x)$, $n(y)$, $n(z)$, etc. and $n(1)$, and for this he gives the following

> RULE. Take one factor from each constituent, and prefix to it the symbol n, add the several terms or results thus formed together, rejecting all repetitions of the same term; the sum thus obtained will be a major limit of the expression, and the least of all such sums will be the major limit to be employed.

It is easy to see that this Rule does give an upper bound to the numerical value of the sum of the constituents but, as we shall see in § 5.7, the rule does not give the best upper bound expressible in terms of $n(x)$, $n(y)$, etc. The fact that Boole ends his description of the major limit with the qualifying phrase "to be employed" seems to indicate that he was aware of the fact that the rule did not necessarily give the best possible bound. It is strange that he cites no reason, as examples to this effect are not immediately obvious.

After giving the corresponding result for the minor numerical limit of a class given as a sum of constituents Boole then proceeds to the most general form of the problem: given the number of individuals in any classes s, t, etc. which are "logically defined" to determine the numerical limits of any other class w also "logically defined". By "logically defined" Boole means expressible as Boolean functions of some set of variables (classes). He first gives a system of major and minor limits (i.e. upper bounds and lower bounds) which, he says, are sufficient for the purposes of his book, and then later in the chapter will discuss the problem of finding narrowest limits.

Let $A + 0B + \frac{0}{0}C + \frac{1}{0}D$ be the complete development of w in terms of s, t, etc. Since some, none, or all of the constituents in the $\frac{0}{0}$ portion may be included in w, Boole has then that a system of major limits of w is given by those of $A + C$. To find a system of minor limits he notes that (i) minor limits are given by $n(1)$ − major limit of the complement, (ii) $1 - w$ has for its development $B + 0A + \frac{0}{0}C + \frac{1}{0}D$, and (iii) $n(1) - n(B + C) = n(A + D)$.

Additionally to these limits on $n(w)$, Boole has the significant result:

> Finally, as the concluding term of the development of w indicates the equation $D = 0$, it is evident that $n(D) = 0$. Hence [since minor limit of $n(D) \leq n(D) = 0$] we have
>
> $$\text{Minor limit of} \quad n(D) \leq 0,$$
>
> and this equation, treated by Prop. III, gives the requisite conditions among the numerical elements $n(s)$, $n(t)$, etc., in order that the problem may be real, and may embody in its data the results of a possible experience.

Thus, interestingly enough, from his development $w = A + 0B + \frac{0}{0}C + \frac{1}{0}D$ Boole derives both limits on the probability of w and also conditions of possible experience.

Boole then summarizes these results in the statement of a Rule (p. 306), observing that to apply it to probability problems "it is only necessary to replace in each of the formulae $n(x)$ by Prob. x, $n(y)$ by Prob. y, etc., and, finally, $n(1)$ by 1." The method is illustrated by application to Ex. 7.

Case III, which we have discussed in our preceding section. But Boole has this to say:

> 13. It is to be observed, that the method developed above does not always assign the narrowest limits which it is possible to determine. But in all cases, I believe, sufficiently limits the solutions of questions in the theory of probabilities.

However nowhere does Boole give an example to show that the method does not give the narrowest limits. He then goes on to state a method which he asserts does (pp. 310–311):

> The problem of the determination of the narrowest limits of numerical extension of a class is, however, always reducible to a purely algebraical form *. Thus, resuming the equation
>
> $$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$
>
> let the highest inferior numerical limit of w be represented by the formula $an(s) + bn(t) \cdots + dn(1)$, wherein $a, b, c, \ldots, d$ are numerical constants to be determined, and s, t, etc., the logical symbols of which A, B, C, D are constituents. Then
>
> $$an(s) + bn(t) \,..\, + dn(1) = \text{minor limit of } A \text{ subject to the condition } \quad D = 0.$$
>
> Hence if we develop the function
>
> $$as + bt \ldots + d,$$
>
> reject from the result all constituents which are found in D, the coefficients of those constituents which remain, and are found also in A, ought not individually to exceed unity in value, and the coefficients of those constituents which remain, and which are not found in A, should individually not exceed 0 in value. Hence we shall have a series of inequalities of the form $f \geq 1$, and another series of the form $g \geq 0$, f and g being linear functions of a, b, c, etc. Then those values of $a, b, \ldots, d$, which while satisfying the above conditions, give to the function
>
> $$an(s) + bn(t) \,..\, dn(1),$$

its highest value must be determined, and the highest value in question will be the highest minor limit of w. To the above we may add the relations similarly formed for the determination of the relations among the given constants $n(s), n(t), \ldots, n(1)$.

* [Boole's footnote] The author regrets the loss of a manuscript, written about four years ago, in which this method, he believes, was developed at considerable length. His recollection of the contents is almost entirely confined to the impression that the principle of the method was the same as above described, and that its sufficiency was proved. The prior methods of this chapter are, it is almost needless to say, easier, though certainly less general.

Although he makes no use of this method for finding narrowest bounds it is nevertheless of particular historical interest, for what Boole has here, clearly formulated, is a linear programming problem (§ 0.8). We shall return to a discussion of it in our § 5.7.

Continuing here with our exposition we turn to the "easy and general" method of BOOLE 1854e. "This object", he says, "was attempted in Chapter XIX of my treatise on the *Laws of Thought*. But the method there developed is somewhat difficult of application, and I am not sure that it is equally general with the one which I am now about to explain." After the introductory remarks Boole enunciates the following:

> *Proposition.* To eliminate any symbol of quantity x from any system of inequations in the expression of which it is involved.

Although not explicitly mentioned, his demonstration of this proposition (by an illustrative example) shows that Boole is thinking of *linear* inequations. Linear equations are admitted along with inequations in the systems he is contemplating. The method which he gives now known as Fourier elimination (§ 0.7) and, since he makes no mention of Fourier, we may suppose that Boole's was an independent development of this idea.

The central problem is now stated (BOOLE [1952], p. 282):

> General proposition. The probabilities of any events whose

> logical expression is known being represented by $p, q, r, \ldots$ respectively, required the conditions to which those quantities are subject.

Using a special case (for two unknowns) of his Challenge Problem[1] as an illustrative example, Boole explains his method of obtaining these conditions as follows, where the illustrative problem (using simplifying notation in place of Boole's) is:

Given

$$P(x) = c_1, \quad P(xz) = c_3,$$
$$P(y) = c_2, \quad P(yz) = c_4,$$
$$P(\bar{x}\bar{y}z) = 0,$$

find w, where $w = P(z)$.

First expand each of the events in the statement of the problem as sums of constituents on a common set of variables i.e., for the example,

$$x = xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z},$$
$$xz = xyz + x\bar{y}z,$$

and similarly for y, yz, and z. Assign a letter (i.e. an unknown) to the probability of each such constituent except those impossible on the data (e.g. $\bar{x}\bar{y}z$ here), viz.

$$P(xyz) = \lambda, \quad P(xy\bar{z}) = \mu, \quad P(x\bar{y}z) = \nu,$$
$$P(x\bar{y}\bar{z}) = \varrho, \quad P(\bar{x}yz) = \sigma, \quad P(\bar{x}y\bar{z}) = \tau,$$
$$P(\bar{x}\bar{y}\bar{z}) = \nu$$

(it will be convenient for us to call these *constituent probabilities*) and for each of the events whose probability is involved in the statement of the problem, express it as a sum of constituent probabilities, thus:

[1] We shall be discussing it below in § 6.2. Our interest here is only in Boole's technique for finding conditions of possible experience for the problem, i.e. the problem as Boole interprets it.

$$\begin{aligned}
\lambda + \mu + \nu + \varrho &= c_1, \\
\lambda + \mu + \sigma + \tau &= c_2, \\
\lambda + \nu \qquad\quad &= c_3, \\
\lambda + \sigma \qquad\quad &= c_4, \\
\lambda + \nu + \sigma \qquad &= w.
\end{aligned}$$

Note here that c_1, c_2, c_3, c_4 are given and w is to be found, except that what Boole is doing here is not solving for w but finding bounds on its value. Additionally, each constituent probability λ, μ, ... is set ≥ 0 and the sum of all, excepting those impossible on the data, is set ≤ 1. The resulting combined system of equations and inequations is then treated as follows. The equations are first used to eliminate as many of the constituent probabilities as possible, then Fourier elimination (§ 0.7) is used to eliminate as many of the remainder as possible. For any given quantity remaining in the system (which may have far more inequations than in the original system) one can solve each of the linear inequations for the given quantity, resulting in a set of upper and lower bounds for the quantity. In the example at hand elimination of all the unknowns except for w gives Boole the inequations

$$w - c_3 \geqq 0 \quad w - c_4 \geqq 0 \quad c_3 + c_4 + w \geqq 0,$$
$$w \leqq 1 - c_2 + c_4, \quad w \leqq 1 - c_1 + c_2,$$

and hence lower bounds of

(7) $$c_3 \text{ and } c_4$$

and upper bounds of

(8) $$c_3 + c_4, \quad 1 - c_2 + c_4, \quad 1 - c_1 + c_2$$

for w. Concerning these Boole says

> These are the conditions assigned in my treatise on the *Laws of Thought*, p. 325. They show that if it is our object to determine Prob. z or w, the solution, to be a correct one, must lead us to a value of that quantity which shall exceed each of the values assigned in (7), and fall short of those assigned in (8). They show also that the data of the problem will only represent a possible

experience when each of the values in (7) shall fall short of, or not exceed each of those in (8).

As we have earlier observed, Boole is not sure of the relative strengths of the two methods for finding bounds, the one just outline and that of the *Laws of Thought*. In specific examples he works out by both methods the results are the same. We shall return to this in our § 5.7 where we shall see that the two methods are alternative ways of solving a linear programming problem.

We should perhaps emphasize that the discussion of this section, being in terms of constituent probabilities, is independent of whether or not the simple events involved are dependent or independent.

§ 4.8. Wilbraham's and Peirce's criticisms

In view of the unusual and complex features of Boole's method of solving probability problems one would think that it would have taken some time for it to be analyzed and understood. But the *Laws of Thought* had hardly appeared when its "doctrine of chances" was subject to acute criticism in WILBRAHAM 1854 (= BOOLE 1952, Appendix B). Wilbraham claimed to show:

(i) that Boole's method tacitly introduced additional assumptions concerning the events in the data of the problems considered, thereby converting a generally indeterminate problem into a determinate one,

(ii) that with these assumptions brought out one could solve Boole's problems by common methods, and finally,

(iii) that the type of problem solved with the additional assumptions was not one of much practical value.

Boole's reply, appearing the next issue of the same journal (BOOLE 1854b = BOOLE 1952, XII) did not address itself to the specifics of Wilbraham's comments; instead Boole contended that he had, in his book, explicitly stated the principles upon which his method depended, had equally explicitly derived from these the algebraic equations which gave the solutions, that if particular assumptions had to be introduced it should be shown where his principles were insufficient, and even if the method does bring in additional equations "there can be no objection so long as the equations in question are consequences of the laws of thought and expectation as applied to the actual data". We undertake in this section an examination of this adverse appraisal of Boole's work in probability.

The first point Wilbraham makes is that when no conditions (i.e. no relationships) among the events of the data are given but only their "absolute chances", then the reasoning in Boole's Chapter XVII shows that the simple events, in terms of which the data are expressed, are assumed to be independent; and that when this independence is expressed in terms of the "ultimate possibilities" (i.e. the constituent probabilities) there results, in the case of n simple events, $2^n - n - 1$ algebraically independent equations (see Theorem 0.915), which equations are, in such cases, a tacit concomitant of Boole's data.

Wilbraham is certainly correct in this assertion, but in fairness to Boole it should be noted that these independence conditions are not surreptitiously or unwittingly introduced by Boole but come rather from his stated general principle:

"VI. The events whose probabilities are given are to be regarded as independent of any connexion but such as expressed, or necessarily implied in the data, ..." together with his identifying absence of logical dependence with stochastic independence. (That lack of knowledge of any connection should imply stochastic independence is, as we have noted in § 4.2, not normally a tenet of probability theory, though one could consider it as a working hypothesis for testing purposes.) So, at least on his part, Boole is justified in contending that the procedure of his method is consequent as to principle (his, that is). The issue between them would have been more clearly joined if Wilbraham had attacked this principle of Boole's. In these cases then, with no given conditions among the events, the assumed (to Wilbraham), or implied (to Boole), independence conditions on the simple events supply enough additional equations so as to determine all constituent probabilities and therewith the probability of any Boolean compound event on these simple events.

Next Wilbraham considers the cases "when certain conditions among the chances of the several events are given". He is here thinking of the circumstances in which, in Boole's treatment, there are "absolute" logical relations among the events as evidenced by the presence of $\frac{1}{0}$ terms. If in such a case there are only two simple events involved then the number of "assumed" independence condition is $2^2 - 2 - 1$, i.e. 1, and hence when an additional condition is "given", then the "assumed" independence condition is replaceable by the new given condition and there still will be sufficiently many algebraic conditions for a solution. But suppose that there are more than two simple events and only one additional given condition. Wilbraham says:

> This new condition does certainly to some extent supersede those previously assumed [i.e. the $2^n - n - 1$ independence conditions]; and it appears to me that Professor Boole's reasoning would lead one to suppose that the former assumptions are entirely banished from the problem, and no others except the said newly given condition assumed in their stead. The fact, however, is that in this case certain additional assumptions are made, otherwise the problem would be indeterminate. The nature of these assumptions, which are different from the assumptions made when no condition besides the absolute chances of the simple events is given, will perhaps, be better seen from the following discussion of an example than from any general reasoning.

The example (essentially PROBLEM V, *Laws of Thought*, p. 335) and solution that now follow is interesting for a number of reasons and we shall comment on it after its presentation. We quote *in extenso* from WILBRAHAM 1854 = BOOLE 1952, pp. 475–479.

> The chances of three events, A, B, and C, are a, b, c, respectively, and the chance of all three happening together is m; what is the chance of A occurring without B?

Wilbraham now shows how to obtain Boole's type of solution for this problem "without the aid of his logical equations".

> Suppose A, B, and C, and a further event S, to be four simple events mutually independent, the absolute chances of which are respectively x, y, z, and s. We suppose for the present no connexion

to exist between the original simple events A, B, and C, and the subsidiary event S. There will be altogether sixteen possible mutually exclusive compound events, the chances of which (since the simple events are independent) are as follows:

(δ) $xyzs$,

(ε) $(1 - x)\, yzs$,

(θ) $x(1 - y)\, zs$,

(ι) $xy(1 - z)\, s$,

($\varkappa$) $xyz(1 - s)$,

(λ) $(1 - x)(1 - y)\, zs$,

(μ) $(1 - x)\, y(1 - z)\, s$,

(ν) $(1 - x)\, yz(1 - s)$,

(o) $x(1 - y)(1 - z)\, s$,

(ϱ) $x(1 - y)\, z(1 - s)$,

(τ) $xy(1 - z)(1 - s)$,

(υ) $x(1 - y)(1 - z)(1 - s)$,

(φ) $(1 - x)\, y(1 - z)(1 - s)$,

(χ) $(1 - x)(1 - y)\, z(1 - s)$,

(ψ) $(1 - x)(1 - y)(1 - z)\, s$,

(ω) $(1 - x)(1 - y)(1 - z)(1 - s)$.

Now a condition is imposed on these new events:

> Let us now make an assumption with respect to the subsidiary event S, viz. that it is never observed except in conjunction with the three other events, and is always observed to happen if they concur. Consequently, those of the above sixteen compound events which represent S occurring while any one or more of the other three events do not occur, and which represent A, B, C, all to occur without S occurring, must be considered as beyond the range of our observation. This does not contradict the former assumption of the mutual independence of the four simple events; for we do not by this last supposition say that such a compound

events are impossible, nor do we make any new assumption as to the probability of their occurrence, but only that, as they are beyond the limits of our observation, we have nothing to do with them. The events therefore, which come within our circle of observation are those marked respectively δ, ν, ϱ, τ, v, φ, χ, ω; and the absolute chance that any event which may occur is an event within the range of our observation is

$$\begin{aligned} xyzs &+ (1-x)\,yz(1-s) + x(1-y)\,z(1-s) \\ &+ xy(1-z)\,(1-s) \\ &+ x(1-y)\,(1-z)\,(1-s) + (1-x)\,y(1-z)\,(1-s) \\ &+ (1-x)\,(1-y)\,z(1-s) \\ &+ (1-x)\,(1-y)\,(1-z)\,(1-s), \end{aligned}$$

which is similar to the quantity called V in Professor Boole's book.

A connection is now established between the originally given events *A*, *B*, *C*, *ABC* and the newly introduced ones, namely by equating the probabilities of the latter, on condition V, with the originally given probabilities:

I must here observe that x, y, and z are not the same as the given quantities a, b, and c; for the latter represent the chances of A, B, and C respectively occurring, provided that the event is one which comes within our range of observation, whereas x, y, and z represent the absolute chances of the same events whether the event be or be not within that range.

Of the eight events δ, ν, ϱ, τ, v, φ, χ, ω, which compose V, four, viz. δ, ϱ, τ, and v, imply the occurrence of A. Consequently, the chance that if the event be within our range of observation A will occur, is the sum of the chances of these last four events divided by the sum of the chances of the eight. This will be equal to the given chance a. Hence

$$\frac{xyzs + \{(1-y)\,z + y(1-z) + (1-y)\,(1-z)\}\,x(1-s)}{V} = a.$$

So also

$$\frac{xyzs + \{(1-x)z + x(1-z) + (1-x)(1-z)\}y(1-s)}{V} = b,$$

$$\frac{xyzs + \{(1-y)z + y(1-z) + (1-y)(1-z)\}z(1-s)}{V} = c.$$

Also as the event S always in cases within our range of observation occurs conjointly with A, B, and C, the chance of S occurring and that of A, B, and C all occurring are the same, and equal to m. Therefore

$$\frac{xyzs}{V} = m.$$

And now for the probability of the desideratum "A, but not B":

> Out of the events represented by V there are two, ϱ and v, which imply that A occurs but not B; consequently, the chance of A occurring but not B, which is the required chance and may be called $u = x(1-y)(1-s)/V$. From these five equations x, y, z, s, may be eliminated, and there remains an equation which gives u. Or the values of x, y, z, and s may be found from the first four equations, and thence the value of any function of them is known.

Wilbraham states that this method of solution is almost identical with that of Boole's and that in introducing the four mutually independent simple events A, B, C, S one is tacitly adding 11 ($= 2^4 - 4 - 1$) equations, namely the equations expressing their mutual independence. Not all of these 11 equations, however, are necessary to the solution but only those referring to constituents present in V.

> It will be found that three only out of the eleven give such relations [i.e. relations involving only the 8 constituents present in V]; and upon the assumptions comprised in these last three equations rests the truth of the solution. The three equations are $v/\omega = \tau/\varphi = \varrho/\chi$, and $\varphi/\omega = \nu/\chi$. The other eight equations, though not contradictory to the data, are not essential to the solution, and need not have been assumed. If these three conditions had been

inserted in the data of the problem, it might have been solved by a simple algebraical process without introducing the subsidiary event S.

Some comments are in order.

By systematically resolving events into sums of constituents Wilbraham indeed manages entirely without Boole's "logical equations" and its associated obscurities. That this could be done is not surprising to us now, but we wonder if this *aperçu* of Wilbraham's, that logical problems concerning events could be so handled, was sufficiently appreciated at that time. Although not abandoning his logical methods, Boole's work subsequent to Wilbraham's paper shows a decided shift towards an emphasis on constituent probabilities.

It cannot be denied that the solution (answer) to Wilbraham's illustrative example is "indeterminate"—a simple Euler-Venn diagram, in which the areas are made proportional to the probabilities of the events, will show that the areas of A, B, C, and ABC can be maintained at a, b, c, and m, respectively, and yet the area of $A\bar{B}$ can still vary. Moreover, Wilbraham has also shown how to obtain Boole's solution by replacing the problem-to-be-solved by another which entails additional conditions ("assumptions") resulting from the mutual independence of the introduced events. Boole's reply to Wilbraham's cirticism was that the additional equations do not represent "hypotheses", but that "they are legitimate deductions from the general principles upon which that method is founded, and it is to those principles that attention ought to be directed."

Another criticism of Wilbraham's concerns the example which Boole used to illustrate his Prop. II, p. 261, discussed by us above in § 4.3. After noting that x, y,... in Prop. II are assumed to be independent Wilbraham remarks (Boole 1952, p. 480):

> How this can be reconciled with Professor Boole's statement with regard to a particular example of the proposition that his reasoning "does not require that the drawings of a white and marble ball should be independent in virtue of the physical constitution of the balls; that the assumption of their

> independence is indeed involved in the solution, but does not rest upon any prior assumption as to the nature of the balls, and their relations or freedom from relations, or form, colour, structure, &c." (page 262), I am at a loss to understand.

In his letter to the *Philosophical Magazine* responding to Wilbraham Boole says nothing about this—it is, however, addressed in the NOTE appearing at the beginning of *Laws of Thought* in which the example is withdrawn as an illustration of Prop. II. Either Wilbraham missed seeing the NOTE or, quite possibly, Boole had it tipped in by his publisher after an initial run. (This might account for the NOTE being absent in the 1916 Open Court reprint, and present in the 1951 Dover reprint.) In the reply at hand Boole attributes Wilbraham's "erroneous judgements" to his [i.e. *W*'s] believing that the events "which in the language of the data appear as *simple events*, are the ultimate [independent] elements of consideration in the problem. These are the elements in terms of which he expresses his equation, overlooking the fact that it is by mere *convention* that such elements are presented as simple, and that the problem might have been expressed quite otherwise".

Another criticism relates to Boole's use of his V to condition the simple independent events. Wilbraham says (BOOLE 1952, pp. 480–481):

> The independence of the events x, $y, \ldots s$, $t, \ldots$ is, as before, assumed in the assumption of the results of Prop. I. Nevertheless, Professor Boole says (page 264) that the events denoted by s, t,&c. whose probabilities are given, have such probabilities not as *independent events*, but as events subject to a certain condition V. He seems throughout to consider V as a condition that does always obtain, and consequently that the chance of any event inconsistent with it is 0, and therefore he ignores the previously assumed independence of the simple events which is inconsistent with such a supposition, instead of considering V as a condition which, if it obtain, the chances of x, $y, \ldots$ are as given in the data of the problem.

Boole's letter contains no direct reply to this. We shall have something to say about it when we undertake a critical examination of Boole's method in § 5.4.

Peirce's cirticism of Boole's probability theory (contained in PEIRCE 1867) shows no awareness of the Wilbraham paper which appeared some 7 years earlier. Whereas Wilbraham's objections are substantial and go to the heart of Boole's enterprise of applying his logical system to probability, Peirce's are of the nature of "improvements" (in his view) or corrections.

Peirce begins by assuming that every expression for a class has "a second meaning, which is its meaning in an equation. Namely, let it denote the proportion of individuals of that class to be found among all individuals in the long run." Thus from the logical *identity* $a =_{,} b$ (for Peirce's notation see our § 1.11) a and b classes, he infers the arithmetical *equation* $a = b$, a and b frequencies. But he also states [1867, p. 255 = 1933, Vol. III, p. 9]:

$$a + b = (a +_{,} b) + (a, b) \tag{28.}$$

without any specifications as to how numerical values accrue to complex logical expressions; e.g. since he uses + both logically and arithmetically the expression on the right hand side in (28.) is ambiguous as between $P(a +_{,} b) + P(a, b)$ and $P((a +_{,} b) + (a, b))$. For the notion *frequency of the b's among the a's* he writes 'b_a', supplying a notation for conditional probability which, as we have remarked, is sorely needed in Boole's theory. But he also considers it to have meaning as a *class*, thereby introducing a confusion similar to that of fraction and rational: a fraction denotes, i.e. determines, a rational, but a rational does not determine a fraction. With Peirce a pair of classes determines a frequency, but a frequency can't determine a pair of classes, let alone a class. Among the "obvious and fundamental properties of the function b_a" such as $ab_a = a, b$ and $ab_a = ba_b$ he includes:

$$\varphi(b_a \text{ and } c_a) = (\varphi(b \text{ and } c))_a \tag{31.}$$

(Peirce uses the verbal 'and' in place of the usual comma separating the arguments of a two-place function since he is using the comma for logical product). It is not clear what Peirce means by this. E.g. if $\varphi(x$ and

y) is $x \mathbin{\dot{+}} y$ then (3.1) becomes (??)

$$b_a \mathbin{\dot{+}} c_a = (b+c)_a.$$

A correct formula with the right-hand side would be

$$b_a + c_a - (b, c)_a = (b \mathbin{\dot{+}} c)_a,$$

in which case the 'φ' on the left-hand side of (31.) would not be the same as on the right-hand side.

Peirce compares his treatment of probability with that of Boole via discussion of several examples. We say a few words about this.

For the problem of finding the probability of the conjunction r of two events with respective probabilities p and q Peirce gives

$$r \mathrel{\dot{=}} p, q = pq_p = qp_q$$

(the symbols 'p' and 'q' being used both for the event and its probability) and then concludes that the answer is an unknown fraction of the least of p and q. Boole, of course, could have arrived at the same result by writing

$$\text{Prob}.xy = \text{Prob}.x \cdot \frac{\text{Prob}.xy}{\text{Prob}.x} = \text{Prob}.y \cdot \frac{\text{Prob}.xy}{\text{Prob}.y}$$

However it is significant to note that Boole's general method applied to this problem would give the answer

$$\text{Prob}.xy = pq$$

i.e. the product of the two probabilities as if the events were independent, since the data indicate no connexion between them and hence by his rule they are simple unconditioned events.

Next Peirce goes on to say that the value for r just given would also be the probability of the conclusion of a hypothetical syllogism whose major premise had probability p and whose minor premise had probability q. He says Boole's answer, namely

$$r = pq + a(1-q), \quad a \text{ arbitrary}$$

is wrong since it implies that if the major premise is false and the minor premise true then the conclusion is (necessarily) false, and he then comments on Boole's "absurd" conclusion (see our discussion in § 4.6) that the propositions which are equivalent when true are not necessarily equivalent when regarded as probable. Boole is then further chided: "Boole, in fact, puts the problem into equations wrongly (an error which it is the chief purpose of a calculus of logic to prevent), and proceeds as if the problem were as follows:—". Peirce then goes on to restate it with conditional probabilities in place of probabilities of conditionals (as in § 4.6). But, amusingly, Peirce's own formulation of the problem of the hypothetical syllogism is itself incorrect, and on two accounts! In the first place the conclusion is only implied by, not equivalent to, the conjunction of the two premises and, secondly, Peirce's data does not include the fact that the premises are logically connected. (For our solution see § 6.6) As for the other version in terms of conditional probabilities, Peirce gives an answer involving 7 parameters. We discuss this version below in § 6.7, where we give our solution (Example 6.72).

Peirce lists three differences between Boole's system and his modification (PEIRCE 1867. pp. 259–260 = 1933, § 3.18). The first of these, threefold in character and relating to the logic, we have mentioned in § 1.11. The other two concern probability:

> Second. Boole uses the ordinary sign of multiplication for logical multiplication. This debars him from converting every logical identity into an equality of probabilities. Before the transformation can be made the equation has to be brought into a particular form, and much labor is wasted in bringing it to that form.
>
> Third. Boole has no such function as a_b. This involves him in two difficulties. When the probability of such a function is required, he can only obtain it by a departure from the strictness of his system. And on account of the absence of that symbol, he is led to declare that, without adopting the principle that simple, unconditioned events whose probabilities are given are independent, a calculus of logic applicable to probabilities would be impossible.

But the ability of Peirce's system to convert "every logical identity into an equality of probabilities" is based on his function b_a (for which $a, b = ab_a$) and, as we have argued, Peirce is mistaken, or at least has given no justification for believing that it has a logical as well as numerical meaning. With regard to the independence of simple unconditioned events, it is a question as to what is meant by such events. Peirce is certainly justified in objecting to calling events independent on the basis of no information. As he says: "But there can be no question that an insurance company, for example, which assumed that events were independent without any reason to think that they really were so, would be subject to great hazard." Our interpretation of 'simple unconditioned event' is given in § 5.1.

Continuing from § 1.11 our discussion of MACFARLANE 1879, we add a few remarks about his attempt to include conditional probability into his formalism.

Macfarlane assumes that any universe U has an "arithmetical value" which is an integer ("generally *plural*, but may be *singular* or *infinite*") and that any character (attribute) x, "when considered as an operation on U", has an arithmetical value $\bar{x}$ [not to be confused with Boole's notation for complement.] Presumably, since he quotes Venn, he is thinking in terms of frequency, but all that he says is: "if x denotes a single positive attribute its value [i.e. $\bar{x}$] is a fraction lying between 0 and 1; but if it is negative, its value lies between 0 and -1." He uses 'Uxy' to denote 'U's which are both x and y' and in practice drops the 'U'—leaving himself open to confusion when more than one universe is involved, as in conditional probability. He seems to have two notions of independence—*real* having to do with frequencies and *formal* having to do with syntactic structure. Real independence corresponds to stochastic independence for it gives, in his notation, $\overline{xy} = \bar{x}\bar{y}$. Concerning formal independence he has (1879, p. 21):

> When the symbols x and y are independent—that is when each refers to U simply—the compound xy has for any given U a definite *arithmetical value*. This value, however, is not determinable from those of x and y; but they give limits to the value [which are, for lower, 0 or $\bar{x} + \bar{y} - \bar{1}$, and for upper, x or y].

This seems to correspond to Boole's "simple unconditioned events" but without his "by definition independent". Continuing the quote:

> But one character may *in its statement* [italics added] involve another character, so as to be formally dependent on the latter. Let U denote the universe of its objects, a any character x a character which is formally dependent on a. Then UA_x denotes U's which are a and of these such as are x; or U's which have a which have x.
>
> The symbol x operates on Ua not on U. Hence
>
> $$\bar{a}_x = \bar{a}\bar{x}.$$

Note that Macfarlane introduces here a new notation, a_x, for a conjunctive compound when the components are "formally dependent". If his multiplication rule is to make sense then the arithmetical value $\bar{a}_x$ should stand for the frequency of the ax's, and $\bar{a}$ that of the a's, both among the U's, while $\bar{x}$ should that of the x's among the a's (of U). While Peirce's symbolization of the multiplication rule (i.e. $a, b = ab_a$) has its drawbacks, Macfarlane's is defective in that one has to keep in mind that for the multiplication rule $\bar{a}$ and $\bar{a}_x$ refer to the same universe, but $\bar{x}$ to the subuniverse which consists of the a's. What a_x means in other contexts isn't clear, although Macfarlane, as Peirce with his 'b_a', attempts to associate a logical (class) notion with his 'a_x'. With no apparent justification he asserts (p. 23): "Since a_x is equivalent to an independent character of arithmetical value $\bar{a}\bar{x}$, the laws of independent characters apply to a_x as a whole", from which he obtains results such as $a_{x+y} = a_x + a_y$. But even if $\bar{a}\bar{x}$ were to determine the arithmetical value of a *character* (which is dubious), it still needn't be something uniquely determined by a and x. Consider, for example, the analogous situations in ordinary arithmetic where $|ab| = |a|\,|b|$, but knowing $|a|\,|b|$ doesn't give us an unambiguous ab.

In § 5.1 below we shall be arguing that the kind of logical notion that Peirce, and Macfarlane in a confused way, were attempting was, in essence, already present in Boole's work.

§ 4.9. Notes to Chapter 4

(for § 4.1)

NOTE 1. The list of probability principles Boole gives is apparently from LAPLACE 1820. The form of the inverse principle (Boole's 6th) is what Laplace initially lists, but then he follows it with the more general form in which the antecedent causes are not necessarily all equal in probability. Although stating only the restricted form, Boole does have in mind in his discussion the more general one. This form still assumes that the causes are mutually exclusive and also are sufficient, i.e. that the event follows from some one of them. Boole forgets to include this latter requirement in his DATA as a third condition.

NOTE 2. Boole, as we have noted, had no clear distinction between the probability of a conditional and conditional probability. Peirce did, but thought that conditional probability carried with it a logical notion which, however, he never made precise. (See our § 4.8) In recent times a number of logicians have explored the idea of a (non-truthfunctional) conditional "If A, B" whose probability would be the conditional probability of B, given A. We cite a few references:

R. Stalnaker. A Theory of Conditionals. *Studies in Logical Theory*, N. Rescher (ed.)(APQ Supplementary Volume). Blackwell 1968.

David Lewis. Probabilities of Conditionals and Conditional Probabilities. *The Philosophical Review*, LXXXV (1976), 297–315.

Bas C. Van Fraassen. Probabilities of Conditionals. *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science.* Vol. I. W. L. Harper and C. A. Hooker, editors. D. Reidel 1976.

(for § 4.2)

NOTE 1. As an indication of the progression of Boole's ideas relative to simple events and their independence, we note that in an early manuscript contained in two of his notebooks, written possibly before 1851 (editor's footnote, p. 141 BOOLE 1952, Essay III), his view is that simple events are [stochastically] independent when they are obtained by independent observations (1952, p. 155):

> It is obviously supposed in the above case that the probabilities of the simple events x, y, z,... are given by independent observations. This is, I apprehend, what is really meant by events being spoken of as independent. For if the events are not independent (according to the ordinary acceptation of that term), the knowledge that they are not so can only be derived from experiences in which they are mutually involved, not from observations upon them as simple and unconnected. And hence if our knowledge is derived from experience, the independence of events is only another name for the independence of our observations of them.

This is before *Laws of Thought*. In *Laws of Thought* events are independent if "all information respecting their dependence is withheld," in as much as "the mind regard them as independent"; additionally, simple unconditioned events, i.e. those which are free to occur in every possible combination, are "by definition" independent. Subsequently to *Laws of Thought*, in BOOLE 1854e, he argues, on the basis of the definition of probability as a ratio of numbers of equally likely cases, that events for which no connexion is known are (stochastically) independent (pp. 433–434):

> Let us, in further illustration of this principle, ["that probability is always relative to our actual state of information and varies with that state of information"] consider the following problem. The probability of an event x is measured by the fraction a/m, that of an event y by the fraction b/n, but of the connexion of the events x and y absolutely nothing is known. Required the probability of the event xy, i.e. of the conjunction of the events x and y.
>
> There are (see definition) a cases in which x happens, to m cases in which it happens or fails; and concerning these cases the mind is in a state of perfect indecision. To no one of them is it entitled to give any preference over any other. There are, in like manner, b cases in which y happens, to n cases in which it happens or fails; and these cases are in the same sense equally balanced. Now the

event xy can only happen through the combination of some one of the a cases in which x happens, with some one of the b cases in which y happens, while *nothing prevents us from supposing any one of the m cases in which x happens or fails from combining with any one of the n cases in which y happens or fails.* [Italics supplied] There are thus ab cases in which the event xy happens, to mn cases which are either favourable or unfavourable to its occurrence. Nor have we any reason to assign a preference to any one of those cases over any other.

Wherefore the probability of the event xy is ab/mn. Or if we represent the probability of the event x by p, that of the event y by q, the probability of the combination xy is pq.

It cannot be disputed that the above is a rigorous consequence of the definition adopted.

Despite Boole's contention, one can dispute that the above is a rigorous consequence of the definition of probability. Referring to the italicized portion, we point out that just because nothing *prevents us* from supposing that any of the m cases may be combined with any of the n cases, does not mean that we *have to* suppose this. We may suppose any connection, or absence of connection, which is not inconsistent with the data (in this case the data is simply that the probability of x is p and that of y is q). And just because nothing is *known* of any connection between x and y does not mean that there can be none. Absence of knowledge of any connection between x and y cannot imply positive knowledge that x and y are stochastically independent. We continue our quote:

That new information might alter the value of Prob. xy is only in accordance with the principle (already exemplified from Laplace) of the relative character of probability. It is only so far forth as they are known, that the connexions, causal or otherwise, of events can affect expectation. Let it be added, that the particular result to which we have been led is perfectly consistent with the well-known theorem, that if x and y are known to be independent events, the probability of the event xy is pq. The difference between the two cases consists not in the numerical value of Prob. xy, but

> in this case, that if we are sure that the events x and y are independent, then are we sure that there exists between them no hidden connexion, the knowledge of which would affect the value of Prob. xy; whereas if we are not sure of their independence, we are sensible that such connexions may exist.

Anent of the last sentence, we could turn Boole's own words against him and say that if, as he is supposing, nothing is known of a connection between x and y then certainly we are not sure of their independence, and hence "we are sensible that such connexions may exist"; but then taking x and y to be stochastically independent is opting for a particular one of many possibilities that may exist.

(for § 4.3)

An early version of Boole's general method in probability is in the manuscript notebooks published as Essay III in BOOLE 1952.

(for § 4.6)

NOTE 1. The answer Boole gives for his Ex. 1 was first stated by him in his 1851b (= 1952, IX), and he there contrasts his solution with the only published solution he is acquainted with, a solution which assumes that the two events are independent.

NOTE 2. Peirce, in his 1867 (= PEIRCE 1933, I = PEIRCE 1984, 2) gives a solution to Boole's Ex. 3 which is essentially the one we give.

NOTE 3. Boole's Ex. 4 is mentioned by him in his 1851a (= 1952, VIII) as a type of problem which his general method could handle. He gives the answer—the same as appears in *Laws of Thought*—but without the method of solution.

We show that the quantity $pq/\bar{r}$, equal to $P(x)P(y)/P(x \vee y \vee z)$, appearing in Ex. 4 lies between the minimum and maximum values obtainable from $P(xy)$ when x and y are allowed to vary, subject only to the constraint that $P(x) = p$, $P(y) = q$.

For when x and y are independent, $P(xy) = P(x)P(y)$, and since $P(x \vee y \vee z) \leq 1$, we then have

(a) $$\min P(xy) \leq P(x)P(y) \leq \frac{P(x)P(y)}{P(x \vee y \vee z)}.$$

Morever,

$$P(x)P(y) \leq P(x)P(x \vee y \vee z)$$

so that

$$\frac{P(x)P(y)}{P(x \vee y \vee z)} \leq P(x).$$

Similarly,

$$\frac{P(x)P(y)}{P(x \vee y \vee z)} \leq P(y).$$

Consequently,

(b) $$\frac{P(x)P(y)}{P(x \vee y \vee z)} \leq \text{smaller of } P(x), P(y) \leq \max P(xy).$$

One readily sees that Boole's solution is correct if the hypothesis of the mutual independence of x, y, and $\bar{x}\bar{y}\bar{z}$ is adjoined. For then

(c) $$P(x)P(y) = P(xy)$$

and

$$P(x)P(\bar{x}\bar{y}\bar{z}) = P(x\bar{x}\bar{y}\bar{z}) = 0$$

so that either

(d) $$P(x) = 0 \text{ or } P(x \vee y \vee z) = 1,$$

and in either case we have, from (c) and (d), the equality of Boole's $pq/\bar{r}$ with $P(xy)$.

