FLUID MECHANICS AND FLUID POWER ENGINEERING

(SI UNITS)

Dr. D.S. KUMAR

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(IN S I UNITS)

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S.K. KATARIA & SONS®
PUBLISHERS & DISTRIBUTORS
4424/6, Guru Nanak Market, Nai Sarak, Delhi-110 006
Ph. : 3937396,
Fax : 011-2412755

Published by :
Sanjeev Kumar Kataria
For S.K. Kataria & Sons®
6, Guru Nanak Market
Nai Sarak, Delhi-6
Ph. : 3937396,
Fax : 011-2412755
Branch Office :
Opp. Clock Tower,
Ludhiana (Pb.)
Ph. : 726401

IMPORTANT

Author and Publishers would welcome suggestions for the improvement of the book or pointing out of any misprint

First Edition : Jan 1987
Sixth Edition : Jan 1998
(Thoroughly Revised & Enlarged)
REPRINT 2000-2001

Price Rs. 250/-

Computer Typesetting By :
Rahul Computers, Delhi-110051

Printed at:
Arora Offset Press
Laxmi Nagar, Delhi 110 092.

"Dedicated to the hallowed memory of

MY REVERED MOTHER

who always impressed upon me the values of hardwork,

honesy & discipline, and these aspects continue to

guide and animate me"

PREFACE TO THE SIXTH EDITION

The author is grateful to the teachers and students of various engineering institutions for their appreciation and healthy criticism of the earlier editions of the book. The details and general scheme of the text remains unchanged in this edition too ; the text provides a balanced treatment of the basic principles of fluid mechanics and their applicability in engineering. While preparing the material for this thoroughly revised and substantially expanded version of the text, a sincere effort has been made to incorporate the suggestions received from the readers ; both teachers and students.

Considerable changes have been made in the text to clear the points that were not apparent or needed additional information for better understanding. Some chapters have been completely overhauled and rewritten with the inclusion of additional articles. These additions have been planned to serve the present need and also the possible future enrichment of the syllabi of the various technological institutes and universities in the country. A wide variety of challenging and stimulating problems set by various examining boards have been selected and solved in the text at appropriate places, and SI units have been consistently used throughout the text. Section on Multiple Choice Questions has been made very exhaustive keeping in view the recent trend of such questions being asked in the various competitive examinations. This will help a student to assess his ability and understanding. Illustrative drawings have been replaced by new ones to improve their presentation and clarity. An effort has been made to rectify the mistakes and misprints appearing in the earlier editions.

The author hopes that this thoroughly revised and updated edition will prove more useful to the students and practicing engineers dealing with fluid mechanics and fluid flow machines. Any suggestions for further updating the book will be highly appreciated and gratefully acknowledged.

I wish to avail myself of this opportunity to thank the entire staff of M/s S.K. Kataria & Sons and especially Mr. Sanjeev Kataria for his co-operation in bringing out this book with a good get-up, nice printing and at reasonable price. The illustrative diagrams of the book have been prepared by Mr. P.K. Mehta and he is to be thanked for the good art work done by him. Much more than thanks is due to my wife, Mrs. Manjit Kaur, for her encouragement and continuous support in this venture.

D.S. Kumar

January, 1998

PREFACE TO THE FIRST EDITION

Basic concepts of fluids and fluid flow are essential in all the engineering disciplines to get better understanding of the courses in the professional programmes, and obviously its importance as a core subject need not be overemphasised. Earlier the subject was tought under the heading hydraulics which is essentially an empirical science that puts little emphasis on the basic fundamentals of fluid flow. Hydraulics is now a defunct subject, and has been replaced by the teaching of fluid mechanics where a balanced treatement is sought to the fundamental and applied aspects of fluid flow.

The author has been teaching the subject of fluid mechanics for the past several years, and this monograph is essentially based on the lectures delivered by him. The lecture notes were prepared because the author felt that there was no single text which could provide a coherent, readily intelligible account and concise exposition of the subject. From the experience gained through useful class discussions and feed back, the notes were revised to improve the clarity and necessary explanatory notes were added during each teaching semester. The subject matter has thus been thoroughly tested in the class room and found suitable.

The book is frankly a compilation and no claim is made of its originality. Acknowledgement is due and is hereby made to all the authors whose work has been freely consulted in the preparation of this text. The sources of information are duly recorded as bibliography to the subject. The author has, however, adopted his own arrangement of the subject and method of presentation that provides a balanced treatment of the subject in a simple, lucid and easily understandable language without sacrificing emphasis on the fundamental aspects of the science of fluid flow. Full use of line diagrams has been made to supplement the text and to explain a particular phenomenon as clearly as possible.

SI units and notations are now being widely adopted and are expected to be universally used in the not very distant future. This aspect has been kept in mind while preparing this text. The text incorporates a number of worked out examples which are of a standard comparable to those set for engineering degree, AMIE and Engineering Services Examinations, and should thus be an invaluable aid to students sitting for these examinations. The author is indebted to universities and engineering institutes whose examination papers have been included in the text by way of illustrations; the author owes the responsibility for their solutions.

The author hopes and believes that this treatise on "Fluid Mechanics and Fluid Power Engineering" will contribute effectively to the existing literature in this field and will be a useful text for the students of engineering curriculum, AMIE and other professional examinations in the subject of fluid mechancis, hydraulics and fluid (hydraulic) machines. In addition, the text would serve as a good referance for practicing engineers and associated workers in the solution of many problems in the application of fluid mechanics and fluid flow.

The author expresses his gratitude to his departmental colleagues with whom he had many hours of useful discussion during the writing and editing of the text. The authoer thanks the publishers also for their considerable patience and good co-operation throughout. Further, the author would be extremely thankful to the readers for their constructive suggestions and healthy criticism with a view to enhance the usefulness of the book. Author and the publishers would gratefully acknowledge if misprints and errors discovered are brought to their notice.

Finally, the author wishes to place on record his apologies and sincere thanks to his wife Mrs. Manjit Kaur and children, Sandeep, Mandeep and Navdeep, who willingly endured certain hardships which resulted from his pre-occupation with this work.

January, 1987

1
Introduction

1.1. INTRODUCTION

Fluid Mechanics is basically a study of the :

(i) physical behaviour of fluids and fluid systems, and of the laws governing this behaviour,

(ii) action of forces on fluids and of the resulting flow pattern.

Fluid mechanics may be divided into three divisions :

1. *Hydro-statics* that studies the mechanics of fluids at absolute and relative rest; the fluid elements are free from shearing stresses.

2. *Kinematics* that deals with translation, rotation and deformation of fluid elements without considering the force and energy causing such a motion.

3. *Dynamics* that prescribes the relations between velocities and accelerations and the forces which are exerted by or upon the moving fluids.

Analysis of fluid flow problems is generally made by considering certain fundamental principles, concepts and laws such as the principles of conservation of mass, momentum and energy; the first and second laws of thermodynamics; equation state relating to fluid properties; Newton's law of viscosity and the restrictions caused by the presence of boundaries.

In this treatise on ''Fluid Mechanics and Fluid Power Engineering'' it is intended to present the fundamentals of fluid mechanics along with relevant portions of experimental hydraulics and hydraulic machines.

1.2. SOLIDS, LIQUIDS AND GASES

Matter exists in two principal forms : solid and fluid. Fluid is further sub-divided into liquid and gas. Distinguishing features amongst these are :

(i) Spacing and the latitude of the motion of molecules is large in a gas, small in a liquid and extremely small in a solid. Accordingly the intermolecular bonds are very weak in a gas, weak in a liquid and very strong in a solid. It is due to these aspects that solid is very compact and rigid in form, liquid accomodates itself to the shape of its container, and gas fills up the whole of the vessel containing it.

(ii) For a given mass, the liquids have a definite volume irrespective of the size of the container. The variation of volume with temperature and pressure is insignificant. Liquid occupies the vessel fully or partially depending on its mass, and that it forms a free surface with the atmosphere. The gas, however, expands to fill any vessel in which it is contained and does not form any free surface. Accordingly, it may be stated : ''A solid has volume and shape; a liquid has volume but no shape; a gas has neither''.

(iii) For all practical purposes, the liquids like solids can be regarded as incompressible. This means that pressure and temperature changes have practically no effect on their volume. The gases are, however,

readily compressible fluids. They expend infinitely in the absence of pressure and contract easily under pressure. Never-the-less when density variation is small, for example in flow of air in a ventilating system, the gas flow can also be treated as incompressible without involving any appreciable error.

When a gas can be readily condensed to a liquid, we call it a *vapour* such as steam and ammonia.

(iv) The deformation due to normal and tangential forces for solids is such that within elastic limits, the deformation disappears and the solid body is restored to its original shape when the stress causing the deformation is removed. A fluid at rest can, however, sustain only normal stresses and deforms continuously when subjected to a shear stress; no matter how small that shear stress may be. Even though the fluid comes to rest when the shear stress is removed, yet there is no tendency to restore the fluid body to its original shape or position.

Thus a fluid can offer no permanent resistance to shear force and possesses a characteristic ability to flow or change its shape. Flow means that the constituent fluid particles continuously change their positions relative to one another. This concept of fluid flow under the application of a shear stress is illustrated in Fig. 1.1. A fluid element occupying the initial position 011 continues to move or deform to new positions 022, 033 etc. when a shear stress τ is applied to it.

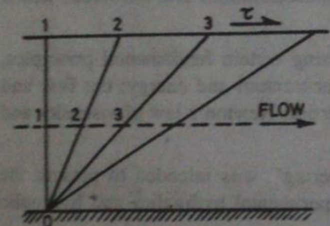


Fig. 1.1. *Fluid flow*

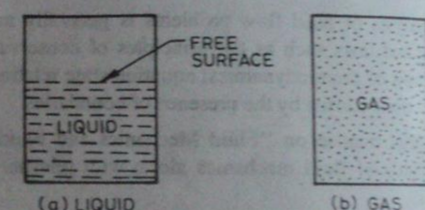


Fig. 1.2 . Behaviour of a fluid in a container

The tendency of continuous deformation of a fluid is called *fluidity*, and the act of continuous deformation is called *flow*.

The above discussion can be summed up as :

Liquid	Gas
(i) A given mass of liquid has a definite volume independent of the size or shape of the container; however, it changes its shape easily and acquires the shape of its container.	*(i)* A given mass of gas has no fixed volume; and it expands continuously to adjust completely fill any container in which it is placed.
(ii) A free surface is formed if the volume of the container is greater than that of the liquid.	*(ii)* No free surface is formed.
(iii) Liquids can be regarded as incompressible for all practical purposes.	*(iii)* Gases are readily compressible.

(iv) Pressure and temperature changes have practically no effect on the volume of a liquid.	(iv) A gas expands infinitely in the absence of pressure and contracts easily under pressure.
(v) Water, kerosene, petrol etc. are liquids.	(v) Air, ammonia, carbon dioxide etc. are gases.

1.3. IDEAL AND REAL FLUIDS

A fluid is said to be ideal if it is assumed to be both incompressible and inviscid (non-viscous). Further an ideal fluid has no surface tension. An inviscid fluid is for which viscosity is zero and no frictional forces are set up even during fluid motion.

$$\mu = 0$$

$$\rho = \text{constant};\ K = \frac{-dp}{dv/v} = \infty$$

$$\sigma = 0$$

Ideal fluids are imaginary and do not exist in nature. However most common fluids such as air and water have very low value of viscosity and can be treated as ideal fluids for all practical purposes without introducing any appreciable error. Since water is incompressible, it is move near to an ideal fluid than air.

Real or practical fluids have viscosity (μ), compressibility (K) and surface tension (σ). Whenever motion takes place, the tangential or shear forces always come into play due to viscosity and some frictional work is done.

1.4. CONTINUUM

Although fluids consist of discrete molecules, analysis of the fluid flow problems is made by a concept that treats fluid as continuous media. All voids or cavities, microscopic or macroscopic, which may occur in the fluids are ignored. The physical properties of the fluids are then continuous from point to point and can be expressed by continuous algebraic functions of space and time co-ordinates. In other words, the fluid properties are treated to be same at a point and identical in all directions from a specified point. Molecular size and the movement of fluid elements about their mean path are generally insignificant with reference to the dimensions of the equipment and as such this assumption of homogeneity and isotropy is quite justifiable except in the field of aerodynamics and rarefield gas dynamics. A continuous and homogenous fluid medium is called *continuum*. From the continuum view point, the overall properties and the behaviour of fluids can be studied without regard for its atomic and molecular structure.

1.5. DEVELOPMENT OF FLUID MECHANICS

The beginning and development of the science of fluid mechanics dates back to the times when the ancient races had their irrigation systems, the Greeks their hydraulic mysteries, the Romans their methods of water supply and disposal, the Middles ages their wind mills and water wheels. It is quite evident from the excavations of Egyptian ruins and Indus Valley Civilization that the concepts of fluid flow and flow resistance, which form the basis of irrigation, drainage and navigation systems, were known to the man who lived at that time about 4000 years ago.

Through their sustained and continued efforts, a host of research workers contributed so extensively to the subject that by the end of 19th century all the essential tools of hydraulics were at hand; the principles of continuity, momentum and energy; the Bernoulli theorem; resistance formulae for pipes and open channels; manometers, Pitot tubes and current meters; towing tanks, wind tunnels and whirling arms; model techniques; and Froude and Reynolds laws of similarity; and the equations of motion of Euler, Navier-Stokes and Reynolds.

Historically the development of fluid mechanics has been influenced by two bodies of scientific knowledge : empirical hydraulics and classical hydrodynamics. *Hydraulics* is an applied science that deals with practical problems of flow of water and is essentially based on empirical formulae deduced from laboratory experiments. *Classical Hydrodynamics* has been hydraulics' mathematical counter part that deals with a hypothetical ideal fluid having no viscosity. However, neither hydraulics not classical hydrodynamics could provide a scientific support to the rapidly developing field of aeronautics; the former because of its strong empirical slant with little regard for reason, and the latter because of its very limited contact with reality. The solution to the dilemma was provided by Ludwieg Prandtl in 1904 who proposed that flow around immersed bodies be approximated by a boundary zone of viscous influence and a surrounding zone of irrotational frictionless motion. This approach has a tremendous effect upon understanding of the motion of real fluids and eventually permitted analysis of lifting vanes, control surfaces and propellers.

The subject of fluid mechanics has grown into a major field in engineering science and its basic principles are embodied in essentially every field involving fluid motion; the subject branches out into various specialities such as hydraulic and marine engineering; aero and gas dynamics, meterology and oceanography; plasma physics and geophysics etc.

1.6. SIGNIFICANCE OF FLUID MECHANICS

The subject of fluid mechanics encompasses a great many fascinating areas like :

- design of a wide range of hydraulic structures (dams, canals, weirs etc.) and machinery (pumps, turbines and fluid couplings)
- design of a complex network of pumping and pipelines for transporting liquids; flow of water through pipes and its distribution to domestic service lines.
- fluidic control devices; both pneumatic and hydraulic
- design and analysis of gas turbines, rocket engines, conventional and supersonic aircrafts
- power generation from conventional methods such as hydroelectric, steam and gas turbines, to newer ones involving magneto fluid dynamics.
- methods and devices or the measurement of various parameters, *e.g.*, the pressure and velocity of a fluid at rest or in motion
- study of man's environment in the subjects like meterology, oceanography and geology
- human circulatory system, *i.e.*, flow of blood in veins and the pumping action of heart.

Fig. 1.3 briefly explains the significance of fluid mechanics and the vital role it plays in a variety of engineering applications.

The significance of fluid mechanics can be well judged by citing just one example of automobile drive where suspension is provided by pneumatic tyres, road shocks are reduced by hydraulic shock absorbers, gasoline is pumped through tubes and later atomized, air resistance creates a drag on the vehicle as whole and the confidence that hydraulic brakes would operate when the vehicle is made to stop.

Undoubtedly, a study of the science of fluid mechanics is a must for an engineer so that he can understand the basic principles of fluid behaviour and apply the same to flow situations encountered in engineering and physical problems.

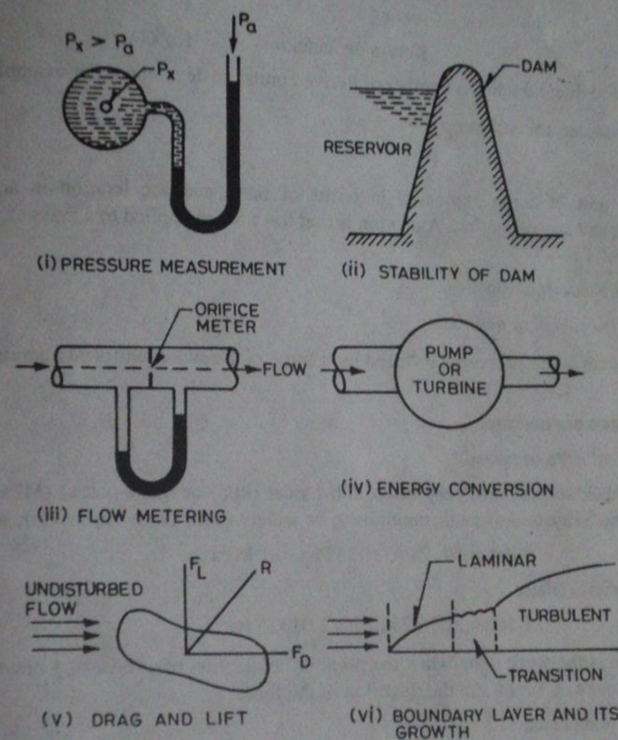


Fig. 1.3. *Fluid mechanics applications*

1.7. UNITS AND DIMENSIONS

A *dimension* is a name which describes the measurable qualities or characteristics of an object such as mass, length, time and temperature etc. A *unit* is an accepted standard for measuring the dimension or quality. For elaborating the difference between unit and dimension, let us consider the distance between two points. The term length when applied to it gives the qualitative concept of this physical quantity. The term unit would, however, indicate the magnitude of the distance. The distance may be quantitatively expressed as a metre or mile. The metre and mile will then be identical dimensions. However, they will be different units as each contains a different amount or quantity of length.

Physical dimensions used in fluid mechanics are expressed in four fundamental dimensions namely mass, length, time and temperature. In the present text, the SI (System International) system of units has been used and the basic units in this system are :

Mass [M]	kilogram	kg
Length [L]	metre	m
Time [T]	second	s
Temperature [θ]	Kelvin or °celcius	K, °C

Based on these fundamental units, a number of derived units are developed. For example :

- **density** = mass per unit volume

  $\equiv kg/m^3$

- **newton** : a unit of force expressed in terms of mass and acceleration in accordance with Newton's second law of motion. A newton is that force when applied to a mass of 1 kg gives it an acceleration of 1 m/s^2

  Force = mass × acceleration

  $\equiv kg \times m/s^2 = N$ or newton

- **pascal** : a pascal is the pressure produced by a force of 1 newton uniformly appiied over an area of 1 m^2

  Pressure = force per unit area

  $\equiv N/m^2 = Pa$ or pascal

The unit of pascal is very small. Very often kilo-pascal (kPa) or mega-pascal (MPa) is used. Two other units, not within the SI system of units, continue to be widely used. These are the *bar*, where

$$1 \text{ bar} = 10^5 \text{ Pa} = 100 \text{ kPa} = 0.1 \text{ MPa}$$

and the standard atmosphere, where

$$1 \text{ atm} = 101.325 \text{ kPa} = 1.01325 \text{ bar}$$

- **joule** : a joule is the work done when the point of application of a force of 1 newton is displaced through a distance of 1 metre in the direction of the force

  Work = force × distance

  ≡ newton × metre = Nm = J or joule

- **watt** : a watt represents a work equivalent of 1 joule done per second

  Power = work done per unit time

  ≡ J/s = W or watt

The commonly used derived terms and their units are listed in Table 1.1.

Table 1.1. Commonly used derived terms

Derived term	Dimension	SI Unit
Area	$[L^2]$	m^2
Volume	$[L^3]$	m^3
Velocity	$[LT^{-1}]$	m/s
Acceleration	$[LT^{-2}]$	m/s^2
Force	$[MLT^{-2}]$	N

Pressure (stress)	$[ML^{-1}T^{-2}]$	N/m^2 Pascal(Pa) = N/m^2
Energy (work)	$[ML^2T^{-2}]$	Nm Joule (J) = Nm
Power	$[ML^2T^{-3}]$	J/s Watt (W) = J/s
Density	$[ML^{-3}]$	kg/m^3
Viscosity	$[ML^{-1}T^{-1}]$	kg/ms (= N s/m^2)
Surface tension	$[MT^{-1}]$	N/m

To facilitate working with magnitudes which may be multiples or submultiples of SI units, we use a prefix before the unit to suit a particular situation. The commonly used prefixes are listed in Table 1.2.

Table 1.2. SI System : Standard Multipliers

Factor	Prefix	Factor	Prefix
10^{12}	tera, T	10^{-3}	milli, m
10^{9}	giga, G	10^{-6}	micro, µ
10^{6}	mega, M	10^{-9}	nano, n
10^{3}	kilo, k	10^{-12}	pico, p
10^{-1}	deci, d	10^{-15}	femto, f
10^{-2}	centi, c	10^{-18}	atto, a

The use of tera, mega, kilo, milli, micro and pico is preferred.

The SI system of units is comprehensive, coherent, rational, brief and the easiest for the numerical and practical purposes, when used all by themselves.

Some of the additional comments on SI units are :

— While writing plurals, do not add s to the symbols. Thus write kg and not kgs.

— Units with name of scientists are not to be written with capital letters. Thus write newton and not Newton as unit of force.

— There are to be no dots and no dashes. For example work is in Nm, not N.m, not N-m etc.

— After every three digits, there is to be a gap but no comma, *e.g.*, 2 000 and not 2,000

— For practical purposes, use of units like tonne, degree celcius, litre, speed in km/hr is permitted, but should be restricted as far as possible.

REVIEW QUESTIONS

1. Define a fluid and distinguish between :
   (i) ideal and real fluids
   (ii) compressible and incompressible fluids.
2. How does :
   (i) a fluid differ from a solid.
   (ii) a liquid differ from a gas.
3. Discuss the concept that "a fluid continues to deform so long as it is acted upon by a shear force".

4. Define the term continuum as applied to the mechanism of flow. Is the continuum model valid in upper atmosphere ?
5. Write a note on the historical development of fluid mechanics.
6. Cite some examples to illustrate the importance of fluid mechanics in the engineering field.
7. Discuss the application of fluid mechanics to the flow of blood in our veins and to the pumping action of heart.
8. Explain the difference between coherent and non-coherent system of units, and comment upon the advantages of SI system of units.
9. Indicate if the following statements are true or false. If false, then write the correct statement :
   (i) A fluid always expands to fill its container.
   (ii) Only normal forces exist in a fluid at rest.
   (iii) Fluids are capable of flowing freely.
   (iv) When subjected to shear force, a fluid deforms continuously no matter how small the shear stress may be.
   (v) An ideal fluid is both frictionless and incompressible.
   (vi) An ideal fluid can sustain tangential forces when set in motion.
   (vii) Vapour refers to a gaseous fluid which can be readily condensed into a liquid.
   (viii) A fluid exhibits equal shear stresses at all points when in motion.
   (ix) Continuum is synonymous with uniformity and homogeneity of fluid medium.
   (x) The concept of continuum in a fluid flow assumes that movement of fluid elements about their mean path is quite significant.
   (xi) Fluid mechanics is a blend of hydraulics and hydrodynamics, and offers rational solution to practical problems.

(**Ans** *(i)* F, *(ii)* T, *(iii)* T, *(iv)*, T, *(v)* T, *(vi)* F, *(vii)* T, *(viii)* F, *(ix)* T, *(x)* F, *(xi)* T)

□□

2

Physical Properties of Fluids

2.1. INTRODUCTION

Every fluid has certain characteristics by means of which its physical condition may be described. Such characteristics are called properties of the fluid. Before an analyst of fluid flow problems can venture to formulate the physical principles governing the flow situation, he has to be thoroughly familiar with the physical properties of fluids. Towards that end, this chapter seeks to provide basic insight into the fluid properties and their behaviour.

2.2. SYSTEM, EXTENSIVE AND INTENSIVE PROPERTIES

System : a prescribed and identifiable quantity of matter, *i.e.*, fluid whose characteristics are under investigation.

Property : any characteristic of a system that can be used to define its state.

Intensive property : a property whose magnitude is independent of the amount of matter like pressure, temperature, mass density etc.

Extensive property : a property whose magnitude is related to the total mass of the system like mass, weight, volume etc.

2.3. SPECIFIC WEIGHT, MASS DENSITY AND SPECIFIC GRAVITY

(a) Specific weight (w) of a fluid is its weight per unit volume.

$$w = \frac{W}{V} \qquad ...(2.1)$$

where W is the weight of the fluid having volume V. The weight of a body is the force with which the body is attracted to the centre of the earth. It is the product of its mass and the local gravitational acceleration, *i.e.*, $W = mg$. The value of g at sea level is 9.807 m/s^2 approximately. Since weight is expressed in newton, the unit of measurement of specific weight is N/m^3. In terms of fundamental units, the dimensional formula of specific weight is $\left[\frac{F}{L^3}\right]$ or $\left[\frac{M}{L^2 T^2}\right]$

For pure water under standard atmospheric pressure of 760 mm of mercury at mean sea level and a temperature of 4°C, the specific weight is 9810 N/m^3. For sea water, the specific weight equals 10000 – 10105 N/m^3. The increased value of specific weight of water is due to the presence of dissolved salts and suspended matter. The specific weight of petroleum and petroleum products varies from 6350 – 8350 N/m^3 and that of mercury at 0°C is 13420 N/m^3. Air has a specific weight of 11.9 N/m^3 at 15°C temperature and at standard atmospheric pressure. The specific weight of a fluid changes from one place to another depending upon changes in the gravitational acceleration.

(b) Density (ρ-pronounced rho) is a measure of the amount of fluid contained in a given volume and is

defined as the mass per unit volume.

$$\rho = \frac{m}{V} \quad ...(2.2)$$

where m is the mass of fluid having volume V. Fluid mass is a measure of the ability of a fluid particle to resist acceleration and is approximately independent of its location on the earth's surface. The units of density correspond to those of mass and volume. The dimensional formula of density in fundamental units is $\left[\frac{M}{L^3}\right]$ or $\left[\frac{FT^2}{L^4}\right]$ and the corresponding units are kg/m^3 or N s^2/m^4.

The density of a fluid diminishes with rise of temperature except for water which has a maximum value at 4°C. The mass density of water at 15.5°C is 1000 kg/m^3, and for air at 20°C and at atmospheric pressure the mass density is 1.24 kg/m^3.

Relations 2.1 and 2.2 are valid only when the fluid medium fills the given volume completely without any blank space, *i.e.*, the fluid is a continuum. For a non-homogenous fluid, these relations give average specific weight and density. To determine the absolute values of w and ρ at any point, the volume is regarded as tending to zero and the limit of the corresponding ratio is calculated.

$$w = \underset{V \to 0}{\text{Lt}} \frac{W}{V} = \frac{dW}{dV} \quad ...(2.3)$$

$$\rho = \underset{V \to 0}{\text{Lt}} \frac{m}{V} = \frac{dm}{dV} \quad ...(2.4)$$

The weight W and the mass m of a fluid are related to each other by the expression $W = mg$. Dividing this expression throughout by volume V of the fluid, we obtain :

$$\frac{W}{V} = \frac{m}{V} g \text{ or } w = \rho g \quad ...(2.5)$$

Equation 2.5 reveals that specific weight w changes with location depending upon gravitational pull.

(c) Specific gravity (s) refers to the ratio of specific weight (or mass density) of a fluid to the specific weight (or mass density) of a standard fluid. For liquids the standards fluid is water at 4°C, and for gases the standard fluid is taken either air at 0°C or hydrogen at the same temperature. Specific gravity is dimensionless and has no units.

A statement that the specific gravity of mercury is 13.6 implies that its weight (or mass) in 13.6 times that of same volume of water. In other words, mercury is 13.6 times heavier than water.

(d) Specific volume (v) represents the volume per unit mass of fluid; specific volume is the inverse of the mass density.

$$v = \frac{V}{m}; \; v = \frac{1}{\rho} \quad ...(2.7)$$

The concept of specific volume is found to be practically more useful in the study of flow of compressible fluids, *i.e.*, gases.

Example 2.1. 2 litre of petrol weighs 14 N. Calculate the specific weight, mass density, specific volume and specific gravity of petrol with respect to water.

Solution : 2 litre = 2×10^{-3} m^3

Specific weight is a measure of the weight per unit volume :

$\therefore$ Specific weight $w = \dfrac{14}{2 \times 10^{-3}} = \mathbf{7000\ N/m^3}$

Mass density is related to specific volume by the relation, $w = \rho g$

Mass density $\rho = \dfrac{w}{g} = \dfrac{7000}{9.81} = \mathbf{713.56\ kg/m^3}$

Specific volume v is the inverse of mass density

$$v = \frac{1}{\rho} = \frac{1}{713.56} = \mathbf{1.4 \times 10^{-3}\ m^3/kg}$$

$$\text{Specific gravity } s = \frac{\text{density of oil}}{\text{density of water}} = \frac{713.56}{1000} \approx \mathbf{0.7136}$$

Example 2.2. If specific gravity of a liquid is 0.80, make calculations for its mass density, specific volume and specific weight (weight density).

Solution : Specific gravity $= \dfrac{\text{mass density of liquid}}{\text{mass density of water}}$

$\therefore$ mass density of liquid $\rho = 0.80 \times 1000 = \mathbf{800\ kg/m^3}$

specific volume $v = \dfrac{1}{\rho} = \dfrac{1}{800} = \mathbf{1.25 \times 10^{-3}\ m^3/kg}$

specific weight (weight density) $w = \rho g = 800 \times 9.81 = \mathbf{7848\ N/m^3}$

2.4. GAS LAWS : THERMODYNAMIC RELATIONS

Density and specific weight are essentially a measure of the molecules contained in a unit volume of the substance. With rise in temperature the spacing between the molecules increases and consequently less number of molecules are contained in the same volume. Consequently the density and specific weight diminish with temperature growth. With pressure rise, more molecules are contained in the same volume and that leads to an increase in these fluid properties. Hence density and specific weight are closely related to thermodynamic properties of pressure and temperature, and these must be quoted in precise calculations of density and specific weight. One of the most important thermodynamic relation is the characteristic equation by which pressure p, volume V occupied by mass m and absolute temperature T of a perfect gas are related by the expression :

$$pV = mRT \quad ; \qquad pv = RT \ ; \qquad \frac{p}{\rho} = RT \qquad \text{...(2.8)}$$

where R is a characteristic gas constant. The units of R as defined by equation 2.8 are :

$$R \equiv \left[\frac{F}{L^2}\right] \div \left[\frac{M}{L^3}\theta\right]$$

$$\equiv \left[\frac{FL}{M\theta}\right] = \frac{\text{Nm}}{\text{kg K}} \text{ or } \frac{\text{J}}{\text{kg K}}$$

The gas constant R for a particular gas is determined from the so called universal gas constant G that is equivalent to :

$$MR = 8314 \text{ Nm/kg mol K} = 8314 \text{ J/kg mol K}$$

where M is the molecular weight of the gas. The value of gas constant R for air may be taken as 287 Nm/kg K.

A change in the density can be brought about by a change in temperature or by a change in pressure. A change in the state of the fluid system at constant pressure constitutes an **isobaric or constant pressure process;** the changing parameters are volume for given mass of gas and temperature and these are related to each other by Charles law :

$$\frac{V}{T} = \text{constant} \; ; \quad \frac{v}{T} = \frac{1}{\rho T} = \text{constant} \qquad ...(2.9)$$

A change in the state of the fluid system at constant temperature constitutes **isothermal process** which is characterised by Boyle's law :

$$pV = \text{constant} \; ; \quad pv = \frac{p}{\rho} = \text{constant} \qquad ...(2.10)$$

When no heat is given to or taken from the fluid system during its change from one state to another, the process is called **adiabatic** and is described by the relation,

$$pV^{\gamma} = \text{constant} \; ; \quad pv^{\gamma} = \frac{p}{\rho^{\gamma}} = \text{constant} \qquad ...(2.11)$$

The exponent γ in the above expression depends upon the molecular structure of the gas; and is the ratio of two specific heats of the gas—that at constant pressure c_p to that at constant volume c_v : $(\gamma = c_p/c_v)$.

Note : For further details regarding expansion or compression of gases, the reader may please refer to Chapter on Compressible Flow.

Example 2.3. A fan delivers 4 m^3 of air per second at 20°C and 1.25 bar. Assuming molecular weight of air as 28.97, calculate the mass of air delivered. Also determine the density, specific volume and specific weight of the air being delivered.

Solution : Universal gas constant $G = 8314$ Nm/kg mol K

$\therefore$ gas constant R for air $= \dfrac{8314}{28.97} = 287$ Nm/kg K

absolute temperature $T = (273 + 20) = 293$ K

pressure $= 1.25$ bar $= 1.25 \times 10^5$ N/m^2, and volume $V = 4$ m^3

The mass of the air being delivered can be determined by using the characteristic gas equation : $pV = mRT$

$$\therefore \text{ mass of air, } m = \frac{pV}{RT} = \frac{1.25 \times 10^5 \times 4}{287 \times 293} = \mathbf{5.94\ kg}$$

density = mass per unit volume

$$\rho = \frac{5.94}{4} = \mathbf{1.485\ kg/m^3}$$

Specific volume is the inverse of density

$$\therefore \text{ specific volume } v = \frac{1}{\rho} = \frac{1}{1.485} = \mathbf{0.673\ m^3/kg}$$

The density ρ and specific weight w are related to each other by the expression $w = \rho g$

$\therefore$ specific weight $w = 1.485 \times 9.81 =$ **14.57 N/m^3**

2.5. COMPRESSIBILITY AND BULK MODULUS

Fluid mechanics deals with both incompressible and compressible fluids. *i.e.*, with fluids of either constant or variable density. When pressure is applied to a fluid, it contracts and when pressure is released it expands. Compressibility of a fluid then characterises its ability to change its volume under pressure. The relative change of volume per unit pressure is given by the *coefficient of compressibility* :

$$\beta_c = \frac{-1}{V}\left(\frac{dV}{dp}\right) \quad ...(2.12)$$

where dp is the small change in pressure applied to the fluid and dV is the incremental volume change in the original volume V. The negative sign implies that a positive pressure increment results in a negative volume increment, *i.e.*, an increase in pressure causes a decrease in volume.

Quite often, the compressibility of fluid is expressed by its *bulk modulus of elasticity K* which is the inverse of the coefficient of compressibility :

$$K = \frac{1}{\beta_c} = -\frac{dp}{dV/V} \quad ...(2.13)$$

The bulk modulus of elasticity measures the compressive stress per unit volumetric strain. Further, recalling that for a given mass m of the fluid;

$$m = \rho V$$

or $$dm = \rho\, dV + V\, d\rho$$

since mass is constant ; $\frac{d\rho}{\rho} = -\frac{dV}{V}$; hence equation 2.13 can be written in the alternate form :

$$K = \frac{dp}{d\rho/\rho} \quad ...(2.14)$$

where $d\rho/\rho$ represents the relative change in density of the fluid.

Further simplification of the compressibility relation (Eqns. 2.13 and 2.14) is possible in view of the process of compression :

(*i*) Isothermal process which is characterised by :

$$pV = \text{constant}$$

$$p\, dV + V\, dp = 0$$

or $$p = -\frac{dp}{dV/V} \quad ; \quad K = p \quad ...(2.15)$$

Thus for an isothermal process, the bulk modulus equals the pressure.

(*ii*) Adiabatic process which is defined by :

$$pV^{\gamma} = \text{constant}$$

or $$p\gamma V^{\gamma-1}\, dV + V^{\gamma}\, dp = 0$$

or $$\gamma p = -\frac{dp}{dV/V} \quad ; \quad K = \gamma p \quad ...(2.16)$$

i.e., for an adiabatic process, K equals γ times the pressure.

Consider volume V of a certain fluid contained in a piston cylinder arrangement (Fig. 2.1) and let the piston of cross- sectional area A move downwards when a force F is applied to it. The fluid is then compressed to volume V_1 and its pressure increases. The variation of volume ratio V_1/V with pressure $p = F/A$ would be as depicted in Fig. 2.1. The slope of this curve at any point gives the bulk modulus of

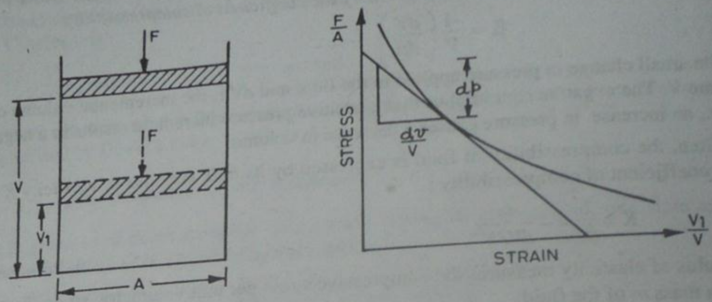


Fig. 2.1. *Compressibility curve*

elasticity of the fluid. An examination of the nature of the curve would reveal that at greater loads the curve becomes steep; it becomes increasingly difficult to compress the gas. An explanation to this aspect lies in the fact that at this stage the spacing between the fluid molecules has been reduced to the extent that it is very difficult to reduce the spacing further.

The bulk modulus of elasticity increases somewhat with temperature and pressure. At ordinary temperatures and pressures $K = 20 \times 10^8$ N/m^2 for water and $K = 1.05 \times 10^5$ N/m^2 for air. That indicates that air is approximately 20,000 times more compressible than water.

To gain some idea about the compressibility of water, imagine that a 1 m^3 of water is subjected to a pressure of 10 bar. Then a change in the volume of water amounts to :

$$-dV = \frac{V\,dp}{K} = \frac{1 \times (10 \times 10^5)}{20 \times 10^8} = \frac{1}{2000}\ \text{m}^3$$

Thus the application of 10 bar pressure to water under ordinary conditions causes its volume to decrease by only 1 part in 2000. Such a volume change is insignificant and as such water is regarded as incompressible fluid for all practical purposes. Exceptions occur only when the water is subjected to severe accelerations such as in water hammer that causes compression waves.

Though the gases in general are compressible, their compressibility becomes important only when the gas velocity becomes more than 20% of the velocity of sound waves in that gas.

Example 2.4. A cylinder contains 0.3 m^3 of air at 200 kPa and is compressed to 0.06 m^3. Calculate the pressure and bulk modulus of compressed air if the compression is achieved. *(i)* isothermally *(ii)* adiabatically. For air take adiabatic exponent $\gamma = 1.4$.

