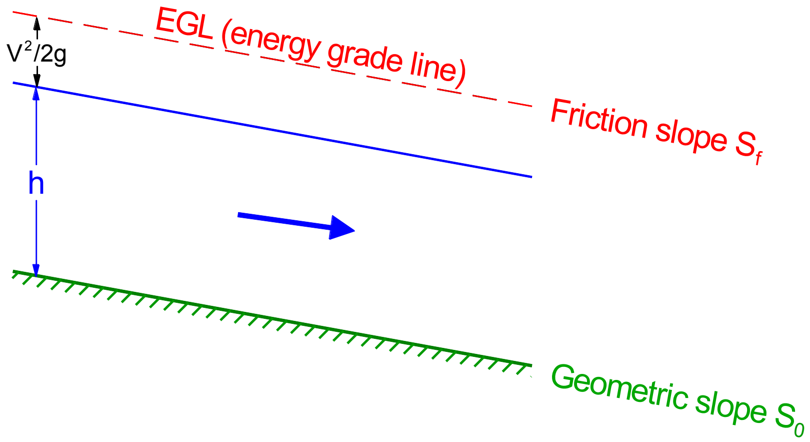
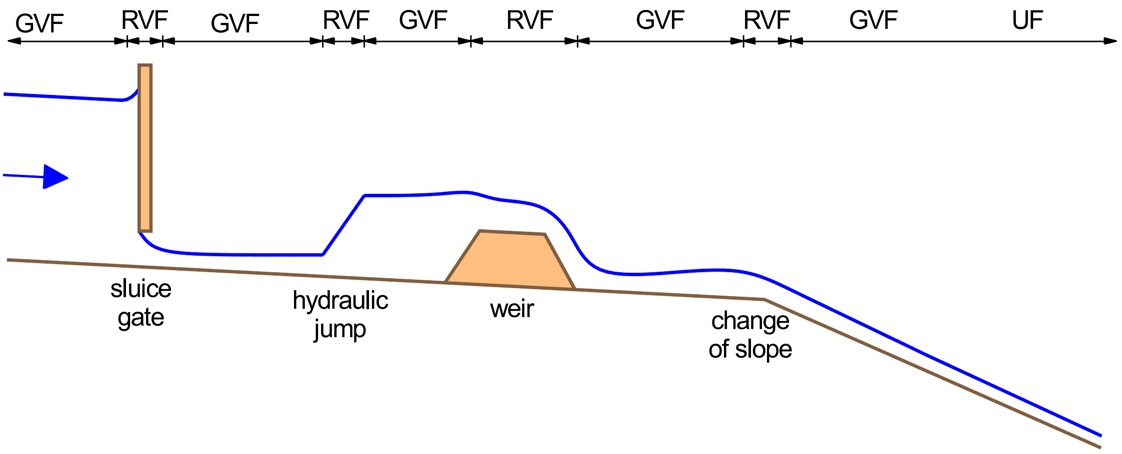
3.1 Normal Flow vs Gradually-Varied Flow



In *normal flow* the downslope component of weight balances bed friction. As a result, the water depth *h* and velocity *V* are constant and the total-head line (or *energy grade line*) is parallel to both the water surface and the channel bed; i.e. the *friction slope* S_f is the same as the *geometric slope* S_0.



As a result of disturbances due to hydraulic structures (weir, sluice, etc.) or changes to channel width, slope or roughness the downslope component of weight may not locally balance bed friction. As a result, the friction slope S_f and bed slope S_0 will be different and the water depth *h* and velocity *V* will change along the channel.

The *gradually-varied-flow equation* gives an expression for d*h*/d*x* and allows one to predict the variation of water depth along the channel.

3.2 Derivation of the Gradually-Varied-Flow Equation

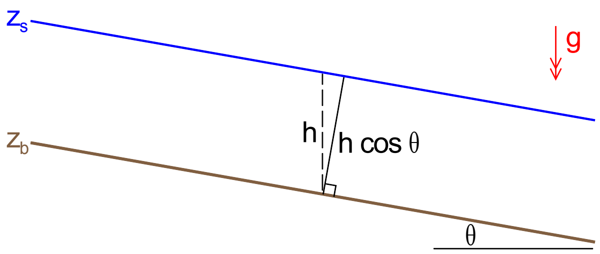
Provided the pressure distribution is hydrostatic then, at any streamwise location x:

$$\frac{p}{\rho g} + z = z_s(x)$$

The total head is then

$$H = z_s + \frac{V^2}{2g} \quad = z_b + h + \frac{V^2}{2g} \qquad (1)$$

where z_s is the level of the free surface and z_b is the level of the bed.



