Ch. VIII Potential Flow and Computational Fluid Dynamics

Review of Velocity-Potential Concepts

This chapter presents examples of problems and their solution for which the assumption of potential flow is appropriate.

For low speed flows where viscous effects are neglected, the flow is irrotational and

$$\nabla \times \mathrm{V} = 0 \qquad \mathrm{V} = \nabla \phi \qquad \mathrm{u} = \frac{\partial \phi}{\partial \mathrm{x}} \qquad \mathrm{v} = \frac{\partial \phi}{\partial \mathrm{y}} \qquad \mathrm{w} = \frac{\partial \phi}{\partial \mathrm{z}}$$

The continuity equation , $\nabla \cdot \mathrm{V} = 0$, now reduces to

$$\nabla^2 \mathrm{V} = \frac{\partial^2 \phi}{\partial \mathrm{x}^2} + \frac{\partial^2 \phi}{\partial \mathrm{y}^2} + \frac{\partial^2 \phi}{\partial \mathrm{z}^2} = 0$$

The momentum equation reduces to Bernoulli's equation:

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2} V^2 + g z = \text{const}$$

Review of Stream Function Concepts

For plane incompressible flow in x-y coordinates a stream function exists such that

$$\mathrm{u} = \frac{\partial \Psi}{\partial \mathrm{y}} \quad \textit{and} \quad \mathrm{v} = -\frac{\partial \Psi}{\partial \mathrm{x}}$$

The condition of irrotationality reduces to Lapace's equation for Ψ and

$$\frac{\partial^2 \Psi}{\partial \mathrm{x}^2} + \frac{\partial^2 \Psi}{\partial \mathrm{y}^2} = 0$$

Elementary Plane-Flow Solutions

Three useful plane-flow solutions that are very useful in developing more complex solutions are:

Uniform stream, **i**U, in the x direction: $\Psi = \mathrm{U\,y}$ $\quad \phi = \mathrm{U\,x}$

Line source or sink: $\Psi = \mathrm{m}\,\theta$ $\quad \phi = m \ln \mathrm{r}$

Line vortex: $\Psi = -\,\mathrm{K} \ln \mathrm{r}$ $\quad \phi = \mathrm{K}\,\theta$

In these expressions, the source strength, 'm' and vortex strength, ' K ', have the dimensions of velocity times length, or $[L^2/t]$.

If the uniform stream is written in plane polar coordinates, we have

Uniform stream, **i**U: $\Psi = \mathrm{U\,r} \sin\theta$ $\quad \phi = \mathrm{U\,r} \cos\theta$

For a uniform stream moving at an angle, a , relative to the x-axis, we can write

$$u = \mathrm{U}\cos\alpha = \frac{\partial \Psi}{\partial \mathrm{y}} = \frac{\partial \phi}{\partial \mathrm{x}} \qquad \mathrm{v} = \mathrm{U}\sin\alpha = -\frac{\partial \Psi}{\partial \mathrm{x}} = \frac{\partial \phi}{\partial \mathrm{y}}$$

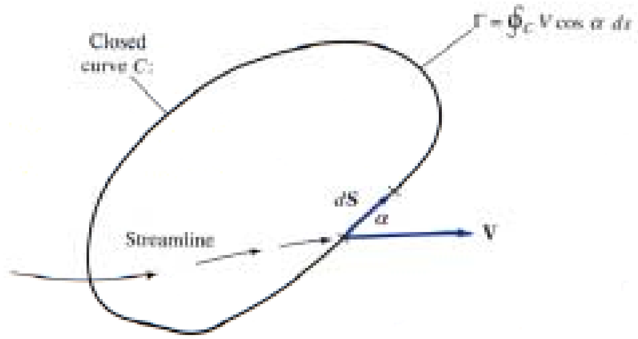
After integration, we obtain the following expressions for the stream function and velocity potential:

$$\Psi = U\left(y\cos\alpha - x\sin\alpha\right) \qquad \phi = U\left(x\cos\alpha + y\sin\alpha\right)$$

Circulation

The concept of fluid circulation is very useful in the analysis of certain potential flows, in particular those useful in aerodynamics analyses. Consider Figure 8.3 shown below:

We define the circulation, Γ , as the counterclockwise line integral of the arc length, ds times the velocity component tangent to the closed curve, C, e.g.



$$\Gamma = \oint_c V \cos \alpha \, d\,s = \int_c V \cdot ds$$

$$\Gamma = \int_c \left(u\,dx + \mathrm{v}\,\mathrm{dy} + \mathrm{w}\,\mathrm{dz} \right)$$

For most flows, this line integral around a closed path, starting and stoping at the same point, yields $\Gamma = 0$. However,

for a vortex flow for which	$\phi = K\,\theta$
the integral yields	$\Gamma = 2\,\pi\,K$

An equivalent calculation can by made by defining a circular path of radius r around the vortex center to yield

$$\Gamma = \int_c \mathrm{v}_\theta \, d\,s = \int_0^{2\pi} \frac{K}{r} r \, d\,\phi = 2\,\pi\,K$$

Superposition of Potential Flows

Due to the mathematical character of the equations governing potential flows, the principle of superposition can be used to determine the solution of the flow which results from combining two individual potential flow solutions.

