AAIT

School of Civil and Environmental Engineering

Engineering Economics (CEng 5211)

Chapter 2:Cost of Money

Content

- **Cost of Money**
  - Interest
  - Time value of money
  - Economic equivalence

Cost of Money

- Interest
- Money borrowed form financial institution is expected to be repaid over time by an amount greater than the amount borrowed.
- It is evident that financial institutions lend money expecting repayment (including interest) greater than the borrowed amount over time.

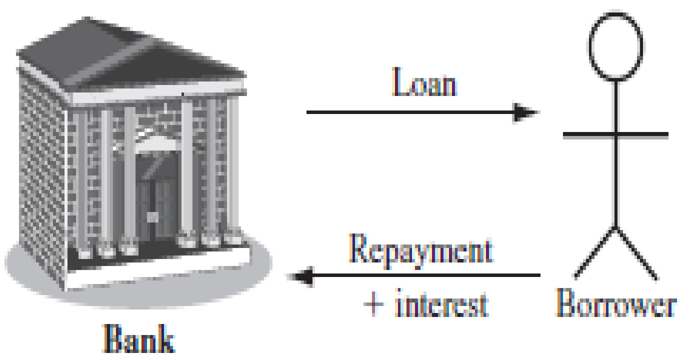
"Interest is the cost of having money available for use."

- In a financial world, money itself is a commodity, and like other goods that are bought and sold, money costs money.
- Interest rate used to measure the cost of money.
- Interest is the fee paid or a fee earned for the use of money. It is return on capital (capital is the invested money and resource).
- Interest= Ending amount - Beginning amount = Amount owed now- Principal
- **Interest rate** is a percentage added to an amount of money over a specified length of time.
- Interest rate (%)= $\frac{\text{Interest added per time unit}}{\text{Principal}} * 100\%$

Interest

- For example if a firm owns the capital and invests it in a project, then the project should return that **capital plus interest** as cost saving or added revenues.
- The interest rate that is appropriate depends on many factors including risks, economic conditions, and time frame.

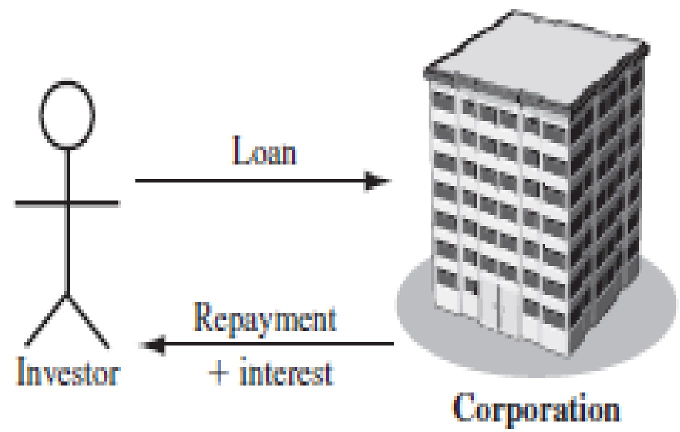
Interest paid



Interest rate

Borrowed money

Interest earned



Rate of return

- Lend money
- Saved “
- Invested

Interest

Definition of Terms

- P=Value or amount of money at a time designated as the present time t=0. Initial deposit, Investment made at t=0.
- F= Value or amount of money at some future time.
- A= Series of equal consecutive end of period amounts of money
- n= Number of interest periods (year, month, day).
- i= interest period per time period(percent per year, percent per month).
- t= stated time period (years, months, days).

Types of Interest

1. **Simple Interest**
2. **Compound Interest**

Interest

Types of Interest

1. **Simple interest:** is computed **only on original sum** (principal), not on prior interest earned and left in the account.

- Interest paid (earned) on only the original amount, or principal, borrowed (lent).
- A bank account, for example, may have its simple interest every year: in this case, an account with $1000 initial principal and 20% interest per year would have a balance of $1200 at the end of the first year, $1400 at the end of the second year, and so on.

$$F=P(1+ ni)$$

End of 1st year: $F = 1000(1+(1*0.2))=1200$

End of 2nd year: $F=1000(1+(2*0.2))=1400$

- Total interest earned (charged) is linearly proportional to:
  - the initial amount of principal (loan)
  - interest rate
  - number of time periods of commitment

Interest

2. Compound Interest: is interest paid (earned) **on any previous interest earned, as well as on the principal borrowed** (lent). It arises when interest is added to the principal of a deposit or loan, so that, from that moment on, the interest that has been added also earns interest. This addition of interest to the principal is called **compounding**.

- A bank account, for example, may have its interest compounded every year: in this case, an account with $1000 initial principal and 20% interest per year would have a balance of $1200 at the end of the first year, $1440 at the end of the second year and soon.

$$F = P(1+i)^n$$

- Development of the equation:

Period	Beginning of period value	End of period value
1	P	P(1+i)
2	P(1+i)	P(1+i) (1+i)
3	$P(1+i)^2$	$P(1+i)^2(1+i)$
…	…	…
N	$P(1+i)^{N-1}$	$P(1+i)^N$

- Interest earned (charged) for a period is based on
  - Remaining principal plus
  - Accumulated (unpaid) interest at the beginning of the period

Period	Beginning of period value	End of period value
1	**1000**	**=1000(1+0.2)=1200**
2	**P(1+i)=1200**	**$=1000(1.2^2)=1440$**

Interest

Example: **i=20%**

- Simple Interest: $\mathbf{F=P(1+ni)}$

t	P_i	End of Year
0		1000
1	200	1200
2	200	1400

- Compound Interest: $F = P(1+i)^n$

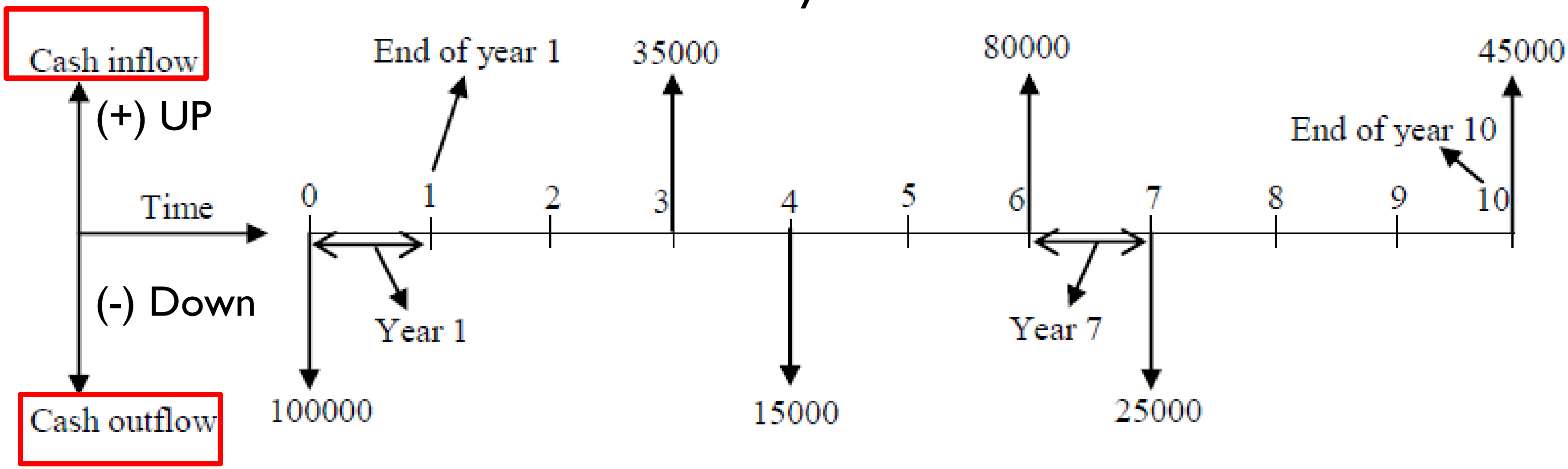
t	Beginning of year (1)	Pi (2)	End of Year =(1)+(2)
0			1,000
1	1,000	200	1,200
2	1,200	240	1,440

- "**Interest is compounded**": means it is computed and then added to the total owed or deposited.