Solution : $p_1 = 200$ kPa $= 2 \times 10^5$ N/m^2 ; $v_1 = 0.3$ m^3 ; $v_2 = 0.06$ m^3

(i) Isothermal compression follows the Boyle's law :

$$p_1 v_1 = p_2 v_2 = \text{constant}$$

∴ pressure of compressed air would be

$$p_2 = \frac{p_1 v_1}{v_2} = \frac{2 \times 10^5 \times 0.3}{0.06} = 10 \times 10^5\ \text{N/m}^2$$

In isothermal change, the bulk modulus of elasticity equals the pressure

$\therefore \quad K = p_2 = \mathbf{10 \times 10^5\ N/m^2}$

(*ii*) Adiabatic compression follows the relation

$p_1 v_1^{\gamma} = p_2 v_2^{\gamma}$

$\therefore$ pressure of air after adiabatic compression would be

$$p_2 = p_1 \left(\frac{v_1}{v_2}\right)^{\gamma} = 2 \times 10^5 \left(\frac{0.3}{0.06}\right)^{1.4} = 19.03 \times 10^5\ \text{N/m}^2$$

In adiabatic change, the bulk modulus of elasticity equals γ times the pressure.

$\therefore \quad K = \gamma p_2 = 1.4 \times (19.03 \times 10^5) = \mathbf{26.65 \times 10^5\ N/m^2}$

Example 2.5. An increase in pressure of a liquid from 7.5 MPa to 15 MPa results into 0.2 percent decrease in its volume. Determine the bulk modulus of elasticity and coefficient of compressibility of the liquid.

Solution : For a finite pressure increase, $K = \dfrac{dp}{(-dV/V)} = \dfrac{dp}{d\rho/\rho}$

Increase in pressure $dp = 15 - 7.5 = 7.5\ \text{MPa} = 7.5 \times 10^6\ \text{N/m}^2$

Decrease in volume $\left(-\dfrac{dV}{V}\right) = 0.2\% = 0.002$

$\therefore$ Bulk modulus of elasticity $K = \dfrac{7.5 \times 10^6}{0.002} = \mathbf{3.75 \times 10^9\ N/m^2}$

Coefficient of compressibility $\beta = \dfrac{1}{K} = \dfrac{1}{3.75 \times 10^9} = \mathbf{0.267 \times 10^{-9}\ m^2/N}$

Example 2.6. Find the increase in pressure required to produce 1 percent reduction in volume of water. Take bulk modulus of elasticity of water $K = 2.16$ GPa.

Solution : For a finite pressure increase, $K = \dfrac{dp}{(-dV/V)} = \dfrac{dp}{(d\rho/\rho)}$

Bulk modulus $K = 2.16\ \text{GPa} = 2.16 \times 10^9\ \text{N/m}^2$

Reduction in volume $\left(-\dfrac{dV}{V}\right) = 1\% = 0.01$

$\therefore$ Increase in pressure $dp = K\left(-\dfrac{dV}{V}\right) = 2.16 \times 10^9 \times 0.01 = \mathbf{2.16 \times 10^7\ N/m^2}$

Example 2.7. Determine the bulk modulus of elasticity of a fluid that has a density increase of 0.002 percent for a pressure increase of 45 kN/m².

Solution : For a finite pressure increase, $K = \dfrac{dp}{(-dV/V)} = \dfrac{dp}{d\rho/\rho}$

$dp = 45\ \text{kN/m}^2 = 45 \times 10^3\ \text{N/m}^2$

$\dfrac{d\rho}{\rho} = 0.002\ \% = 2 \times 10^{-5}$

$\therefore$ Bulk modulus $K = \frac{45 \times 10^3}{2 \times 10^{-5}} = \mathbf{22.5 \times 10^8\ N/m^2}$

Example 2.8. At a depth of 8 km from the surface of ocean, the pressure is stated to be 82 MN/m². Determine the mass density, weight density and specific volume of water at this depth. Take density at surface $\rho = 1025\ kg/m^3$ and bulk modulus of elasticity K = 2350 MPa for the indicated pressure range.

Solution : For a finite pressure increase, $K = \frac{dp}{(-dV/V)} = \frac{dp}{d\rho/\rho}$

$$dp = (82 - 0) = 82\ MN/m^2 = 82 \times 10^6\ N/m^2$$

$$K = 2350\ MPa = 2350 \times 10^6\ N/m^2$$

$$\text{Increase in density } d\rho = \rho \frac{dp}{K} = 1025 \times \frac{82 \times 10^6}{2350 \times 10^6} = 35.76\ kg/m^3$$

Therefore, at a depth of 8 km from the free surface, mass density $\rho = 1025 + 35.76 = \mathbf{1060.76\ kg/m^3}$

Weight density (specific weight) $= \rho g = 1060.76 \times 9.81 = \mathbf{10406\ N/m^3}$

$$\text{Specific volume } v = \frac{1}{\rho} = \frac{1}{1060.76} = \mathbf{0.000943\ m^3/kg}$$

Example 2.9. Find the percentage reduction in volume of water if there occurs an increase in pressure by 10^4 kN/m² over the atmospheric pressure of 101.3 kN/m². If the same reduction in volume of air is to be attained by an isothermal process, work out the necessary increase in pressure. Comment on your results. For water : Bulk modulus of elasticity = 219×10^4 kN/m².

Solution : For a finite pressure rise, $K = \frac{dp}{(-dV/V)}$

$$-\frac{dV}{V} = \frac{dp}{K} = \frac{10^4}{219 \times 10^4} \approx 4.566 \times 10^{-3} \approx \mathbf{0.457\ \%}$$

Water will thus decrease in volume by 0.457 percent.

For an isothermal process, the bulk modulus equals the pressure. That is,

$$K = p = 101.3\ kN/m^2$$

$$\therefore \quad dp = K\left(\frac{-dV}{V}\right) = 101.3 \times (4.566 \times 10^{-3}) = \mathbf{0.463\ kN/m^2}$$

The calculations show that a reduction in volume of air can be accomplished by only a small pressure increase and that air is approximately $\frac{10^4}{0.463} = 21598$ times more compressible than water.

Example 2.10. A reservoir of capacity 0.01 m³ is completely filled with a fluid of coefficient of compressibility 0.75×10^{-9} m²/N. Calculate the amount of fluid that will spill over if pressure in the reservoir is reduced by 2×10^7 N/m².

Solution : For a finite pressure change, the relative change of volume per unit pressure is prescribed by the coefficient of compressibility;

$$\beta_c = -\left(\frac{dV/V}{dp}\right)$$

The change in the fluid volume can be worked out by substituting the given data in the above expression. That is

$$dV = -\beta_c\, V\, dp = -0.75 \times 10^{-9}\,(0.01)\,(-2 \times 10^7) = \mathbf{1.5 \times 10^{-4}\ m^3} \text{ (an increase)}$$

A negative sign with the pressure value accounts for the decrease in pressure during the process. Further, since the reservoir was initially full to the entire capacity, the volume increase resulting from pressure drop will flow out of the reservoir.

2.5.1. Thermal Expansion

On heating, the fluids expand and the relative increase in volume per degree rise in temperature during an isobaric (constant pressure) process is called the *coefficient of thermal expansion*.

$$\beta_{ex} = \frac{1}{V}\left(\frac{dV}{dt}\right) \qquad ...(2.17)$$

This coefficient increases both with temperature and pressure; the typical values for water are :

$$\beta_{ex} = 14 \times 10^{-6} \text{ at } 0^\circ \text{ C and 1 bar}$$

$$= 700 \times 10^{-6} \text{ at } 100^\circ \text{ C and 100 bar}$$

For slight temperature variations, 0–20°C, β_{ex} is generally assumed to be zero for water.

Example 2.11. A hydraulic system is filled with 0.2 m^3 of fluid of which 0.02 m^3 is in an open tank. The system was filled at 20°C and during operation the temperature rises to 50°C in the tank. Assuming coefficient of thermal expansion of the fluid β_{ex} = 0.00070 per ° C, find the volume of fluid in the tank.

Solution : Substituting the given data in the relation :

$\beta_{ex} = \frac{1}{V}\left(\frac{dV}{dt}\right)$, we can find the increase in total volume, *i.e.*,

$$dV = \beta_{ex} \times V \times dt = 0.00070 \times 0.2 \times (50 - 20) = 0.0042 \text{ m}^2$$

This volume increase will have to be accommodated in the open tank. Consequently, the volume of fluid in the tank would rise to :

$$0.02 + 0.0042 = \mathbf{0.0242\ m^3}$$

2.6. VISCOSITY

Viscosity is a property of the fluid by virtue of which it offers resistance to shear or angular deformation.

Experimental evidence indicates that when any fluid flows over a solid surface the velocity is not uniform at any cross section; it is zero (no slip) at the solid surface and progressively approaches the free stream velocity in the fluid layers far away from the solid surface. This aspect of the velocity profile (a curve connecting the tips of velocity vectors) indicates the existence of some resistance to flow due to friction between a fluid layer and the solid surface, and between adjacent layers of fluid itself. Again the velocity gradient (the spatial rate of change of velocity du/dy) is large at the solid surface and gradually diminishes to zero with distance from the wall. Evidently the resistance between the fluid and surface is greater when compared to that between the fluid layers themselves.

The resistance to flow because of internal friction is called *viscous resistance*, and the property which enables the fluid to offer resistance to relative motion between adjacent layers is called the *viscosity* of fluid. Viscosity is thus a measure of resistance to relative translational motion of adjacent layers of a fluid. This

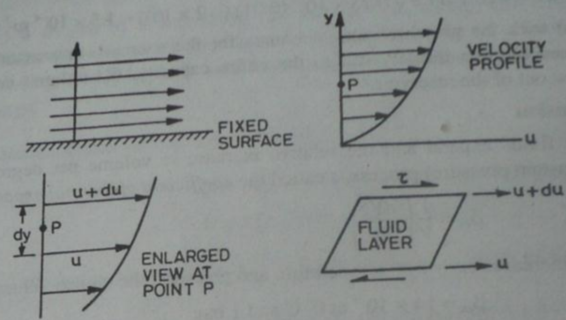


Fig. 2.2. *Velocity profile and viscosity concept*

property is manifested by all the real fluids, and it distinguishes them from ideal or non-viscous fluids. Mollases, tar and glycerine are examples of highly viscous liquids; the intermolecular force of attraction between their molecules is very large and consequently they cannot be easily poured or stirred. Fluids like water, air and petrol have a very small viscosity; they flow much more easily and rapidly and are called thin fluids.

2.6.1. Newton's Law of Viscosity

Consider two adjacent layers at an infinitesimal distance dy apart and moving with velocity u and $(u + du)$, respectively. The upper layer moving with velocity $(u + du)$ drags the lower layer along with it by exerting a force F. However, the lower layer tires to retard or restrict the motion of upper layer by exerting a force equal and opposite to F. These two equal and opposite forces induce a shear or viscous resistance τ (pronounced tau) given by F/A where A is the contact area between the two layers. Experimental measurements have shown that the shear stress is proportional to the spatial rate of change of velocity normal to the flow :

$$\tau \propto \frac{du}{dy} \; ; \; \tau = \mu \frac{du}{dy} \qquad ...(2.18)$$

The term $\frac{du}{dy}$ is more usually called the velocity gradient at right angles to the direction of velocity itself. The proportionality constant μ (pronounced mew) in equation 2.18 is a function of the fluid involved and is called the **coefficient of viscosity, absolute viscosity or dynamic viscosity**. Equation 2.18 was first suggested by Newton and is referred to as the **Newton's viscosity equation** or **Newton's law of viscosity**.

The following observations help to appreciate the interaction between viscosity and velocity distribution :

- Maximum shear stresses occur where the velocity gradient is the largest, and the shear stresses disappear where the velocity gradient is zero.
- Velocity gradient at the solid boundary has a finite value. The velocity profile cannot be asymptotic to the boundary because that would imply an infinite velocity gradient and, in turn, an infinite shear stress.

- Velocity gradient becomes less steep (du/dy becomes small) with distance from the boundary. Consequently maximum value of shear stress occurs at the boundary and it progressively decreases with distance from the boundary.

Deformation of fluid elements can be prescribed in terms of the angle of shear strain $d\theta$. Fig. 2.3 indicates a thin sheet of fluid element $ABCD$ placed between two plates distance dy apart. The length and the width of the plates are much larger than the thickness dy so that the edge effects can be neglected. When force F is applied to the upper plate, it causes it to move at a small speed du relative to the bottom plate. Velocity gradient sets up a shear stress $\tau = F/A$ which makes the fluid element distort to position $AB'C'D$ after a short time interval dt.

Distance $BB' = CC'$

$= \text{speed} \times \text{time} = du \times dt$

For small angular displacement $d\theta$, $BB' = dy \times d\theta$

$$\therefore\ du \times dt = dy \times d\theta\ ;\quad \frac{du}{dy} = \frac{d\theta}{dt} \qquad ...(2.19)$$

Invoke Newton's law of viscosity, *i.e.*, express the shear stress in terms of velocity gradient :

$$\tau = \mu \frac{du}{dy}\ ;\ \tau = \mu \frac{d\theta}{dt} \qquad ...(2.20)$$

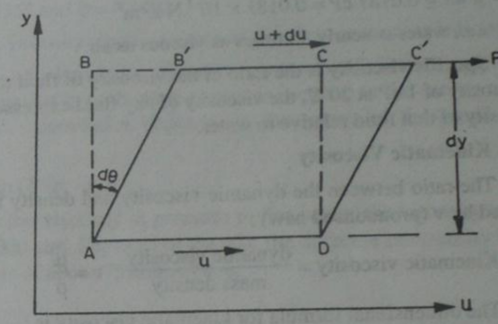


Fig. 2.3. *Shear stress and time rate of shear strain*

Apparently the shear stress in fluids is dependent on the rate of fluid deformation $\frac{d\theta}{dt}$. This characteristic serves to distinguish a solid from a fluid. Whereas the shear stress in a solid material is generally proportional to shear strain; the shear stress in a viscous fluid is proportional to time rate of strain.

2.6.2. Dimensional Formula and Units of Viscosity

The units of viscosity can be worked out from Newton's equation of viscosity; $\tau = \mu\, du/dy$. Solving for the viscosity μ and inserting dimensions F, L, T for force, length and time :

$$\mu = \frac{\tau}{du/dy} = \left[\frac{F}{L^2}\right] \div \left[\frac{L}{T} \times \frac{1}{L}\right] = \left[\frac{FT}{L^2}\right]$$

When the force dimension is expressed in terms of mass, $F = \left[\frac{ML}{T^2}\right]$, the dimensions for viscosity in terms of mass, length and time become $\left[\frac{M}{LT}\right]$.

When appropriate units are inserted for force, length and time, the dynamic viscosity will have the units :

$$\mu = \frac{\tau}{(du/dy)} = \frac{N/m^2}{\left(\frac{m}{s} \times \frac{1}{m}\right)} = \frac{N\,s}{m^2} = \text{Pa s}$$

Sometimes, the coefficient of dynamic viscosity μ is designated by poise (P)

$$1 \text{ poise} = \frac{1 \text{ gm}}{\text{cm sec}} = \frac{1 \text{ dyne sec}}{\text{cm}^2}$$

$$= \frac{10^{-5}}{(10^{-2})^2} \frac{\text{N s}}{\text{m}^2} = \frac{0.1 \text{ N s}}{\text{m}^2} = 0.1 \text{ Pa s}$$

A poise turns out to be a relatively large unit, hence the unit centipoise (cP) is generally used : 1 cP = 0.01 P. Typical values of viscosity for water and air at 20°C and at standard atmospheric pressure are :

μ water = 1.0 cP = 10^{-3} N s/m^2

μ air = 0.0181 cP = 0.0181 × 10^{-3} N s/m^2

i.e., water is nearly 55 times as viscous as air.

Specific viscosity is the ratio of the viscosity of fluid to the viscosity of water at 20°C. Since water has a viscosity of 1 cP at 20°C, the viscosity of any fluid expressed in centipoise units would be a measure of the viscosity of that fluid relative to water.

2.6.3. Kinematic Viscosity

The ratio between the dynamic viscosity and density is defined as *kinematic viscosity* of fluid and is denoted by ν (pronounced new) :

$$\text{Kinematic viscosity} = \frac{\text{dynamic viscosity}}{\text{mass density}} ; \; \nu = \frac{\mu}{\rho} \qquad ...(2.21)$$

The dimensional formula for kinematic viscosity is :

$$\nu \equiv \left[\frac{M}{LT}\right] \div \left[\frac{M}{L^3}\right] = \left[\frac{L^2}{T}\right]$$

The kinematic viscosity does not involve force; its only dimensions being length and time as in kinematics of fluid flow. Typical units of ν are m^2/s or cm^2/s, the latter being referred to as stoke (St) to perpetuate the name of the English physicist Sir George Stokes. A centistoke (c St) is one-hundredth of a stoke : 1c St = 0.01 St. Typical values of kinematic viscosity at 20°C and at standard atmospheric pressure are :

$$\nu \; water = 1.0 \text{ cSt} = 1 \times 10^{-6} \text{ m}^2/\text{s}$$

$$\nu \; air = 15.0 \text{ cSt} = 15 \times 10^{-6} \text{ m}^2/\text{s}$$

i.e., the kinematic viscosity of air is about 15 times greater than the corresponding value of water.

2.6.4. Effect of Temperature on Viscosity

There exists a distinct difference between fluids of liquid and gaseous nature in the effect of temperature on the value of their dynamic viscosity. Increase of temperature causes a decrease in the viscosity of a liquid, whereas viscosity of gases increase with temperature growth. This difference in behaviour can be explained by considering the basic mechanism that gives rise to viscosity. The viscous forces in a fluid are the outcome of intermolecular cohesion and molecular momentum transfer. In liquids the molecules are comparatively more closely packed; molecular activity is rather small and so the viscosity is primarily due to molecular cohesion. The molecular cohesion decreases with growth of temperature and consequently the viscosity of liquids drops at elevated temperatures. In gases the molecular cohesive forces are negligibly small and the viscosity results primarily from the molecular momentum transfer. This molecular activity increases with a rise in temperature and so does the gas viscosity.

The following empirical relations have been suggested for variation of viscosity with temperature and pressure :

(i) *For liquids :*

$$\mu_t = \frac{\mu_0}{1 + At + Bt^2}$$

where μ_t is the viscosity at t°C, μ_0 is the viscosity at 0°C and A and B are the constants depending upon the liquid. For water

$$\mu_0 = 0.0179 \text{ poise}$$

$$A = 0.03368 \text{ and } B = 0.000221$$

The above correlation represents a hyperbola; viscosity tending to zero as temperature tends to infinity.

High pressures also affect the viscosity of a liquid; the viscosity increases with increasing pressure. This may be attributed to the fact that with pressure growth, there occurs an increase in the energy required for the relative movement of molecules. The correlation depends on the nature of the liquid and is exponential.

$$\mu_p = \mu_o \exp\left[K\,(p - p_0)\right]$$

where K is a constant for the liquid and μ is the viscosity at pressure p. For water, the viscosity becomes two-fold when pressure increases from 1 to 1000 atm. For most of the oils, the increase in viscosity is of the order of 10 to 15 percent for a pressure increase of about 75 atmosphere.

(ii) *For gases :*

The reduction in viscosity of gases with increasing temperature is prescribed by the relationship :

$$\mu_t = \mu_0 + \alpha t - \beta t^2$$

where μ_t is the viscosity at t°C, μ_0 is the viscosity at 0°C, and α and β are the constants depending upon the gas. For air

$$\mu_0 = 1.7 \times 10^{-5} \text{ N s/m}^2$$

$$\alpha = 0.56 \times 10^{-7} \text{ and } \beta = 0.1189 \times 10^{-9}$$

Example 2.12. *(i)* A lubricating oil of viscosity μ undergoes steady shear between a fixed lower plate and an upper plate moving at speed *V*. The clearance between the plates is *h*. Show that a linear velocity profile results if the fluid does not slip at either plate.

***(ii)* Two horizontal plates are placed 1.25 cm apart, the space between them being filled with oil of viscosity 14 poise. Compute the shear in the oil if the upper plate is moved with a velocity of 2.5 m/s.**

Solution : The shear stress τ is constant throughout the fluid for the given geometry and motion, and, therefore, from Newton's law of viscosity

$$\frac{du}{dy} = \frac{\tau}{\mu} = \text{constant}$$

or

$$u = a + by$$

The constants a and b are evaluated from the no slip conditions at the upper and lower plates.

$$u = 0 \text{ at } = y = 0\ ;\ 0 = a$$

$$u = V \text{ at } y = h\ ;\ V = a + bh$$

Hence $a = 0$ and $b = V/h$. The velocity profile between the plates is then given by $u = \frac{Vy}{h}$ and is linear as indicated in Fig. 2.4.

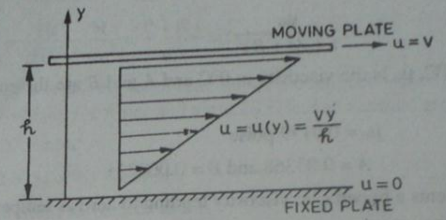


Fig. 2.4.

(ii) Viscous shear stress is given by the Newton's law of viscosity

$$\tau = \mu \frac{du}{dy}$$

Given $\mu = 14$ poise $= 1.4$ N s/m^2

$$du = 2.5 \text{ m/s and } dy = 1.25 \times 10^{-2} \text{ m}$$

$$\therefore \quad \tau = 1.4 \times \frac{2.5}{1.24 \times 10^{-2}} = 280 \text{ N/m}^2 = \mathbf{280 \text{ Pa}}$$

Although oil is very viscous, this is a modest shear stress; about 360 times less than atmospheric pressure.

Example 2.13. The clearance space between a shaft and a concentric sleeve has been filled with a Newtonian fluid. The sleeve attains a speed of 60 cm/s when a force of 500 N is applied to it parallel to the shaft. What force is needed if it is desired to move the sleeve with a speed of 300 cm/s ?

Solution : For a Newtonian fluid, $\tau = \mu \frac{du}{dy}$. Since the space between the shaft and the sleeve is very small, *i.e.*, the oil film is thin, it can be presumed that $\frac{du}{dy} = \frac{u}{t}$ where u is the sleeve speed and t is the oil film thickness. Further

$$\text{Shear stress } \tau = \frac{\text{force}}{\text{area}} = \frac{F}{A}$$

$$\therefore \quad \frac{F}{A} = \mu \frac{u}{t} \text{ or } F = A \mu \frac{u}{t}$$

A, μ and t are constant and therefore $F \propto u$ and accordingly $\frac{F_1}{u_1} = \frac{F_2}{u_2}$

Inserting the appropriate values,

$$\frac{500}{60} = \frac{F_2}{300} \; ; \; F_2 = \mathbf{2500 \text{ N}}$$

Example 2.14. Two horizontal flat plates are placed 0.15 mm apart and the space between them

is filled with an oil of viscosity 1 poise. The upper plate of area 1.5 m² is required to move with a speed of 0.5 m/s relative to the lower plate. Determine the necessary force and power required to maintain this speed.

Solution : Viscous shear stress $\tau = u \dfrac{du}{dy}$

Given : $\mu = 1$ poise $= 0.1$ N s/m² ; $du = 0.5$ m/s

$dy = 0.15$ mm $= 0.15 \times 10^{-3}$ m

$\therefore$ Shear stress $\tau = \dfrac{0.1 \times 0.5}{0.15 \times 10^{-3}} = 333.3$ N/m²

(i) Shear resistance or force, F = shear stress × area

$= 333.3 \times 1.5 =$ **500 N**

(ii) Power required to move the upper plate at a speed of 0.5 m/s,

$= Fu = (500 \times 0.5)$ Nm/s $= 250$ W $=$ **0.25 kW**

Example 2.15. A dash pot 10 cm diameter and 12.5 cm long slides vertically down in a 10.05 cm diameter cylinder. The oil filling the annular space has a viscosity of 0.80 poise. Find the speed with which the piston slides down if load on the piston is 10 N.

Solution : Since the space between the dash pot and the cylinder is very small, *i.e.*, the oil film is thin, we can presume that $\dfrac{du}{dy} = \dfrac{u}{t}$ where u is the piston speed and t is the oil film thickness.

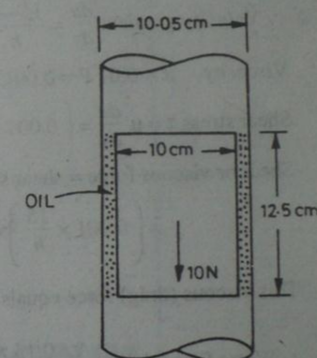


Fig. 2.5.

Shear stress $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{u}{t}$

Shear or viscous force = shear stress × area $= \mu \dfrac{u}{t} (2 \pi r l)$

Given : $r = \dfrac{10}{2} = 5$ cm $= 0.05$ m ; $u = 0.8$ poise $= 0.08$ N s/m²

$t = \dfrac{10.05 - 10}{2} = 0.025$ cm $= 0.00025$ m

Viscous force equals the load of 10 N

$\therefore \quad 10 = 0.08 \times \dfrac{u}{0.00025} \times (2 \pi \times 0.05 \times 0.125)$

Hence piston speed u = **0.796 m/s**

Example 2.16. A cylinder of diameter 15 cm and weight 90 N slides a distance of 12.5 cm in a lubricated pipe. The clearance between the cylinder and pipe is 2.5 × 10⁻³ cm. The cylinder is noted to decelerate at a rate of 0.6 m/s² when the speed is 6 m/s. Calculate the viscosity of the oil used for lubricating the pipe.

Solution : Viscous shear stress $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{u}{t}$

Viscous resistance or force = shear stress × area

$$= \mu \frac{u}{t} \times \pi\, dl$$

$$= \frac{\mu \times 6}{2.5 \times 10^{-5}} \times \pi\,(0.15)\,(0.125) = 14130\,\mu\ \text{N}$$

Invoking Newton's second law : ΣF = mass × acceleration

$$\therefore \qquad 90 - 14130\,\mu = \frac{90}{9.81}\,(-0.6)$$

$$90 - 14130\,\mu = -5.5$$

$$\therefore \qquad \mu = \frac{95.5}{14130} = \mathbf{6.76 \times 10^{-3}\ N\ s/m^2}$$

Example 2.17. A skater weighing 750 N skates at 15 m/s and is supported by an average skating area of 10 cm². Determine the average thickness of thin film of water that exists between the skates and the ice. Take the viscosity of water as 0.01 poise and the effective coefficient of friction between skates and ice as 0.02.

Solution : Normal reaction by ice on skates = 750 N

Frictional reaction = coefficient of friction × normal reaction

$$= 0.02 \times 750 = 15\ \text{N}$$

Within the water film thickness of h metre, the velocity varies linearly from 0 m/s to 15 m/s.

$\therefore$ Velocity gradient $\frac{du}{dy} = \frac{15-0}{h} = \frac{15}{h}$ per second

Viscosity $\mu = 0.01\ \text{P} = 0.001\ \text{N s/m}^2$

Shear stress $\tau = \mu \frac{du}{dy} = \left(0.001 \times \frac{15}{h}\right)\ \text{N/m}^2$

Shear or viscous force = shear stress × area

$$= \left(0.001 \times \frac{15}{h}\right) \times 10 \times 10^{-4} = \left(\frac{0.15 \times 10^{-4}}{h}\right)\ \text{N}$$

This viscous (drag) force equals the frictional reaction of 15 N. That is : $\frac{0.15 \times 10^{-4}}{h} = 15$

$\therefore$ Water film thickness $= \frac{0.15 \times 10^{-4}}{h} = \mathbf{1 \times 10^{-6}\ m}$

Example 2.18. Find the kinematic viscosity of a liquid in stokes whose specific gravity is 0.95 and dynamic viscosity is 0.012 poise.

Solution : $\mu = 0.012$ poise $= 0.012 \times 0.1 = 1.2 \times 10^{-3}$ N s/m²

Mass density of liquid = specific gravity × mass density of water

$$\rho = 0{,}95 \times 1000 = 950\ \text{kg/m}^3$$

$\therefore$ Kinematic viscosity $\nu = \frac{\mu}{\rho} = \frac{1.2 \times 10^{-3}}{950} = 1.263 \times 10^{-6}\ \text{m}^2/\text{s}$

$$= 1.263 \times 10^{-2}\ \text{cm}^2/\text{s} = \mathbf{1.263 \times 10^{-2}\ stokes}$$

Example 2.19. A hydraulic lift used for lifting automobiles has 20 cm diameter ram which slides in a 20.016 cm diameter cylinder. The annular space between the cylinder and ram is filled with an oil of kinematic viscosity 3.5 stokes and relative density 0.85. If the travel of 3.2 m long ram has a uniform rate of 15 cm/s, estimate the frictional resistance experienced by the ram.

Solution : Kinematic viscosity $\nu = 3.5$ stokes $= 3.5\ \text{cm}^2/\text{s} = 3.5 \times 10^{-4}\ \text{m}^2/\text{s}$

Mass density $\rho = 0.85 \times 1000 = 850\ \text{kg/m}^3$

$\therefore$ Dynamic viscosity $\mu = \rho\,\nu$

$$= 850 \times (3.5 \times 10^{-4}) = 0.2975\ \text{N s/m}^2$$

$$\text{Thickness of oil film} = \frac{(20.016 - 20)}{2} \times 10^{-2} = 0.00008\ \text{m}$$

$$\text{Shear stress } \tau = \mu \frac{du}{dy} = \mu \frac{V}{t} = \frac{0.2975 \times 0.15}{0.00008\ \text{m}} = 557.81\ \text{N/m}^2$$

Frictional resistance = shear stress × area

$$= 557.81 \times (\pi \times 0.20 \times 3.2) = 1121\ \text{N} \approx \mathbf{1.12\ kN}$$

Example 2.20. Two square flat plates with each side 60 cm are spaced 12.5 mm apart. The lower plate is stationary and the upper plate requires a force of 100 N to keep it moving with a velocity of 2.5 m/s. The oil film between the plates has the same velocity as that of plates at the surface of contact. Assuming a linear velocity distribution, determine :

***(i)* the dynamic viscosity of the oil poise, and**

***(ii)* the kinematic viscosity of the oil in stokes if the specific gravity of the oil is 0.95.**

Solution : Shearing stress $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{u}{t} = \mu \times \dfrac{2.5}{0.0125} = 200\,\mu$

$$\text{Shearing area} = 0.6 \times 0.6 = 0.36\ \text{m}^2$$

$$\text{Shearing force} = \tau \times A = 200\,\mu \times 0.36 = 72\,\mu$$

Equating it to the given force,

$$72\,\mu = 100\ ;\ \mu = \frac{100}{72} = 1.39\ \text{Ns/m}^2 = \mathbf{13.9\ poise}$$

(ii) Kinematic viscosity of the oil,

$$\nu = \frac{\mu}{\rho} = \frac{1.39}{0.95 \times 1000} = 14.63 \times 10^{-4}\ \text{m}^2/\text{s}$$

$$= 14.63\ \text{cm}^2/\text{s} = \mathbf{14.63\ stokes}$$

Example 2.21. A cubical block weighing 4.5 N and having a 40 cm edge is allowed to slide down an inclined plane surface making an angle of 30° with the horizontal on which there is a uniform layer of oil 0.005 cm thick. If the expected steady state velocity of the block is 12.5 cm/s, determine the viscosity of the oil. Also express the kinematic viscosity in stokes if the oil has a mass density of 800 kg/m³.

Solution : Component of weight of block along the plane = $W \sin \alpha$

Assuming a linear velocity profile in the oil film, Shear stress $\tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$

Shearing force opposing motion $= \mu \frac{u}{t} A$

Under equilibrium conditions;

$$W \sin \alpha = \mu \frac{u}{t} A$$

$\therefore$ Dynamic viscosity $\mu = \frac{W \sin \alpha \times t}{u A}$

$$= \frac{4.5 \sin 30^\circ \times (0.005 \times 10^{-2})}{(12.5 \times 10^{-2}) \times (0.4 \times 0.4)}$$

$= 0.0056$ N s/m^2 = **0.056 poise**

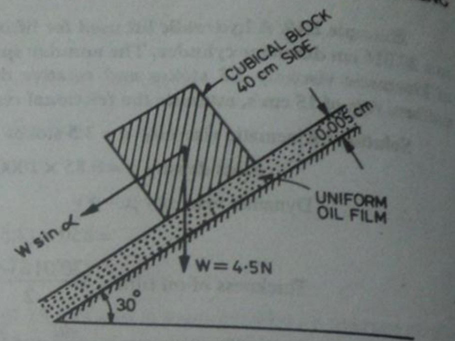


Fig. 2.6.

Kinematic viscosity $\nu = \frac{\mu}{\rho} = \frac{0.0056}{800} = 7 \times 10^{-6}$ m^2/s

$= 7 \times 10^{-2}$ cm^2/s = **7×10^{-2} stokes**

Example 2.22. A square plate 50 cm × 50 cm weighing 200 N is allowed to slide down an inclined plane which is laid at a slope of 1 vertical to 2.5 horizontal. What terminal velocity will be attained by the plate if 0.02 mm thickness of oil film lies interposed between the inclined plane and the plate ? The oil has a dynamic viscosity of 2.25×10^{-3} N s/m^2.

Solution : Let the plane be inclined at an angle α with the horizontal.

$$\tan \alpha = \frac{1}{2.5} = 0.4 \ ; \ \alpha = 21.8^\circ$$

Component of weight of plate along the plane = $W \sin \alpha$

Assuming a linear velocity profile in the oil fim,

Shear stress $\tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$

Shearing force opposing motion $= \mu \frac{u}{t} A$

Under equilibrium conditions : $W \sin \alpha = \mu \frac{u}{t} A$

$\therefore$ Speed $u = \frac{W \sin \alpha \times t}{\mu A} = \frac{200 \sin 21.8^\circ \times (0.02 \times 10^{-3})}{2.25 \times 10^{-3} \times (0.5 \times 0.5)}$ = **2.64 m/s**

Example 2.23. A 15 cm diameter vertical cylinder rotates concentrically inside another cylinder of diameter 15.10 cm. Both cylinders are 25 cm high. The space between the cylinders is filled with a liquid whose viscosity is unknown. If a torque of 12 Nm is required to rotate the inner cylinder at 100 rpm, determine the viscosity of the fluid.

Solution : For a linear viscosity profile within the thin oil film :

Viscous shear stress $\tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$

Viscous resistacne or force = shear stress × area = $\mu \frac{u}{t} \times (2\pi rl)$

Viscous torque = viscous force × torque arm

The torque arm equals the radius of the cylinder which is rotating.

$\therefore$ Torque $T = \mu \frac{u}{t} \times (2\pi rl) \times r$

Given : $t = \frac{15.10 - 15.0}{2} = 0.05 \text{ cm} = 0.0005 \text{ m}$

$r = \frac{15}{2} = 7.5 \text{ cm} = 0.075 \text{ m}$

$u = \frac{2\pi r N}{60} = \frac{2\pi \times 0.075 \times 100}{60} = 0.785 \text{ m/s}$

$\therefore \quad 12 = \mu \times \frac{0.785}{0.0005} \times (2\pi \times 0.075 \times 0.25) \times 0.075$

Dynamic viscosity $\mu = 0.865 \text{ N s/m}^2 = 8.65$ **poise**

Example 2.24. In a 50 mm long journal-bearing arrangement, the clearance between the two at concentric condition is 0.1 mm. The shaft is 20 mm in diameter and rotates at 3000 rpm. The dynamic viscosity of the lubricant used is 0.01 Pa s and the velocity variation in the lubricant is linear. Considering the lubricant to be Newtonian, calculate the frictional torque the journal has to overcome, and the corresponding power loss.

Solution : For a linear velocity profile $\frac{du}{dy} = \frac{u}{t}$ where u is the tangential velocity of shaft and t is the thickness of oil fim.

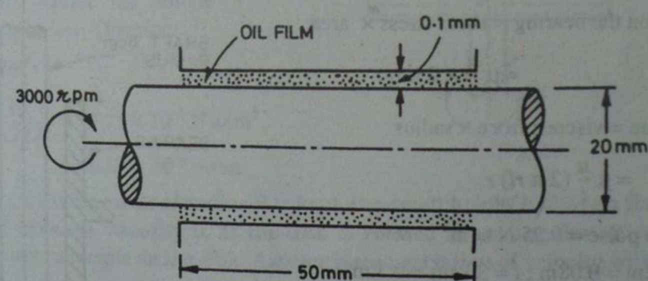


Fig. 2.7.

Viscous shear stress $\tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$

Viscous resistance or force on the bearing,

$$= \text{shear stress} \times \text{force} = \mu \frac{u}{t} \times 2\pi rl$$

$$\text{Viscous torque} = \text{viscous force} \times \text{torque arm} = \mu \frac{u}{t} (2\pi rl)\, r$$

Given :
$$\mu = 0.01 \text{ Pa s} = 0.01 \text{ N s/m}^2$$
$$r = 10 \text{ mm} = 0.01 \text{ m} \; ; \; l = 50 \text{ mm} = 0.05 \text{ m}$$
$$u = \frac{2\pi r N}{60} = \frac{2\pi \times 0.01 \times 3000}{60} = 3.14 \text{ m/s}$$
$$t = 0.1 \text{ mm} = 0.0001 \text{ m}$$

$$\therefore \text{ Frictional torque } T = 0.1 \times \frac{3.14}{0.0001} \times (2\pi \times 0.01 \times 0.05) \times 0.01 = 9.85 \times 10^{-3} \text{ Nm}$$

If the shaft rotates with angular velocity ω, than

$$\omega = \frac{2\pi N}{60} = 2\pi \times \frac{3000}{60} = 314 \text{ rad/s}$$

∴ Power utilized in overcoming the frictional resistance,

$$P = T\omega = (9.85 \times 10^{-3}) \times 314 = 3.09 \text{ Nm/s} = \mathbf{3.09 \text{ W}}$$

Example 2.25. A stationary bearing of length 30 cm and internal radius 8.025 cm has been used to provide lateral stability to a 8 cm radius shaft rotating at a constant speed of 200 revolutions per minute. The space between the shaft and bearing is filled with a lubricant having a viscosity 2.5 poise. Find the torque required to overcome the friction in bearing. Take the velocity profile as linear.