Although not crucial, we make the *small-slope* assumption and make no distinction between the vertical depth h (which forms part of the total head) and that perpendicular to the bed, $h \cos\theta$ (which is used to get the flow rate).

The total head may be written

$$H = z_b + E \qquad (2)$$

where E is the *specific energy*, or head relative to the bed:

$$E = h + \frac{V^2}{2g} \qquad (3)$$

In frictionless flow, $H = constant$; i.e. the energy grade line is horizontal. In reality, H decreases over large distances due to bed friction; the energy grade line slopes downward.

Differentiate (2):

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\mathrm{d}z_b}{\mathrm{d}x} + \frac{\mathrm{d}E}{\mathrm{d}x} \qquad (4)$$

Define:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -S_f \qquad (5)$$

$$\frac{\mathrm{d}z_b}{\mathrm{d}x} = -S_0 \qquad (6)$$

S_f is the downward slope of the energy grade line, or *friction slope*; (more about how this is calculated later). S_0 is the actual geometric slope. Then

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}x} = S_0 - S_f} \qquad (7)$$

Thus, the specific energy only changes if there is a difference between the geometric and friction slopes, i.e. between the rates at which gravity drives the flow and friction retards it. Otherwise we would have normal flow, in which the depth and specific energy are constant.

Equations (5) and (7) are two forms of the *gradually-varied-flow equation*. However, the third, and most common, form rewrites dE/dx in terms of the rate of change of depth, dh/dx.

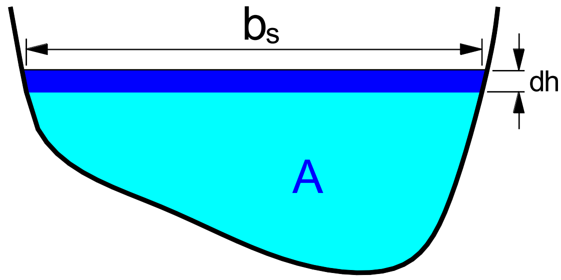
$$E = h + \frac{V^2}{2g} \qquad \text{where} \qquad V = \frac{Q}{A} \tag{8}$$

$$\Rightarrow \qquad E = h + \frac{Q^2}{2gA^2}$$

Differentiating with respect to streamwise distance x (using the chain rule for the last term):

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}x} - \frac{Q^2}{gA^3}\frac{\mathrm{d}A}{\mathrm{d}x}$$

If b_s is the width of the channel at the surface:



$$\mathrm{d}A = b_s\,\mathrm{d}h \qquad \text{and} \qquad \bar{h} = \frac{A}{b_s}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= \frac{\mathrm{d}h}{\mathrm{d}x}\left(1 - \frac{Q^2 b_s}{gA^3}\right)\\ &= \frac{\mathrm{d}h}{\mathrm{d}x}\left(1 - \frac{V^2}{g\bar{h}}\right)\\ &= \frac{\mathrm{d}h}{\mathrm{d}x}(1 - \mathrm{Fr}^2)\end{aligned}$$

Combining this with (7) gives, finally,

Gradually-Varied-Flow Equation

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2} \tag{9}$$

3.3 Finding the Friction Slope

Since the flow (and hence velocity profile) is only gradually-varying with distance, friction is primarily determined by the local bulk velocity V. The local friction slope S_f can then be evaluated on the "quasi-uniform-flow" assumption that there is the same rate of energy loss as in normal flow of the same depth; e.g. using Manning's equation:

$$V = \frac{1}{n} R_h^{2/3} S_f^{1/2}$$

Inverting for the friction slope:

$$S_f = \frac{n^2 V^2}{R_h^{4/3}} = \frac{n^2 Q^2}{R_h^{4/3} A^2} \tag{10}$$

Both A and R_h (which depend on the channel shape) should be written in terms of depth h.

In general, the deeper the flow then the smaller the velocity and friction losses. Qualitatively,

greater depth $\Rightarrow$ lower velocity $\Rightarrow$ smaller S_f;
smaller depth $\Rightarrow$ higher velocity $\Rightarrow$ greater S_f.

3.4 Profile Classification

For any given discharge (but independent of slope) there is a particular *critical depth* h_c, at which Fr = 1. For example, in a wide or rectangular channel, $h_c = (q^2/g)^{1/3}$.

For any given discharge and slope there is a particular *normal depth* h_n, associated with uniform flow. For example, in a wide channel, $h_n = (nq/\sqrt{S})^{3/5}$. The normal depth is that to which the flow would tend given a long enough undisturbed fetch.