- **Define:**
  - Nominal Interest rates
  - Real Interest rates: factors **inflation** into the equation.
    =Nominal Interest Rate − Inflation Rate
  - Effective Interest rates: concept of compounding into account.

Interest

- **Cash flow diagram**
  - Describes inflow and outflow of money overtime.

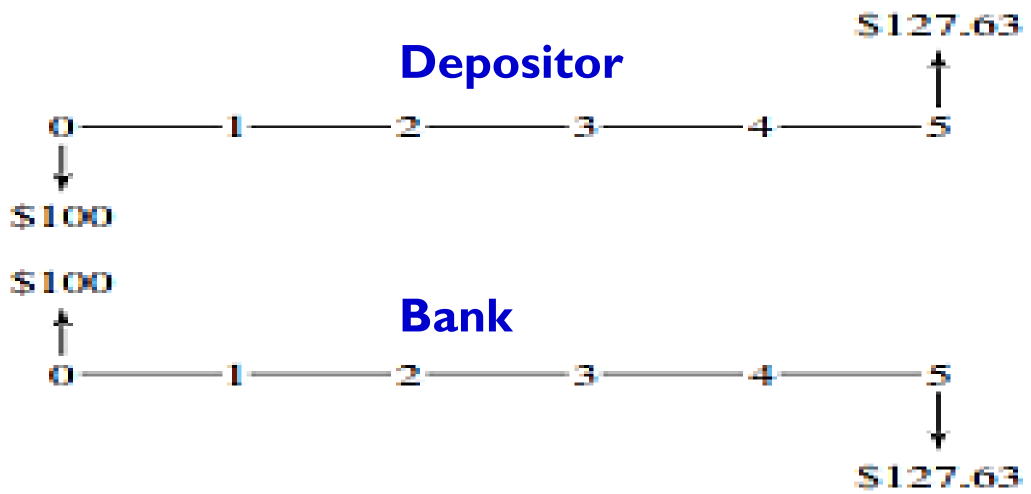


- The horizontal line is **time scale**. Moving from **left to right** with progression of time. Beginning of first year is traditionally defined as "**Time 0**".
- Arrow placed at the end of the period signify cash flows.
  - **Downward arrow** represent expense, disbursement= Negative cash flow, **outflow** "–"
  - **Upward arrow** represent Receipts= Positive cash flow, **Inflow** "+"
- Cash flow diagram is dependent on the point of view.

- **Cash flow diagram**
- **Cash flow diagram is dependent on the point of view.**

Example:

Cash flow diagram for banks and depositor: depending on which viewpoint is taken the diagram can simply be **reversed.** The depositor and the bank have opposite perspectives on cash in and cash out for the initial deposit and the final withdrawal.

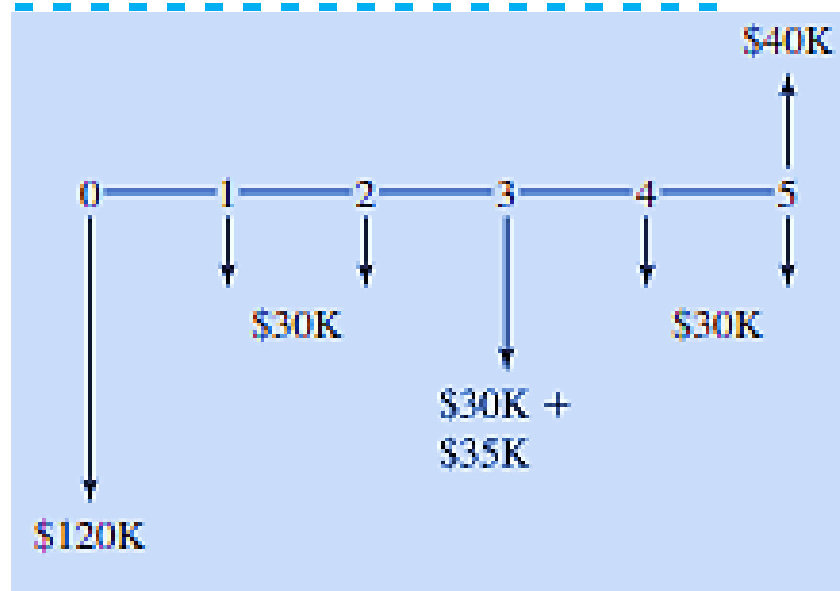


Interest

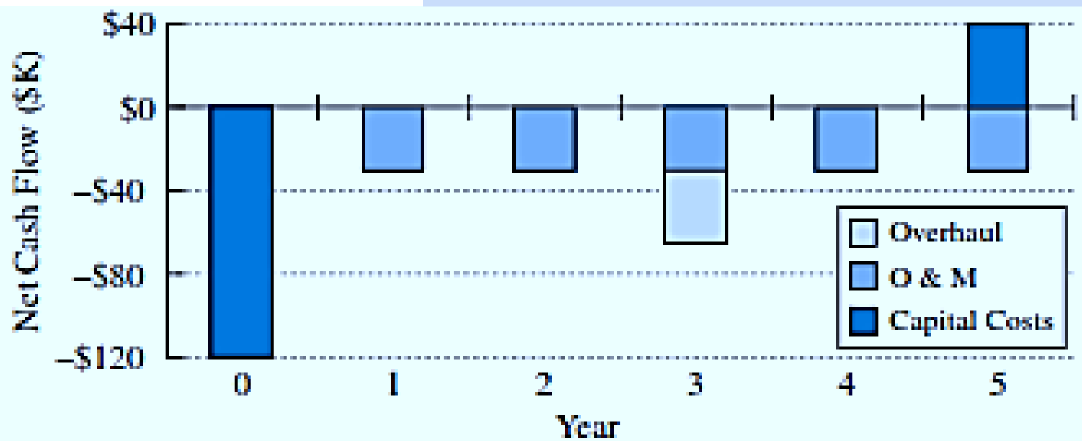
- Example

1. An earth moving company is considering purchase of a of heavy equipment. The cash flow diagram for the following anticipated cash flows:

   First cost=$120,000
   Operating & maintenance cost=$30,000 per year
   Overhaul cost=$35,000 in year 3
   Salvage value=$40,000 after 5 years



Year	Capital Costs	O&M	Overhaul
0	−120	−30	
1		−30	
2		−30	
3		−30	−35
4		−30	
5	40	−30	

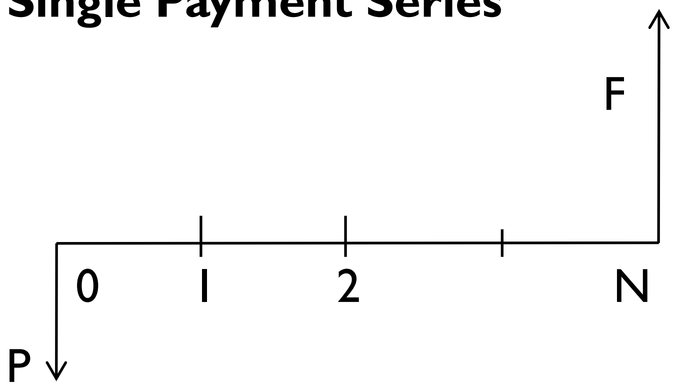


2. An amount P is deposited now so that an equal annual amount of A_1=2000 per year for the first 5 years, starting 1year after the deposit. And a different annual withdrawal of A_2= 3000 per year for the following 3 years. Draw the cash flow diagram if i=8.5% per year. **[Assignment]**