Solution : For a linear velocity profile $\frac{du}{dy} = \frac{u}{t}$ where u is the tangential velocity of the shaft and t is the thickness of oil film.

$$\text{Shear stress } \tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$$

Viscous force on the bearing = shear stress × area

$$= \mu \frac{u}{t} \times 2\pi rl$$

Frictional torque = viscous froce × radius

$$= \mu \frac{u}{t} (2\pi rl)\, r$$

Given : $\mu = 2.5 \text{ poise} = 0.25 \text{ N s/m}^2$
$$r = 8 \text{ cm} = 0.08\text{m} \; ; \; l = 30 \text{ cm} = 0.3 \text{ m}$$
$$u = \frac{2\pi r N}{60} = \frac{2\pi \times 0.08 \times 200}{60} = 1.675 \text{ m/s}$$
$$t = 8.025 - 8 = 0.025 \text{ cm} = 0.00025 \text{ m}$$

∴ Frictional torque T

$$= 0.25 \times \frac{1.675}{0.00025} \times (2\pi \times 0.08 \times 0.3) \times 0.08$$
$$= 20.19 \text{ Nm}$$

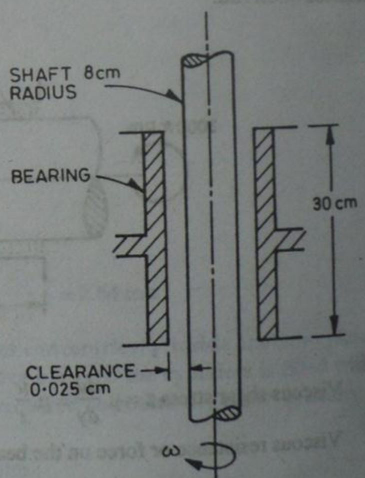


Fig. 2.8.

Rotational speed of the shaft, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 200}{60} = 20.93$ rad/s

$\therefore$ Power utilized in overcoming the frictional torque, $P = T\omega$

$$= 20.19 \times 20.93 = 422.5 \text{ Nm/s} = \mathbf{422.5\ W}$$

Example 2.26. In a rotating cylinder viscometer, the radii of the cylinders are 3.2 cm and 3 cm, and the outer cylinder is rotated steadily at 180 rpm. For a certain liquid filled in the annular space to a depth of 7.5 cm, the torque produced on the inner cylinder is 10^{-4} Nm. Assuming velocity distribution to be linear, calculate the viscosity of the liquid.

Solution : For the given cylinder speed, the tangential velocity is,

$$u = \frac{2\pi r_2 N}{60} = \frac{2\pi \times 0.032 \times 180}{60} = 0.603 \text{ m/s}$$

With assumption of linear velocity distribution,

$$\frac{du}{dy} = \frac{u}{t} = \frac{0.603}{(3.2-3)\times 10^{-2}} = 301.5 \text{ per sec}$$

Viscous shear $\tau = \mu \frac{u}{t} = 301.5\,\mu$

Viscous force F = viscous shear $\times$ area

$= 301.5\,\mu \times 2\pi r_1 l$

$= 301.5\,\mu \times (2\pi \times 0.03 \times 0.075) = 4.26\,\mu$

Viscous torque = viscous force $\times$ radius

$= 4.26\,\mu \times r_1 = 4.26\,\mu \times 0.03 = 0.1278\,\mu$

Viscous torque equals the torque T measured by the torque meter. Therefore,

$0.1278\,\mu = 10^{-4}$

$$\mu = \frac{10^{-4}}{0.1278} = 7.825 \times 10^{-4} \text{ N s/m}^2$$

$$= \mathbf{78.25 \times 10^{-4}\ poise}$$

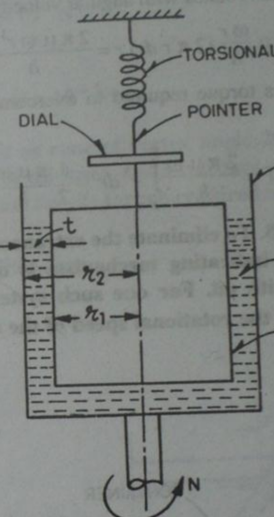


Fig. 2.9.

Example 2.27. A circular disk of radius R is kept at a small height h above a fixed bed by means of a layer of oil of dynamic viscosity μ. If the disk is rotated at an angular velocity ω, obtain an expression for the viscous torque on the disk. Assume linear variation of velocity within the oil film.

Solution : Consider an element of disk of width dr at a radial distance r. For linear variation of velocity with depth h of oil film.

Shear stress $\tau = \mu \frac{du}{dy} = \mu \frac{u}{h}$

Shear force = shear stress $\times$ area $= \mu \frac{u}{h} 2\pi r\, dr$

Viscous torque on the element, $dT = \mu \frac{u}{h}(2\pi r\, dr)\, r$

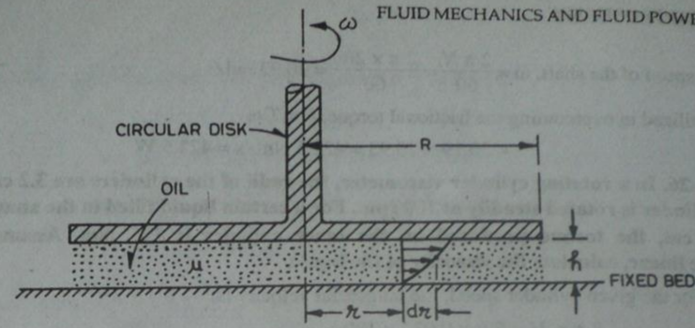


Fig. 2.10.

Since the shaft rotates with angular velocity ω radians per second, the peripheral velocity $u = \omega r$

$$dT = \mu \frac{\omega r}{h} (2 \pi r \, dr) \, r = \frac{2 \pi \mu \omega r^3 \, dr}{h}$$

Total viscous torque required to overcome the shearing force can be obtained by integrating from $r = 0$ to $r = R$

$$\text{Total torque } T = \frac{2 \pi \mu \omega}{h} \int_0^R r^3 \, dr = \frac{1}{2} \frac{\pi \mu \omega}{h} R^4$$

Example 2.28. To eliminate the extraneous rotations in electrical measuring devices, the motion of the pointer of indicating mechanism is often dampened by having a circular disk turn in a container filled with oil. For one such system illustrated in Fig. 2.11, make calculations for the damping torque if the rotational speed of the disk in an oil of viscosity 7.5×10^{-3} Pa s is stated to be 0.2 rad/sec.

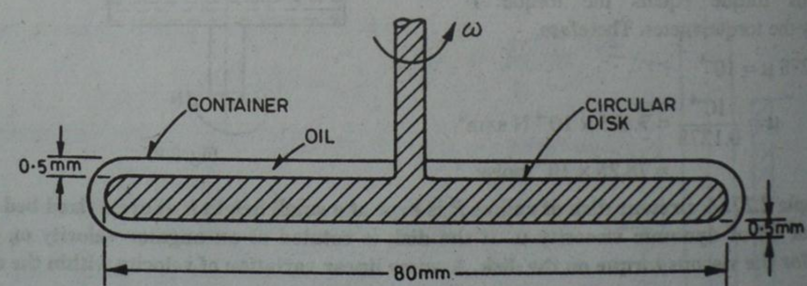


Fig. 2.11.

Solution : Since the oil film is thin, a linear velocity profile can be assumed. Consider an element of width dr at a radial distance r. Shear stress acting on this element,

$$\tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$$

Shear force = shear stress × area = $\mu \frac{u}{t} (2 \pi r \, dr)$

Viscous torque on one face of the element, $dT = \mu \frac{u}{t}(2\pi r\,dr)\,r$

Since the disk rotates with angular velocity ω rad/sec, the tangential velocity $u = \omega r$

$$dT = \mu \frac{\omega r}{t}(2\pi r\,dr)\,r = \frac{2\pi\mu\omega}{t} r^3\,dr$$

Total viscous torque providing dampling effect can be obtained by integrating from $r = 0$ to $r = R$

Total torque $T = \frac{2\pi\mu\omega}{t}\int_0^R r^3\,dr = \frac{1}{2}\frac{\pi\mu\omega}{t}R^4$

Thus the torque acting on both the faces would equal $\frac{\pi\mu\omega}{t}R^4$

Given : $\omega = 0.2$ rad/sec ; $\mu = 7.5 \times 10^{-3}$ Pa s $= 7.5 \times 10^{-3}$ N s/m^2

$t = 0.5 \times 10^{-3}$ m ; $R = 40$ mm $= 0.04$ m

$$\therefore \text{Damping torque} = \frac{\pi(7.5 \times 10^{-3})\,0.2}{0.5 \times 10^{-3}} \times (0.04)^4 = \mathbf{2.41 \times 10^{-5}\ Nm}$$

Example 2.29. A conical thrust bearing idealised as cone of vertex angle 2θ, maximum cone radius R rests and revolves over a uniform fluid layer of thickness *t* at a constant angular velocity ω. If the fluid has a dynamic viscosity μ, set up an expression for the torque required and the rate of heat dissipation in the bearing.

Solution : Consider an elementary area at radius *r* of the cone

$$dA = 2\pi r\,dr = 2\pi r \frac{dr}{\sin\theta}$$

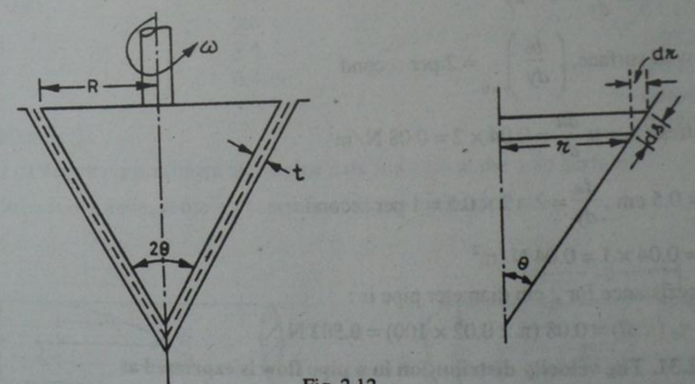


Fig. 2.12.

Shear stress $\tau = \mu \frac{du}{dy} = \mu \frac{u}{t}$

Shear force = shear stress × area of the element $= \mu \frac{u}{t}\left(2\pi r \frac{dr}{\sin\theta}\right)$

Viscous torque on the element, $dT = \mu \frac{u}{t}\left(2\pi r \frac{dr}{\sin\theta}\right) r$

Since the cone rotates with angular velocity ω rad/sec, the tangential velocity $u = \omega r$

$$dT = \mu \frac{\omega r}{t}\left(2\pi r \frac{dr}{\sin\theta}\right) r = \frac{2\pi\mu\omega}{t\sin\theta} r^3\, dr$$

Total viscous torque can be worked out by integrating from $r = 0$ to $r = R$

$$\text{Total torque } T = \frac{2\pi\mu\omega}{t\sin\theta}\int_0^R r^3\, dr = \frac{\pi\mu\omega}{2t\sin\theta} R^4$$

Power utilized in overcoming the viscous resistance,

$$P = T\omega = \frac{\pi\mu\omega^2}{2t\sin\theta} R^4$$

Example 2.30. The velocity distribution in a pipe line is prescribed by the relation :

$$u = 2y - y^2$$

where *u* denotes the velocity at a distance *y* from the solid boundary. Calculate

(i) **shear stress at the wall,**

(ii) **shear stress at 0.5 cm from the wall and**

(iii) **total resistance for a 2 cm diameter pipe over a length of 100 m.**

Assume coefficient of viscosity $\mu = 0.4$ poise.

Solution : $\mu = 0.4$ P $= 0.04$ N s/m^2

Velocity gradient $\frac{du}{dy} = 2 - 2y$

(i) At the wall surface, $\left(\frac{du}{dy}\right)_{y=0} = 2$ per second

$\therefore$ Shear stress $\tau_0 = \mu \frac{du}{dy} = 0.04 \times 2 = 0.08$ N/m^2

(ii) For $y = 0.5$ cm, $\frac{du}{dy} = 2 - 2 \times 0.5 = 1$ per second

$$\tau = 0.04 \times 1 = 0.04 \text{ N/m}^2$$

(iii) Total resistance for 2 cm diameter pipe is :

$$F = \tau_0 (\pi dl) = 0.08 (\pi \times 0.02 \times 100) = \mathbf{0.502\ N}$$

Example 2.31. The velocity distribution in a pipe flow is expressed as

$$\frac{v}{V} = 1 - \left(\frac{r}{R}\right)^2$$

where *V* represents the maximum velocity at the centre line, *v* the velocity at radial distance r from the pipe axis and *R* is the pipe radius. During a particular test-run, the maximum velocity of 10 m/s was recorded when a liquid of viscosity 2 centipoise was made to flow in a pipe of 50 cm radius. Calculate and draw the velocity and shear stress distribution for a particular cross-section. Also work

out the drag per km length of pipe, and the power required to overcome this drag and thus maintain the flow.

Solution : The prescribed velocity distribution is

$$v = V\left\{1-\left(\frac{r}{R}\right)^2\right\}$$

$$= 10\left\{1-\left(\frac{r}{0.5}\right)^2\right\} = 10 - 40\, r^2 \qquad ...(i)$$

and $$\frac{dv}{dr} = -40 \times 2r = -80\, r$$

From Newton's law of viscosity, $\tau = \mu \frac{du}{dy}$ where y is the distance from the wall. For a pipe of radius R,

$$y = R - r \text{ and } dy = -dr$$

$$\therefore \quad \tau = -\mu \frac{dv}{dr} = 80\, \mu\, r$$

$$\mu = 2 \text{ centipoise} = 0.02 \text{ poise} = 0.002 \text{ N s/m}^2$$

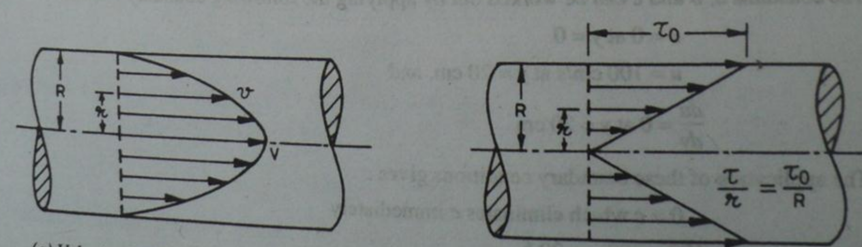
$$\tau = 80 \times 0.002\, r = 0.16\, r \qquad ...(ii)$$

The velocity and shear stress at various points at a particular cross-section can be calculated from expression *(i)* and *(ii)*

r (m)	$v = 10 - 40\, r^2$ (m/s)	$\tau = 0.16\, r$ (N/m^2)
0 (pipe axis)	10	0
0.1	9.6	0.016
0.2	8.4	0.032
0.3	6.4	0.048
0.4	3.6	0.064
0.5 (wall surface)	0	0.080

Remarks : *(i)* Velocity is maximum at the pipe axis and zero at the wall surface.

(ii) Shear stress is maximum at the wall surface and zero at the pipe axis.



(a) Velocity distribution

(b) Shear stress distribution

Fig. 2.13.

(iii) The velocity distribution is parabolic and the corresponding shear stress is linear over the cross-section. Such is the case for laminar flow through a pipe.

$$\text{Drag force } F_d = \tau_0 \times 2\,\pi\, Rl$$
$$= 0.08\,(2\,\pi \times 0.5 \times 1000) = \mathbf{251.2\ N}$$

For laminar flow, average flow velocity over the cross-section equals half the maximum velocity at the centre.

$$V_{av} = \frac{10}{2} = 5 \text{ m/s}$$

∴ Power required to overcome the drag, *i.e.*, to maintain the flow

$$= 251.2 \times 5 = 12560 \text{ Nm/s} = \mathbf{1.256\ kW}$$

Example 2.32. A fluid of absolute viscosity 8 poise flows past a flat plate and has a velocity 100 cm/s at the vertex which is at 20 cm from the plate surface. Make calculations for the velocity gradients and shear stress at points 5, 10 and 15 cm from the boundary. Assume *(i)* a straight line velocity distribution and *(ii)* a parabolic velocity distribution. Comment upon the results.

Solution : $\mu = 8$ poise $= 0.8$ N s/m^2

(i) For a straight line velocity distribution, the velocity gradient at the boundary *i.e.*, at $y = 0$ is

$$\frac{du}{dy} = \frac{100 - 0}{20 - 0} = 5 \text{ per sec}$$

$$\text{Shear stress } \tau = \mu \frac{du}{dy} = 0.8 \times 5 = 4 \text{ N/m}^2$$

For $y = 5$, 10 and 15 cm the velocity gradient $\frac{du}{dy}$ and the shear stress τ would be also 5 per sec and 4 N/m^2.

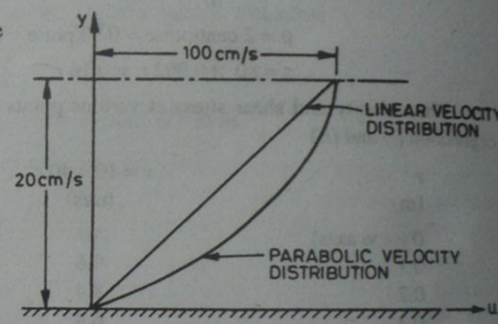


Fig. 2.14.

(ii) The parabolic velocity distribution can be prescribed by the relation

$$u = ay^2 + by + c$$

and
$$\frac{du}{dy} = 2\,ay + b$$

The constants *a*, *b* and *c* can be worked out by applying the following boundary conditions :

$$u = 0 \text{ at } y = 0$$
$$u = 100 \text{ cm/s at } y = 20 \text{ cm, and}$$
$$\frac{du}{dy} = 0 \text{ at } y = 20 \text{ cm}$$

The application of these boundary conditions gives :

$$0 = c \text{ which eliminates } c \text{ immediately}$$
$$100 = 400\,a + 20\,b$$
$$0 = 40\,a + b$$

The last two equalities are solved for *a* and *b*, and we obtain

$$a = -0.25 \text{ and } b = 10$$

That gives the velocity profile as

$$u = -0.25\, y^2 + 10\, y$$

and $$\frac{du}{dy} = -0.5\, y + 10$$

It is to be noted that in these expressions, the velocity is in cm/s and distance y is in cm.

The velocity gradients and shear stresses at the specified points are as given below :

Location	Velocity gradients	Shear stress
(i) $y = 5$ cm	$\frac{du}{dy} = -0.5 \times 5 + 10$ $= 7.5$ per sec	$\tau = \mu\, du/dy$ $= 0.8 \times 7.5 = 6 \text{ N/m}^2$
(ii) $y = 10$ cm	$\frac{du}{dy} = -0.5 \times 10 + 10$ $= 5$ per sec	$\tau = 0.8 \times 5 = 4 \text{ N/m}^2$
(iii) $y = 15$ cm	$\frac{du}{dy} = -0.5 \times 15 + 10$ $= 2.5$ per sec	$\tau = 0.8 \times 2.5 = 2 \text{ N/m}^2$
(iv) $y = 0$ (*i.e.*, at the boundary)	$\frac{du}{dy} = -0.5 \times 0 + 10$ $= 10$ per sec	$\tau = 0.8 \times 10 = 8 \text{ N/m}^2$

Comments : With parabolic velocity distribution, the velocity gradient becomes less steep (*du/dy* becomes small) with distance from the boundary. Consequently maximum value of shear stress occurs at the boundary and it progressively decreases with distance from the boundary.

Example 2.33. A flat plate 0.3 m^2 in area moves edgewise through oil between large fixed parallel planes 10 cm apart. If the velocity of plate is 0.6 m/s and the oil has a kinematic viscosity of 0.45×10^{-4} m^2/s and specific gravity 0.8, calculate the drag force when *(i)* the plate is 2.5 cm from one of the planes *(ii)* the plate is equidistant from both the planes.

Solution : Dynamic viscosity of oil,

$$\mu = \rho\, v = (0.8 \times 1000) \times (0.45 \times 10^{-4}) = 0.036 \text{ N s/m}^2$$

Let t_1 and t_2 represent the distances of the flat plate from the plane horizontal surfaces. Then the shear stresses on the two sides of the plate are :

$$\tau_1 = \mu \frac{du}{dy} = \mu \frac{V}{t_1} \quad \text{and} \quad \tau_2 = \mu \frac{V}{t_2}$$

Drag force or viscous resistance against the motion of plate

$$F = \text{shear stress} \times \text{area}$$

$$= \left(\mu \frac{V}{t_1} + \mu \frac{V}{t_2} \right) A = \mu A V \left[\frac{1}{t_1} + \frac{1}{t_2} \right]$$

(i) When $t_1 = 2.5$ cm and $t_2 = 7.5$ cm

Drag force $F = 0.036 \times 0.3 \times 0.6 \left[\frac{1}{0.025} + \frac{1}{0.075} \right] = \mathbf{0.345\ N}$

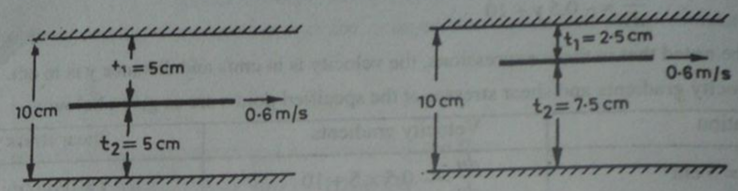


Fig. 2.15.

(ii) When the plate is placed mid way

$$t_1 = t_2 = 0.05 \text{ m}$$

$$\text{Drag force} \quad F = 0.036 \times 0.3 \times 0.6 \left[\frac{1}{0.05} + \frac{1}{0.05} \right] = \mathbf{0.259\ N}$$

Example 2.34. A thin plate of area A is placed mid way in a gap of height h between two horizontal plane surfaces. The gap is filled with oil of viscosity μ_1 and the plate is pulled edgewise with a constant velocity V. The gap is next filled with a lighter oil of viscosity μ_2 and the plate is located unsymmetrically in the gap but parallel to the walls. Experiments indicate that for the same velocity V, the force required was same. Establish a relation for μ_2 in terms of μ_1 and the distance from the near wall to the plate.

Solution : Let t_1 and t_2 represent the distances of the flat plate from the plane horizontal surfaces. Then the shear stresses on the two sides of the plate are :

$$\tau_1 = \mu \frac{du}{dy} = \mu \frac{V}{t_1} \quad \text{and} \quad \tau_2 = \mu \frac{V}{t_2}$$

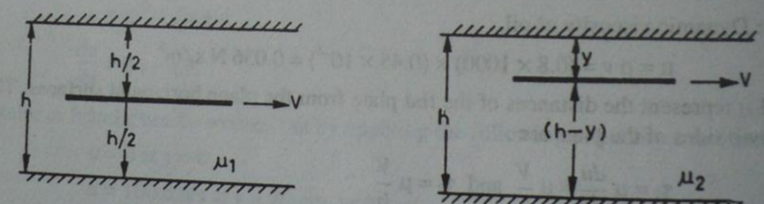


Fig. 2.16.

Drag force or viscous resistance against the motion of plate,

$$F = \text{shear stress} \times \text{area}$$

$$= \left(\mu \frac{V}{t_1} + \mu \frac{V}{t_2} \right) A = \mu A V \left[\frac{1}{t_1} + \frac{1}{t_2} \right]$$

Case *(a)* Liquid of viscosity μ_1 fills the gap and the plate is placed mid-way. That is

$$t_1 = t_2 = \frac{h}{2}$$

$$\therefore \text{ Drag force } F_1 = \mu_1 AV\left[\frac{1}{h/2}+\frac{1}{h/2}\right] = \frac{4\,\mu_1 AV}{h}$$

Case *(b)* Liquid of viscosity μ_2 fills the gap and the plate is placed unsymmetrically. Let,

$$t_1 = y \text{ and } t_2 = (h-y)$$

$$\therefore \text{ Drag force } F_2 = \mu_2 AV\left[\frac{1}{y}+\frac{1}{h-y}\right] = \frac{\mu_2 AV}{y\,(h-y)}$$

Since the drag forces are equal, we have :

$$4\,\frac{\mu_1 AV}{h} = \frac{\mu_2 AVh}{y\,(h-y)}$$

or
$$\mu_2 = 4\,\mu_1\,\frac{y\,(h-y)}{h^2} = 4\,\mu_1\,\frac{y}{h}\left(1-\frac{y}{h}\right)$$

Example 2.35. A thin plate of very large area is placed in a gap of height *h* with oils of viscosities μ_1 and μ_2 on the two sides of the plate. The plate is pulled at a constant velocity *V*. Calculate the position of plate so that *(a)* the shear force on the two sides of the plate is equal, *(b)* the force required to drag the plate is minimum. Assume viscous flow and neglect all end effects.

Solution : Let *y* be the distance of thin plate from one of the surfaces of the gap.

Force on upper side of plate,

$$F_u = \mu\,\frac{du}{dy}\times A = \mu_1\,\frac{V}{h-y}\,A$$

Force on lower side of plate,

$$F_l = \mu_2\,\frac{V}{y}\,A$$

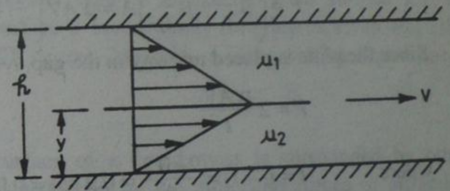


Fig. 2.17.

(a) Since the forces on the two sides of the plate are stated to be equal, we have

$$\mu_1\,\frac{V}{h-y}\,A = \mu_2\,\frac{V}{y}\,A$$

$$\frac{\mu_1}{h-y} = \frac{\mu_2}{y} \quad \text{or} \quad \mu_1\,y = \mu_2\,h - \mu_2\,y$$

or
$$y = \frac{\mu_2\,h}{\mu_1+\mu_2}$$

(b) Total drag force = sum of forces on the upper and lower surfaces of the plate

$$F = \mu_1\,\frac{V}{h-y}\,A + \mu_2\,\frac{V}{y}\,A$$

For the drag force to be minimum, $\dfrac{dF}{dy} = 0$

That is
$$\frac{\mu_1\,VA}{(h-y)^2} - \frac{\mu_2\,VA}{y^2} = 0$$

or $$\frac{\mu_1}{\mu_2}=\frac{(h-y)^2}{y^2}=\frac{h^2+y^2-2hy}{y^2}=\frac{h^2}{y^2}+1-\frac{2h}{y}$$

$$\therefore \quad \frac{h^2}{y^2}-2\frac{h}{y}+\left(1-\frac{\mu_1}{\mu_2}\right)=0$$

Solving this quadratic equation for $\frac{h}{y}$, we get

$$\frac{h}{y}=2\pm\frac{\sqrt{4-4(1-\mu_1/\mu_2)^2}}{2}=1\pm\sqrt{\mu_1/\mu_2}$$

Since $\frac{h}{y}$ cannot be less than unity,

$$\frac{h}{y}=1+\sqrt{\mu_1/\mu_2} \text{ or } y=\frac{h}{1+\sqrt{\mu_1/\mu_2}}$$

Example 2.36. A 2 cm wide gap between two vertical plane surfaces is filled with an oil of specific gravity 0.85 and dynamic viscosity 2.5 N s/m². A metal plate 1.25 m × 1.25 m × 0.2 cm thick and weighing 30 N is placed midway in the gap. Find the force required if the plate is to be lifted up with a constant velocity of 0.12 m/s.

Solution : The shear stresses on two sides of the plate are :

$$\tau_1=\mu\frac{du}{dy}=\mu\frac{V}{t_1} \text{ and } \tau_2=\mu\frac{V}{t_2}$$

Drag force or viscous resistance against the motion of plate,

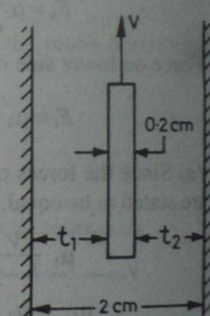


Fig. 2.18.

$$F=\left(\mu\frac{V}{t_1}+\mu\frac{V}{t_2}\right)A=\mu AV\left[\frac{1}{t_1}+\frac{1}{t_2}\right]$$

Since the plate is placed midway in the gap, $t_1=t_2$ and therefore,

$$F=2\frac{\mu AV}{t}$$

where $$t=t_1=t_2=\frac{2-0.2}{2}=0.9 \text{ cm}=0.009 \text{ m}$$

$$\therefore \quad F=\frac{2\times2.5\times(1.25\times1.25)\times0.12}{0.009}=104.17 \text{ N}$$

Upthrust or buoyant force on the plate

= specific weight × volume of oil displaced

= (0.85 × 9810) × (1.25 × 1.25 × 0.002) = 26.06 N

Effective weight of the plate = 30 – 26.06 = 3.94 N

∴ Total force required to lift the plate at the given velocity,

= 104.17 + 3.94 = **108.11 N**

. SURFACE TENSION AND CAPILLARITY

1. Cohesion and Adhesion

Liquids have characteristic properties of cohesion and adhesion. *Cohesion* refers to the force with which the neighbouring or adjacent fluid molecules are attracted towards each other. *Adhesion* represents the adhering or clinging of the fluid molecules to the solid surface with which they come in contact. In brief, forces between like molecules are cohesive and the forces between unlike molecules are adhesive. When a liquid like, mercury, is spilled on a smooth horizontal surface, it tends to gather into droplets because the cohesive molecular forces are greater than the adhesive forces between the mercury molecules and the material of the surface. Mercury tends to stay away from the surface and is said to be a *not-wetting* liquid. In case of water, adhesive forces are greater than cohesive forces. Naturally when water is poured on the same smooth horizontal surface, it would spread out and wet the horizontal surface. The wetting and non-wetting of the surface is dictated by the angle of contact between the liquid and the surface material.

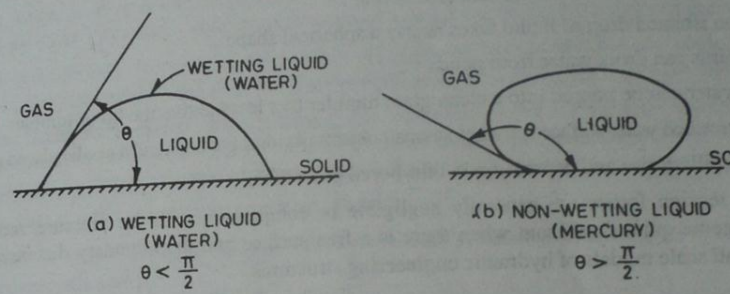


Fig. 2.19. *Wetting and non-wetting liquids*

Refer to Fig. 2.19 which illustrates the liquid-gas interface with a solid surface. Liquid would wet the surface when $\theta < \pi/2$ and the degree of wetting increases as θ decreases to zero. For a non- wetting liquid $\theta > \pi/2$. The contact angle is dependent on the nature and type of liquid, the solid surface and its cleanliness. For pure water in contact with a clean glass surface θ is essentially zero degree. Even when the water is slightly contaminated, θ becomes as high as 25 degree. Mercury, a non- wetting liquid has θ between 130 to 150 degree.

2.7.2. Surface Tension

A liquid molecule lying well beneath the free surface of a liquid mass is surrounded by other molecules all around it. Consequently the molecule is acted upon by the molecular forces of attraction

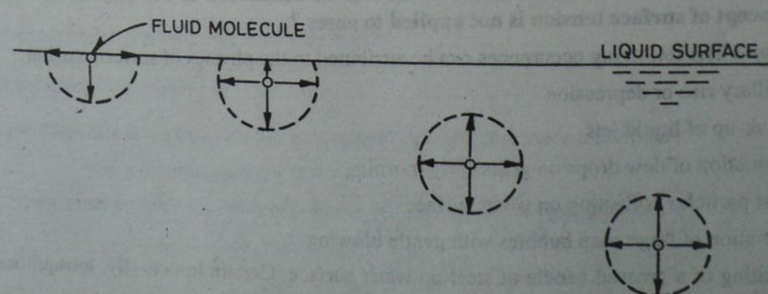


Fig. 2.20. *Forces of attraction on a liquid molecule*

(cohesion) that are equal in all directions. These equal and opposite forces cancel out; there is no resultant force acting upon the molecule within the fluid mass and this aspect keeps the liquid mass in equilibrium. Never-the-less, a liquid molecule at the free surface has no liquid molecules above it to counteract the forces due to molecules below it. Consequently as depicted in the molecular arrangement of Fig. 2.20, the molecules lying at the surface have a net attraction tending to pull them into the interior of the liquid mass. A quantum of energy/work is thus expended to bring the molecule to the free liquid surface which then acts like an elastic or stretched membrane. Energy expended per unit area of the surface is called *surface tension*; designated by sigma σ. Surface tension occurs at the interface of a liquid and a gas or at the interface of two liquids, and is essentially due to inter-molecular forces of cohesion.

It is primarily due to surface tension effects that :

— an isolated drop of liquid takes nearly a spherical shape
— birds can drink water from ponds
— water can be poured into a clean glass tumbler to a level above the lip of tumbler
— stretched water surface can support small objects like dust particles and a needle placed gently upon it
— capillary rise and depression in thin-bored glass tubes

Surface tension forces are generally negligible in comparison with the pressure and gravitational forces, but become quite significant when there is a free surface and the boundary dimensions are small, *e.g.*, in the small scale models of hydraulic engineering structures.

The dimensional formula for surface tension is $\left[\frac{F}{L}\right]$ or $\left[\frac{M}{T^2}\right]$; it is usually expressed in N/m. The value of surface tension depends upon *(i)* nature of the liquid, *(ii)* nature of the surrounding matter which may be a solid, liquid or a gas, *(iii)* kinetic energy and hence the temperature of liquid molecules. Growth in temperature results in a reduction of the inter-molecular cohesive forces and hence in a reduction of the surface tension force. At a critical point where the liquid and vapour phase become indistinguishable, the surface tension becomes zero. Surface tension values for liquids are generally quoted when in contact with air as the surrounding medium.

$\sigma = 0.073$ N/m for air-water interface

$\sigma = 0.480$ N/m for air-mercury interface

The surface tension values drop with rise in temperature.

Example 2.37. List some occurences which can be attributed to the physics of surface tension. Why the concept of surface tension is not applied to gases ?

Solution : The following occurences can be attributed to the physics of surface tension :

(i) capillary rise or depression

(ii) break-up of liquid jets

(iii) formation of dew drops on grass early morning

(iv) dust particles collecting on water surface

(v) formation of large soap bubbles with gentle blowing

(vi) floating of a greased needle of steel on water surface. Certain insects (fly, mosquitoes) can creep freely on the water surface.

(vii) spherical shape of a droplet of liquid. When a molten metal is poured into water from a suitable

height, the falling stream of molten metal breaks up and the detached portions acquire spherical shape. This technique is used for preparing lead shots and glass marbles.

Small drops of mercury are always spherical, but larger ones are somewhat flattened. Shape of the drop is governed by the combined influence of the surface tension and the gravitational force due to weight. For tiny drop, the gravitational force in negligible and surface tension makes the drop spherical. However in case of larger drops, force of gravity is appreciable and tries to flatten the drop so as to lower the position of centre of gravity. Eventually the drop acquires an oval or elliptical shape.

Rain drops, tiny dew drops and small water drops ejected from a burette are always spherical.

(viii) On adding soap or detergent powder into water, its surface tension is lowered considerably. This soap-water solution can more easily seep into the pores of clothes and remove the dirt and grease etc.

(ix) Antiseptic creams are generally prepared in oil or greasy base of low surface tension so that even small amount of cream spreads on the whole of cut (wound) and prevents oozing out of the blood.

For gases, the inter-molecular distance among gas molecules is very large and consequently there is no appreciable force of cohesion and as such the characteristic property of surface tension is not exhibited/manifested by gases even though a gas is also a fluid.

2.7.3. Pressure Inside a Water Droplet and Soap Bubble

Due to surface tension acting at the interface, the pressure p_i inside a small droplet or bubble becomes greater than ambient pressure p_0.

(a) For a liquid droplet : Consider a small spherical droplet of liquid (say a rain drop) of diameter d and let it be cut into two halves.

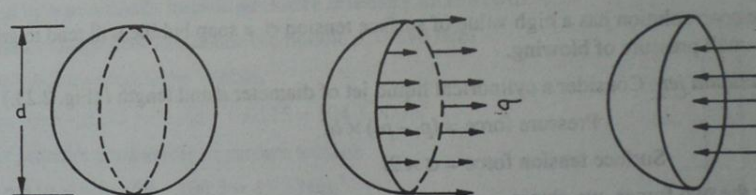


Fig. 2.21.

The forces acting on one half (say left half) will be

(i) pressure force $= (p_i - p_0)\,\frac{\pi}{4}\,d^2$

(ii) tensile force due to surface tension acts around the circumference and equals

$$= \sigma \times \text{circumference} = \sigma \times \pi\, d$$

Under equilibrium conditions, these two forces will be equal and opposite, *i.e.*,

$$(p_i - p_0)\,\frac{\pi}{4}\,d^2 = \sigma \times \pi\, d$$

$$(p_i - p_0) = \frac{\sigma \times \pi\, d}{\frac{\pi}{4}\,d^2} = \frac{4\,\sigma}{d} \qquad \text{...(2.23)}$$

Evidently the pressure within a liquid droplet varies inversely as its diameter ; pressure intensity decreases with an increase in the size of the droplet.

(b) For a soap bubble : A soap bubble has two surfaces in contact with air ; one inside and other outside. Surface tension force will act on both the surfaces and accordingly

$$(p_i - p_0)\ \frac{\pi}{4} d^2 = 2\,(\sigma \times \pi\, d)$$

$$(p_i - p_0) = \frac{2\,(\sigma \times \pi\, d)}{\frac{\pi}{4} d^2} = \frac{8\,\sigma}{d} \qquad ...(2.24)$$

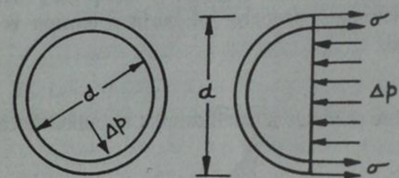


Fig. 2.22.

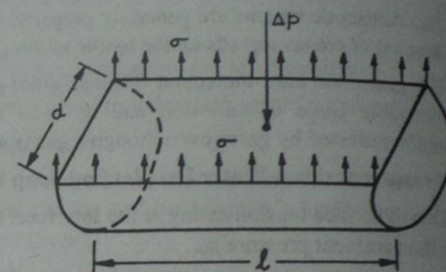


Fig. 2.23.

Since the soap solution has a high value of surface tension σ, a soap bubble will tend to grow larger in diameter with small pressure of blowing.

(c) For a liquid jet : Consider a cylindrical liquid jet of diameter d and length l (Fig. 2.23.)

$$\text{Pressure force} = (p_i - p_0) \times ld$$

$$\text{Surface tension force} = \sigma \times 2l$$

Equating the two forces, we obtain

$$(p_i - p_0) \times ld = \sigma \times 2l$$

$$(p_i - p_0) = \frac{\sigma \times 2l}{ld} = \frac{2\,\sigma}{d} \qquad ...(2.25)$$

Example 2.38. Set up a relationship between surface tension and pressure intensity (in excess of outside pressure) for a liquid droplet. How does the relationship differ from that of a hollow soap bubble ?

The pressure inside a soap bubble of 50 mm diameter is 2.5 N/m² above the atmosphere. Estimate the surface tension of the soap film.