A downward slope is classed as *steep* if the normal depth is less than the critical depth (i.e. the normal flow is supercritical) and *mild* if the normal depth is greater than the critical depth (i.e. the normal flow is subcritical).[1] Note that, in principle at least, a particular slope could be described as either steep or mild depending on the flow rate.

In general, given the actual depth h, critical depth h_c and normal depth h_n, simple inspection of signs of numerator and denominator on the RHS of the GVF equation

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2}$$

will tell us whether depth is increasing or decreasing with distance. In the special case $S_f = S_0$ we have $\mathrm{d}h/\mathrm{d}x = 0$; i.e. normal flow. Otherwise:

- $S_0 - S_f > 0$ if and only if h is greater than normal depth; (slower, hence less friction)
- $1 - \mathrm{Fr}^2 > 0$ if and only if h is greater than critical depth

Hence,

$$\frac{\mathrm{d}h}{\mathrm{d}x} < 0 \quad \text{(depth \textbf{decreasing}) if and only if } h \text{ lies \textbf{between} normal and critical depths.}$$

Considering the whole range of possibilities allows a two-character classification of slopes (S1, M3 etc) where:

- the first character is S, C, M, H, A for Steep, Critical, Mild, Horizontal, Adverse;
- the second character is 1, 2, 3 depending on where h lies with respect to h_c and h_n.

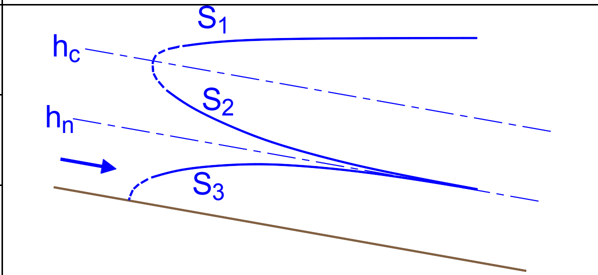
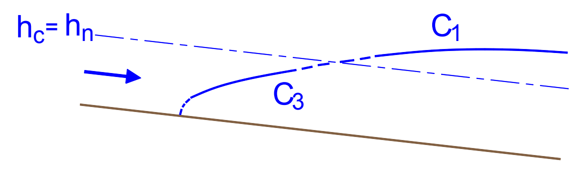
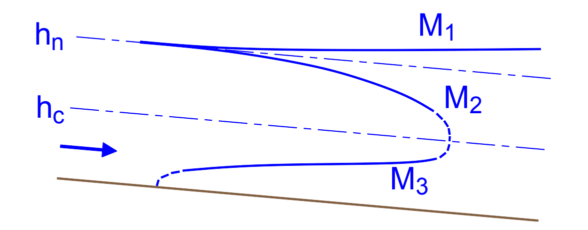
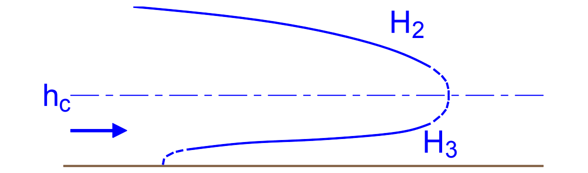
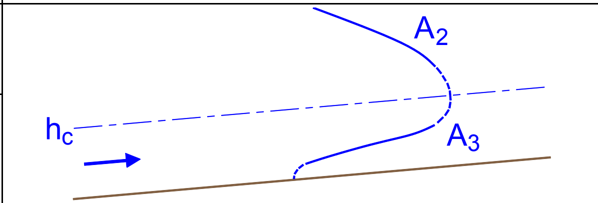
Typical profiles and examples of where they occur are given in the table overleaf.

A *backwater* curve is a streamwise increase of depth because of a downstream obstruction or because the slope is insufficient to maintain that rate of flow. A reduction in depth is called a *drawdown* curve. Profiles with second character 1 or 3 are backwater curves (numerator and denominator of the GVF equation have the same sign); profiles with second character 2 are drawdown curves (numerator and denominator of the GVF equation have opposite signs).

Note that:

- the normal depth (where $S_0 = S_f$) is approached asymptotically ($\mathrm{d}h/\mathrm{d}x \to 0$);
- the critical depth (where Fr = 1) is approached at right angles ($\mathrm{d}h/\mathrm{d}x \to \infty$).
- above normal depth a backwater asymptotes a horizontal surface (M1, S1)

[1] An alternative statement is that a downward slope is *steep* if it exceeds the *critical slope* (that slope at which the normal flow is critical).