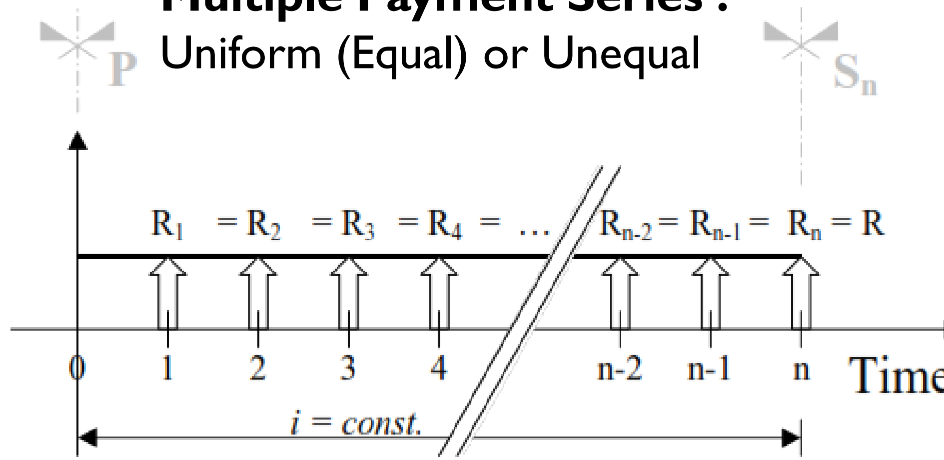
Interest

- **Cash flow diagram**

Single Payment Series

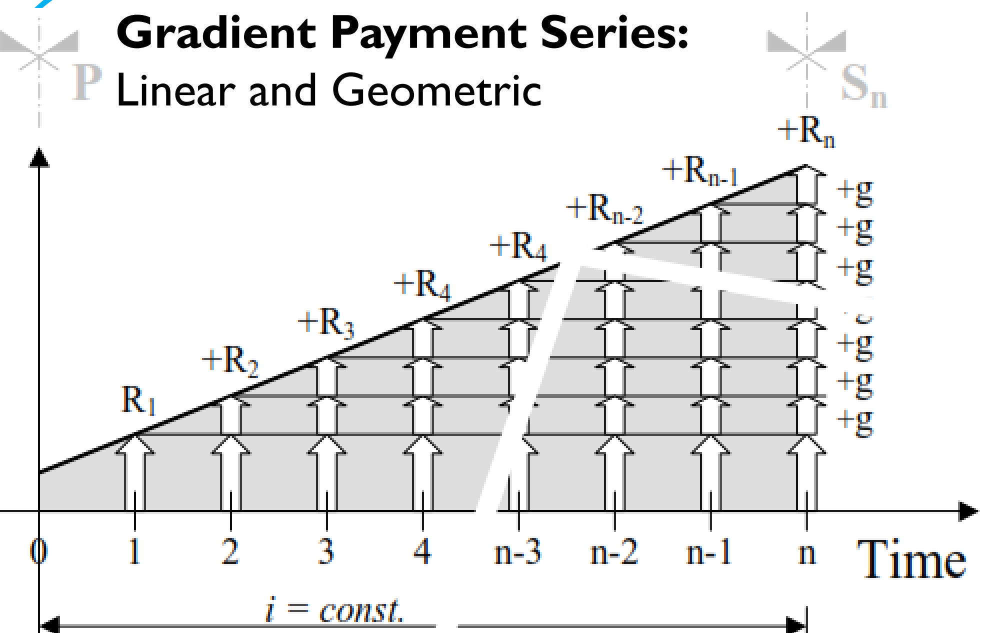


Multiple Payment Series :
Uniform (Equal) or Unequal

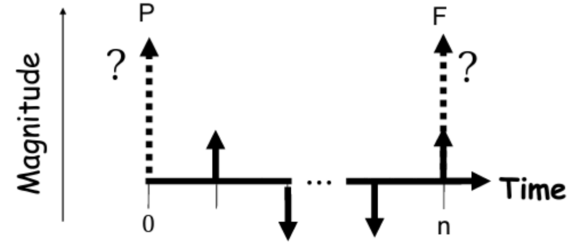


$A=R_1=R_2=\ldots R_n$

Gradient Payment Series:
Linear and Geometric



1. Single Payment Series

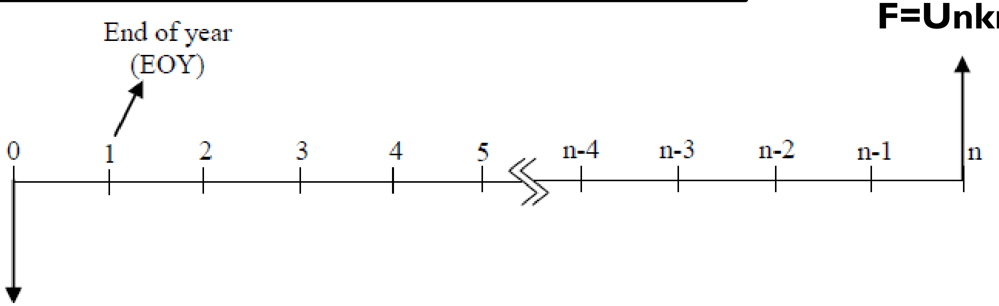


P→ F

$$F = P(1+i)^n$$

$(1+i)^n$: Single payment compound amount factor

F=Unknown



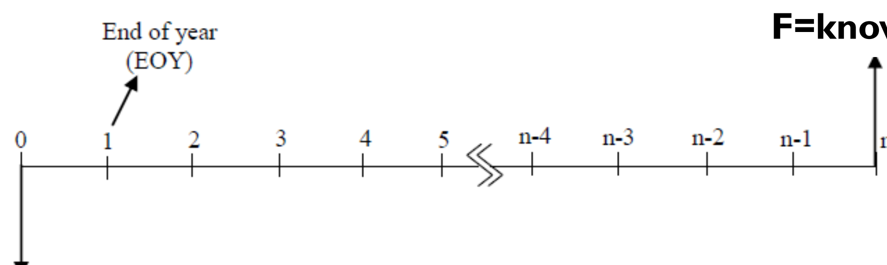
P=Known

F→P

$$P = F\frac{1}{(1+i)^n}$$

$\frac{1}{(1+i)^n}$: Single payment present worth factor

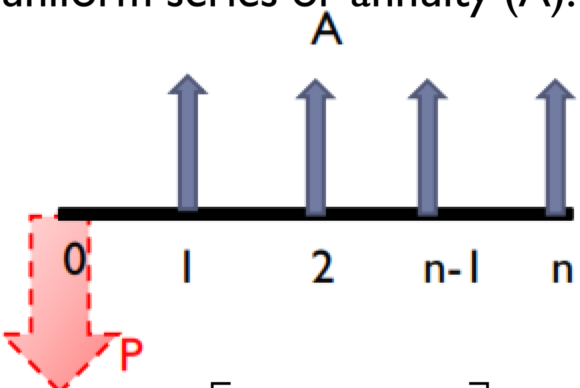
F=known



P=Unknown

2. Multiple Payment Series: Uniform/ Even

- To convert from a present worth (P) to a uniform series or annuity (A).