NOTE 4. The problem on the hypothetical syllogism (Boole's Ex. 5) makes its first appearance in his 1852b (= 1952, IX). Again only the answer is given, and he contrasts it with the only other solution known to him, namely one in which the answer for the probability of the conclusion is the product of the probabilities of the two premises ("... a result which manifestly involves the hypothesis that the conclusion cannot be true on any other grounds than are supplied by the premises"). We note further that the form Boole considers here has as premises categoricals ("all *Ys* are *Xs*") whereas in *Laws of Thought* they are conditionals ("If the proposition *Y* is true *X* is true").

PEIRCE 1867 (= 1933, I = 1984, 2) shows that Boole's solution to Ex. 5, i.e. $P(x|z) = pq + a(1-q)$, can't be correct by giving an example (x = a certain man is a negro, y = he was born in Massachusetts, z = he is a white man) for which $pq > 0$ and yet $P(x|z) = 0$. The solution Peirce gives, involving seven introduced parameters, is incomprehensible to us.

(for § 4.8)

Although Wilbraham's place in the history of mathematics can hardly compare with that of Boole's—one would look in vain for his name in any book on the subject—it appears that he was no inconsiderable mathematician. He was, in fact, the discoverer of the Gibbs phenomenon for Fourier series in 1848, a half-century before Gibbs. (See H.S. Carslaw, *Fourier Series and Integrals*, 3rd edition, Dover Publications, p. 294.) It is of some interest to note that Wilbraham's paper reporting the (Gibbs) phenomenon appears in an issue of volume 3 of *The Cambridge and Dublin Mathematical Journal* immediately following an article of Boole's in which Boole expounds his newly discovered calculus of logic.

CHAPTER 5

BOOLE'S PROBABILITY MADE RIGOROUS

In this chapter we present a formal theory of probability which provides a basis for understanding Boole's general method in probability. By a formal theory we mean a mathematical structure, or framework, which can model, in this case, stochastic situations. We do not define probability but only provide, axiomatically, relationships from which the probabilities of some events are obtained when others are specified. The question of the initial determination of probabilities is, in our view, an epistemological one and is not considered. As it covers only finite stochastic situations the theory is a limited one, but this is all that is needed to show what is involved in Boole's ideas and to compare them with current conceptions of the subject. Here too, as with his logical system, we shall find that despite erroneous ideas and unnecessarily complicated methods there is much of interest and value to be gleaned.

In § 5.3 we analyze in detail Boole's method of solving the "general problem" in probability and determine for the method the circumstances and extent of its validity. In succeeding sections we give contemporary treatments of (i) his method for finding conditions of possible experience and (ii) of his solution of the problem of absolute probabilities—in neither of these two topics does our treatment depend on the peculiarities of Boole's probability theory, although they both are essential features of it. Our discussion of the problem of absolute probabilities is long and tedious, but we felt it worthy of attention as it is a key feature of Boole's method, a proof of which (in the general case) baffled him a long time. Involved is a rare example of a mathematical theorem which gives necessary and sufficient conditions for a rather general system of algebraic equations to have a unique solution. Finally, using our meaning of a solution, we solve Boole's general probability problem by modern linear programming methods.

§ 5.1. Simple and Boole probability algebras, calculi, models

Basic to Boole's probability theory was the tenet that (compound) events could be treated as simple events which may be unconditioned, and by definition independent, or may be conditioned by logically expressed (or implied) conditions. Boole never made these ideas really clear—justification for his views being based on a combination of linguistic and psychological notions (see § 4.2) and, while there were not direct criticisms, there were telling indirect ones (§ 4.8). As a key step in explicating Boole's ideas we introduce in this section the notion of a Boole probability model and we illustrate its use as a valid, though complicated, way of handling finite stochastic situations.

A probability algebra $\langle \mathfrak{B}, P \rangle$ (§ 0.9) is a *simple probability algebra* if $\mathfrak{B}$ is a Boolean algebra with n algebraically free generators $x_1, \ldots, x_n$ (§ 0.5.4), and such that for these generators[1] one has $0 < P(x_i) < 1$ for $i = 1, \ldots, n$. If, in addition, with respect to the probability function P the set of generators is a (stochastically) independent set, then we say we have a *Boole probability algebra.*

In any probability algebra $\langle \mathfrak{B}, P \rangle$ the function P assigns to each element of $\mathfrak{B}$ a real number in the interval $[0, 1]$; in a Boole probability algebra this number is, moreover, explicitly obtainable in terms of the probabilities of the generators when an explicit expression for the element in terms of the generators is known. This corresponds to Boole's procedure of going from $P(V)$ to $[V]$ (Boole also uses 'V', with numerical significance, in place of $[V]$) when V is expressed in terms of simple unconditioned events (see §§ 0.9, 4.3). If we think of V in disjunctive normal form we readily see, since for no generator x_i is $P(x_i) = 0$ or $P(\bar{x}_i) = 0$, that $[V] = 0$ if and only if $V = 0$.

[1] We are using the letters "x_i" to accord with Boole's notation. They are, of course, not to be thought of as variables which can be assigned any element of the Boolean algebra.

Corresponding to a simple probability algebra we define a *simple probability calculus* as a probability calculus $\langle S, P\rangle$ (§ 0.9) in which the formulas of S are those obtainable by application of the propositional connectives to the *finite* set of propositional variables $X_1, \ldots, X_n$, and such that for no X_i is $P(X_i) = 0$ or $P(X_i) = 1$. A *Boole probability calculus* is then a simple probability calculus in which to $P1 - P3$ (of § 0.9) are adjoined the $2^n - n - 1$ equations expressing that $\{X_1, \ldots, X_n\}$ is a mutually independent set (with respect to P). If $\langle S(X_1, \ldots, X_n), P\rangle$ is a simple (respectively Boole) probability calculus then the probability model $\langle S(A_1, \ldots, A_n), P\rangle$ will be called a *simple* (respectively *Boole*) *probability model*, and the atomic sentences $A_1, \ldots, A_n$ its *generators*. We identify Boole's "simple unconditioned events which are by definition independent" with the generators of a Boole probability model.

It will be convenient for us to think of $S(A_1, \ldots, A_n)$ as a Boolean algebra. We do this by going over to its Lindenbaum algebra (equivalence classes under logical equivalence), and then identifying each equivalence class with one of its members, e.g. the one in complete disjunctive normal form. Thus (changing from Roman to German font) $\mathfrak{S}(A_1, A_2)$ is the Boolean algebra with the 16 elements

$$0(= A_1\bar{A}_1),\ A_1A_2,\ A_1A_2 \vee A_1\bar{A}_2,\ \ldots,$$
$$A_1A_2 \vee A_1\bar{A}_2 \vee \bar{A}_1A_2,\ 1(= A_1A_2 \vee A_1A_2 \vee \bar{A}_1A_2 \vee \bar{A}_1\bar{A}_2)$$

For such an algebra the Boolean operations, while syntactically complicated, are semantically obvious—e.g. the complement of A_1A_2 would first be thought of as $\bar{A}_1 \vee \bar{A}_2$, then converted to normal form as $\bar{A}_1A_2 \vee \bar{A}_1\bar{A}_2 \vee A_1\bar{A}_2$. As a consequence of this, in what follows we need only refer to algebras, but can nevertheless, with Boole, think of events as propositions.

If one's interest were only in mutually independent events and their logical combinations—such as, for example, the two events H and A, where H is *Heads comes up* (on the toss of a coin) and A is *An ace is drawn* (from a deck of cards)—then one could take these events as generators of a Boole probability model and, with probabilities assigned (from experimental or other considerations) to the generators the situation would be adequately modeled. However, such a theory can have but limited application. Even as trivial a stochastic experiment as a coin-toss needs more than this since the events *Heads comes up* and *Tails comes up* (referring to the same coin) are not independent and, although Boole would refer to them as simple events (i.e. not explicitly compounded of others), they couldn't be simple *unconditioned* events. What is needed is a way of bringing into the situation that if Heads comes up Tails does not, and if Tails comes up Heads does not, and that these are the only possibilities. Contemporary probability theory does this be modeling the experiment with a four-element Boolean algebra $\{0, \{H\}, \{T\}, \{H, T\}\}$, representing all the possible events, with the set $1 = \{H, T\}$ containing the two distinct possible outcomes of the toss. But we can also accomplish the same result by making use of Boole's idea of a conditioned event—a (compound) event being for him conditioned when some possibilities are excluded from happening. For the example at hand we would assume[1] simple unconditoned events H and T (i.e. H and T are free generators of a Boolean algebra $\mathfrak{S}(H, T)$) which are then conditioned by the exclusion of the possibilities HT and $\bar{H}\bar{T}$. To use such conditioned events, which are clearly not elements of $\mathfrak{S}(H, T)$, requires a means of formal representation.

Boole had probability assignments for what we are referring to as conditioned events, but no explicit notation for them. Yet a suitable notation is readily extractable from his technique for solving problems in probability where he treats constituents in the $\frac{1}{0}$ part of an expansion as events which are excluded from happening. This suggests taking conditioned events as Boolean quotients namely, returning to our example, as quotients whose expansions are

$$(1) \qquad \begin{aligned} &H\bar{T} + \tfrac{1}{0}(HT + \bar{H}\bar{T}) \\ &\bar{H}T + \tfrac{1}{0}(HT + \bar{H}\bar{T}), \end{aligned}$$

[1] Recall Boole's Principle VI (in *Laws of Thought*, p. 256), or its later version in BOOLE 1854e (= BOOLE 1952, p. 296): "Principle II.—Any events which suffice simply, or by combination, for the expression of the data may be assumed as simple [unconditioned] events and symbolized accordingly, provided that we explicitly determine the whole of the relations which implicitly connect them."

with the readings "Heads (but not Tails), given that HT and $\bar{H}\bar{T}$ cannot happen" and "Tails (but not heads), given that HT and $\bar{H}\bar{T}$ cannot happen". By virtue of the isomorphisms expressed in Theorem 0.62 these conditioned events can be thought of not only as Boolean quotients, but equally well as residue classes (as written in (1)), or as elements of an algebra of subelements. In general, for a fixed element $D(\neq 1)$ of a Boolean algebra $\mathfrak{B}$, the residue classes of the form

$$A\bar{D} + \tfrac{1}{0}D,$$

as A ranges over $\mathfrak{B}$, consitute (with the operation appropriately defined) a Boolean algebra isomorphic to the algebra of subelements of $\bar{D}$. In our example, with $\bar{D} = V = HT + \bar{H}\bar{T}$, this algebra would be the four-element algebra $\mathfrak{S}(H, T)|V$ with universe $\{0, H\bar{T}, \bar{H}T, V\}$. Thus as far as having the requisite set of outcomes representing chance events is concerned, an algebra of quotients with $\frac{1}{0}$ part $HT + \bar{H}\bar{T}$ can adequately model the coin-toss experiment. But what of the probabilities for the elements of this algebra?

Before addressing ourselves to this question let us introduce a notation for the special kind of Boolean quotient representing a conditioned event, namely

$$E|F \quad \text{for} \quad \frac{\bar{F} +_B E}{F}\left[= \frac{\bar{F} + EF}{F} = \frac{F \to E}{F}\right].$$

Note that the residue class isomorphic to $E|F$ is $EF + \frac{1}{0}\bar{F}$, and hence that there would be no $\frac{0}{0}$ constituents in the expansion of such a quotient (expansion, that is, with respect to the generators of the Boolean algebra of which E and F are elements). In terms of this notation the two outcomes of the coin-toss would be represented by $H|V$ and $T|V$, V being $H\bar{T} + \bar{H}T$. Now for the question asked at the end of the preceding paragraph.

Boole took the probability of a conditioned event as a conditional probability, that is, using our introduced notation, for him

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

For the example at hand,

$$(2) \qquad \begin{aligned} P(H \mid V) &= \frac{P(HV)}{P(V)} \quad (= p, \text{ say}) \\ P(T \mid V) &= \frac{P(TV)}{P(V)} \quad (= q, \text{ say}) \end{aligned}$$

(since it is $H|V$ and $T|V$ that represent outcomes of an experiment it would be to these events that probabilities, p and q, would be given, assigned or determined). But (2) presumes that one can meaningfully talk about the probabilities present on the right-hand sides, namely $P(HV)$, $P(TV)$ and $P(V)$; that is, that we have P-values for elements of $\mathfrak{S}(H, T)$.[1] For this latter condition it suffices that $\langle \mathfrak{S}(H, T), P \rangle$ be a probability algebra—but not any probability algebra will do for, p and q being given, we need to have P-values for HT, $\bar{H}T$, $H\bar{T}$ and $\bar{H}\bar{T}$ such that (2) holds. Designating these four P-values by λ_1, λ_2, λ_3, λ_4, we see that the conditions for (2) holding are that

$$(3) \qquad \begin{gathered} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \qquad \lambda_i \geq 0, \\ \frac{\lambda_2}{\lambda_2 + \lambda_3} = p \\ \frac{\lambda_3}{\lambda_2 + \lambda_3} = q \end{gathered}$$

This system can be satisfied only if $p + q = 1$. Assuming this to be the case there are then only two independent equations (plus the inequalities $\lambda_i \geq 0$) for this system, and infinitely many solution sets for the λ_i's, any one of which defines a probability function P (see § 0.9). With any such P the probability model $\langle \mathfrak{S}(H, T), P \rangle$ then serves to specify an

[1] There is an abuse of notation here. Since we are talking about the (in general) non-isomorphic algebras $\mathfrak{S}(H, T)$ and $\mathfrak{S}(H, T)|V$ there should be, when considering probability models, different symbols used for their associated probability functions. However we shall here in (2) use the same symbol P and informally recognize the difference by the kinds of expressions used in designating the arguments.

appropriate $\langle \mathfrak{S}(H, T) | V, P \rangle$, the P here being defined via (1). Note that although there are many allowable P functions for $\langle \mathfrak{S}(H, T), P \rangle$, all meeting conditions (3), that for $\langle \mathfrak{S}(H, T) | V, P \rangle$ is unique, i.e. is the same no matter which $\langle \mathfrak{S}(H, T), P \rangle$ is selected. For equations (3) only specify the ratio λ_2/λ_3 and since

$$P(H | V) = \frac{\lambda_2/\lambda_3}{1 + \lambda_2/\lambda_3}, \quad P(T | V) = \frac{1}{1 + \lambda_2/\lambda_3}$$

the P-values in $\langle \mathfrak{S}(H, T) | V, P \rangle$ are the same for all such solutions.

Exactly how Boole would have viewed the coin-toss experiment in terms of his basic ideas isn't known, but very likely the H and T, the "prior" or "absolute" events before the exclusion of HT and $\bar{H}\bar{T}$, would be for him simple unconditioned, and by definition, independent events.[1] Translated into our formulation this means assuming that $\langle \mathfrak{S}(H, T), P \rangle$ is a *Boole* probability algebra (model). Such an assumption is stronger than necessary since, as just seen, an ordinary probability algebra suffices. And Boole's stronger assumption still does not impose enough conditions to specify a unique $\langle \mathfrak{S}(H, T), P \rangle$: from our discussion of Case (ii) on §4.4 we know that equations (2), which for a Boole probability algebra are now,

$$(4) \qquad \begin{aligned} \frac{[HV]}{[V]} &= \frac{h\bar{t}}{h\bar{t} + \bar{h}t} = \frac{h/\bar{h}}{h/\bar{h} + t/\bar{t}} = p \\ \frac{[TV]}{[V]} &= \frac{\bar{h}t}{h\bar{t} + \bar{h}t} = \frac{t/\bar{t}}{h/\bar{h} + t/\bar{t}} = q \quad (p + q = 1) \end{aligned}$$

have infinitely many solution sets (h, t), $0 < h, t, < 1$, any one of which will define a Boole probability algebra $\langle \mathfrak{S}(H, T), P \rangle$ providing a (unique) $\langle \mathfrak{S}(H, T) | V, P \rangle$.

Our discussion of the coin tossing problem is generalizable to the case of n events $E_1, \ldots, E_n$ of which one and only one can occur—corresponding to the two events $H\bar{T}$ and $\bar{H}T$ we have the n

[1] "If, ..., any events s, t &c so enter as that nothing is known or can be inferred respecting their connection, they must be treated (Principle I) as if they were independent, ..." BOOLE 1854e, p. 440.

outcomes

$$(5) \qquad E_1\bar{E}_2\ldots\bar{E}_n,\ \bar{E}_1E_2\bar{E}_3\ldots\bar{E}_n,\ldots,\bar{E}_1\bar{E}_2\ldots\bar{E}_{n-1}E_n,$$

and, with V as their logical sum, the algebra $\mathfrak{S}(E_1,\ldots,E_n)|V$ then corresponds to the algebra of subsets of the n-point sample (or event) space of the usual approach now current. If probabilities p_i ($i = 1,\ldots,n$) are assigned to the conditioned events of $\mathfrak{S}(E_1,\ldots,E_n)|V$ which correspond to (5), then it can be readily seen that the problem of determining a $\langle\mathfrak{S}(E_1,\ldots,E_n), P\rangle$ so as to have a $\langle\mathfrak{S}(E_1,\ldots,E_n)|V, P\rangle$ with the assigned probabilities has a solution so long as Boole's "conditions of possible experience", namely

$$p_1 + \cdots + p_n = 1, \quad p_i \geq 0\ (i = 1,\ldots,n)$$

are met. (Normally for (5) one would have $0 < p_i < 1$, $i = 1,\ldots,n$. We omit discussion of the special case of some of the p_i being 0 or 1.)

What do the "prior" events H and T in the coin-toss discussion stand for if $H|V$ and $T|V$ represent the outcomes of the toss? Intuition is no help since we never think of Heads and Tails apart from their being opposite faces of a coin. In view of the infinitely many probability values which can be used for H and T in $\langle\mathfrak{S}(H,T), P\rangle$ it can hardly be that they stand for anything well-determined. Evidently we have a feature of the formal apparatus being used to which nothing in the experiment corresponds. Accordingly we may use an arbitrary freely generated Boolean algebra $\mathfrak{B}(s,t)$ in place of $\mathfrak{S}(H,T)$ and, with P appropriately defined on $\mathfrak{B}(s,t)$, use $\langle\mathfrak{B}(s,t)|s\bar{t}+\bar{s}t, P\rangle$ to model the coin-toss experiment by associating $s|s\bar{t}+\bar{s}t$ with Heads and $t|s\bar{t}+\bar{s}t$ with Tails. In his last paper on probability [1862] Boole seems to express a similar opinion in the context of his solution of the General Problem in Probabilities. We bring this up again in § 5.4.

Clearly, for the handling of stochastic situations the probability theory based on Boole's ideas which we have presented is unnecessarily complicated—e.g. any four element Boolean algebra suffices to model the coin-toss—but its introduction will be of service in helping us understand Boole's procedure for solving the General Problem in Probability (in § 5.4).

§ 5.2. Conditioned-events probability realm

In the preceding section we associated probability values with Boolean quotients whose expansions were free of $\frac{0}{0}$ terms. But Boole's scheme of things involves a more general association of probability with any quotient (more exactly, any expansion), and we wish to do the same—but from our point of view and precisely formulated.

In our discussion of $E|F$ we wrote its expansion as $EF + \frac{1}{0}\bar{F}$ rather than the fuller $EF + \frac{0}{0}0 + \frac{1}{0}\bar{F}$, i.e. in the notation of § 2.7, rather than $[EF] + \frac{1}{0}[\bar{F}]$. Here the brackets indicate a residue class modulo the 0 ideal, and hence can be omitted since there is no need to distinguish between X and $X + (0)$. But in the general case of a Boolean expansion $A + 0B + \frac{0}{0}C + \frac{1}{0}D$ which, by virtue of § 2.7, we take to be

$$[A] + \tfrac{1}{0}[\bar{V}], \quad (\bar{V} = D, A + B + C + D = 1)$$

the brackets now indicating a residue class modulo the ideal (C) $(= \frac{0}{0}C)$, so that

$$[A] = A + \tfrac{0}{0}C = \text{set of elements } A + vC, v \in \mathfrak{B}.$$

Thus the natural probability notion to associate with $A + 0B + \frac{0}{0}C + \frac{1}{0}D$ would seem to be the *set* of probability values $P((A + vC)|V)$ as v ranges over $\mathfrak{B}$.

We bring together the informal discussion on probability values for Boolean quotients of this and the preceding section and define a *conditioned-events probability realm* (for a free Boolean algebra), a concept which we believe provides a formal framework for Boole's use of probability in connection with (expansions of) quotients. It will be noticed that what the definition does, in essence, is to assign the notion of probability to successively wider classes of Boolean quotients: first it is attributed to elements whose developments have neither $\frac{0}{0}$ nor $\frac{1}{0}$ constituents, next to those having $\frac{1}{0}$ but no $\frac{0}{0}$ constituents and finally to those having both $\frac{0}{0}$ and $\frac{1}{0}$ constituents. (We talk about constituents here as we are thinking of the elements of the free Boolean algebra as sums of constituents on the free generators—each element being uniquely representable as such a sum, and each quotient having a unique development in terms of constituents on the free generators.)

Consider $BQ(\mathfrak{B}_n)$, the partial algebra of Boolean quotients of a free Boolean algebra $\mathfrak{B}_n$ (with n free generators). For any two elements E and F of $\mathfrak{B}_n$ we define, as in the preceding section, the *conditioned-event* $E|F$ to be the Boolean quotient

$$\frac{F \to E}{F}\left(= \frac{\bar{F} +_B E}{F} = \frac{\bar{F} + EF}{F}\right).$$

As has already been noted, the expansion of a conditioned event has no $\frac{0}{0}$ constituents. Note also that for any E,

$$E|0 = \frac{0 \to E}{0} = \frac{0 \to 1}{0} = 1|0.$$

For fixed $V \neq 0$, the set of all quotients of the form $b|V$, $b \in B_n$, forms (with appropriate operations) a Boolean algebra $\mathfrak{B}_n(V +_B B_n)^{-1}$ (see end of § 2.7), and this algebra is isomorphic to $\mathfrak{B}_n|V$. At the risk of some confusion we now change notation and use '$\mathfrak{B}_n|V$' likewise to designate the isomorphic $\mathfrak{B}_n(V +_B B_n)^{-1}$—not only is the notation '$\mathfrak{B}_n|V$' shorter, but by virtue of a widespread convention it looks like it ought to be the set of all $b|V$, $b \in B_n$. For the special case of V being 1 we have the algebra $\mathfrak{B}_n|1$, all of whose elements have neither $\frac{0}{0}$ nor $\frac{1}{0}$ consituents; it is isomorphic to $\mathfrak{B}_n$ and we shall at times not distinguish between them. Now for the definition.

A *conditoned-events probability realm* is an ordered pair $\langle BQ(\mathfrak{B}_n), P\rangle$, where $BQ(\mathfrak{B}_n)$ is the partial algebra of Boolean quotients for a free Boolean algebra $\mathfrak{B}_n$ (with n free generators) and P is a real-valued function defined on all elements of $BQ(\mathfrak{B}_n)$, except for $1|0$, such that the following properties hold:

(CP1) $\langle \mathfrak{B}_n|1, P\rangle$ is an Boole probability algebra.

(CP2) For any E and F, $F \neq 0$, of B_n

$$P(E|F)P(F|1) = P(EF|1).$$

(CP3) For any $\omega \in BQ(\mathfrak{B}_n)$, except $\omega = 1|0$, whose expansion in terms of constituents on the generators of $\mathfrak{B}_n$ is $A + 0B + \frac{0}{0}C + \frac{1}{0}D$,

$$P(\omega) \in \{p : \exists v \in B_n,\ p = P((A + vC)|V)\}$$

$(V = \bar{D} \neq 0)$.

By virtue of (CP1) elements of $BQ(\mathfrak{B}_n)$ whose developed form lacks $\frac{0}{0}$ and $\frac{1}{0}$ constituents are accorded a P value. Further, since

$$(1) \qquad A + 0B + \tfrac{1}{0}D \approx \frac{\bar{D} \to A}{\bar{D}} = A \mid V \qquad (V = \bar{D} \neq 0)$$

we have that any element (other than $1 \mid 0$) whose development lacks $\frac{0}{0}$ constituents has its P value determined by (CP2) since

$$(2) \qquad P(A \mid V) = \frac{P(AV \mid 1)}{P(V \mid 1)} = \frac{P(A \mid 1)}{P(V \mid 1)}.$$

(Recall that in a Boole probability algebra $P(V) = 0$ if and only if $V = 0$.) Hence $P(V|1) \neq 0$ since $\bar{D} = V \neq 0$.) Finally, (CP3) determines the P value of an element with $\frac{0}{0}$ constituents only to the extent of its being a member of a specified set of P values.

The progressive enlargements of the probability notion given by (CP1)–(CP3) is a consistent one, for in (CP2) when $F = 1$ the result is an identity, and in (CP3) when $C = 0$ we have

$$P(\omega) = P(A \mid V),$$

also an identity, since if ω's expansion is $A + \frac{1}{0}D$ we have

$$\omega = \frac{\bar{D} \to A}{\bar{D}} = A|\bar{D} = A \mid V.$$

Allowing ourselves the use of an indefinite term we can express the membership relation of (CP3) in the form of an equation:

$$(3) \qquad P(\omega) = P((A + vC) \mid V), \quad \text{for some } v \in B_n.$$

Further, as A and vC are mutually exclusive, we have (Theorem 0.95 (ii))

$$(4) \qquad P((A + vC) \mid V) = P(A \mid V) + P(vC \mid V)$$

and hence

$$P(\omega) = P(A \mid V) + P(vC \mid V).$$

We also have (Theorem 0.95 (i))

$$P(vC \mid V) = P(v|C)P(C \mid V),$$

so that

$$(5)\qquad P(\omega) = P(A|V) + P(v|C)P(C|V), \qquad \text{for some } v \in B_n.$$

$$= \frac{P(AV|1)}{P(V|1)} + \frac{P(vC|1)}{P(C|1)} \cdot \frac{P(CV|1)}{P(V|1)} \qquad \text{(by (CP2))}$$

$$= \frac{P(A|1)}{P(V|1)} + \frac{P(vC|1)}{P(C|1)} \cdot \frac{P(C|1)}{P(V|1)}. \qquad (AV = A, CV = C)$$

Dropping the distinction between $\mathfrak{B}_n|1$ and $\mathfrak{B}_n$, we then have

$$(6)\qquad P(\omega) = \frac{P(A)}{P(V)} + c\frac{P(C)}{P(V)}$$

where $c = \dfrac{P(vC)}{P(C)}$ for some $v \in B_n$.

Up to this point no essential use has been made of the assumption (in (CP1)) that $\mathfrak{B}_n|1$ was a Boole probability algebra. Employment of this assumption enables us to follow Boole and write (6) in the form

$$(7)\qquad P(\omega) = \frac{A}{C} + c\frac{C}{V}$$

where A, C, V are now numerical expressions obtained from the correspondingly symbolized logical expressions by considering x_i $(= x_i|1)$ as standing for the probability it has when representing an element of $\mathfrak{B}_n(= \mathfrak{B}_n|1)$, and the logical symbols as symbolizing the like-name arithmetical operations.

5.3. Reprise of Boole's General Problem in Probability

Having formulated in terms of the notion of conditioned event a comprehendable account of Boole's association of probability with expansions involving the four types of constituents, we now return to his method, described in § 4.3, for solving "any" problem in probability. As compared with that in the *Laws of Thought* a later statement of the method (in BOOLE 1854e) shows an improvement in clarity and moreover is "completed" with an additional rule. Here Boole formulates the problem in the following manner (*Boole's General Problem*).

The events whose probabilities are given and the event whose probability is sought are supposed to be expressed as [Boolean] functions of a common set of simple events $x, y, z, \ldots$ Additionally, these events may be subject to purely logical conditions expressed by a set of equations (which can be combined into a single logical equation). Using his rudimentary symbolism Boole writes this as

Probabilities given:

$$\text{Prob. } \varphi(x, y, z \ldots) = p, \quad \text{Prob. } \psi(x, y, z \ldots) = q, \text{ etc.} \tag{1}$$

Annexed absolute conditions:

$$\theta(x, y, z \ldots) = 0, \text{ etc.} \tag{2}$$

Quaesitum, or probability sought:

$$\text{Prob. } F(x, y, z \ldots). \tag{3}$$

Now Boole wishes to express the event whose probability is sought in terms of the events whose probabilities are given. He contends, by virtue of what he calls Principle II in his paper, that one may assume that all these events (i.e. the one whose probability is given and the one whose probability is sought) are simple events provided one "explicitly determines the whole of the relations which implicitly connect them". Accordingly he introduces new logical symbols w, s, t, etc. standing for simple events and puts

$$\varphi(x, y, z \ldots) = s, \quad \psi(x, y, z \ldots) = t, \ \ldots \text{ etc.}$$

$$F(x, y, z \ldots) = w. \tag{4}$$

Combining all the logical equations of (2) and (4) into a single system, Boole eliminates by his method x, y, z, ... and obtains a single equation in $w, s, t, \ldots$ of the form $Ew = G$, E and G containing no occurrence of w. The solution for w is expressed in the form

$$w = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D \tag{5}$$

where A, B, C, D are the sums of constituents on the simple events $s, t, \ldots$

obtained by expanding the quotient G/E. For these four sums Boole give the following interpretation in terms of the occurrence and non-occurrence of events:

> 1st. A represents those combinations of the events s, t, etc. which must happen if w happen.
>
> 2nd. B those combinations which cannot happen if w happen, but may otherwise happen.
>
> 3rd. C those combinations which may or may not happen if w happen.
>
> 4th. D those combinations which cannot happen at all.
>
> And the above representing all possible combinations, we have
>
> $$A+B+C+D = 1. \qquad (6)$$

Under the simplifying assumption that constituents of type C are absent Boole then has that

> ... The event w, then, consists solely of that combination of the simple events s, t, etc. which is denoted by A, and the sole condition to which these events are subject is
>
> $$D = 0, \qquad \text{or} \qquad A+B = 1; \qquad (7)$$
>
> these logical equations being, by virtue of the necessary equation (6), strictly equivalent when C does not make its appearance in the development.
>
> The problem may now be briefly stated as follows: the events s, t, etc. are subject to condition (7) [i.e. they are conditioned events] and at the same time their respective [conditioned by (7)] probabilities are
>
> $$\text{Prob. } s = p, \qquad \text{Prob. } t = q, \text{ etc.}$$
>
> Required the value of Prob. A [conditioned by (7)].

At this stage we see that by eliminating the original simple events x, y, z, ... Boole has converted the problem over to one solely in terms of the introduced simple [unconditioned] events s, t, ... For the problem as thus reformulated Boole gives a concrete illustration in terms of the "familiar notion of an urn containing balls". The urn is to initially contain balls of the kind s, t, etc. so that if a ball with an s-quality is selected out then the event s has happened, and if a ball selected has both s- ant t-qualities then the event st has happened. If nothing is known about the connection of the events s, t, ... then Boole maintains (by what he calls Principle I in his paper) that the events must be taken as independent. But if the events are subject to a condition e.g. $D = 0$, then this is introduced into the situation by supposing that balls of type D are somehow attached by strings to the urn so that none of this kind are ever extracted.

> The general problem may therefore be represented as follows: An urn contains balls whose species are expressed by means of the qualities s, t, etc. and their opposites, concerning the connexion of which qualities nothing is known. Suddenly all balls of the species D are attached by threads to the walls of the urn, and this being done, there is a probability p that any ball drawn is of the species s, a probability q that it is of the species t, and so on. What is the probability that it is of the species A, supposing that A and D denote mutually exclusive species of balls, each defined by means of the properties s, t, and their opposites?

Having taken the prior unconditioned events (i.e. s, t, ... before the nexus) to be independent Boole then establishes that these prior probabilities are subject to

$$\frac{V_s}{V} = p, \qquad \frac{V_t}{V} = q, \qquad \text{etc.} \tag{9}$$

i.e. have, when conditioned by $D = 0$, the probabilities of the events to which s, t, ... have been equated as in (4), and that the probability sought is given by

$$\text{Prob.}\, w = \frac{A}{V} \tag{10}$$

where now the symbols V_s, V_t, V, A, etc. are taken in their numerical significance as algebraic expressions in the probabilities of the events.

Boole states without proof ("It will hereafter be shown,") that the system (9) furnishes one and only one solution when the problem is a *real* one; it is this assertion which Boole wishes to have included as an additional rule to his method as originally stated in the *Laws of Thought*. The values of the prior probabilities s, t, ... determined from (9) are then substituted into (10) to obtain that of w.

In the general case, i.e. when consituents of the $\frac{0}{0}$ type are present, Boole has in place of (10)

$$\text{(11)} \qquad \text{Prob.}\, w = \frac{A + cC}{V} = \frac{A}{V} + c\frac{C}{V},$$

with c being interpreted as a conditional probability, namely

$$c = \frac{\text{Prob.}\, Cw}{\text{Prob.}\, C},$$

which for Boole indicates "the new experience requisite to complete the solution of the problem" (i.e. to have an answer not containing an indefinite term). As in the special case of $\frac{0}{0}$ terms absent, equations (9) are to be solved for the prior probabilities s, t, ... and the results substituted into (11) so as to obtain a value for Prob. w [$=$ Prob. $F(x, y, z \ldots)$] in terms of the given values p, q, ...

A number of questions arise:

(i) Exactly what is the status of the introduced events w, s, t, ...?

(ii) Supposing that the logical relations among the events in the data do imply $Ew = G$, what justification is there for equating Prob. w with the conditional probability value associated with the expansion of G/E as in the preceding section (where we codified Boole's ideas, not justified them)?

(iii) Is Boole's reformulation of the probability problem in terms of the w, s, t, ... equivalent to the original one in terms of x, y, z, ... or even if not equivalent, does it yet imply the correct probability value for the "quaesitum"?

(iv) Is Boole correct in claiming that the problem is a real one if and only if the system (9) has a unique solution for the prior probabilities?

Indirect responses to (i)–(iii) will come out of our results in the next section where we show that Boole's solution to the General Problem is in a certain, though not his, sense correct. Discussion of (iv) is postponed to the end of § 5.6.

§ 5.4. Justification for Boole's solution of the General Problem

Our analysis of Boole's solution of the General Problem is carried out within the framework of the precisely defined notions of § 5.1, none of which depend on the peculiarities of Boole's logical or probability ideas.

We assume that there is a Boolean algebra $\mathfrak{A}$ having $x, y, z, \ldots$ (and hence $\varphi(x, y, z, \ldots)$, $\psi(x, y, z, \ldots)$, $\ldots$ $\chi(x, y, z, \ldots)$, $F(x, y, z, \ldots)$) as elements. Now when Boole's problem posits probabilities (P-values) for φ, ψ, ..., it may or may not be the case that there exists a probability algebra $\langle\mathfrak{A}, P\rangle$ in which φ, ψ, ..., have the posited values. This question, examined by Boole under the topic "conditions of possible experience", is connected with the fourth of the questions raised in the preceding section, for a by a "real problem" Boole means one for which the conditions of possible experience are met—that is that the assignment of probabilities is a possible one. As we are postponing our discussion of this to the next section, we continue under the assumption (for deductive purposes) that there is a probability algebra $\langle\mathfrak{A}, P\rangle$ in which the P-values of φ, ψ, ..., χ are the stated probabilities in the General Problem, and also for which the "absolute" conditions $\theta(x, y, z, \ldots) = 0$ hold.

Note that the data, even if the conditions of possible experience are met, do not necessarily fully determine $\langle\mathfrak{A}, P\rangle$ for all we have is the information that $x, y, z, \ldots$ are elements of $\mathfrak{A}$ and that there are assigned P-values for φ, ψ, ..., χ and this may or may not determine the P-values of all the elements of $\mathfrak{A}$—in particular that of $F(x, y, z, \ldots)$. Accordingly we reformulate the problem as follows: for any $\langle\mathfrak{A}, P\rangle$ meeting the conditions described at the end of the last paragraph what, to the extent to which they are determined, are the possible P-values for $F(x, y, z, \ldots)$? Here in this section we give an answer hewing closely to Boole's ideas, which will turn out to be only a partial solution. In § 5.7 we give another type of answer.

We first distinguish two uses to which Boole (without consciously realizing it) puts the equations

$$(1) \qquad s = \varphi(x, y, z, \ldots),\ t = \psi(x, y, z, \ldots),\ \ldots,\ u = \chi(x, y, z, \ldots),$$
$$w = F(x, y, z, \ldots).$$

In the one he uses these equations to establish the relation obtaining among $s, t, \ldots, u, w$, considering them as elements of $\mathfrak{A}(x, y, z, \ldots)$ which are related to $x, y, z, \ldots$ as specified by these equations. But he also invokes his Principle II to justify treating $s, t, \ldots, u, w$ as simple [unconditioned] events. We interpret this latter use to mean that what is being considered is a mapping h from a free Boolean algebra $\mathfrak{B}(s, t, \ldots, u)$ with free generators $s, t, \ldots, u$ into a Boolean algebra $\mathfrak{A}(x, y, z, \ldots)$, that is that equations (1) (without $w = F$) now means

$$(2) \qquad h(s) = \varphi(x, y, z, \ldots),\ h(t) = \psi(x, y, z, \ldots), \ldots,\ h(u) = \chi(x, y, z, \ldots).$$

(Differing from Boole, we are here dropping mention of w as it will be expressed in terms of $s, t, \ldots, u$.) Although (2) specifies the value of h only for the free generators of $\mathfrak{B}(s, t, \ldots, u)$ any such mapping can be extended, uniquely to a homomorphism of $\mathfrak{B}(s, t, \ldots, u)$ into $\mathfrak{A}(x, y, z, \ldots)$ (See SIKORSKI 1969, 12.2). Assume that h stands for that homomorphism.