Type	Symbol	Definition	Sketches	Examples
STEEP (normal flow is supercritical)	S1	$h > h_c > h_n$		Hydraulic jump upstream with obstruction or reservoir controlling water level downstream.
	S2	$h_c > h > h_n$		Change to steeper slope.
	S3	$h_c > h_n > h$		Change to less steep slope.
CRITICAL (undesirable; undular unsteady flow)	C1	$h > h_c = h_n$		
	C3	$h_c = h_n > h$		
MILD (normal flow is subcritical)	M1	$h > h_n > h_c$		Obstruction or reservoir controlling water level downstream.
	M2	$h_n > h > h_c$		Approach to free overfall.
	M3	$h_n > h_c > h$		Hydraulic jump downstream; change from steep to mild slope or downstream of sluice gate.
HORIZONTAL (limiting mild slope; $h_n \to \infty$)	H2	$h > h_c$		Approach to free overfall.
	H3	$h_c > h$		Hydraulic jump downstream; change from steep to horizontal or downstream of sluice gate.
ADVERSE (upslope)	A2	$h > h_c$		
	A3	$h_c > h$		

3.5 Qualitative Examples of Open-Channel-Flow Behaviour

A *control point* is a location where there is a known relationship between water depth and discharge (aka "*stage-discharge relation*"). Examples include critical-flow points (weirs, venturi flumes, sudden changes in slope, free overfall), sluice gates, entry or discharge to a reservoir. A hydraulic jump can also be classed as a control point. Control points often provide a location where one can start a GVF calculation; i.e. a boundary condition.

Some general rules:

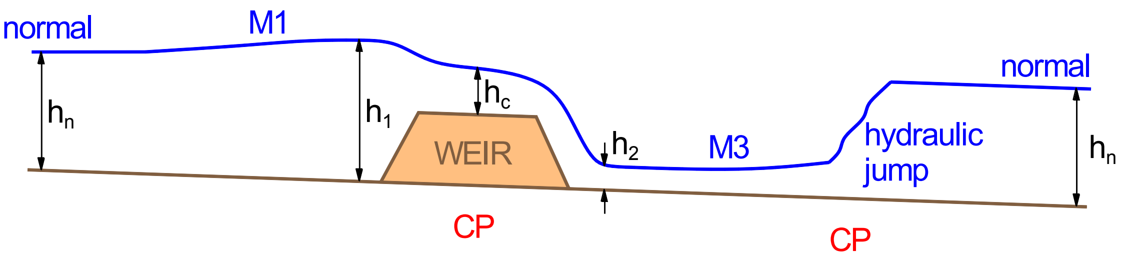
(i) *Supercritical* ⇒ controlled by *upstream* conditions.
Subcritical ⇒ controlled by *downstream* conditions.

(ii) Given a long-enough undisturbed fetch the flow will try to revert to normal flow.

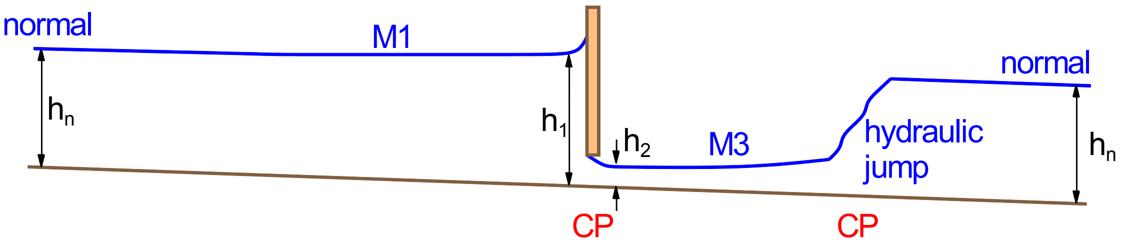
(iii) A hydraulic jump occurs between regions of supercritical and subcritical gradually-varied flow at the point where the jump condition for the sequent depths is correct.

(iv) Where the slope is mild (i.e. the normal flow is subcritical), and any downstream control is a long way away, a hydraulic jump can be assumed to jump directly to the normal depth.