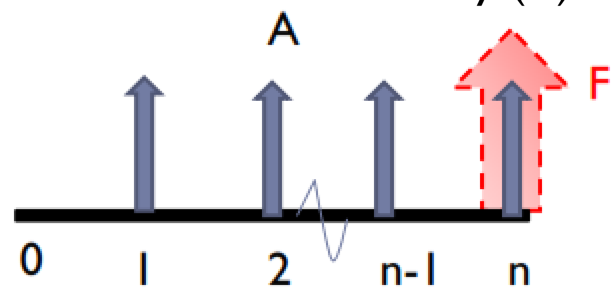
$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

Capital recovery factor

$$\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

$$= P[A/P, i\%, n]$$

- To convert from a future value (F) to a uniform series or annuity (A).



$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

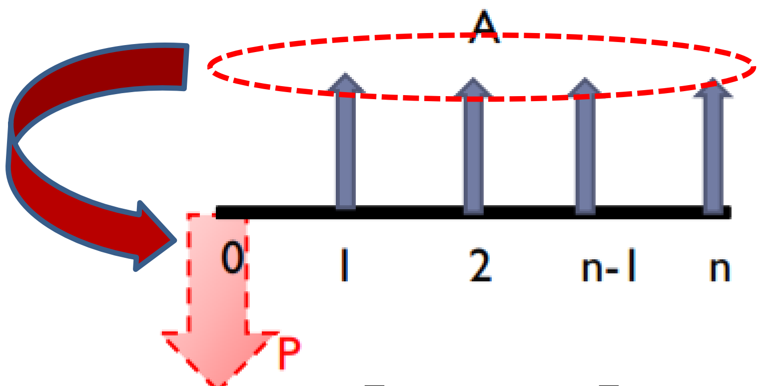
$$\left[\frac{i}{(1+i)^n - 1}\right]$$

Sinking fund factor

$$= F[A/F, i\%, n]$$

2. Multiple Payment Series: Uniform/ Even

- To convert from a uniform series or annuity (A) to a present worth (P)



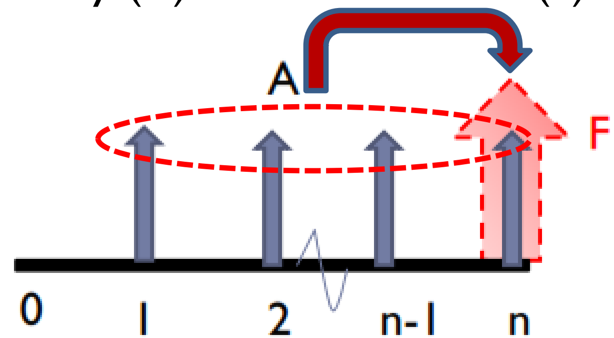
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

Equal payment series present worth factor

$$\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

$$= A[P / A, i\%, n]$$

- To convert from a uniform series or annuity (A) to a future value (F).



$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\left[\frac{(1+i)^n - 1}{i}\right]$$

Equal payment series compound amount factor

$$= A[F / A, i\%, n]$$

2. Multiple Payment Series: Uniform/ Even

- Example: If $10,000 is borrowed and payments of $2000 are made each year for 9 years, what is the interest rate?

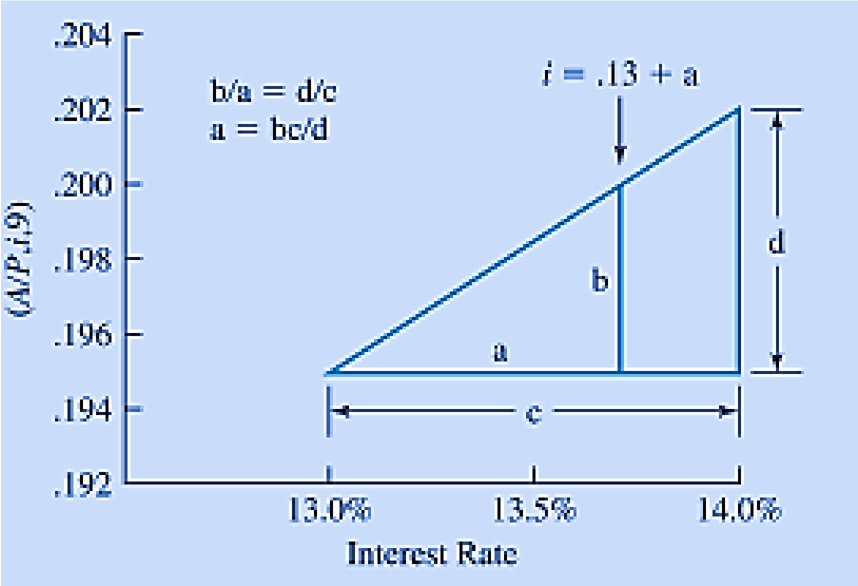
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] = A[P/A, i\%, n]$$

$$10{,}000 = 2{,}000\left[\frac{(1+i)^9 - 1}{i(1+i)^9}\right]$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = P[A/P, i\%, n]$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

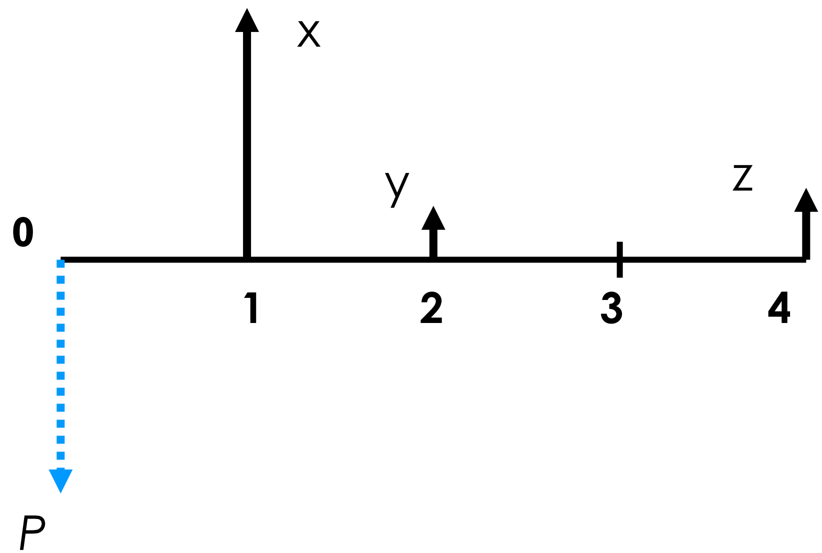
$$0.2 = [A/P, i\%, 9]$$



- Either solve the equation or use tables for interest factors and find that the interest rate is between 13% and 14%. These capital recovery factors, (A/P,.13,9) = 0.1949 and (A/P,.14,9) = 0.2022 include the value of .2.
- We interpolate for the value of i.