Solution : For a soap bubble, the pressure in excess of outside pressure is given by :

$$(p_i - p_0) = \frac{8\,\sigma}{d}\ ;\ 2.5 = \frac{8\,\sigma}{50 \times 10^{-3}}$$

$\therefore$ Surface tension $\sigma = \frac{2.5 \times (50 \times 10^{-3})}{8} = \mathbf{0.01562\ N/m}$

Example 2.39. Air is introduced through a nozzle into a tank of water to form a stream of bubbles. If the bubbles are intended to have a diameter of 2 mm, calculate by how much the pressure of the air at the nozzle must exceed that of the surrounding water. Assume that surface tension of water is 0.073 N/m. What would be the absolute pressure inside the bubble if the surrounding water is at 100 kPa ?

Solution : For the water bubbles (droplets), the excess pressure intensity of air over that of surrounding water is

$$p_i - p_0 = \frac{4\sigma}{d} = \frac{4 \times 0.073}{2 \times 10^{-3}} = 146\ \text{N/m}^2$$

Pressure outside the droplet $p_0 = 100$ kPa $= 100000$ N/m^2

$\therefore$ pressure inside the droplet $p_i = 146 + 100000$

$= 100146$ N/m^2 = **100.146 kPa** (absolute)

Example 2.40. (*i*) Some insects can walk on water. Why ?

(*ii*) Soap bubbles rise up in air. Why ? Will the pressure within the soap bubble be more than the surrounding atmosphere.

In measuring the unit energy of a mineral oil (specific gravity = 0.85) by the bubble method, a tube having an internal diameter of 1.5 mm is immersed to a depth of 1.25 cm in the oil. Air is forced through the tube forming a bubble at the lower end. What magnitude of the unit surface energy will be indicated by a maximum bubble pressure intensity of 150 N/m² ?

Solution : Gauge pressure inside the bubble $p_i = 150$ N/m^2

Gauge pressure outside the bubble,

$$p_0 = wh = (0.85 \times 9810) \times 0.0125 = 104.3\ \text{N/m}^2$$

$\therefore$ Net pressure attributable to surface tension

$$p = (p_i - p_0) = 150 - 104.3 = 45.7\ \text{N/m}^2$$

Recollecting that for a bubble,

$$(p_i - p_0) = \frac{4\sigma}{d}$$

Taking diameter of bubble equal to that of the tube,

$$45.77 = \frac{4\sigma}{0.0015}\ ;\ \sigma = \mathbf{0.0172\ N/m}$$

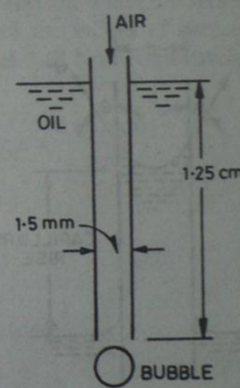


Fig. 2.24.

2.7.4. Capillary or Meniscus Effect

When a small diameter glass tube, called the capillary tube is dipped into a water container, water rises in the tube to a level that stands higher than the level of water in the container. Conversely the surface of mercury is depressed down in the capillary tubing when it is dipped in mercury. The phenomenon of liquid rise or fall in a capillary tube is called the *capillary or meniscus effect.* Capillary is a surface tension effect that depends upon the relative inter-molecular attraction between different substances; it is due to both cohesion and adhesion.

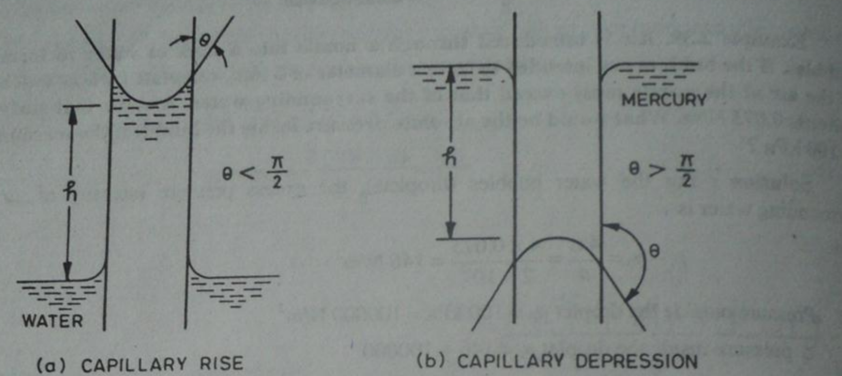


Fig. 2.25. *Capillary rise and depression*

Adhesion between glass and water molecules is greater than cohesion between water molecules. Consequently the water molecules spread over the glass surface and form a concave meniscus with small angle of contact. Opposite conditions hold good for mercury *i.e.*, cohesion between mercury molecules is greater than adhesion of mercury to glass. Mercury then displays a convex meniscus with the angle of contact greater than 90-degree.

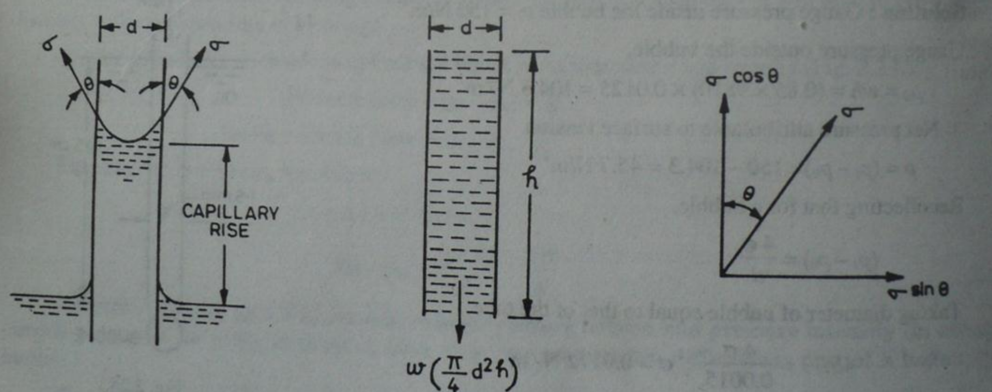


Fig. 2.26. *Rise or depression of liquid in a capillary tube*

Knowing the surface tension σ, angle of contact θ, tube diameter *d* and specific weight of liquid *w*, the rise (for water) or depression (for mercury) of the liquid in the capillary tube can be worked out by the following analysis :

Weight of liquid raised or lowered in the capillary tube

$$= \text{(area of tube} \times \text{ rise or fall)} \times \text{specific weight}$$

$$= \left(\frac{\pi}{4} d^2 h \right) w$$

Vertical component of surface tension force

$$= \sigma \cos \theta \times \text{circumference}$$

$$= \sigma \cos \theta \times \pi d = \pi d \sigma \cos \theta$$

When in equilibrium, the downward weight of the liquid column h is balanced by the vertical component of the force of surface tension.

Hence $$\frac{\pi}{4} d^2 h w = \pi d \sigma \cos \theta$$

or $$h = \frac{4 \sigma \cos \theta}{wd} \quad \text{...(2.26)}$$

It is to be noticed that for $0 \leq \theta < 90^\circ$, h is positive (concave meniscus and capillary rise) and that for $90 \leq \theta < 180^\circ$, h is negative (convex meniscus and capillary depression).

Evidently the capillary action is inversely proportional to the tube diameter. For precise work, the small diameter tubes are to be avoided ; recommended minimum tube diameter for water and mercury is 6 mm. Further since the presence of dirt affects the surface tension and hence the capillary rise or depression, the interior surface of the tube is to be kept clean.

Example 2.41. Explain why in a capillary tube, the meniscus of water is concave upwards while the meniscus of mercury is convex upwards.

Calculate the capillary effects in millimeters in a glass tube of 4 mm diameter, when immersed in *(i)* water and *(ii)* in mercury. The temperature of the liquid is 20°C and the values of surface tension of water and mercury at 20°C in contact with air are 0.0735 N/m and 0.48 N/m respectively. The contact angle of water $\theta = 0^\circ$ and for mercury $\theta = 130^\circ$.

Solution : The rise or depression h of a liquid in a capillary tube is given by : $h = \dfrac{4 \sigma \cos \theta}{wd}$.

Case (i) : *Capillary effect in water :*

$$\sigma = 0.0735 \text{ N/m} \; ; \text{ angle of contact } \theta = 0$$

$$w = 9800 \text{ N/m}^3 \text{ at } 20^\circ\text{C (say)}$$

$$\therefore \quad h = \frac{4 \times 0.0735 \times \cos 0^\circ}{9800 \times 0.004} = 7.50 \times 10^{-3} \text{ m} = \mathbf{7.50 \text{ mm}} \text{ (rise)}$$

Case (ii) : *Capillary effect in mercury :*

$$\sigma = 0.48 \text{ N/m} \; ; \text{ angle of contact } \theta = 130^\circ$$

$$w = (9800 \times 13.6) \text{ N/m}^3$$

$$\therefore \quad h = \frac{4 \times 0.48 \times \cos 130^\circ}{(9800 \times 13.6) \times 0.004} = -2.31 \times 10^{-3} \text{ m} = \mathbf{2.31 \text{ mm}} \text{ (depression).}$$

Example 2.42. Why should a mercury column in a thin glass tube be depressed while a water column be lifted up ?

Calculate the size of glass tube if the capillary rise is limited to 2.2 mm of water. Assume

suitable values of required data at 20°C and 1 atm pressure.

Solution : Capillary rise $h = 2.2 \text{ mm} = 2.2 \times 10^{-3}$ m

The rise or depression h of a liquid in capillary tubing is given by

$$h = \frac{4\,\sigma \cos\theta}{wd}$$

The most appropriate values of desired data at 20°C and 1 atm pressure are :

Surface tension $\sigma = 0.073$ N/m for air water interface

Angle of contact $\theta = 0$ degree for water-clean glass surface

Density of water $\rho = 998$ kg/m^3

$$\therefore \quad 2.2 \times 10^{-3} = \frac{4 \times 0.073 \times \cos 0^\circ}{(998 \times 9.81) \times d} \; ; \; d \approx 0.0136 \text{ m}$$

Thus minimum diameter of tube should be **1.36 cm**

Example 2.43. Find the smallest diameter of a manometer tube such that error due to capillary action in the measured gauge pressure of 100 N/m² is less than 5 percent. The manometric liquid is water.

Solution : For water $\sigma = 0.073$ N/m , $\theta = 0^\circ$ and $w = 9810$ N/m^3

From hydrostatic equation, $p = wH$, the gauge pressure in terms of height of water column is

$$H = \frac{p}{w} = \frac{100}{9810} = 0.01019 \text{ m}$$

The rise of water in a capillary tube is given by,

$$h = \frac{4\,\sigma \cos\theta}{wd} = \frac{4 \times 0.073 \times \cos 0^\circ}{9810 \times d} = \frac{2.976 \times 10^{-5}}{d}$$

$$\text{Percentage error} = \frac{2.976 \times 10^{-5}}{d \times 0.01019} \times 100 \le 5$$

$$\therefore \quad d = \frac{2.976 \times 10^{-5}}{5 \times 0.01019} \times 100 = 0.0584 \text{ m} \approx 5.84 \text{ cm}$$

The minimum diameter of the manometer tube should be **5.84 cm**

Example 2.44. ***(a)*** **Water peeps up through sand columns. Why ?**

Estimate the height to which water would rise in a clay soil of average grain diameter 0.06 mm. It may be assumed that surface tension at air-water interface is 0.0735 N/m and interspaces in clay are of size equal to one-fifth of mean diameter of clay grain. Take angle of contact $\theta = 0$–degree .

(b) **What should be the average diameter of capillary tubes in a tree if the sap is carried to a height of 8 m ? It may be assumed that sap in trees has the same characteristics as water and it rises purely due to capillary phenomenon.**

Solution : The rise of liquid in a capillary tubing is given by;

$$h = \frac{4\,\sigma \cos\theta}{wd}$$

(a) Size of interspaces in clay, $d = 1/5$ of soil grain diameter = 0.012 mm = 0.012×10^{-3} m surface tension $\sigma = 0.0735$N/m

Angle of contact = 0-degree

$$\therefore \quad h = \frac{4 \times 0.0735 \times \cos 0^{o}}{9810 \times (0.012 \times 10^{-3})} \approx 2.5 \text{ m}$$

(b) Sap is stated to have the characteristics of water

$$\sigma = 0.0735 \text{ N/m} \;;\; \theta = 0\text{–degree}$$

$$\therefore \quad d = \frac{4\,\sigma \cos\theta}{wh} = \frac{4 \times 0.0735 \times \cos 0^{o}}{9810 \times 8} = 3.746 \times 10^{-6} \text{ m} = \mathbf{3.746 \times 10^{-3} \text{ mm}}$$

Example 2.45. A U-tube is made up of two capillaries of bore 1 mm and 2 mm respectively. The tube is held vertically and is partially filled with liquid of surface tension 0.05 N/m and zero contact angle. Calculate the mass density of the liquid if the estimated difference in the level of the two menisci is 1.25 cm.

Solution : Let h_1 and h_2 be the heights of liquid columns in the two limbs of bore d_1 and d_2 respectively. Then

$$h_1 = \frac{4\,\sigma\cos\theta}{wd_1} = \frac{4\,\sigma}{wd_1} \;;\quad h_2 = \frac{4\,\sigma}{wd_2}$$

$$h_1 - h_2 = \frac{4\,\sigma}{w}\left[\frac{1}{d_1} - \frac{1}{d_2}\right] = \frac{4\,\sigma}{\rho\, g}\left[\frac{1}{d_1} - \frac{1}{d_2}\right]$$

Substituting $h_1 - h_2 = 0.0125$ m ; $d_1 = 0.001$ m ; $d_2 = 0.002$ m and $\sigma = 0.05$ N/m

$$0.0125 = \frac{4 \times 0.05}{\rho \times 9.807}\left[\frac{1}{0.001} - \frac{1}{0.002}\right] \;;\quad \rho = 816 \text{ kg/m}^3$$

Hence mass density $\rho = \mathbf{816 \text{ kg/m}^3}$

Example 2.46. A single column U-tube manometer, made of glass tubing having a nominal inside diameter of 2.5 mm, has been used to measure pressure in a pipe or vessel containing air. If the limb opened to atmosphere is 10 percent oversize, find the error in mm of mercury in the measurement of air pressure due to surface tension effects. It is stated that mercury is the manometric fluid for which surface tension $\sigma = 0.514$ N/m and angle of contact $\alpha = 140$ degree.

Solution : The surface tension manifests the phenomenon of capillary action due to which rise or depression of manometric liquid in a tube is given by

$$h = \frac{4\,\sigma\cos\theta}{wd}$$

For the given case : $d_1 = 2.5$ mm

$$d_2 = 2.5 \times 1.1 = 2.75 \text{ mm}$$

$$\therefore \quad h_1 = \frac{4 \times 0.514 \times \cos 140^{o}}{(9810 \times 13.6) \times (2.5 \times 10^{-3})} = 4.72 \times 10^{-3} \text{ m}$$

$$h_2 = \frac{4 \times 0.514 \times \cos 140^{o}}{(9810 \times 13.6) \times 92.75 \times 10^{-3})} = 4.29 \times 10^{-3} \text{ m}$$

Hence, error in measurement due to surface tension effects

$$= (4.72 - 4.29) \times 10^{-3} = 0.43 \times 10^{-3} \text{ m} = \mathbf{0.43 \text{ mm}}$$

Example 2.47. How the rise of liquid level in a capillary tube gets affected when *(i)* the tube is of insufficient length, *(ii)* the top of tube is closed.

Glass tubing to be used in a different U-tube manometer is not of uniform diameter and the two limbs are stated to be 4 mm and 5 mm in diameter. Assuming that the manometer is to be used to measure readings in the range of 25 mm to 150 mm, make calculations for the percentage error that can creep in the highest and lowest readings. Comment on your findings.

Take surface tension $\sigma = 0.0735$ N/m and angle of contact $\theta = 0$-degree.

Solution : The radius of curvature R of the meniscus and the radius r of the capillary tube are related by the expression

$$r = R \cos \theta \quad \text{or} \quad R = \frac{r}{\cos \theta}$$

Height to which liquid would rise in the capillary tube is

$$h = \frac{4\,\sigma \cos \theta}{wd} = \frac{4\,\sigma}{wd} = \frac{2\,\sigma}{wr} \qquad \text{...}(i)$$

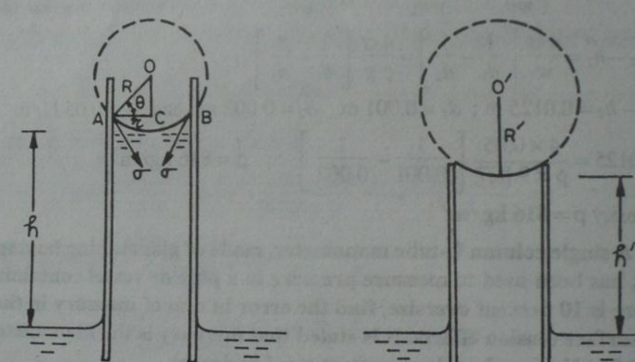


Fig. 2.27.

If the length of capillary tube h' is less than the expected height h, the meniscus will change its radius of curvature to R' such that

$$h' = \frac{2\,\sigma}{w\,R'} \qquad \text{...}(ii)$$

From expressions *(i)* and *(ii)*

$$Rh = R'\,h'$$

Evidently if the tube is of insufficient length, the radius of curvature of the liquid menisus will go on increasing making it more and more flat till liquid may be in equilibrium.

(ii) Air enclosed above the liquid in a closed capillary tube would exert downward pressure, and

accordingly the rise of liquid level in it would be less than the normal rise.

(b) The height to which a liquid rises or depresses in a capillary tube is given by

$$h = \frac{4\,\sigma \cos\theta}{wd}$$

In 4 mm limb : $h_1 = \dfrac{4 \times 0.0735 \cos 0^\circ}{9810 \times (4 \times 10^{-3})} = 0.0075$ m

In 5 mm limb : $h_2 = \dfrac{4 \times 0.0735 \cos 0^\circ}{9810 \times (5 \times 10^{-3})} = 0.006$ m

∴ Error in pressure measurement due to capillary action,

$$= 0.0075 - 0.006 = 0.0015 \text{ m} = 1.5 \text{ mm}$$

Percentage error :

at the highest reading $= \dfrac{1.5}{150} \times 100 = \mathbf{1\%}$

at the lowest reading $= \dfrac{1.5}{25} \times 100 = \mathbf{6\%}$

Apparently the capillary error is relatively much higher at the low pressure readings.

Example 2.48. Develop a formula for capillary rise of a fluid having surface tension σ and contact angle θ between :

(i) two concentric glass tubes of radii r_0 and r_i

(ii) two vertical glass plates set parallel to each other and having a gap t between them.

Solution : (i) Equating the weight of water risen to the vertical component of surface tension force,

$$wh\,\pi\,(r_0^2 - r_i^2) = \sigma \cos\theta \times 2\,\pi\,(r_0 + r_i)$$

$$\therefore \quad h = \frac{\sigma \cos\theta \times 2\,\pi\,(r_0 + r_i)}{w\,\pi\,(r_0^2 - r_i^2)} = \frac{2\,\sigma \cos\theta}{w\,(r_0 - r_i)}$$

(ii) When in equilibrium, the downward weight of the liquid column h is balanced by the vertical component of the force of surface tension. That is

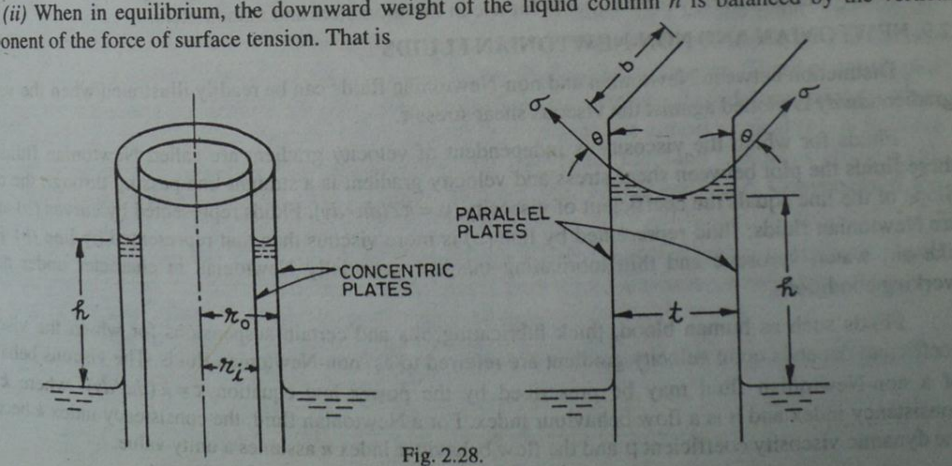


Fig. 2.28.

$$whbt = \sigma \cos \theta \times 2b$$

$$\therefore \quad h = \frac{\sigma \cos \theta \times 2b}{wbt} = \frac{2\sigma \cos \theta}{wt}$$

2.8. EVAPORABILITY AND VAPOUR PRESSURE

Evaporability means a change from liquid to gaseous phase. The evaporation rate varies for different liquids and depends upon the prevailing pressure and temperature conditions.

Consider a liquid enclosed in a closed space. The liquid molecules having high energy leave the liquid space in the vapour state. The vapour molecules are in gaseous nature and exert their own pressure, called *vapour pressure*. Some of the ejected molecules have a tendency to rebound back and get absorbed in the liquid surface. Thus there may be an interchange of molecules between the liquid and the gaseous space above it. The vapour pressure will have a constant value when the molecules leave and enter the liquid at the same rate. The constant vapour pressure is called the *saturated vapour pressure*. Further evaporation of the liquid ceases, once the saturated vapour pressure has been reached. Never-the-less if the vapour pressure above the liquid surface gets reduced, say by evacuation, re-evaporation starts and continues until new equilibrium conditions are attained. If the vapour pressure falls considerably, then the molecules leave the liquid surface very rapidly and this phenomenon is called *boiling*. During boiling vapours are formed in the liquid space itself and then these vapours rise to the surface. When these bubbles move towards a zone of high pressure, they collapse. The collapsing pressure of bubbles may be as high as 100 atmospheres and this may cause a local mechanical failure of the solid surface. This effect is called *cavitation*. The growth and decay of the vapour bubbles adversely affects the performance of a hydrodynamic machine and the ultimate effects may be the break-down of the machine itself due to severe pitting and the erosion of blade surfaces in the region of cavitation. Similar effects occur when the gases dissolved in a liquid are liberated at sufficiently low pressures.

Molecular activity and hence temperature influences the evaporation rate, consequently the vapour pressure of a liquid depends upon temperature; it increases with temperature growth. At 20°C water has a vapour pressure of 0.235 N/cm^2 and mercury has a vapour pressure of 1.72×10^{-5} N/cm^2. To arrest cavitation the pressure at any point in the fluid phenomenon should not be allowed to fall below the saturated vapour pressure at the local temperature.

Mercury has got the lowest value of vapour pressure, and this lower vapour pressure combined with high density makes mercury most suitable for use in thermometers and manometers.

2.9. NEWTONIAN AND NON-NEWTONIAN FLUIDS

Distinction between Newtonian and non-Newtonian fluids can be readily illustrated when the velocity gradient du/dy is plotted against the viscous shear stress τ.

Fluids for which the viscosity is independent of velocity gradient are called Newtonian fluids. For these fluids the plot between shear stress and velocity gradient is a straight line passing through the origin. Slope of the line equals the coefficient of viscosity, $\mu = \tau/(du/dy)$. Fluids represented by curves *(a)* and *(b)* are Newtonian fluids; fluid represented by line *(a)* is more viscous than that represented by line *(b)*. Fluids like air, water, kerosene and thin lubricating oils are essentially Newtonian in character under normal working conditions.

Fluids such as human blood, thick lubricating oils and certain suspensions for which the viscosity coefficient depends upon velocity gradient are referred to as non-Newtonian fluids. The viscous behaviour of a non-Newtonian fluid may be prescribed by the power law equation $\tau = k\,(du/dy)^n$ where k is a consistency index and n is a flow behaviour index. For a Newtonian fluid, the consistency index k becomes the dynamic viscosity coefficient μ and the flow behaviour index n assumes a unity value.

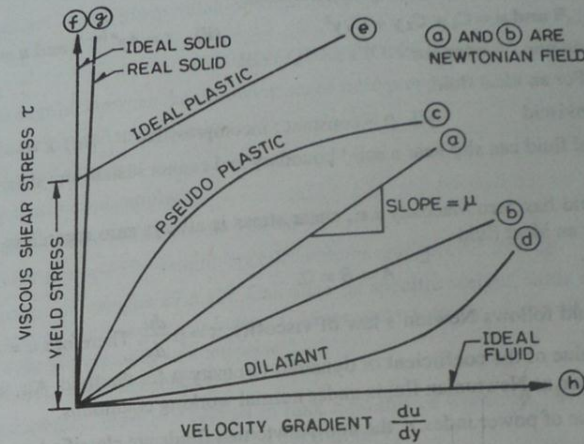


Fig. 2.29. *Variation of shear stress with velocity gradient (time rate of deformation)*

Fluids for which the flow behaviour index n is less than unity are called *pseudo*-plastic. Viscosity coefficient is smaller at greater rates of velocity gradient and the curve becomes flatter as the shear rate (*i.e.*, velocity gradient) increases (curve c). Examples of pseudo-plastic fluids are the milk, blood, clay and liquid cement. Fluids for which the index n is greater than unity are called *dilatant*. Viscosity coefficient is more at greater rates of viscosity gradient and the flow curve steeps with increasing shear rate (curve d). Concentrated solution of sugar and aqueous suspension of rice starch are examples of dilatant fluids.

An *ideal plastic* substance indicates no deformation when stressed upto a certain point (yield stress) and beyond that it behaves like a Newtonian fluid and hence is represented by line (e). For certain substances, there is finite deformation for a given load, *i.e.*, rate of deformation is zero. These materials plot as ordinate (curve f) and are called ***elastic materials*** or ***ideal solids***. Actual solids deform slightly when subjected to shear stress of larger magnitude and hence plot as a straight line almost vertical (g). A fluid for which shear stress is zero (even if there is velocity gradient) is the ***ideal fluid*** and it plots as abscissa (h). Fluids which show an apparent increase in viscosity with time are called *thixotropic*. Conversely if the apparent viscosity decreases with time, the fluid is called *rheopectic*.

This text deals with Newtonian fluids for which viscosity is independent of velocity gradient as well as duration of stress. Non-Newtonian fluids come under the purview of another science called *Rheology*.

Example 2.49. (*a*) State the characteristics of an ideal fluid.

(*b*) The general relation between shear stress and velocity gradient of a fluid can be written as

$$\tau = A\left(\frac{du}{dy}\right)^n + B$$

where A, B and n are constants that depend upon the type of fluid and conditions imposed on the flow. Comment on the value of these constants so that the fluid may behave as *(i)* an ideal fluid, *(ii)* a Newtonian fluid and *(iii)* a non- Newtonian fluid.

(c) Indicate whether the fluid with characteristics

(i) $\tau = Ay + B$ **and** $u = C_1 + C_2 y + C_3 y^2$ **(ii)** $\tau = Ay^{n(n-1)}$ **and** $u = Cy^n$

is Newtonian or non-Newtonian.

Solution : *(a)* For an ideal fluid,

(i) $\mu = 0$: inviscid *(ii)* $\rho =$ constant : incompressible *(iii)* $k = \infty$

Further, an ideal fluid can slip near a solid boundary and cannot sustain any shear force however small it may be.

(b) An ideal fluid has zero viscosity *i.e.*, shear stress is always zero regardless of the motion of the fluid. Accordingly for an ideal fluid

$$A = B = 0.$$

A Newtonian fluid follows Newton's law of viscosity; $\tau = \mu \dfrac{du}{dy}$. Therefore $n = 1$ and $B = 0$, and the constant A takes the value of the coefficient of dynamic viscosity μ for the fluid. Air, water and many other engineering fluids behave as Newtonian fluids under normal working conditions.

Based on the value of power index n, the non-Newtonian fluids are classified as :

- *Dilatant fluids* if $n > 1$ and $B = 0$.

For example concentrated solution of sugar, aqueous suspension and printing ink.

- *Pseudo plastic fluids* if $n < 1$ and $B = 0$

For example milk, blood, clay and liquid cement.

- *Bingham fluid* or *ideal plastic* if $n = 1$ and $B = \tau_0$

An ideal plastic has a definite yield stress and a constant linear relation between shear stress developed and rate of deformation. That is

$$\tau = \tau_0 + \mu \frac{du}{dy}$$

Water suspension of clay and flyash, sewage sludge are common examples of Bingham fluids.

(c) : (i) $\dfrac{du}{dy} = C_2 + 2\,C_3\,y$

For a Newtonian fluid $\tau = \mu \dfrac{du}{dy}$

$$\therefore \quad \tau = \mu\,(C_2 + 2\,C_3\,y) = 2\,C_3\,\mu\,y + \mu\,C_2$$

which can be recast as

$$\tau = Ay + B \text{ where } A = 2\,C_3\,\mu \text{ and } B = \mu\,C_2$$

This has the same form as the given shear stress and as such the fluid characteristics correspond to that of an ideal fluid.

(ii) $\dfrac{du}{dy} = C\,n\,y^{n-1}$

For a Newtonian fluid $\tau = \mu \dfrac{du}{dy}$

$$\therefore \qquad \tau = \mu\, C\, n\, y^{n-1}$$

which does not conform to the given value of shear stress and obviously the fluid is non-Newtonian in character.

REVIEW QUESTIONS

1. Define and distinguish between the following set of fluid properties :
   - *(i)* specific weight and mass density
   - *(ii)* cohesion and adhesion
   - *(iii)* surface tension and capillarity
   - *(iv)* dynamic viscosity and kinematic viscosity.

2. Define : mass density, specific weight, specific volume and specific gravity.

   3.2 m^3 of a certain oil weighs 27.5 kN. Calculate its specific weight, mass density, specific volume and specific gravity with respect to water. If kinematic viscosity of the oil is 7×10^{-3} stokes, what would be its dynamic viscosity in centipoise ? **(Ans.** 8.59 kN/m^3, 876 kg/m^3, 0.1164 m^3/kN, 0.876, 6.132 cP)

3. Define compressibility. How is it related to bulk modulus of elasticity ? Which of the following two gases is more compressible under the stated conditions ?
   - *(i)* Gas A at 125 kN/m^2 absolute pressure is compressed isothermally
   - *(ii)* Gas B at 100 kN/m^2 absolute pressure is compressed isentropically ($\gamma = 1.4$)

   (**Ans.** $K_A = p = 125$ kN/m^2 ; $K_B = \gamma p = 140$ kN/m^2 ; $K_A < K_B$ and so gas A is more compressible)

4. A cylinder contains 0.75 m^3 of gas at 20°C and 2.5 bar pressure. After compression, the volume gets reduced to 0.15 bar. Determine final pressure and bulk modulus of compressed air if compression takes place under :

   (i) isothermal conditions *(ii)* adiabatic conditions ($\gamma = 1.4$)

   (**Ans.** *(i)* 12.5 bar, 1.25 MN/m^2; *(ii)* 23.75 bar, 3.325 MN/m^2)

5. Mention some examples where compressibility of water is taken into account.

   A high pressure steel vessel is stated to contain 1.153 units of liquid at 8 atmosphere. When the pressure is raised to 20 atmosphere, the volume of liquid equals 1.152 units. Estimate the average value of the bulk modulus of elasticity and the coefficient of compressibility of the liquid over the given range of pressure. Take 1 atm = 101.3 kN/m^3. (**Ans.** 1440 MN/m^2; 0.000714 m^2/MN)

6. The bulk modulus of elasticity of a fluid is defined as $K = \dfrac{-dp}{dv/v}$. Show that it is equivalent to $K = \dfrac{dp}{d\rho/\rho}$.

   A gas at pressure p is compressed under isothermal conditions. Calculate the change in pressure required to compress the gas to one-third of its volume. $\left(\textbf{Ans. } \frac{2}{3}p\right)$

7. What is meant by viscosity of a liquid, how does it manifest and in what units it is measured ?

Does the viscosity of liquids and gases increase or decrease with temperature growth ? Suggest reasons for the difference in behaviour, if any.

8. (a) Justify the statement : There is a one-to-one correspondence between Hooke's law of elasticity and Newton's law of viscosity ?

(b) Explain viscosity on the basis of molecular motion.

(c) Define the terms poise and stokes and derive expression for their dimensions in terms of M, L and T. (**Ans.** Both give linear relations between stress and strain or rate of strain)

9. Elaborate the difference between dynamic viscosity and kinematic viscosity.
Find the kinematic viscosity in stokes of a liquid whose specific gravity is 0.95 and viscosity is 0.011 poise. (**Ans.** 0.0116 stokes)

10. The space between two parallel plates 5 mm apart is filled with an oil of relative density 0.9. A force of 2 N is required to drag the upper plate of area 900 cm^2 at a constant velocity of 0.8 m/s. Assume straight line velocity distribution and calculate the velocity gradient, the dynamic viscosity and kinematic viscosity of the oil. (**Ans.** 160 cm/s/cm, 1.39 poise, 1.52 stokes)

11. A thin plate of area A is placed midway in a gap of height h filled with a liquid of dynamic viscosity μ_1. The plate requires a force F to move with a constant uniform velocity V. The same gap is subsequently filled with another liquid of viscosity μ_2 and the same plate is positioned at a distance $\frac{h}{4}$ from one wall. Experiments indicated that for the same velocity V, the force required was same. Prove that $\mu_1 = \frac{4}{3}\mu_2$.

12. A steel shaft 25 mm diameter and 30 cm long falls of its own weight inside a vertical open tube 25.2 mm diameter. The clearance, assumed uniform, is filled with glycerine of viscosity 1.5 Pa s. Calculate how fast will the cylinder fall at terminal conditions. Take density of steel $\rho = 7850$ kg/m^3 (**Ans.** 0.032 m/s)

13. A rectangular solid block of base area 0.3 m^2, weight 90 N is allowed to slide down an inclined plane surface making an angle of 30-degree with the horizontal. The surface of the plane is smeared with 3 mm thick layer of lubricant oil of viscosity 8 poise. Workout the terminal velocity with which the block slides down the plane. (**Ans.** 0.0562 m/s)

14. A 50 mm diameter shaft rotates at 500 revolutions per minute in a 80 mm long journal bearing with 51 mm internal diameter. The annular space between the shaft and bearing is filled with a lubricating oil of dynamic viscosity 1.0 poise. Determine the torque required to overcome friction in bearing and the rate at which heat is generated at the bearing. (**Ans.** 0.082 Nm, 4.29 W)

15. A 60 mm diameter cylinder rotates concentrically inside another cylinder of diameter 60 mm. Both the cylinders are 80 mm long. The annular space between the cylinders is filled with a liquid of dynamic viscosity 3 poise. What would be the rotational speed of the inner shaft when an external torque of 1.5 Nm is applied to it ? (**Ans.** 1760 rpm)

16. A vertical gap 25 mm wide and of infinite extent contains oil of specific gravity 0.95 and dynamic viscosity 2.5 Pa s. A square metal plate, 1.5 m side and 1.5 mm thick, weighing 50 N is to be lifted through the gap at a constant speed of 100 mm/s. Calculate the viscous resistance to be overcome and

the power required in lifting the plate. Assume the plate to be placed centrally inside the gap. (**Ans.** 95.85 N, 14.58 W)

17. The velocity distribution near the solid wall at a section in a laminar flow is given by

$$u = 3.0 \sin (4 \pi y) \text{ for } y \leq 0.125 \text{ m}$$

Make calculations for the shear stress at distance 0, 6 and 12.5 cm from the wall; given the dynamic viscosity of the fluid $\mu = 5$ poise. (**Ans.** 18.84 N/m^2, 13.73 N/m^2, 0)

18. In a torsion viscometer, the outer cylinder of 150.5 mm diameter is rotated by turning the shaft at a constant speed of 100 rev/min. Owing to its viscosity, the liquid under test transmits a torque of 540 Nm to the inner cylinder of 150 mm diameter which is suspended by torsion wire fixed at its upper end. If the liquid is 130 mm deep, find its viscosity.

19. A flat disc of diameter 10 cm rotates on a table separated by an oil film of 1.5 mm thickness. If the torque required to rotate the disc at 50 rev/min is 3×10^{-4} Nm, find the viscosity of oil. It may be presumed that velocity gradient in the oil film is linear. (**Ans.** 0.0877 poise)

20. Explain how the surface tension accounts for *(i)* formation of a drople' and *(ii)* rise of liquid in a capillary ?

The pressure inside an air bubble of diameter 0.01 mm is 29.2 kPa in excess of ambient pressure. Workout the surface tension at air-water interface. (**Ans.** 0.073 N/m)

21. Why does oil spread when it is poured on water surface ?

A glass tube of 2 mm internal diameter is immersed in oil of mass density 950 kg/m^3 to a depth of 12 mm. If the oil has a surface tension of 0.036 N/m, what pressure is needed in the formation of a just released bubble? (**Ans.** 183.83 N/m^2)

22. A clean glass tube of 2.5 mm internal diameter is immersed in mercury (specific gravity = 13.6). Determine the level of mercury in the tube in relation to the free surface of mercury outside the tube. Presume that for mercury-clean glass angle of contact $\theta = 130^\circ$ and for air-mercury interface surface tension $\sigma = 0.48$ N/m. (**Ans.** 3.7 mm depression)

23. Calculate the maximum capillary rise of water to be expected between two vertical clean glass plates spaced 1 mm apart. If the water is replaced by mercury, what would be the maximum capillary depression of mercury in the same space. Assume appropriate values for the surface tension and angle of contact. (**Ans.** 15 mm rise; 4.9 mm depression)

24. A U-tube is made of two capillaries of bore 4 mm and 5 mm respectively. The tube is held vertically and partially filled with a liquid of mass density 1000 kg/m^3 and angle of contact 0-degree. Calculate the surface tension of the liquid if the estimated difference in the level of two menisci is 1.5 mm.

25. Derive an expression for the capillary rise of a fluid having surface tension σ and contact angle θ between two vertical parallel plates a distance t apart. If the plates are of glass, what will be the capillary rise of water having $\sigma = 0.073$ N/m, $\theta = 0^\circ$ and $t = 1$ mm. (**Ans.** 14.9 mm)

26. What is vapour pressure ? How can water boil at room temperature ? Discuss the significance of vapour pressure in problems related to liquids in motion.

27. *(a)* Draw the stress-strain relationship for the following fluids and discuss the behaviour of each fluid under an external shear force : an ideal fluid, a Newtonian fluid, a pseudoplastic fluid, a dilatant fluid, a Bingham fluid, and a plastic.

(b) Classify the following fluids : air, sugar solution, glycerine, printer's ink, molten metal and water.