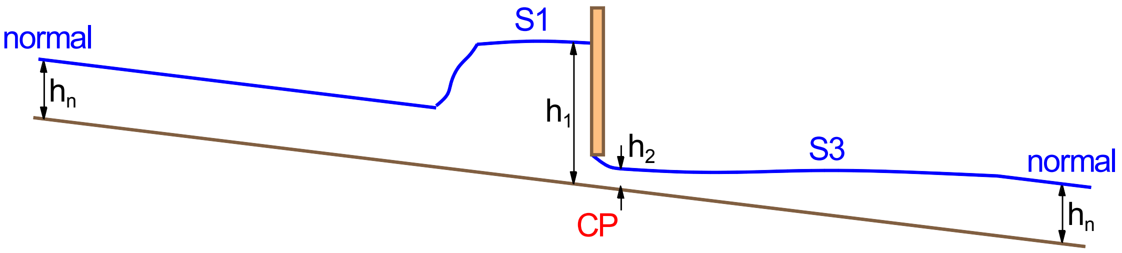
Flow over a weir (mild slope)



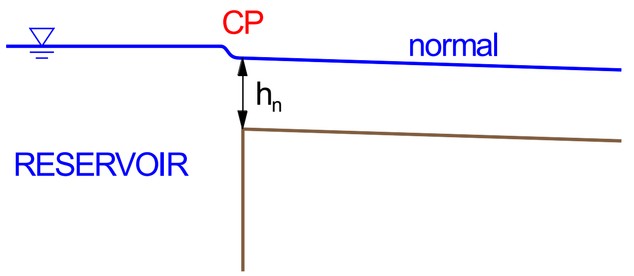
Flow under a sluice gate – (a) mild slope



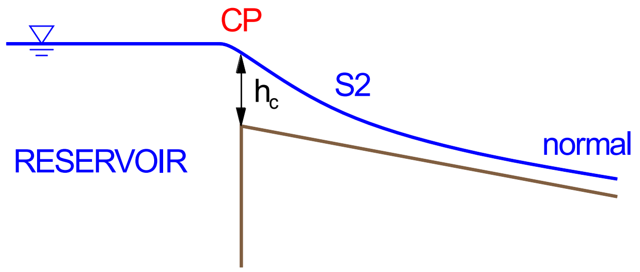
Flow under a sluice gate – (b) steep slope



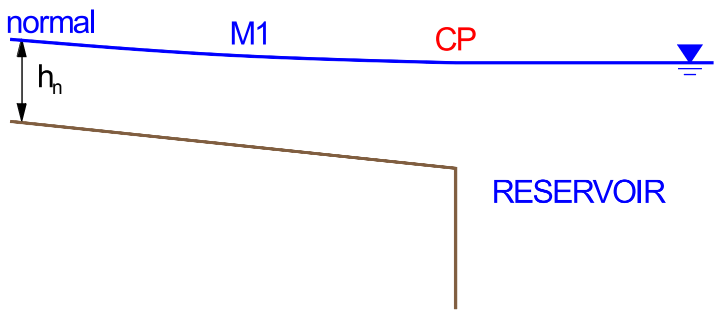
Flow from a reservoir – (a) mild slope



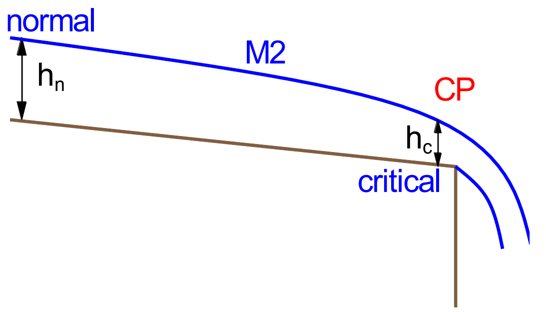
Flow from a reservoir – (b) steep slope



Flow into a reservoir (mild slope)



Free overfall (mild slope)



3.6 Numerical Solution of the GVF Equation

Analytical solutions of the GVF equation are very rare and it is usual to solve it numerically. The process yields a series of discrete pairs of distance x_i and depth h_i. Intermediate points can be determined, if required, by interpolation.

All methods employ a discrete approximation to one of the following forms of GVF equation:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -S_f \qquad \textbf{(total head } H = z_s + \frac{V^2}{2g}\textbf{)} \tag{11}$$

$$\frac{\mathrm{d}E}{\mathrm{d}x} = S_0 - S_f \qquad \textbf{(specific energy } E = h + \frac{V^2}{2g}\textbf{)} \tag{12}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2} \qquad \textbf{(depth } h\textbf{)} \tag{13}$$

In any of these the friction slope can be obtained by inverting Manning's equation:

$$S_f = \frac{n^2 V^2}{R_h^{4/3}} = \frac{n^2 Q^2}{R_h^{4/3} A^2} \tag{14}$$

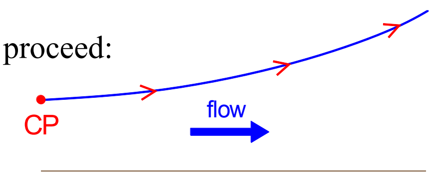
and the Froude number is

$$\mathrm{Fr} = \frac{V}{\sqrt{g\bar{h}}} \tag{15}$$

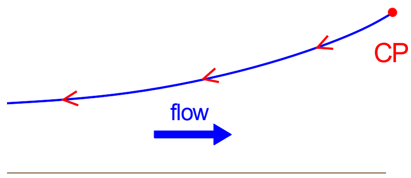
where $\bar{h} = A / b_s$ is the mean depth (= actual depth for a rectangular or wide channel).