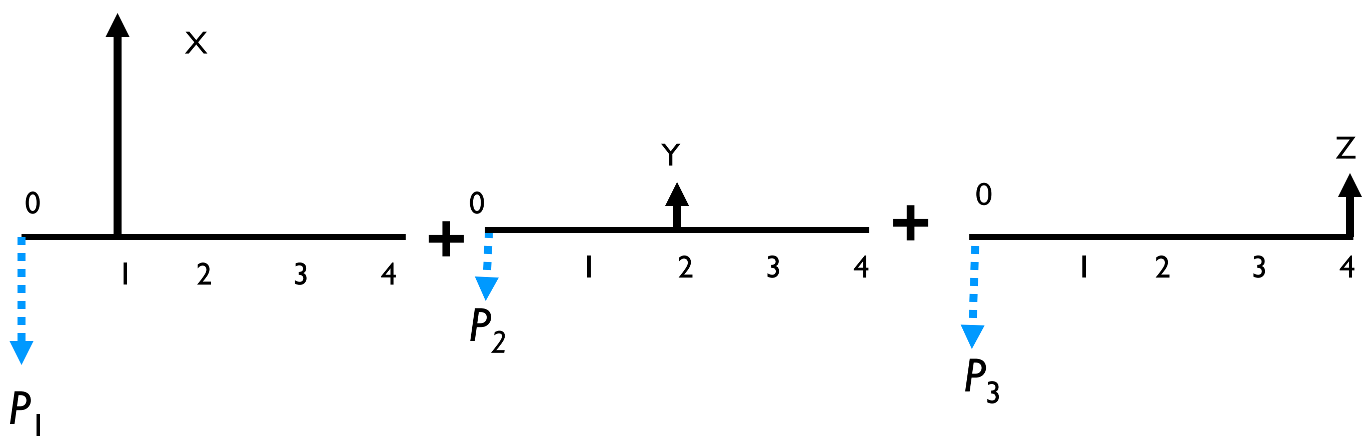
$$i = 0.13 + (0.2 - 0.1949)(0.14 - 0.13)/(0.2022 - 0.1949) = 13.7\%$$

2. Multiple Payment Series: Uneven



Uneven payment series → P

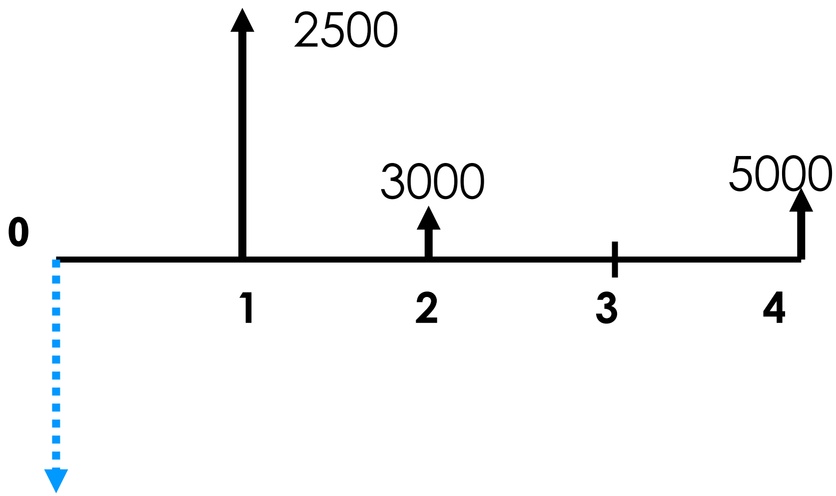
Uneven payment series → F



$P=P_1+P_2+P_3$

2. Multiple Payment Series: Uneven

Exercise: Determine the present value for the given uneven payment series. Take i=8%

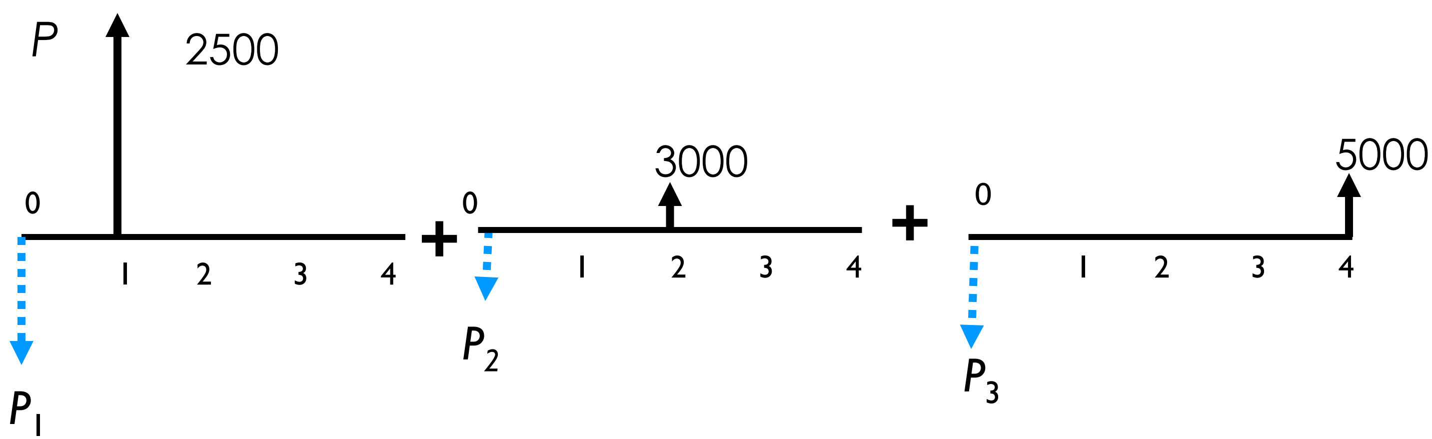


P_1= 2500 (P/F, 8%, 1)= 2,314.81

P_2=3000 (P/F, 8%, 2)= 2,572.02

P_3= 5000 (P/F, 8%, 4)=3,675.15

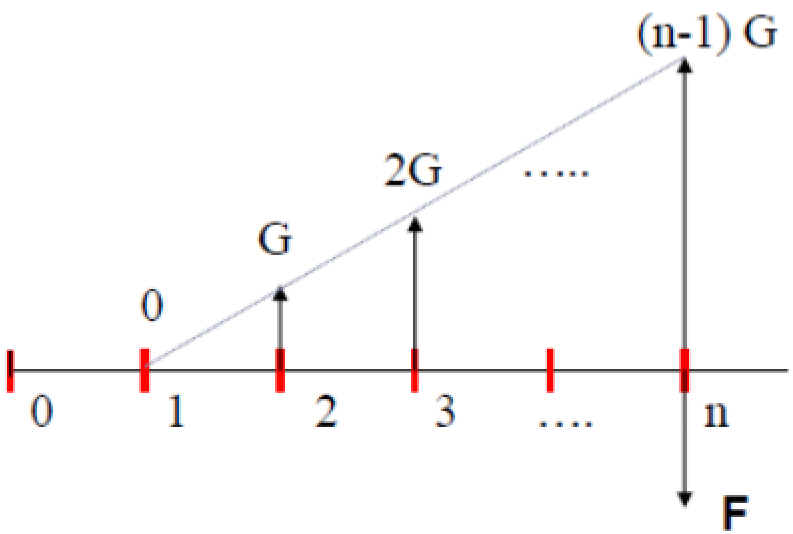
$P=P_1+P_2+P_3=8,562$

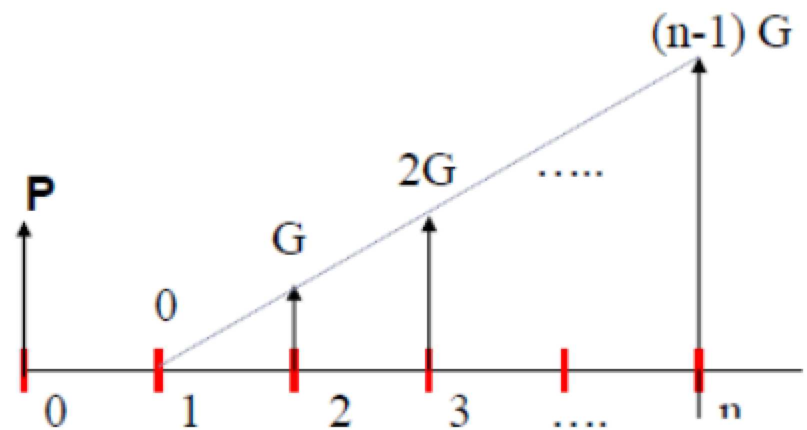


3. Gradient Payment Series: Linear and Geometric

- A gradient series of cash flows occurs when the value of a give cash flow is greater than the previous cash flow by a constant amount, G, gradient step.