28. Explain the characteristics of fluid properties to which the following fluid phenomenon are attributable :
    - *(i)* formation of bubbles and discontinuity in flow systems
    - *(ii)* water hammer in pipe flow
    - *(iii)* petrol evaporates more readily than water at ordinary temperature
    - *(iv)* breaking up of liquid jet.
29. State true of false; if false give the correct statement :
    - *(i)* For an ideal fluid, bulk modulus of elasticity $k = \infty$
    - *(ii)* The velocity gradient can be interpreted as rate of angular deformation.
    - *(iii)* If cohesion predominates, the liquid gets lifted up at the line of contact.
    - *(iv)* In laminar flow with parabolic velocity distribution, shear stresses are greatest near walls.
    - *(v)* The dynamic viscosity of a perfect gas increases when it is compressed isothermally.
    - *(vi)* Newton's law of viscosity is valid for a turbulent flow with linear velocity distribution.
    - *(vii)* There is one-to-one correspondence between Hooke's law of elasticity and Newton's law of viscosity.
    - *(viii)* An oil jet flows without breakup due to viscosity.

(**Ans.** *(i)* T, *(ii)* T, *(iii)* F, *(iv)* T, *(v)* F, *(vi)* F, *(vii)* T, *(viii)* T)

□ □

3

Fluid Statics

3.1. INTRODUCTION

Fluid statics is the study of a fluid at rest; the concept includes situations where fluids are either actually at rest or undergo uniform acceleration in a container or rotate as a solid mass. No shear force is then present as the fluid particles do not move with respect to one another. This aspect of fluid behaviour has been studied in this chapter with a view to :

- establish a relation for pressure variation along a vertical depth in fluid; the relation has been applied to the measurement of pressure with manometers;
- compute the hydrostatic forces and pressure distribution on submerged bodies; that helps in the design of structures and equipment like dams, ship and hydraulic actuators (gates) etc;
- analyse the stability of floating bodies; importance of metacentric height under static condition.

The mathematical relations for fluids at rest or moving as solid mass are obtained by consideration of the equilibrium of a fluid element (an infinitesimal region of the continuum) under the influence of normal pressure force and the self weight of the fluid mass acting in the vertical direction.

SECTION A : Pressure and its relationship with height

3.2. PRESSURE

A fluid element or mass is essentially acted upon by two categories of forces : body forces and surface forces. *Body forces* on fluid elements are caused by agencies such as gravitational, electric or magnetic fields. The magnitude of these forces is proportional to the mass of the fluid. *Surface forces* represent the action of the surrounding fluid on the element under consideration through direct contact. These forces are due to surface stresses like pressure (normal force) and shear (tangential force). In fluids at rest, there is no relative motion between the layers of the fluid. The velocity gradient is zero and hence there is no shear in the fluid. Consequently there is no tangential component of force and hence for a stationary fluid, the force exerted is normal to the surface of the containing vessel. This normal surface force is called the pressure force. The mathematical definition of *intensity of pressure* (or simply pressure), in the absence of shearing stress, is

$$p = \frac{dF}{dA} \qquad \text{...(3.1)}$$

where dF represents the resultant force acting normal to an infinitesimal area dA. If the total force F acts uniformly over the entire area A, then $p = F/A$. Pressure has the dimensions of $\left[FL^{-2} \right]$ and is usually expressed in N/m^2 (pascal), bar or atmosphere.

1 bar = 10^5 N/m^2 = 100 kPa

1 atm = 101.3 kPa

3.3. PASCAL'S LAW

An important and unique property of hydrostatic pressure is reflected in Pascal's law, which states that:

"Intensity of pressure at a point in a fluid at rest is same in all directions".

Consider a small wedge shaped element of stationary fluid and assume that the element has a unit depth perpendicular to the plane of the paper (Fig. 3.1). The element is acted upon by the normal pressure forces and the vertical force due to weight. Let p_x, p_y and p_θ be the pressure intensities on the faces AB, BC and AC respectively. Then :

Force on face AB $= p_x \times$ area of face $AB = p_x\,(dy \times 1) = p_x\,dy$

Like wise,

force on face BC $= p_y\,dx$

force on face AC $= p_\theta\,ds$

The weight of fluid element is,

= (area of triangular element × depth) × specific weight

$$= \left(\frac{1}{2}\,dx\,dy \times 1\right) \times w = \frac{1}{2}\,w\,dx\,dy$$

and it acts through the centre of gravity. Since the fluid element is in equilibrium, the forces in the horizontal and vertical directions must balance.

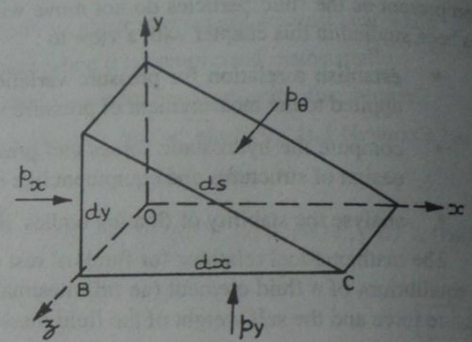


Fig. 3.1.

Resolving the forces in x-direction,

$$p_x\,dy = p_\theta\,ds \sin\theta$$

From Fig. 3.1 : $dy = ds \sin\theta$

$\therefore$ $p_x\,dy = p_\theta\,dy \;;\; p_x = p_\theta$ ...(3.2)

Resolving the forces in y-direction,

$$p_y\,dx = \frac{1}{2}\,w\,dx\,dy + p_\theta\,ds\cos\theta$$

Let the size of the elemental system approach smaller and smaller dimensions; then the gravitation force (weight) which diminishes as the product of two dimensions (dx and dy) can be neglected in comparison with the pressure forces for which the diminishing effect is proportional to be reduction in single dimension (dx). This in the limit

$$p_y\,dx = p_\theta\,ds\cos\theta$$

From Fig. 3.1 : $dx = ds\cos\theta$

$\therefore$ $p_y\,dx = p_\theta\,dx \;;\; p_y = p_\theta$ ...(3.3)

From equations (3.2) and (3.3), we have

$$p_x = p_y = p_\theta \qquad ...(3.4)$$

This result is independent of the angle θ and, therefore, it follows that pressure acts equally in all directions in a stationary fluid. Pressure at a point has only one value regardless of the orientation of the area upon which it is determined. Independence of direction implies that pressure is a scalar quantity.

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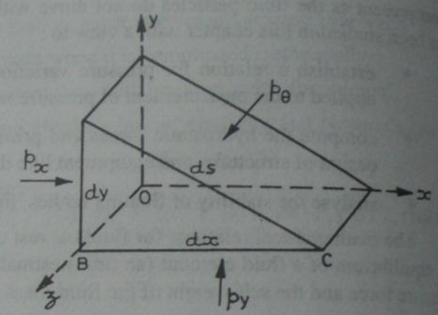

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3.4. PRESSURE-DENSITY-HEIGHT RELATIONSHIP : HYDROSTATIC LAW

The fundamental equation relating pressure, density and vertical distance can be established by considering the equilibrium of an imaginary cylindrical element in a body of fluid at rest. The cylindrical element is of cross-sectional area dA and height dy.

The pressure forces acting on the fluid element are :

(*i*) pressure force on bottom face $AB = p\,dA$ acting in the upward direction.

(*ii*) pressure force on top face BC

$= \left(p + \frac{\partial p}{\partial y} dy\right) dA$ acing in the downward direction.

(*iii*) weight of fluid element = specific weight × volume = $w\,dA\,dy$

(*iv*) pressure forces on surface AC and BD are equal and opposite and hence cancel out.

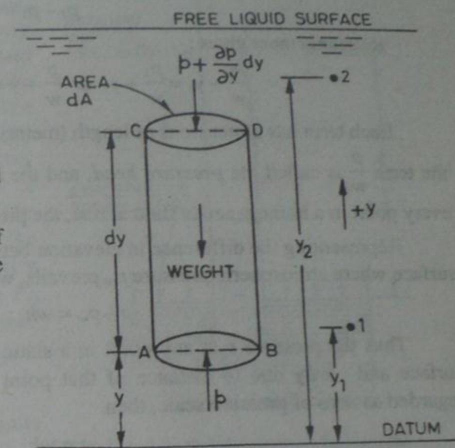

Fig. 3.2. *Pressure-density-height relationship*

For the element to be in equilibrium, the sum of downward forces on the element must be equal to the upward forces. That is

$$p\,dA - \left(p + \frac{\partial p}{\partial y} dy\right) dA - w\,dA\,dy$$

or $\quad \frac{\partial p}{\partial y} = -w$

Since we are considering the variation of pressure only in the y-direction, the partial differential $\frac{\partial p}{\partial y}$ can be replaced by exact differential $\frac{dp}{dy}$.

$$\frac{dp}{dy} = -w \text{ or } dp = -w\,dy \qquad ...(3.5)$$

This is the fundamental equation of fluid statics and indicates that a negative pressure gradient exists upward along any vertical. Thus the pressure decreases in the upward direction and increases in the downward direction with magnitude equal to specific weight.

Letting height dy to be zero, then $dp = 0$, *i.e.*, the pressure intensity remains the same if there is no change in elevation. Thus the pressure will be constant everywhere over the same level surface in a continuous body of static fluid. A surface layer where pressure is same at all points is called an *isobaric surface* or *equipotential surface*. The equipotential surfaces are horizontal planes; the free surface (a surface separating the liquid from the atmosphere) being one of them.

Equation 3.5 can be integrated directly to determine the difference in pressure between any two points in the fluid mass

$$\int_1^2 dp = -\int_1^2 w\, dy \qquad \text{...(3.6)}$$

Evaluation of this integral is possible only for a given dependence of specific weight w with elevation y.

3.5. PRESSURE VARIATION FOR AN INCOMPRESSIBLE FLUID

For incompressible fluid, the specific weight is independent of pressure intensity and it remains fairly constant with height. Then for a homogeneous fluid of constant specific weight, equation 3.6 becomes :

$$p_2 - p_1 = -w\,(y_2 - y_1)$$

Rearrangement gives :

$$\frac{p_1}{w} + y_1 = \frac{p_2}{w} + y_2 \quad \text{or} \quad \frac{p}{w} + y = \text{constant} \qquad \text{...(3.7)}$$

Each term has dimensions of length (metre). The co-ordinate y is called the ***position or elevation head***, the term $\frac{p}{w}$ is called the *pressure head*, and the sum $\left(\frac{p}{w} + y\right)$ is called the ***piezometric head***. Evidently at every point in a homogeneous fluid at rest, the piezometric head is constant.

Representing the difference in elevation between two points by h and measuring it from the free liquid surface where atmospheric pressure p_{at} prevails, we have :

$$p - p_{at} = wh \;;\quad p = p_{at} + wh \qquad \text{...(3.8)}$$

Thus the pressure p at any point in a static fluid is partly due to atmospheric pressure p_{at} at the free surface and partly due to distance of that point beneath the free surface. If atmospheric pressure p_{at} is regarded as zero of pressure scale, then

$$p = wh \qquad \text{...(3.9)}$$

This pressure p at any point in a static fluid when expressed above atmospheric pressure is called the *positive or gauge pressure*. Equation 3.9 demonstrates that pressure at any point submerged below a free liquid surface exposed to atmosphere is equal to the product of the vertical distance below the surface and the specific weight of the liquid.

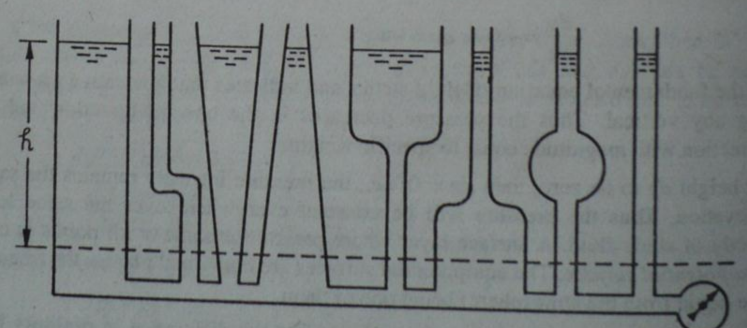

Fig. 3.3. ***Pressure for containers of various shapes***

Apparently the pressure intensity depends only upon the height of column and not at all upon the size of column. Accordingly in vessels of different shapes and configurations, the same unit pressure would be exerted against the bottom of the container as shown in Fig. 3.3. Again the section z — z is at the same distance from the free surface and as such the points lying on this section in each of the container would experience the same pressure intensity.

3.6. ATMOSPHERIC, ABSOLUTE, GAUGE AND NEGATIVE PRESSURE

Following terms are generally associated with pressure and its measurement :

(i) Atmospheric Pressure (P_{at}) : This is the pressure exerted by the envelope of air surrounding the earth's surface. Atmospheric pressure is usually determined by a mercury column barometer shown in Fig. 3.4. A long, clean thick glass tube closed at one end is filled with pure mercury. The tube diameter is such that capillarity effects are minimum. The open end is stoppered and the tube is inserted into a mercury container; the stoppered end kept well beneath the mercury surface. When the stopper is removed, mercury runs out of the tube into the container and eventually mercury level in the tube settles at height h above mercury level in the container. Atmospheric pressure P_{at} acts at the mercury surface in the container, and mercury vapour pressure P_{vp} exists at the top of mercury column in the tube. From hydrostatic equation.

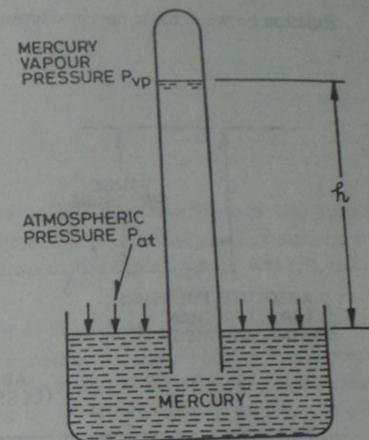

Fig. 3.4. *Simple mercury barometer*

$$P_{at} - P_{vp} = wh$$

Mercury has a low vapour pressure (≈ 0.17 N/m^2 at 20° C) and thus for all intents and purposes it can be neglected in comparison with P_{at} which is about 10^5 N/m^2 at mean sea level. Then

$$P_{at} = wh \qquad \text{...(3.10)}$$

Mercury is preferred as other liquids possess sufficient vapour pressure at room temperature and that creates error in pressure indication. Atmospheric pressure varies with altitude, because the air nearer the earth's surface is compressed by the air above. At sea level, value of atmospheric pressure is close to 1.01325 bar or 760 mm of Hg column or 10.33 m of water column.

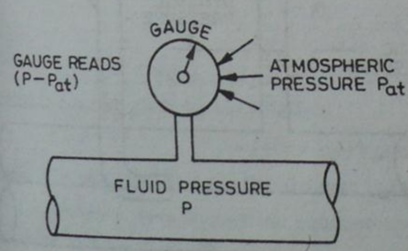

Fig. 3.5. *Gauge pressure*

(ii) Absolute Pressure (P_{abs}) : Pressure has been defined as the force per unit area due to interaction of fluid particles amongst themselves. A zero pressure intensity will occur when molecular momentum is zero. Such a situation can occur only when there is a perfect vacuum, *i.e.*, a vanishingly small population of gas molecules or of molecular velocity. Pressure intensity measured from this state of vacuum or zero pressure is called absolute pressure.

(iii) Gauge Pressure (P_g) *and Vacuum* (P_{vac}) : Instruments and gauges used to measure fluid pressure generally measure the difference between the unknown

pressure P and the existing atmospheric pressure p_{at} as shown in Fig. 3.5.

When the unknown pressure is more than atmospheric pressure, the pressure recorded by the instrument is called *gauge pressure*. A pressure reading below the atmospheric pressure is known as *vacuum, rarefaction or negative pressure*. Actual absolute pressure is then the algebraic sum of the gauge indication and the atmospheric pressure

$$P_{abs} = P_{at} + P_g$$

$$P_{abs} = P_{at} - P_{vac} \qquad ...(3.11)$$

Relation between these pressure terms is illustrated in Fig. 3.6.

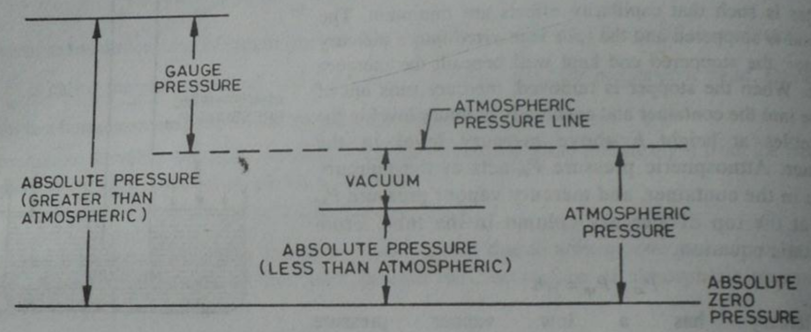

Fig. 3.6. *Relation between absolute, gauge and atmospheric pressure*

(iv) Static (P_s) and total pressure (P_t) : Static pressure is defined as the force per unit area acting on the wall by a fluid at rest or flowing parallel to the wall in a pipe line. Static pressure of a moving fluid is measured with an instrument which is at rest relative to the fluid. The instrument should theoretically move with the same speed as that of the fluid particle itself. As it is not possible to move a pressure transducer along in a flowing fluid, static pressure is measured by inserting a tube into the flow path. Care is taken to ensure that the tube does not protrude into the pipe line and causes no errors due to impact and eddy formation. When the tube protrudes into the stream, there would be local speeding up of the flow due to its deflection around the tube; hence an erroneous reading of the static pressure would be observed.

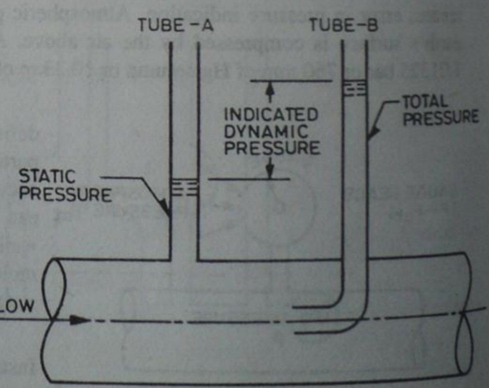

Fig. 3.7. *Static and total pressure*

Total or stagnation pressure is defined as the pressure that would be obtained if the fluid stream were brought to rest isentropically. In Fig. 3.7, tube B

senses the total pressure while tube A senses only the static component of pressure.

The difference between the total and the static pressure gives the pressure due to fluid velocity; referred to as the dynamic pressure. The *dynamic pressure* is due to flow speed and is also known as the *velocity or impact* pressure. For an incompressible fluid or for a gas flowing at low velocities, the dynamic pressure equals $V^2/2g$ where V is the velocity of fluid flow.

Velocity pressure = total pressure – static pressure

$$\frac{V^2}{2g} = (P_t - P_s)$$

Some of the commonly used pressure units are :

1 bar = 10^5 N/m² = 100 kPa = 750.06 mm of Hg

1 micron = 1μ = 10^{-3} mm of Hg

1 torr = 1 mm of Hg

1μ bar = 1 dyne/cm²

Quite often pressure is expressed in the unit *atmosphere*. This unit simply uses the standard atmospheric value of 101.3 kPa and is defined as one atmosphere. "Two atmospheres" would be then 202.6 kPa. Sometimes, we assume atmospheric pressure equivalent to a rounded figure of 100 kPa and call it a *technical atmosphere*.

Selection of one or another of the various units of pressure or head depends on the use made of the measurement. Head and pressure are related by the hydrostatic equation $p = \rho gh$, where p is the pressure either absolute or gauge, h is the height of a liquid column and ρ is the density of the liquid. Since density of a liquid varies considerable with temperature, hence the need to state temperature when pressure is expressed in units of liquid column head. Conversions to standard conditions may also be made by : $w_s h_s = w_m h_m$ where subscript m refers to conditions at the local measured temperature and subscript s refers to conditions at the desired temperature.

Note : Please refer to chapter 14 for the description of various methods used for measuring pressure exerted by fluids.

Example 3.1. A piston of area 150 cm² exerts a force of 600 N. Find the intensity of pressure on the fluid in contact with the underside of piston if the piston is in equilibrium.

Solution : Under equilibrium conditions, pressure exerted by the fluid equals the piston force

$$\therefore \text{Intensity of pressure} = \frac{\text{force}}{\text{area}}$$

$$= \frac{600}{(150 \times 10^{-4})} = \mathbf{4 \times 10^4\ N/m^2}$$

Example 3.2. Determine the pressure at a point 8 m below the free surface in a fluid that has a variable density given by the relation

$$\rho = (400 + Ay)\ \text{kg/m}^3$$

where A = 10 kg/m⁴ and y is the distance in metres measured from the free surface.

Solution : From hydrostatic equation $dp = -w\,dy$. Taking distance y to be positive when measured down the free water surface,

$$dp = w\,dy = \rho\,g\,dy = g\,(400 + Ay)\,dy$$

Upon integration,

$$p = g\left(400\,y + \frac{Ay^2}{2}\right)$$

For $y = 8$ m and $A = 10$ kg/m^4

$$p = 9.81\left(400 \times 8 + \frac{10 \times 8^2}{2}\right)$$

$$= 34531 \text{ N/m}^2 \approx \mathbf{34.53 \text{ kN/m}^2}$$

Example 3.3. Explain the terms intensity of pressure and pressure head.

Convert a pressure of 1.5 bar to *(i)* metres of water *(ii)* cm of mercury (sp. gr. 13.6).

Solution : From hydrostatic equation, $p = wh$

(i) $h = \dfrac{p}{w} = \dfrac{1.5 \times 10^5}{9810} = \mathbf{15.29\ m}$ of water

(ii) $h = \dfrac{1.5 \times 10^5}{(13.6 \times 9810)} = 1.124$ m of mercury

$= \mathbf{112.4\ cm}$ of mercury

Example 3.4. At a certain location in the flow field, pressure equals 50 m of water column. Obtain the equivalent pressure head in terms of *(i)* kerosene of specific gravity 0.8 *(ii)* carbon tetrachloride of specific gravity 1.5.

Solution : Invoking hydrostatic equation $p = wh$

∴ Pressure intensity corresponding to 50 m of water column

$$p = 9810 \times 50 = 490500 \text{ N/m}^2$$

Equivalent pressure head in terms of

(i) kerosene of specific gravity 0.8

$$= \frac{490500}{0.8 \times 9810} = \mathbf{62.5\ m} \text{ of kerosene}$$

(ii) Carbon tetrachloride of specific gravity 1.5

$$= \frac{490500}{1.5 \times 9810} = \mathbf{33.33\ m} \text{ of carbon tetrachloride}$$

Example 3.5. Measurements of pressure at the base and top of a mountain are 74 cm and 60 cm of mercury respectively. Workout the height of mountain if air has a specific weight of 11.97 N/m^3.

Solution : $p_{base} = 74$ cm of mercury $= \dfrac{74}{100} \times (13.6 \times 9810)$ N/m^2

$$p_{top} = \frac{60}{100} \times (13.6 \times 9810) \text{ N/m}^2$$

Now, $p_{base} - p_{top} = wh$

$$\frac{(74-60)}{100} \times 13.6 \times 9810 = 11.97\,h\,; \qquad h = 1560.4 \text{ m}$$

∴ Mountain has a height of **1560.4 m**

Example 3.6. A gauge on the suction side of a pump shows a negative pressure of 0.285 bar. Express this pressure in terms of (*i*) pressure intensity kPa (*ii*) N/m² absolute (*iii*) m of water gauge (*iv*) m of oil (specific gravity 0.85) absolute and (*v*) cm of mercury gauge. Take atmospheric pressure as 76 cm of mercury and relative density of mercury as 13.6.

Solution : (*i*) Gauge reading $= 0.285 \text{ bar} = 0.285 \times 10^5 \text{ N/m}^2$

$$= 0.285 \times 10^5 \text{ Pa} = 28.5 \text{ kPa gauge (vacuum)}$$

(*ii*) Atmospheric pressure $p_{at} = 76$ cm of mercury

$$= (13.6 \times 9810) \times 0.76 = 101396 \text{ N/m}^2$$

Absolute pressure = atmospheric pressure – vacuum pressure

$$p_{abs} = p_{at} - p_{vac}$$

$$= 101396 - 28500 = 72896 \text{ N/m}^2 \text{ absolute}$$

(*iii*) Equivalent heads of water, oil and mercury can be worked out by applying the hydrostatic equation, $p = wh$

$$\text{Head of water (gauge)} = \frac{0.285 \times 10^5}{9810} = 2.905 \text{ m of water (gauge)}$$

$$\text{Head of oil (absolute)} = \frac{72896}{0.85 \times 9810} = 8.742 \text{ m of water (absolute)}$$

$$\text{Head of mercury (gauge)} = \frac{0.285 \times 10^5}{13.6 \times 9810} = 0.2136 \text{ m of mercury}$$

$$= 21.36 \text{ cm of mercury (gauge)}$$

Example 3.7. A diver is working at a depth of 20 m below the surface of sea water (sp. wt. = 10 kN/m³). Calculate the pressure intensity at this depth. What would be the absolute pressure if barometer reads 760 mm of mercury column at the sea level ?

Solution : From hydrostatic equation, $p = wh$

∴ Pressure intensity at the given depth,

$$p_g = 10\,000 \times 20 = 200\,000 \text{ N/m}^2 = \mathbf{200 \text{ kPa}}$$

Atmospheric pressure $p_{at} = 760$ mm of mercury

$$= (13.6 \times 9810) \times 0.76 = 101396 \text{ N/m}^2 \approx 101.4 \text{ kPa}$$

∴ Absolute pressure $p_{abs} = p_{at} + p_g$

$$= (101.4 + 200) = \mathbf{301.4 \text{ kPa}}$$

Example 3.8. The inlet to pump is 10 m above the bottom of sump from which it draws water through a suction pipe. If the pressure at the pump inlet is not to fall below 30 kN/m² absolute, workout the minimum depth of water in the tank. Take atmospheric pressure as 100 kN/m².

Solution : Let p_{vac} be the vacuum (suction) pressure at the pump intel. Then

$$p_{vac} = p_{at} - p_{abs}$$

$$= (100 - 30) = 70 \text{ kN/m}^2$$

$$= 70000 \text{ N/m}^2$$

Further let h represent the distance between the pump inlet and the free water surface in the sump.

Applying hydrostatic equation, $p = wh$, we have

$$70000 = 9810\,h\,; \quad h = 7.136 \text{ m}$$

$\therefore$ Minimum depth of water in the sump,

$$= (10 - 7.136) = \mathbf{2.864 \text{ m}}$$

Example 3 9. The tyre of an airplane is inflated to 420 kN/m² at sea level where atmospheric pressure is 760 mm of mercury and the temperature is 15°C. Workout the pressure at an elevation of 10 000 m where atmospheric pressure is 225 mm of mercury and the temperature is – 45°C. Express this pressure in kN/m² and kN/m² absolute.

Solution : Let subscripts 1 and 2 refer to the conditions at sea level and at an elevation of 10 000 m respectively.

At sea level,

$$p_{at} = 760 \text{ mm of mercury}$$

$$= (9.81 \times 13.6) \times 0.76 = 101.396 \text{ kN/m}^2$$

$$p_g = 420 \text{ kN/m}^2$$

$$(p_1)_{abs} = 101.396 + 420 \approx 521.4 \text{ kN/m}^2$$

From characteristic gas equation,

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

$$\frac{521.4 \times V}{(273 + 15)} = \frac{p_2 V}{(273 - 45)} \qquad (\because V \text{ is constant})$$

$$\therefore \quad (p_2)_{abs} = \mathbf{412.78 \text{ kN/m}^2}$$

$$p_{at} = (9.81 \times 13.6) \times 0.225 = \mathbf{30.02 \text{ kN/m}^2}$$

$$(p_2)_{gauge} = 412.78 - 30.02 = \mathbf{382.76 \text{ kN/m}^2}$$

Example 3.10. The cylindrical fuel tank of a motor car, 20 cm in diameter and with its axis horizontal is filled with a petrol of mass density 800 kg/m³. A 35 mm diameter filler pipe rises from the top of the tank to a height of 50 cm. Calculate the force on one end of the fuel tank when the filler pipe is full.

Solution : Pressure intensity at any point in a flow field varies directly with depth and is prescribed by the hydrostatic equation, $p = wh$

(i) Pressure intensity at the tank top is due to 50 cm head of petrol in the filler pipe and it equals

$$p_t = wh = (800 \times 9.81) \times 0.5 = 3924 \text{ N/m}^2$$

(ii) Pressure intensity at the tank bottom is due to 70 cm head of petrol in the filler pipe and the tank and it equals

$$p_b = (800 \times 9.81) \times 0.7 = 5494 \text{ N/m}^2$$

Average pressure intensity on end of the tank,

$$p_{av} = \frac{3924 + 5494}{2} = 4709 \text{ N/m}^2$$

Force on one end of the tank,

$$F = p_{av} \times \text{area} = 4709 \times \frac{\pi}{4}(0.2)^2 = \mathbf{147.86\ N}$$

Example 3.11. What forces act on a fluid element in static equilibrium ?

A cylindrical tank 5 m diameter × 10 m height is completely filled with water. Find *(a)* the intensity of pressure and the total force on the bottom of the tank *(b)* minimum, maximum and average pressure intensities on the vertical surface *(c)* the total force on the vertical surface.

Solution : Pressure intensity at the bottom,

$$p = wh = 9810 \times 10 = 98100\ \text{N/m}^2$$

Total pressure force at the bottom,

$$F = pA = 98100 \times \frac{\pi}{4}(5)^2 = 1925212\ \text{N}$$

(b) Pressure intensity varies directly with the depth.

Minimum pressure intensity occurs at the top and is equal to

$$p_{min} = 0 \qquad (\because h = 0 \text{ at the top})$$

Maximum pressure intensity occurs at the bottom and is equal to

$$p_{max} = wh = 9810 \times 10 = 98100\ \text{N/m}^2$$

Average pressure intensity is at the middle,

$$p_{av} = \frac{0 + 98100}{2} = 49050\ \text{N/m}^2$$

(c) Total force on the wall

$$= p_{av} \times \text{ lateral surface area}$$

$$= 49050 \times (\pi \times 5 \times 10) = 7704756\ \text{N} \approx \mathbf{7705\ kN}$$

Example 3.12. A cylindrical tank of 3 m height and 5 cm^2 cross-sectional area is filled with water upto a height of 2 m and remaining with oil of specific gravity 0.8. The vessel is open to atmosphere. Calculate :

(*i*) pressure intensity at the interface, (*ii*) absolute and gauge pressure on the base of the tank in terms of water head, oil head and N/m^2.

Also workout the net force experienced by the base of the tank. Take atmospheric pressure as 1.0132 bar.

Solution : Applying hydrostatic equation, the pressure intensity at any point in the flow field is given by $p = wh$

(*i*) Pressure at the interface between the oil and water is due to 1 m of oil and it equals

$$p_{interface} = (0.8 \times 9810) \times 1 = \mathbf{7848\ N/m^2}$$

(*ii*) Pressure at the base of tank equals the sum of pressures at the interface (due to 1 m of oil) and pressure due to 2 m of water.

$$p_{base} = 7848 + (9810 \times 2) = 27468\ \text{N/m}^2 \quad \text{(gauge)}$$

$$= \frac{27468}{9810} = 2.8\ \text{m of water (gauge)}$$

$$= \frac{27468}{0.8 \times 981} = 3.5 \text{ m of oil (gauge)}$$

Atmospheric pressure, $p_{at} = 1.0132 \text{ bar} = 1.0132 \times 10^5 \text{ N/m}^2$

$$= \frac{1.0132 \times 10^5}{9810} = 10.328 \text{ m of water}$$

$$= \frac{1.0132 \times 10^5}{0.8 \times 981} = 12.910 \text{ m of oil}$$

Absolute pressure = atmospheric pressure + gauge pressure

$\therefore$ p_{base} absoute $= 10.328 + 2.8 =$ **13.128 m** of water

$= 12.910 + 3.5 =$ **16.410 m** of oil

$= 101320 + 27468 =$ **128788 N/m²**

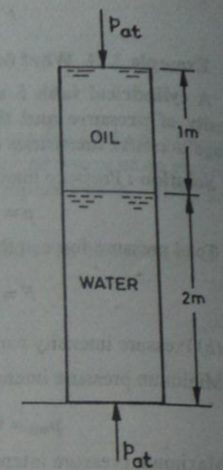

Fig. 3.8.

The surface of the base exposed to water is acted upon by p_{base} absolute whereas atmospheric pressure acts on the other surface. As such, the net force experienced by the base of the tank would be due to gauge pressure at the base.

$$F = (p_{base})_{gauge} \times \text{cross–sectional area}$$

$$= 27468 \times (5 \times 10^{-4}) = \mathbf{13.734 \text{ N}}$$

Example 3.13. The fuel gauge for a gasoline tank in a car reads proportional to the bottom gauge as shown in Fig. 3.9. The tank is 25 cm deep and accidently constains 1.5 cm of water in addition to the gasoline. Estimate the height of air remaining at the top when the gauge erroneously reads 'full'. Take specific weight of gasoline = 6.65 kN/m³ and specific weight of air = 0.0118 kN/m³.

Solution : When full of gasoline,

$$p_{gauge} = wh$$

$$= 6.65 \times 0.25 = 1.6625 \text{ kN/m}^2$$

The gauge would erroneously read 1.6625 kN/m² even when h centimeters of air remain at the top. Obviously when water is also accidently present,

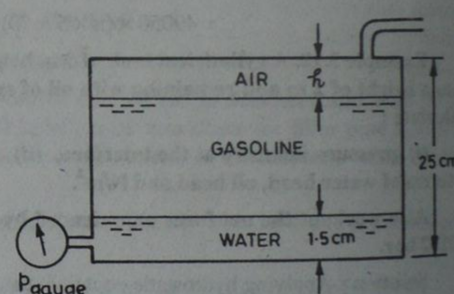

Fig. 3.9.

1.6625 = pressure due to 1.5 cm height of water
+ pressure due to $(25 - 1.5 - h)$ cm height of gasoline
+ pressure due to h cm height of air

$$= 9.81 \times \frac{1.5}{100} + 6.65\left(\frac{25 - 1.5 - h}{100}\right) + 0.0118 \times \frac{h}{100}$$

Solution gives : $h =$ **0.714 cm** of air

Example 3.14. A spherical container 2 m in diameter is made up of two hemispheres, one resting on the other with interface horizontal. The sphere is completely filled with oil of specific gravity 0.7, through a small hole on the top. What should be the minimum weight of the upper hemisphere to prevent it from lifting ?

Solution : Weight of oil in the upper hemisphere

= specific weight × volume of oil in the hemisphere

$$=(0.7 \times 9810) \times \frac{1}{2}\left\{\frac{4}{3}\pi (1)^3\right\} = 14375 \text{ N}$$

This equals the downward force at the plane of hemisphere interface.

From hydrostatic equation, the pressure intensity at the interface is

$$p = wh = (0.7 \times 9810) \times 1 = 6867 \text{ N/m}^2$$

and this acts uniformly over an area of $\frac{\pi}{4} d^2$

$$\text{Vertical upward force} = \frac{\pi}{4} \times (2)^2 \times 6867 = 21562 \text{ N}$$

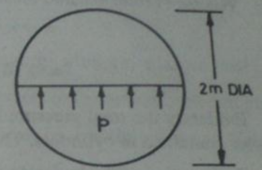

Fig. 3.10.

$$\text{Net vertical upward force} = 21562 - 14375$$

$$= 7188 \text{ N} = 7.188 \text{ kN}$$

∴ Minimum weight of upper hemisphere = **7.188 kN**

Example 3.15. Explain the concept of hydrostatic paradox.

A cylinder 30 cm in diameter and 100 cm in height is fixed centrally on the top of a large cylinder of 100 cm diameter and 60 cm length. Both the cylinders are filled with water. Calculate *(i)* total pressure at the bottom of bigger cylinder *(ii)* weight of total volume of water. What is the hydrostatic paradox between the two results and how this difference can be reconciled ?

Solution : All the three different vessels shown in Fig. 3.11 are filled with a liquid upto the same height h and have the same area A at the bottom. From hydrostatic equation, $p = wh$, the pressure density depends only upon the height of column and not at all upon the size of column. Accordingly in these vessels of different shapes and sizes, the same unit pressure would be exerted against the bottom of the vessels. Since each vessel has the same area A at the bottom, pressure force $F = pA$ on the base of each vessel would be same. This is independent of the fact that the weight of water in each vessel is different. This situation is referred to as *hydrostatic paradox*.

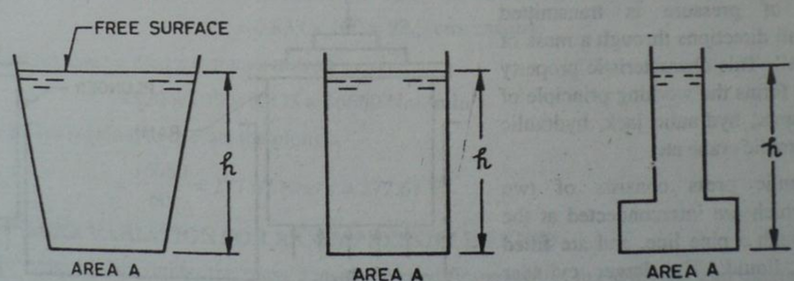

Fig. 3.11. *Hydrostatic paradox*

(b) Depth of water on the bottom of bigger tank,

$$h = 1 + 0.6 = 1.6 \text{ m}$$

Area at the bottom, $A = \frac{\pi}{4}(1)^2 = 0.785 \text{ m}^2$

Pressure intensity at the bottom, $p = wh = 9810 \times 1.6 = 15696 \text{ N/m}^2$

Total pressure force at the bottom, $F = pA = 15696 \times 0.785 =$ **12321 N**

Weight of total volume of water contained in the cylinders,

$$= w \times \text{volume of water}$$

$$= 9810\left[\frac{\pi}{4}(1)^2 \times 0.6 + \frac{\pi}{4}(0.3)^2 \times 1\right] = \mathbf{5314\ N}$$

Evidently the total pressure force at the bottom of cylinder is greater than the weight of total volume of water contained in cylinders. This is the hydrostatic paradox.

Explanation of the hydrostatic paradox :

Total pressure force on the bottom of bigger tank,

$$= 12321 \text{ N (downward)}$$

A reaction at the roof of the lower tank is caused by an upward force which equals

$$w\,A\,y = 9810 \times \frac{\pi}{4}(1^2 - 0.3^2) \times 1 = 7007 \text{ N (upward)}$$

The distance y corresponds to depth of water in the cylinder fixed centrally on the top of the larger cylinder.

Net downward force exerted by water = 12321 − 7007 = 5314 N

and it equals the weight of water in the two cylinders.

Example 3.16. State Pascal's law and discuss its applications.

A hydraulic press has a ram 30 cm diameter and plunger 5 cm diameter. Neglecting all losses due to friction etc., calculate the force to be applied at the plunger to lift a load of 20 kN. If the plunger has a stroke of 30 cm and if it executes 100 strokes per minute, calculate the distance through which the load is raised per minute. Also make calculations for the power expended in driving the plunger.