We now explore implications of each of the two uses.

Adjoin to (1)—with $s, t, \ldots, u, w$ considered to be variables taking as values elements of $\mathfrak{A}(x, y, z, \ldots)$—the equation $\theta(x, y, z, \ldots) = 0$ (if present in the data). From this conjoined set of equations, i.e.

$$(3) \qquad s = \varphi,\ t = \psi, \ldots,\ u = \chi,\ w = F,\ 0 = \theta,$$

eliminate $x, y, z, \ldots$ and let the resulting equation be written as

$$(4) \qquad Ew = G,$$

with E and G free of w. This represents the necessary condition relating $s, t, \ldots, u, w$ implied by (3) (see end of § 2.6). Let $A + 0B + \frac{0}{0}C + \frac{1}{0}D$ be the complete expansion of the quotient G/E with respect to $s, t, \ldots, u$. By § 2.6, (4) is equivalent to

$$(5) \qquad \begin{cases} w = A(s, t, \ldots, u) + vC(s, t, \ldots, u), \text{ for some } v \in \mathfrak{A} \\ D(s, t, \ldots, u) = 0 \end{cases}$$

and hence (3) implies (5). In this implication replace w, s, ..., u, respectively, by (the formulas in (1) abbreviated to) F, φ, ψ, ..., χ. Then supposing $\theta(x, y, z, \ldots) = 0$ we have from equations (5)

$$(6)\quad \begin{cases} F(x, y, z, \ldots) = A(\varphi, \psi, \ldots, \chi) + vC(\varphi, \psi, \ldots, \chi) \text{ for some } v \in \mathfrak{A} \\ D(\varphi, \psi, \ldots, \chi) = 0 \end{cases}$$

For any probability algebra $\langle \mathfrak{A}, P \rangle$, $\mathfrak{A}$ being the Boolean algebra under consideration, the first equation of (6) and the mutual exclusiveness of A and C imply

$$(7)\quad P(F(x, y, z, \ldots)) = P(A(\varphi, \psi, \ldots, \chi)) + P(vC(\varphi, \psi, \ldots, \chi))$$

$$(8)\quad = P(A(\varphi, \psi, \ldots, \chi)) + cP(C(\varphi, \psi, \ldots, \chi))$$

where $$c = \frac{P(vC(\varphi, \psi, \ldots, \chi))}{P(C(\varphi, \psi, \ldots, \chi))} \quad \text{for some } v \in \mathfrak{A}$$

(assuming, for the form with c, that $P(C) \neq 0$).

We next consider what we are taking as Boole's other use of $s, t, \ldots, u$, namely as simple events (generators) of a free Boolean algebra $\mathfrak{B}(s, t, \ldots, u)$ and that this Boolean algebra is mapped by h (the h determined by (2)) homomorphically into the $\mathfrak{A}(x, y, z, \ldots)$ under consideration. We determine the kernel (set of elements mapped into 0) of h.

Reproducing the above deduction of (5) from (3), but with '$h(s)$', '$h(t)$', ..., '$h(u)$' replacing 's', 't', ..., 'u' we have for the second equation of (5)

$$(9)\quad D(h(s), h(t), \ldots, h(u)) = 0,$$

this being the necessary condition relating $h(s)$, $h(t)$, ..., $h(u)$ (see end of § 2.6). Since h is a homomorphism (9) gives

$$h(D(s, t, \ldots, u)) = 0$$

and $$\forall X(X \subseteq D(s, t, \ldots, u) \rightarrow h(X) = 0).$$

Since (9) is the strongest condition obtainable relating $h(s)$, $h(t)$, ..., $h(u)$, (see end of § 2.6), no element except one contained in $D(s, t, \ldots, u)$ gets mapped into 0. Thus h's kernel is the set $\{X : X \subseteq D(s, t, \ldots, u)\}$.

We are now ready to begin a derivation of what corresponds to Boole's solution of the General Problem.

Let $\mathfrak{A}^*$ be the subalgebra of $\mathfrak{A}$ whose universe is the set of images of elements of $\mathfrak{B}(s, t, \ldots, u)$ under h.

By the Homomorphism Theorem 0.21 $\mathfrak{A}^*$ is isomorphic to $\mathfrak{B}/\equiv_{\theta_h}$, where θ_h is the induced congruence relation on $\mathfrak{B}$ which "identifies" (puts into the same equivalence class) elements of $\mathfrak{B}$ whose images under h are the same. For b_1, $b_2 \in \mathfrak{B}$, we have

$$\begin{aligned} h(b_1) = h(b_2) &\leftrightarrow h(b_1) +_\Delta h(b_2) = 0 && (+_\Delta \text{ is sym. difference}) \\ &\leftrightarrow h(b_1 +_\Delta b_2) = 0 && (h \text{ is a homomorphism}) \\ &\leftrightarrow b_1 +_\Delta b_2 \subseteq D && (h\text{'s kernel is } \{X : X \subseteq D\}) \\ &\leftrightarrow (b_1 +_\Delta b_2)V = 0 && (V = \bar{D}) \\ &\leftrightarrow b_1 V = b_2 V. \end{aligned}$$

Thus we have the isomorphisms:

$$\begin{aligned} \mathfrak{A}^* &\simeq \mathfrak{B}/\equiv_{\theta_h} && && \text{(Theorem 0.21)} \\ &\simeq \mathfrak{B}/(D) && [= \mathfrak{B}/\tfrac{1}{0}D] \\ &\simeq \mathfrak{B}|V && && \text{(the cut-down algebra)} \\ &\simeq \mathfrak{B}|V && [= \mathfrak{B}(V +_B B)^{-1}] && \text{(Theorem 0.62)} \end{aligned}$$

In particular the isomorphic images of φ, ψ, ..., χ are, respectively, $s|V$, $t|V, \ldots, u|V$ since (using '$\approx$' to indicate the relation of isomorphism)

$$\begin{aligned} \varphi &\approx [s] = s + (D) \approx s|V \\ \psi &\approx [t] = t + (D) \approx t|V \\ &\vdots \\ \chi &\approx [u] = u + (D) \approx u|V \end{aligned}$$

Boole's conception of the General Problem involves no specified Boolean algebra and simply considers probability values $p, q, \ldots, r$ for φ, ψ, ..., χ and asks for that of F. To pursue his argument as we are formulating it we need to have defined on $\mathfrak{A}(x, y, z, \ldots)$, assumed to be a Boolean algebra having $\varphi, \psi, \ldots, \chi, F$ as elements, a probability function.

As the method of obtaining this function requires a number of successive steps it might be helpful to outline the procedure in advance: First we specify a probability function P^0 on $\mathfrak{B}(s, t, \ldots, u)$, which then results in having a probability function P^* defined on $\mathfrak{B}|V$. Next P^* is transferred to $\mathfrak{B}|V$'s isomorph $\mathfrak{A}^*$, a subalgebra of $\mathfrak{A}$, and finally this function is transferred from $\mathfrak{A}^*$ to some of $\mathfrak{A}$.

Let $\langle \mathfrak{B}(s, t, \ldots, u), P^0 \rangle$ be a *Boole* probability algebra ($s, t, \ldots, u$ stochastically as well as algebraically independent) such that

$$(10) \qquad \frac{P^0(sV)}{P^0(V)} = p, \quad \frac{P^0(tV)}{P^0(V)} = q, \ldots, \frac{P^0(uV)}{P^0(V)} = r,$$

where $V = \bar{D}$, $D(\neq 1)$ being the $\frac{1}{0}$ part in the expansion of G/E of equation (4). Conditions as to when a probability algebra meeting these requirements exists are discussed in § 5.6. Making use of the stochastic independence of $s, t, \ldots, u$ we obtain from (10) (using Boole's notations)

$$(11) \qquad \frac{[sV]}{[V]} = \frac{V_s}{V} = p, \quad \frac{[tV]}{[V]} = \frac{V_t}{V} = q, \ldots, \frac{[uV]}{[V]} = \frac{V_u}{V} = r,$$

these equations being expressed solely in terms of the "prior" probabilities $P^0(s)$, $P^0(t)$, $\ldots$, $P^0(u)$.

In terms of P^0 we define a probability function P^* on $\mathfrak{B}|V$, namely as a conditional probability with respect to V by setting, for each $b \in \mathfrak{B}$,

$$(12) \qquad P^*(b|V) = \frac{P^0(bV)}{P^0(V)}.$$

By Theorem 0.96, $\langle \mathfrak{B}|V, P^* \rangle$ is a probability algebra. Attributing to an element of $\mathfrak{A}^*$ (defined above) the same P^* value which its isomorphic image in $\mathfrak{B}|V$ has, we obtain a probability algebra $\langle \mathfrak{A}^*, P^* \rangle$ isomorphic to $\langle \mathfrak{B}|V, P^* \rangle$. Note that $\varphi, \psi, \ldots, \chi$, being elements of $\mathfrak{A}^*$, have P^* values which, by (10), are the same as their hypothesized P values. Replacing 'P' by 'P^*' in the right in (8), and assuming the formulas refer to $\langle A^*, P^* \rangle$[1], we have

[1] Thus, in effect, specifying additional properties of P besides the given values on $\varphi, \psi, \ldots, \chi$; for it is not true that for *any* P such that $P(\varphi) = P^*(\varphi)$, $P(\psi) = P^*(\psi), \ldots$, is it the case that $P(A(\varphi, \psi, \ldots, \chi)) = P^*(A(\varphi, \psi, \ldots, \chi))$. A sufficient condition is that $\varphi, \psi, \ldots, \chi$ be mutually independent (with respect to P and P^*).

$$(13) \qquad P(F(x,y,z,\ldots)) = P^*(A(\varphi,\psi,\ldots,\chi)) + cP^*(C(\varphi,\psi,\ldots,\chi))$$

where
$$c = \frac{P(vC(\varphi,\psi,\ldots,\chi))}{P^*(C(\varphi,\psi,\ldots,\chi))} \quad \text{for some } v \in \mathfrak{A}.$$

(Note the absence of the asterisk on the P in the numerator—though we have $C \in \mathfrak{A}^*$, the product vC need not be in $\mathfrak{A}^*$.) Making use of the isometry of $\langle \mathfrak{B} | V, P^* \rangle$ and $\langle \mathfrak{A}^*, P^* \rangle$ we can, on replacing $\varphi, \psi, \ldots, \chi$ by their isomorphic images, rewrite (13) as

$$(14) \qquad P(F(x,y,z,\ldots)) = P^*(A(s|V, t|V, \ldots, u|V)) + cP^*(C(s|V, t|V, \ldots, u|V)).$$

By virtue of the fact that the mapping $b \to b|V$ is a homomorphism (14) can be converted to

$$P(F(x,y,z,\ldots)) = P^*(A(s,t,\ldots,u)|V) + cP^*(C(s,t,\ldots,u)|V).$$

Using the definition of P^* as a conditioned probability and also that $AV = A$, $CV = C$, we obtain from this

$$(15) \qquad P(F(x,y,z,\ldots)) = \frac{P^0(A(s,t,\ldots,u)V)}{P^0(V)} + c\frac{P^0(C(s,t,\ldots,u)V)}{P^0(V)}$$
$$= \frac{P^0(A(s,t,\ldots,u))}{P^0(V)} + c\frac{P^0(C(s,t,\ldots,u))}{P^0(V)}$$

Since $\langle \mathfrak{B}(s,t,\ldots,u), P^0 \rangle$ is a *Boole* probability algebra, we may convert (15) to

$$(16) \qquad P(F(x,y,z,\ldots)) = \frac{[A(s,t,\ldots,u)]}{[V]} + c\frac{[C(s,t,\ldots,u)]}{[V]},$$

or in Boole's notation

$$P(F(x,y,z,\ldots)) = \frac{A}{V} + c\frac{C}{V}.$$

Note that on the right hand side of (16), apart from c, the only probabilities appearing are $P^0(s)$, $P^0(t)$, …, $P^0(u)$. If now equations (11) are solvable for the $P^0(s)$, $P^0(t)$, …, $P^0(u)$—this question will be examined in § 5.6—the results could be substituted into (16) to obtain the sought for solution of the General Problem. This is essentially Boole's answer to the General Problem.

One notes the astonishing feature of this solution, namely that the only occurrence of 'P' on the right-hand side of (16) is in the term c. If, with Boole, we take c as un undetermined [conditional] probability, then the result for $P(F(x, y, z, \ldots))$ implied by (16) and (10) does not depend on any knowledge of P except for the given values on $\varphi, \psi, \ldots, \chi$.

With this result we believe we have established the rationale for Boole's solution to the General Problem—as a partial answer. We differ with him in his belief that the expression he gives for $P(F(x, y, z, \ldots))$ in terms of $p, q, \ldots, r$ (when equations (11) can be explicitly solved) is the only one meeting the conditions of the problem. Note that in the course of the demonstration it was premised, or tacitly assumed, that $\langle \mathfrak{A}^*, P^* \rangle = \langle \mathfrak{A}^*, P \rangle$. This corresponds to Boole's adoption of his Principle II (so called in his [1854e]), or Principle VI in *Laws of Thought* (p. 256), against which as a necessary principle of probability we have argued in § 4.9, Note 1 for § 4.2. It will be noted that in Boole's examples Ex. 1 and Ex. 2 (discussed above in § 4.5), stochastic independence of what corresponds to $s, t, \ldots, u$ plays no role, and Boole's solution agrees with the correct one. However in his Ex. 4 one obtains his solution from the correct (full) one only by adding the assumption that $s(= x)$, $t(= y)$, and $u(= \bar{x}\bar{y}\bar{z})$ are independent (see § 4.9, Note 3 for § 4.6). Wilbraham's criticism (§ 4.8) did bring out that in particular cases Boole's "determinate" solution can be obtained by "common" methods provided additional independence assumptions are adjoined to the data. He did not, however, analyse Boole's method sufficiently so as to show exactly how and why these came in.

We return to the questions asked at the end of the preceding section, couching our responses in terms of the analysis of this section.

With regard to (i) concerning the status or nature of the events $s, t, \ldots, u$, one notes that Boole's views underwent some change. E.g. in 1857 he says:

> 10. I postulate that when the data are not the probabilities of simple events, we must, in order to apply them to the calculation of probability, regard them not as primary, but as derived from

> some anterior hypothesis, which presents the probabilities of simple events as its system of data, and exhibits our actual data as flowing out of that system, in accordance with those principles which have already been shown to be involved in the very definition of probability. [1952, p. 315]

But then later in 1862 [his last paper on his probability method] he writes:

> I have but one further observation on Principle II. to make. It is that in the general problem we are not called upon to interpret the ideal events. The whole procedure is, like every other procedure of abstract thought, formal. We do not say that the ideal events exist, but that the events in the translated form of the actual problem are to be considered to have such relations with respect to happening or not happening as a certain system of ideal events would have if conceived first as free, and then subjected, without their freedom being otherwise affected, to relations formally agreeing with those to which the events in the translated problem are subject. [1952, p. 391]

Our analysis has brought out that a dual usage is involved—on the one hand 's', 't', ..., 'u' symbolize the given events of the data, i.e. elements of $\mathfrak{A}(x, y, \ldots, z)$, and on the other they stand for stochastically independent events of the *introduced* auxiliary algebra $\langle \mathfrak{B}(s, t, \ldots, u), P \rangle$; in this latter use they are the "ideal" elements of Boole's later conception, whose employment was justified (in his view) by the adoption of a new principle.

Question (ii) concerns Boole's attribution of the conditional probability $A/V + cC/V$ to "Prob. w", where w is given by $Ew = G$, E and G functions of s, t, ..., u. We know (§ 2.6) that the equation for w is equivalent to

$$(17) \qquad \begin{cases} A \subseteq w \subseteq A + C \\ D = 0 \quad (V = 1) \end{cases}$$

However, there is no way of assigning probabilities to stochastically independent s, t, ..., u (in $\langle \mathfrak{B}(s, t, \ldots, u), P \rangle$) so as to accord with (17) (save if D is the empty sum of constituents on s, t, ..., u). What Boole does is to consider $s, t, \ldots, u$ as events conditioned by $D = 0$, in effect going over to $\langle \mathfrak{B} | V, P \rangle$, and assigns probabilities to the [conditioned] events $s, t, \ldots, u$ to accord with the data. Boole never really made it clear why $P(s|V)$ should equal $P(\varphi(x, y, \ldots, z))$ when it is s that is equal to $\varphi(x, y, \ldots, z)$. Our analysis shows this by using the homomorphism $h(s) = \varphi(x, y, \ldots, z)$ from $\mathfrak{B}(s, t, \ldots, u)$ into $\mathfrak{A}(x, y, \ldots, z)$ and (ignoring the distinction between P and P^*) the isometry of $\langle \mathfrak{B} | V, P \rangle$ and $\langle \mathfrak{A}^*, P \rangle$. Assuming that one has a $\langle \mathfrak{B}(s, t, \ldots, u), P \rangle$ we obtain from (17)

$$\frac{P(AV)}{P(V)} \leq \frac{P(wV)}{P(V)} \leq \frac{P((A+C)V)}{P(V)}$$

i.e., in $\langle \mathfrak{B} | V, P \rangle$

$$P(w|V) = P(A|V) + cP(C|V) \qquad c \in [0, 1]$$

so that, as in § 5.2,

$$P(w|V) = \frac{A}{V} + c\frac{C}{V}$$

with A, C, V now having numerical significance and s, t, $\ldots, u$ representing (unconditioned) probabilities of $\langle \mathfrak{B}(s, t, \ldots, u), P \rangle$.

Re (iii). Wilbraham's example, discussed in § 4.8, implicitly shows that Boole's reformulation of the General Problem in terms of the ideal events can't be in general an equivalent one. (A simpler example is present in our discussion, in § 4.6 plus Note 3, of Boole's Ex. 4.) The core of Boole's error in thinking that it was general is his belief that (to put it in terms of our version) the representation of the data in the probability algebra $\langle \mathfrak{B} | V, P \rangle$ necessarily captures all the probability relationship that exists connecting the probabilities of φ, ψ, $\ldots, \chi$ and F. It so happens that under appropriate circumstances one can find a unique Boole probability algebra $\langle \mathfrak{B}(s, t, \ldots, u), P \rangle$ for which $\langle \mathfrak{B} | V, P \rangle$ is isometric to $\langle \mathfrak{A}^*, P \rangle$, but Boole never showed that there couldn't be any other probability algebra in which one could represent the data, e.g. one having perhaps more generators and not necessarily stochastically independent. As our correct (full) solution of Boole's Ex. 4 shows, there can be a range of values for $P(F(x, y, \ldots, z))$ meeting the conditions of the problem not all of which are included in Boole's form even with the indefinite c ranging from 0 to 1.

The next sections are devoted to an examination of argument that the equations (11) for the "prior" or "absolute" probabilities of $s, t, \ldots, u$ have a unique solution if and only if the problem is a "real" one. In the first of these sections we interpret this notion and give a necessary and sufficient condition for a problem to be, in Boole's sense, a real one.

§ 5.5. Conditions of possible experience—a consistency (solvability problem)

We continue our discussion of this topic from § 4.7. Boole's most thoroughgoing exposition of this topic—already referred to in § 4.7—is in his paper "On the Conditions by which the Solutions of Questions in the Theory of Probabilities are Limited" [1854c = 1952, XIII] where the problem is expressed as follows:

> The probabilities of any event whose logical expression is known being represented by p, q, r, ... respectively, required the conditions to which these quantities are subject.

Our first task is to rephrase this in precise terms. We do this first for the simplest situation and then later indicate extensions.

Problem on Conditions of Possible Experience (Simple Case)

Given n Boolean functions $\varphi_1, \ldots, \varphi_n$ on m arguments $x_1, \ldots, x_m$, determine conditions in terms of the parameters $p_1, \ldots, p_n$ so that the system of equations

$$(1)\qquad \begin{aligned} P(\varphi_1(x_1, \ldots, x_m)) &= p_1 \\ P(\varphi_2(x_1, \ldots, x_m)) &= p_2 \\ &\vdots \\ P(\varphi_n(x_1, \ldots, x_m)) &= p_n \end{aligned}$$

shall be consistent in the sense of having a solution for the $x_1, \ldots, x_m$ in some probability algebra $\langle \mathfrak{A}, P \rangle$.

The conditions on $p_1, \ldots, p_n$ for which consistency holds are the *conditions of possible experience* for the problem. Boole considers a problem to be a "real" one only if the data, assumed to be of the form (1), meet the conditions of possible experience.

Following Boole's example we convert this conditions-of-possible-experience problem over to a problem on the consistency (solvability) of a linear system. Let $C_1, \ldots, C_{2^m}$ be a listing of the 2^m consituents on letters $x_1, \ldots, x_m$ and let $c_{ij} = 1$ if the j-th one of the constituents is present in the expansion of the Boolean function φ_i and otherwise let $c_{ij} = 0$. In terms of 2^m variables $\lambda_1, \ldots, \lambda_{2^m}$ we write a linear system

$$\text{(2)} \qquad \begin{aligned} c_{i1}\lambda_1 + \cdots + c_{i2^m}\lambda_{2^m} &= p_i \qquad (i = 1, \ldots, n), \\ \lambda_j &\geq 0 \qquad (j = 1, \ldots, 2^m) \\ \sum_{1 \leq j \leq 2^m} \lambda_j &= 1, \end{aligned}$$

where the first n equations correspond to the n equations of (1) and the last two conditions are obvious normative conditions. It is readily seen that the consistency of (2) is a necessary condition for that of (1)—for if there is a solution for $x_1, \ldots, x_m$ in some probability algebra $\langle \mathfrak{A}, P \rangle$ then to have λ's satisfying (2) we merely take $\lambda_j = P(C_j)$. Does the converse condition hold, i.e. is consistency of (2) a sufficient condition for that of (1)? Boole says nothing explicit about this.

For us, that the consistency of (2) is a sufficient condition for that of (1) follows from our Theorem 0.94. By that theorem if $\lambda_1, \ldots, \lambda_{2^m}$ are any set of numbers between 0 and 1 which add up to 1, then there is a probability algebra $\langle \mathfrak{A}, P \rangle$ and 2^m events which are the constituents on events $E_1, \ldots, E_m$ in $\mathfrak{A}$ whose probabilities are respectively the λ_j's. Thus, if in addition the λ_j's satisfy

$$c_{i1}\lambda_1 + \cdots + c_{i2^m}\lambda_{2^m} = p_i \qquad (i = 1, \ldots, n)$$

it readily follows that

$$P(\varphi_i(E_1, \ldots, E_m)) = p_i \qquad (i = 1, \ldots, n).$$

By using the short-hand '$\exists\langle \mathfrak{A}, P \rangle$' for 'There is a probability algebra $\langle \mathfrak{A}, P \rangle$ and events $x_1, \ldots, x_m$ in $\mathfrak{A}$ such that' and '$L(\lambda, p)$' as short for the conjunction of conditions listed in (2), the equivalence just discussed can be written:

$$\exists\langle\mathfrak{A}, P\rangle \bigwedge_{i=1}^{n} [P(\varphi_i(x_1, \ldots, x_m)) = p_i] \leftrightarrow \exists\lambda_1, \ldots, \lambda_{2^m} L(\lambda, p).$$

Having established that the Problem on Conditions of Possible Experience is equivalent to one on solvability of a linear system, we turn our attention to this equivalent formulation. The question of solvability of such systems has been discussed in our § 0.7—we can apply Fourier elimination to such a linear system, as Boole does, and obtain necessary and sufficient conditions that the linear system have a solution. The procedure of the demonstration of Theorem 0.71 applied to the system (2) with parameters $p_1, \ldots, p_n$ would result in a series of linear inequalities (or equalities) relating these parameters; and the relations so obtained are exactly Boole's "conditions of possible experience".

The conditions of possible experience were also used by Boole to obtain bounds on probabilities. In such a system of inequalities—whose parameters we now wish to call $p_1, \ldots, p_n, w$—any one of the parameters, say w, can be treated as an unknown and be solved for in each linear relation of the system. In general the solutions for w will be of the form

$$R_i(p_1, \ldots, p_n) \leq w \qquad (i = 1, \ldots, r)$$

and

$$w \leq S_j(p_1, \ldots, p_n) \qquad (j = 1, \ldots, s)$$

which may be combined into the equivalent form

$$\max_i (R_i) \leq w \leq \min_j (S_j),$$

giving, as with Boole, *possible bounds* on the value of w.

Some of Boole's examples (e.g. Ex. 5 in our § 4.6) have data referring to conditional probabilities, e.g. $P(\varphi|\psi) = p$. If this were treated as

$$\text{(4)} \qquad \frac{P(\varphi\psi)}{P(\psi)} = p$$

and the probabilities of the constituents in numerator and denominator were replaced by λ's (as above), then a nonlinear expression in the λ's results. Or, if written in the form

(5) $$P(\varphi\psi) - pP(\psi) = 0,$$

then the parameter p appears in the coefficient(s) of the λ's and not in the constant term position. Boole makes no mention of this, circumspectly avoids either alternative, and introduces a new parameter, e.g. c, and replaces a datum (4) by a pair of equations

(6) $$P(\varphi\psi) = cp \qquad P(\psi) = c,$$

which fundamentally alters the nature of the datum from a one-parameter to a two-parameter one.

Of more significance is Boole's treatment of conditional probabilities in connection with his solution of the General Problem. For data in the form (1) we have seen that the conditions of possible experience is a system of inequalities linear in $p_1, \ldots, p_n$. In his solution of the General Problem Boole has a translated form for the data which he asserts leads to the same system. Recall that Boole introduces new variables $s_1, \ldots, s_n$ (as we shall call them here), setting

$$s_1 = \varphi_1(x_1, \ldots, x_m), s_2 = \varphi_2(x_1, \ldots, x_m), \ldots, s_n = \varphi_n(x_1, \ldots, x_m)$$

and in terms of the new symbols the data are expressed (as in (10) of § 5.4) by

(7) $$\frac{P(Vs_1)}{P(V)} = p_1, \quad \frac{P(Vs_2)}{P(V)} = p_2, \ldots, \frac{P(Vs_n)}{P(V)} = p_n.$$

It is Boole's contention that an instance of the General Problem is a real one if and only if the conditions of experience for (7) are satisfied, and he observes that these conditions are the same as for the original data (as given e.g. in (1)) [1952, p. 380, and also p. 398]. But what he means by the conditions of experience here is not (expressing it in our formulation) with reference to a probability algebra $\langle \mathfrak{B}(s_1, \ldots, s_n), P \rangle$ but to a conditioned-events probability algebra $\langle \mathfrak{B} | V, P \rangle$. For he expresses the left-hand sides of the equations in (7) not in terms of the probabilities of constituents on $s_1, \ldots, s_n$ but on the ratios of these probabilities to that of V—each left-hand side of the equations then becoming a sum of the form $\sum v_j$, where the v_j are the described quotients of probabilities (conditional probabilities). The resulting equations are linear in the v's. Adjoining the usual normative conditions and eliminating the v's results in a set of inequations linear in $p_1, \ldots, p_n$. That these conditions of experience are the same as for the original data was "*a priori* evident" to Boole, though he does illustrate the result with an example which shows the relationship between the v's and the λ's. We present a brief general argument using notions of the preceding section.

Referring to (7), let $v_1, \ldots, v_e$ $(e \leq 2^m)$ be all the quotients of the described kind corresponding to constituents appearing anywhere in $Vs_1, \ldots, Vs_n$. Rewriting (7) as

$$(8) \qquad P(s_1 | V) = p_1, P(s_2 | V) = p_2, \ldots, P(s_n | V) = p_n$$

and with

$$P(s_i | V) = \sum\nolimits^{(i)} v_j \qquad (i = 1, \ldots, n)$$

we see that we have a system of equations in $\langle \mathfrak{B} | V, P\rangle$ of the same nature as (1) is for $\langle \mathfrak{A}, P\rangle$. Our earlier argument now leads to the equivalence

$$(9) \qquad \exists\langle \mathfrak{B} | V, P\rangle[\ \wedge_{i=1}^{n} P(s_i | V) = p_i] \\ \leftrightarrow \exists v_1 \ldots v_e N(v, p)$$

where $N(v, p)$ is the linear system for the v's corresponding to (2) [but with $\sum_{j=1}^{e} v_j \leq 1$ in place of $\sum_{j=1}^{2^m} \lambda_j = 1$]. We also have the equivalents:

$$\exists\langle \mathfrak{B} | V, P\rangle[\wedge_{i=1}^{n} P(s_i | V) = p_i] \\ \leftrightarrow \exists\langle \mathfrak{A}^*, P\rangle[\wedge_{i=1}^{n} P(\varphi_i) = p_i] \\ \leftrightarrow \exists\langle \mathfrak{A}, P\rangle[\wedge_{i=1}^{n} P(\varphi_i(x_1, \ldots, x_m) = p_i],$$

the first of these by virtue of the isometry of a $\langle \mathfrak{B} | V, P\rangle$ and its related $\langle \mathfrak{A}^*, P\rangle$ as described in the preceding section, while the second becomes apparent if one thinks of $\varphi_1, \ldots, \varphi_n$ as fundamental regions in a Venn diagram (for $\mathfrak{A}^*$) which are then subdivided so as to be $\varphi_1(x_1, \ldots, x_m)$, $\ldots, \varphi_n(x_1, \ldots, x_m)$ (for $\mathfrak{A}$) with $x_1, \ldots, x_m$ as the fundamental regions.

We conclude this section with a remark on the narrowness of Boole's view of what a probability problem could be. In his formulation of the General Problem it is assumed that the data are given in terms of equations (as in (1)). We see no reason why a datum couldn't be of the form

$$P(\varphi(x_1, \ldots, x_m))\rho p,$$

where ρ is any one of the order relations

$$<, \quad \leq, \quad >, \quad \geq$$

(equality can be omitted since a statement of the form $a = b$ is expressible as a conjunction $a \leq b$ and $b \leq a$), or even more generally as

$$f(P(\varphi_1), P(\varphi_2), \ldots, P(\varphi_n))\rho p \tag{11}$$

(φ_i abbreviating $\varphi_i(x_1, \ldots, x_m)$) for an n-ary function f. In the case of functions f which are linear in their arguments we can readily show that Fourier elimination can be applied to a system of relations of the form (11) in terms of the λ's (probabilities of constituents on $x_1, \ldots, x_m$), resulting in a system of order relations between linear combinations of the parameters $p_1, \ldots, p_n$, thus considerably generalizing Boole's concept of conditions of possible experience.

§ 5.6. The problem of absolute probabilities II

Our initial presentation of this topic in § 4.3 adhered closely to the original one given in the *Laws of Thought* and we there in § 4.3 deliberately disregarded much of Boole's later ideas on the subject. It is clear that in the *Laws of Thought* he had not yet foreseen the difficulties involved, for in Chapter XVII, when he first comes out with the equations for determining the absolute probabilities, he merely states: "from which equations equal in number to the quantities $p', q', r', \ldots$ [i.e. the absolute probabilities sought] the values of these quantities may be determined". Throughout the book the equations are solved on an *ad hoc* basis for each individual problem. Subsequently Boole realized the need for a general proof of the existence and uniqueness of solutions for these equations so as to justify the general applicability of his method.

In an article appearing in *The Philosophical Magazine* (Boole 1854e = Boole 1952, XV) he announces that the rules for his probability method given in the *Laws of Thought* are in need of an addition, namely by a necessary and sufficient condition which he states as follows:

> If the problem be a real one, the system (I.) [the equations for the absolute probabilities] will furnish one set, and only one set, of positive fractional values of s, t, etc., which substituted in (II.) [i.e. the equation Prob. $w = (A + cC)/V$] will determine Prob. w.
>
> If the system (I.) does not furnish a single system of positive fractional values of s, t, etc., the problem is not a real one, and does not in its statement represent a possible experience.

At the end of this paper he says: "The verification of these results will be considered in my next paper". The topic is resumed in Boole 1855 in which the core of the mathematical difficulties is brought out, but no successful resolution. There is further discussion in the appendix of his Keith Prize memoir (Boole 1857 = Boole 1952, XVI). Finally in Boole 1862 (= Boole 1952, XVII) he claims to have overcome the "analytic difficulties" which impeded him and we have the promised general demonstration. The present section is devoted to a detailed examination of Boole's proof.

Recall that for Boole's general problem in probability, when transformed so as to be in terms of the "ideal" events, $s, t, \ldots$, one needs to determine the solution of a system of algebraic equations

$$\frac{V_s}{V} = p, \qquad \frac{V_t}{V} = q, \quad \ldots, \quad \text{etc.}$$

where V is a rational integral form in products of $s, t, \ldots \bar{s}, \bar{t}, \ldots$ (i.e. algebraic products homonymous with the constituents of the logical V), and where V_s is the sum of those terms of V having s as a factor, V_t those having t as a factor, etc. Boole converts these equations to a standard form by dividing, on the left sides, numerator and denominator by the product $\bar{s}\bar{t}, \ldots$, which doesn't appear in any of the numerators, replacing $s/\bar{s}$ by $x_1, t/\bar{t}$ by $x_2, \ldots$ and renaming $p, q, r, \ldots$ to $p_1, p_2, p_3, \ldots$ so that the equations appear as

$$(1) \qquad \frac{V_1}{V} = p_1, \quad \frac{V_2}{V} = p_2, \quad \ldots, \quad \frac{V_n}{V} = p_n,$$

where now V is a rational integral form in the variables $x_1, \ldots, x_n$, no variable appearing with exponent other than 1, and where V_i $(i = 1, \ldots, n)$ is the sum of the terms in V having x_i present. As originating from a logical problem the coefficients of any term present in V is 1, but in what follows, for the purposes of his proof, Boole allows these coefficients to have any positive value. Since the original variables $s, t, \ldots$ range from 0 to 1, the x_i will then range from 0 to ∞.

Boole's paper then continues with a purely algebraic result to the effect that a determinant of a certain specified kind can have in its expansion [if non-empty] only positive terms. A statement and proof of this result we have placed in our § 0.10 (Theorem 0.101). As an immediate particular application of this result Boole has

PROPOSITION II

> If V be any rational entire function of the n variables $x_1, x_2, \ldots, \ldots, x_n$, but involving no powers of those variables above the first, and if, further, all the different terms of V have positive signs, then the determinant
>
> $$\begin{vmatrix} V & V_1 & V_2 & \cdots & V_n \\ V_1 & V_1 & V_{12} & \cdots & V_{1n} \\ V_2 & V_{21} & V_2 & \cdots & V_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ V_n & V_{n1} & V_{n2} & \cdots & V_n \end{vmatrix}$$
>
> in which V_i denotes the sum of the terms in V which contain x_i, and V_{ij} the sum of the terms in V which contain x_i, x_j, will, when developed as a rational and entire function of $x_1, x_2, \ldots, x_n$, consist wholly of terms with positive coefficients.

As we have already remarked in § 0.10 it is a simple matter to verify that, under the hypotheses of this Proposition, the conditions (i)–(iii) of

Theorem 0.101 hold and hence the displayed determinant can't have negative terms in its expansion (as a rational integral function of $x_1, \ldots, x_n$). However Boole overlooks that it could very well be 0, as for example in the case of $V = x_1x_2$, $V_1 = V_2 = V_{12} = V_{21} = x_1x_2$. When V is "complete in form" (having all possible 2^n terms present), Boole finds the determinant impressive:

> The rapidity with which the complexity of the determinant increases as the number of variables increases, is remarkable. For example, if $n = 2$ and $V = axy + bx + cy + d$, the determinant is
>
> $$\begin{vmatrix} axy + bx + cy + d & axy + bx & axy + cy \\ axy + bx & axy + bx & axy \\ axy + cy & axy & acy + cy \end{vmatrix};$$
>
> and its calculated value will be found to be
>
> $$abcx^2y^2 + ab\,dx^2y + ac\,dxy^2 + bc\,dxy,$$
>
> consisting of four positive terms.
>
> But if $n = 3$ and
>
> $$V = axyz + byz + cxz + dxy + ex + fy + gz + h,$$
>
> the developed determinant will consist of fifty eight positive terms. Its calculated value will be found in the Memoir on Testimonies and Judgements.

While it is true that these two computed examples of Boole's show that the determinant of Proposition II is not identically 0 for a V complete in form, Boole gives no proof for general n and tacitly assumes that it is true for such V's (as well as for others). However, in § 0.10 we have already seen that this is indeed the case:

REMARK (to Proposition II). *If V is complete in form then the determinant is not identically zero and is positive for positive values of x_1, $x_2, \ldots, x_n$.*

Boole first turns his attention to establishing his principal result for the case of a V complete in form:

PROPOSITION III

The functions $V, V_1, V_2, \ldots, V_n$ being defined as above, if V be complete in form, i.e. if it consist of all the terms which according to definition it can contain, each with a positive coefficient, then the system of equations

$$\frac{V_1}{V} = p_1, \qquad \frac{V_2}{V} = p_2, \qquad \ldots, \qquad \frac{V_n}{V} = p_n, \qquad \ldots (1.)$$

will, when $p_1, p_2, \ldots, p_n$ are [positive] proper fractions, admit of one solution, and only one, in positive values of $x_1, x_2, \ldots, x_n$.

The proof, by induction on n, is quite ingenious. We follow Boole's demonstration, supplementing or emending it where needed to accord with present-day standards of rigor.

For $n = 1$ we have $V = ax_1 + b$ with $a > 0$ and $b > 0$, and the single equation

$$\frac{V_1}{V} = \frac{ax_1}{ax_1 + b} = p_1$$

has the unique positive solution

$$x_1 = \frac{bp_1}{a(1 - p_1)};$$

and here $x_1 > 0$ since $0 < p_1 < 1$. (Boole isn't always clear about whether he is taking 0 to be a positive number or not. We shall exclude it, i.e. "positive" shall mean strictly greater than 0. Note that $p_1 = 1$ is excluded, otherwise x_1 would be undefined.) Assuming as hypothesis of induction that the proposition holds for V's with $n - 1$ variables, Boole proposes to establish it for a V with n variables by showing that as x_1 varies from 0 to ∞, with the values of

$$x_2 \quad [= x_2(x_1)], \quad \ldots, \quad x_n \quad [= x_n(x_1)],$$

being determined by hypothesis of induction as unique positive values satisfying the last $n - 1$ equations of (1.), the value of

$$\frac{V_1}{V} \quad [= F(x_1, x_2(x_1), \ldots, x_n(x_1))]$$

varies continuously and monotonically from 0 to 1 and hence takes on the value p_1 exactly once in the course of its variation. We take a close look at the details of this part of the proof.