- **Linear Gradient Series**





$$F = G\left[\frac{(1+i)^n - in - 1}{i^2}\right]$$

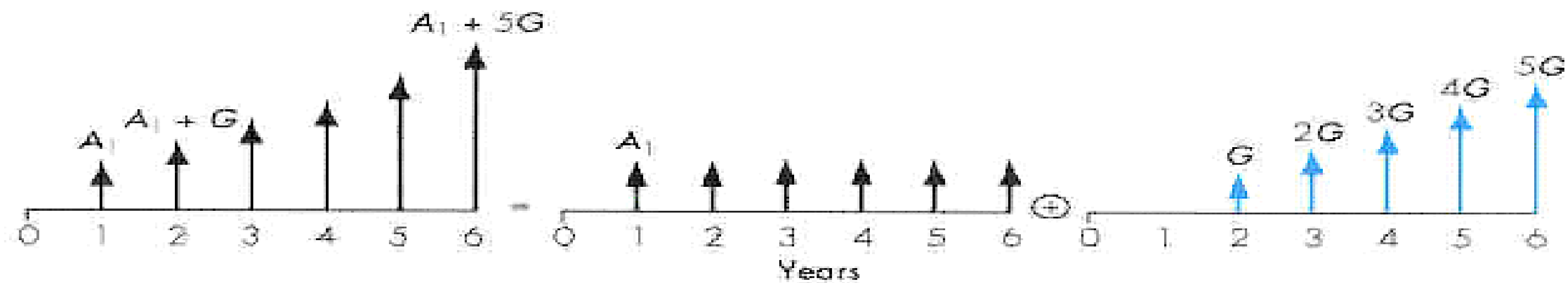
$$= G[F/G, i\%, n]$$

$$P = G\left[\frac{(1+i)^n - in - 1}{(1+i)^n i^2}\right]$$

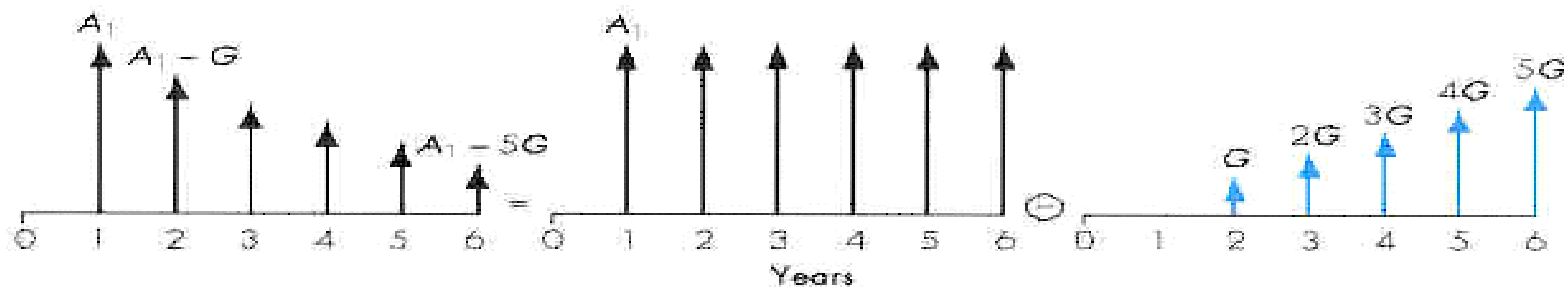
$$= G[P/G, i\%, n]$$

3. Gradient Payment Series: Linear and Geometric

- **Linear Gradient Series**
  - Gradient as composite



(a)Increasing gradient series

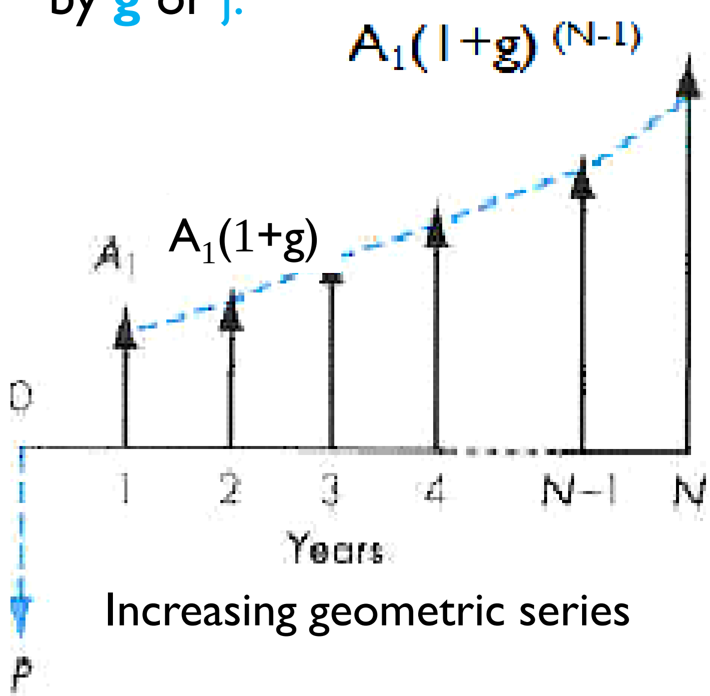


(b)Decreasing gradient series

3. Gradient Payment Series: Linear and Geometric

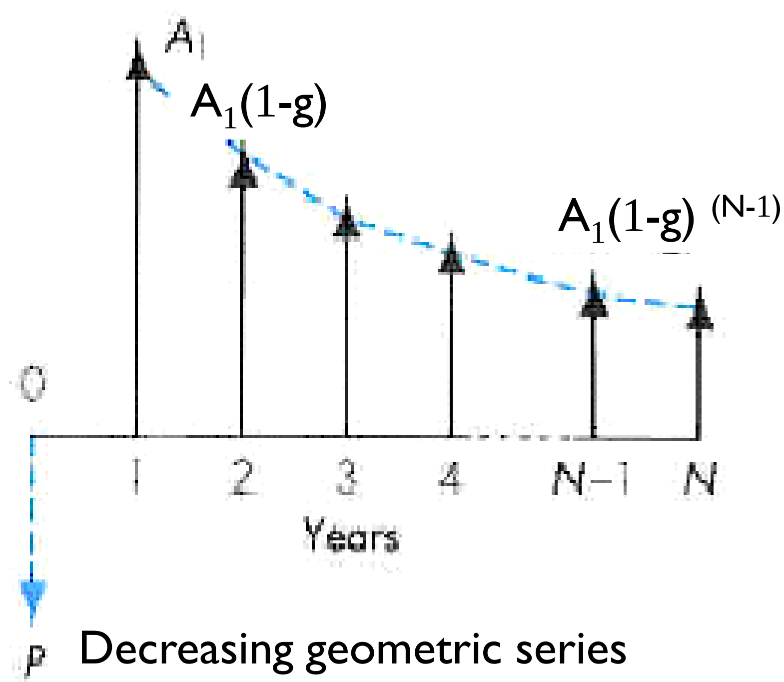
- **Geometric Gradient Series**
- The geometric cash flow series occurs when the size of a cash flow increases or decreases by a fixed percent from one point to the next.
- **Percent change** in a cash flow's size from one point to the next is denoted by g or j.

$$P=\begin{cases} A_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] & \text{if } i\neq g \\ A_1\left(\dfrac{N}{1+i}\right) & \text{if } i=g \end{cases}$$