Several classic examples of this are presented as follows:

1. Source m at (-a,0) added to an equal sink at (+a, 0).

$$\psi = -m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2} \qquad \phi = \frac{1}{2} m \ln \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

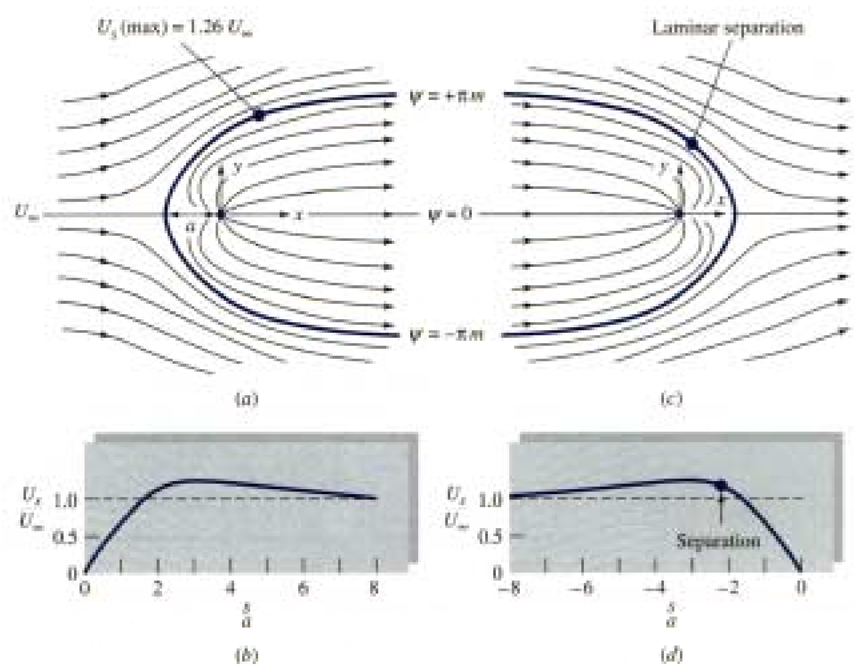
The streamlines and potential lines are two families of orthogonal circles (Fig. 4.13).

2. Sink m plus a vortex K, both at the origin.

$$\psi = m\theta - K \ln r \qquad \phi = m \ln r + K\theta$$

The streamlines are logarithmic spirals swirling into the origin (Fig. 4.14). They resemble a tornado or a bathtub vortex.

3. Uniform steam **i** U_∞ plus a source m at the origin (Fig. 4.15), the Ranking half body. If the origin contains a source, a plane half-body is formed with its nose to the left as shown below. If the origin contains a sink, m < 0, the half-body nose is to the right.. For both cases, the stagnation point is at a position $a = m / U_\infty$ away from the origin.

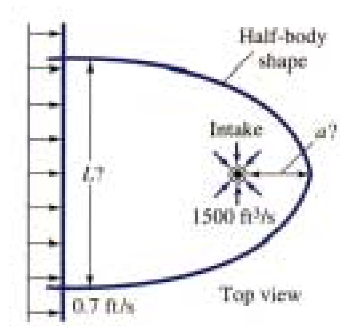


Example 8.1

An offshore power plant cooling water intake has a flow rate of 1500 ft^3/s in water 30 ft deep as in Fig. E8.1. If the tidal velocity approaching the intake is 0.7 ft/s, (a) how far downstream does the intake effect extend and (b) how much width of tidal flow in entrained into the intake?

The sink strength is related to the volume flow, Q and water depth by

$$m = \frac{Q}{2\pi b} = \frac{1500\ ft^3/s}{2\pi 30\ ft} = 7.96\ ft^2/s$$



E8.1

The lengths a and L are given by

$$a = \frac{m}{U_\infty} = \frac{7.96\ ft^2/s}{0.7\ ft/s} = 11.4\ ft$$

$$L = 2\pi a = 2\pi 11.4\ ft = 71\ ft$$

Flow Past a Vortex

Consider a uniform stream, U_∞ flowing in the x direction past a vortex of strength K with the center at the origin. By superposition the combined stream function is

$$\psi = \psi_{stream} + \psi_{vortex} = U_\infty r \sin\theta - K \ln r$$

The velocity components of this flow are given by

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta \qquad v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta + \frac{K}{r}$$

Setting v_r and v_θ = 0, we find the stagnation point at θ = 90°, r = a = K/ U_∞ or (x,y) = (0,a).