It is a noteworthy fact, and a circumstance which Boole takes advantage of, that for any $x_1 \geq 0$ the last $n - 1$ equations of (1.) are again of the form (1.) for a V with $n - 1$ variables and with the hypotheses of the proposition holding. Hence, by the inductive hypothesis, there is for each such value of x_1 a unique set of positive values for $x_2, \ldots, x_n$ which satisfy the last $n - 1$ equations of (1.). To indicate the dependence of the thus defined functions on x_1 we denote them by

$$x_2(x_1), \ldots, x_n(x_1).$$

When these are substituted for $x_2, \ldots, x_n$ into the left-hand side of the first equation in (1.) we obtain for V_1/V the function

$$(2) \quad \frac{A(x_2(x_1), \ldots, x_n(x_1))x_1}{A(x_2(x_1), \ldots, x_n(x_1))x_1 + B(x_2(x_1), \ldots, x_n(x_1))} \quad \left[= \frac{Ax_1}{Ax_1 + B} \right].$$

From the positiveness of $x_2(x_1), \ldots, x_n(x_1)$ and the nature of V_1 and V we see that A is non-negative and B is positive (even for $x_1 = 0$). Abbreviating the expression in (2) by

$$F(x_1, x_2(x_1), \ldots, x_n(x_1))$$

or, even briefer, by $\Phi(x_1)$, we see that for $x_1 > 0$ the value of $\Phi(x_1)$ lies between 0 and 1 and that on substituting 0 for x_1, we obtain

$$F(0, x_2(0), \ldots, x_n(0)) = \Phi(0) = 0,$$

except if $x_2(0) = x_3(0) = \cdots = x_n(0) = 0$, for then (2) reduces to 0/0 and determines no value for x_1. This could happen if $p_2 = p_3 = \cdots = p_n = 0$. Validation of this step in the proof requires interpreting, in Proposition III, "positive" as meaning greater than 0 and "proper fraction" as excluding 0 and 1. Boole now repeats the argument for $x_1 = \infty$ and concludes that $\Phi(\infty) = 1$. For a 20-th century mathematician for whom the last $n - 1$ equations are meaningless when "$x_1 = \infty$" the argument lacks cogency. Actually all one needs to have is

$$\lim_{x_1 \to \infty} \Phi(x_1) = 1$$

and this does follow—looking at (2) we know from the form of V that term for term $B(x_2,\ldots,x_n)$ is algebraically of the same degree as $A(x_2,\ldots,x_n)$ and each corresponding term with the same variables, differing only in the coefficients, hence the values of the fraction

$$\frac{B}{A}=\frac{B(x_2(x_1),\ldots,x_n(x_1))}{A(x_2(x_1),\ldots,x_n(x_1))}$$

remains bounded as $x_1\to\infty$, so that the quotient

$$\frac{Ax_1}{Ax_1+B}=\frac{x_1}{x_1+(B/A)}$$

has limit 1.

Boole next asserts: "It is manifest too, that it [i.e. V_1/V with $x_2,\ldots,x_n$ determined by the last $n-1$ equations for $x_1\geq 0$] varies continuously". This may be manifest to Boole, but we think it should have more justification than this. What is manifest is that for values of $x_1,\ldots,x_n$ in the positive range $F(x_1,x_2,\ldots,x_n)$ is continuous in the indicated arguments and hence if in addition the positive functions $x_2(x_1),\ldots,x_n(x_1)$ are continuous then so is $\Phi(x_1)=F(x_1,x_2(x_1),\ldots,x_n(x_1))$. Justification for the continuity of $x_2(x_1),\ldots,x_n(x_1)$ comes by virtue of the implicit function theorem (§ 0.10) and requires that the system have a non-vanishing Jacobian. Calculation of the Jacobian is simplified by introducing an auxiliary variable t and adding the equation $V-e^{-t}=0$, so that our system becomes

$$\begin{aligned}&V-e^{-t}=0\\&V_2-p_2e^{-t}=0\\(3)\qquad&\quad\vdots\\&V_n-p_ne^{-t}=0,\end{aligned}$$

implicitly defining $x_2,\ldots,x_n,t$ as functions of x_1 by the inductive hypothesis (which, while it gives us the functions, does not give us the continuity of these functions). Now the functions on the left-hand side of the equations in (3) are certainly of class C' (continuous derivative) so that if the Jacobian is $\neq 0$ for $x_1>0$ we can conclude by the implicit function theorem that $x_2(x_1),\ldots,x_n(x_1),t(x_1)$ are continuous (indeed of class C'). Making use of the fact that

$$(4)\qquad \frac{\partial V}{\partial x_i}=\frac{V_i}{x_i},\qquad \frac{\partial V_i}{\partial x_i}=\frac{V_i}{x_i},\qquad \frac{\partial V_j}{\partial x_i}=\frac{V_{ji}}{x_i},$$

we find the Jacobian to be

$$\begin{vmatrix} e^{-t} & \frac{V_2}{x_2} & \frac{V_3}{x_3} & \cdots & \frac{V_n}{x_n} \\ p_2 e^{-t} & \frac{V_2}{x_2} & \frac{V_{23}}{x_3} & \cdots & \frac{V_{2n}}{x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ p_n e^{-t} & \frac{V_{n2}}{x_2} & \frac{V_{n3}}{x_3} & \cdots & \frac{V_n}{x_n} \end{vmatrix}$$

which, on making use of (3) and taking out columnar factors, is

$$(5)\qquad \frac{1}{x_2x_3\cdots x_n}\begin{vmatrix} V & V_2 & V_3 & \cdots & V_n \\ V_2 & V_2 & V_{23} & \cdots & V_{2n} \\ \vdots & \vdots & & & \vdots \\ V_n & V_{n2} & \cdots & & V_n \end{vmatrix}$$

and is $\neq 0$ by Proposition II. and the Remark following it, inasmuch as for fixed x_1 the expression V is complete in form with respect to $x_2, \ldots, x_n$. Thus with the continuity of $x_2(x_1), \ldots, x_n(x_1)$ for $x_1 > 0$ established that of

$$\frac{V_1}{V} = F(x_1, x_2(x_1), \ldots, x_n(x_1)) = \Phi(x_1)$$

is assured; moreover, from the implicit function theorem it also follows that V_1/V has a continuous derivative. Returning to Boole's demonstration:

> If then it [i.e. V_1/V] vary by continuous *increase*, it will once, and only once in its change, become equal to p_1, and the whole system of equations thus be satisfied together. I shall show that it does vary by continuous *increase*.
>
> If it vary continuously from 0 to 1 and not by continuous

increase, it must in the course of its variation, assume once a maximum or minimum value.

Boole then goes on to determine if the derivative of V_1/V could be 0. This tacitly assumes that the existence of an extremum implies the existence of a zero derivative value—which it does here since, as just mentioned, V_1/V has a continuous derivative.

By adjoining the equation

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\frac{V_1}{V}\right) = \frac{\mathrm{d}}{\mathrm{d}x_1}(\mathrm{e}^t V_1) = 0$$

to the n equations obtained by differentiating with respect to x_1 each of the equations

$$\mathrm{e}^t V - 1 = 0, \quad \mathrm{e}^t V_2 - p_2 = 0, \ldots, \quad \mathrm{e}^t V_n - p_n = 0,$$

and then making use of (4), one obtains the system of equations

$$\begin{aligned}
&V\frac{\mathrm{d}t}{\mathrm{d}x_1} + \frac{V_1}{x_1}\frac{\mathrm{d}x_1}{\mathrm{d}x_1} + \frac{V_2}{x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x_1} + \cdots + \frac{V_n}{x_n}\frac{\mathrm{d}x_n}{\mathrm{d}x_1} = 0\\
&V_1\frac{\mathrm{d}t}{\mathrm{d}x_1} + \frac{V_1}{x_1}\frac{\mathrm{d}x_1}{\mathrm{d}x_1} + \frac{V_{12}}{x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x_1} + \cdots + \frac{V_{1n}}{x_n}\frac{\mathrm{d}x_n}{\mathrm{d}x_1} = 0\\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&V_n\frac{\mathrm{d}t}{\mathrm{d}x_1} + \frac{V_{n1}}{x_1}\frac{\mathrm{d}x_1}{\mathrm{d}x_1} + \frac{V_{n2}}{x_2}\frac{\mathrm{d}x_2}{\mathrm{d}x_1} + \cdots + \frac{V_n}{x_n}\frac{\mathrm{d}x_n}{\mathrm{d}x_1} = 0
\end{aligned} \tag{6}$$

constituting a necessary condition for an extremum. But by Proposition II. above the determinant of the coefficients, namely

$$\frac{1}{x_1 x_2 \cdots x_n}\begin{vmatrix} V & V_1 & V_2 & \cdots & V_n \\ V_1 & V_1 & V_{12} & \cdots & V_{1n} \\ V_2 & V_{21} & V_2 & \cdots & V_{2n} \\ \vdots & & & & \vdots \\ V_n & V_{n1} & \cdots & & V_n \end{vmatrix},$$

is $\neq 0$. Hence if V_1/V were to have an extremum then the system (6) would have only the trivial solution—but this is impossible since $\mathrm{d}x_1/\mathrm{d}x_1$ can't be 0. Thus V_1/V does not have an extremum and herewith Boole's Proposition III. is established.

Having disposed of this case Boole then considers V's which are incomplete in form (less than 2^n terms). Here the "conditions of possible experience" play an important role—in this context they are necessary conditions (on $p_1, \ldots, p_n$) which are implied by equations (1.) and which are obtained as follows (Proposition IV. in Boole's paper).

For each distinct term appearing in any of $V_1, \ldots, V_n$ let λ_i ($i = 1, \ldots, l$) be the ratio of that term to V. Then we have either

$$\lambda_1 + \lambda_2 + \cdots + \lambda_l = 1,$$

or

$$\lambda_1 + \lambda_2 + \cdots + \lambda_l \leq 1,$$

according as the distinct terms in $V_1, \ldots, V_n$ make up all of V or not; to this condition adjoin the n equations

$$\lambda_{k_1} + \lambda_{k_2} + \cdots + \lambda_{k_{r(i)}} = p_i \quad (i = 1, \ldots, n) \tag{7}$$

corresponding respectively to the n equations of (1.) (the values of the indices $k_1, \ldots, k_{r(i)}$ being such as to select the appropriate λ's for the terms in the i-th equation $V_i/V = p_i$); and finally to all of these add the inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_l \geq 0.$$

To this combined system of equations and inequations Boole now applies his technique of eliminating the λ's (Fourier elimination, § 0.7) so obtaining a system of linear inequations in $p_1, \ldots, p_n$ of the form

$$\begin{aligned} a_{11}p_1 + a_{12}p_2 + \cdots + a_{1n}p_n + b_1 &\geq 0 \\ a_{21}p_1 + a_{22}p_2 + \cdots + a_{2n}p_n + b_2 &\geq 0 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{m1}p_1 + a_{m2}p_2 + \cdots + a_{mn}p_n + b_m &\geq 0. \end{aligned} \tag{8}$$

These are the *conditions of possible experience* (or "inequations of condition") associated with equations (1.), constituting a necessary condition for the existence of a solution to these equations.

Boole observes that in the elimination process *equations* "present themselves" only if the equations (1.) are not independent. Indeed, if an equation,

$$a_1p_1 + a_2p_2 + \cdots + a_np_n + b = 0, \tag{9}$$

is deducible then it must have come from (7) by having

$$a_1 \frac{V_1}{V} + a_2 \frac{V_2}{V} + \cdots + a_n \frac{V_n}{V} + b = 0, \tag{10}$$

in which case, from (9) and (10),

$$a_1\left(\frac{V_1}{V} - p_1\right) + a_2\left(\frac{V_2}{V} - p_2\right) + \cdots + a_n\left(\frac{V_n}{V} - p_n\right) = 0,$$

so that for any $a_i \neq 0$, if the equations of (1.) other than the i-th are satisfied then so is the i-th.

Equation (10) can be written

$$a_1 V_1 + a_2 V_2 + \cdots + a_n V_n + bV = 0, \tag{11}$$

in which form it expresses the linear dependence of the functions $V_1, V_2, \ldots, V_n, V$. As we shall presently see, for his principal theorem Boole will assume that the equations (1.) are "independent with respect to $x_1, \ldots, x_n$" and, as a consequence of this, the linear dependence cannot happen if the hypotheses are satisfied. Likewise, all of the inequations in (8) would, if holding, have to be strict.

As described in § 0.7 one can obtain from (8) upper and lower bounds for any one of the p_i's in terms of the others. Thus for each inequation of (8) for which $a_{i1} \neq 0$ one can obtain an upper bound

$$p_1 \leq \frac{-1}{a_{i1}}(a_{i2}p_2 + \cdots + a_{in}p_n + b_i) \tag{12a}$$

or a lower bound

$$\frac{-1}{a_{i1}}(a_{i2}p_2 + \cdots + a_{in}p_n + b_i) \leq p_1 \tag{12b}$$

according as a_{i1} is negative or positive and hence, calling $\min_1(p_2, \ldots, p_n)$ the smallest of the upper bounds and $\max_1(p_2, \ldots, p_n)$ the largest of the lower bounds, we have

$$\max_1(p_2, \ldots, p_n) \leq p_1 \leq \min_1(p_2, \ldots, p_n). \tag{13}$$

Boole notes that the conditions of possible experience are independent of the particular values of the coefficients of the terms of V—so long as they are positive—since these do not enter into the equations or inequations for the λ's.

Having introduced the conditions of possible experience which he will need Boole is now ready to tackle the case of V's incomplete in form[1].

PROPOSITION V

> Let V be incomplete in form; then provided that the equations
>
> $$\frac{V_1}{V} = p_1, \quad \frac{V_2}{V} = p_2, \quad \ldots, \quad \frac{V_n}{V} = p_n, \qquad \ldots (1.)$$
>
> are independent with respect to the quantities $x_1, x_2, \ldots, x_n$ and that the inequations of condition deducible by the last proposition are satisfied, the equations will admit of one solution, and only one, in positive finite values of $x_1, x_2, \ldots, x_n$.

Here in this statement of his result Boole includes an hypothesis not mentioned in earlier versions, namely that the equations (1.) "are independent with respect to the quantities $x_1, x_2, \ldots, x_n$"—this restriction has the important implication of narrowing down the class of V's for which the system (1.) has a unique solution, and thus reducing Boole's earlier unspoken, but implied, claims of general applicability of his probability method no matter what V may arise. As to what "independent with respect to the quantities $x_1, x_2, \ldots, x_n$" means is not explained by Boole, but what he explicitly uses this hypothesis for is to assure that no variable cancels out from numerator and denominator of all the V_i/V in (1.), so reducing the system to one of n equations in $n - 1$ unknowns, and that there does not exist a linear relationship holding identically among the V_i/V. We interpret Boole's independence assumption to mean that the equations (1.) are not degenerate, i.e. that the n

[1] In the remainder of this section we shall be running two systems of numberings of formulas—those which appear on the right hand side of the page are part of the quotations from Boole's paper and are distinguished in the text from our numbering of formulas by having a period after the numeral (as was originally printed in Boole's paper).

functions an the n variables are essential. Referring to the Sufficient Condition for Functional Independence in §0.10 we see that if the Jacobian of $\{V_1/V, \ldots, V_n/V\}$ with respect to $x_1, \ldots, x_n$ is not identically 0 (over a given region) then no continuous, *a fortiori* no linear, relationship can exist among the $V_i/V - p_i$ (alternatively, among $V_1, \ldots, V_n, V$); moreover, if a variable were absent from all the functions then the column in the Jacobian having the derivatives of the V_i/V with respect to this variable would be a column of all 0's, making the Jacobian identically 0. Thus the Jacobian not identically 0 is sufficient for Boole's stated requirements. Additionally, although Boole didn't realize it one also needs such a condition to insure the continuity (which he had tacitly assumed) of the solution of the last $n-1$ equations as functions of x_1 occuring in the course of the proof. In the analogous situation in the proof of Proposition III, just examined, this assumption (Jacobian not identically 0) was not needed as it is automatically fulfilled for a V complete in form. However, as noted earlier, for a V incomplete in form the Jacobian could be identically 0. Accordingly, we shall adopt the non-vanishing of the Jacobian as an explication of Boole's requirement that the equations be "independent with respect to the quantities $x_1, \ldots, x_n$."

Instead of directly obtaining the Jacobian of these functions we can, as before, simplify the calculations by introducing an auxiliary variable t and adding the equation $V - e^{-t} = 0$ to (1.). For this equivalent system the Jacobian is

$$\text{(14)} \qquad \frac{1}{x_1 x_2 \cdots x_n} \begin{vmatrix} V & V_1 & V_2 & \cdots & V_n \\ V_1 & V_1 & V_{12} & \cdots & V_{1n} \\ V_2 & V_{21} & V_2 & \cdots & V_{2n} \\ \vdots & \vdots & & & \vdots \\ V_n & V_{n1} & \cdots & & V_n \end{vmatrix},$$

where, but for the independence requirement, this Jacobian could be identically 0, unlike for the case of V complete in form.

In broad outline Boole's proof of Proposition V. is similar to that of Proposition III. (in which V is complete in form). He claims to show that when x_1 is assigned a value between 0 and ∞ the $x_2, \ldots, x_n$ are determined as finite positive values satisfying the last $n-1$ equations; that [as a function of x_1] V_1/V will be between $\max_1(p_2, \ldots, p_n)$ and $\min_1(p_2, \ldots, p_n)$; that when x_1 is 0 or ∞ the values of $x_2, \ldots, x_n$ (which may be 0 or ∞ in this circumstance) determined by the last $n - 1$ equations give to V_1/V the extreme values $\max_1(p_2, \ldots, p_n)$ and $\min_1(p_2, \ldots, p_n)$; and finally that as x_1 varies from 0 to ∞ the function V_1/V varies "by continuous increase" between the extreme values. Since the inequations of condition, assumed to hold by hypothesis, imply that p_1 is between these values, the function V_1/V will take on this value once and only once at the same time the last $n - 1$ equations are satisfied, and all this for finite positive values of $x_1, \ldots, x_n$. We proceed to the details.

We suppose that the system (1.) satisfies the "independence" hypothesis, which we have taken to mean that the Jacobian (14) is not identically 0, and also that the "conditions of inequation" hold. Further, as hypothesis of induction we are supposing that the Proposition is true for systems with $n - 1$ equations.

> In the system (1.) let x_1 receive any finite positive value, and let V by the substitution of this value become U; the last $n - 1$ equations of (1.) will thus assume the form
>
> $$\frac{U_2}{U} = p_2, \quad \frac{U_3}{U} = p_3, \quad \ldots, \quad \frac{U_n}{U} = p_n, \qquad \ldots(2.)$$
>
> in which the quantities $p_2, p_3, \ldots, p_n$ satisfy the conditions [of inequation] to which the direct application of Proposition IV. to this reduced system of equations would lead.

The assertion here is readily verified for, as Boole notes, the change from V to U, resulting from assigning a value to x_1, entails the coalescing of some pairs of terms which have the same variables present except that one of the pair has x_1 and the other doesn't. Thus the λ system for U (call the variables here "μ") may be obtained from that for V by omitting the equation for $V_1/V = p_1$ and in the remaining equations or inequations replacing, where appropriate, sums of two λ's corresponding to coalesced terms by a single μ, e.g. $\lambda_i + \lambda_j$ by μ_k, and replacing the pair of inequations $\lambda_i \geq 0$, $\lambda_j \geq 0$ by $\mu_k \geq 0$. Thus any relationship concerning $p_2, \ldots, p_n$

derivable from the μ system which is true independently of the μ's can equally as well be derived from the λ system by replacing in the derivation each μ by the corresponding λ, or sum of two λ's, according as the ratio designated by μ corresponds to an uncoalesced or a coalesced term of V and, in the case of a coalescing, the condition $\mu_k \geq 0$ by $\lambda_i \geq 0$, which implies it. This replacement results in a subsystem of the λ system—hence the inequations of condition for (2.) hold if that for (1.) does.

Boole now concludes, by hypothesis of induction, that the system (2.) for the U's is satisfied by a unique set of finite positive values for $x_2, x_3, \ldots, \ldots, x_n$ for the assigned positive value to x_1. Although Boole has verified one of the two hypotheses in Proposition V. for application to the case of $n - 1$ equations, he says nothing about the other, the independence condition. What needs to be shown is that if the Jacobian for $\{V_1/V, \ldots, V_n/V\}$ with respect to $x_1, \ldots, x_n$ is not identically 0, then neither is that for $\{U_2/U, \ldots, U_n/U\}$ with respect to $x_2, \ldots, x_n$. A proof is not immediately obvious. We supply one in an endnote to this section.

As in the case of Proposition III. one has that the positive functions $x_2(x_1), \ldots, x_n(x_1)$ which satisfy the last $n - 1$ equations of (1.) are continuously differentiable and, when substituted for $x_2, \ldots, x_n$ in the function V_1/V, results in a function of x_1, call it also

$$\Phi(x_1), \quad \text{or} \quad F(x_1, x_2(x_1) \ldots, x_n(x_1)),$$

which is defined for all $x_1 > 0$, which is continuously differentiable, and monotonic. What remains to be shown now is that p_1 lies within the extremes of the range of values of $\Phi(x_1)$. But at $x_1 = 0$ and $x_1 = \infty$, where $\Phi(x_1)$ has its extreme values, the equations (1.) need no longer retain the appropriate from for the application of the theorem and to handle this situation Boole introduces, to use his own words ,"a remarkable transformation". (Boole speaks of the "value" of V_1/V at 0 and at ∞ but it is only $\lim_{x_1 \to 0} V_1/V$ and $\lim_{x_1 \to \infty} V_1/V$ that we need and which we shall concern ourselves with.) In what follows we continue to follow Boole's ideas, though at times we go our own way in adherence to current mathematical standards, or for the sake of clarity.

Since

$$\max_1 (p_2, \ldots, p_n) \leq p_1 \leq \min_1 (p_2, \ldots, p_n),$$

the objective, to show that p_1 lies within the extremes of the values of $\Phi(x_1)$, would be attained if one could show that

$$\lim_{x_1 \to 0} F(x_1, x_2(x_1), \ldots, x_n(x_1)) = \max_1 (p_2, \ldots, p_n)$$

and

$$\lim_{x_1 \to \infty} F(x_1, x_2(x_1), \ldots, x_n(x_1)) = \min_1 (p_2, \ldots, p_n).$$

Direct evaluation of these limits (as for the case of V complete in form) is not easy to come by and Boole uses his remarkable transformation for this purpose.

Equations (1.) are first replaced by an equivalent system in the following manner. Let

$$a_{i1}p_1 + a_{i2}p_2 + \cdots + a_{in}p_n + b_i \geq 0 \tag{3.}$$

be one of the inequations of condition for the V at hand. Then from this and (1.) we have that the sum

$$a_{i1}V_1 + a_{i2}V_2 + \cdots + a_{in}V_n + b_iV \tag{4.}$$

(also ≥ 0) when viewed as a rational integral form in $x_1, \ldots, x_n$ consists solely of positive terms[1]. Any term $Ax_{r_1}x_{r_2} \ldots x_{r_\rho}$ present in V has as its coefficient in (4).

$$a_{ir_1}A + a_{ir_2}A + \cdots + a_{ir_\rho}A + b_iA$$

[1] To see this let $Bx_rx_s \ldots$ be any one of the terms in (4.) and divide the entire expression through by the product $x_rx_s \ldots$, so converting this term to B. Any variable appearing in some numerator cannot also appear in a denominator. For if a term originally had x_i as a factor it would either be cancelled or be left as it was, and if x_i were absent it still would be absent or else be present in the denominator. Since the terms of (4.) are distinct the only constant term appearing after the division is B. Let each x_i with numerator occurrence approach 0 and each x_j with denominator occurrence approach ∞. Then, as (4.) is non-negative, its limit after the division, namely B, is also non-negative and hence, being different from 0, is positive.

and hence, as A is positive by hypothesis on V, we have that either

(17a) $$a_{ir_1} + a_{ir_2} + \cdots + a_{ir_\varrho} + b_i > 0.$$

or

(17b) $$a_{ir_1} + a_{ir_2} + \cdots + a_{ir_\varrho} + b_i = 0,$$

depending on whether

$$Ax_{r_1}x_{r_2}\ldots x_{r_\varrho}$$

is present or absent in (4.). Now effect a separation of the terms of V into two classes, placing a term in one or the other of the classes according as (17a) or (17b) is the case for the term, and let H and K be the respective sums of the terms in these two classes. Then

$$V = H + K$$

and

$$H = a_{i1}V_1 + a_{i2}V_2 + \cdots + a_{in}V_n + b_iV.$$

We observe that K cannot be an empty sum (i.e. H cannot be V) since the second of these equations would then imply that the functions $V_1, V_2, \ldots, V_n, V$ are linearly dependent, in violation of the independence hypothesis. For the same reason H cannot be an empty sum. The following linear combination of the equations of (1.), namely

$$\frac{a_{i1}V_1 + \cdots + a_{in}V_n + b_iV}{V} = a_{i1}p_1 + \cdots + a_{in}p_n + b_i$$

i.e.

$$\frac{H}{V} = a_{i1}p_1 + \cdots + a_{in}p_n + b_i,$$

is an equation which, in the presence of the other equations, can equivalently replace any one of the equations of (1.). Boole replaces the first equation and writes the system as

$$\begin{aligned} \frac{H}{H+K} &= a_{i1}p_1 + \cdots + a_{in}p_n + b_i \\ \frac{H_2+K_2}{H+K} &= p_2 \\ &\vdots \\ \frac{H_n+K_n}{H+K} &= p_n \end{aligned} \tag{6.}$$

where $H_i + K_i$ $(i = 2, \ldots, n)$ is the separation of V_i into terms of the H and K types corresponding to that of V.

Up to now we have been supposing that (3.) was any one of the inequations of condition. Now we select the (or an) i which gives the maximum lower bound on p_1, i.e. an i for which

$$-\frac{1}{a_{i1}}(a_{i2}p_2 + \cdots + a_{in}p_n + b_i) = \max_1(p_2, \ldots, p_n)$$

—in particular this implies that $a_{i1} > 0$ (see §0.7). If we abbreviate $\max_1(p_2, \ldots, p_n)$ by p_1^* then we may rewrite this condition as

$$a_{i1}p_1^* + a_{i2}p_2 + \cdots + a_{in}p_n + b_i = 0. \tag{18}$$

The systems (1.) and (6.) are still equivalent if we replace p_1 in both by p_1^*. Call the resulting systems (1.)* and (6.)* and note that, by virtue of (18), the first equation of (6.)* is

$$\frac{H}{H+K} = 0.$$

Since the last $n-1$ equations of both systems as well as that of (1.) are the same and, by the inductive hypothesis are satisfied by the functions $x_2(x_1), \ldots, x_n(x_1)$, the substitution of these functions for $x_2, \ldots, x_n$ in the systems reduces them respectively to the single equations

$$F(x_1, x_2(x_1), \ldots, x_n(x_1)) = p_1^*$$

and

$$G(x_1, x_2(x_1), \ldots, x_n(x_1)) = 0,$$

where

$$G(x_1, x_2, \ldots, x_n) = \frac{H}{H+K}.$$

Since the two systems are equivalent the question as to whether

$$\lim_{x_1 \to 0} F(x_1, x_2(x_1), \ldots, x_n(x_1)) = p_1^*$$

is equivalent to the question as to whether

(19) $$\lim_{x_1 \to 0} G(x_1, x_2(x_1), \ldots, x_n(x_1)) = 0.$$

Now we bring Boole's transformation into play, introducing new variables $y_1 (= 1), y_2, \ldots, y_n$ by stipulating that

$$x_j = x_1^{a_{ij}/a_{i1}} y_j \qquad (j = 1, \ldots, n).$$

where i is the above chosen index.

Under this transformation a term of V of the form $Ax_{r_1} x_{r_2} \cdots x_{r_\varrho}$ becomes

$$Ax_1^{(a_{ir_1} + \cdots + a_{ir_\varrho})/a_{i1}} y_{r_1} y_{r_2} \cdots y_{r_\varrho}.$$

If we make this transformation throughout (6.)* and multiply numerator and denominator on the left sides of the equations by $x_1^{b_i/a_{i1}}$, then the net effect is to replace each term like $Ax_{r_1} x_{r_2} \cdots x_{r_\varrho}$ by

(21) $$Ax_1^{\delta} y_{r_1} y_{r_2} \cdots y_{r_\varrho}, \quad \text{where } \delta = \frac{a_{ir_1} + \cdots + a_{ir_\varrho} + b_i}{a_{i1}}$$

By our choice of i above $a_{i1} > 0$, and so by (17a, b) and (21) each term in H (and in $H_2, \ldots, H_n$) has x_1 as a factor with positive exponent and each term in K (and in $K_2, \ldots, K_n$) has x_1 with exponent 0, i.e. x_1 is not present. Furthermore, if $x_2(x_1), \ldots, x_n(x_1)$ are the positive functions (defined for $x_1 > 0$) satisfying the last $n - 1$ equations of (1.) then the functions $y_2(x_1), \ldots, y_j(x_1), \ldots, y_n(x_1)$ defined by

$$x_j(x_1) = x_1^{a_{ij}/a_{i1}} y_j(x_1) \quad (j = 2, \ldots, n)$$

for $x_1 > 0$, will satisfy the last $n - 1$ equations of (6.)* expressed in terms of the variables $y_2, \ldots, y_n$. Then in terms of the functions $y_j(x_1)$ condition (19) becomes

(22) $$\lim_{x_1 \to 0} G\,(x_1, x_1^{a_{i2}/a_{i1}} y_2(x_1), \ldots, x_1^{a_{in}/a_{i1}} y_n(x_1)) = 0.$$

Boole establishes (22)—according to his lights—by showing that the last $n-1$ equations of (6.)* expressed in terms of $y_2, \ldots, y_n$ have, when $x_1 = 0$, a unique solution, say $y_2^0, \ldots, y_n^0$, in positive values. For if this is the case then when these values along with 0 for x_1 are substituted in for $y_2, \ldots, y_n$, one obtains $H = 0$ and $K \neq 0$ so that the equation $H/(H+K) = 0$ is satisfied at the same time as the last $n-1$ equations. In order to justify this we need to verify that

(i) for $x_1 = 0$ the last $n-1$ equations of (6.)*, which for this value take the form

$$\frac{K_2}{K} = p_2, \ldots, \frac{K_n}{K} = p_n, \tag{23}$$

in variables $y_2, \ldots, y_n$) have indeed a unique positive solution, $y_2^0, \ldots, y_n^0$, and

(ii) the solution $y_2(x_1), \ldots, y_n(x_1)$ of the last $n-1$ equations of (6.)*, which are continuously differentiable for $x_1 > 0$ are also continuous at $x_1 = 0$, so that

$$\lim_{x_1 \to 0} y_j(x_1) = y_j^0.$$

Then, as $G\,(= H/(H+K))$ is continuous for non-zero values of the denominator, the limit in (22) is indeed equal to $H/(H+K)$ with $x_1 = 0$ and $y_j = y_j^0$.

Item (i) would follow by the inductive hypothesis if one could show that the hypotheses of the theorem (Prop. V.) hold for (23). Observe that the inequations of condition for (1.) still hold if p_1 is replaced by p_1^* since the only necessary condition on p_1 is

$$\max_1(p_2, \ldots, p_n) \le p_1 \le \min_1(p_2, \ldots, p_n),$$

and this still holds true if p_1 is replaced by $p_1^* = \max_1(p_2, \ldots, p_n)$. Hence we may take the inequations of condition for (1.) with p_1 replaced by p_1^* as inequations of condition for (1.)* or, since (1.)* is equivalent to (6.)*, as inequations of condition for (6.)*. Now the first equation of (6.)* namely,

$$\frac{H}{H+K} = 0,$$

implies that all those λ's in the λ system for (6.)*, which correspond to terms of V present in H, are to be made equal to 0. If these λ's are put equal to 0 in the remaining equations of the λ system for (6.)*, the resulting system will be the same as that for (23). It is then evident that the inequations of conditions for (23) hold since those for (6.)* do. As for the independence requirement for (23) again Boole says nothing. Attempts on our part to prove, under the inductive hypothesis, that equations (23) do satisfy the independence condition were unsuccessful. We also examined all possible types of 3-variable V's and turned up no counter-example to the theorem. Thus there is a gap here in Boole's proof that we were unable to close or to show that it was unclosable.

Turning next to item (ii), the continuity of the solution $y_2(x_1),\ldots,y_n(x_1)$ of the last $n-1$ equations of (6.)* can be extended to $x_1 = 0$ if the Jacobian of the system is $\neq 0$ at $x_1 = 0$ and $y_j = y_j^0$ $(j = 2,\ldots,n)$. This would be the case if (23) satisfied the independence conditions, i.e. if the Jacobian for (23) were not identically 0. For by Proposition II it is equal to a sum of positive terms in the variables $y_2,\ldots, y_n$ and hence its values $\neq 0$ for the positive values $y_j = y_j^0$. Thus item (ii) as well as (i) makes use of (23) satisfying the independence conditions.

We may briefly dispose of the other desideratum, the establishing of

$$\lim_{x_1\to\infty} F(x_1, x_2(x_1), \ldots, x_n(x_1)) = \min{}_1 (p_2, \ldots, p_n),$$

by noting that the coefficient a_{i1} in the inequation producing $\min_1(p_2,\ldots,p_n)$ is negative, so that the role played by x_1 in the case of the $\max_1(p_2, \ldots, p_n)$ discussion is now played by $1/x_1$, and the established result for $x_1 \to 0$ is paralleled by that for $x_1 \to \infty$. (A general result about logical variables still holds if these are replaced by their negations, i.e. $\bar{x}$ can stand for any arbitrary value just as x. In the present context this would correspond to replacing the numerical quotient of probabilities $x_1 = x/\bar{x}$ by its reciprocal $\bar{x}/x$.)

Boole concludes his demonstration of Prop. V. with:

> The above reasoning established rigorously that if the proposition is true for $n - 1$ variables, it is true for n variables. It remains then to consider the limiting case of $n = 1$.

Here, however, only the complete form of V, viz. $V = ax + b$, leads to a definite value of x, and this, as has been seen, is finite and positive. If we give to V the particular form ax, the equation $V_x/V = p$ becomes

$$\frac{ax}{ax} = p, \quad \text{or} \quad p = 1,$$

which determines p, but leaves x indefinite. If we employ the other particular form $V = b$, we obtain no equation whatever, and here again x is indefinite. But as the reducing transformations are all definite, the above indefinite forms cannot present themselves in the last stage of the problem when the original equations are independent and admit of definite solution.

The proposition is therefore established.

We would establish the $n = 1$ case as follows. Having $a \neq 0$, $b = 0$ is precluded since then

$$\frac{V_1}{V} = \frac{ax_1}{ax_1} = 1,$$

and the hypothesis of the theorem on $p_1, \ldots, p_n$ (carried over from Prop. III.) is that $0 < p_i < 1$ for each i. Similarly $b \neq 0$, $a = 0$ is precluded as then $V_1/V = 0$. So $V = ax_1 + b$ with both $a, b \neq 0$ is the only possibility and, as we have seen in Prop. III., the equation $V_1/V = p_1$ has, for this V, a unique positive solution for x_1.

Discussion. The extraordinary effort which Boole obviously must have expended to find a proof of his Proposition V betokens the importance which he attached to it. As here shown, we were able to verify on contemporary standards of rigor all of the steps in the proof save one; if the theorem is wrong it can only be shown to be so by a counter-example with a V of at least four variables since we (the author) have verified it for all 2- and 3-variable V's. Supposing for the sake of discussion that the theorem is correct, is it as important to the theory as Boole considered it to be?

Before replying to this question we should answer to (iv) left over from § 5.3: Is a probability problem a real one if and only if there is a unique solution in positive values for the equations (1) [(9) in § 5.3] for it? If we take "problem" here in Boole's sense as described in § 5.3 and "real" as defined in § 5.5 then a problem is a real one if

$$\exists \langle \mathfrak{B} | V, P \rangle \bigwedge_i [P(s_i | V) = p_i], \tag{24}$$

and this is equivalent to the conditions of possible experience. Clearly if equations (1) have a solution as described in Proposition V then there is a Boole probability algebra $\langle \mathfrak{B}(s_1, \ldots, s_n), P \rangle$ in which the equations

$$\frac{P(Vs_i)}{P(V)} = p_i \qquad (i = 1, \ldots, n)$$

are satisfied and hence there is a $\langle \mathfrak{B} | V, P \rangle$ as stated in (24). In the other direction (24), being equivalent to the conditions of possible experience, does imply, by virtue of Proposition V, that there is a unique positive set of values satisfying (1) *if*, in addition, the independence requirements are satisfied. If we enlarge the meaning of a "real problem" to include this condition then the answer to (iv), on the basis of Proposition V is "yes", and apparently Boole then has succeeded in establishing that his method is general, providing the answer to any problem in probability. Being able to "embed" the problem into a Boole probability algebra via Proposition V so as to obtain probabilities for the ideal events $s_1, \ldots, s_n$ is basic to the method. Hence Boole's concern to prove Proposition V.

To us the achievement, ingenious as it is, does not have the fundamental character Boole thought it did. One can certainly envision a far wider class of problems than specified in Boole's General Problem—even retaining the general feature of his conditions of possible experience as a set of linear inequalities (see § 5.5). But of more serious import is the failure of the method to give a full solution set to the question of finding the sought for probability, even in some cases when the method gives a one-parameter family of solutions. And, as Wilbraham in essence pointed out (§ 4.8), though the method does give a correct value (or class of values) the utility of it is limited, since when made explicit, the additional conditions required to produce Boole's particular solution are not "natural" ones connected with the problem but with Boole's technical device.

§ 5.7. Boole's General Problem linearly programmed

In connection with his general probability problem Boole considered it essential that the events whose probabilities were given and the event whose probability was sought (the *objective event* as we shall call it) could be treated as simple events conditioned by the logical relations implicit or explicit in the data of the problem, and that as events so conditioned their probabilities were the given ones of the original formulation. Without this assumption—principle, to Boole—the problem was deemed unsolvable, or to have no "determinate" solution. As we have seen in this chapter, on the basis of this principle Boole did attain his determinate solution goal.