Increasing geometric series

or



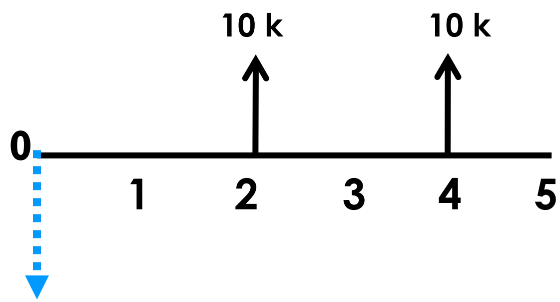
Decreasing geometric series

Economic Factors

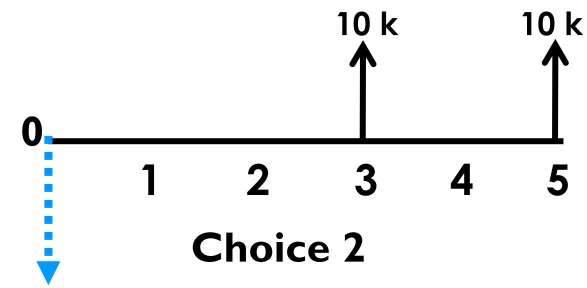
Find	Given	Symbol	Factor	Name
P	F	(P/F, i ,n)	$(1+i)^{-n}$	Single payment present worth factor
F	P	(F/P, i ,n)	$(1+i)^{n}$	Single payment compound amount factor
P	A	(P/A, i ,n)	$\frac{(1+i)^n-1}{i(1+i)^n}$	Uniform series present worth factor
A	P	(A/P, i ,n)	$\frac{i(1+i)^n}{(1+i)^n-1}$	Uniform series capital recovery factor
F	A	(F/A, i ,n)	$\frac{(1+i)^n-1}{i}$	Uniform series compound amount factor
A	F	(A/F, i ,n)	$\frac{i}{(1+i)^n-1}$	Uniform series sinking fund factor
P	G	(P/G, i ,n)	$\frac{[1-(1+ni)(1+i)^{-n}]}{i^2}$	Gradient series present worth factor
A	G	(A/G, i ,n)	$\frac{(1+i)^n-(1+ni)}{i[(1+i)^n-1]}$	Gradient series to uniform series conversion factor
P	A_1,j	(P/A_1,i, j, n)	$\frac{1-(1+j)^n-(1+i)^{-n}}{i-j}$, $i\neq j$	Geometric series present worth factor
F	A_1,j	(F/A_1,i, j, n)	$\frac{(1+i)^n-(1+j)^n}{i-j}$, $i\neq j$	Geometric series future worth factor

Multiple Payment Series: Uniform/ Even

- **Example**: Consider two investment choices that both require an initial out flow of 20, 000 birr and an expected revenue as shown by respective cash flow diagrams. Which one should be chosen?



Choice 1



Choice 2

- PV of expected future revenue

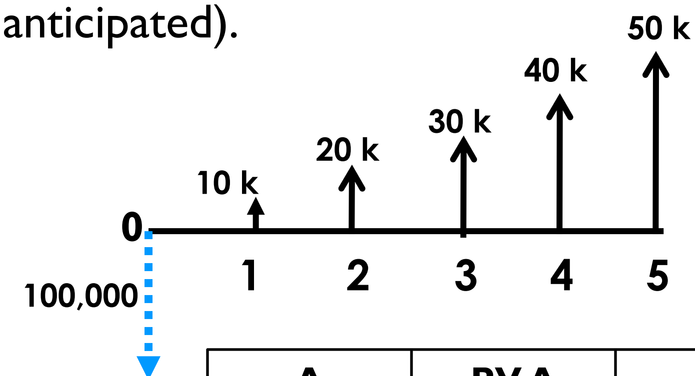
$$P = F \frac{1}{(1+i)^n}$$

	1	PV 1	2	PV 2
0	-20,000	-20,000	-20, 000	-20,000
1				
2	10,000	8,264.46		
3			10,000	7,513.15
4	10,000	6,830.13		
5			10,000	6,209.21
		-4,905		-6,278

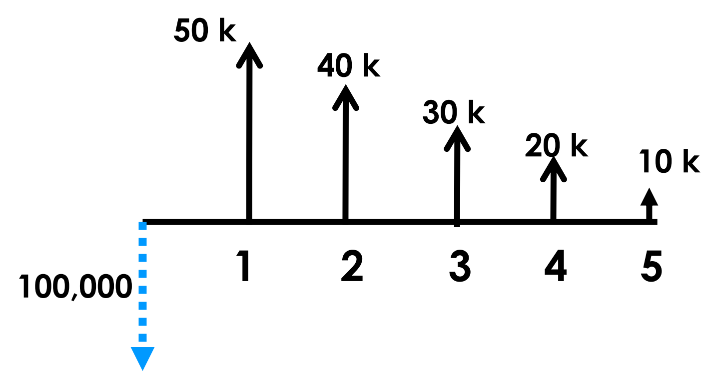
Example: A consulting engineer is considering two investment alternatives (A and B) having cash flow alternative shown below.

Assume an equivalent i=10%

Alternative A: is an investment in a land development venture. Several other limited partners are considering purchasing land, subdividing it, and selling land parcels over a 5 yr period (an increase in land value is anticipated).



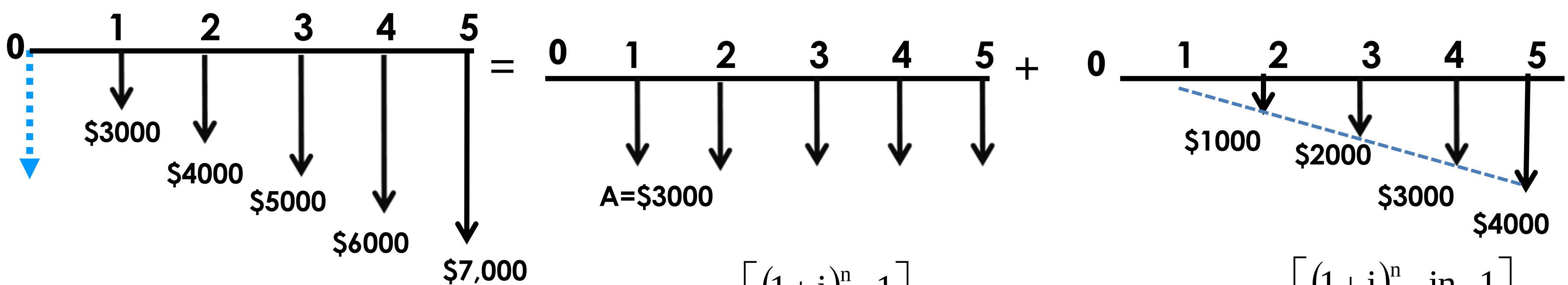
Alternative B: is for computer and the software required to provide specialized computer-design capabilities for clients. The engineer anticipate that competition will develop quickly if his plan proves successful, a declining revenue profile is anticipated.



	A	PV A	B	PV B
0	(100,000)	(100,000)	(100,000)	(100,000)
1	10,000	9,090.91	50,000	45,454.55
2	20,000	16,528.93	40,000	33,057.85
3	30,000	22,539.44	30,000	22,539.44
4	40,000	27,320.54	20,000	13,660.27
5	50,000	31,046.07	10,000	6,209.21
		6,526		**20,921**

Gradient Payment Series: Linear

- Example: Maintenance cost for a particular production machine increase by $1000/yr over the 5 year life of the equipment. The initial maintenance cost is $3000. Using an interest rate of 8 % compounded annually, determine the present worth equivalent for the maintenance cost.