Solution : Pascal's law states that "intensity of pressure is transmitted equally in all directions through a mass of fluid at rest". This characteristic property of the fluid forms the working principle of hydraulic press, hydraulic jack, hydraulic lift and hydraulic crane etc.

Hydraulic press consists of two cylinders which are interconnected at the bottom through a pipe line, and are filled with some liquid. The larger cylinder contains a ram of area A and a plunger of area a reciprocates inside the smaller cylinder. A force F applied to the plunger produces an intensity of pressure p_1 which is transmitted equally in all directions

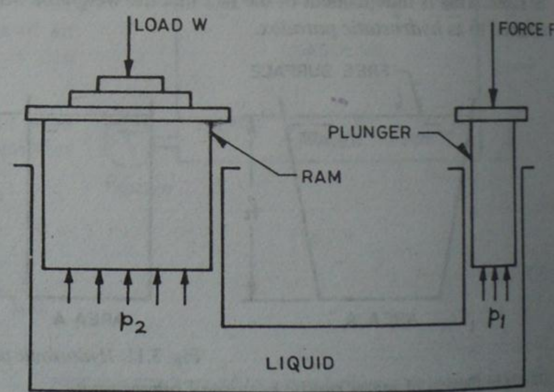

Fig. 3.12. *Working principle of hydraulic press*

through the liquid. If the plunger and the ram are at the same level and if their weights are neglected, then pressure intensity p_2 acting on the ram must equal p_1.

Now $p_1 = \frac{F}{a}$ and $p_2 = \frac{W}{A}$ where W is the weight to be lifted by the ram. If $p_1 = p_2$, then

$$\frac{F}{a} = \frac{W}{A} \; ; \quad W = F\left(\frac{A}{a}\right)$$

The above expression indicates that by applying a small force F on the plunger, a larger load can be lifted by the ram. Mechanical advantage of the system equals $\left(\frac{A}{a}\right)$, the ratio of the area of ram and plunger. By proper adjustment of plunger and ram areas, even a small force can be multiplied many times; the hydraulic press is often called "a machine of multiplying forces."

Given : $W = 20 \times 10^3$ N ; $a = \frac{\pi}{4}(5)^2$ cm^2 and $A = \frac{\pi}{4}(30)^2$ cm^2

Substituting this data in the expression $\frac{F}{a} = \frac{W}{A}$,

$$\frac{F}{\frac{\pi}{4}(5)^2} = \frac{20 \times 10^3}{\frac{\pi}{4}(30)^2} \; ; \quad \mathbf{F = 555.6\ N}$$

(*b*) Stroke volume on the plunger side = stroke × area $= 30 \times \frac{\pi}{4}(5)^2$ cm^3

If S_r denotes the distance moved by the ram in one stroke, then stroke volume on ram side equals $S_r \times \frac{\pi}{4}(30)^2$. Stroke volumes on the plunger and ram side equal each other. Therefore :

$$S_r \times \frac{\pi}{4}(30)^2 = 30 \times \frac{\pi}{4}(5)^2 \; ; \quad S_r = 0.833 \text{ cm}$$

∴ Distance through which load is raised on the ram side

= (distance moved by the ram in one stroke) × number of strokes/minute

= 0.833 × 100 = **83.3 cm/minute**

(*c*) Work done = load × distance moved

$= (20 \times 10^3) \times 0.833 = 16660$ Nm/min

∴ Power required to operate the plunger

$= \frac{16660}{60} = 277.67$ Nm/s = **277.67 W**

3.7. PRESSURE VARIATION FOR A COMPRESSIBLE FLUID

For compressible fluids, the mass density or specific weight is not constant and changes both with pressure and elevation. Apparently, while evaluating the integral $\int dp = -\int w\,dy$ use has to made of some functional relationship between pressure and specific weight. This aspect assumes significance when dealing with atmosphere in the field of Aeronautics and Meteorology. The atmosphere envelops the earth in a series or concentric layers of shells. Both pressure and density decrease with height. On the basis of changes in temperature in the atmosphere, two major layers may be recognised : troposphere and stratosphere. The lowest layer *troposphere* extends upto height of about 11 km above sea level and in this region the temperature of the air decreases at an average rate of 6.5°C per kilometer of height above sea level. Functional relation between pressure and density is polytropic with index $n = 1.24$. The *stratosphere* extends beyond 11 km and is characterised by the existence of a fairly constant temperature –56.5°C for an altitude upon 32 km. Functional relation between pressure and density in this region is isothermal. It may, however, be pointed out that transition from one region to another is gradual and the zone of transition is called *tropopause*.

Now we proceed to establish relations for the variation of pressure, density and temperature with altitude in the regions of troposphere and stratosphere. The conditions at any height above sea level shall be evaluated with respect to conditions at sea level which are designated by the subscript o.

***Troposphere or Polytropic Atmosphere :** Here the functional relationship between p and ρ is given by :

$$\frac{p_0}{\rho_0^n} = \frac{p_1}{\rho_1^n} = \frac{p}{\rho^n} = \text{constant}$$

or
$$\rho = \rho_0 \left(\frac{p}{p_0} \right)^{1/n}$$

Substituting the above value of density ρ in the hydrostatic equation :

$$dp = -w\,dy = -\rho\,g\,dy = -\rho_0 \left(\frac{p}{p_0} \right)^{1/n} g\,dy$$

After separating the variables and integrating :

$$\int_0^1 p^{-\frac{1}{n}}\,dp = \frac{-\rho_0}{p_0}\,g\,p_0^{\frac{n-1}{n}} \int_0^1 dy$$

or
$$\frac{n}{n-1}\left[p_1^{\frac{n-1}{n}} - p_0^{\frac{n-1}{n}} \right] = -\frac{\rho_0}{p_0}\,g\,p_0^{\frac{n-1}{n}}\,(y_1 - y_0)$$

Substituting $\dfrac{p_0}{\rho_0} = RT_0$ and simplifying :

$$\frac{p_1}{p_0} = \left[1 - g\,\frac{n-1}{n\,RT_0}\,(y_1 - y_0) \right]^{n/(n-1)} \quad \text{...(3.12)}$$

Recognising that for a polytropic process,

$$\frac{\rho_1}{\rho_0} = \left(\frac{p_1}{p_0} \right)^{1/n} \quad \text{and} \quad \frac{T_1}{T_0} = \left(\frac{p_1}{p_0} \right)^{\frac{n-1}{n}}$$

we obtain the following expressions representing variation of density and temperature with altitude.

$$\frac{\rho_1}{\rho_0} = \left[1 - g\,\frac{n-1}{nRT_0}\,(y_1 - y_0) \right]^{1/(n-1)} \quad \text{...(3.13)}$$

$$\frac{T_1}{T_0} = \left[1 - g\,\frac{n-1}{nRT_0}\,(y_1 - y_0) \right] \quad \text{...(3.14)}$$

or
$$T_1 = T_0 - \lambda\,(y_1 - y_0) \quad \text{...(3.15)}$$

where the coefficient λ (pronounced 'lamda') $= g\dfrac{n-1}{nR}$ is the temperature lapse rate, *i.e.*, the rate of decrease of temperature with altitude.

***Stratosphere or Isothermal Atmosphere :** Here the functional relation between pressure p and density ρ is defined by the Boyle's law, *i.e.*,

$$\frac{p}{\rho}=\frac{p_1}{\rho_1}=\text{constant}\ ;\quad \rho=p\frac{\rho_1}{p_1}$$

Substituting the above value of ρ in hydrostatic equation :

$$dp=-w\,dy=-\rho\,g\,dy=-p\frac{\rho_1}{p_1}g\,dy$$

After separating the variables and integrating :

$$\int_1^2\frac{dp}{p}=-\frac{\rho_1}{p_1}g\int_1^2 dy$$

or

$$\log_e\frac{p_2}{p_1}=-\frac{\rho_1}{p_1}g\,(y_2-y_1)$$

Further, the atmosphere can be treated as perfect gas for which the characteristic gas equation $p=\rho RT$ applies. Hence $p_1/\rho_1=RT_s$ where T_s is the constant temperature in isothermal atmosphere.

$\therefore$

$$\log_e\frac{p_2}{p_1}=-\frac{g}{RT_s}(y_2-y_1)$$

or

$$\frac{p_2}{p_1}=\exp\left[\frac{-g}{RT_S}(y_2-y_1)\right]\qquad\text{...(3.16)}$$

Since, from Boyle's law $p_2/p_1=\rho_2/\rho_1$, the expression for density variation with altitude in the stratosphere would be given by :

$$\frac{\rho_2}{\rho_1}=\exp\left[\frac{-g}{RT_S}(y_2-y_1)\right]\qquad\text{...(3.17)}$$

Note : The gravitational acceleration g varies with elevation y from the sea level :

$$g=g_0\left(\frac{r_0}{r_0+y}\right)^2$$

where g_0 is acceleration at sea level at the standard latitude of 40° : g is local acceleration at any altitude, and r_0 is the earth's mean radius (about 6400 km). However its variation in the stratosphere is less than 0.5% and accordingly it can be assumed constant without introducing any appreciable error.

Example 3.17. Calculate the pressure of air 7200 m above sea level assuming *(i)* air is incompressible *(ii)* pressure variation follows isothermal law, and *(iii)* pressure variation follows adiabatic law. Assume the following data at sea level and neglect variation of gravity with altitude.

Atmospheric pressure and temperature = 1.0132 bar and 15°C

Specific weight of air = 11.85 N/m³

Solution : Atmospheric pressure at sea level,

$$p_0=1.0132\text{ bar}=101320\text{ N/m}^2$$

(i) Air is incompressible : From hydrostatic law,

$$dp = -w\,dy$$

$$\int_p^{p_0} dp = -w \int_{y_0}^{y} dy$$

$$p - p_0 = -w\,(y - y_0)$$

$\therefore$
$$p = p_0 - wy \qquad (\because\ y_0 = \text{datum} = 0)$$

$$= 101320 - 11.85 \times 7200 = \mathbf{16000\ N/m^2}$$

(ii) Pressure variation follows isothermal level :

$$p = p_0 \exp\left(\frac{-gy}{RT_0}\right)$$

$$= p_0 \exp\left(\frac{-gy\,\rho_0}{p_0}\right) \qquad \left(\because\ \frac{p_0}{\rho_0} = RT_0\right)$$

$$= p_0 \exp\left(\frac{-w_0\,y}{p_0}\right)$$

$$= 101320 \exp\left(\frac{-11.85 \times 7200}{101320}\right) = 101320 \exp(-0.8421) = 43649\ \text{N/m}^2$$

(iii) Pressure variation follows adiabatic law ($n = 1.4$) :

$$p_0 = p_0\left[1 - g\frac{n-1}{nRT_0}(y - y_0)\right]^{n/(n-1)}$$

$$= p_0\left[1 - \frac{n-1}{n}\frac{g\,\rho_0}{p_0}(y - y_0)\right]^{n/(n-1)}$$

$$= p_0\left[1 - \frac{n-1}{n}\frac{w_0}{p_0}\,y\right]^{n/(n-1)} \qquad (\because\ y_0 = 0 \text{ and } w_0 = \rho_0\, g)$$

$$= 101320\left[1 - \frac{1.4-1}{1.4} \times \frac{11.85}{101320} \times 7200\right]^{1.4/(1.4-1)}$$

$$= 101320\,[1 - 0.2406]^{3.5} = 101320\,(0.7594)^{3.5} = \mathbf{38667\ N/m^2}$$

Example 3.18. Establish the following relation between temperature *T* and altitude *H*.

$$gH = \frac{nR}{n-1}(T_0 - T)$$

where T_0 is the temperature at zero altitude, *n* is the polytropic index and *R* is the gas constant for air.

At sea level, the air has the following properties :

pressure = 100 kN/m², temperature = 290 K, density = 1.225 kg/m³

Make calculations for pressure and temperature at an altitude of 3500 m. Take n = 1.25.

Solution : From characteristic gas equation, Gas constant $R = \frac{p_0}{\rho_0 T_0} = \frac{100 \times 10^3}{1.225 \times 290} = 281.49$ J/kg K

For polytropic atmosphere,

$$T = T_0 - \lambda (y - y_0) = T_0 - g\,\frac{n-1}{nR}\,(y - y_0)$$

$$= 200 - \frac{9.81 \times (1.25 - 1) \times 3500}{1.25 \times 281.49} = 265.61 \text{ K}$$

Further,

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{n-1}{n}} \; ; \; \frac{265.61}{290} = \left(\frac{p}{100}\right)^{\frac{1.25-1}{1.25}}$$

$$p = 100 \times \left(\frac{265.61}{290}\right)^{\frac{1.25}{1.25-1}} = 64.45 \text{ kN/m}^2$$

and

$$\frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{1/n} \; ; \; \frac{\rho}{1.225} = \left(\frac{64.45}{100}\right)^{1/1.25}$$

$$\therefore \quad \rho = 1.225 \times \left(\frac{64.45}{100}\right)^{1/1.25} = 0.862 \text{ kg/m}^3$$

Example 3.19. The measuring instruments fitted inside an airplane indicate pressure 1.032×10^5 Pa, temperature $T_0 = 288°$ K and density $\rho_0 = 1.285$ kg/m^3 at take off. If a standard temperature lapse rate of 0.0065° K/m is assumed, at what elevation is the plane when a pressure of 0.53×10^5 Pa is recorded ? Neglect variation of g with altitude and take airport elevation as 600 m.

(*b*) A person must breathe a constant mass rate of air to maintain his metabolic processes. If he inhales 20 times per minute at the airport level of 600 m, what would you expect his breathing rate at the calculated altitude of the plane ?

Solution : From characteristic gas equation,

Gas constant $R = \frac{p_0}{\rho_0 T_0} = \frac{1.032 \times 10^5}{1.285 \times 288} = 288.8$ Nm/kg K

Temperature lapse rate $\lambda = g\,\frac{n-1}{nR}$

$$0.0065 = 9.81 \times \frac{n-1}{278.8n} \; ; \quad n = 1.226$$

For polytropic atmosphere, $\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{n-1}{n}}$

∴ Temperature at the altitude of flight,

$$T = 288 \times \left(\frac{0.53}{1.032}\right)^{\frac{1.226-1}{1.226}} = 225.37° \text{ K}$$

Also

$$T = T_0 - \lambda (y - y_0)$$

$$y - y_0 = \frac{T_0 - T}{\lambda} = \frac{288 - 255.37}{0.0065} = 5020 \text{ m}$$

$\therefore$ Altitude of plane, $y = 5020 + 600 = \mathbf{5620\ m}$

(*b*) Density of air at the altitude of flight works out as

$$\rho = \rho_0 \left(\frac{p}{p_0}\right)^{\frac{1}{n}} = 1.285 \times \left(\frac{0.53}{1.032}\right)^{\frac{1}{1.226}} = 0.744 \text{ kg/m}^3$$

For the same mass of air, the inhaling rate is inversly proportional to density.

$\therefore$ Rate of inhaling at the flight altitude,

$$= 20 \times \frac{1.285}{0.744} = \mathbf{34.54} \text{ times per minute.}$$

SECTION B. Hydrostatic Forces On Submerged Surfaces

Uptil now we engaged our attention to the computation of pressure intensity at a point in a static mass of fluid. However, in many practical applications an engineer is required to determine the pressure forces on the entire surface rather than the pressure intensity at a point. The noteable examples are :

— forces on submerged objects such as submarines, ships and balloons

— forces on walls of containers such as pipes, tanks and dams

— forces on gates in the walls of containers, submerged bodies and many other hydraulic structures.

This section concentrates on the methods employed to compute the magnitude, location and direction of a resultant pressure force acting on a plane or curved surface.

3.8. FORCE ON A HORIZONTAL SUBMERGED PLANE SURFACE

Fig. 3.13 shows a plane surface submerged and held in a horizontal position at depth y below the free surface of the liquid. Since every point on the surface is at the same depth, the pressure intensity is constant over the entire plane surface. From hydrostatic equation $p = wh$, and if A is the total area of the surface then total pressure force on the horizontal surface is :

$$F = p\,A = w\,y\,A = A\,(wy) \qquad ...(3.18)$$

For the given configuration, the depth $y = y_c$ depth of the centre of gravity (centroid) of the submerged surface below the free surface of the liquid:

$$\therefore \quad F = w\,A\,y_c \qquad ...(3.19)$$

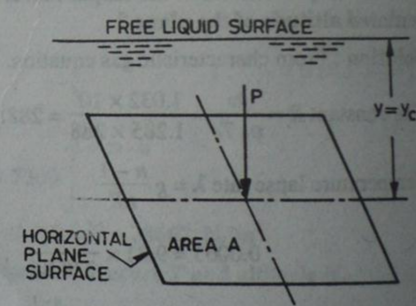

Fig. 3.13. *Force on a horizontal submerged plane surface*

3.9. FORCE ON A VERTICAL PLANE SUBMERGED SURFACE

Consider a plane surface of arbitrary shape immersed vertically in a static mass of fluid (Fig. 3.14). Depth of the liquid varies from point to point ; pressure intensity is thus not constant over the entire surface. Analysis for the total pressure is then made by dividing the entire surface into a number of small parallel strips. The force on a small strip is calculated and the total pressure force on the whole area is then obtained by integrating the force on small strip.

Consider an elementary strip of area dA at a depth y from the free surface of liquid and parallel to it.

Pressure intensity on the strip, $p = wy$

Area of the strip $= dA$

Pressure force on the strip,

$$dF = p\, dA = w\, y\, dA$$

Total pressure force on the whole surface,

$$F = \int wy\, dA = w \int y\, dA$$

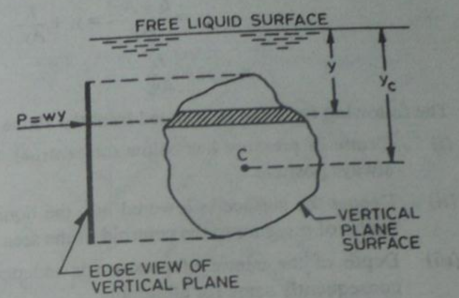

Fig. 3.14. *Force on a vertical plane submerged surface*

But $\int y\, dA$ = moment of the surface about the free liquid surface

= area of the surface × distance of centre of gravity of the immersed surface from the free liquid surface

$$= A\, y_c \quad \text{...(3.20)}$$

$$\therefore \quad F = w\, A\, y_c$$

Depth of centre of pressure : Centre of pressure defines the point of application of the total pressure force on the surface and its location can be calculated by applying the principle of moments which states that :

"sum of the moment of the resultant force about an axis is equal to the sum of the components about the same axis".

Total pressure force on the strip, $dF = w\, y\, dA$

Moment of this pressure force about the free liquid surface $= w\, y\, dA \times y = w\, y^2\, dA$

Sum of moments of all such pressure forces about the free liquid surface

$$= \int w\, y^2\, dA = w \int y^2\, dA$$

But $\int y^2\, dA$ = moment of inertia of the entire surface about the free liquid surface $= I_0$

$\therefore$ Sum of moments about the free liquid surface $= wI_0$ ...(3.21)

Let the total pressure force $F = w\, A\, y_c$ act at point P which lies at distance y_p from the free liquid surface. Moment of force F about the free surface of liquid

$$= w\, A\, y_c \times y_p \quad \text{...(3.22)}$$

Equating equations (3.21) and (3.22),

$$w\, A\, y_c \times y_p = w\, I_0\ ; \quad y_p = \frac{I_0}{A y_c}$$

From the theorem of parallel axis, $I_0 = I_c + A y_c^2$ where I_c is the moment of inertia of the area about an axis passing through the centroid

$$\therefore \quad y_P = \frac{I_c + A y_c^2}{A y_c} = y_c + \frac{I_c}{A y_c} \quad \text{...(3.23)}$$

$$y_P - y_c = \frac{I_c}{A y_c}$$

The following facts can be gleaned from the above equation :

(i) Centre of pressure lies below the centroid, because for any plane surface, the factor $I_c / A y_c$ is always positive

(ii) Deeper the surface is lowered into the liquid (*i.e.*, greater is the value of y_c) closer comes the centre of pressure to the centroid of the area

(iii) Depth of the centre of pressure is independent of the specific weight of the liquid and is consequently same for all liquids.

If a vertical rectangular surface has breadth b and depth d, then

$$y_P = \frac{d}{2} + \frac{bd^3/12}{(bd)\, d/2} = \frac{2}{3} d$$

i.e., the centre of pressure is at a depth equal to two- third of the submerged height of the surface below the liquid level.

3.10. FORCE ON AN INCLINED SUBMERGED PLANE SURFACE

Consider a plane surface of arbitrary shape immersed entirely in a static mass of liquid (Fig. 3.15). The plane of the surface intersects the free liquid surface at axis $0 - 0'$ making an arbitrary angle α. The axis $0 - 0'$ is perpendicular to the plane of the surface.

The differential force dF on an elementary strip of area dA, where the pressure intensity is p, is given by

$$dF = p\, dA = w\, y\, dA = w\, l \sin\alpha\, dA \quad \text{...(3.24)}$$

where y is the vertical depth of the elementary area from the free liquid surface, and l is its distance from the axis $0 - 0'$.

Force on the entire immersed surface can be computed by integrating the differential force dF over the entire area A.

$$F = \int p dA = w \sin\alpha \int l\, dA$$

Integral $\int l\, dA$ is the first moment of area A about the axis $0 - 0'$ and is equal to $A l_c$ where l_c is the distance of centroid of the immersed surface from axis $0 - 0'$.

$$F = w \sin\alpha\, A\, l_c$$

$$= w A (l_c \sin\alpha) = w A y_c \quad \text{...(3.25)}$$

Evidently through equations (3.19), (3.20) and (3.25), we conclude that

"Whatever may be the inclination of the submerged plane surface to the free liquid surface, the magnitude of the resultant hydrostatic force equals the product of the area and the pressure at the centroid of the area".

Centre of pressure defines the point of application of total hydrostatic pressure force on the surface, and its location is determined by taking moments of relevant forces about the axis $0 - 0'$.

$$\therefore \quad y_p = \frac{I_c + Ay_c^2}{Ay_c} = y_c + \frac{I_c}{Ay_c} \qquad ...(3.23)$$

$$y_p - y_c = \frac{I_c}{Ay_c}$$

The following facts can be gleaned from the above equation :

(i) Centre of pressure lies below the centroid, because for any plane surface, the factor I_c / Ay_c is always positive

(ii) Deeper the surface is lowered into the liquid (*i.e.*, greater is the value of y_c) closer comes the centre of pressure to the centroid of the area

(iii) Depth of the centre of pressure is independent of the specific weight of the liquid and is consequently same for all liquids.

If a vertical rectangular surface has breadth b and depth d, then

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where y is the vertical depth of the elementary area from the free liquid surface, and l is its distance from the axis $0 - 0'$.

Force on the entire immersed surface can be computed by integrating the differential force dF over the entire area A.

$$F = \int p dA = w \sin\alpha \int l\, dA$$

Integral $\int l\, dA$ is the first moment of area A about the axis $0 - 0'$ and is equal to Al_c where l_c is the distance of centroid of the immersed surface from axis $0 - 0'$.

$$F = w \sin\alpha\, A\, l_c$$

$$= w\, A\, (l_c \sin\alpha) = w\, A\, y_c \qquad ...(3.25)$$

Evidently through equations (3.19), (3.20) and (3.25), we conclude that

"Whatever may be the inclination of the submerged plane surface to the free liquid surface, the magnitude of the resultant hydrostatic force equals the product of the area and the pressure at the centroid of the area".

Centre of pressure defines the point of application of total hydrostatic pressure force on the surface, and its location is determined by taking moments of relevant forces about the axis $0 - 0'$.

Moment of pressure force dF on the elementary strip about axis $0-0'$.

$= dF \times l$

$= w\,y\,dA \times l = w\,(l \sin\alpha)\,dA \times l$

Sum of the moments of all such elementary forces about axis $0-0'$.

$= \int w \sin\alpha\, l^2\, dA = w \sin\alpha \int l^2\, dA$

The integral $\int l^2\, dA = I_0$ represents the second moment of area (moment of inertia) of the surface about axis $0-0'$.

$\therefore$ Sum of moments of all the elementary forces about axis $0-0'$.

$$= w \sin\alpha\, I_0 \qquad \text{...(3.26)}$$

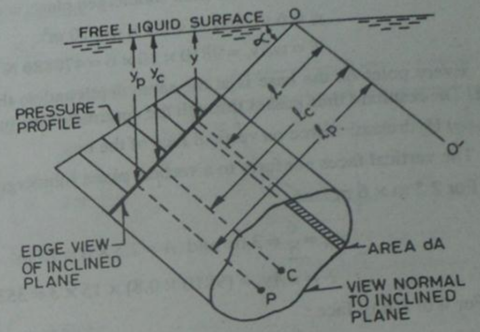

Fig. 3.15. *Force on an inclined submerged plane surface*

If l_p denotes the distance of centre of pressure of the total hydrostatic pressure force, $F = w\,A\,y_c$, from the axis $0-0'$, then the moment of total force about axis $0-0'$ is also given by,

$$= w\,A\,y_c \times l_p \qquad \text{...(3.27)}$$

Equating the two expressions for the total moments given by equations (3.26) and (3.27), we get

$$w\,A\,y_c \times l_p = w \sin\alpha\, I_0$$

From the theorem of parallel axis, $I_0 = I_c + A\,l_c^2$, where I_c is the moment of inertia of the surface about an axis passing through centroid and parallel to axis $0-0'$. Then

$$w\,A\,y_c \times l_p = w \sin\alpha\,(I_c + A l_c^2)$$

But
$$l_p = \frac{y_p}{\sin\alpha} \quad \text{and} \quad l_c = \frac{y_c}{\sin\alpha}$$

$$\therefore \quad w\,A\,y_c \times \frac{y_p}{\sin\alpha} = w \sin\alpha \left[I_c + A\,\frac{y_c^2}{\sin^2\alpha} \right]$$

Upon simplification, the depth of centre of pressure from the free liquid surface is

$$y_p = \frac{I_c \sin^2\alpha}{A\,y_c} + y_c$$

$$y_p - y_c = \frac{I_c \sin^2\alpha}{A\,y_c} \qquad \text{...(3.28)}$$

For a horizontal surface $\alpha = 0$ and so $y_p = y_c$ *i.e.*, the centre of pressure coincides with the centroid.

For a vertical surface $\alpha = 90°$ and so

$$y_p - y_c = \frac{I_c}{A y_c} \qquad \text{same as equation (3.23)}$$

Example 3.20. A rectangular box with base 2.5 m × 4 m is filled with kerosene oil of specific gravity 0.8 to a depth of 6 m. Determine the resultant pressure and its point of application on the base and on each vertical face of the box.

Solution : *(a)* Hydrostatic force on the base of the box

The base conforms to a horizontal submerged plane with

$$y_c = 6 \text{ m} \text{ and } A = 2.5 \times 4 = 10 \text{ m}^2$$

$$\therefore \quad F = wAy_c = 9810 \times 10 \times 6 = \mathbf{470880 \ N}$$

Every point on the base is at the same depth and so the pressure intensity is constant over the entire plane. The resultant thus passes through the centroid of the base and is directed vertically downwards.

(b) Hydrostatic force on vertical face of the box.

The vertical faces conform to a vertical plane submerged surface

For 2.5 m × 6 m face :

$$y_c = \frac{6}{2} = 3 \text{ m} \text{ and } A = 2.5 \times 6 = 15 \text{ m}^2$$

$$\therefore \quad F = wAy_c = (9810 \times 0.8) \times 15 \times 3 = \mathbf{353160 \ N}$$

For 4 m × 6 m face :

$$y_c = \frac{6}{2} = 3 \text{ m} \text{ and } A = 4 \times 6 = 24 \text{ m}^2$$

$$\therefore \quad F = wAy_c = (9810 \times 0.8) \times 24 \times 3 = \mathbf{565056 \ N}$$

The centre of pressure for each vertical face lies at a depth equal to $\frac{2}{3}$ time the submerged height, *i.e.*, at $\frac{2}{3} \times 6 = 4$ m below the free surface.

Example 3.21. ***(a)* A rectangular sluice gate of breadh *b* and depth *d* has been provided in the vertical side of a lock. If the centroid of the area is *h* units below the free water surface, show that the depth of centre of pressure is given by**

$$y_p = \left(h + \frac{d^2}{12h} \right)$$

***(b)* A vertical semicircular area of diameter *d* and radius *r* is submerged and has its diameter in a liquid surface. Derive an expression for the depth to its centre of pressure.**

Solution : *(a)* Area of sluice gate = $b \times d$

Depth of centroid from the free surface, $y_c = h$

Moment of inertia of the area about centre of gravity,

$$I_c = \frac{bd^3}{12}$$

Position of centre of pressure is given by

$$y_p = y_c + \frac{I_c}{Ay_c}$$

$$= h + \frac{bd^3/12}{bd \times h} = \left(h + \frac{d^2}{12h} \right)$$

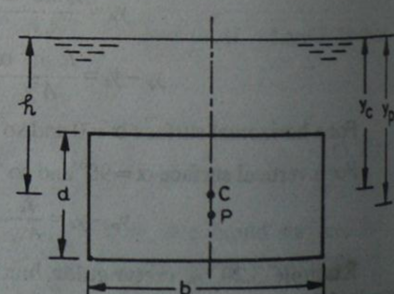

Fig. 3.16.

(b) Area of semicircle = $\frac{\pi r^2}{2}$

Depth of centroid from the free surface,

$$y_c = \frac{4r}{3\pi}$$

Moment of inertia of the area about the free surface,

$$I_0 = \frac{1}{2}\left(\frac{\pi d^4}{64}\right) = \frac{1}{2}\left[\frac{\pi (2r)^2}{64}\right] = \frac{\pi r^2}{8}$$

From the theorem of parallel axis, $I_0 = I_c + Ay_c^2$ where I_c is the moment of inertia of the area about an axis passing through the centroid : $I_c = I_0 - Ay_c^2$

$$= \frac{\pi r^2}{8} - \frac{\pi r^2}{2} \times \left(\frac{4r}{3\pi}\right)^2 = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$$

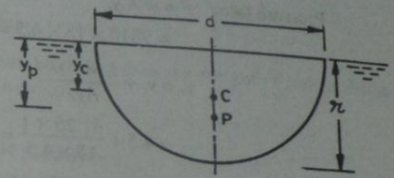

Fig. 3.17.

Position of centre of pressure in given by,

$$y_P = y_c + \frac{I_c}{Ay_c}$$

$$= \frac{4r}{3\pi} + \frac{[\pi/8 - 8/(9\pi)]\, r^4}{\pi r^2/2 \times \frac{4r}{3\pi}} = 0.589\, r$$

Example 3.22. A rectangular plate 3 m × 5 m is immersed vertically in water such that the 3 m side is parallel to the water surface. Determine the hydrostatic force and the centre of pressure if the top edge of the surface is

(i) **Flush with the water surface, *(ii)* 2 m below the water surface**

Comment on the results.

Solution : The situation corresponds to a vertical plane submerged surface.

Case (i) : Upper edge coincides with the water surface

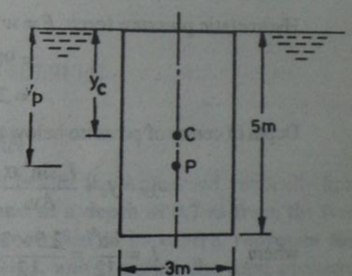

Fig. 3.18.

Depth to centroid, $y_c = \frac{5}{2} = 2.5$ m

Area of the plate, $A = 3 \times 5 = 15$ m^2

∴ Pressure force $F = w\, A\, y_c$

$= 9810 \times 15 \times 2.5 =$ **367875 N**

Depth of Centre of pressure below the water surface is given by

$$y_P = y_c + \frac{I_c \sin^2 \alpha}{Ay_c}$$

where $I_c = \frac{bd^3}{12} = \frac{3 \times 5^3}{12} = 31.25$ m^4 and $\sin\alpha = \sin 90° = 1$

$$\therefore \quad y_P = 2.5 + \frac{31.25 \times 1}{15 \times 2.5} = \mathbf{3.333\ m}$$

Centre of pressure is $(3.333 - 2.5) = 0.8333$ m below the centroid.

Case (ii) : Upper edge is 2 m below the water surface

$$y_c = 2 + \frac{5}{2} = 4.5 \text{ m} \; ; \; A = 3 \times 5 = 15 \text{ m}^2$$

$$I_c = 31.25 \text{ m}^4 \text{ and } \sin\alpha = \sin 90^\circ = 1$$

Pressure force $F = w\,A\,y_c$

$$= 9810 \times 15 \times 4.5 = \mathbf{662175\ N}$$

$$y_P = y_c + \frac{I_c \sin^2\alpha}{A y_c}$$

$$= 4.5 + \frac{31.25 \times 1}{15 \times 4.5} = \mathbf{4.963\ m}$$

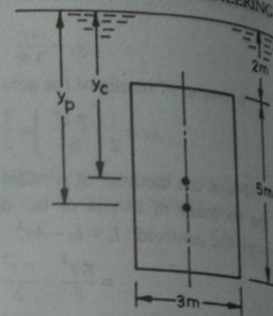

Fig. 3.19.

Centre of pressure is $(4.963 - 4.5) = 0.463$ m below the centroid.

Comparing the positions of centre of pressure for the two cases we note that as the depth of immersion is increased, closer comes the centre of pressure of the centroid of the area.

Example 3.23. Water rises to level E in the pipe attached to tank ABCD as shown in Fig. 3.20. Neglecting the weight of the tank and riser pipe, determine : *(a)* magnitude and location of the resultant force acting on area AB which is 2.5 m wide, *(b)* total force on the bottom of the tank. Compare the total weight of water with the result in *(b)* and explain the difference.

Solution : *(a)* Hydrostatic force on vertical face AB of the tank.

The vertical face conforms to a vertical plane submerged surface.

Depth of centroid below the free surface of water at E,

$$y_c = 3.7 + \frac{2.0}{2} = 4.7 \text{ m}$$

Area of the face, $A = 2.5 \times 2 = 5 \text{ m}^2$

Hydrostatic pressure force, $F = w\,A\,y_c$

$$= 9810 \times 5 \times 4.7$$

$$= \mathbf{230535\ N/m^2}$$

Depth of centre of pressure below the force water surface is

$$y_P = y_c + \frac{I_c \sin^2\alpha}{A y_c}$$

where $$I_c = \frac{bd^3}{12} = \frac{2.5 \times 2^3}{12} = 1.67 \text{ m}^2$$

$$\sin\alpha = \sin 90^\circ = 1$$

$$\therefore \quad y_P = 4.7 + \frac{1.67 \times 1}{5 \times 4.7} = 4.771 \text{ m}$$

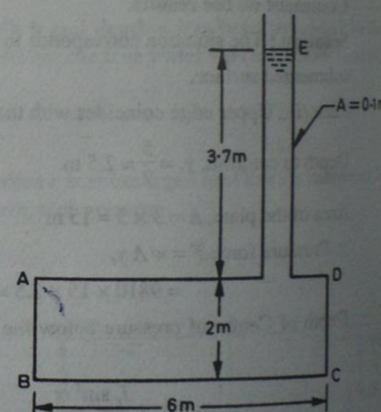

Fig. 3.20.

Centre of pressure is $(4.771 - 4.7) = 0.071$ m below the centroid.

(b) Hydrostatic force on the base BC of the box

The base conforms to a horizontal submerged plane with

$$y_c = (3.7 + 2.0) = 5.7 \text{ m and } A = 6 \times 2.5 = 15 \text{ m}^2$$

$$\therefore \quad F = w\,A\,y_c = 9810 \times 15 \times 5.7 = \mathbf{838755\ N}$$

This force passes through centroid of the base and is directed vertically downwards.

(*c*) Total weight of water in the tank and pipe

$$= \text{specific weight} \times \text{volume of water}$$

$$= 9810 \times (6 \times 1 \times 2.5 + 3.7 \times 0.1) = \mathbf{297930\ N}$$

Evidently the total pressure force at the bottom of the tank is greater than the weight of total volume of water contained in the arrangement. This is the hydrostatic paradox.

Explanation of hydrostatic paradox :

Total pressure force on the bottom of the tank = 838755 N (downward)

A tension at the roof of the tank (face AD) is caused by an upward force which equals

$$w\,A\,y = 9810\,(6 \times 2.5 - 0.1) \times 3.7 = 540825 \text{ N (upward)}$$

Net downward force exerted by water

$$= 838755 - 540825 = 297930 \text{ N}$$

and it equals the total weight of water in the tank and pipe.

Example 3.24. A sliding gate 2 m wide × 1 m high and weighing 20 kN has been located in a vertical plane with its upper edge at a depth of 5 m from the surface of water. Calculate the vertical force required to raise the gate. It may be presumed that the gate has a coefficient of friction of 0.12 between itself and the guides. Neglect buoyancy effect on the gate.

Solution : Area of the gate, $A = 2 \times 1 = 2 \text{ m}^2$

Depth of centre of gate from the free water surface,

$$y_c = 5 + \left(\frac{1}{2} \times 1\right) = 5.5 \text{ m}$$

Pressure force on the gate, $F = w\,A\,y_c$

$$= 9810 \times 2 \times 5.5 = 107910 \text{ N} \approx 107.91 \text{ kN}$$

Force required to raise the gate = $\mu\,F$ + weight of gate

$$= 0.12 \times 107.91 + 20 = \mathbf{32.95\ kN}$$

Example 3.25. A rectangular plate of 2 m length and 1 m height lies immersed vertically in a liquid of relative density 0.75 such that 2 m side is parallel to and at a depth of 0.7 m from the free liquid surface. If the plate has a circular hole of 0.5 m diameter drilled at its centre, represent the problem in a clear line sketch with all the given data incorporated, and compute the total pressure exerted by the liquid on the plate and the depth of the centre of pressure.

Solution : Specific weight of liquid

$$= 0.75 \times 9810 = 7357.5 \text{ N/m}^3$$

Area of plate, $A = 2 \times 1 - \frac{\pi}{4}(0.5)^2 = 1.804 \text{ m}^2$

Depth of centroid of plate from the free surface,

$$y_c = 0.7 + \frac{1}{2} = 1.2 \text{ m}$$

$\because$ Total pressure force $F = w\,A\,y_c$

$= 7357.5 \times 1.804 \times 1.2 = \mathbf{15927\ N}$

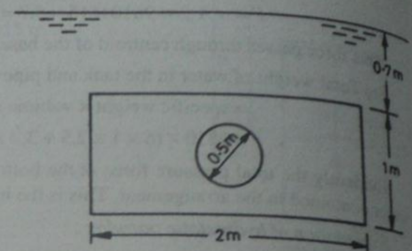

Fig. 3.21.