In this section we are going to show that Boole's general problem can be solved without additional assumptions, solved not in his sense but in the sense of giving best possible upper and lower bounds on the probability of the objective event, these bounds being determined by the given data. Conceivably such a solution could have been presented by Boole for, as mentioned in § 4.7, the idea of finding bounds on the probability of an event originated with Boole, as well as the idea of obtaining, in his phrase, "the narrowest limits". Applying this technique of finding narrowest limits for the probability of the objective event would then be solving the general problem as we view it. But as Boole never conceived of this as a solution to the general problem it is left for us to present it this way. We are, of course, bound to current standards of rigor but, on the other hand, we shall be able to take advantage of the 20-th century development of linear programming theory.

We consider a slightly simplified version of the general problem in which there are no given explicit logical relations—briefly put the problem then is: Given

(1) $$P(\psi_i(A_1, \ldots, A_m)) = p_i \qquad (i = 1, \ldots, n)$$

$$\text{find } P(\varphi(A_1, \ldots, A_m)).$$

By way of clarifying the notion of a best possible bound as we here use it, let $\varphi, \psi_1, \ldots, \psi_n$ be m-ary Boolean functions (more exactly—Boolean polynomial expressions with m variables, hence interpretable in any

Boolean algebra.) We say that the real-valued function

$$\mathrm{BUB}_{\varphi}(\psi_1, \ldots, \psi_n; x_1, \ldots, x_n),$$

(abbreviated to $\mathrm{BUB}_{\varphi}(x_1, \ldots, x_n)$) whose arguments x_i range over the unit interval [0, 1], is a *best possible upper bound for the probability of* φ *relative to* $\psi_1, \ldots, \psi_n$ if

(i) for any probability algebra $\langle \mathfrak{A}, P \rangle$, any events $A_1, \ldots, A_m$ in $\mathfrak{A}$, and any real numbers $p_1, \ldots, p_n$ in [0, 1] such that $P(\psi_i(A_1, \ldots, A_m)) = p_i$ $(i = 1, \ldots, n)$ we have

$$P(\varphi(A_1, \ldots, A_m)) \leq \mathrm{BUB}_{\varphi}(p_1, \ldots, p_n)$$

and

(ii) under the same hypothesis as (i) if f is a function for which

$$P(\varphi(A_1, \ldots, A_m)) \leq f(p_1, \ldots, p_n) \leq \mathrm{BUB}_{\varphi}(p_1, \ldots, p_n)$$

then $f = \mathrm{BUB}_{\varphi}$.

We define the best lower bound BLB_{φ} similarly, the sense of the inequality being reversed throughout. The two kinds of bounds are connected through the relation

$$\mathrm{BLB}_{\varphi}(x_1, \ldots, x_n) = 1 - \mathrm{BUB}_{\bar{\varphi}}(x_1, \ldots, x_n).$$

It should be noted that $\mathrm{BUB}_{\varphi}(p_1, \ldots, p_n)$ need not be defined for all sets of values of $p_1, \ldots, p_n$—as it would not be, for example, if

$$P(\psi_1(A_1)) = P(A_1) = p_1,$$

$$P(\psi_2(A_1)) = P(\bar{A}_1) = p_2$$

and $p_1 + p_2 \neq 1$. We can now say that our way of solving Boole's general problem is to give explicit expressions for the functions BLB_{φ} and BUB_{φ} and a specification of their range of definition. To this we now turn.

The constituents on Boolean elements $A_1, \ldots, A_m$ will be denoted by $K_j(A_1, \ldots, A_m)$ $(j = 1, \ldots, 2^m)$. We assume that these 2^m constituents are always arranged in a fixed order, say by stipulating that the j-th one, $K_j(A_1, \ldots, A_m)$, is the one in which the variables A_{m-i+1} $(i = 1, \ldots, m)$ appear negated or unnegated according as the binary expansion of the integer $2^m - j$ has a 0 or 1 as the coefficient of 2^{i-1}. For example if $m = 3$ then this stipulation gives the arrangement

j	coefficients			$K_j(A_1, A_2, A_3)$
1	1	1	1	$A_1A_2A_3$
2	1	1	0	$A_1A_2\bar{A}_3$
3	1	0	1	$A_1\bar{A}_2A_3$
4	1	0	0	$A_1\bar{A}_2\bar{A}_3$
5	0	1	1	$\bar{A}_1A_2A_3$
6	0	1	0	$\bar{A}_1A_2\bar{A}_3$
7	0	0	1	$\bar{A}_1\bar{A}_2A_3$
8	0	0	0	$\bar{A}_1\bar{A}_2\bar{A}_3$

corresponding to the oft-encountered standard truth-table arrangement.

For an m-ary Boolean function φ we shall say "K_j implies φ" if $K_j(A_1, \ldots, A_m)$ is a constituent present in the complete expansion of $\varphi(A_1, \ldots, A_m)$, and otherwise that "$K_j$ implies $\bar{\varphi}$". From the fact that constituents (on a given set of variables) are mutually exclusive we may write the probability of any compound event $\varphi(A_1, \ldots, A_m)$ as the sum of the probabilities of constituents present in φ's expansion and then, using matrix notation, as a product of a row vector with a column vector, thus:

(2)
$$P(\varphi(A_1, \ldots, A_m)) = \sum_{j=1}^{2^m} \delta_j^{(\varphi)} k_j = \boldsymbol{\delta}^{(\varphi)} \boldsymbol{k},$$

where

$$k_j = P(K_j(A_1, \ldots, A_m)),$$

$$\delta_j^{(\varphi)} = \begin{cases} 1 & \text{if } K_j \text{ implies } \varphi, \\ 0 & \text{if } K_j \text{ implies } \bar{\varphi}, \end{cases}$$

and

$$\boldsymbol{\delta}^{(\varphi)} = [\delta_1^{(\varphi)} \cdots \delta_{2^m}^{(\varphi)}],$$

$$\boldsymbol{k} = [k_1 \ldots k_{2^m}]^{\mathrm{T}} = \begin{bmatrix} k_1 \\ \vdots \\ k_{2^m} \end{bmatrix}.$$

Note that the vector $\boldsymbol{\delta}^{(\varphi)}$ is independent of the events A_i, depending only on the nature of the function φ, not on its arguments. Now, under the hypotheses of the general problem, the probabilities k_j are subject to constraints coming from the conditions $P(\psi_i(A_1, \ldots, A_m)) = p_i$. We express these conditions in terms of the k_j by substituting for each $\psi_i(A_1, \ldots, A_m)$ its expansion in terms of the $K_j(A_1, \ldots, A_m)$ and, by distributing the probability operator P over the exclusive disjuncts, obtain

$$(3_1)\text{–}(3_n) \qquad \sum_{j=1}^{2^m} a_{ij}k_i = p_i, \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } K_j \text{ implies } \psi_i, \\ 0 & \text{if } K_j \text{ implies } \bar{\psi}_i, \end{cases}$$

i ranging from 1 to n.

To these we must further add

$$(4) \qquad \sum_{j=1}^{2^m} k_j = 1,$$

$$(5_1)\text{–}(5_{2^m}) \qquad k_j \geq 0 \quad (j = 1, \ldots, 2^m).$$

Conditions (3_1)–(3_n) and (4) can be combined and expressed as a matrix equation

$$(6) \qquad \boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+,$$

where $\boldsymbol{p}^+$ is the $(n + 1)$-component column vector $[p_1 \ldots p_n \; 1]^{\mathrm{T}}$, where $\boldsymbol{k}$ is the above introduced column vector of constituent probabilities, and where $\boldsymbol{A}^{(\psi)}$ is the $(n + 1) \times 2^m$ matrix.

$$\boldsymbol{A}^{(\psi)} = \begin{bmatrix} a_{11} \cdots a_{12^m} \\ a_{21} \cdots a_{22^m} \\ \vdots \qquad \vdots \\ a_{n1} \qquad a_{n2^m} \\ 1 \cdots 1 \end{bmatrix}, \qquad a_{ij} = \begin{cases} 1 & \text{if } K_j \text{ implies } \psi_i, \\ 0 & \text{if } K_j \text{ implies } \bar{\psi}_i, \end{cases}$$

consisting of the $n \times 2^m$ matrix $[a_{ij}]$ with an additional $(n + 1)$-st row of all 1's. The matrix $\boldsymbol{A}^{(\psi)}$ is determined solely by the functions $\psi_1, \ldots, \psi_n$. Conditions (5_1)–(5_{2^m}) can be expressed by the simple matrix inequality

$$(7) \qquad \boldsymbol{k} \geq \boldsymbol{0}.$$

The notion of a probability algebra occurs in our definition of a best possible bound. But as the definition entails a universal quantification over all probability algebras it isn't surprising that there is an equivalent formulation not using this notion—as we now show.

Given $\varphi, \psi_1, \ldots, \psi_n, p_1 \ldots, p_n$ define $\alpha(p_1, \ldots, p_n)$ to be the set of numbers $P(\varphi(A_1, \ldots, A_m))$ obtained by using any events $A_1, \ldots, A_m$ in any probability algebra $\langle \mathfrak{A}, P \rangle$ for which $P(\psi_i(A_1, \ldots, A_m)) = p_i$ $(i = 1, \ldots, n)$, that is let $\alpha(p_1, \ldots, p_n)$ be

$$\{q\colon \text{There is a } \langle \mathfrak{A}, P \rangle \text{ and events } A_1, \ldots, A_m \text{ for which } P(\psi_i(A_1, \ldots, A_m)) = p_i \, (i = 1, \ldots, n) \text{ and } q = P(\varphi(A_1, \ldots, A_m))\}.$$

We show that BLB_φ and BUB_φ are, respectively, the greatest lower bound (infimum) and least upper bound (supremum) of α.

LEMMA 5.61. *If $\alpha(p_1, \ldots, p_n)$ is non-empty, then*

$$\sup \alpha(p_1, \ldots, p_n) = \mathrm{BUB}_\varphi(p_1, \ldots, p_n),$$
$$\inf \alpha(p_1, \ldots, p_n) = \mathrm{BLB}_\varphi(p_1, \ldots, p_n).$$

PROOF. Suppose $\alpha(p_1, \ldots, p_n)$ is non-empty. Then since it is a bounded set its sup exists. Clearly for any $\langle \mathfrak{A}, P \rangle$ and any events $A_1, \ldots, A_m$ for which $P(\psi_i(A_1, \ldots, A_m)) = p_i$ we have

$$P(\varphi(A_1, \ldots, A_m) \leq \sup \alpha(p_1, \ldots, p_n);$$

and if f were a function for which

$$P(\varphi(A_1, \ldots, A_m)) \leq f(p_1, \ldots, p_n) \leq \sup \alpha(p_1, \ldots, p_n)$$

but

$$f(p_1, \ldots, p_n) < \sup \alpha(p_1, \ldots, p_n),$$

then $\sup \alpha(p_1, \ldots, p_n)$ would not be the least upper bound. A similar argument establishes the result for inf.

Let $\beta(\boldsymbol{p}^+)$ be the set of numbers $\boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ for any $\boldsymbol{k}$ satisfying conditions (6) and (7), that is let $\beta(\boldsymbol{p}^+)$ be

$$\{q\colon \text{There is a } \boldsymbol{k} \geq \boldsymbol{0} \text{ such that } A^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+ \text{ and } q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}\}.$$

LEMMA 5.62. $\alpha(p_1, \ldots, p_n) = \beta(\boldsymbol{p}^+)$.

PROOF. Suppose $q \in \alpha(p_1, \ldots, p_n)$. Then by the discussion connected with (2)–(7) there is a $\boldsymbol{k} \geq \boldsymbol{0}$ such that $\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+$ and $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$, i.e. $q \in \beta(\boldsymbol{p}^+)$. Now suppose $q \in \beta(\boldsymbol{p}^+)$, i.e. there is a vector $\boldsymbol{k} \geq \boldsymbol{0}$ such that $\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+$ and $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$. Since each component k_j of $\boldsymbol{k}$ is ≥ 0 and the sum of all is $= 1$ (by the last equation in $\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+$) we have then by Theorem 0.94 that there is a probability algebra with events $A_1, \ldots, A_m$ such that the constituents $K_j(A_1, \ldots, A_m)$ have probabilities k_j $(j = 1, \ldots, 2^m)$. For this probability algebra the first n equations of $\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+$ tells us that $P(\psi_i(A_1, \ldots, A_m)) = p_i$, and $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ tells us that $q = P(\varphi(A_1, \ldots, A_m))$. Thus $q \in \alpha(p_1, \ldots, p_n)$.

THEOREM 5.63. *If $\beta(\boldsymbol{p}^+)$ is non-empty, then*

$$\sup \alpha(p_1, \ldots, p_n) = \sup \beta(\boldsymbol{p}^+) = \max \beta(\boldsymbol{p}^+),$$
$$\inf \alpha(p_1, \ldots, p_n) = \inf \beta(\boldsymbol{p}^+) = \min \beta(\boldsymbol{p}^+).$$

PROOF. By Lemma 5.62 we need only establish the equality for $\beta(\boldsymbol{p}^+)$. Suppose $\beta(\boldsymbol{p}^+)$ is non-empty. The set of points k in E^{2^m} (Euclidean 2^m-space) satisfying the conditions $\boldsymbol{k} \geq \boldsymbol{0}$ and $\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+$ consists of points lying in the unit hypercube $\boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{1}$ and on the (closed) intersection of the $n + 1$ hyperplanes $\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+$ and hence is a bounded and closed set. Since $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ is a continuous function of $\boldsymbol{k}$ the corresponding set of values of q, as a continuous image of a compact set, is also closed. Thus $\sup \beta(\boldsymbol{p}^+)$ is in $\beta(\boldsymbol{p}^+)$ and is its maximum.

On the basis of this theorem we see that finding $\mathrm{BLB}_\varphi(p_1, \ldots, p_n)$ and $\mathrm{BUB}_\varphi(p_1, \ldots, p_n)$ is equivalent to finding the minimum and maximum of the linear form $\boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ subject to linear constraints, that is to the following *Linear Programming Problem*:

Find
(i) minimum of $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$,
(ii) maximum of $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$,
subject to

(8) $$\boldsymbol{A}^{(\psi)}\boldsymbol{k} = \boldsymbol{p}^+, \qquad \boldsymbol{k} \geq 0.$$

Since for $\boldsymbol{k}$'s satisfying the constraints (8) the values of $\boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ are bounded, the linear programming problem has a solution if and only if the set of such values (the set of feasible solutions) is non-empty. But what the set is depends upon the values of the parameters $p_1, \ldots, p_n$. Thus it is of particular interest to us to know under what conditions, i.e. for what values of $p_1, \ldots, p_n$ we have at least one feasible solution—these conditions being precisely Boole's conditions of possible experience for the probability problem. To obtain these conditions we follow one of Boole's methods. By Theorem 0.71 a necessary and sufficient condition for the linear system (8) to have a solution is that the set of relations obtained by Fourier-eliminating the k_j be satisfied (i.e. be true). As we have observed in § 0.7, with the $p_1, \ldots, p_n$ as parameters the set of relations resulting after the elimination of the k_j comes out to be a linear system (equations and/or inequations) in the p_i. So it is these relations which determine under what conditions we have values for $\mathrm{BLB}_\varphi(p_1, \ldots, p_n)$ and $\mathrm{BUB}_\varphi(p_1, \ldots, p_n)$.

Given that the conditions of experience are satisfied, how does one obtain the optimal values of $\boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$? We discuss various possibilities.

In case one has specific numerical values for $p_1, \ldots, p_n$ the efficiency of the simplex method (Dantzig 1963) for finding optimal values of the objective function is hard to better, proceeding as one does from one basic solution (extremal point of the convex polytope) to another in highly directive fashion until optimality is found. However, if we were to retain $p_1, \ldots, p_n$ as parameters in order to obtain explicit expressions for the functions BLB_φ and BUB_φ, then the outstanding advantage of the simplex method is unavailable since when found the coordinates of the basic solutions would be (linear) functions of the parameters and one couldn't tell if the value of $\boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ were optimal or not. Even worse, the character of the convex polytope, i.e. what its extremal points are, depends on $p_1, \ldots, p_n$ (via $\boldsymbol{p}^+$ in $A^{(\varphi)}\boldsymbol{k} = \boldsymbol{p}^+$) and this character could change from one set of values of $p_1, \ldots, p_n$ to another. We have to look for other means.

It turns out that the Fourier-Motzkin elimination method (§ 0.8) can be used to solve our parameter form of a linear programming problem. Here this simply amounts to adding the equation $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$ to the system (8) and applying Fourier elimination to remove the k_i's. This results in a set of linear inequations for q in terms of the

p_i which falls into two classes of the form

$$L_\mu(p_1, \ldots, p_n) \leq q$$

and

$$q \leq U_\nu(p_1, \ldots, p_n).$$

By setting

$$\mathrm{BLB}_\varphi(p_1, \ldots, p_n) = \max_\mu \, [L_\mu(p_1, \ldots, p_n)],$$

$$\mathrm{BUB}_\varphi(p_1, \ldots, p_n) = \min_\nu \, [U_\nu(p_1, \ldots, p_n)],$$

we then have the sought for explicit expressions. This is essentially Boole's "easy and general" method, described in our § 4.7, and illustrated by Boole with his Challenge Problem.

The foregoing procedure for finding BLB_φ and BUB_φ, while theoretically unobjectionable, is not without some practical disadvantages. As one procedes with the Fourier elimination the number of inequations grows very rapidly; also, since one has to work symbolically, i.e. non-numerically, because of the presence of the parameters, it is not easy to program computers to do the job. Happily, help in this respect comes by going over to the dual form of the linear program—from this form, as we now see, the functions BLB_φ and BUB_φ can be found by a purely numerical process.

By Theorem 0.81 we know that our linear programming problem has the two equivalent forms

Primal:

$$\boldsymbol{A}^{(\varphi)}\boldsymbol{k} = \boldsymbol{p}^+, \qquad \boldsymbol{k} \geq \boldsymbol{0},$$

(i) minimize $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$,

(ii) maximize $q = \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k}$.

Dual:

(i) $\boldsymbol{A}^{(\varphi)\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{\delta}^{(\varphi)\mathrm{T}}$, $\boldsymbol{y}$ unrestricted; maximize $z = (\boldsymbol{p}^+)^{\mathrm{T}}\boldsymbol{y}$,

(ii) $\boldsymbol{A}^{(\varphi)\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{\delta}^{(\varphi)\mathrm{T}}$, $\boldsymbol{x}$ unrestricted; minimize $w = (\boldsymbol{p}^+)^{\mathrm{T}}\boldsymbol{x}$.

Here we observe that in the dual form the constraints specifying the region of feasibility are independent of the parameters (which appear only in the objective function), so that basic feasible solutions can be found purely numerically. Supposing these basic solutions to be denoted by $\boldsymbol{y}^{(1)}, \ldots, \boldsymbol{y}^{(r)}$ and $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(s)}$ then

$$\mathrm{BLB}_{\varphi}(p_1, \ldots, p_n) = \max\,[(\boldsymbol{p}^{+})^{\mathrm{T}}\,\boldsymbol{y}^{(1)},, \ldots, (\boldsymbol{p}^{+})^{\mathrm{T}}\boldsymbol{y}^{(r)}],$$

$$\mathrm{BUB}_{\varphi}(p_1, \ldots, p_n) = \min\,[(\boldsymbol{p}^{+})^{\mathrm{T}}\,\boldsymbol{x}^{(1)}, \ldots, (\boldsymbol{p}^{+})^{\mathrm{T}}\,\boldsymbol{x}^{(s)}]$$

are the explicit expressions for the best lower and upper bound functions for the probability of φ relative to $\psi_1, \ldots, \psi_n$. The following example, a slight generalization of the one arising from Boole's Challenge Problem, illustrates this method.

Given

$$P(A_1) = c_1$$

$$P(A_2) = c_2$$

$$P(A_1A_3) = c_3$$

$$P(A_2A_3) = c_4$$

$$P(\bar{A}_1\bar{A}_2A_3) = c_5,$$

find the best possible upper bound for $P(A_3)$.

This differs from the Challenge Problem by having as the value for $P(\bar{A}_1\bar{A}_2A_3)$ a parameter c_5 instead of 0. For this problem the matrix equation $\boldsymbol{Ak} = \boldsymbol{c}^{+}$ is:

$$\begin{matrix} A_1 \\ A_2 \\ A_1A_3 \\ A_2A_3 \\ \bar{A}_1\bar{A}_2A_3 \\ \\ \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ 1 \end{bmatrix}$$

(on the left are the Boolean functions which generate the corresponding rows of ones and zeros of the coefficient matrix). Going over to the equivalent dual problem, we wish to

$$\text{minimize} \quad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + x_6$$

subject to

$$(9) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where the column on the right side of the inequality sign is the $\boldsymbol{\delta}^{(\varphi)}$ corresponding to the function $\varphi(A_1, A_2, A_3) = A_3$. The basic feasible solutions (corner points of the polyhedron) specified by (9) are

$$(0,0,1,1,1,0)$$
$$(-1,0,1,0,0,1)$$
$$(0,-1,0,1,0,1)$$

giving a best upper bound for $P(A_3)$ of

$$\min[c_3 + c_4 + c_5, \bar{c}_1 + c_3, \bar{c}_2 + c_4].$$

(We obtain Boole's result for the Challenge Problem by setting $c_5 = 0$.)

Elsewhere (HAILPERIN 1965) we have reported on the results of using a computer to systematically obtain best possible bounds on the probability of Boolean functions with respect to their argument variables (the type of problem which Boole largely concerned himself with in his Chapter XIX). To cite a specific example of one of these, we found best bounds on the probability of

$$\varphi(A_1, A_2, A_3) = A_1A_2\bar{A}_3 + A_1\bar{A}_2A_3 + \bar{A}_1A_2A_3$$

given that

$$P(\psi_1(A_1, A_2, A_3)) = P(A_1) = p_1,$$

$$P(\psi_2(A_1, A_2, A_3)) = P(A_2) = p_2,$$

$$P(\psi_3(A_1, A_2, A_3)) = P(A_3) = p_3.$$

Here we have $\boldsymbol{\delta}^{(\varphi)} = [0\ 1\ 1\ 0\ 1\ 0\ 0\ 0]$, since the three constituents of φ's expansion are the second, third, and fifth in the standard arrangement (1). The matrix $\boldsymbol{A}^{(\psi)}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

in which the first three lines correspond in an obvious way with the expansions of A_1, A_2, and A_3 in terms of constituents. Written out explicitly the primal form of the linear programming problem for the best upper bound is:

$$\text{maximize } \boldsymbol{\delta}^{(\varphi)}\boldsymbol{k} = k_2 + k_3 + k_6$$

subject to the constraints

$$(9) \qquad \begin{aligned} &k_1 + k_2 + k_3 + k_4 &&= p_1, \\ &k_1 + k_2 \qquad\qquad + k_5 + k_6 &&= p_2, \\ &k_1 \qquad + k_3 \qquad + k_5 \qquad + k_7 &&= p_3, \\ &k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 &&= 1, \\ &\qquad k_i \geq 0 \quad (i = 1, \ldots, 8). \end{aligned}$$

The dual form for this problem would be:

$$\text{minimize } (\boldsymbol{p}^+)^{\mathrm{T}}\boldsymbol{x} = p_1x_1 + p_2x_2 + p_3x_3 + x_4,$$

subject to the constraints

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\geq 0,\\
x_1 + x_2 \qquad\;\; + x_4 &\geq 1,\\
x_1 + \qquad + x_3 + x_4 &\geq 1,\\
x_1 \qquad\qquad\quad + x_4 &\geq 0,\\
x_2 + x_3 + x_4 &\geq 1,\\
x_2 \qquad + x_4 &\geq 0,\\
x_3 + x_4 &\geq 0,\\
+ x_4 &\geq 0.
\end{aligned}$$

It is this dual form which corresponds to Boole's "purely algebraic form" which he mentions in Chapter XIX as giving the narrowest limits, though without justifying his assertion. The example here is of particular interest in that the computed result, namely

$$\mathrm{BUB}_\varphi(p_1, p_2, p_3)$$

$$= \min\left[p_1 + p_2,\ p_1 + p_3, p_2 + p_3, \bar{p}_1 + \bar{p}_2 + \bar{p}_3, \frac{p_1 + p_2 + p_3}{2}\right]$$

shows that Boole's Rule for obtaining an upper bound (by taking one factor from each constituent, rejecting any duplicate) would not give the narrowest limit in that it omits the possibility of $\frac{1}{2}(p_1 + p_2 + p_3)$.

As we have mentioned earlier, Boole recognized that this simple rule for finding bounds on the probability of a logical function did not always give the narrowest bounds. That he formulated the problem of finding such narrowest bounds as a linear programming problem, in both dual and primal forms and used Fourier elimination to find the bounds in the latter case, should be counted among his interesting and noteworthy achievements. True, he lacked the insight of the duality theorem of linear programming to connect up the two formulations, but that theorem did not appear until some 100 years later.

In concluding this section we mention a natural extension of the result described in this section which again shows that Boole's General Problem does not include "any" problem in probability as he claims.

Instead of having a set of data for the problem in the form of equations, as in (1), one could have two-sided inequalities, e.g.

$$a_i \leq P(\psi_i(A_1, \ldots, A_m)) \leq b_i$$

where the a_i, b_i are given numbers in the interval $[0, 1]$. The method outlined in this section is readily adapted to this more general situation (see HAILPERIN 1965, section 6). This extended result has been used in connection with development of a probability logic in HAILPERIN 1984 (see § 6.6 below).

§ 5.8. Notes to Chapter 5

(for § 5.1)

We believe that the notion of a Boole probability algebra provides a suitable mathematical basis for investigations of fault analysis in digital circuits in which one inquires concerning the probability of a signal at an output $f(X_1, \ldots, X_n)$ (f Boolean) on random assignment of signals to inputs $X_1, \ldots, X_n$. For the reader interested in pursuing the matter further we offer the following starting references:

K. P. Parker and E. J. McCluskey. Analysis of Logic with Faults Using Input Signal Probabilities. *IEEE Transactions on Computers*, Vol. C-24, May 1975, 573–578.

——. Probabilistic Treatment of General Combinational Networks. *ibid.*, June 1975, 668–670.

In these papers the authors supply two algorithms for computing the probability of a signal at an output for given probabilities of a signal at the inputs. To their two algorithms we add another employing techniques which Boole might conceivably have used:

0. Start with the function in the form $C_1 \vee C_2 \vee C_3 \vee \cdots$, with each C_i a product of literals.
1. Replace it by $C_1 + \bar{C}_1 C_2 + \bar{C}_1 \bar{C}_2 C_3 + \cdots$ (The $+$ signs are used to

indicate mutual exclusiveness of the terms, over which terms the probability operator distributes.)

2. (a) For each term $\bar{C}_1\bar{C}_2\ldots\bar{C}_rC_{r+1}$ replace each $\bar{C}_i$ by a disjunction (use '$\vee$') of literals, deleting any whose opposite is a factor of C_{r+1}. Multiply out and delete any repeated factor or term.

(b) If there are any $\vee$'s, repeat 1. and 2.(a) to disjunctions until only $+$'s remain.

3. Replace each X_i by x_i (its probability) and each $\bar{X}_i$ by $1-x_i$.

Example:

$$X_1(X_2 \vee X_3) \vee X_2X_3$$

$$X_1X_2 \vee X_1X_3 \vee X_2X_3$$

$$X_1X_2 + \overline{X_1X_2}X_1X_3 + \overline{X_1X_2}\,\overline{X_1X_3}X_2X_3$$

$$X_1X_2 + (\bar{X}_1 \vee \bar{X}_2)X_1X_3 + (\bar{X}_1 \vee \bar{X}_2)(\bar{X}_1 \vee \bar{X}_3)X_2X_3$$

$$X_1X_2 + \bar{X}_2X_1X_3 + \bar{X}_1X_2X_3$$

$$x_1x_2 + (1-x_2)x_1x_3 + (1-x_1)x_2x_3.$$

The algebraic operations involved are easily automated.

(for § 5.6)

NOTE 1. In addition to the objective evidence of Boole's preoccupation with the matter of this section, we have the following personal statement in a letter to De Morgan:

> ... but the correspondence [with Cayley on Boole's Challenge Problem] has led me to resume the analytical discussion of my method which I had vainly attempted to complete before—this time with success. I have proved that in all cases the conditions of analytical validity of the method are simply the conditions of consistency in the data—what I have elsewhere termed the conditions of possible experience.
>
> I do not think I have ever engaged in as difficult a mathematical investigation. The most important part of it consists in proving

that a certain functional determinant is always positive whatever the number of the variables n. [SMITH 1982, p. 101]

NOTE 2. We prove that if the Jacobian of $\{V_1/V, \ldots, V_n/V\}$ with respect to $x_1, \ldots, x_n$ is not identically zero, then neither is that of $\{U_2/U, \ldots, U_n/U\}$ with respect to $x_2, \ldots, x_n$.

Suppose that the determinant in (14) is not identically zero (in the positive range of the variables). Then by Proposition II its value for a fixed positive value of x_1 will be positive for all positive values of $x_2, \ldots, x_n$. Thus

$$\begin{vmatrix} U & U_1 & U_2 & \ldots & U_n \\ U_1 & U_1 & U_{12} & \ldots & U_{1n} \\ U_2 & U_{21} & U_2 & \ldots & U_{2n} \\ \vdots & & & & \\ U_n & U_{n1} & U_{n2} & \ldots & U_n \end{vmatrix} \tag{15}$$

which arises from (14) by fixing x_1 as some positive value, is positive for all positive values of $x_2, \ldots, x_n$. By interchanging the first two rows and then the first two columns we convert (15) to the equal determinant

$$\begin{vmatrix} U_1 & U_1 & U_{12} & \ldots & U_{1n} \\ U_1 & U & U_2 & \ldots & U_n \\ U_{21} & U_2 & U_2 & \ldots & U_{2n} \\ U_{31} & U_3 & U_{32} & \ldots & U_{3n} \\ \vdots & & & & \\ U_{n1} & U_n & U_{n2} & \ldots & U_n \end{vmatrix}$$

in which A_{11}, the minor of the a_{11} element, is the Jacobian for $\{U_2/U, \ldots, U_n/U\}$. We show that if this determinant (i.e. A_{11}) were identically 0, then one obtains a contradiction with the assumption that the value of (15) is positive for all values in the positive range of the variables. For if A_{11} were identically 0 then the value of (16) is unchanged if the element in the a_{11} position is replaced by any value

M whatever. (Think of the determinant as being expanded by minors of the first row or column.) We shall specify M presently. By elementary operations (16) with the a_{11} element replaced by M may be converted to upper-right triangular form resulting in a determinant of the form

$$\begin{vmatrix} M-Q & 0 \ldots\ldots 0 \\ & d_1 \quad 0 \,.\,.\, 0 \\ & \quad d_2 \,.\, 0 \\ & \qquad d_n \end{vmatrix}$$

in which Q represents the sum of the quantities subtracted from M in producing all 0's in the upper triangle, $M - Q, d_1, \ldots, d_n$ are the elements along the main diagonal, and the remaining elements are irrelevant, since the value of the determinant is the product

$$(M - Q)d_1 d_2 \ldots d_n,$$

and is independent of M. By choosing M to be first $Q - 1$, then $Q + 1$, we see that this product must be 0, otherwise it would change in sign. But having the value 0 is in contradiction with its being positive. Thus A_{11}, the Jacobian for $\{U_2/U, \ldots, U_n/U\}$, is not identically 0.

(for § 5.7)

The linear programming approach to finding probability bounds has also been developed (apparently independently of HAILPERIN 1965) in

S. Kounias and J. Marin. Best Linear Bonferroni Bounds. *SIAM Journal of Applied Mathematics*, Vol. 30 (1976), 307–323.

Another, and less general, approach is presented in § 2.6 of

Alfred Rényi. *Foundations of Probability*. Holden-Day, Inc. 1970.

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CHAPTER 6

APPLICATIONS. PROBABILITY LOGIC

In the applications portion of this chapter we shall be including consideration of material not only from *Laws of Thought* but also from various papers which Boole published prior and subsequent to it. Throughout this material there are, in connection with conditional probability, frequent references to events as "causes" or "effects". This language was in common use at the time and Boole followed the practice, though he was aware that the relationship of "cause" to "effect" in these contexts was not identical with that of physical causation. Generally, a "cause" is the, or a, conditioning event, and an "effect" the, or a, event conditioned. For example, in $P(A|B)$ A is referred to as "effect" and B as "cause".

The notion of a probability logic is not found in Boole, although there are many instances of probabilistic inferences. The logic we present in §§ 6.6, 6.7 is such a natural outgrowth of his ideas that we feel obliged to include it in this treatise.

§ 6.1. Michell's argument and inverse probability

Our topic for this section has its origin in a *Philosophical Transactions* paper "An Inquiry into the Probable Parallax and Magnitude of the Fixed Stars from the Quantity of Light which they afford us, and the Particular Circumstance of their Situation" (MICHELL 1767). Among other items in the paper, Michell has an argument for there being a compelling reason, "either by the original act of the Creator, or in consequence of some general law, such perhaps as gravity" accounting for the stars being in groups, on the basis of "the greatness of the odds against things having been in the present situation, if it was not owing to some such cause." On the assumption of a random distribution of the stars over the celestial sphere [with equal probability of a star being in any of the 13,131 subregions of 2° diameter], he computes the odds that, of the 230 stars comparable in brightness to the double star β Capricorni, two should fall within that angular distance and finds it to be 80 to 1. When more stars are taken into account, e.g. the six brightest stars of the Pleiades, the odds against such a close grouping amount to 500,000 to 1. Since there are a very large number of such groupings the conclusion is that it is "next to a certainty" that there is a cause for these groupings and that it is not a matter of chance. To outline Michell's argument, let S = present situation of the stars, L = existence of a general law. Then

$$P(S|\bar{L}) \approx 0 \to P(\bar{S}|\bar{L}) \approx 1$$
$$\to P(L|S) \approx 1,$$

and since $P(S|\bar{L}) \approx 0$ (β Capricorni, Pleiades, etc.), then $P(L|S) \approx 1$. Note the tacit use of $P(\bar{S}|\bar{L}) = P(L|S)$.

Michell's argument was widely accepted. Not until some 80 years later was it (the argument, that is, not the conclusion) vehemently objected to by J. D. Forbes [1850]. His paper analyzes Michell's argument at great length, describing fallacies he finds and, incidentally, some errors in the probability calculations (e.g. that the β Capricorni odds should, on Michell's assumptions, be 160 to 1). We shall simply state Forbes' two principal objections: (i) Michell takes the high improbability of an event's happening, when it is one of a great many possibilities as that of the event where it is already the case. ("The improbability, for instance, of a given deal producing a given hand at whist is so immense, that were we to assume Mitchell's[1] principle, we should be compelled to assign to it as the result of an active cause with far more probability then even

[1] Forbes, and Boole likewise, spell the name with a "*t*". The two papers of Michell's I have seen, and the *Dictionary of National Biography*, spell it without a "*t*".

found by him for the physical connexion of the six stars of the Pleiades.") (ii) Michell's assumption of a uniform probability distribution (to use a present-day term) for any star and any of the subregions of the celestial sphere "leads to conclusions obviously at variance with the idea of random or lawless distribution, and is therefore not the expression of that Idea." Forbes likens it to assuming that any face of a die has an equal chance of coming up before one knows whether the die is loaded or not.

Forbes' paper stimulated the writing of BOOLE 1851a. Although it is Boole's first published paper on probability he says in it that the subject of Forbes' discussion "is closely related to a class of speculations in the pursuit of which I have been long engaged..." and hints at having had a general method for a "considerable period." In contrast to Forbes' prolixity Boole, in a couple of paragraphs, using a different explanation, exposes the core of Michell's fallacious argument (BOOLE 1851a = 1952, pp. 249–250):

> The proper statement of Mr. Mitchell's problem, as relates to β Capricorni, would therefore, be the following:–
>
> 1. Upon the hypothesis that a given number of stars have been distributed over the heavens according to a law or manner whose consequences we should be altogether unable to foretell, what is the probability that such a star as β Capricorni would nowhere be found?
>
> 2. Such a star as β Capricorni having been found, what is the probability that the law or manner of distribution was not one whose consequences we should be altogether unable to foretell?
>
> The first of the above questions certainly admits of a perfectly definite numerical answer. [Forbes denied that this was possible unless one had a specific probability distribution. Michell, in effect, does assume one.] Let the value of the probability in question be p. It has then generally been maintained that the answer to the second question is also p, and against this view Prof. Forbes justly contends. [Forbes, who is not explicit on this point, is being given more credit than is warranted.]

Boole then goes over to an abstract formulation:

Let us state Mr. Mitchell's problem, as we may now do, in the following manner:—There is a calculated probability p in favour of the truth in a particular instance of the proposition. If a condition A has prevailed, a consequence B has not occurred. Required the similar probability for the proposition, if a consequence B has occurred, the condition A has not prevailed.

Now, the two propositions are logically connected. The one is the "negative conversion" of the other; and hence, if either is *true* universally, the other is so. It seems hence to have been inferred, that if there is a probability p in a special instance in favour of the former, there is the same probability p in favour of the latter. But this inference would be quite erroneous. It would be an error of the same kind as to assert that whatever probability there is that a stone arbitrarily selected is a mineral, there is the same probability that a non-mineral arbitrarily selected is a non-stone. But that these probabilities are different will be evident from their fractional expressions, which are—

1. $$\frac{\text{Number of stones which are minerals}}{\text{Number of stones}}$$

2. $$\frac{\text{Number of non-minerals which are not stones}}{\text{Number of non-minerals}}$$

It is true that if either of these fractions rises to 1, the other does also; but otherwise, they will, in general, differ in value.