$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$
$$= A[P/A, i\%, n]$$
$$= 3000[P/A, 8\%, 5]$$
$$= 3000 * 3.9927$$
$$P_u = \$11{,}978.13$$

$$P = G\left[\frac{(1+i)^n - in - 1}{(1+i)^n i^2}\right]$$
$$= G[P/G, i\%, n]$$
$$= 1000[P/G, 8\%, 5]$$
$$= 1000 * 7.37243$$
$$P_G = \$7{,}372.43$$

$P = P_U + P_G = 19{,}350.56$

Gradient Payment Series: Geometric

- **Example**: Assume you receive an annual bonus and deposit it in a saving account that pays 8 percent compounded annually. Your initial bonus is 500 birr and the size of your bonus increases by 10% each year. Determine how much will be in the fund immediately after your 10th deposit.

A1=500, i=8%, g=10%, and n=10 years

End of Year(n)	Cash flow
0	0
1	500
2	550
3	605
4	666
5	732
6	805
7	886
8	974
9	1072
10	1179

$$P = \begin{cases} A_1\left[\dfrac{1-(1+g)^n(1+i)^{-n}}{i-g}\right] & \text{if } i \neq g \\ A_1\left(\dfrac{n}{1+i}\right) & \text{if } i = g \end{cases}$$

$$\rightarrow F = P(1+i)^n$$

$$F = A_1\left[\frac{(1+i)^n-(1+g)^n}{i-g}\right]$$

= 500(F/A_1,8%,10%,10)

= 500* 21.74

F= 10,870.44

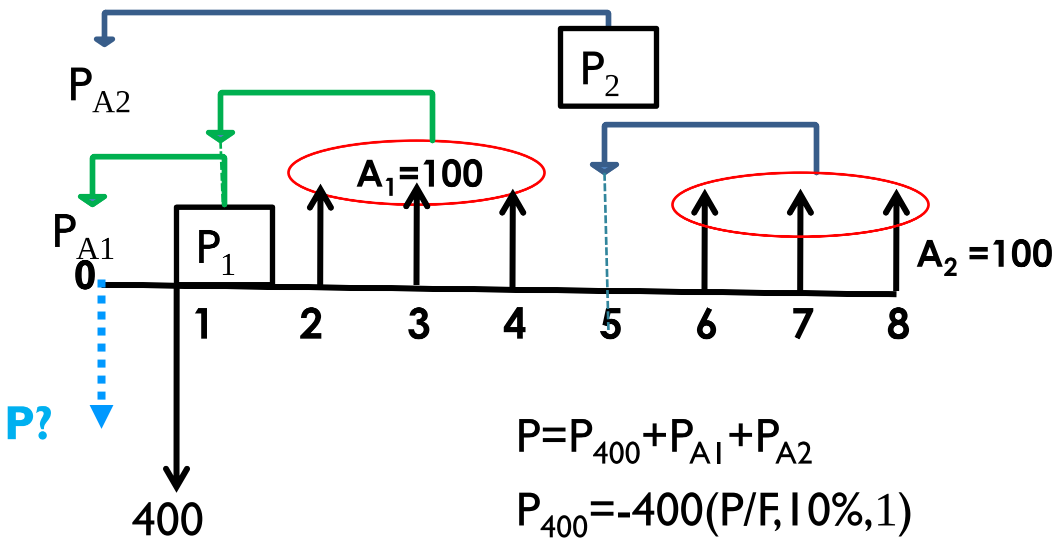
P= 10,870.44/(1.08^{10})=5,035.12

Economic Equivalence

- Equivalence in engineering economics analysis means " the state of being equal in value."
- Two cash flows series are equivalent at some specified interest rate i%, if the present worth are equal using an **interest rate of i%.**
- Two or more cash flow profiles are equivalent if their time value of money worth **at a common point in time are equal**.
- There are certain rules that one should follow to make these calculations.
  - They need to **have** a **common time basis**;
  - **Equivalence** is **dependent** on **interest** rate; and
  - **Equivalence** is **maintained regardless** of **anything**.

Economic Equivalence

- Example : What single sum of money at t=0 is equivalent to the cash table below.



$P=P_{400}+P_{A1}+P_{A2}$

P_{400}=-400(P/F,10%,1)

P_{A1}=[100(P/A,10%,3)(P/F,10%,1)]

P_{A2}=[100(P/A,10%,3)(P/F,10%,5)]

P=16.85

- Exercise : What single sum of money at t=6 is equivalent to the cash table below.

Assignment-1

To be submitted next class

1. Using an 10% discount rate, what uniform series over five periods, [1,5], is equivalent to the cash flow given in Figure 1.

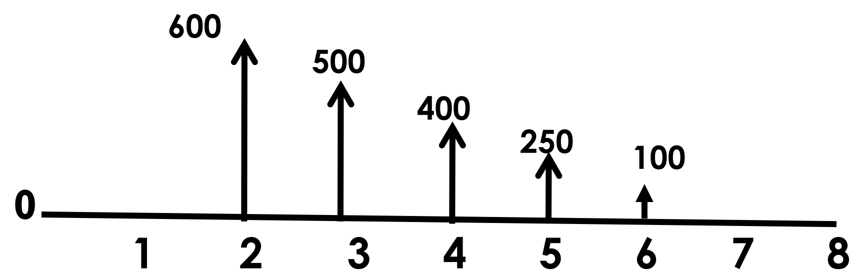


Figure 1

2. For what interest rate are the two cash flows shown in Figure 2 equivalent ?

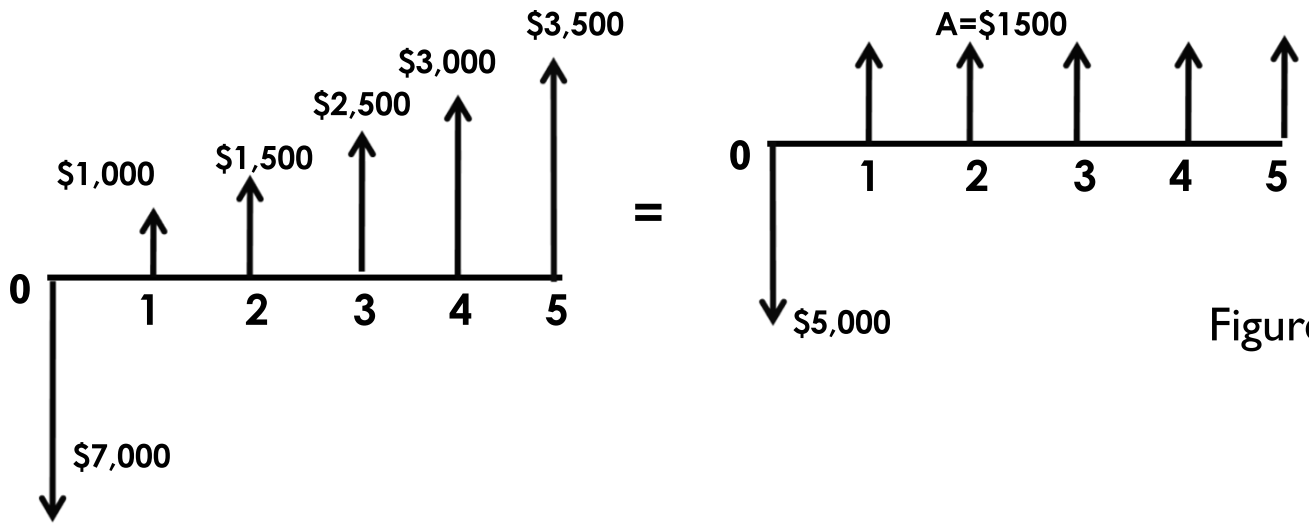


Figure 2

3. Define and clarify with examples
   - Nominal interest rates and
   - Effective Interest rates

Thank You