Moment of inertia about the centroid axis parallel to the free surface,

$$I_c = \frac{2 \times 1^3}{12} - \frac{\pi}{64}(0.5)^2 = 0.1636 \text{ m}^4$$

$\therefore$ Depth of centre of pressure,

$$y_p = y_c + \frac{I_c}{A y_c}$$

$$= 1.2 + \frac{0.1636}{1.804 \times 1.2} = \mathbf{1.2756\ m}$$

The centre of pressure is $(1.2756 - 1.2) = 0.0756$ m below the centroid.

Example 3.26. An isosceles triangular plate of base 4 m and altitude 6 m is immersed vertically in water. Its axis of symmetry is parallel to and at a depth of 6m from the free water surface. Calculate the magnitude and location of total pressure force.

Solution : Area of the surface immersed,

$$A = \frac{1}{2} \times BC \times AD$$

$$= \frac{1}{2} \times 4 \times 6 = 12 \text{ m}^2$$

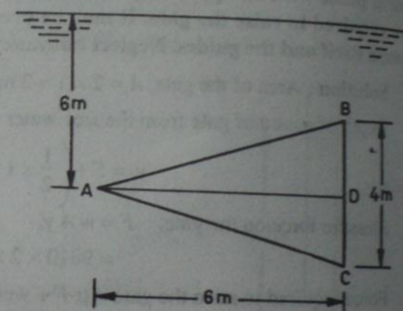

Fig. 3.22.

Depth of centroid from the free surface,

$$y_c = 6 \text{ m}$$

Total pressure force $= w\,A\,y_c$

$= 9.81 \times 12 \times 6 = \mathbf{706.32\ kN}$

The triangular plate lies immersed with its axis AD parallel to the free water surface and its moment of inertia about the centroidal axis is given by,

I_c = moment of inertia of triangle ABD about AD

+ moment of inertia of triangle ACD about AD

$$= \frac{AD \times (BD)^3}{12} + \frac{AD \times (CD)^3}{12}$$

$$= 2\,\frac{AD \times (BD)^3}{12} \qquad (\because\ BD = CD = 2 \text{ m})$$

$$= 2\left(\frac{6 \times 2^3}{12}\right) = 8 \text{ m}^4$$

$\therefore$ Depth of centre of pressure, $y_p = y_c + \dfrac{I_c}{A y_c}$

$$= 6 + \frac{8}{12 \times 6} = \mathbf{6.111\ m}$$

The centre of pressure is (6.111 – 6) = 0.111 m below the centroid.

Example 3.27. A triangular plate of base width 1.5 m and height 2 m lies immersed in water with the apex downwards. The base of the plate is 1 m below and parallel to the free water surface. Calculate the total pressure on the plate and the depth of centre of pressure.

Solution : Area of the plate, $A = \frac{1}{2} \times 1.5 \times 2 = 1.5\ m^2$

Depth of centre of plate from the free water surface,

$$y_c = 1 + \left(\frac{1}{3} \times 2\right) = 1.667\ m$$

Pressure force on the plate, $F = w\,A\,y_c$

$$= 9810 \times 1.5 \times 1.667$$
$$= \mathbf{24530\ N}$$

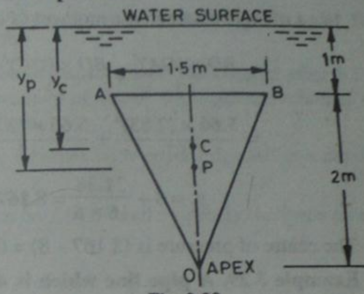

Fig. 3.23.

Depth of centre of pressure below the water surface is given by

$$y_p = y_c + \frac{I_c}{A y_c}$$

For a triangular element, the moment of inertia about its centroid is $\frac{bh^3}{36}$. Therefore,

$$I_c = \frac{1.5 \times 2^3}{36} = 0.333\ m^4$$

$$\therefore \quad y_p = 1.667 + \frac{0.333}{1.5 \times 1.667} = \mathbf{1.80\ m}$$

Centre of pressure is (1.80 – 1.667) = 0.133 m below the centroid.

Example 3.28. A square plate 4 m × 4 m hangs in water from one of its corners and its centroid lies at a depth of 8 m from the free water surface. Workout the total pressure on the plate and locate the position of centre of pressure with respect to the plate centroid.

Solution : Area of the plate, $A = 4 \times 4 = 16\ m^2$

Depth to centroid, $y_c = 8$ m

$\therefore$ Pressure force $F = w\,A\,y_c$

$$= 9810 \times 16 \times 8$$
$$= \mathbf{1255680\ N} \approx 1.26\ MN$$

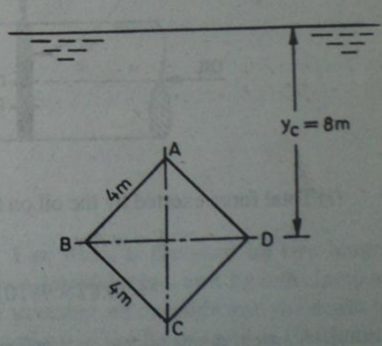

Fig. 3.24.

Depth of centre of pressure below the water surface is given by

$$y_p = y_c + \frac{I_c}{A y_c}$$

The plate lies with its diagonal *BD* parallel to the free water surface and its moment of inertia about the centre is given by

I_c = moment of inertia of triangle ABD about the line BD + moment of inertia of triangle BCD about the line BD

Now $AC = BD = \sqrt{4^2+4^2} = 5.66$ m and

$$OA = OC = \frac{1}{2} \times 5.66 = 2.83 \text{ m}$$

For a triangle element, the moment of inertia about the base is $\frac{bh^3}{12}$. Therefore,

$$I_c = \frac{BD \times (OA)^3}{12} + \frac{BD \times (OC)^3}{12}$$

$$= \frac{5.66 \times (2.83)^3}{12} + \frac{5.66 \times (2.83)^3}{12} = 21.38 \text{ m}^4$$

$$\therefore \quad y_P = 8 + \frac{21.38}{16 \times 8} = 8.167 \text{ m}$$

The centre of pressure is (8.167 – 8) = 0.167 m below the centroid.

Example 3.29. A pipe line which is 4 m in diameter contains a gate valve. The pressure at the centre of pipe is 1.962 Pa. If the pipe is filled with oil of specific gravity 0.87, find the force exerted by the oil upon the gate and the position of the centre of pressure.

Solution : From hydrostatic equation, $p = wh$

$\therefore$ pressure head at the centre of pipe,

$$h = \frac{p}{w} = \frac{1.962 \times 10^5}{0.87 \times 9810} = 22.98 \text{ m}$$

Thus the height of equivalent free oil surface above the centre of pipe is, $y_c = 22.98$ m

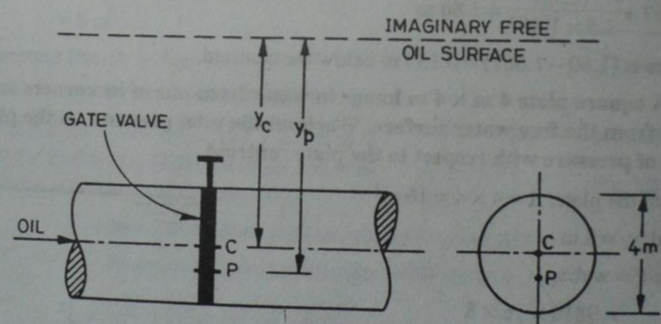

Fig. 3.25.

(i) Total force exerted by the oil on the gate is,

$$F = w\,A\,y_c$$

$$= (0.87 \times 9810) \times \frac{\pi}{4}(4)^2 \times 22.98 = 2.46 \times 10^6 \text{ N}$$

(ii) Vertical depth of centre of pressure,

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{A y_c} \quad ; \sin \alpha = \sin 90° = 1$$

$$\therefore \quad y_p = 22.98 + \frac{\frac{\pi}{64}(4)^4 \times 1}{\frac{\pi}{4}(4)^2 \times 22.98} = \mathbf{23.033\ m}$$

This distance is measured from the equivalent level surface the oil. Apparently this point lies (23.033 – 22.98) = 0.043 m below the centre of pipe.

Example 3.30. A vertical square area 1 m × 1 m is submerged in water with upper edge 0.5 m below the water surface. Locate the horizontal line on the surface of the of the square such that the force on the upper portion equals the force on the lower portion.

Solution : *ABCD* is the square plate submerged vertically in water with upper edge *AB* at a depth of 0.5 m below the free water surface.

Let *EF* be the line such that force on *AEFB* equals the force on *EDCF*, and evidently the force on each portion equals half the total force on the entire plate *ABCD*.

Total pressure force on the entire plate *ABCD*

$$= w A y_c = w \times (1 \times b) \times \left(0.5 + \frac{1.0}{2}\right) = wb$$

where *b* represents the width of plate

Total pressure force on portion *AEFB*

$$= w \times (y \times b) \times \left(0.5 + \frac{y}{2}\right) = wyb\left(0.5 + \frac{y}{2}\right)$$

Now, pressure force on *AEFB*

$$= \frac{1}{2} \times \text{ pressure force on } ABCD$$

or $$wyb\left(0.5 + \frac{y}{2}\right) = \frac{1}{2} \times wb$$

or $$2y\left(0.5 + \frac{y}{2}\right) = 1$$

$$\therefore \quad y = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2} = 0.618 \text{ m} \quad \text{or} -1.618 \text{ m}$$

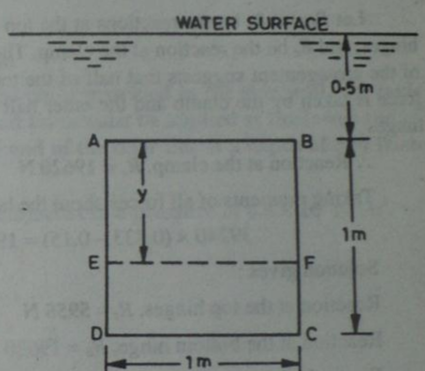

Fig. 3.26.

Evidently, $y = \mathbf{0.618\ m}$

Example 3.31. A rectangular vertical door, 2 m high × 1 m wide, is fastened by two hinges situated 15 cm below the top and 15 cm above the bottom on one vertical edge, and by one clamp at the centre of other vertical edge. The door is subjected to water pressure on one side and the depth of water above the top of door is 1 m. Make calculations for the reactions at the hinges and at the clamp.

Solution : Depth of centroid of the door,

$$y_c = 1 + \left(\frac{1}{2} \times 2\right) = 2 \text{ m}$$

Area of the door, $A = 2 \times 1 = 2 \text{ m}^2$

Total pressure force on the door,

$$F = w\,A\,y_c = 9810 \times 2 \times 2 = 39240 \text{ N}$$

The above resultant horizontal pressure force acts at the centre of pressure.

Depth of centre of pressure

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{A y_c} \quad ; \quad \sin \alpha = \sin 90^\circ = 1$$

$$I_c = \frac{bd^3}{12} = \frac{1 \times 2^3}{12} = 0.667 \text{ m}^4$$

$$\therefore \quad y_p = 2.0 + \frac{0.667 \times 1}{2 \times 2} = 2.167 \text{ m}$$

This corresponds to $(3 - 2.167) = 0.833$ m above the base.

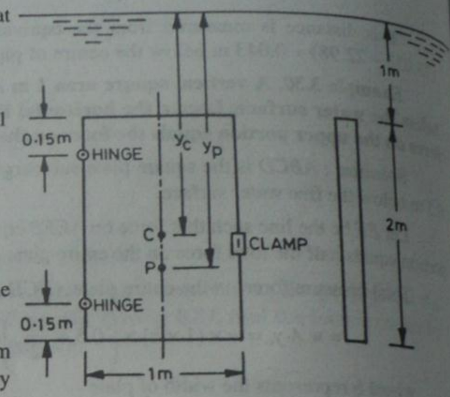

Fig. 3.27.

Let R_t and R_b be the reactions at the top and bottom hinges, and R_c be the reaction at the clamp. The symmetry of the arrangement suggests that half of the total pressure force is taken by the clamp and the other half by the two hinges

$\therefore$ Reaction at the clamp, $R_c =$ **19620 N**

Taking moments of all forces about the horizontal axis through the bottom hinge, we have

$$39240 \times (0.833 - 0.15) = 19620 \times 0.85 + R \times (2 - 0.15 - 0.15)$$

Solution gives :

Reaction at the top hinges, $R_t =$ **5955 N**

Reaction at the bottom hinge, $R_b = 19620 - 5955 =$ **13665 N**

Example 3.32. A circular opening, 3 metres in diameter, in the vertical side of a water tank is closed by a disc of 3 meters diameter, which can rotate about a horizontal diameter. Calculate *(i)* the force on the disc *(ii)* torque required to maintain the disc in the vertical position when the head of water above the horizontal diameter is 4 metres.

Solution : Area of opening, $A = \frac{\pi}{4} \times 3^2 = 7.068 \text{ m}^2$

Depth of centroid from the free surface, $y_c = 4$ m

(i) Total force on the disc, $F = w\,A\,y_c = 9810 \times 7.068 \times 4 =$ **277348 N**

(ii) Depth of centre of pressure,

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{A y_c} \quad ; \quad \sin \alpha = \sin 90^\circ = 1 \quad ; \quad I_c = \frac{\pi}{64}(3)^4 = 3.976 \text{ m}^4$$

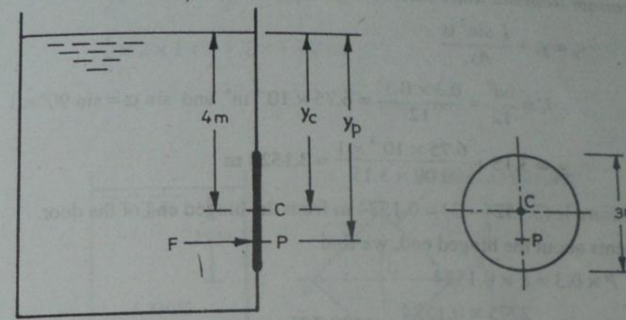

Fig. 3.28.

∴ Distance between the centre of pressure and the centroid,

$$y_p - y_c = \frac{3.976 \times 1}{7.068 \times 4} = 0.1406 \text{ m}$$

∴ Torque on the disc, T = force × force arm

$$= 277348 \times 0.1406 = \mathbf{38995\ Nm}$$

Example 3.33. A square door with side dimensions 30 cm is provided in the side wall of a tank which is filled with water of specific weight 9790 N/m³. What force must be applied at the lower end of the gate so as to hold the hinged door closed ? The hinged end of the door lies at a depth of 3 m from the free water surface.

How this force would change if the water surface is subjected to a pressure of 0.5×10^5 N/m².

Solution : Depth of centroid, $y_c = 3 + \dfrac{0.3}{2} = 3.15$ m

Area of the door, $A = 0.3 \times 0.3 = 0.09$ m²

Hydrostatic force on the door,

$$F = w\,A\,y_c = 9790 \times 0.09 \times 3.15 = 2775 \text{ N}$$

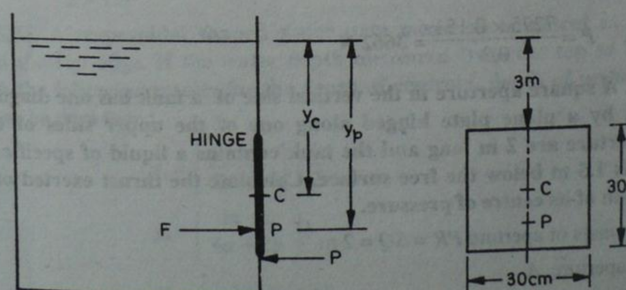

Fig. 3.29.

Depth of pressure from the water surface is given by :

$$y_P = y_c + \frac{I_c \sin^2 \alpha}{A y_c}$$

where $I_c = \frac{bd^3}{12} = \frac{0.3 \times 0.3^3}{12} = 6.75 \times 10^{-4}\ \text{m}^4$ and $\sin\alpha = \sin 90^\circ = 1$

$$\therefore \quad y_P = 3.15 + \frac{6.75 \times 10^{-4} \times 1}{0.09 \times 3.15} = \mathbf{3.1524\ m}$$

Centre of pressure is (3.1524 – 3) = 0.1524 m from the hinged end of the door.

Taking moments about the hinged end, we find

$$P \times 0.3 = F \times 0.1524$$

$$\therefore \quad P = \frac{2775 \times 0.1524}{0.3} = \mathbf{1409.7\ N}$$

Thus a horizontal force equivalent to 1409.7 N should be applied at the lower end to keep the door closed.

(b) From hydrostatic equation, $p = wh$

$\therefore$ Water head equivalent to the given pressure of $0.5 \times 10^5\ \text{N/m}^2$ above the water surface is

$$h = \frac{p}{w} = \frac{0.5 \times 10^5}{9790} = 5.107 \text{ m of water}$$

Now calculations for the magnitude and location of the pressure force are to be made corresponding to a water depth (3 + 5.107) = 8.107 m above the hinged end of the door.

$$y_c = 8.107 + \frac{0.3}{2} = 8.257 \text{ m}$$

$$F = w A y_c = 9790 \times 0.09 \times 8.257 = 7275 \text{ N}$$

$$y_P = y_c + \frac{I_c \sin^2 \alpha}{A y_c} = 8.257 + \frac{6.75 \times 10^{-4} \times 1}{0.09 \times 8.257} = 8.258 \text{ m}$$

Centre of pressure is thus (8.258 – 8.107) = 0.151 m from the hinged end of door.

Taking moments about the hinged end,

$$P \times 0.3 = F \times 0.151$$

$$P = \frac{7275 \times 0.151}{0.3} = \mathbf{3662\ N}$$

Example 3.34. A square aperture in the vertical side of a tank has one diagonal vertical and is completely covered by a plane plate hinged along one of the upper sides of the aperture. The diagonals of the aperture are 2 m long and the tank contains a liquid of specific gravity 1.15. The centre of aperture is 1.5 m below the free surface. Calculate the thrust exerted on the plate by the liquid and the position of its centre of pressure.

Solution : Diagonals of aperture $PR = SQ = 2$ m

Area of square aperture, A

$$= \text{area of } \Delta\, PRS + \text{area of } \Delta\, PRQ$$

$$= \frac{1}{2} PR \times SO + \frac{1}{2} PR \times QO$$

$$= \frac{1}{2} \times 2 \times 1 + \frac{1}{2} \times 2 \times 1 = 2 \text{ m}^2$$

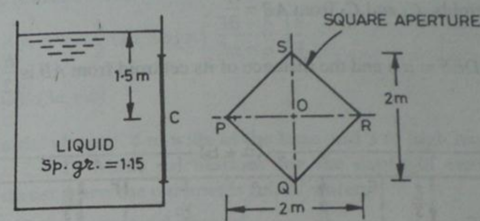

Fig. 3.30.

Depth of centre of aperture plate from the free water surface, $y_c = 1.5$ m

Pressure force or thrust on the plate,

$$F = wAy_c = (1.15 \times 9810) \times 2 \times 1.5 = \mathbf{33844.5\ N}$$

Depth of centre of pressure below the free water surface is given by

$$y_p = y_c + \frac{I_c}{Ay_c}$$

where I_c = moment of inertia of $PQRS$ about diagonal PR

= MOI of $\Delta\, PSR$ about PR + MOI of $\Delta\, PQR$ about PR

The momnet of inertia of a triangle about its base equals base × (height)^3/12

$$\therefore \quad I_c = \frac{2 \times 1^3}{12} + \frac{2 \times 1^3}{12} = 0.333 \text{ m}^4$$

$$\text{Hence} \quad y_p = 1.5 + \frac{0.333}{2 \times 1.5} = \mathbf{1.611\ m}$$

Example 3.35. A trapezoidal shaped water dam measures *a* metres at the bottom edge and $(a + b)$ metres at the top edge. If the water depth measured from the top to bottom edge stands *h* metres, establish the following results for the depth of centroid, depth of pressure and the resultant pressure force on the dam face :

$$y_c = \left(\frac{3a + b}{2a + b}\right) \times \frac{h}{3}$$

$$y_p = \left(\frac{4a + b}{6a + 2b}\right) h$$

$$F = \frac{1}{6} wh^2 (3a + b)$$

Solution : Area of the trapezium $A = \frac{1}{2}$ [top length AB + bottom length DC] × height

$$= \frac{1}{2}[(a+b)+a]\,h = \frac{h}{2}(2a+b)$$

Area of the two triangles ADE and $BCF = 2\left[\frac{1}{2}\times\frac{b}{2}\times h\right] = \frac{1}{2}bh$

and distance of centroids, C_1 and C_2 from $AB = \frac{h}{3}$

Area of rectangle $CDEF = a\,h$ and the distance of its centroid from AB is $\frac{h}{2}$

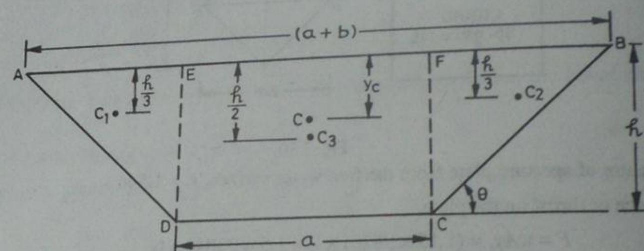

Fig. 3.31.

(i) Position of the centroid of trapezium can be determined by taking moments of these areas about AB.

Moment of $ABCD$ = moment of rectangle $CDEF$ + moment of triangles ADE and BCF

$$\frac{h}{2}(2a+b)\,y_c = ah\times\frac{h}{2} + \frac{1}{2}bh\times\frac{h}{3}$$

Simplification would give, $y_c = \left(\dfrac{3a+b}{2a+b}\right)\dfrac{h}{3}$

(ii) Depth of centre of pressure,

$$y_p = y_c + \frac{I_c \sin^2\alpha}{Ay_c}\;;\quad \sin\alpha = \sin 90^\circ = 1$$

I_c = momentum of inertia of the trapezium $ABCD$ about its centroid C

= moment of inertia of the rectangle $CDEF$ about G_3 + moment of inertia of the triangles ADE and BCF about their centroids C_1 and C_2

$$= \frac{ah^3}{12} + 2\left(\frac{b}{2}\times\frac{h^3}{36}\right) = \frac{h^3}{36}(3a+b)$$

$$\therefore \quad y_P = \left(\frac{3a+b}{2a+b}\right)\frac{h}{3} + \frac{\frac{h^3}{36}(3a+b)}{\frac{h}{2}(2a+b)\times\left(\frac{3a+b}{2a+b}\right)\frac{h}{3}}$$

$$= \left(\frac{3a+b}{2a+b}\right)\frac{h}{3} + \frac{h}{6} = \frac{h}{6}\left[\frac{6a+b}{2a+b}+1\right] = \left(\frac{4a+b}{6a+2b}\right)h$$

(iii) Resultant hydrostatic pressure force,

$$F = w\,A\,y_c = w \times \frac{h}{2}(2a+b)\times\left(\frac{3a+b}{2a+b}\right)\frac{h}{3}$$

$$= \frac{1}{6}wh^2(3a+b)$$

Example 3.36. A trapezoidal channel 4 m wide at the base and 3 m high has sides inclined at 45° to the horizontal (side slopes 1 : 1). Make calculations for the depth of centroid and centre of pressure and the hydrostatic thrust when the channel is full of water.

Solution : Refer Fig. 3.31 for the trapezoidal channel.

From the given data,

$$a = 4 \text{ m}; \qquad h = 3 \text{ m} \quad \text{and} \quad \theta = 45^\circ$$

the top width works out to be 10 m, $\quad \therefore \quad b = 6$ m

(i) Depth of centroid below the surface of water level,

$$y_c = \left(\frac{3a+b}{2a+b}\right)\frac{h}{3} = \left(\frac{3\times4+6}{2\times4+6}\right)\times\frac{3}{3} = \mathbf{1.286\ m}$$

(ii) Depth of centre of pressure,

$$y_P = \left(\frac{4a+b}{6a+2b}\right)h = \left(\frac{4\times4+6}{6\times4+2\times6}\right)\times 3 = \mathbf{1.833\ m}$$

(iii) Hydrostatic pressure force,

$$F = \frac{1}{6}wh^2(3a+b)$$

$$= \frac{1}{6}\times 9810\times 3^2(3\times4+6) = \mathbf{264870\ N}$$

Example 3.37. A vertical dock gate separates two water reservoirs of depth H_1 and H_2. Find the resultant pressure exerted on the gate and the point of its application if $H_1 : H_2 = 2$. To what position does this line tend as the depth of water in both sides becomes equal ?

Solution : Total pressure on the left side of the gate,

$$F_1 = wAy_c$$

$$= w(BH_1)\times\frac{H_1}{2} = \frac{wBH_1^2}{2}$$

where B is the breadth of the gate. This force acts at a distance of $\frac{H_1}{3}$ from the bottom.

Total pressure on the right side of the gate

$$F_2 = w\,(BH_2) \times \frac{H_2}{2} = \frac{wBH_2^2}{2}$$

and this acts at a distance of $\frac{H_2}{3}$ from the bottom

∴ Resultant horizontal force on the gate,

$$F = F_1 - F_2 = \frac{wB}{2}\,(H_1^2 - H_2^2)$$

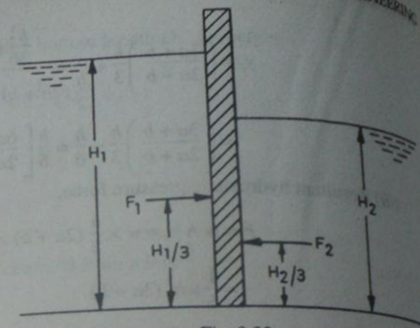

Fig. 3.32.

Let the resultant horizontal force act at a distance x from the bottom. Then taking moments about the bottom,

$$F \times x = F_1 \times \frac{H_1}{3} - F_2 \times \frac{H_2}{3}$$

$$= \frac{wBH_1^3}{6} - \frac{wBH_2^3}{6}$$

$$\therefore \quad x = \frac{\frac{wB}{6}\,(H_1^3 - H_2^3)}{\frac{wB}{2}\,(H_1^2 - H_2^2)}$$

$$= \frac{(H_1 - H_2)\,(H_1^2 + H_2^2 + H_1 H_2)}{3\,(H_1 - H_2)\,(H_1 + H_2)} = \frac{H_1^2 + H_2^2 + H_1 H_2}{3\,(H_1 + H_2)}$$

(i) When $H_1 : H_2 = 2$

$$x = \frac{H_1^2 + (H_1/2)^2 + H_1 \times \frac{H_1}{2}}{3\,(H_1 + H_1/2)} = \frac{7}{18}\,H_1$$

(ii) When depth of water is same on both sides, $H_1 = H_2$ and therefore,

$$x = \frac{H_1^2 + H_1^2 + H_1 \times H_1}{3\,(H_1 + H_1)} = \frac{H_1}{2}$$

Apparently the line of action of the resultant pressure force tends to reach the middle point of the gate when the depth of water on both sides becomes equal.

Example 3.38. A vertical rectangular gate, 6 m high and 4 m wide, has water on one side to a depth of 3 m and a liquid of specific gravity 0.85 m to a depth of 2 m on the other side. Calculate (i) total pressure exerted on each side of the gate and *(ii)* resultant hydrostatic pressure both in magnitude and point of application with respect to the bottom.

Solution : Total pressure on left side of the gate,

$$F_1 = w\,A\,y_c$$

$$= 9.81 \times (4 \times 3) \times \frac{3}{2} = 176.58 \text{ kN}$$

This force acts at a distance of $y_1 = \frac{3}{3} = 1$ m from the bottom.

Total pressure on the right side of the gate,

$$F_2 = (9.81 \times 0.85) \times (4 \times 2) \times \frac{2}{2} = 66.71 \text{ kN}$$

and this acts at a distance of $y_2 = \frac{2}{3} = 0.667$ m from the bottom.

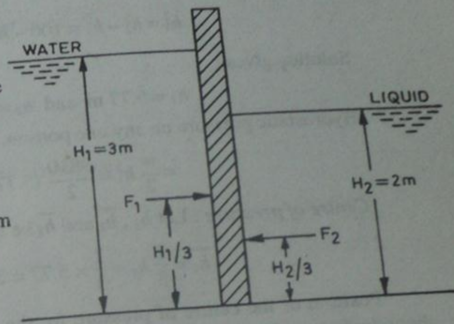

Fig. 3.33.

Resultant pressure force on the gate,

$$F = F_1 - F_2 = 176.58 - 66.71 = \mathbf{109.87 \text{ kN}}$$

Let the resultant pressure force act at a distance y from the bottom. Then taking moments about the bottom.

$$F \times y = F_1 y_1 - F_2 y_2$$

$$109.87\, y = 176.58 \times 1 - 66.71 \times 0.667 = 132.11$$

$$\therefore \quad y = \frac{132.11}{109.87} = 1.202 \text{ m}$$

The resultant force acts at distance of **1.202 m** from the bottom.

Example 3.39. The hydrostatic water pressure acts only on one side and to a depth of 10 m from the top of a dock gate which is reinforced with three horizontal beams. Locate the position of beams in order that each carries an equal load. Also determine the load on each beam.

Solution : The gate is divided into three sections so that the water pressure on each section is the same. The beams are then provided in the horizontal planes at the centre of pressure of each section.

Consider an elementary strip of thickness dh at depth h. Then for a unit width of the gate,

Force on the element, $dF = w\, h\, dh$ (hydrostatic equation)

Force on section 1, $F_1 = \int_0^{h_1} w\, h\, dh = \frac{w}{2} h_1^2$

Force on section 2, $F_2 = \int_{h_1}^{h_2} w\, h\, dh$

$$= \frac{w}{2}(h_2^2 - h_1^2)$$

Force on section 3, $F_3 = \int_{h_2}^{h_3} w\, h\, dh = \frac{w}{2}(h_3^2 - h_2^2)$

Total force on the gate, $F = \int_0^{h_3} w\, h\, dh = \frac{w}{2} h_3^2$

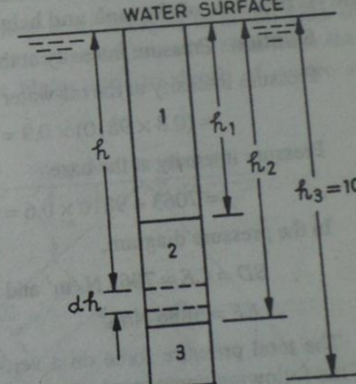

Fig. 3.34.

Load carried by each section is same and it equals one-third of total force on the gate. Thus

$$\frac{w}{2} h_1^2 = \frac{w}{2}(h_2^2 - h_1^2) = \frac{w}{2}(h_3^2 - h_2^2) = \frac{w}{6} h_3^2$$

Given $h_3 = 10$ m

$$\therefore \quad h_1^2 = h_2^2 - h_1^2 = 100 - h_2^2 = \frac{100}{3}$$

Solution gives :

$$h_1 = 5.77 \text{ m and } h_2 = 8.16 \text{ m}$$

Hydrostatic pressure on any one portion,

$$= \frac{w}{2} h_1^2 = \frac{9810}{2} (5.77)^2 = 163302 \text{ N}$$

Centre of pressure : Let $\overline{h_1}$, $\overline{h_2}$ and $\overline{h_3}$ be the centres of pressure of the three sections.

$$\overline{h_1} = \frac{2}{3} h_1 = \frac{2}{3} \times 5.77 = 3.846 \text{ m}$$

Position of the centre of pressure for the section 2 is obtained by taking moments of relevant forces about the free surface. That is

$$\frac{w}{2}(h_2^2 - h_1^2) \times \overline{h_2} = \left(\frac{w}{2} h_2^2 \times \frac{2}{3} h_2\right) - \left(\frac{w}{2} h_1^2 \times \frac{2}{3} h_1\right)$$

$$\overline{h_2} = \frac{2}{3} \frac{h_2^3 - h_1^3}{h_2^2 - h_1^2} = \frac{2}{3} \frac{(8.16)^3 - (5.77)^3}{(8.16)^2 - (5.77)^2} = 7.046 \text{ m}$$

Similarly for the bottom portion, the centre of pressure from the free surface is

$$\overline{h_3} = \frac{2}{3} \frac{h_3^3 - h_2^3}{h_3^2 - h_2^2} = \frac{2}{3} \frac{(10)^3 - (8.16)^3}{(10)^2 - (8.16)^2} = 9.084 \text{ m}$$

Example 3.40. A tank with vertical sides is square in plan with sides 2.0 m × 2.0 m and depth 1.5 m. The tank contains water upto a height of 0.6 m and an immiscible oil of specific gravity 0.8 floats on the water top for the remaining 0.9 m height. Calculate the hydrostatic pressure force on one vertical side of the tank and height of its centre of pressure above the base.

Solution : Pressure intensity at the top = 0

Pressure intensity at the oil-water interface

$$= (0.8 \times 9810) \times 0.9 = 7063 \text{ N}$$

Pressure intensity at the base

$$= 7063 + 9810 \times 0.6 = 12949 \text{ N/m}^2$$

In the pressure diagram,

$BD = CE = 7063 \text{ N/m}^2$ and

$EF = 5886 \text{ N/m}^2$

The total pressure force on a vertical force is built up of the following components :

(*i*) Pressure force F_1

$$= (\text{area of } \Delta \text{ ABD}) \times \text{ tank width}$$

$$= \left(\frac{1}{2} \times 7063 \times 0.9\right) \times 2 = 6356.7 \text{ N}$$

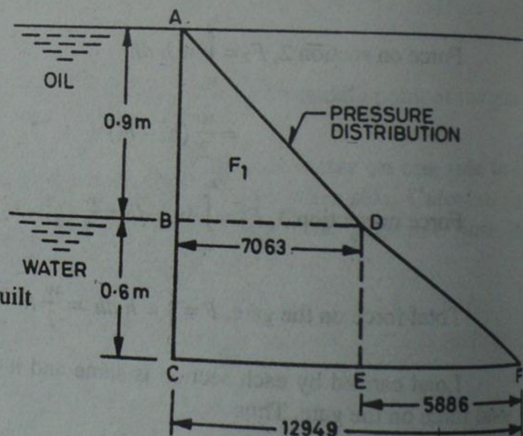

Fig. 3.35.

and this force acts at $\frac{2}{3} \times 0.9 = 0.6$ m below the free oil surface.

(ii) Pressure force, F_2

$= $ (area of rectangle BDEC) $\times$ tank width

$= (7063 \times 0.6) \times 2 = 8475.6$ N

and this force acts at $0.9 \times \frac{1}{2} + 0.6 = 1.2$ m below the free oil surface.

(iii) Pressure force, F_3

$= $ (area of Δ DEF) $\times$ tank width

$$= \left(\frac{1}{2} \times 5886 \times 0.6\right) \times 2 = 3531.6 \text{ N}$$

and this force acts at $0.9 + \frac{2}{3} \times 0.6 = 1.3$ m below the free oil surface.

$\therefore$ Total pressure force on one vertical face of the tank.

$$F = F_1 + F_2 + F_3$$

$$= 6356.7 + 8475.6 + 3531.6 = \mathbf{18364 \text{ N}}$$

The position of the resultant pressure force F can be obtained by taking moments of relevant forces about the free oil surface.

$$\bar{h} = \frac{6356.1 \times 0.6 + 8475.7 \times 1.2 + 3531.6 \times 1.3}{18364}$$

$= 1.011$ m below the free oil surface

Obviously the resultant pressure force acts at $(1.5 - 1.011) = 0.489$ m above the box.

Example 3.41. A rectangular plate 0.6 m wide and 1.2 m deep is submerged in an oil bath of specific gravity 0.8. The maximum and minimum depths of the plate are 1.6 m and 0.75 m from the free surface. Calculate the hydrostatic force on one face of the plate, and the depth of centre of pressure.

Solution :

$$\sin\theta = \frac{1.6 - 0.75}{1.2} = 0.7083 \ ; \ \theta = 45.1^\circ$$

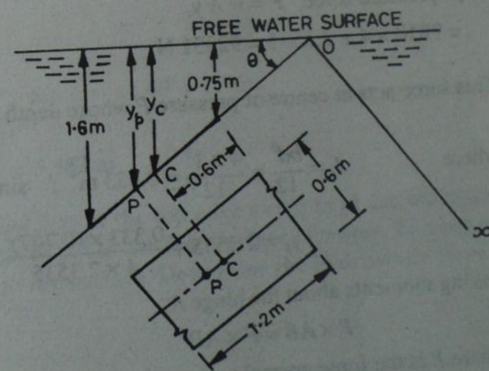

Fig. 3.36.

Depth of centroid of plate,

$y_c = 0.75 + 0.6 \sin 45.1 = 1.175$ m

Area of plate , $A = 1.2 \times 0.6 = 0.72 \text{ m}^2$

Total hydrostatic force,

$$F = w A y_c$$

$$= (9.81 \times 0.8) \times 0.72 \times 1.175$$

$$= \mathbf{6.639 \text{ kN}}$$

This force acts at centre of pressure P whose depth from the free water surface is

$$y_P = y_c + \frac{I_c \sin^2 \alpha}{A y_c}$$

where $$I_c = \frac{bd^3}{12} = \frac{0.6 \times 1.2^3}{12} = 0.0864 \; ; \quad \sin\theta = \sin 45.1^\circ = 0.7083$$

$\therefore$ $$y_P = 1.175 + \frac{0.0864 \times (0.7083)^2}{0.72 \times 1.175} = \mathbf{1.226\ m}$$

If centre of pressure is measured along the plane of the area from the axis OX, then

$$y_P = \frac{1.226}{\sin 45.1^\circ} = 1.731 \text{ m}$$

Example 3.42. An inclined rectangular sluice gate 4 m wide × 1 m deep has been installed to control the discharge of water (Fig. 3.37). The upper end A is hinged and lies at a distance of 2 m from the free surface of water. What force normal to the gate be applied at the lower end B to open it ?

Solution :

Area of the gate, $A = 4 \times 1 = 4 \text{ m}^2$

The distance of centre of gravity of a rectangular element from its either face is

$$AC = \frac{1}{2} \times \text{depth of the gate}$$

$$= \frac{1}{2} \times 1 = 0.5 \text{ m}$$

$\therefore$ Depth of centroid of the gate from the free water surface is

$$y_c = 2 + AC \sin 45^\circ$$

$$= 2 + 0.5 \times 0.707 = 2.3535 \text{ m}$$

Total pressure force, $F = w A y_c$

$$= 9810 \times 4 \times 2.3535 = \mathbf{92351\ N}$$

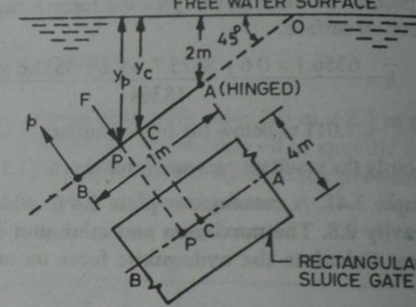

Fig. 3.37.