As already noted and discussed in §§ 4.1, 4.6, Boole is here mistakenly taking a conditional probability as the probability of a conditional proposition. However the upshot of his objection is sound since he correctly treats the probability of his "conditional" as a conditional probability. He then goes on to state his answer to the problem of finding the probability P of "the Proposition, if the event B does happen, the condition A has not been satisfied," given "the probability (p) of the truth of the proposition, If the condition A is satisfied, the event B will not happen", namely

(1) $$P = \frac{c(1-a)}{c(1-a)+a(1-p)},$$

"where c and a are arbitrary constants, whose interpretation is a follows: viz. a is the probability of the fulfilment of the condition A, c the probability that the event B would happen if the condition A were not satisfied." What Boole has derived by his special methods (details were given later in *Laws of Thought*, p. 366) is a simple case of what is now dubbed "Bayes' Rule"; in modern notation (1) is

$$(2) \qquad P(A|B) = \frac{P(B|\bar{A})P(\bar{A})}{P(B|\bar{A})P(\bar{A}) + P(B|A)P(A)}$$

and is a straightforward consequence of the multiplication rule (or definition of conditional probability). Boole stresses the indefiniteness of the value which the formula (1) gives if there is no further information concerning c and a. He also discusses results obtained by taking various specific values for c and a here, and also in a follow-up letter [1851b]. In this follow-up letter he mentions arguments of a reviewer in the Edinburgh Review, Laplace, and De Morgan as having the same erroneous notion as that of Michell, but tempers his criticism in the cases of Laplace and De Morgan by saying that the *reasoning* is not in error, as they are tacitly assuming values for c and a. These remarks of Boole's are later repeated, with some elaboration, in Chapter XX of *Laws of Thought*.

§ 6.2. Boole's Challenge Problem

Boole's letter to the *Philosophical Magazine* on Michell's Problem has a follow-up one [1851b = 1952 IX] in which, among other comments, he mentions the inquiry of a "correspondent" (= W. F. Donkin) as to whether the general method which he had alluded to in his discussion of Michell's Problem "involves any fundamentally different idea of probability from that which is commonly accepted." Boole doesn't address himself directly to this question, but shortly thereafter there appeared in *The Cambridge and Dublin Mathematical Journal* [1851c = 1952 X] a problem proposed by him "as a test of the sufficiency of received methods."

> The question is the following:—If an event C can only happen as a consequence of some one or more of certain causes A_1, $A_2, \ldots, A_n$, and if generally c_i represent the probability of the cause A_i, and p_i the probability that if the cause A_i exist the event E will exist, then the series of values $c_1, c_2, \ldots, c_n, p_1, p_2, \ldots, p_n$, being given, required the probability of the event E.
>
> It is to be noted that in this question the quantity c_i represents the total probability of the existence of the cause A_i, not the probability of its exclusive existence; and p_i the total probability of the existence of the event E when A_i is known to exist, not the probability of E's existing as a *consequence* of A_i. By the cause A_i is indeed meant the event A_i with which in a proportion p_i of the cases of its occurrence the event E has been associated.

Some two years later there is a note by Cayley [1853] presenting a solution for the case of $n = 2$, which he states as:

> Given the probability α that a cause A will act, and the probability p that A acting the effect will happen; also the probability β that a cause B will act, and the probability q that B acting the effect will happen; required the total probability of the effect.

After illustrating the problem with an example he simply asserts:

> The sense of the terms being clearly understood, the problem presents of course no difficulty. Let λ be the probability that the cause A acting will act efficaciously; μ the probability that the cause B acting will act efficaciously; then
>
> $$p = \lambda + (1 - \lambda)\mu\beta$$
>
> $$q = \mu + (1 - \mu)\alpha\lambda,$$
>
> which determine λ, μ; and the total probability ρ of the effect is given by
>
> $$\rho = \lambda\alpha + \mu\beta - \lambda\mu\alpha\beta.$$

Exactly what Cayley means by "acting efficaciously" isn't stated, but from the example given it appears that, letting E be the event of the effect taking place, A_e that A acts efficaciously, B_e that B acts efficaciously, he is assuming

$$E = A_e \vee \bar{A}_e B_e, \qquad A_e \to A, \qquad B_e \to B.$$

The introduced quantities λ and μ are the

$$\lambda = P(A_e|A), \qquad \mu = P(B_e|B),$$

and his equations relating λ and μ to $p[= P(E|A)]$, $q[= P(E|B)]$, $\alpha[= P(A)]$, $\beta[= P(B)]$ are then

$$\begin{aligned} &P(E|A) = P(A_e|A) + (1 - P(A_e|A))P(B_e|B)P(B) \\ &P(E|B) = P(B_e|B) + (1 - P(B_e|B))P(A_e|A)P(A) \end{aligned} \tag{1}$$

Now if $E = A_e \vee \bar{A}_e B_e$, then

$$\begin{aligned} P(E|A) &= P(A_e|A) + P(\bar{A}_e B_e|A) \\ &= P(A_e|A) + P(\bar{A}_e|A)P(B_e|\bar{A}_e A), \end{aligned}$$

which doesn't coincide with the first equation in (1) except if

$$P(B_e|\bar{A}_e A) = P(B_e|B)P(B) \quad [= P(B_e B) = P(B_e)],$$

i.e. except if B_e and $\bar{A}_e A$ are independent. Similarly, to obtain the second of Cayley's equations requires A_e and $\bar{B}_e B$ to be independent. Moreover, the formula he gives to obtain $P(E)$ from λ and μ, namely

$$\rho = \lambda\alpha + \mu\beta - \lambda\mu\alpha\beta$$

holds good only if A_e and B_e are independent; and although A_e and B_e independent implies $\bar{A}_e$ and B_e independent, it does not imply that $\bar{A}_e A$ and B_e are independent. Thus three additional assumptions are needed to justify Cayley's solution as an answer to the Challenge Problem.

Boole in [1854a = 1952 XI] contends that Cayley's solution can't be valid since it doesn't check out when $p = P(E|A) = 1$ and $q = P(E|B) = 0$. For these values the quadratic equation for $P(E)$ which results when λ and μ are eliminated from Cayley's three equations yields roots 1 or $\alpha(1 - \beta)$, neither of which, Boole says, can be correct since the answer ought to be α. He presents the quadratic formula which he says his solution leads to and that his "investigation" of the problem is in "a treatise now on the eve of publication" [i.e. *Laws of Thought*].

Dedekind [1855, written July 1854], accepting Cayley's analysis of the problem, defends it against Boole's charge by pointing out the obvious error in Boole's counter-example, namely that for $p = 1$, $q = 0$, Cayley's equations for λ and μ imply that $\alpha = 0$. Thus $\alpha(1-\beta)$ is then the correct value. Dedekind goes on to stress the necessity of specifying the ranges of values for parameters for which the problem is meaningful and shows, in the case at hand, that having the differences $p - \beta q$ and $q - \alpha p$ non-negative is a necessary and sufficient condition; and that, geometrically measured, only $\frac{1}{2}$ the possible values for the parameters p, q, α, β provide meaningful values for the problem.

The question, for the case $n = 2$, appears as PROBLEM I in Chapter XX (Problems Relating to the Connexion of Causes and Effects) of *Laws of Thought*. The solution Boole gives uses the full panoply of his probability method and is quite long. We describe the highlights.

Using x, y, z to represent, respectively, the events of the causes A_1, A_2 and the effect E happening, Boole then has the given numerical data

$$P(x) = c_1, \qquad P(y) = c_2$$
$$P(xy) = c_1 p_1, \qquad P(yz) = c_2 p_2$$

together with the logical condition

$$z\bar{x}\bar{y} = 0,$$

expressing that E happens only if A_1 or A_2 happen. Setting $s = xz$, $t = yz$ and combining these two equations with the given logical condition, results in a logical equation whose solution for z is

$$z = \frac{s+t}{s+t-(x\bar{s}+s\bar{x}+y\bar{t}+t\bar{y}+\bar{x}\bar{y})}.$$

Development of the right-hand side results in a $V = stxy + s\bar{t}x\bar{y} + \bar{s}t\bar{x}y + \bar{s}\bar{t}$, and provides the equation for the absolute probabilities of the independent events x, y, x, t (represented by x, y, s, t

taken in the numerical sense):

$$c_1[= P(x|V)] = (stxy + s\bar{t}x\bar{y} + \bar{s}\bar{t}x)/V$$

$$c_1p_1[= P(s|V)] = (stxy + s\bar{t}x\bar{y})/V$$

$$c_2[= P(y|V)] = (stxy + \bar{s}t\bar{x}y + \bar{s}\bar{t}y)/V$$

$$c_2p_2[= P(t|V) = (stxy + \bar{s}t\bar{x}y)/V$$

and the equation for the sought-for probability

$$u[= P(z|V)] = (stxy + s\bar{t}x\bar{y} + \bar{s}t\bar{x}y)/V.$$

Boole adroitly eliminates the quantities x, y, s, t, V from these five algebraic equations to obtain his equation for u,

$$(1) \qquad \frac{(u - c_1p_1)(u - c_2p_2)}{c_1p_1 + c_2p_2 - u} = \frac{(1 - c_1\bar{p}_1 - u)(1 - c_2\bar{p}_2 - u)}{1 - u}$$

which he had cited in his note [1854a = 1952 XI] disputing Cayley's solution. The question of which root should be taken is next considered.

Boole had already shown (in Chapter XIX) that the conditions of possible experience for this problem are:

lower limits: c_1p_1, c_2p_2

upper limits: $1 - c_1(1 - p_1)$, $1 - c_2(1 - p_2)$, $c_1p_1 + c_2p_2$

(see our discussion in § 4.7). He then shows that the equation for u has one and only one root between the smaller of the upper limits and the larger of the lower limits. This is, of course, establishing in a particular instance the result whose general case we discussed in § 5.6.

Wilbraham's paper [1854] (discussed in part in § 4.8) contains a critical analysis of Boole's solution of the Challenge Problem. He expresses the given conditions and also that for $P(E)$ as linear equations in the probabilities of constituents and points out that there are only six equations and seven unknowns that have to be eliminated, so that the problem cannot (in general) have a determinate solution. He notes that of the independence relations introduced by Boole in his use of x, y, s, t as independent events only two, namely

$$\frac{P(A_1A_2E)}{P(\bar{A}_1A_2E)} = \frac{P(A_1\bar{A}_2E)}{P(\bar{A}_1\bar{A}_2\bar{E})}$$

$$\frac{P(A_1A_2\bar{E})}{P(\bar{A}_1A_2\bar{E})} = \frac{P(A_1\bar{A}_2\bar{E})}{P(\bar{A}_1\bar{A}_2\bar{E})}$$

are relevant in this particular instance. He compares this with the results of a similar analysis of Cayley's solution which likewise involves (somewhat different) independence conditions.

The substance of Boole's reply [1854b = 1952 XII and part of 1854d = 1952 XIV] to Wilbraham's criticism centers on the Challenge Problem solutions. He asserts that there is no doubt that Cayley's is "erroneous", repeating his error we have already referred to. In his counter-example to Cayley's result he mistakenly attributes the same conditions of possible experience as for his solution. (That these conditions would not be the same is apparent from Dedekind's note, written in the same month as Boole's reply to Wilbraham, but Dedekind's note did not appear in print until the next year.) In contrast to the (asserted) erroneousness of Cayley's solution Boole says: "On the other hand, there are no cases whatever in which the problem is solvable by other methods, which do not furnish a verification of the solution I have given." Surprisingly, a candidate for proving Boole's solution to be wrong doesn't appear until MacColl 1880. We shall discuss it presently.

Some six years later Cayley returns to the question. His communication [1862] contains full expositions of his solution (taking into consideration Dedekind's contribution) and also of Boole's. These are followed by comments by Boole (quoted from a letter), a response from Cayley, and then a further reply by Boole.

Cayley begins by recognizing the difference between his interpretation of the Challenge Problem which the calls the *Causation* form and Boole's which he denominates the *Concomitance* form. Before presenting Boole's solution Cayley says [1862, 255]:

> Although given as a solution of the causation statement of the question, as already remarked, it seems to be (and I think Prof. Boole would say that it is) a solution of the concomitance

> statement of the question. It is certainly a most remarkable and suggestive one; I am strongly inclined to believe that it is correct; which of course does not interfere with the correctness of my solution, if the two really belong to distinct questions.
>
> I reproduce Prof. Boole's solution, without attempting to explain (indeed I do not understand to my own satisfaction) the logical principles upon which it is based. It is conducted by means of auxiliary quantities x, y, s, t, which are quantities replacing logical symbols originally represented by the same letters.

Boole, as quoted by Cayley, no longer considers Cayley's solution erroneous [1862, p. 361]:

> 1st. "I think that your solution is correct under conditions partly expressed and partly implied. The one to which you direct attention is the assumed independence of the causes denoted by A and B. Now I am not sure that I can state precisely what the others are; but one at least appears to me to be the assumed independence of the events of which the probabilities according to your hypothesis are $\alpha\lambda$, $\beta\mu$. Assuming the independence of the causes as to *happening*, I do not think that you are entitled *on that ground* to assume their independence as to *acting*;..."

and concerning Cayley's difficulties in understanding the "ideal events" Boole remarks [p. 363]:

> 6thly. The x, s, *&c.*, about the interpretation of which you inquire, are the probabilities of ideal events in an ideal problem connected by a formal relation with the real one. I should fully concede that the auxiliary probabilities which are employed in my method always refer to an ideal problem; but it is one, the form of which, as given by the calculus of logic, is not arbitrary. Nor does its connexion with the real problem appear to me arbitrary. It involves an extension, but as it seems to me a perfectly scientific extension, of the principles of the ordinary theory of probabilities...."

To this point Cayley replies [p. 364]:

> 6thly. I do not in anywise assert, or even suppose, that the ideal problem is arbitrary, or that its connexion with the real problem is arbitrary. I simply do not know what the ideal problem is; I do not know the point of view, or the assumed mental state of knowledge or ignorance according to which x, y, s, t are the probabilities of A, B, AE, BE. It is to be borne in mind that x, y, s, t are, in Prof. Boole's solution, determined as numerical quantities included between the limits 0 and 1, i.e. as quantities which are or may be actual probabilities. What I desiderate is, that Prof. Boole should give for his auxiliary quantities x, y, s, t such an explanation of the meaning as I have given for my auxiliary quantities λ, μ. I do not find any such explanation in the memoir referred to.

And in an addendum to his communication [pp. 364–365]:

> Prof. Boole, in his reply, dated April 2, writes, "No such explanation as you desiderate of the interpretation of the auxiliary quantities in my method of solution is possible; because they are not of the nature of additional data, and their introduction does not limit the problem as any hypotheses which are of that nature do. I do not see any difficulty whatever in the conception of the ideal problem."
>
> We join issue as follows: Prof. Boole says that there is no difficulty in understanding, I say that I do not understand the *rationale* of his solution.

We believe that our § 5.4 provides the rationale that Cayley was asking for.

Hugh MacColl's *The Calculus of Equivalent Statements* (*Fourth Paper*) [1880] is noteworthy for its introduction of the notation 'x_a' for "the *chance* that the statement x is true *on the assumption that the statement a is true*". (Confer Peirce's 'b_a' of his [1867] described in our § 4.8.) In this paper MacColl solves Boole's Challenge Problem as follows (we use modern notation):

$$
\begin{aligned}
P(E) &= P(A_1E + A_2E) \qquad (E \to A_1 \vee A_2) \\
&= P(A_1E) + P(A_2E) - P(A_1A_2E) \\
&= P(A_1)P(E|A_1) + P(A_2)P(E|A_2) - P(A_1A_2)P(E|A_1A_2) \\
&= c_1p_1 + c_2p_2 - P(A_1A_2)P(E|A_1A_2).
\end{aligned}
$$

He then goes on to specialize to the case of A_1 and A_2 being independent and the event E more probable when both causes exist than when only one of them does. Then $P(A_1A_2) = c_1c_2$ and, $P(E|A_1A_2)$ being greater than p_1 or p_2, one has $c_1p_1 + c_2p_2 - c_1c_2p_1$ and $c_1p_1 + c_2p_2 - c_1c_2p_2$ as upper limits and $c_1p_1 + c_2p_2 - c_1c_2$ as a lower limit. Taking numerical values of $c_1 = .1$, $c_2 = .2$, $p_1 = .6$, and $p_2 = .7$ MacColl computes limits of .18 and .186 between which $P(E)$ lies. But using these values to compute the exact value for $P(E)$ which Boole's solution gives results in the value .19069, which lies outside the interval, and thus his conclusion that Boole's solution is wrong.

While Boole didn't live to see this example of MacColl's, it is easy to surmise the kind of rejoinder he would have made if he had. Namely, that in effect MacColl is changing the problem by having additional conditions which should be included in the data. Thus one should add

$$
\begin{aligned}
&P(A_1A_2)[= P(A_1)P(A_2)] = c_1c_2 \\
&P[A_1A_2E] \geq p_ic_1c_2, \qquad (i = 1, 2)
\end{aligned}
$$

the latter coming from $P(E|A_1A_2) \geq P(E|A_i)$. Given these additional conditions Boole's technique for finding limits within which his solution would be then gives:

upper limits:	$c_1p_1 + c_2p_2 - c_1c_2p_1$	(as with MacColl)
	$c_1p_1 + c_2p_2 - c_1c_2p_2$	(as with MacColl)
	$c_1p_1 + (1 - c_1)c_2$	
	$c_2p_2 + (1 - c_2)c_1$	
lower limits:	$c_1p_1 + c_2p_2 - c_1c_2$	(as with MacColl)
	c_1p_1	
	c_2p_2,	

so that the computed value for Boole's solution of this problem would indeed lie within MacColl's limits.

A similar misunderstanding of Boole's method occurs in KEYNES 1921, p. 187. After citing Boole's equation (1) for determining $u[= P(E)]$, and the upper and lower limits for obtaining the proper root, Keynes says:

> This solution can easily be seen to be wrong. For in the case where A_1 and A_2 cannot both occur, the solution is $c_1 p_1 + c_2 p_2$; whereas Boole's equations do not reduce to this simplified form.

But Boole's method would require adjoining $A_1 A_2 = 0$ to the data. If one does this then the answer does come out to be $u = c_1 p_1 + c_2 p_2$. (The easiest way to see this is to take Boole's derivation, pp. 322–323, and delete the constituents containing the product xy.) The discrepancy between Boole's and Keynes' result can be explained by noting that the concomitant (tacitly assumed in Wilbraham's view) independence conditions of Boole's method are not the same when one alters the data.

§ 6.3. Further problems on causes

The problems discussed in the preceding sections of this chapter are but two of ten treated by Boole in his Chapter XX (Problems Relating to the Connexion of Causes and Effects). The first of the ten is the $n = 2$ case of the Challenge Problem, to which we have just devoted the preceding section. However, since certain aspects of the discussion are in other places we here collect everything together in a brief summary.

> PROBLEM I.—The probabilities of two causes A_1 and A_2 are c_1 and c_2 respectively. The probability that if the cause A_1 present itself, an event E will accompany it (whether as a consequence of the cause A_1 or not) is p_1, and the probability that if the cause A_2 present itself, that event E will accompany it, whether as a consequence of it or not, is p_2. Moreover, the event E cannot appear in the absence of both causes A_1 and A_2.* [Footnote omitted] Required the probability of the event E. [p. 321]

That Boole's interpretation of the notion of cause is, in Cayley's phrase, that of *Concomitance* and not *Causation* is here apparent from the parenthetical phrase, and also from the example in Boole's footnote (flooding of a field by upper sources of the River Lee, or tides from the ocean). The footnote example shows him to be considering probability as idealized frequency and conditional probability as simply subfrequency of the "given" event. Although the statement of the problem refers to conditional probabilities, i.e. $P(E|A_1)$ and $P(E|A_2)$, by virtue of having the probabilities of the causes given one can express the problem solely in terms of absolute probabilities, namely:

$$\text{given:}\quad P(A_1) = c_1, \qquad P(A_2) = c_2,$$

$$P(A_1E) = c_1p_1, \quad P(A_2E) = c_2p_2,$$

$$E\bar{A}_1\bar{A}_2 = 0,$$

$$\text{find:}\quad P(E).$$

As we have noted in the preceding section, Boole's adopted principles enable him to deduce that the value of $P(E)$ satisfies a quadratic equation (obtained from the data and his principles) and that only one of the roots meets the conditions of the possible experience (conditions of limitation) for the problem, i.e. lies between the

$$\text{lower limits:}\quad c_1p_1, \quad c_2p_2$$

and the

$$\text{upper limits:}\quad 1-c_1(1-p_1), \quad 1-c_2(1-p_2), \quad c_1p_1+c_2p_2.$$

Moreover, these conditions of limitation give consistency conditions for the parameters; these are that each lower limit shall not exceed any upper limit (0 as a lower limit and 1 as an upper limit are assumed, but not explicitly stated, by Boole).

Wilbraham's formulation of the problem "without Boole's assumptions" [1854 = BOOLE 1952, p. 481] leads to a set of linear equations in 7 quantities (probabilities of constituents) and gives for $P(E)$

$$c_1p_1+c_2p_2-P(EA_1A_2).$$

He says: "We can get no further in the solution without further assumptions or data, having only six equations from which to eliminate seven unknown quantities. Without such the question is indeterminate."

However recognizing (in agreement with Wilbraham) that a value of $P(E)$ need not be determined, but looking at the problem as one of finding, subject to the given conditions on A_1, A_2, and E, best possible upper and lower bounds for $P(E)$ good in all probability algebras, we found (§ 5.7) that these bounds coincided with the minimum of Boole's upper limits and the maximum of his lower limits. This, then, is our solution to Boole's Challenge Problem.

PROBLEM II (p. 326) differs in no essential respects from PROBLEM I, merely having the side condition $\bar{A}_1\bar{A}_2 = 0$ in place of $E\bar{A}_1\bar{A}_2 = 0$. Likewise PROBLEM III (p. 327) amounts to nothing much different. It drops the side condition, has just

$$P(A_1) = c_1, \qquad P(A_2) = c_2$$

$$P(EA_1) = c_1 p_1, \quad P(EA_2) = c_2 p_2$$

as given data, and asks for the conditional probability $P(A_1 | A_2)$ [or of $P(A_2 | A_1)$]. But since

$$P(A_1 | A_2) = \frac{P(A_1 A_2)}{P(A_2)} = \frac{1}{c_2} P(A_1 A_2)$$

the question reduces to one of finding an absolute probability, i.e. $P(A_1 A_2)$.

PROBLEM IV, continuing in the same vein as III, adjoins the probability of E as a datum.

In PROBLEM V, given are the probabilities of three events and also of their conjunction and asked for is the probability of the conjunction of two of them:

given: $P(A_1) = p$, $P(A_2) = q$, $P(A_3) = r$, $P(A_1 A_2 A_3) = m$

find: $P(A_1 A_2)$

Wilbraham's paper [1854] discusses this problem in detail (inessentially modified in asking for $P(A_1\bar{A}_2)$ in place of $P(A_1 A_2)$) to illustrate what he contends is going on in Boole's method. (See our

§ 4.8.) Boole's stated solution is

$$P(A_1A_2) = \frac{H - \sqrt{(H^2 - 4pq\bar{r}^2 - 4\bar{p}\bar{q}\bar{r}m)}}{2\bar{r}}$$

where $H = \bar{p}\bar{q} + (p+q)\bar{r}$. Details of the computation are not given, nor the conditions of limitation on the parameters. One can readily show (or see from a Venn diagram) that $P(A_1A_2)$ is not fully determined and that, under the given constraints, m and $p+q-1$ are lower limits and p, q, $1+r+m$ are upper limits and that in any probability algebra

$$\max\{m, p+q-1\} \leq P(A_1A_2) \leq \min\{p, q, 1-r+m\},$$

with appropriate consistency conditions on the constants (e.g. $0 \leq m \leq r$, $q \leq 1$, etc.).

Problem VI is the Challenge Problem for general n. Here, unlike for the case of $n = 2$, no explicit expression for the value which Boole's method leads to can be given since it is a root of an n-th degree algebraic equation. With $c_i = P(A_i)$ and $p_i = P(E|A_i)$, the limits he obtains are

lower limits: $c_1p_1, c_2p_2, \ldots, c_np_n$

upper limits: $\bar{c}_1 + c_1p_1, \bar{c}_2 + c_2p_2, \ldots, \bar{c}_n + c_np_n$,

$c_1p_1 + c_2p_2 + \cdots + c_np_n$.

By introducing $n-1$ additional constants d_i,

$$d_i = P(E\overline{\bar{A}_1 \ldots \bar{A}_{i-1}}A_i),$$

Keynes [1921, pp. 189–190] derives a formula for $P(E)$:

$$P(E) = \sum_{i=1}^{n} c_ip_i - \sum_{i=2}^{n} d_i.$$

(For a shorter proof than Keynes' see our endnote 1 for § 6.3.)

Problem VII brings in a new aspect. We have the same data as for the Challenge Problem, namely the probabilities $P(A_i) = c_i$ of the causes and the conditional probabilities $P(E|A_i) = p_i$, but required now is "the probability, that if any definite and given combination of the causes A_1, $A_2, \ldots, A_n$ presents itself, the event E will be realized." If $\varphi(A_1, \ldots, A_n)$ represents the combination, then what is being sought is the conditional probability $P(E|\varphi(A_1,\dots,A_n))$, i.e. the quotient

$$\frac{P(E\varphi(A_1,\dots A_n))}{P(\varphi(A_1,\dots A_n))}.$$

Boole's approach to this problem is to use his method to determine, separately, the value of the numerator and of the denominator. But this procedure can't be generally valid. For example, if $\varphi(A_1, A_2) = A_1A_2$ then

$$P(E|A_1A_2) = \frac{P(EA_1A_2)}{P(A_1A_2)}$$

and a Venn type diagram with area representing probability suffices to show that the area representing the probabilities of A_1, A_2, EA_1, EA_2 can be held fixed but those for EA_1A_2 and A_1A_2 can be made to vary independently (though not completely, since one has to have $P(EA_1A_2) \leq P(A_1A_2)$). This particular case of PROBLEM VII with $\varphi(A_1, A_2) = A_1A_2$ comes up in Boole's Keith Prize Memoir and is discussed in our next section. We shall see that the special methods of § 6.7 enable us to obtain best possible upper and lower bounds on $P(E|A_1A_2)$ (Example 6.74 in § 6.7).

PROBLEM VIII likewise asks for $P(E)$ given the probabilities of the causes ($P(A_i) = c_i$) and of the conditional events ($P(E|A_i) = p_i$) but additionally has the general side conditions

$$\bar{A}_1\bar{A}_2\dots,\bar{A}_n = 0, \quad \varphi(A_1,\dots,A_n) = 1$$

for arbitrary φ. He gives his typical implicitly expressed result but shows that it reduces to the standard $P(E) = c_1p_1 + \cdots + c_np_n$ when the A_i are exclusive ($\varphi(A_1,\dots,A_n) = A_1\bar{A}_2\dots\bar{A}_n + \cdots + \bar{A}_1\bar{A}_2\dots A_n$) as well as exhaustive ($\bar{A}_1\bar{A}_2\dots\bar{A}_n = 0$).

PROBLEM IX considers what would result if, in each of the preceding problems, the desideratum were the *a posteriori* $P(A_r|E)$ in place of $P(E)$. Since he believes that in each of these problems the data imply a "determinate" value for $P(E)$, the result would then be the quotient of $P(EA_r)$ by $P(E)$, i.e., $c_rp_r/P(E)$.

Boole's final problem in Chapter XX looks superficially like the Law of Succession problem (to be presently discussed) but differs significantly from it by assuming known the probability of the occurrence of the recurring event and also that of a "permanent" cause of the event (p. 358):

> The following problem is of a much easier description than the previous ones.
>
> PROBLEM X.—The probability of the occurrence of a certain natural phenomenon under given circumstances is p. Observation has also recorded a probability a of the existence of a permanent cause of that phenomenon, i.e. of a cause which would always produce the event under the circumstances supposed. What is the probability that if the phenomenon is observed to occur n times in succession under the given circumstances, it will occur the $n+1$th time? What also is the probability, after such observation, of the existence of the permanent cause referred to?

If we let C represent Boole's "permanent cause" and $A_1, \ldots, A_n, A_{n+1}$ the successive occurrences of the phenomenon then we may write his interpretation of this problem as:

given: (i) $P(C) = a, \quad P(A_1) = P(A_2) = \ldots = P(A_{n+1}) = p$

(ii) $C \to A_i \qquad (i = 1, \ldots, n+1)$

find: (I) $P(A_{n+1} | A_1 A_2 \ldots A_n)$, and

(II) $P(C | A_1 A_2 \ldots A_n)$.

To solve (I) Boole applies his method to the separate probabilities $P(A_1 A_2 \ldots A_n A_{n+1})$ and $(P(A_1 A_2 \ldots A_n)$ and then forms their quotient, obtaining

$$\frac{P(A_1 A_2 \ldots A_n A_{n+1})}{P(A_1 A_2 \ldots A_n)} = \frac{a + (p-a)\left(\dfrac{p-a}{1-a}\right)^n}{a + (p-a)\left(\dfrac{p-a}{1-a}\right)^{n-1}}$$

and then the solution for (II) is

$$\frac{P(A_1 A_2 \ldots A_n C)}{P(A_1 A_2 \ldots A_n)} = \frac{P(C)}{P(A_1 A_2 \ldots A_n)} = \frac{a}{a + (p-a)\left(\dfrac{p-a}{1-a}\right)^{n-1}}$$

Keynes [1921, pp. 192–193] shows that these results which Boole has for (I) and (II) can be obtained by ordinary techniques if one adjoins to Boole's data the independence conditions

$$P(A_i|A_1A_2\ldots A_{i-1}\bar{C}) = P(A_i|C),$$

i.e., the conditions that if C does not hold, then preceding occurrences of the phenomenon are irrelevant to the next occurrence. Our endnote 2 for § 6.3 presents an easier to read version of Keynes' derivation.

Boole comments on the circumstance that none of the ten problems on causes which he has so far considered involve any "arbitrary element"—meaning that the solution of the logical equation for the unknown doesn't have $\frac{0}{0}$ terms in its expansion—and thus furnishes a "determinate" probability. (We have seen, however, that even in such cases it is only by virtue of his special approach that definite values come out.) He then goes on to consider examples in which this is not the case, in particular Michell's problem on the distribution of stars and Laplace's on the inclination of the planetary orbits. These we have already discussed in § 6.1.

The chapter continues with a trenchant criticism of the "most usual mode of endeavoring to evade the *necessary* arbitrariness of the solution of problems in the theory of probabilities which rest upon insufficient data,...", which is to assume that the unknown probability is—to use present-day language—a random variable uniformly distributed over [0, 1], an idea going back to the model used in Bayes' memoir (see MAISTROV 1974, p. 100). As an example of the use of this assumption, Boole derives the well-known (Laplace) Rule of Succession: If an event (of unknown probability) has occurred m times in succession, the probability of its occurring an $(m+1)$st time is the conditional probability

$$\frac{\int_0^1 p^{m+1}\mathrm{d}p}{\int_0^1 p^m\mathrm{d}p} = \frac{m+1}{m+2}.$$

(The argument has a lot that's wrong with it—see KEYNES 1921, pp. 372–

378 for a history and critique.) In addition to thus calling attention to the assigning of "all possible degrees of probability" assumption, Boole remarks that there are other ways of looking at the problem, e.g. as a black-and-white-balls-in-an-urn situation and cites TERROT 1953. (For a modern treatment of the Rule of Succession *via* balls-in-an-urn see FELLER 1965, p. 113.) Boole points out that in this latter approach one assumes equal probabilities for all possible ratios of black to white balls, but that one could also, on the basis of "equal distribution of knowledge or ignorance", assume that "*all possible constitutions of the system of balls are equally probable.*" For an urn having μ *distinguishable* balls which can be either black or white there are 2^{μ} different *constitutions* possible, but only $\mu + 1$ different *ratios* of black to white.) On the basis of equally probable constitutions Boole gives an intuitive argument for the probability of the $(m + 1)$st occurrence of drawing a white ball being $\frac{1}{2}$, independently of m—that is, that past information has no relevance to the outcome. He then follows it with a mathematical proof which is lengthy and complicated because of the need to evaluate (as $\mu \to \infty$) a certain limit. It so happen that this limit is now recognizable [1] as an immediate consequence of the well-known proof of the Weierstrass Approximation Theorem using Bernstein polynomials. (See our endnote 3 for § 6.3.)

Boole's chapter on causes concludes with the comment (p. 375):

> 26. These results [on the Rule of Succession] only illustrate the fact, that when the defect of data is supplied by hypothesis [on the kind of distribution involved], the solutions will, in general, vary with the nature of the hypotheses assumed: so that the question still remains, only more definite in form, whether the principles of the theory of probabilities serve to guide us in the election of such hypotheses. I have already expressed my conviction that they do not—a conviction strengthened by other reasons than those above stated.

[1] As pointed out to me by my colleague, Bennett Eisenberg.

He goes on to mention an interesting attempt:

> Thus, a definite solution of a problem having been found by the method of this work, an equally definite solution is sometimes attainable by the same method when one of the data, suppose Prob. $x = p$, is omitted. But I have not been able to discover any mode of deducing the second solution from the first by *integration*, with respect to p supposed variable within limits determined by Chap. XIX. This deduction would, however, I conceive, be possible, were the principle adverted to in Art. 23 [uniform distribution over [0, 1] for an unknown probability] valid.

and ends on an autobiographical note:

> Still it is with diffidence that I express my dissent on these points from mathematicians generally, and more especially from one who, of English writers, has most fully entered into the spirit and the methods of Laplace; and I venture to hope, that a question, second to none other in the Theory of Probabilities in importance, will receive the careful attention which it deserves.

§ 6.4. Probability of judgements

Even in Boole's time the subject of probability of judgements (relating to decisions of judges, juries, assemblies, etc.), though developed by such able mathematicians as Condorcet, Laplace and Poisson, was no longer considered a viable theory, having been severely criticized in the 1830's and 1840's—severely because of the sensitive political and sociological ("moral sciences") implications of its use (Daston 1981). Boole's chapter on the subject in *Laws of Thought* (Chapter XXI, Particular Application of the Previous General Method to the Question of the Probability of Judgements) constitutes a relatively sympathetic critique which adopts the (to us, dubious) fundamental notions and general approach of the subject, extends its methodology, yet is pessimistic about useful results but only for want of data to determine certain key parameters. We summarize Boole's presentation.

Assuming that it is meaningful to speak of the probability that a member of a jury, A_i, will form a correct opinion on the case ($= x_i$) and of the probability that the accused party is guilty ($= k$), then for the probability of condemnation by A_i, one has

$$kx_i + (1-k)(1-x_i),$$

since either the accused is guilty and A_i judges correctly, or the accused is innocent and A_i judges incorrectly.

> In like manner, if there be n jurymen whose separate probabilities of correct judgement are $x_1, x_2 \ldots x_n$, the probability of an unanimous verdict of condemnation will be
>
> $$X = kx_1x_2\ldots x_n + (1-x_1)(1-x_2)\ldots(1-x_n).$$
>
> Whence, if the several probabilities $x_1, x_2 \ldots x_n$ are equal, and are each represented by x, we have
>
> $$X = kx^n + (1-k)(1-x)^n. \qquad (2)$$
>
> The [conditional] probability in the latter case, that the accused person is guilty [if condemned], will be
>
> $$\frac{kn^n}{kx^n + (1-k)(1-x)^n}.$$
>
> All these results assume, that the events whose probabilities are denoted by k, x_1, x_2 &c, are independent, an assumption which however so far as we are concerned is involved in the fact that those events are the only ones of which the probabilities are given. (pp. 377–378)

Note here that Boole justifies the independence of the events involved not by assumption but by an appeal to his principle which accords independence to simple unconditioned events.

In similar fashion one has for the probability, X_i, that i out of n of the jurymen declare for condemnation.

$$X_i = \binom{n}{i}\{kx^i(1-x)^{n-i} + (1-k)x^{n-i}(1-x)^i\} \qquad (3)$$

In Boole's view:

> It is apparent that the whole inquiry is of a very speculative character. The values of x and k cannot be determined by direct observation. We can only presume that they must both in general exceed the value of $\frac{1}{2}$; that the former, x, must increase with the progress of public intelligence; while the latter, k, must depend much upon those preliminary steps in the administration of the law by which persons suspected of crime are brought before the tribunal of their country. It has been remarked by Poisson, that in periods of revolution, as during the Reign of Terror in France, the value of k may fall, if account be taken of political offences, far below the limit $\frac{1}{2}$. [pp. 379–380]

Examining Laplace's assumption that all values of x between $\frac{1}{2}$ and 1 be considered equally probable [and exhaustive] he concludes: "This hypothesis is entirely arbitrary, and it would be unavailing here to examine its consequences." He then goes on to describe Poisson's method of deducing values of x and k from court records.

Using one set of records (for 1825–1830) Poisson found (with X_i as in (3))

$$X_7 + X_8 + \cdots + X_{12} = .4782$$

and for the year 1831, there having been a change of rule requiring a majority of four,

$$X_8 + X_9 + \cdots + X_{12} = .3631.$$

On the assumption that x and k were the same for both. The two equations then yield pairs of values for x and for k, the sum of the values in each pair being 1. (It is clear that there would have to be these mirror-image solutions since the X_i are invariant under the interchange of x and $1-x$, and of k and $1-k$. Ambiguity in the deduced values for x and k can only be resolved by a presumption as described by Boole.

Boole comments on the poor quality of the statistical data Poisson had available to him. More significantly, he notes that one could, if appropriate records were kept, have many more equations involving x

and k than just those resulting from the two kinds of majority votes, and briefly discusses the question of handling such a "supernumerary system of data". The contribution which Boole's *general method* makes to the subject then follows in a series of propositions. (In these the 'k' is omitted as the questions now considered concern only the correctness of judgment of members of a body and not guilt of an accused.)

PROPOSITION I.