Physically, integration should start at a control point and proceed:

- **forward** in x if the flow is **supercritical** (upstream control)

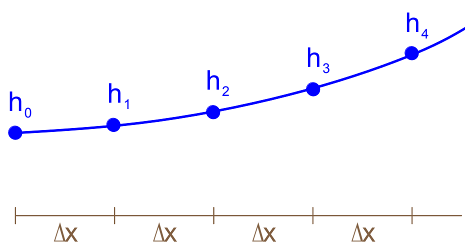


- **backward** in x if the flow is **subcritical** (downstream control).

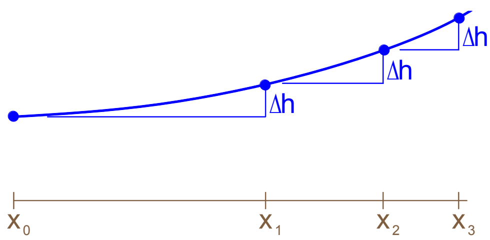


There are two main classes of method:

- *Standard-step* methods: solve for depth h at specified distance intervals Δx.



- *Direct-step* methods: solve for distance x at specified depth intervals Δh. (One advantage is that they can calculate profiles starting from a critical point, where $1 - \mathrm{Fr}^2 = 0$ and standard-step methods would fail).



3.6.1 Total-Head Form of the GVF Equation

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -S_f$$

This is solved as a standard-step method (find depth h at specified distance intervals Δx). The equation is discretised as

$$\frac{H_{i+1} - H_i}{\Delta x} = -(\frac{S_{f,i} + S_{f,i+1}}{2}) \qquad (16)$$

solving sequentially for h_1, h_2, h_3, … starting with the depth at the control point h_0.

Since both H and S_f are functions of h, the method operates by adjusting h_{i+1} iteratively at each step so that the LHS and RHS of (16) are equal.

This is a good method, but since it requires iterative solution at each step it is better suited to a computer program than hand or spreadsheet calculation.

3.6.2 Specific-Energy Form of the GVF Equation

$$\frac{\mathrm{d}E}{\mathrm{d}x} = S_0 - S_f \qquad \text{(where } E = h + \frac{V^2}{2g}\text{)}$$

Since increments in E are determined by successive values of h, this is solved as a direct-step method (find displacement x at specified depth intervals Δh).

First invert to make E the independent variable:

$$\frac{\mathrm{d}x}{\mathrm{d}E} = \frac{1}{S_0 - S_f}$$

The equation is then discretised ($\frac{\Delta x}{\Delta E} \approx \frac{\mathrm{d}x}{\mathrm{d}E}$) and rearranged for distance increments as:

$$\Delta x = \frac{\Delta E}{(S_0 - S_f)_{av}}, \qquad \text{where} \qquad \Delta E = E_{i+1} - E_i \qquad (17)$$

There are various ways of estimating the average slope difference: the example to follow uses the average of values at depths h_i and h_{i+1}.

3.6.3 Depth Form of the GVF Equation

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2}$$

Here we shall solve this by a direct-step method (find displacement x at specified depth intervals Δh).

First, invert to make h the independent variable:

$$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{1 - \mathrm{Fr}^2}{S_0 - S_f}$$

The function on the RHS is first written as a function of h. The equation is then discretised ($\frac{\Delta x}{\Delta h} \approx \frac{\mathrm{d}x}{\mathrm{d}h}$) and rearranged for distance increments as:

$$\Delta x = \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)_{av} \Delta h \qquad (18)$$

As before, the bracketed term on the RHS can be taken as the average of values at the start and end of an interval or (my own preference) by evaluation at the interval mid-depth $h_{mid} = \frac{1}{2}(h_i + h_{i+1})$.

Comment. Different authors adopt different ways of solving the GVF equation numerically, particularly in choosing whether to use the specific-energy or depth form, and how to form the average derivative (e.g., average of values at the ends of the interval or simply the single value at the midpoint). All should give the same answer when the step size Δh becomes very small, but may differ for the larger step sizes typical of hand calculations. The specific-energy form seems to be slightly more common in the literature, but my own tests suggest that the depth form, with derivative evaluated just once at the mid-point of the interval, gives slightly better results for large step sizes. Note that surface profiles become highly curved near critical points and more steps, with a smaller Δh, should be used there.