An Infinite Row of Vortices

Consider an infinite row of vortices of equal strength K and equal spacing a. A single vortex, i , has a stream function given by

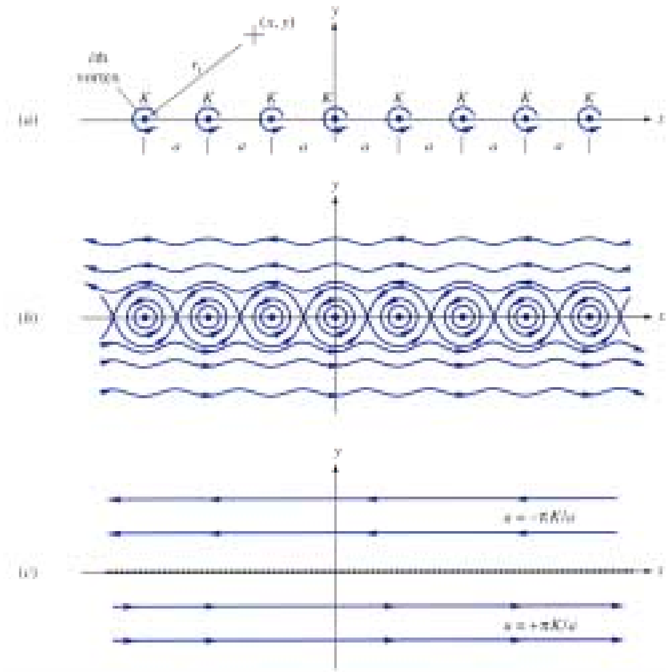


Fig. 8.7 Superposition of vortices

$$\psi_i = -K \sum_{i=1}^{\infty} \ln r_i$$

This infinite sum can also be expressed as

$$\psi = -\frac{1}{2} K \ln\left[\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cosh\frac{2\pi x}{a}\right)\right]$$

The resulting left and right flow above and below the row of vortices is given by

$$u = \left.\frac{\partial \psi}{\partial y}\right|_{|y|>a} = \pm\frac{\pi K}{a}$$

Plane flow past Closed-Body Shapes

Various types of external flows over a closed-body can be constructed by superimposing a uniform stream with sources, sinks, and vortices.

Key Point: The body shape will be closed only if the net source of the outflow equals the net sink inflow.

Two examples of this are presented below.

The Rankine Oval

A Rankine Oval is a cylindrical shape which is long compared to its height. It is formed by a source-sink pair aligned parallel to a uniform stream.

The individual flows used to produce the final result and the combined flow field are shown in Fig. 8.9. The combined stream function is given by

$$\psi = U_\infty y - m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2}$$

or

$$\psi = U_\infty r \sin\theta + m(\theta_1 - \theta_2)$$

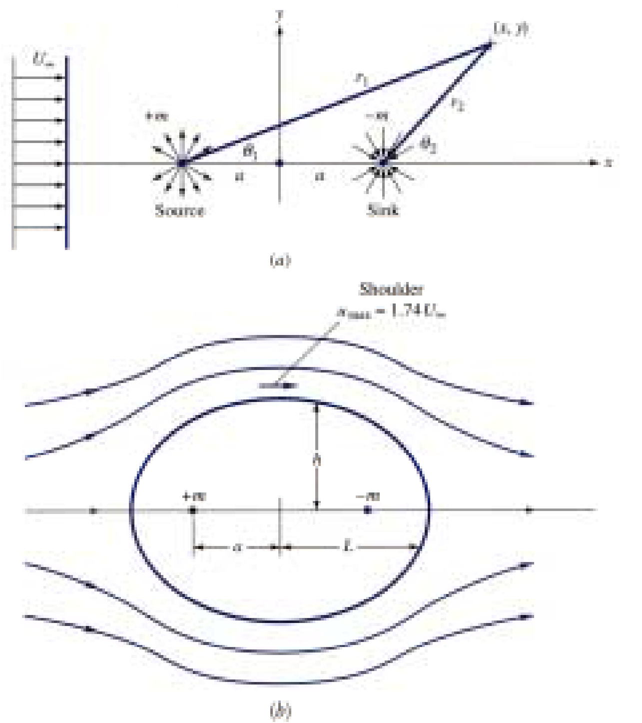


Fig. 8.9 The Rankine Oval

The oval shaped closed body is the streamline, $\psi = 0$. Stagnation points occur at the front and rear of the oval, $x = \pm L, y = 0$. Points of maximum velocity and minimum pressure occur at the shoulders, $x = 0, y = \pm h$. Key geometric and flow parameters of the Rankine Oval can be expressed as follows:

$$\frac{h}{a} = \cot \frac{h/a}{2m/(U_\infty a)} \qquad \frac{L}{a} = \left(1 + \frac{2m}{U_\infty a}\right)^{1/2}$$

$$\frac{u_{\max}}{U_\infty} = 1 + \frac{2m/(U_\infty a)}{1 + h^2/a^2}$$

As the value of the parameter $m/(U_\infty a)$ is increased from zero, the oval shape increases in size and transforms from a flat plate to a circular cylinder at the limiting case of $m/(U_\infty a) = \infty$.