> *From the mere records of the decisions of a court or deliberative assembly, it is not possible to deduce any definite conclusion respecting the correctness of the individual judgments of its members.*
>
> Though this Proposition may appear to express but the conviction of unassisted good sense, it will not be without interest to show that it admits of rigorous demonstration. [p. 382]

The question here is formulated as that of finding $P(x_i)$, for $i = 1, \ldots, n$, where x_i is the event of a member A_i uttering a correct opinion, and given as data is the set of equations.

$$P(X_j) = a_j \quad (j = 1, \ldots, m)$$

where X_j is a [Boolean, not numerical, as in the preceding usage] function of the events $x_1, \ldots, x_n$. The X_j are taken to be of the special kind which is invariant under interchange of x_i and $\bar{x}_i$; Boole's reason is that the court or assembly does not presume to know whether the decision or opinion was correct, and hence can provide only relative opinions of correctness, e.g. the frequency [probability] of unanimous votes (here $X = x_1x_2\ldots x_n + \bar{x}_1\bar{x}_2\ldots\bar{x}_n$) or the frequency with which A_1 differed from all other members (here $X = x_1x_2\ldots x_n + \bar{x}_1x_2\ldots x_n$), etc. To continue, Boole's general method starts with the equations

(1) $$X_1 = t_2, \quad X_2 = t_2, \quad \ldots, \quad X_m = t_m$$

and on elimination of, say, $x_2, \ldots, x_m$, obtains

(2) $$Ex_1 + E'(1 - x_1) = 0$$

leading to

$$x_1 = \frac{E'}{E' - E} = \frac{E}{0},$$

the latter by virtue of $E' = E$, coming from the special form of the X_j.

Thus x_1 (and likewise any other x_i) can have no specific probability value determined. We can verify this result of Boole's by noting that the elimination result (2) is equivalent to (§ 2.3)

(3) $$\exists x_2 \ldots x_n (X_1 = t_1 \wedge X_2 = t_2 \wedge \cdots \wedge X_m = t_m),$$

which may be written in the form

(4) $$\Phi(t_1, \ldots, t_m, x_i) = 0$$

so that

$$\Phi(t_1, \ldots, t_m, 1)x_1 + \Phi(t_1, \ldots, t_m, 0)\bar{x}_1 = 0.$$

But as an X_j with x_1 replaced by 1 is equal to that X_j with x_1 replaced by 0, we have

$$\Phi(t_1, \ldots, t_m, 1) = \Phi(t_1, \ldots, t_m, 0),$$

and hence (3) is independent of x_1.

Boole next considers what ensues with regard to a compound X of $x_1, \ldots, x_n$ if, in addition to the probabilities of the X_j [assumed distinct and mutually exclusive], one also has those of the x_i given. Note that here (as well as in Proposition I) the x_i, although simple, are not unconditioned events, being conditioned by the data involving the X_j:

PROPOSITION II.

8. Given the probabilities of n simple events $x_1, x_2, \ldots, x_n$, viz.:—

Prob. $x_1 = c_1$, Prob. $x_2 = c_2, \ldots$, Prob. $x_n = c_n$; (1)

also the probabilities of the $m-1$ compound events $X_1, X_2, \ldots, X_{m-1}$, viz.:—

Prob. $X_1 = a_1$, Prob. $X_2 = a_2, \ldots$, Prob. $X_{m-1} = a_{m-1}$; (2)

the latter events $X_1 \ldots X_{m-1}$ being distinct and mutually exclusive;

required the probability of any other compound event X. [pp. 386–387]

Despite the generality of the conditions in this Proposition Boole succeeds in showing (on the assumptions of his method) that the solution for $P(X)$ is given by the set of equations

$$(5) \qquad P(X) = \frac{a_1(XX_1)}{X_1} + \frac{a_2(XX_2)}{X_2} + \cdots + \frac{a_m(XX_m)}{X_m}$$

$$c_i = \frac{a_1(x_iX_1)}{X_1} + \frac{a_2(x_iX_2)}{X_2} + \cdots + \frac{a_m(x_iX_m)}{X_m} \quad (i = 1,\ldots,n)$$

where

$$X_m = 1 - X_1 - X_2 - \cdots - X_{m-1}$$

$$a_m = 1 - a_1 - a_2 - \cdots - a_{m-1}$$

and (XX_j) indicates the intersection (common constituents) of X and X_j, with the resulting expression in $x_1,\ldots,x_n$ taken in the numerical sense; and similarly for (x_iX_j). The values of $x_1,\ldots,x_n$ are found from the last n equations (Boole glosses over difficulties with regard to the solution of a non-linear algebraic system) and then substituted into the first to give $P(X)$ in terms of the parameters $c_1,\ldots,c_n$, $a_1,\ldots,a_{m-1}$.

For use with the next proposition the solution system (5) is converted to a symmetric form: (i) the index enumerating the given functions is reduced by one so as to have $X_1,\ldots,X_{m-2}$ (ii) X is then included as X_{m-1} (iii) $P(X_{m-1})$, formerly $P(X)$, is now a_{m-1}, and (iv) X_m and a_m are defined as before. After some algebraic manipulations the system (5) then takes the form

$$\frac{X_{m-1}}{a_{m-1}} = \frac{X_m}{a_m}$$

$$(6) \qquad c_i = \frac{a_1(X_1X_1)}{X_1} + \frac{a_2(x_2X_2)}{X_2} + \cdots + \frac{a_m(x_mX_m)}{X_m} \quad (i = 1,\ldots,n)$$

Although Boole's Proposition I rules out the possibility of obtaining, solely on the basis of records of decisions, the probabilities $P(x_i)$ of correct judgement of members of an assembly, he now claims that, by use of an hypothesis less restricted that Laplace's and Poisson's assumption of independence of the x_i, one can come out with a value for the "mean probability of correct judgement", this being the common value for $P(x_i)$, assuming these to be all equal.

PROPOSITION III

10. Given any system of probabilities drawn from recorded instances of unanimity, or of assigned numerical majority in the decisions of a deliberative assembly; required, upon a certain determinate hypothesis, the mean probability of correct judgment for a member of the assembly. [p. 392]

Boole's argument for this proposition, if not cogent, is at least quite ingenious. He divides the "immediate data of experience" into two groups

$$\text{(7)} \qquad \text{(i)} \quad P(X_1) = a_1, \quad P(X_2) = a_2, \ldots, \quad P(X_{m-2}) = a_{m-2}$$

$$\text{(ii)} \quad P(X_{m-1}) = a_{m-1}$$

where, as before, the X_j are logical functions of $x_1, \ldots, x_n$ and it is now assumed that

$$\text{(8)} \qquad P(x_1) = P(x_2) = \cdots = P(x_n) = c,$$

for some "intermediate" value c. Boole takes it that the X_j are mutually exclusive ("from the very nature of the case") and applies Proposition II to (7), whose solution would then be given by equations (6) in which the last n of them collapse into a single equation by virtue of (8) producing

$$\text{(9)} \qquad \frac{X_{m-1}}{a_{m-1}} = \frac{X_m}{a_m}$$

$$c = \frac{a_1(xX_1)}{X_1} + \frac{a_2(xX_2)}{X_2} + \cdots + \frac{a_m(xX_m)}{X_m}$$

But now instead of c being given and a_{m-1} an unknown to be determined, Boole takes a_{m-1} to be the value given by the datum (7) (ii), and c the quantity determined by equations (9). As to which of the $m - 1$ data equations is to be thus sequestered and used as the $P(X_{m-1}) = a_{m-1}$, Boole says: "As to the principle of selection, I apprehend that the equation (2) [(7)(ii)] reserved for final comparison should be that which, from the magnitude of its numerical element a_{m-1}, is esteemed the most important of the primary series furnished by experience." Clearly Boole's notion of the mean probability of correct judgement of members of an assembly is much less precise than, say, the notion of the mean height of members of an assembly.

After extending his results of Proposition III to a similar treatment which includes considerations of guilt (k) as well as correctness of judgement, Boole concludes his chapter with a summary and some general reflections. After Boole nothing further seems to have been written about the subject.

§ 6.5. Combination of testimonies

Although at one time actively pursued, the combination of testimonies (or of evidence) is now no longer a standard item in the repertoire of probability applications. There is a brief history of the topic and a critique in KEYNES 1921, pp. 180–185. Our interest here is primarily, though not exclusively, in the conditional probability problems which Boole formulated as a result of his analysis of these questions.

Boole takes the notion of testimony in a broad sense. For example, in *Problem* I of his Keith Prize memoir [1857 = 1952 XVI] the event of an astronomical observation being made is considered to be a testimony. In this memoir there is an analysis of the question of combining the several measurements of a quantity so as to obtain a single ("most probable") value. The treatment is markedly different from that now current, which is based on statistical theory. We devote some discussion to Boole's ideas on combining of measurements, beginning with his statement of the problem.

PROBLEM I

Two simultaneous observations of a physical magnitude, as the

> elevation of a star, assign to it the respective values, p_1 and p_2. The probability, when the first observation has been made that it is correct, is c_1, the corresponding probability for the second observation is c_2. Required the most probable value of the physical magnitude hence resulting. [1952, p. 333]

Boole's first step is to consider a measurement p, e.g. the elevation of a star, as a probability, namely as the probability "that a pointer, directed at random to that quadrant of elevation in which the star, regarded as a physical point, is situated, will point below the star [assuming the measures in an entire quadrant range from 0 to 1]." In modern parlance: if X is a random variable uniformly distributed over $[0, 1]$, then $P(0 \leq X \leq p) = p$. Such a random variable can always be introduced by normalizing measures to the interval $[0, 1]$. To express the problem formally Boole introduces the following events:

> The event which consists in the first observation, such as it is, being made $= x$.
>
> The event which consists in the second observation, such as it is, being made $= y$.
>
> The event which consists in the first observation being correct, $= w$.
>
> The event which consists in the second observation being correct, $= v$.
>
> The event which consists in a pointer, directed at random to the quadrant in which the star is situated, pointing below that star, $= z$.

It isn't exactly clear what Boole means by his desideratum, "the most probable value of the physical magnitude hence resulting"; but what he takes it to be in this context is the conditional probability $P(z|xy)$. In terms of the above introduced random variable X this would be

$$P(0 \leq X \leq s \mid 0 \leq X \leq p_1, 0 \leq X \leq p_2).$$

Here $0 \leq X \leq s$, s being the elevation of the star, expresses the event z and $0 \leq X \leq p_1, 0 \leq X \leq p_2$ express the events x and y. The conditions in the data concerning the correctness of the observations are taken to be conditional probabilities:

$$(7) \qquad c_1 = P(w|x), \qquad c_2 = P(v|y)$$

(If x and y were testimonies of witnesses then c_1 and c_2 would be their credibilities. See KEYNES 1921, p. 181.) As usual, Boole believes he has to have his data in terms of absolute (non-conditional) probabilities, and hence is forced to introduce two additional parameters a_1, a_2 to express (7), which then becomes:

$$(8) \qquad \begin{array}{ll} P(x) = a_1 & P(y) = a_2 \\ P(wx) = a_1 c_1 & P(vy) = a_2 c_2. \end{array}$$

He adjoins the reasonable conditions that w implies x (if the observation is correct, then it has been made), and that v implies y, but inexplicably, also requires $wv = 0$ (not both observations can be correct), which implies that if $p_1 = p_2$ then the observations must be incorrect. Finally the condition that the observations result in p_1 and p_2 is taken as

$$(9) \qquad P(z|w) = p_1, \qquad P(z|v) = p_2.$$

Applying his general method to the problem as so formulated he comes out (after extensive calculation) with a result which one can write as (the bar indicating complementation with respect to 1):

$$P(z|xy) = \frac{(\overline{a_1 c_1}/\overline{c_1})c_1 p_1 + (\overline{a_2 c_2}/\overline{c_2})c_2 p_2 + c(\overline{a_1 c_1 + a_2 c_2})}{1 + (\overline{a_1}/\overline{c_1})c_1 + (\overline{a_2}/\overline{c_2})c_2}$$

The presence of the arbitrary constant c in the result indicates to Boole that "those principles of probability which relate to the combination of *event* do not *alone* suffice to enable us to combine into a definite result the conflicting measures of an astronomical observation."

This is Boole's *First Solution* to the above-cited *Problem* I. Naturally one doesn't expect a treatment with the sophistication of contemporary statistical estimation theory, where one looks for an estimator (function of the observations) for an unknown parameter (e.g. the mean) of a population (with a given type of probability distribution). Even so, his analysis of the problem seems shallow—the formal representation shows no (internal) connection between x and y, which are events depicting measurements of the same physical entity. Similarly for w and x which are only related by the requirement that $w \rightarrow x$. Introduction of the two additional parameters a_1 and a_2 is forced, and the requirement implying $p_1 \neq p_2$ seems needlessly restrictive. With regard to these animadversions Boole's *Second Solution* offers no improvement. There is, however, an ingenious idea introduced which merits mention. The idea is embodied in the following definition:

> *Definition.* The mean strength of any probabilities of an event which are founded upon different judgements or observations is to be measured by that supposed probability of the event *a priori* which those judgements or observations following thereupon would not tend to alter. [1952, p. 339]

Recall that Boole is substituting a probability value in place of a measurement. Thus his definition is specifying a single value to represent a combination of measurements, and corresponds to what current statistical theory would call an estimator—but without a specification of what parameter of the population is being estimated. For the problem at hand (*Problem* I) this amounts to finding a value for $P(z)$ in terms of the parameters such that

$$P(z|xy) = P(z)$$

Boole accomplishes this by adjoining $P(z) = r$ to the data, applying his method to find $P(z|xy)$ (now in terms of r also) and equates this to r. Solution for r results in

$$r = \frac{(\bar{a}_1/\bar{c}_1)c_1p_1 + (\bar{a}_2/\bar{c}_2)c_2p_2}{(\bar{a}_1/\bar{c}_1)c_1 + (\bar{a}_2/\bar{c}_2)c_2}, \tag{10}$$

which, he notes, is of the form of a weighted average,

$$r = W_1p_1 + W_2p_2$$

of p_1 and p_2. He also observes that when $a_1 = a_2$ (uniform prior probability of each measurement) and $c_1 = c_2$ (equal "credibilities") then (10) reduces to the ordinary average

$$r = \frac{p_1 + p_2}{2}.$$

We shall not present an (upper and lower bound) solution for Boole's *Problem* I (i.e. given (8) and (9), find $P(z|xy)$ as we shall be presently doing it for the similar but less complicated *Problem* II.

The next general problem taken up in the Keith Prize memoir is that of determining "the combined force of two testimonies or judgements in support of a fact, the strength of each testimony being given." Boole formulates this verbally as:

PROBLEM II

> 34. Required the probability of an event z, when two circumstances x and y are known to be present—the probability of the event z, when we only know of the existence of the circumstance x being p,—and its probability when we only know of the existence of y being q. [1952, p. 355]

Here, too, believing he has to express the problem in terms of absolute probabilities Boole introduces two additional parameters for the respective probabilities of x and y. We then have the following formal statement:

given: $P(x) = c_1, \qquad P(y) = c_2$

$P(z|x) = p_1\ [P(xz) = c_1 p_1]$

$P(z|y) = p_2\ [P(yz) = c_2 p_2]$

find: $P(z|xy)$

Writing the required conditional probability $P(z|xy)$, namely $P(xyz)/P(xy)$, in the form

$$\frac{P(xyz)}{P(xyz) + P(xy\bar{z})},$$

the separate values for $P(xyz)$ $[= u]$ and $P(xy\bar{z})$ $[= t]$ are then determined (but only implicitly) by the general method and the quotient $u/(u+t)$ taken to be the answer. (We have earlier, in § 6.3, commented on the illegitimacy of this procedure.) There is a detailed discussion of the (implicitly given) solution for various specializations of the parameters.

Methods we shall develop in § 6.7, extending those of § 5.7, will enable us to obtain best possible bounds on conditional probabilities subject to linearly expressible constraints. Viewing Boole's *Problem* II in this light, i.e. as that of finding best possible upper and lower bounds on $P(z|xy)$ subject to $P(x) = c_1$, $P(y) = c_2$, $P(z|x) = p_1$, $P(z|y) = p_2$, we shall there find (Example 6.74):

Case 1. $c_1 + c_2 \leq 1$

upper bound: 1

lower bound: 0

Case 2. $c_1 + c_2 > 1$

upper bound: $\dfrac{c_1 p_1}{c_1 + c_2 - 1}, \quad \dfrac{c_2 p_2}{c_1 + c_2 - 1}, \quad 1$

lower bound: $\dfrac{c_1 + c_2 p_2 - 1}{c_1 + c_2 - 1}, \quad \dfrac{c_2 + c_1 p_1 - 1}{c_1 + c_2 - 1}$

(Even without the general theory of § 6.7 one can by relatively easy algebra show directly, in *Case* 1, that $P(z|xy)$ can take on any value between 0 and 1, thus contravening Boole's "determinate" solution. See our endnote in § 6.8.)

We can also give a solution to Boole's *Problem* II as literally stated, that is without the introduction of the two parameters (for $P(x)$ and $P(y)$). Applying the methods of § 6.7 to the problem

given: $P(z|x) = p_1, \quad P(z|y) = p_2$

find: $P(z|xy)$

one finds (*Example* 6.75):

upper bounds: 1

lower bounds: p_1, p_2

so that $P(z|xy)$ is capable of taking on any value between $\max\{p_1, p_2\}$ and 1.

§ 6.6. Probability logic

Although the idea of having probability inferences included as part of logic was then not unknown—for example, in his *Formal Logic* De Morgan has, subsequent to a preliminary chapter on Probability, an entire chapter devoted to Probable Inference—Boole, surprisingly, makes little of the idea. The conception which he had of his *general method* was that of a problem-solving algorithm which produces the probability of an event (to the extent it is determined), given the probabilities of other events. The few clear instances of probable inference that occur in his work are only incidental examples, not part of a coherent development. Yet viewed from a modern semantic standpoint, his development of probability theory in close connection with propositional logic is suggestive of a logic of probability (values) extending the logic of truth (values). There is, first of all, the replacement of the physical notion of *event* by the logical *proposition*. Further, probability values accrue (more or less) to propositional compounds from values of their atomic propositions. Additionally, the solution of a problem by Boole's *general method*, when expressed in the form

(*given*) implies (*required*, assumed found)

is in effect a general inference form since no specific probability values are involved, nor is there any reference to the material content of the propositions, only their logical form.

On the other hand there are difficulties. Only in the case of a compound build up from (stochastically) independent propositions do we obtain a uniquely corresponding value-computing function (akin to a truth-function in two-valued logic) which then gives the value of the compound. This inadequacy is, of course, not a peculiarity of Boole's system—there is no way of expressing, for example, $P(x \vee y)$ solely in terms of $P(x)$ and $P(y)$ for arbitrary x and y. Moreover, the seeming inference form associated with Boole's *general method* furnishes, in what would be the conclusion, either only one out of a possible set of values—and that one under special (independence) conditions—or else a range of values implicitly given by an expression containing an undetermined constant.

The logic we shall now present, a new type of propositional logic, circumvents the difficulties indicated in the preceding paragraph by making use of two changes in the usual conception of a logic, in addition to enlarging the set of assignable values to propositions. The first is to have constituents (basic conjunctions), rather than simple (atomic) propositions, as the fundamental components determining the (probability) value of a compound; and the second is to have the (probability) value of a formula in the conclusion of a general inference form be determined only to within an interval of values

Probability Logic

Syntax. The definition of *formula* will be the same as that in ordinary two-valued logic, i.e. we have

(i) propositional variables $A_1, \ldots, A_n, \ldots$
(ii) logical connectives: $\neg$, $\wedge$, $\vee$
(iii) parentheses: (,)

from which formulas are constructed in customary fashion. Conventions and abbreviations in general use will be assumed.

Semantics: Here we have new definitions of *model* and *logical consequence.*

Let $k_j (j = 1, \ldots, 2^n)$ be real numbers such that for each $j, k_j \in [0, 1]$, and such that the sum of all is 1. A *probability model M* (adequate for a set of formulas $\chi_1, \ldots, \chi_N$) is an assignment of the numbers k_j to the basic conjunctions (constituents) K_j on variables $A_1, \ldots, A_n$, where this list of variables is long enough to include all those occuring in $\chi_1, \ldots, \chi_N$.

We write $P_M(K_j)$ for the value assignment by M to K_j (i.e.,

$P_M(K_j) = k_j$) and extend P_M to all formulas on $A_1, \ldots, A_n$ by setting $P_M(\varphi) = \sum^{(\varphi)} k_j$, where the summation is over all j for which K_j implies φ or, lacking such, we set $P_M(\varphi) = 0$. By 'K_j implies φ' we mean with K_j and φ taken as formulas in the ordinary two-valued sense. Thus we are including two-valued logic as a part of our semantic apparatus. The so-defined function P_M satisfies the following general properties of probability:

P1. (i) $P(\varphi) = 0$, if φ implies $A_1 \wedge \neg A_1$
(ii) $P(\varphi) \leq P(\psi)$, if implies ψ

P2. $P(\neg \varphi) = 1 - P(\varphi)$

P3. $P(\varphi \vee \psi) = P(\varphi) + P(\psi)$, if $\varphi \wedge \psi$ implies $A_1 \wedge \neg A_1$.

Note that if in a probability model M some $k_{j_0} = 1$ (and hence all other k_j's are 0) then for $i = 1, \ldots, n$, the formula A_i has P_M value 0, or 1, according as A_i appears negated, or unnegated, in K_{j_0}. Such models then coincide with two valued models.

We define the notion of logical consequence in terms of probability models. Although our interest will be in subintervals of $[0, 1]$, the definition we now give is formulable just as well with arbitrary subsets of $[0, 1]$ and we so state it. Note that we allow ranges of values for formulas in the antecedent as well as in the consequent.

Let $\alpha_1, \ldots, \alpha_m$, β be subsets of $[0, 1]$. The *(probability) logical consequence* relation, denoted by

(1) $$P(\varphi_1) \in \alpha_1, \ldots, \quad P(\varphi_m) \in \alpha_m \models P(\psi) \in \beta,$$

holds if and only if

(2) For all probability models M (adequate for $\varphi_1, \ldots, \varphi_m, \psi$): if

$$P_M(\varphi_1) \in \alpha_1, \ldots, \quad P_M(\varphi_m) \in \alpha_m, \text{ then } P_M(\psi) \in \beta.$$

The intuitive picture here is that of a "truth"-table entered from basic conjunctions $K_1, \ldots, K_{2^n}$ with additional columns headed $\varphi_1, \ldots, \varphi_m, \psi$. The $K_1, \ldots, K_{2^n}$ are assigned all possible sets of 2^n real numbers from $[0, 1]$, the sum of such numbers in each set being 1. Each assignment (row) determines a probability model and corresponding values for $\varphi_1, \ldots, \varphi_m, \psi$. The premise conditions in (1) select out of the $2^{\aleph_0}$ rows of the table those in which the probabilities of the φ_i are, respectively, in the sets α_i, the relation (1) then holding if for each of these rows ψ has a probability in β.

As is well-known, two-valued propositional logic can be developed either syntactically (via formal deducibility) or semantically (via models, or truth-tables), the latter being generally recognized as the primary mode. For our probability logic we shall adhere to the semantic approach, using logical consequence ($\models$) as the fundamental notion. Note that the symbol 'P' in the definition has been given a meaning only in connection with '$\models$'.

When a subset α of $[0, 1]$ is a singleton, say, $\alpha = \{a\}$, then we shall write $P(\varphi) = a$ in place of $P(\varphi) \in \alpha$.

THEOREM 6.51. $\models P(\varphi) = 1$ *iff* φ *is a (two-valued) tautology.*

PROOF (a) If φ is a tautology then, for any M, $P_M(\varphi) = 1$. (b) If φ is not a tautology then its expansion as an alternation of basic conjunctions (we assume it has at least on, otherwise it is equivalent to $A_1 \wedge \neg A_1$ and hence $P_M(\varphi) = 0$) is missing at least one conjunction, say K_{j_1}. There are models in which $k_{j_1} \neq 0$: in any such $P_M(\varphi) \neq 1$.

Since properties P1–P3 hold for any P_M we can list any of the simple identities derivable from these properties as probability consequence relations. For example, $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi)$ can be rephrased as

$$P(\varphi) = a,\ P(\psi) = b,\ P(\varphi \wedge \psi) = c \models P(\varphi \vee \psi) = a + b - c.$$

However, for more substantial results, including an effective procedure for deciding relations of logical consequence, we need some general theorems on probability algebras. These are results on best possible upper and lower bounds (from HAILPERIN 1965) which we have already referred to in § 5.7. They happen to be couched in terms of probability algebras, whereas what we have been using in this section are probability measures on sets of formulas built up from $A_1, \ldots, A_n$ and the connectives $\neg$, $\wedge$, $\vee$. However the transition is easily made. Since by P1 (ii), logically equivalent formulas have the same probability value, we can simplify the structure carrying the probability measure by "identifying" equivalent formulas, i.e., by going over to the Lindenbaum algebra (§ 0.5). Supposing this to be done, we shall not bother to change notation, using $\neg$, $\wedge$, $\vee$ in place of corresponding Boolean operations, and having a formula stand for the equivalence class of which it is a member.

GENERAL THEOREM (on upper and lower probability bounds). *Let $\alpha_1, \ldots, \alpha_m$ ($\alpha_i = [a_i, b_i]$) be probability intervals (subintervals of $[0, 1]$). Let. $\varphi_1, \ldots, \varphi_m$, ψ be Boolean polynomials in variables $A_1, \ldots, A_n$. Then*:

(i) *there are* $2m$-ary *numerical-valued functions $L_\psi^{(\varphi)}(a_1, \ldots, a_m, b_1, \ldots, b_m)$ $[= L_\psi^{(\varphi)}$ for short$]$ and $U_\psi^{(\varphi)}(a_1, \ldots, a_m, b_1, \ldots, b_m)$ $[= U_\psi^{(\varphi)}]$ depending only on the Boolean structures of $\varphi_1, \ldots, \varphi_m, \psi$, such that the two-sided inequality*

$$L_\psi^{(\varphi)} \le P(\psi) \le U_\psi^{(\varphi)} \tag{3}$$

holds in any probability algebra for which

$$P(\varphi_i) \in \alpha_i \qquad (i = 1, \ldots, m) \tag{4}$$

(ii) *the bounds given in* (3) *are best possible, and*

(iii) *there is a linear programming problem specified by* (4) *and the structure of ψ such that carrying out its solution effectively determines whether or not* (4) *is consistent and, if it is, then the explicit forms for $L_\psi^{(\varphi)}$ and $U_\psi^{(\varphi)}$ are provided by the solution.*

Examples of upper and lower bounds as described in the General Theorem have been given in § 5.7. If one examines the proof one sees that the full generality of "all probability algebras" is not needed—all that is used is all possible assignments of real numbers k_j ($j = 1, \ldots, 2^n$) to constituents on $A_1, \ldots, A_n$, with $k_j \in [0, 1]$ and $\sum_{j=1}^{2^n} k_j = 1$; in other words, just probability models adequate for $\varphi_1, \ldots, \varphi_m$, ψ. This enables one to rephrase the General Theorem on bounds as a probability logical consequence result. Namely, introducing the interval-valued function

$$Pf_\psi^{(\varphi)}(\alpha_1, \ldots, \alpha_n) = [L_\psi^{(\varphi)}, U_\psi^{(\varphi)}],$$

and making use of the fact that $[L_\psi^{(\varphi)}, U_\psi^{(\varphi)}]$ is best possible, we have the

GENERAL THEOREM (on probability logical consequence). *Let* $\alpha_1, \dots, \alpha_m$ *be probability intervals and* $\varphi_1, \dots, \varphi_m$, ψ *propositional formulas on variables* $A_1, \dots, A_n$. *Then*

$$P(\varphi_1) \in \alpha_1, \dots, P(\varphi_m) \in \alpha_m \models P(\psi) \in \beta$$

for any β *such that* $Pf_\psi^{(\varphi)}(\alpha_1, \dots, \alpha_m) \subseteq \beta$. *Consistency conditions on the premises (antecedent of* $\models$ *) and the explicit form of* $Pf_\psi^{(\varphi)}$ *are effectively determinable in terms of* $\varphi_1, \dots, \varphi_m$, ψ *and* $\alpha_1, \dots, \alpha_m$.

We illustrate the General Theorem with a number of examples, in all of which the strongest possible conclusions will be given, i.e. one with $\beta = Pf_\psi^{(\varphi)}$. In what follows we shall tacitly assume that parameters designating probabilities lie between 0 and 1.

Example 6.61. (Generalizing modus ponens)

$$P(A_1) = p, \quad P(A_1 \to A_2) = q \models P(A_2) \in [p + q - 1, q].$$

Consistency condition: $p + q \geq 1$.

We work this simple example out in some detail so as to illustrate the ideas. Let k_1, k_2, k_3, k_4 be the values associated with the probabilities of the basic conjuctions on A_1, A_2. On expressing the conditions in the antecedent in terms of the k's, and adjoining the probability requirements on the k's, one obtains

$$\text{(5)} \qquad \begin{aligned} k_1 + k_2 \qquad\qquad &= p \\ k_1 \qquad + k_3 + k_4 &= q \\ k_1 + k_2 + k_3 + k_4 &= 1 \\ k_1, k_2, k_3, k_4 \geq 0 & \end{aligned}$$

From the three equations in (5) one has $p + q - 1 = k_1$, and hence $p + q \geq 1$. (For a general technique of determining consistency, i.e. solvability, of a system of inequations see Theorem 0.71.) The interval for $P(A_2)$ in the consequent may be found in either of two ways (see § 5.7):

(a) (Polytope corners). The best upper bound for $P(A_2) = k_1 + k_3$

subject to conditions (5) is given by maximizing $k_1 + k_3$, subject to

$$(6)\qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}, \quad \text{and} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \geq 0.$$

The dual form of this linear programming problem is to minimize $px_1 + qx_2 + x_3$ subject to

$$(7)\qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

A corner point of the polytope specified by (7) is found by converting three out of the four inequations into equations, solving, and testing to see if the solution satisfies the other inequality. For (7) we find only one corner point (0, 1, 0). Hence $p \cdot 0 + q \cdot 1 + 0[= q]$ is the minimum value. The best lower bound is obtained by complementary techniques, and gives the result stated.

(b) (Fourier elimination). Here we adjoin to the system (5) the equation

$$w = k_1 + k_2 [= P(A_2)]$$

and eliminate all variables but w. This results in the relations

$$p + q - 1 \leq w, \qquad w \leq q$$

$$0 \leq q \leq 1, \qquad 0 \leq p \leq 1, \qquad 1 \leq p + q,$$

which gives the bounds on w and the consistency conditions.

In the following examples we shall omit the calculations and just state the results.

A slightly more general form of probabilistic modus ponens occurs in

SUPPES 1966, formula (7) p. 53:

$$\frac{\begin{array}{c}P(B \to A) \geq r\\ P(B) \geq \rho\end{array}}{P(A) \geq r + \rho - 1}$$

For us this becomes

Example 6.62. (More general modus ponens)

$$P(A_1 \to A_2) \in [p, 1],\ P(A_1) \in [q, 1] \models P(A_2) \in [p + q - 1, 1]$$

Consistency condition: $p + q \geq 1$.

Example 6.63. (Generalized hypothetical syllogism)

$$P(A_1 \to A_2) = p,\ P(A_2 \to A_3) = q \models P(A_1 \to A_3) \in [p + q - 1, 1]$$

Consistency condition: $1 \leq p + q$.

The next example illustrates the effect of additional information. In Example 6.61 the premises show no relationship between A_1 and A_2. If we replace A_1 by $A_1 \vee A_2$ the result is still true but a sharper result can be given, namely (replacing the formula $A_1 \vee A_2 \to A_2$ by the equivalent $A_1 \to A_2$):

Example 6.64.

$$P(A_1 \vee A_2) = p, P(A_1 \to A_2) = q \models P(A_2) = p + q - 1$$

with consistency conditions as in 6.61.

In the preceding three examples the formula involved in the conclusion was a necessary consequence (in the two-valued sense) of the formulas in the premises. Here is an example where this is not the case:

Example 6.65.

$$P(A_1 \to A_2) = p \models P(A_2 \to A_1) \in [1 - p, 1].$$

Although $P(A_1 \vee A_2)$ cannot be expressed as a function only of $P(A_1)$ and $P(A_2)$ it is the case that the probability interval can be so expressed:

Example 6.66.

$$P(A_1) = p,\ P(A_2) = q \models P(A_1 \vee A_2) \in [\max\{p, q\}, \min\{1, p + q\}].$$

The following is the logical consequence form of Boole's Ex. 1 (§4.6):

Example 6.67.

$$P(A_1) = p, P(A_1 A_2) = q \models P(A_2) \in [q, q + 1 - p].$$

Consistency condition: $q \leq p$.

On comparing this with Boole's solution of Ex. 1.,

$$P(A_2) = q + c(1 - p), \quad c = P(A_2 | \bar{A}_1),$$

one notes that in one respect Example 6.67 gives less information about $P(A_2)$, but on the other hand it isn't known from Boole's solution that c could range between 0 and 1.

Example 6.68.

$$P(A_1 \vee A_2) = p, P(\bar{A}_1 \vee \bar{A}_2) = q \models P(A_1 \bar{A}_2 \vee A_2 \bar{A}_1) = p + q - 1.$$

This is Boole's Ex. 2 (§4.6). It results from Example 6.64 on replacing A_2 by $A_1 \bar{A}_2 \vee \bar{A}_1 A_2$ and simplifying

$$A_1 \vee (A_1 \bar{A}_2 \vee \bar{A}_1 A_2) \quad \text{to} \quad A_1 \vee A_2$$

and

$$A_1 \rightarrow (A_1 \bar{A}_2 \vee \bar{A}_1 A_2) \quad \text{to} \quad \bar{A}_1 \vee \bar{A}_2.$$

Boole's Ex. 4 (§4.6) takes the form

Example 6.69.

$$P(A_1) = p, P(A_2) = q, P(\bar{A}_1 \bar{A}_2 \bar{A}_2) = r \models P(A_3) \in [L, U],$$

where

$$L = \max\{0, \bar{r} - p - q\} \text{ and } U = \min\{\bar{r}, 2\bar{r} - p, 2\bar{r} - q, 3\bar{r} - p - q\}.$$

Consistency conditions: $p + r \leq 1$, $q + r \leq 1$.

Boole's Ex. 5 (§4.6) is the probabilistic form of the hypothetical syllogism, which he there formulates as though the problem were about conditional probabilities. Our Example 6.63 gives the solution when the conditionals are taken in the usual sense (i.e. 'If A_1 then A_2' is taken as '$\bar{A}_1 \vee A_2$'). For our solution of the problem as Boole formulates it see Example 6.72 below.

We conclude our section on probability logic with an oft-encountered problem in probable inference: Given the probabilities of two or more arguments for a conclusion what probability should be assigned to the conclusion? Here is De Morgan's version of this problem:

> *Problem* 3. Arguments being supposed logically good, and the probabilities of their conclusions (that is, of all their premises being *true*) being called their validities, let there be a conclusion for which a number of arguments are presented, of validities a, b, c, *&c*. Required the probability that the conclusion is proved [1854, p. 201]

De Morgan's language we interpret as follows:

'A is an argument for C' means '$A \to C$'
'A is logically good (for C)' means '$A \to C$ is valid'
'A proves C' means '$A \wedge (A \to C)$'
'the validity of A' is '$P(A)$'

Note that, in accordance with this, the validity of A is equal to the probability that A proves C if the argument A is logically good. The problem considered then is: given arguments $A_1, \ldots, A_n$ (for C), all being logically good, and with respective validities $a_1, \ldots, a_n$, what is the probability that the conclusion is proved? Since $(A_1 \to C) \wedge \cdots \wedge (A_n \to C)$ is logically equivalent to $(A_1 \vee \cdots \vee A_n) \to C$ the question is equivalent to asking for $P(A_1 \vee \cdots \vee A_n)$ given that $P(A_i) = a_i$ $(i = 1, \ldots, n)$. De Morgan gives the result

$$1 - (1 - a_1)(1 - a_2) \ldots (1 - a_n),$$

which is correct if

$$P(\neg A_1 \neg A_2 \ldots \neg A_n) = P(\neg A_1)P(\neg A_2) \ldots P(\neg A_n),$$

i.e., if the $A_1, \ldots, A_n$ are stochastically independent. Without this assumption we have

Example 6.610.

$$P(A_1) = a_1, \ldots, P(A_n) = a_n \models P(A_1 \vee A_2 \vee \cdots \vee A_n) \in [\max\{a_1, \ldots, a_n\}, \min\{1, a_1 + a_2 + \cdots + a_n\}].$$

§ 6.7. An extension of probability logic

Many of the problems involving conditional probabilities which Boole dealt with can also be viewed as inferential, i.e. logical, in nature. The theory of the preceding section does not directly cover the situation, but a slight extension of it does. However, before turning to this we would like to outline the discussion of a particular probabilistic inference question at issue between Bishop Terrot and Archbishop Whately, to which Boole contributed an important clarification.

Terrot begins his paper [1857] with the following excerpt from Whately:

> "As in the case of two probable premises, the conclusion is not established except upon the supposition of their being *both* true, so in the case of two (and the like holds good for any number) distinct and independent indications of the truth of some proposition, unless *both* of them *fail*, the proposition must be true: we therefore multiply together the fractions indicating the probability of the failure of each—the chances against it—and the result being the total chances against the establishment of the conclusion by these arguments, this fraction being deducted from unity, the remainder gives the probability *for* it. "E.g. A certain book is conjectured to be such an author, partly, 1*st*, from its resemblance in style to his known works; partly, 2*nd*, from its being attributed to him by someone likely to be pretty well informed. Let the probability of the conclusion, as deduced from these arguments by itself, be supposed $\frac{2}{5}$, and in the other case $\frac{3}{7}$; then the *opposite* probabilities will be respectively $\frac{3}{5}$ and $\frac{4}{7}$, which multiplied together give $\frac{12}{35}$ as the probability against the conclusion; i.e. the chance that the work may *not* be his, notwithstanding the reasons for believing that it is; and, consequently, the probability in *favour* of the conclusion will be $\frac{23}{35}$, or nearly $\frac{2}{3}$ "(Whately's *Logic*, 8th Ed., p. 211).

By applying Whately's rule to the negative of the conclusion in the example give, Terrot comes out with a probability against the conclusion which isn't 1 minus that which Whately has obtained for the conclusion. This incompatibility, he claims, shows that "the principle and method must be erroneous." He goes on to discuss the only other attempt at a solution which he is acquainted with (De Morgan 1837, section 15) and declares, since the rule there given is the same as that of Whately, that it too is wrong. (Terrot's criticism of De Morgan's strange derivation is justified, but apparently he was unaware that it was replaced by a different one in De Morgan 1854.) Terrot then goes on to present his attempt at a solution, leading him to a doubly infinite series of terms of which he says: "This infinite series of infinite series I cannot sum. If they can be summed, then their sum divided by the infinite of the second order n^2, is the probability required."