Example (Examination, January 2007 – modified).

A long rectangular channel of width 4 m has a slope of 1:5000 and a Manning's n of 0.015 $m^{-1/3}$ s. The total discharge is 8 $m^3 s^{-1}$. The channel narrows to a width of 1 m as a venturi flume over a short length.

(a) Determine the normal depth for the 4 m wide channel.

(b) Show that critical conditions occur at the narrow 1 m wide section.

(c) Determine the depth just upstream of the venturi where the width is 4 m.

(d) Determine the distance upstream to where the depth is 5% greater than the normal depth using two steps in the gradually-varied flow equation given below; (you may use either form).

Data
In standard notation,

$$\frac{dh}{dx} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2} \qquad \text{or} \qquad \frac{dE}{dx} = S_0 - S_f$$

Solution.
(a) For the normal depth,

$$Q = VA \qquad \text{where} \qquad V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad R_h = \frac{h}{1 + 2h/b}, \qquad A = bh$$

$$\Rightarrow \qquad Q = \frac{b\sqrt{S}}{n} \frac{h^{5/3}}{(1 + 2h/b)^{2/3}}$$

Rearranging as an iterative formula for h to find teh normal depth at the *channel* slope S_0:

$$h = \left(\frac{nQ}{b\sqrt{S_0}} \right)^{3/5} (1 + 2h/b)^{2/5}$$

Here, with lengths in metres:

$$h = 1.570(1 + 0.5h)^{2/5}$$

Iteration (from, e.g., h = 1.570) gives normal depth:

$$h_n = 2.090 \text{ m}$$

Answer: 2.09 m.

(b) To determine whether critical conditions occur, compare the total head in the approach flow with that assuming critical conditions at the throat.

The total head, assuming normal flow and measuring heights from the bed of the channel is

$$H_a = E_a \quad = h_n + \frac{V_n^2}{2g} \quad = h_n + \frac{Q^2}{2gb^2h_n^2} \quad = 2.137 \text{ m}$$

At the throat the discharge per unit width is

$$q_m = \frac{Q}{b_{\min}} \quad = 8 \text{ m}^2 \text{ s}^{-1}$$

The critical depth and critical specific energy at the throat are

$$h_c = \left(\frac{q_m^2}{g}\right)^{1/3} \quad = 1.869 \text{ m}$$

$$E_c = \frac{3}{2} h_c \quad = 2.804 \text{ m}$$

Since the bed of the flume is flat ($z_b = 0$), the critical head $H_c = E_c$.

Since the approach-flow head H_a is less than the critical head H_c (the minimum head required to pass this flow rate through the venturi, the flow must back up and increase in depth just upstream to supply this minimum head. It will then undergo a subcritical to supercritical transition through the throat. The total head throughout the venturi is $H = H_c = 2.804$ m.

(c) In the vicinity of the venturi the total head is $H = 2.804$ m. Upstream (where width $b = 4$ m), we seek the subcritical solution of

$$H = z_s + \frac{V^2}{2g} \quad = h + \frac{Q^2}{2gb^2h^2}$$

Rearrange for the deeper solution:

$$h = H - \frac{Q^2}{2gb^2h^2}$$

Here, with lengths in metres:

$$h = 2.804 - \frac{0.2039}{h^2}$$

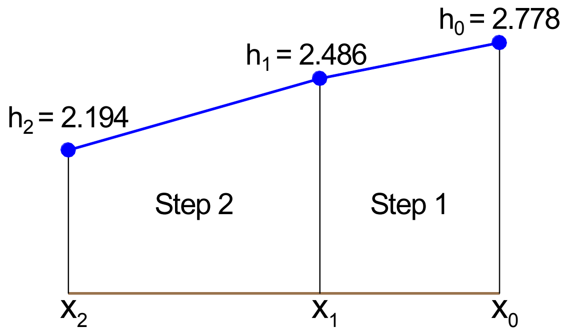
Iterate (from, e.g., $h = 2.804$) to get the depth just upstream of the venturi:

$$h = 2.778 \text{ m}$$

Answer: 2.78 m

(d) Do a GVF calculation (subcritical, so physically it should start at the fixed downstream control and work upstream, although mathematically it can be done the other way) from the pre-venturi depth ($h = 2.778$ m) to where $h =$ 2.194 m (i.e. $1.05 \times h_n$). Using two steps the depth increment per step is

$$\Delta h = \frac{2.194 - 2.778}{2} \quad = -0.292 \text{ m}$$



Both depth and specific-energy methods are shown on the following pages.