Specific values of these parameters are presented in Table 8.1 for four different values of the dimensionless vortex strength, $K/(U_\infty a)$.

Table 8.1 Rankine-Oval Parameters

$m/(U_\infty a)$	h/a	L/a	L/h	$u_{\max}/U_\infty$
0.0	0.0	1.0	∞	1.0
0.01	0.31	1.10	32.79	1.020
0.1	0.263	1.095	4.169	1.187
1.0	1.307	1.732	1.326	1.739
10.0	4.435	4.458	1.033	1.968
10.0	14.130	14.177	1.003	1.997
∞	∞	∞	1.000	2.000

Flow Past a Circular Cylinder with Circulation

It is seen from Table 8.1 that as source strength m becomes large, the Rankine Oval becomes a large circle, much greater in diameter than the source-sink spacing 2a. Viewed, from the scale of the cylinder, this is equivalent to a uniform stream plus a doublet. To add circulation, without changing the shape of the cylinder, we place a vortex at the doublet center. For these conditions the stream function is given by

$$\psi = U_\infty \sin\theta \left(r - \frac{a^2}{r} \right) - K \ln \frac{r}{a}$$

Typical resulting flows are shown in Fig. 8.10 for increasing values of non-dimensional vortex strength $K / U_\infty a$.

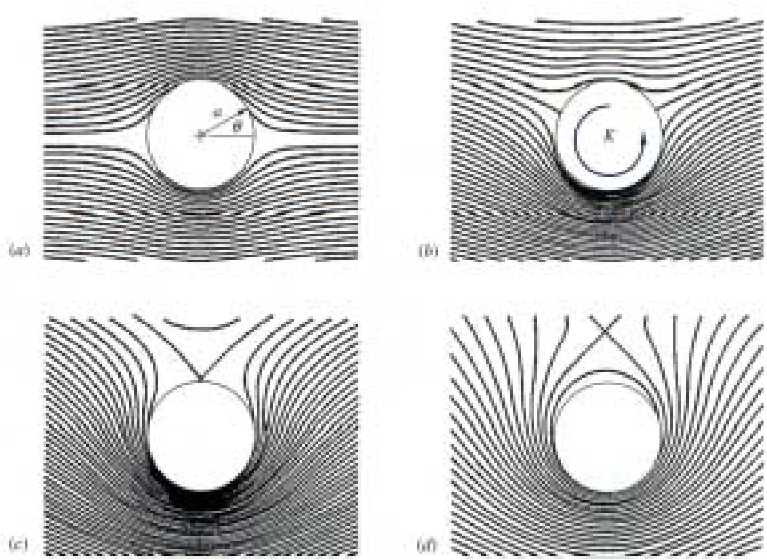


Fig. 8.10 Flow past a cylinder with circulation for values of $K / U_\infty a$ of (a) 0, (b) 1.0, (c) 2.0, and (d) 3.0

Again the streamline $\psi = 0$ is corresponds to the circle r = a. As the counter-clockwise circulation $\Gamma = 2\pi K$ increases, velocities below the cylinder increase and velocities above the cylinder decrease. In polar coordinates, the velocity components are given by

$$\text{v}_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta \left(1 - \frac{a^2}{r^2} \right)$$

$$\text{v}_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta \left(1 + \frac{a^2}{r^2} \right) + \frac{K}{r}$$

For small K, two stagnation points appear on the surface at angles θ_s or for which

$$\sin\theta_s = \frac{K}{2U_\infty a}$$

Thus for K = 0, θ_s = 0 and 180°. For $K/U_\infty a = 1$, θ_s = 30 and 150°. Figure 8.10c is the limiting case for which with $K/U_\infty a = 2$, $\theta_s = 90^\circ$ and the two stagnation points meet at the top of the cylinder.

The Kutta-Joukowski Lift Theorem

The development in the text shows that from inviscid flow theory,

> The lift per unit depth of any cylinder of any shape immersed in a uniform stream equals to $\rho U_\infty \Gamma$ where Γ is the total net circulation contained within the body shape. The direction of the lift is 90o from the stream direction, rotating opposite to the circulation.

This is the well known Kutta-Joukowski lift theorem.