Boole's contribution to the "controversy" (a defense of Whately had appeared in the United Church Journal) is in an Appendix to the Keith Prize memoir [1857].

Boole first notes that the result he has given as the solution of his *Problem* II (see § 6.5)—which he believes contains a rule for "computing the joint force of two probabilities in favour of a conclusion"—is not the same as Whately's. After presenting Whately's argument (that quoted above) he makes the point:

> A confusion may here be noted between the probability that a conclusion is proved, and the probability in favour of a conclusion furnished by evidence which does not prove it. In the proof and statement of his rule, Archbishop Whately adopts the former view of the nature of the probabilities concerned in the data. In the exemplification of it, he adopts the latter. He thus applies the rule to a case for which it was not intended, and to which it is, in fact, inapplicable. [1952, p. 383]

Boole then remarks that the rule and "the conditions for its just application" are to be found in De Morgan's *Formal Logic*, p. 201. This we have discussed at the end of our preceding section. Interpreted in this manner Whately's rule is then

$$(1)\qquad \frac{A_1 \to C, A_2 \to C, P(A_1) = a_1, P(A_2) = a_2}{P((\overline{\bar{A}_1\bar{A}_2})C) = P(\overline{\bar{A}_1\bar{A}_2}) = 1 - \bar{a}_1\bar{a}_2},$$

and is correct if A_1 and A_2 are stochastically independent. (Whately states this independence condition, but De Morgan neglected to do so.) But, as Boole notes, the exemplification Whately has involves conditional probabilities and the premises are not as in (1) but should be

$$P(C|A_1) = a_1, P(C|A_2) = a_2.$$

Thus the formula $1 - \bar{a}_1\bar{a}_2$ is inappropriate as the value of $P(C|A_1A_2)$, and accounts for the incompatibility when Terrot computes $P(C|A_1A_2)$ and $P(\bar{C}|A_1A_2)$ by its use. Boole believes his solution to *Problem* II (§ 6.5) gives the answer, i.e. the value of $P(C|A_1A_2)$. Our solution, which is different, is given below in Example 6.75.

PROBABILITY LOGIC (extended)

The definition of the logical consequence relation

$$(2) \qquad P(\varphi_1|\chi_1) \in \alpha_1, \ldots, P(\varphi_m|\chi_m) \in \alpha_m \models P(\psi|\rho) \in \beta$$

will be formally the same as that for absolute probabilities. However two important differences arise from the introduction of conditional probabilities.

(i) Since a premise of the form $P(\varphi|\chi) \in \alpha, \alpha = [a, b]$, is equivalent to

$$a \leq \frac{P(\varphi\chi)}{P(\chi)} \leq b,$$

i.e. to a pair of inequalities

$$aP(\chi) \leq P(\varphi\chi)$$

$$P(\varphi\chi) \leq bP(\chi),$$

the premises in (2) still translate as a system of linear inequalities in constituent probabilities (the k's); but the parameters (a and b in the example) will no longer be isolated in the constant terms of the inequalities but would appear also in the coefficients of the k's. Thus although we still have a system of linear constraints which specifies the region of feasible solutions, it is no longer possible to have a purely *numerically* defined region of feasible solutions by going over to the dual form (as we did in § 5.7); for in the dual form the coefficients will still contain occurrences of the parameters. This is only a practical obstacle, not a theoretical one, as the corner points of the polytope could still be found (in terms of the parameters). It will turn out, however, that Fourier-Motzkin elimination will be the method of choice in most of the examples we look at.

(ii) As in probability logic with absolute probabilities, determining whether a conclusion follows from the premises resolves into the determination of best possible upper and lower bounds. However now it is not of a linear form in the k's but rather of a linear fractional form since a conditional probability, as a quotient of absolute probabilities will, in terms of the k's, be of the form

$$\frac{\sum^{(1)} k_i}{\sum^{(1)} k_i + \sum^{(2)} k_i}.$$

The logical question then reduces to an algebraic one of *optimizing a linear fractional form subject to linear contraints.* A theory for this has been developed and a principal result, in a form sufficient for our purposes, is stated in §0.8 (Theorem 0.82).[1]

As a simple example illustrating the ideas and methods involved, consider the problem (as Boole would have formulated it): given the probabilities of two events, determine the probability of their conjunction if it is known that one or the other (or both) has happened. I.e.

given: $P(A_1) = p_1, \quad P(A_2) = p_2$

find: $P(A_1 A_2 | A_1 \vee A_2)$.

As a logical inference problem this would be:

Find L and U such that

$$P(A_1) = p_1, P(A_2) = p_2 \models P(A_1 A_2 | A_1 \vee A_2) \in \beta.$$

if and only if $[L, U] \subseteq \beta$.

[1] I am indebted to my colleague Murray Schechter for calling my attention to linear fractional programming, which then enabled me to develop an extension of probability logic which includes conditional probabilities.

In terms of constituent probabilities the question is that of

$$\text{optimizing } \frac{k_1}{k_1 + k_2 + k_3} = \frac{P(A_1 A_2 (A_1 \vee A_2))}{P(A_1 \vee A_2)}$$

subject to the conditions

$$(1) \qquad \begin{aligned} k_1 + k_2 \qquad\qquad &= p_1 \\ k_1 \qquad + k_3 \qquad &= p_2 \\ k_1 + k_2 + k_3 + k_4 &= 1 \\ k_1, k_2, k_3, k_4 \geq 0, \quad k_1 + k_2 + k_3 &> 0 \end{aligned}$$

By Theorem 0.82 this is equivalent to

$$\text{optimizing } y_1$$

subject to

$$(2) \qquad \begin{aligned} y_1 + y_2 \qquad\qquad &= \lambda p_1 \\ y_1 + \qquad + y_3 \qquad &= \lambda p_2 \\ y_1 + y_2 + y_3 + y_4 &= \lambda \\ y_1 + y_2 + y_3 \qquad &= 1 \\ \lambda \geq 0, \quad y_1, y_2, y_3, y_4 &\geq 0 \end{aligned}$$

The simplest way to solve this linear programming problem is to eliminate (§0.7) λ, y_4, y_3, y_2, so obtaining inequalities on y_1 implied by (2). One readily finds

$$0 \leq y_1, \; p_1 + p_2 - 1 \leq y_1$$

$$y_1 \leq \frac{p_1}{p_2}, \quad y_1 \leq \frac{p_2}{p_1}.$$

Thus we have

Example 6.71. For $p_1 p_2 \neq 0$,

$$P(A_1) = p_1, P(A_2) = p_2 \models P(A_1 A_2 | A_1 \vee A_2) \in [L, U],$$

where

$$L = \max\{0, p_1 + p_2 - 1\} \quad \text{and} \quad U = \min\{p_1/p_2, p_2/p_1\}.$$

As another example we consider Boole's interpretation of the probabilistic hypothetical syllogism with conditional probabilities, but without the introduction of extra parameters, so that the conditional probabilities are essential. Here, it turns out, the conclusion being "empty", the consequence relation is of little interest:

Example 6.72.

$$P(A_1|A_2) = p, P(A_2|A_3) = q \models P(A_1|A_3) \in [0, 1].$$

For verification of this see our endnote in § 6.8 where we show that for any p, q in the open interval (0,1) there are probability models satisfying the premises and in which $P(A_1|A_3) = 0$, and also there are some in which $P(A_1|A_3) = 1$. Thus no interval short of [0, 1] can include all possible values of $P(A_1|A_3)$.

An example of a probabilistic inference scheme which involves a conditional probability occurs in Suppes 1966, p. 50,

$$\begin{array}{l} P(A|B) = r \\ \underline{P(B) = \rho} \\ \therefore\ P(A) \geq r\rho. \end{array}$$

Using our linear programming approach we come up, automatically, with the stronger conclusion:

Example 6.73.

$$P(A_1|A_2) = p, P(A_2) = q \models \mathrm{P}(\mathrm{A}_1) \in [pq, 1 - \bar{p}q]$$

The following two results have already been referred to in § 6.5 in connection with Boole's *Problem* II.

Example 6.74.

$$P(A_1) = c_1, P(A_2) = c_2, P(A_3|A_1) = p_1, P(A_3|A_2) = p_2$$

$$\models P(A_3|A_1A_2) \in [0, 1], \qquad \text{if } c_1 + c_2 \leq 1$$

$$\models P(A_3|A_1A_2) \in [L, U], \qquad \text{if } c_1 + c_2 > 1$$

where

$$L = \max\left\{0, \frac{c_1 + c_2p_2 - 1}{c_1 + c_2 - 1}, \frac{c_2 + c_1p_1 - 1}{c_1 + c_2 - 1}\right\}$$

$$U = \min\left\{\frac{c_1p_1}{c_1 + c_2 - 1}, \frac{c_2p_2}{c_1 + c_2 - 1}, 1\right\}.$$

Example 6.75.

$$P(A_3|A_1) = p_1, P(A_3|A_2) = p_2 \models P(A_3|A_1A_2) \in [\max\{p_1, p_2\}, 1]$$

§ 6.8. Notes to Chapter 6

(for § 6.1)

NOTE 1. Discussion of Michell's Problem in connection with Boole's ideas may be found in KEYNES 1921, pp. 294–296, and also in FISCHER 1973, pp. 40–42.

NOTE 2. With regard to Boole's observation that one cannot "contrapose" a conditional probability, the following result may be of interest. Suppose $p, 0 < p < 1$, is some fixed value and q any arbitrarily selected value in $[0, 1]$. Then there is a probability algebra and events A and B such that $P(A|B) = p$ and $P(\bar{B}|\bar{A}) = q$. Thus having just the value of $P(A|B)$ determined imposes no restriction whatever on $P(\bar{B}|\bar{A})$.

To see this let k_1, k_2, k_3, k_4 be, respectively, the probabilities of AB, $\bar{A}B$, $A\bar{B}$ and $\bar{A}\bar{B}$. Then

(1) $$\frac{k_1}{k_1 + k_2} = p, \qquad \sum_{i=1}^{4} k_i = 1, \qquad k_1, k_2, k_3, k_4 \geq 0.$$

Subject to these conditions we investigate the value of

(2) $$q = P(\bar{B}|\bar{A}) = \frac{k_4}{k_2 + k_4}.$$

Eliminating k_1 between the two equations of (1), solving for k_2 and substituting the value in (2) yields

$$q = \frac{k_4}{\bar{p}(1-k_3)+pk_4}.$$

This function of two variables is continuous at all points (k_3, k_4) except $(1,0)$. In the (k_3, k_4)-plane (with $k_3, k_4 \in [0,1]$) the value of q on the line $k_3 + k_4 = 1$ is 1, and on the k_3-axis its value is 0. Hence on any line connecting the two the value of q ranges continuously from 0 to 1.

(for § 6.3)

NOTE 1. Keynes' solution for the Challenge Problem can be derived as follows. Since $E \to A_1 \vee \cdots \vee A_n$, we have

(1) $$E = EA_1 + E\bar{A}_1A_2 + E\bar{A}_1\bar{A}_2A_3 + \cdots + E\bar{A}_1\bar{A}_2\ldots\bar{A}_{n-1}A_n,$$

and $P(E)$ is then the sum of the probabilities of the separate terms.

Now

$$P(EA_i) = P(A_i)P(E|A_i) = c_ip_i,$$

and letting, for $i \geq 2$,

$$d_i = P(E\overline{\bar{A}_1\ldots\bar{A}_{i-1}}A_i),$$

we have for each term on the right in (1) after the first,

$$P(E\bar{A}_1\ldots\bar{A}_{i-1}A_i) = P(EA_i) - P(E\overline{\bar{A}_1\ldots\bar{A}_{i-1}}A_i)$$
$$= c_ip_i - d_i.$$

Hence

$$P(E) = \sum_{i=1}^{n} c_ip_i - \sum_{i=2}^{n} d_i.$$

NOTE 2. The following is a somewhat clearer derivation of the results of pp. 192–193 in KEYNES 1921.

We have $P(C) = a$, $[a \neq 1]$ and for $i = 1, \ldots, n$, $P(A_i) = p$, $C \to A_i$, and $P(A_i|A_1A_2\ldots A_{i-1}\bar{C}) = P(A_i|\bar{C})$. Then

(a) $$P(A_1A_2\ldots A_i|C) = 1$$

and

$$P(A_i|A_1A_2\ldots A_{i-1}\bar{C}) = P(A_i|\bar{C})$$
$$= \frac{P(A_i) - P(A_iC)}{P(\bar{C})}$$
$$= \frac{p-a}{1-a},$$

so that

(b) $$P(A_1A_2\ldots A_r|\bar{C}) = P(A_1|\bar{C})P(A_2|A_1\bar{C})\ldots P(A_r|A_1A_2\ldots A_{r-1}\bar{C})$$
$$= \left(\frac{p-a}{1-a}\right)^r.$$

Thus

$$P(A_1A_2\ldots A_r) = P(A_1A_2\ldots A_rC) + P(A_1A_2\ldots A_r\bar{C})$$
$$= P(C)P(A_1A_2\ldots A_r|C) + P(\bar{C})P(A_1A_2\ldots A_r|\bar{C})$$
$$= a\cdot 1 + (1-r)\left(\frac{p-a}{1-a}\right)^r$$

and hence

$$P(A_{n+1}|A_1A_2\ldots A_n) = \frac{a + (1-a)\left(\frac{p-a}{1-a}\right)^{n+1}}{a + (1-a)\left(\frac{p-a}{1-a}\right)^n}$$

and also

$$P(C|A_1A_2\ldots A_n) = \frac{a}{a + (1-a)\left(\frac{p-a}{1-a}\right)^n}$$

Note that, since $0 \leq p - a \leq 1 - a$, as $n \to \infty$ the value of

$$P(C|A_1A_2\ldots A_n) \to 1 \qquad (= 1 \text{ if } p = a).$$

NOTE 3. The following is a brief derivation, using 20th century mathematics, of the limit which Boole works out in *Laws of Thought*, pp. 373–374.

For any continuous real-valued function f on $[0, 1]$ the sequence of Bernstein polynomials

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{k}{n} x^k(1-x)^{n-k} \qquad (n = 1, 2, \dots)$$

has $f(x)$ as its uniform limit (Weierstrass Approximation Theorem). By taking

$$f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{p-r},$$

passing to the limit and setting $x = \frac{1}{2}$, one has Boole's result

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{p-r}\binom{k}{n}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^p.$$

NOTE 4. Peirce [1883, pp. 171–173] describes the Rule of Succession and, approvingly, Boole's critism of it centering on his introduction of the alternative hypothesis of equally probable constitutions instead of equally probable ratios. Peirce makes the amusing observation that, contrary to Boole's tolerating either assumption, it would be "far better" to assume equally probable constitutions, for if the Rule resulting from this assumption were applied to more than one unknown event no inconsistency could arise, as would be the case if the Rule were based on the assumption of equally probable ratios.

(for § 6.5)

NOTE 1. When translated into probabilities of constituents (§ 5.7) Boole's *Problem* II (as he formulates it) is that of finding values of $k_1/(k_1 + k_2)$ subject to the conditions

$$(*) \qquad \begin{aligned} k_1 + k_2 + k_3 + k_4 + k_5 + k_6 &\le 1 \\ k_1 + k_2 + k_3 + k_4 \qquad\qquad &= c_1 \\ k_1 + k_2 \qquad\qquad + k_5 + k_6 &= c_2 \\ k_1 \quad + k_3 \qquad\qquad\qquad &= c_1 p_1 \\ k_1 \qquad\qquad + k_5 \qquad &= c_2 p_2, \quad k_i \ge 0. \end{aligned}$$

We show that, for $c_1, c_2, p_1, p_2 \in [0, 1]$, if $c_1 + c_2 \leq 1$, then subject to the constraints $(*)$ one can have $k_1/(k_1 + k_2) = r$ for any r, $0 < r < 1$.

Set

$$k_2 = k_1\left(\frac{1-r}{r}\right)$$

$$k_3 = c_1 p_1 - k_1$$

$$k_4 = c_1 - c_1 p_1 - k_1\left(\frac{1-r}{r}\right)$$

$$k_5 = c_2 p_2 - k_1$$

$$k_6 = c_2 - c_2 p_2 - k_1\left(\frac{1-r}{r}\right)$$

and choose k_1 so that

$$0 \leq k_1 \leq Q$$

where

$$Q = \min\left\{\frac{r}{1-r}, c_1 p_1, (c_1 - c_1 p_1)\frac{r}{1-r}, c_2 p_2, (c_2 - c_2 p_2)\frac{r}{1-r}\right\}$$

Then one can readily show:

(i) each $k_i \geq 0$

(ii) each of the equations in $(*)$ is satisfied, and

(iii) the inequality in $(*)$ reduces to $c_1 + c_2 \leq 1 + k_1 r$, which is satisfied since we have by assumption, $c_1 + c_2 \leq 1$.

Thus Boole's $P(z|xy)$ of *Problem* II (his version) can take on any value between 0 and 1.

(for § 6.6)

Note 1. For some historical background material on probabilistic inference, and for more details on the probability logic here presented, see Hailperin 1984b.

Note 2. For the conditional probability form of the hypothetical syllogism (*Example* 6.62) the linear programming problem is than of

$$\text{optimizing} \quad \frac{k_1 + k_3}{k_1 + k_3 + k_5 + k_7} \quad [= P(A_1 | A_3)]$$

subject to the conditions

$$(*) \qquad \begin{aligned} k_1 + k_2 &= p(k_1 + k_2 + k_5 + k_6) \\ k_1 + k_5 &= q(k_1 + k_3 + k_5 + k_7) \\ k_1 + k_2 + k_3 + k_5 + k_6 + k_7 &\leq 1 \\ k_i \geq 0 \quad (i &= 1, 2, 3, 5, 6, 7) \end{aligned}$$

We shall show (for $p, q > 0$) that there are solutions of $(*)$ in which $k_5 = k_7 = 0$ (and thus for which $P(A_1|A_3) = 1$). When $k_5 = k_7 = 0$ the system $(*)$ becomes

$$(**) \qquad \begin{aligned} k_1 + k_2 &= p(k_1 + k_2 + k_6) \\ k_1 \qquad &= q(k_1 + k_2) \\ k_1 + k_2 + k_3 + k_6 &\leq 1 \quad (k_i \geq 0) \end{aligned}$$

Solving for k_3 and k_6 and substituting into the inequality results in an equivalent system

$$(***) \qquad \begin{gathered} k_3 = \frac{1-q}{q} k_1, \quad k_6 = \frac{1-p}{p}(k_1 + k_2) \\ \left(1 + \frac{1-q}{q} + \frac{1-p}{p}\right) k_1 + \left(1 + \frac{1-p}{p}\right) k_2 \leq 1 \quad (k_i \geq 0) \end{gathered}$$

Since the coefficients on k_1 and k_2 are both greater than 1 there are clearly arbitrarily small positive values satisfying the inequalities and the entire system.

A similar argument (with $p, q < 1$) shows that there are solutions for $(*)$ in which $k_1 = k_3 = 0$, so that there are probability models in which $P(A_1|A_3) = 0$.

BIBLIOGRAPHY

BETH, EVERT W.

1959 *The foundations of mathematics.* North-Holland Publishing Company, Amsterdam.

BOOLE, GEORGE

1841 On the integration of linear differential equations with constant coefficients. *Cambridge mathematical journal,* Vol. 2, pp. 114–119.

1847 *The mathematical analysis of logic, being an essay towards a calculus of deductive reasoning.* Macmillan, Barclay, & Macmillan, Cambridge; George Bell, London. (Also in BOOLE 1952)

1851a On the theory of probabilities, and in particular Mitchell's problem of the distribution of the fixed stars. *The London, Edinburgh, and Dublin philosophical magazine and journal of science,* series 4, vol. 1, pp. 521–530. (Reprinted BOOLE 1952, VIII)

1851b Further observations on the theory of probabilities. *Ibid.,* series 4, vol. 2, pp. 96–101. (Reprinted BOOLE 1952, IX)

1851c Proposed question in the theory of probabilities. *The Cambridge and Dublin mathematical journal,* vol. vi, p. 186. (Reprinted BOOLE 1952, X)

1854 *An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities.* Walton and Maberley, London. (See also BOOLE 1916, 1951)

1854a Solution of a question in the theory of probabilities. *The London, Edinburgh, and Dublin philosophical magazine and journal of science*, series 4, vol. 7, pp. 29–32. (Reprinted BOOLE 1952, XI)

1854b Reply to some observations by Mr. Wilbraham in the *Philosophical Magazine* vii p. 465, on the theory of chances developed in Professor Boole's "Laws of Thought". *Ibid.*, series 4, vol. 8, pp, 87–91. (Reprinted BOOLE 1952, XII)

1854c On the conditions by which solutions of questions in the theory of probabilities are limited. *Ibid.*, series 4, vol. 8, pp. 91–98. (Reprinted BOOLE 1952 XIII)

1854d Further observations in reply to Mr. Wilbraham. *Ibid.*, series 4, vol. 8, pp. 175–176. (Reprinted BOOLE 1952, XIV)

1854e On a general method in the theory of probabilities. *Ibid.*, series 4, vol. 8, pp. 431–444. (Reprinted BOOLE 1952, XV)

1855 On certain propositions in algebra connected with the theory of probabilities. *Ibid.*, series 4, vol. 9, pp. 165–179.

1857 On the application of the theory of probabilities to the question of the combination of testimonies or judgements. *Transactions of the Royal Society of Edinburgh*, vol. 21, part 4, pp. 597–652. (Reprinted BOOLE 1952, XVI)

1862 On the theory of probabilities. *Ibid.*, vol. 152, pp. 225–252. (Reprinted BOOLE 1952, XVII)

1916 *George Boole's collected logical works. Volume* II. *The laws of thought* (1854). ed. by P. E. B. Jourdain. Open Court, Chicago and New York. (No Volume I published) Reprinted 1940.

1951 *An investigation of the laws of thought, on which are founded the mathematical theories of logic and probabilities.* Dover Publications, Inc., New York. (Reprint of BOOLE 1854)

1868 Of propositions numerically definite (read posthumously by De Morgan 16 March 1868), *Transactions of the Cambridge Philosophical Society*, vol. 11 (1871) pp. 396–411. (Reprinted BOOLE 1952, IV)

1952 *Studies in logic and probability*, Watts & Co., London, and the Open Court Publishing Company, LaSalle, Illinois. (Editor, R. Rhees)

BROAD, C. D.

1917 Review of Boole's *Collected logical works*, vol. 2. *Mind*, n.s. vol. 26, pp. 81–99.

BRODY, BORUCH A.

1967 *The rise of the algebra of logic.* (A dissertation presented to the faculty of Princeton University) University Microfilms, Inc. Ann Arbor, Michigan

CARNAP, RUDOLF

1958 *Introduction to symbolic logic and its applications.* Dover Publications, Inc., New York.

CAYLEY, ARTHUR

1853 Note on a question in the theory of probabilities. *The London, Edinburgh and Dublin philosophical magazine and journal of science.* series 4, vol. vi, p. 259.

1854 On the theory of groups, as depending on the symbolic equation $\vartheta^n = 1$. *Ibid.* series 4, vol. 7, p. 40–47. (Also paper 125 in *The Collected Mathematical Papers of Arthur Cayley.* vol. 2, pp. 123–130. Cambridge 1889. Reprinted 1963, Johnson Reprint Corporation, New York.)

1862 On a question in the theory of probabilities. *Ibid.* series 4, vol. 23, pp. 352–470. Postscript, pp. 470–471.

1871 Note on the calculus of logic. *Quarterly journal of pure and applied mathematics*, vol. 11, pp. 282–283.

CHARNES, A.

1962 Programming with linear fractional functionals. *Naval research logistics quarterly*, vol. 9, pp. 181–186.

CORCORAN, J. and WOOD, S.

1980 Boole's criteria for validity and invalidity. *Notre Dame journal of formal logic*, vol. 21, pp. 609–638.

DANTZIG, GEORGE

1963 *Linear programming and extensions.* Princeton University Press, Princeton, N.J.

DASTON, LORRAINE J.

1981 Mathematics and the moral sciences: the rise and fall of the probability of judgements. *Epistemological and social problems in the early nineteenth century 1785–1840.* ed. H. N. Jahnke and M. Otte. D. Reidel. Dordrecht/Boston/London.

DEDEKIND, RICHARD

1855 Bemerkungen zu eine Aufgabe der Wahrscheinlichkeitsrechnung. *Journal für die reine und angewandte Mathematik.* (Crelle), vol. 50, pp. 268–271.

DE MORGAN AUGUSTUS

1837 Theory of probabilities. [contributed to *Encyclopedia Metropolitana*] London.

1846 On the structure of the syllogism, and on the application of the theory of probabilities to questions of argument and authority (read 9 Nov. 1846) [also "On the syllogism I"]. *Transactions of the Cambridge Philosophical Society,* vol. 8 (1849), pp. 379–408. (Reprinted in DE MORGAN 1966, pp. 1–21.)

1847 *Formal logic,* Taylor and Walton, London. (Reprinted 1926, Editor A. E. Taylor)

1850 On the symbols of logic, the theory of the syllogism, and in particular of the copula [also "On the syllogism II"], *Transactions of the Cambridge Philosophical Society,* vol. 9 (1856), pp. 79–127. (read Feb. 1850)

1966 *On the syllogism and other logical writings.* Edited, with an introduction by PETER HEATH. Yale University Press, New Haven.

DIPERT, RANDALL R.

1981 Peirce's propositional logic. *Review of Metaphysics*, vol. 34, pp. 569–595.

FELLER, WILLIAM

1957 *An introduction to probability theory and its applications.* Vol. I. Second edition. John Wiley & Sons, Inc. New York-London-Sydney.

FISCHER, RONALD A.

1973 *Statistical methods and scientific inference.* Third edition. Macmillan Publishing Co., New York.

FORBES, JAMES D.

1850 On the alleged evidence for a physical connection between stars forming binary or multiple groups, deduced from the doctrine of chances. *The London, Edinburgh and Dublin philosophical magazine and journal*, series 3, vol. 37, pp. 401–427.

FOSTER, A. L.

1945 The idempotent elements of a commutative ring form a Boolean algebra; ring-duality and transformation theory. *Duke mathematical journal*, vol. 12, pp. 143–152.

FREDE, MICHAEL

1947 *Die Stoische Logik*. Van den Hoeck & Ruprecht.

FREGE, GOTTLOB

1879 *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens.* Halle.

GOULD, JOSIAH B.

1970 *The philosophy of Chrysippus.* State University of New York Press.

GREGORY, DUNCAN F.

1840 On the real nature of symbolical algebra. *Transactions of the Royal Society of Edinburgh*, vol. 14, pp. 208–216.

GRÄTZER, GEORGE

1968 *Universal algebra*. D. Van Nostrand Company Inc. Princeton, N.J.

HAILPERIN, THEODORE

1965 Best possible inequalities for the probability of a logical function of events. *American mathematical monthly*, vol. 72, pp. 343–359.

1984a Boole's abandoned propositional logic. *History and philosophy of logic*, vol. 5, pp. 39–48.

1984b Probability logic. *Notre Dame journal of formal logic*, vol. 25, pp. 198–212.

HAMILTON, WILLIAM ROWAN

1837 Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time. *Transactions of the Royal Irish Academy*, vol. XVII, pp. 293–422, (read 1833, 1835). (Reprinted as Paper I 1967)

1844 On a new species of imaginary quantities connected with the theory of quaternions. *Proceedings of the Royal Irish Academy*, vol. II, pp. 424–434. (read 1843) (Reprinted as Paper V in HAMILTON 1967)

1967 *The mathematical papers of Sir William Rowan Hamilton*. Vol. III, edited by H. Halberstam and R. E. Ingram. Cambridge.

HARLEY, ROBERT

1866 George Boole, F.R.S. *The British quarterly review*, vol. 44, pp. 141–181. (Reprinted as Appendix A in BOOLE 1952)

HOOLEY, JOHN

1966 Boole's method for solving Boolean equations. *The Mathematical Gazette*, vol. 50, pp. 114–118.

JEVONS, W. STANLEY

1864 *Pure logic or the logic of quality apart from quantity*. London. (Reprinted in JEVONS 1890)

1870 On the mechanical performance of logical inference. *Philosophical transactions of the Royal Society of London*, vol. 160, pp. 497–518. (Reprinted in JEVONS 1890)

1890 *Pure logic and other minor works*. Ed. by Robert Adamson and Harriet A. Jevons, London and New York.

JØRGENSEN, JØRGEN

1931 *A treatise of formal logic. Its evolution and main branches, with its relations to mathematics and philosophy*. Levin and Munksgaard, Copenhagen. vols. 1–3.

JOURDAIN, PHILIP E. B.

1913 The development of the theories of mathematical logic and the principles of mathematics. William Stanley Jevons. *The quarterly journal of pure and applied mathematics*, vol. 44, pp. 113–128.

KAPPOS, D. A.

1960 *Strukturtheorie der Wahrscheinlichkeitsfelder und -räume*. Springer-Verlag, Berlin-Gottingen-Heidelberg. (Ergebnisse der Mathematik und ihre Grenzgebiete. Neue Folge, Heft 24)

1969 *Probability algebras and stochastic spaces*. Academic Press, New York and London.

KEYNES, JOHN MAYNARD

1921 *A treatise on probability*. Macmillan and Co. London.

KNEALE, W. and KNEALE, M.

1962 *The development of logic*. Oxford University Press. London.

KNOBLOCH, E.

1981 Symbolik und Formalismus in mathematischen Denken des 19. und beginnenden 20. Jahrhunderts. *Mathematical Perspectives*. Ed. J. W. Dauben. Academic Press.

KOPPELMAN, E.

1971 The calculus of operations and the rise of abstract algebra. *Archive for History of Exact Sciences*, vol. 8, pp. 155–242.

LAITA, LUIS M.

1977 The influence on Boole's search for a universal method in analysis on the creation of his logic. *Annals of Science*, vol. 34, pp. 163–176.

1979 Influences on Boole's logic: The controversy between William Hamilton and Augustus De Morgan. *Ibid.* vol. 36, pp. 45–65.

1980 Boolean algebra and its extra-logical sources: The testimony of Mary Everest Boole. *History and philosophy of logic*, vol. 1, pp. 37–60.

LAMBERT, J. H.

1782 *Johann Heinrich Lamberts logische und philosophische Abhandlungen*. Ed. Johann Bernoulli, vol. 1. (Berlin)

LAPLACE, P. S.

1820 *Théorie analytique des probabilités*. 3rd edition. Paris.

LEWIS, C. I.

1918 *A survey of symbolic logic*. University of California Press, Berkeley, Calif. (Reprinted with corrections and omission of Chapters V and VI by Dover Publications, New York, 1960.)

MacColl, Hugh

1880 The calculus of equivalent statements (fourth paper.) *Proceedings of the London Mathematical Society*, vol. 11, pp. 113–121.

Macfarlane, Alexander

1879 *Principles of the algebra of logic, with examples.* Edinburgh.

MacHale, Desmond

1985 *George Boole. His life and work.* Boole Press Ltd. Dún Laoghaire.

Maistrov, L. E.

1974 *Probability theory. A historical sketch.* Academic Press. New York and London. (Translated and edited by Samuel Kotz)

McCoy, N. H.

1948 *Rings and ideals.* The Carus mathematical monographs Number 8. The Mathematical Association of America.

Mendelson, Elliot

1979 *Introduction to Mathematical Logic.* Second Edition. Van Nostrand. New York.

Michell, John

1767 An inquiry into the probable parallax and magnitude of the fixed stars, from the quantity of light which they afford us, and the particular circumstances of their situation. *Philosophical transactions of the Royal Society*, vol. 12, pp. 423–440.

Muir, Thomas

1920 *The theory of determinants*, Vol. III, St. Martins Press.

(Republished by Dover Publications, Inc., New York. Four volumes bound as two. No date.)

MUSGRAVE, A.

1972 George Boole and psychologism. *Scientia* (Milano), vol. 107, pp. 593–608.

OSTROWSKI, A.

1951 *Vorlesungen über Differential – und Integralrechnung.* Zweiter Band. Verlag Birkauser, Basel.

PEACOCK, GEORGE

1833 Report on the recent progress and present state of certain branches of analysis. *British Association for the Advancement of Science.* Report 3. pp. 185–352.

PEIRCE, C. S.

1867 On an improvement in Boole's calculus of logic. *Proceedings of the American Academy of Arts and Sciences*, vol. 7 (1868), pp. 250–261. Presented 12 March 1867. (Reprinted in PEIRCE 1933, Paper I, and in PEIRCE 1984, 2.)

1867a Upon the logic of mathematics. *Ibid.* vol. 7 (1868) (by title, 10 September 1867) pp. 402–412. (Reprinted in PEIRCE 1933, II and in PEIRCE 1984, 5.)

1870 Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic. *Memoirs of the American Academy of Arts and Sciences*, n.s. vol. 9 (1873), pp. 317–378. Communicated 26 January 1870. (Reprinted in PEIRCE 1933, III and PEIRCE 1984, 39.)

1880 On the algebra of logic. *American Journal of Mathematics*, vol. 3, pp. 15–57. (Reprinted in PEIRCE 1933, XIII)

1933 *Collected papers of Charles Saunders Peirce.* Edited by Charles Hartshorne and Paul Weiss. Volume III. Harvard University Press, Cambridge, Mass.

1984 *Writings of Charles S. Peirce. A Chronological Edition.* Volume II 1867–1871. Indiana University Press, Bloomington.

PRIOR, ARTHUR N.

1948 Categoricals and hypotheticals in George Boole and his successors. *The australasian journal of philosophy*, vol. 27, pp. 171–196.

1957 *Time and modality.* Oxford University Press. New York.

PYCIOR, H. M.

1981 George Peacock and the British origins of symbolical algebra. *Historia Mathematica*, vol. 8, pp. 23–45.

QUINE, W. V.

1972 *Methods of logic.* Third edition. Holt, Rinehart and Winston, Inc., New York.

RADO, R.

1975 The cardinal module and some theorems on families of sets. *Annali di matematica pura ed applicata*, (4), vol. 102, pp. 135–154.

RÉNYI, ALFRED

1970 *Foundations of probability.* Holden-Day Inc. San Francisco, Cambridge, London, Amsterdam.

RICHARDS, JOHN

1980 Boole and Mill: Differing perspectives on logical psychologism. *History and philosophy of logic*, vol. 1, pp. 19–36.

SCHRÖDER, ERNST

1891 *Vorlesungen über die Algebra der Logik (exakte Logik).* Vol. 2, part 1. B. G. Teubner, Leipzig.

SHOENFIELD, J. R.

1967 *Mathematical logic.* Addison-Wesley, Reading, Mass.

SIKORSKI, ROMAN

1969 *Boolean algebras.* Third edition. Ergebnisse der Mathematik und ihre Grenzgebiete. Neue Folge. Band 25. Berlin-Göttingen-Heidelberg-New York.

SIMMONS, H.

1970 The solution of a decision problem for several classes of rings. *Pacific journal of mathematics,* vol. 34, pp. 547–557.

SMITH, G. C.

1982 *The Boole-De Morgan correspondence 1842–1864.* Oxford. New York.

STOER, J. and WITZGALL, C.

1970 *Convexity and optimization in finite dimensions I.,* Springer-Verlag. New York-Heidelberg-Berlin (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Band 163)

STONE, M. H.

1935 Subsumption of the theory of Boolean algebras under the theory of rings. *Proceedings of the National Academy of Sciences,* vol. 21, pp. 103–105.

STYAZHKIN, N. I.

1969 *History of mathematical logic from Leibniz to Peano.* The M.I.T. Press, Cambridge, Mass., and London, England. Translation of the Russian *Stanovleniye idey matematicheskoy logiki.* Nauka, Moscow, 1964.

SUPPES, P.

1966 Probabilistic inference and the concept of total evidence. *Aspects of inductive logic*. Ed. J. Hintikka and P. Suppes)

TARSKI, A., MOSTOWSKI, A. and ROBINSON, R. M.

1953 Undecidability and essential undecidability in arithmetic. *Undecidable theories*, by A. TARSKI in collaboration with A. MOSTOWSKI and R. M. ROBINSON. Studies in logic and foundations, North-Holland Publishing Company, Amsterdam.

TAYLOR, GEOFFREY

1956 George Boole, F.R.S. 1815–1864. *Notes and records of the Royal Society of London*, vol. 12, pp. 44–52.

TERROT, CHARLES HUGHES

1857 On the possibility of combining two or more probabilities of the same event, so as to form one definite probability. *Transactions of the Royal Society of Edinburgh*, vol. xxi, pp. 369–376.

THOMSON, WILLIAM

1853 *An outline of the necessary laws of thought*. Third edition. William Pickering, London, W. Graham, Oxford.

UZKOV, A. I.

1948 On rings of quotients of commutative rings (Russian) *Matematicheskii sbornik* (N.S.) 22(64), pp. 439–441. (Translated in *American mathematical society translation number 3*, pp. 1–5, American Mathematical Society, New York, 1949)

VAN EVRA, JAMES W.

1977 A reassessment of George Boole's theory of logic. *Notre Dame journal of formal logic*, vol. 28, pp. 363–377.

1984 Richard Whately and the rise of modern logic. *History and philosophy of logic*, vol. 5, pp. 1–18.

VENN, JOHN

1894 *Symbolic logic*. Second edition. Macmillan and Co., London and New York. (Reprinted 1971 by Chelsea Publishing Co., Bronx, N.Y.)

WHATELY, RICHARD

1852 *Elements of logic: comprising the substance of the article in the Encyclopedia Metropolitana: with additions, etc.* (This work, dating from 1827, appeared in many printings, editions, and by various publishers, too numerous to list.)

WHITNEY, H.

1933 Characteristic functions and the algebra of logic. *Annals of mathematics*, vol. 34, pp. 405–414.

WILBRAHAM, HENRY

1854 On the theory of chances developed in Professor Boole's "Laws of Thought". *The London, Edinburgh, and Dublin philosophical magazine and journal of science*, series 4, vol. 7, pp. 465–476. (Reprinted BOOLE 1952, Appendix B)

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