METHOD 1: using the depth form of the GVF equation

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - \mathrm{Fr}^2}$$

$$\Rightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}h} = \frac{1 - \mathrm{Fr}^2}{S_0 - S_f}$$

$$\frac{\Delta x}{\Delta h} \approx \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)_{mid}$$

$$\Rightarrow \qquad \Delta x \approx \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)_{mid} \Delta h$$

('*mid*' means mid-point of the interval: half way between h_i and h_{i+1}; sometimes written $h_{i+½}$.)

For convenience, work out numerical expressions for Fr^2 and S_f in terms of h:

$$\mathrm{Fr}^2 = \frac{V^2}{gh} \quad = \frac{Q^2/b^2}{gh^3} \quad = \frac{0.4077}{h^3}$$

Manning's equation (see earlier) gave $Q = \dfrac{b\sqrt{S}}{n} \dfrac{h^{5/3}}{(1 + 2h/b)^{2/3}}$. Assuming that the rate of loss of energy (S_f) at a general depth h is the same as the channel slope that would give normal flow at that depth, rearrangement for the slope gives

$$S_f = \left(\frac{nQ}{b}\right)^2 \frac{(1 + 2h/b)^{4/3}}{h^{10/3}} \quad = 9 \times 10^{-4} \frac{(1 + 0.5h)^{4/3}}{h^{10/3}}$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{1 - \mathrm{Fr}^2}{S_0 - S_f} \quad = \frac{1 - \dfrac{0.4077}{h^3}}{\left(2 - 9 \times \dfrac{(1 + 0.5h)^{4/3}}{h^{10/3}}\right) \times 10^{-4}}$$

With

$$\Delta x = \left(\frac{\mathrm{d}x}{\mathrm{d}h}\right)_{mid} \Delta h \qquad \text{and} \qquad \Delta h = -0.292 \text{ m}$$

working may then be set out in tabular form. (All depths assumed to be in metres.)

i	h_i	x_i	h_{mid}	$(\mathrm{d}x/\mathrm{d}h)_{mid}$	Δx
0	2.778	0			
			2.632	10810	–3157
1	2.486	–3157			
			2.340	18850	–5504
2	2.194	**–8661**			

This gives a distance of about 8.7 km upstream.

METHOD 2: Using the specific-energy form of the GVF equation

$$\frac{\mathrm{d}E}{\mathrm{d}x} = S_0 - S_f$$

$$\Rightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}E} = \frac{1}{S_0 - S_f}$$

$$\Rightarrow \qquad \frac{\Delta x}{\Delta E} \approx \frac{1}{(S_0 - S_f)_{av}}$$

$$\Rightarrow \qquad \Delta x \approx \frac{\Delta E}{(S_0 - S_f)_{av}}$$

('*av*' is taken as the average of values calculated at start and end of each interval in *h*.)

Here:

$$E = h + \frac{V^2}{2g} \quad = h + \frac{Q^2/b^2}{2gh^2} \quad = h + \frac{0.2039}{h^2}$$

and the same expression as before may be used for S_f, so that:

$$S_0 - S_f = 2\times10^{-4} - 9\times10^{-4}\,\frac{(1+0.5h)^{4/3}}{h^{10/3}}$$

With

$$\Delta x = \frac{\Delta E}{(S_0 - S_f)_{av}}$$

working may then be set out in tabular form. (All depths assumed to be in metres.)

i	h_i	x_i	E_i	$S_0 - S_f$	ΔE	$(S_0 - S_f)_{\text{av}}$	Δx
0	2.778	0	2.804	1.046×10^{-4}			
					–0.285	8.883×10^{-5}	–3208
1	2.486	–3208	2.519	7.304×10^{-5}			
					–0.283	4.851×10^{-5}	–5834
2	2.194	**–9042**	2.236				

This gives a distance of about 9.0 km upstream.

Smaller steps Δh will give more 'accurate' results (and closer agreement between the two methods).

Example (Examination, January 2008 – reworded)

A long, wide channel has a slope of 1:2747 with a Manning's *n* of 0.015 $m^{-1/3}$ s. It carries a discharge of 2.5 m^3 s^{-1} per metre width, and there is a free overfall at the downstream end. An undershot sluice is placed a certain distance upstream of the free overfall which determines the nature of the flow between sluice and overfall. The depth just downstream of the sluice is 0.5 m.

(a) Determine the critical depth and normal depth.

(b) Sketch, with explanation, the two possible gradually-varied flows between sluice and overfall.

(c) Calculate the particular distance between sluice and overfall which determines the boundary between these two flows. Use one step in the gradually-varied-flow equation.