This force acts at centre of pressure P whose depth from the free water surface is $y_p = y_c + \dfrac{I_c \sin^2 \alpha}{A y_c}$

where $$I_c = \frac{bd^3}{12} = \frac{4 \times 1^3}{12} = 0.333 \text{ m}^4 \; ; \quad \sin\alpha = \sin 45 = 0.707$$

$\therefore$ $$y_P = 2.3535 + \frac{0.333 \times (0.707)^2}{4 \times 2.3535} = 2.3712 \text{ m}$$

Taking moments about the hinge A.

$$P \times AB = F \times AP$$

where P is the force normal to the gate applied at B to open it.

Now $$AP = OP - OA$$

$$= \frac{y_P}{\sin 45} - \frac{2}{\sin 45} = \frac{2.3712}{0.707} - \frac{2}{0.707} = 0.525 \text{ m}$$

$$\therefore \quad P \times 1 = 92351 \times 0.525 \quad ; \quad P = \frac{92351 \times 0.525}{1} = \mathbf{48484 \text{ N}}$$

Example 3.43. A triangular plate of base width 2 m and height 3 m is immersed in water with its plan making an angle of 60° with the free surface of water. Determine the hydrostatic pressure force and the position of centre of pressure when the apex of the triangle lies 5 m below the free water surface.

Solution :

Area of the plate,

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 2 \times 3 = 3 \text{ m}^2$$

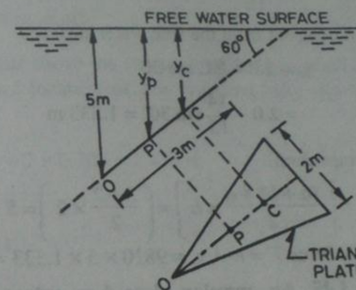

Fig. 3.38.

The distance of centre of gravity of a triangular element from its apex is :

$$OC = \frac{2}{3} \text{ of the height of triangle}$$

$$= \frac{2}{3} \times 3 = 2 \text{ m}$$

$\therefore$ Depth of centroid from the free water surface is,

$$y_c = 5 - OC \times \sin 60^\circ = 5 - 2 \times 0.866 = 3.268 \text{ m}$$

Total pressure force $F = w\,A\,y_c$

$$= (1000 \times 9.81) \times 3 \times 3.268 = \mathbf{96177 \text{ N}}$$

Depth of centre of pressure from the water surface is

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{A y_c}$$

where $\quad I_c = \frac{bh^3}{36} = \frac{2 \times 3^3}{36} = 1.5 \text{ m}^4 \quad ; \quad \sin \alpha = \sin 60^\circ = 0.866$

$$\therefore \quad y_p = 3.268 + \frac{1.5 \times (0.866)^2}{3 \times 3.268} = \mathbf{3.382 \text{ m}}$$

Example 3.44. A trapezoidal plate measuring 2 m at the top edge and 3 m at the bottom edge is immersed in water with the plan making an angle of 30° to the free surface of water. The top and the bottom edges lie at 1 m and 2 m respectively from the surface. Determine the hydrostatic force on the plate.

Solution : From the given data :

$$a = 2 \text{ m} \; ; \; a + b = 3 \text{ m} \; ; \quad \therefore b = 1 \text{ m}$$

and height of trapezium, $h = AB = \dfrac{2-1}{\sin 30} = 2$ m

Distance of centroid of a trapezium plate from its base,

$$y = \frac{h}{3}\left(\frac{3a+b}{2a+b}\right)$$

$$= \frac{2}{3}\left(\frac{3\times 2+1}{2\times 2+1}\right) = \frac{14}{15}\text{ m}$$

$$BC = \frac{14}{15}\text{ m}$$

Depth of centroid from the free water surface,

$$y_c = 2.0 - BC\sin 30^\circ$$

$$= 2.0 - \frac{14}{15}\sin 30^\circ = 1.533\text{ m}$$

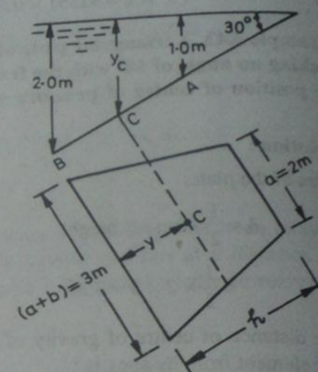

Fig. 3.39.

Area of trapezium,

$$A = \left[\frac{(a+b)+a}{2}\times h\right] = \left(\frac{3+2}{2}\times 2\right) = 5\text{ m}^2$$

Hydrostatic force $F = w\,A\,y_c = 9810 \times 5 \times 1.533 =$ **75194 N**

Example 3.45. An annular plate 4 m external diameter and 2 m internal diameter with its greatest and least depths below the surface being 3 m and 1.5 m respectively. Calculate the magnitude, direction and location of the force acting upon one side of the plate due to water pressure.

Solution : If α is the inclination of the plate with the water surface, then from the geometry of Fig. 3.40.

$$\sin\alpha = \frac{3-1.5}{4} = \frac{1.5}{4} = 0.375\ ;\quad \alpha = 22^\circ$$

Area of the plate

$$A = \frac{\pi}{4}(4^2 - 2^2) = 9.42\text{ m}^2$$

Depth to centroid

$$y_c = \frac{3+1.5}{2} = 2.25\text{ m}$$

Total pressure force $= w\,A\,y_c$

$= (9810) \times 9.42 \times 2.25 =$ **207860 N**

This force acts perpendicular to the plate so it is acting in a direction which is $(90 - 22) = 68^\circ$ to the vertical.

Vertical depth of centre of pressure,

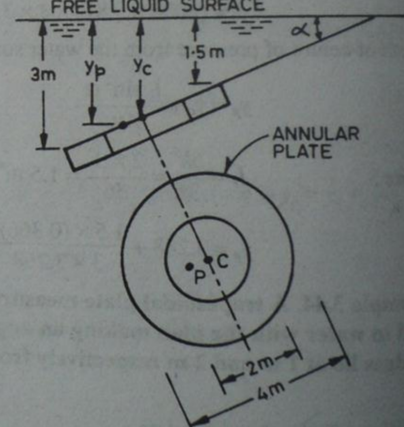

Fig. 3.40.

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{A y_c}, \quad I_c = \frac{\pi}{64}(4^4 - 2^4) = 11.78 \text{ m}^4$$

$$= 2.25 + \frac{11.78 \times 0.375^2}{9.42 \times 2.25} = \mathbf{2.328 \text{ m}}$$

Example 3.46. A 4 m square gate provided in an oil tank is hinged at its top edge (Fig. 3.41). The tank contains gasoline (specific gravity 0.7) upto a height of 2 m above the top edge of the gate. The space above the oil is subjected to a negative pressure of 6865 N/m². Determine the necessary vertical pull to be applied at the lower edge to open the gate.

Solution : Oil head equivalent to the given negative pressure above the oil surface is :

$$h = \frac{p}{w} = \frac{6865}{0.7 \times 9810} = 1.0 \text{ m of oil}$$

This negative pressure will reduce the oil head above the top edge of the gate from 2 m to (2 – 1) = 1.0 m of oil. Calculations for the magnitude and location of the pressure force are thus to be made corresponding to a head of 1.0 m of oil.

$$y_c = 1.0 + \frac{4}{2}\sin 45^\circ = 1.0 + 2 \times 0.707 = 2.414 \text{ m}$$

$$A = 4 \times 4 = 16 \text{ m}^2$$

$$F = w A y_c = (9810 \times 0.7) \times 16 \times 2.414$$

$$= 265150 \text{ N}$$

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{A y_c}$$

$$= 2.414 + \frac{\frac{1}{12} \times 4 \times 4^3 + (0.707)^2}{16 \times 2.414} = 2.804 \text{ m}$$

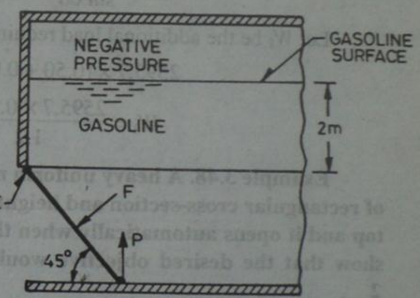

Fig. 3.41.

∴ Vertical distance of centre of pressure below top edge of the gate

$$= 2.804 - 1 = 1.804 \text{ m}$$

Taking moments about the hinge,

$$P \sin 45^\circ \times 4 = 265150 \times \frac{1.804}{\sin 45^\circ}$$

$$P \times 0.707 \times 4 = 265150 \times \frac{1.804}{0.707}; \quad P = \mathbf{239237.5 \text{ N}}$$

Thus a vertical force equivalent to 239237.5 N should be applied at the lower edge to open the gate.

Example 3.47. A circular opening in the sloping wall of a reservoir is closed by a disc valve 75 cm in diameter as indicated in the accompanying figure. The side is hinged at point A and a balance weight W is just sufficient to hold the valve close when the reservoir is empty. How much additional weight need to be placed in order that the valve remains closed until the water level is 60 cm above the centre of the valve ?

Solution : Area of the disc valve

$$A = \frac{\pi}{4}(0.75)^2 = 0.441 \text{ m}^2$$

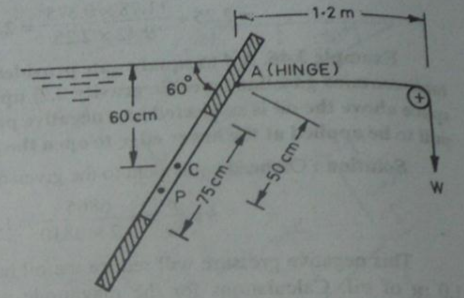

Fig. 3.42.

Depth of centroid of valve,

$$y_c = 60 \text{ cm} = 0.6 \text{ m}$$

(i) Total force on the valve,

$$F = wAy_c = 9810 \times 0.441 \times 0.6$$
$$= \mathbf{2595.7\ N}$$

(ii) Depth of centre pressure,

$$y_p = y_c + \frac{I_c \sin^2 \alpha}{Ay_c}$$

where $I_c = \frac{\pi}{64}(0.75)^4 = 0.0155 \text{ m}^4$

$$\sin^2 \alpha = \sin^2 60 = \frac{3}{4} = 0.75$$

∴ Distance between the centre of pressure and the centroid,

$$y_p - y_c = \frac{0.0155 \times 0.75}{0.441 \times 0.6} = 0.0439 \text{ m vertically below the centroid}$$

$$= \frac{0.0439}{\sin 60} = 0.0507 \text{ m below the centroid but along the sloping wall}$$

Let W_1 be the additional load required on the arm. Taking moments about the hinge, we obtain :

$$2595.7 \times (0.50 + 0.0507) = W_1 \times 1.2$$

$$W_1 = \frac{2595.7 \times 0.5507}{1.2} = \mathbf{1191.2\ N}$$

Example 3.48. A heavy uniform metal door, inclined at an angle α to the vertical, closes a tunnel of rectangular cross-section and height *h*. The door is to be so designed that it swings about the tunnel top and it opens automatically when the tunnel is just full of water. Draw the pressure diagram, and show that the desired objective would be achieved when weight per square metre of the door is $\frac{2}{3}wh \operatorname{cosec} \alpha$..

Solution :

Length of the metal door, $AB = \frac{h}{\cos \alpha} = h \sec \alpha$

The pressure intensity varies from zero at the hinge point A to *wh* at point B.

Total hydrostatic pressure force on the door,

$$F = (\text{area of pressure diagram}) \times \text{ width of door}$$

$$= \left(\frac{1}{2} h \sec \alpha \times wh\right) \times 1 = \frac{1}{2} wh^2 \sec \alpha$$

This force acts at point *C* located at distance

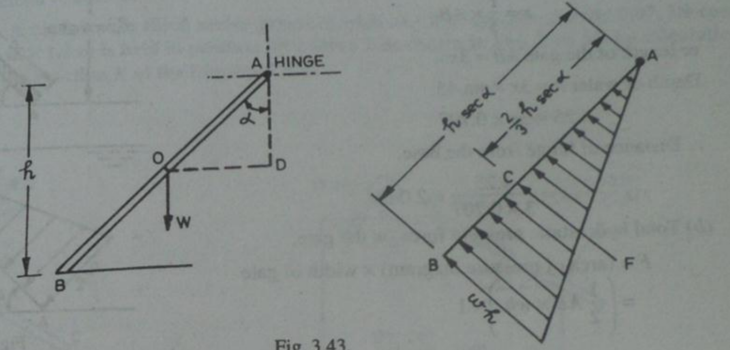

Fig. 3.43.

$$AC = \frac{2}{3}\ AB = \frac{2}{3}\ h \sec \alpha$$

from the hinge point A.

The load W (weight of metal door per unit width) acts at its centre point O.

$$OD = AO \sin \alpha = \frac{1}{2} AB \sin \alpha = \frac{h}{2} \sec \alpha \sin \alpha$$

Taking moments about the hinge point A,

$$W \times OD = F \times AC$$

$$W \times \frac{h}{2} \sec \alpha \sin \alpha = \frac{1}{2} wh^2 \sec \alpha \times \frac{2}{3} h \sec \alpha$$

$$\therefore \qquad W = \frac{2}{3} wh^2 \sec \alpha \operatorname{cosec} \alpha$$

Area of door per unit width,

$$= AB \times \text{unit width} = h \sec \alpha$$

$\therefore$ Weight per unit area of the door

$$= \frac{\frac{2}{3} wh^2 \sec \alpha \operatorname{cosec} \alpha}{h \sec \alpha} = \frac{2}{3} wh \operatorname{cosec} \alpha$$

Example 3.49. A gate supporting water takes the form of an inclined shield which swings around a hinged axis O (Fig. 3.44). Find the position x of the hinge at which a water level of $h = 4.25$ m on the left would cause the gate to tip over the hinge. (Also calculate the magnitude of hydrostatic force on the gate just before it opens (tips about the hinge) automatically. Neglect frictional effects.

Solution : The gate would tip about the hinge point O when the line of action of the resultant pressure force F lies from O to B anywhere on the gate; the limiting condition being the situation when the resultant force passes through the hinged point O. The resultant also passes through the centroid of the pressure diagram, and the centroid lies at a distance $\frac{1}{3} \times AB$ from the bottom point A.

$$\therefore \quad x = \frac{1}{3} \times AB$$

or length of the gate $AB = 3x$

Depth of water $h = 3x \times \sin 45$

$$4.25 = 3x \times 0.707$$

$\therefore$ Distance of hinge from the base,

$$x = \frac{4.25}{3 \times 0.707} = 2.0 \text{ m}$$

(*b*) Total hydrostatic pressure force on the gate,

$$F = (\text{area of pressure diagram}) \times \text{width of gate}$$

$$= \left(\frac{1}{2} AB \times wh\right) \times 1$$

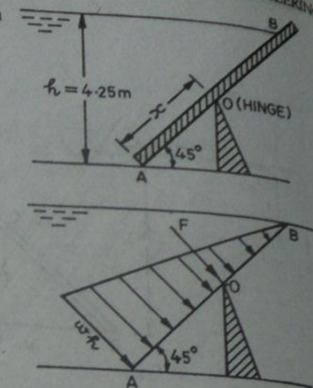

Fig. 3.44.

Length of gate, $AB = 3x = 3 \times 2.0 = 6.0$ m

$$\therefore \quad F = \left(\frac{1}{2} \times 6 \times 9810 \times 4.25\right) \times 1 = \mathbf{125077 \text{ N}} \text{ per unit width}$$

Example 3.50. A rectangular gate of dimensions 5 m × 1.5 m is hinged at its base and inclined at 60 degree with the horizontal. The gate is kept in stable position by hanging 50 kN weight at its upper end through a pulley system as shown in Fig. 3.45. Find the height *h* of water at which the gate begins to fall. Neglect the weight of gate and frictional effects at the hinge and pulley.

Solution : The pressure intensity varies from zero at point D to wh at the hinge point A.

Side AD of the pressure diagram,

$$AD = \frac{h}{\sin 60} = 1.1547\, h$$

Total hydrostatic pressure force on the gate,

$$F = (\text{area of pressure diagram}) \times \text{width of the gate}$$

$$= \left(\frac{1}{2} \times 1.1547\, h \times wh\right) \times 1.5$$

$$= 0.866\, wh^2$$

This force acts at point C located at distance

$$AC = \frac{1}{3} AD = \frac{1}{3} \times 1.1547\, h = 0.3849\, h$$

from the hinge point A.

Taking moments about the hinge point A, we obtain:

$$50 \times 10^3 \times 5 = 0.866\, wh^2 \times 0.3849\, h$$

$$= 0.333\, wh^3$$

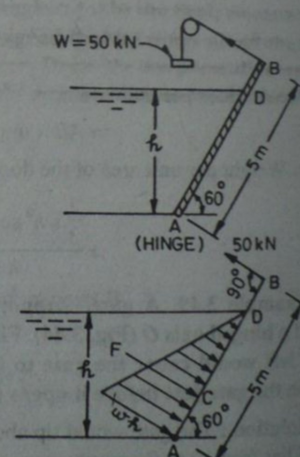

Fig. 3.45.

$$h^3 = \frac{50 \times 10^3 \times 5}{0.333 \times 9810} = 75.075$$

$\therefore$ Height of water $h = \mathbf{4.239\ m}$

Example 3.51. A container is filled under pressure with an oil of specific gravity 0.87. Its cover plate measuring 1.2 m × 1.2 m is held in position by a force Z as shown in Fig. 3.46. Make calculations for this force Z and the reaction R at the hinge point A.

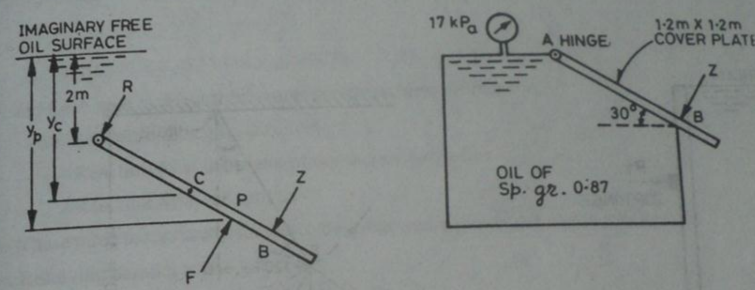

Fig. 3.46.

Solution : The oil head equivalent to the given gauge pressure of 17×10^3 N/m² is

$$h = \frac{17 \times 10^3}{9810 \times 0.87} \approx 2.0 \text{ m}$$

Hence the free water surface can be imagined to be 2 m above the hinge point A.

Depth of centre of cover plate, $(AC = 0.6 \text{ m})$

$$y_c = 2.0 + 0.6 \sin 30^\circ = 2.3 \text{ m}$$

Pressure force on the cover plate, $F = wAy_c$

$$= (9810 \times 0.87) \times (1.2 \times 1.2) \times 2.3 = 28267 \text{ N}$$

Vertical depth of centre of pressure, $y_p = y_c + \dfrac{I_c \sin^2 \alpha}{Ay_c}$

$$= 2.3 + \frac{\dfrac{1.2 \times 1.2^3}{12} \times \sin^2 30^\circ}{(1.2 \times 1.2) \times 2.3} = 2.313 \text{ m}$$

Taking moments about the hinge point A,

$$Z \times AB = F \times AP$$

$$Z \times 1.2 = 28267 \times \frac{2.313 - 2.0}{\sin 30}$$

Solution gives : Force $Z = 14746$ N

Since $R + Z = F$; Reaction $R = F - Z = 28267 - 14746 = \mathbf{13521\ N}$

Example 3.52. The gates of a lock which is 8 m wide include an angle of 120° when in the closed position. Each gate is held on two hinges, one placed at 0.8 m and the other at 6 m from the bottom of lock. If the water levels are 9 m and 3 m on the upstream and downstream sides respectively, determine

(i) resultant horizontal force due to water pressure and its line of action and

(ii) magnitude of forces on the hinges.

Solution : With reference to Fig. 3.47, *AB* and *BC* are the two gates which are held in contact at *B* by the water pressure. These gates open towards the upstream side (shown by dotted lines) and remain closed due to higher hydrostatic pressure on upstream than on the downstream.

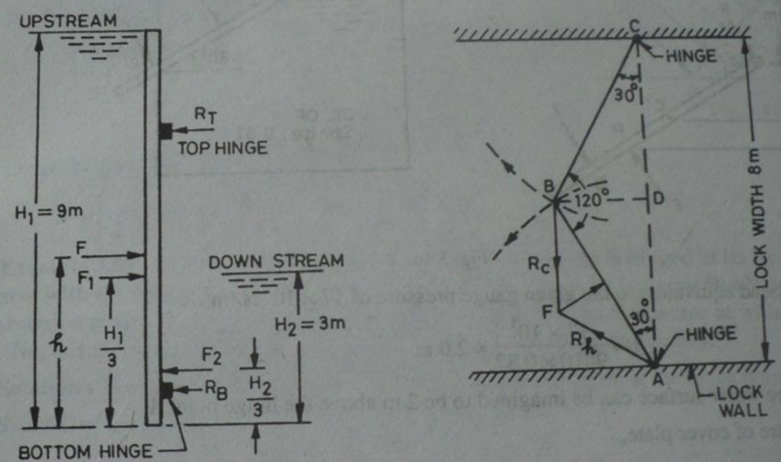

Fig. 3.47.

Width *B* of lock $= BC = \frac{CD}{\cos 30} = \frac{4}{0.866} = 4.62$ m

Area of upstream water face, $A_1 = BH_1$

Depth to centroid of upstream face $y_{C1} = \frac{H_1}{2}$

Water pressure on the upstream face

$$F_1 = wA_1\, y_{C1} = \frac{1}{2}\, w\, BH_1^2$$

$$= \frac{1}{2} \times 9810 \times 4.62 \times 9^2 = 1835549 \text{ N}$$

and it acts at $\frac{H_1}{3} = \frac{9}{3} = 3$ m from the bottom of the gate.

Likewise water pressure on the downstream side of the gate.

$$F_2 = \frac{1}{2} w \, BH_2^2 = \frac{1}{2} \times 9810 \times 4.62 \times 3^2 = 203950 \text{ N}$$

and it acts at $\frac{H_2}{3} = \frac{3}{3} = 1$ m from the bottom of the gate

∴ Resultant horizontal force due to water pressure,

$$F = (F_1 - F_2) = (1835549 - 203950) = \mathbf{1631599 \text{ N}}$$

If the resultant force F acts at a height h from the bottom, then

$$F\,h = F_1 \times \frac{H_1}{3} - F_2 \times \frac{H_2}{3}$$

$$1631599\, h = 1835549 \times 3 - 203950 \times 1 \; ; \quad h = 3.25 \text{ m}$$

(ii) Each gate is in equilibrium under the action of three forces :

— net hydrostatic pressure force F

— Reaction R_c at the plane of contact of two gates

— Reaction R_h at the hinges

These three forces are assumed to be in the same plane and so meet at a point.

Resolving parallel to the gate,

$$R_c \cos 30 = R_h \cos 30 \quad ; \quad R_c = R_h \qquad ...(i)$$

Resolving normal to the gate,

$$F = R_c \sin 30 + R_h \sin 30$$

Substituting $R_c = R_h$

$$F = 2\, R_h \sin 30 = 2\, R_h \times 0.5 = R_h \qquad ...(ii)$$

From *(i)* and *(ii)* ; $R_c = R_h = F = 1631599$ N

If R_T and R_B represent the reactions at the top and bottom hinges respectively, then

$$R_T + R_B = R_h$$

Taking moments about the bottom hinge,

$$R_T (6 - 0.8) = F\,(3.25 - 0.8)$$

$$R_T = \frac{1631599 \times (3.25 - 0.8)}{(6 - 0.8)} = 768734 \text{ N}$$

$$\therefore \quad R_B = 1631599 - 768734 = 862865 \text{ N}$$

$$\text{Ratio} \quad \frac{R_B}{R_T} = \frac{862865}{768734} = \mathbf{1.122}$$

3.11. FORCE ON CURVED SUBMERGED SURFACE

Let Fig. 3.48 represent the trace of a curved surface submerged wholly in a static mass of liquid. Consider on the curved surface an elementary area dA lying at a vertical depth y below the free surface of the liquid. If p is the normal pressure intensity at the elementary area, then the differential force acting in a direction normal to the surface is $dF = p\,dA = wy\,dA$ and the total force on the entire curved surface is $F = \int wy\,dA$. Since for a curved surface direction of the pressure force varies from point to point, straight-forward integration procedure can no longer be applied. Computation of total pressure on a curved surface is then made possible by assessing the pressure forces acting on projected horizontal and vertical planes.

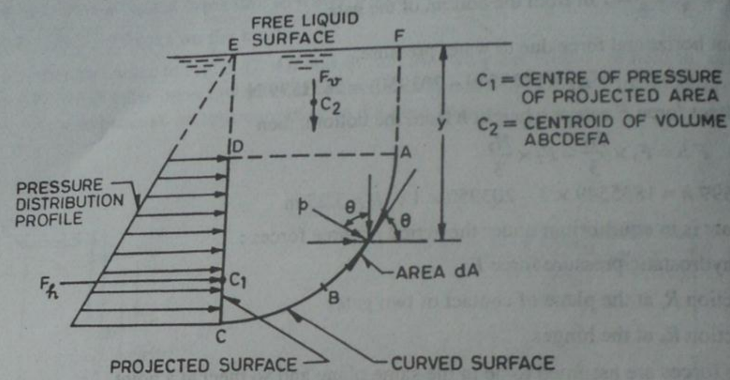

Fig. 3.48. *Force on curved submerged surface*

For the elementary area :

$$dF_h = dF \sin\theta = p\, dA \sin\theta$$

$$dF_v = dF \cos\theta = p\, dA \cos\theta$$

where θ is the inclination of the elementary area dA with the horizontal. Substituting $p = wy$ in the above expressions and subsequent integration yields :

$$F_h = \int dF_h = w\int y\, dA \sin\theta \quad \text{...(3.26 a)}$$

$$F_v = \int dF_v = w\int y\, dA \cos\theta \quad \text{...(3.26 b)}$$

In these expressions $dA \sin\theta$ and $dA \cos\theta$ represent respectively the vertical and horizontal projections of the elementary area dA.

Consequently,

$w\int y\, dA \sin\theta$ represents the total pressure force on projected area of curved surface on the vertical plane. The point of application of the horizontal component F_h is at the centre of pressure of the projected area.

$w\int y\, dA \cos\theta$ represents the total pressure on projected area of the curved surface on the horizontal plane, and it equals the weight of liquid lying in the portion *ABCDEFA*; weight of liquid extending from the curved surface to the free surface of the liquid. The point of application of the component F_v acting vertically downward is at the centroid of the liquid volume above the curved surface.

The resultant pressure force F is then equal to $\sqrt{F_v^2 + F_h^2}$; acting at angle $\tan^{-1}\left(\frac{F_v}{F_h}\right)$ with the horizontal.

In some engineering applications, the liquid acts from below the curved surface. In that case, the vertical component of pressure force acts upwards, and equals the weight of an imaginary column of liquid above the curved surface upto the free surface.

Example 3.53. The profile of a vessel is quadrant of a circle of radius *r*. Obtain from first principles the horizontal and vertical components of the total pressure force.

Solution : Consider an elementary strip of radius r at depth h and subtending an angle $d\theta$. If the vessel has a unit depth perpendicular to the plane of paper, then

Area of the element $dA = r\,d\theta \times \text{unit depth} = r d\theta$

Depth $h = r \sin\theta$

Pressure intensity $p = wh = w\,r \sin\theta$

Pressure force $dF = p\,dA = w\,r \sin\theta \times r d\theta$

$$= w\,r^2 \sin\theta\, d\theta$$

Vertical component of elementary pressure force,

$$dF_v = w\,r^2 \sin^2\theta\, d\theta$$

Horizontal component of elementary pressure force,

$$dF_h = w\,r^2 \sin\theta \cos\theta\, d\theta$$

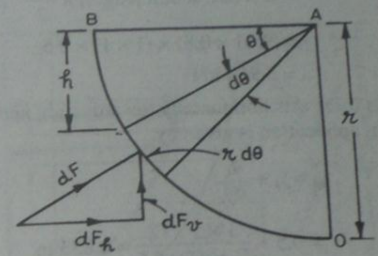

Fig. 3.49.

∴ Total vertical pressure force,

$$F_v = \int_0^{\pi/2} wr^2 \sin^2\theta\, d\theta = wr^2 \int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{wr^2}{2}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} \cos 2\theta\, d\theta\right]$$

$$= \frac{wr^2}{2}\left[\,|\,\theta\,|_0^{\pi/2} - |\sin 2\theta/2\,|_0^{\pi/2}\right] = \frac{wr^2\pi}{4}$$

$$= w\left(\frac{\pi}{4} r^2 \times \text{unit length}\right)$$

= specific weight × (volume of liquid contained in the curved surface)

Thus the vertical component of pressure force on a curved surface equals the weight of the volume of liquid extending vertically from the curved surface to the free surface of liquid.

Total horizontal pressure force,

$$P_h = \int_0^{\pi/2} wr^2 \sin\theta \cos\theta\, d\theta = \frac{wr^2}{2}\int_0^{\pi/2} 2 \sin\theta \cos\theta\, d\theta$$

$$= \frac{wr^2}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta = \frac{wr^2}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{wr^2}{2}$$

$$= w\,(r \times \text{unit depth}) \times \frac{r}{2} \equiv w\,A\,y_c$$

Thus the horizontal component of pressure force on a curved surface equals the pressure force on projected of curved surface on a vertical plane.

Example 3.54. A container of geometry shown in Fig. 3.50, contains a liquid of specific gravity 0.8 upto a depth of 2 m. Determine the magnitude and direction of hydrostatic pressure force per unit length of container exerted on its vertical face BC, curved corner CD and horizontal bottom DG.

Solution : *(i)* For the vertical face *BC* :

$$F = w\,A\,y_c$$

$$= w \times (BC \times \text{unit length}) \times \frac{BC}{2}$$

$$= (9.81 \times 0.8) \times (1 \times 1) \times 0.5$$

$$= \mathbf{3.924\ kN}$$

This force acts horizontally towards right and its point of application is given by,

$$y_p = y_c + \frac{I_c}{A y_c}$$

$$= 0.5 + \frac{1 \times 1^3/12}{(1 \times 1) \times 0.5} = \mathbf{0.833\ m}$$

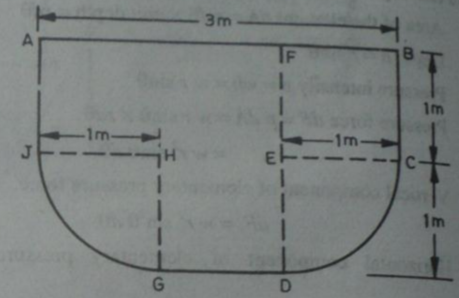

Fig. 3.50.

(ii) For the curved surface *CD* :

The horizontal component of hydrostatic pressure force on the curved corner *CD* equals the pressure force on its projected area on a vertical plane.

$$F_h = \text{specific weight} \times (\text{vertical projected area}) \times (\text{depth of centre of vertical projection})$$

$$= w \times (DE \times \text{unit length}) \times \left(FE + \frac{ED}{2}\right)$$

$$= (9.81 \times 0.8) \times (1 \times 1) \times (1 + 1/2) = 11.772\ \text{kN}$$

The vertical component of hydrostatic pressure force on the curved corner equals the weight of liquid supported by curve *CD* upto the free surface of liquid *FB*

F_v = weight of liquid contained in portion *BCDEF*

= weight of liquid in portion *BCEF* + weight of liquid in portion *CDE*

= specific weight [volume of liquid in portion *BCEF* + volume of liquid in portion *CDE*]

$$= w \times \left[BC \times CE \times \text{unit length} + \frac{1}{4}\pi (CE)^2 \times \text{unit length} \right]$$

$$= 9.81 \times 0.8 \left[1 \times 1 \times 1 + \frac{1}{4}\pi (1)^2 \times 1 \right] = 14\ \text{kN}$$

Resultant pressure force on the curved corner CD,

$$= \sqrt{F_h^2 + F_v^2} = \sqrt{(11.772)^2 + (14)^2} = \mathbf{18.29\ kN}$$

The angle made by the resultant with horizontal is given by

$$\theta = \tan^{-1}\left(\frac{F_v}{F_h}\right) = \tan^{-1}\left(\frac{14}{11.772}\right) = \tan^{-1}(1.189) = \mathbf{49.93^\circ}$$

Example 3.55. Calculate the magnitude and direction of the resultant hydrostatic pressure force acting on the curved face of a dam which has a shape prescribed by the relation $y = y_0 (x/x_0)^2$. With apex having the co-ordinates $x_0 = 4$ m and $y_0 = 8$ m. The dam has a width of 2 m and the fluid is water with mass density 1000 kg/m^3.

Solution : Equation of the curve *AB* is,

$$y = y_0\left(\frac{x}{x_0}\right)^2$$

$$= 8\left(\frac{x}{4}\right)^2 = \frac{x^2}{2}$$

or $\quad x^2 = 2y$

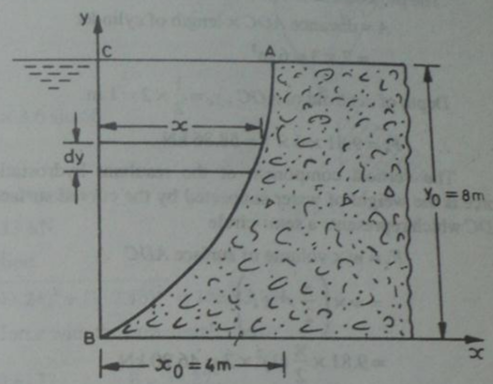

Fig. 3.51.

The horizontal component of hydrostatic pressure force acting on a curved surface equals the pressure force on its projected area onto a vertical plane

F_h = specific weight × (vertical projected area) × (depth of centre of vertical projection)

$$= w \times (BC \times \text{width of dam}) \times \frac{BC}{2}$$

$$= (1000 \times 9.81) \times (8 \times 2) \times \frac{8}{2}$$

$$= 627840 \text{ N} = 627.84 \text{ kN}$$

The vertical component of hydrostatic pressure force on the curved surface *AB* equals the weight of liquid supported by the curve *AB* upto the free surface of liquid *AC*.

F_v = weight of water in portion *ABC*

= specific weight × volume of liquid in portion *ABC*

= specific weight × (area of portion *ABC* × width of dam)

$$= w\left[\int_o^8 x\,dy \times 2\right] = w\left[\int_o^8 \sqrt{2}\, y^{1/2}\,dy\right] \times 2$$

$$= 2\sqrt{2}\, w\left[\frac{y^{3/2}}{3/2}\right]^8 = 2\sqrt{2}\,(9.81 \times 1000) \times \frac{2}{3}(8)^{3/2} = 418552 \text{ N} = 481.55 \text{ kN}$$

Pressure thrust exerted by water on the dam,

$$= \sqrt{F_v^2 + F_h^2} = \sqrt{(418.55)^2 + (627.84)^2} = \mathbf{754.56\ kN}$$

The angle made by the resultant with horizontal is given by

$$\theta = \tan^{-1}\left(\frac{F_v}{F_h}\right) = \tan^{-1}\left(\frac{418.55}{627.84}\right) = \tan^{-1}(0.677) = \mathbf{33.69^\circ}$$

Example 3.56. A cylinder 2 m in diameter × 3 m in length and supported as shown in Fig. 3.52

retains water on one side. If the cylinder weighs 150 kN make calculations for the vertical reaction at A and the horizontal reaction at B. Ignore the frictional effects.

Solution : The horizontal component of the resultant hydrostatic force acting on the gate is the horizontal force on the projected area of the curved surface on a vertical plane.

F_h = hydrostatic pressure force on the curved area ADC projected on the vertical plane AOC

$= wAy_c$

The projected area of the curved surface ADC on the vertical plane AOC is

A = distance $AOC \times$ length of cylinder

$= 2 \times 3 = 6 \text{ m}^2$

Depth of centroid of AOC, $y_c = \frac{1}{2} \times 2 = 1$ m

$\therefore \quad F_h = 9.81 \times 6 \times 1 = 58.86$ kN

The vertical component of the resultant hydrostatic force is the weight of water supported by the curved surface ADC which represents a semi-circle

$F_v = w \times$ volume of surface ADC

$= w \times \left(\frac{\pi}{2} r^2 \times l \right)$

$= 9.81 \times \frac{\pi}{2} (1)^2 \times 3 = 46.20$ kN

The vertical component F_v is acting in the upward direction.

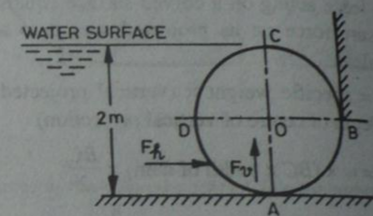

Fig. 3.52.

Therefore for the equilibrium of cylinder

Vertical reaction at A = weight of cylinder – F_v

= 150 – 46.20 = **103.8 kN**

Horizontal reaction at B = **58.86 kN**

Example 3.57. A 60° sector gate of 3.6 metre radius is mounted on the spillway of a dam as shown in Fig. 3.53. Its hinge and one of its end radial arms are at the same horizontal level as the water surface. What is the magnitude and direction of the resultant pressure on the gate, if the length of the gate is 3 metres.

Solution : Height of water surface above B,

$h = BD = BC \sin 60° = 3.60 \sin 60° = 3.12$ m

Horizontal component of hydrostatic force acting on the curved surface AB

= hydrostatic force acting on area projected on to the vertical plane BD

$= w A y_c$

$\therefore \; F_h = 9.81 \times (3 \times 3.12) \times \frac{3.12}{2} = 143.24$ kN

This force acts horizontally towards the right.

Vertical component of hydrostatic pressure is the weight of imaginary volume of water on the curved surface *AB*.

Area *ABD* = area of section *ABC* – area of triangle *BCD*

The sector (included angle 60°) represents $\frac{1}{6}$th of a circle.

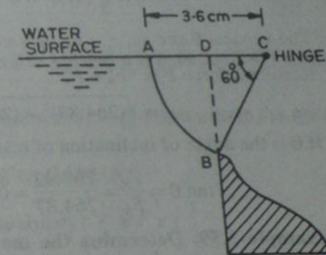

Fig. 3.53.

$$\therefore \quad \text{Area } ABD = \frac{\pi}{6}r^2 - \frac{1}{2} \times DC \times DB$$

$$= \frac{\pi}{6}(3.6)^2 - \frac{1}{2} \times 3.6 \cos 60 \times 3.6 \sin 60$$

$$= 6.78 - 2.80 = 3.98 \text{ m}^2$$

$$\therefore \quad F_v = \text{specific weight} \times \text{volume of water}$$

$$= 9.81 \times (3.98 \times 3) = \mathbf{117.13 \text{ kN}}$$

The vertical force acts in the upward direction.

Resultant force $\quad F = \sqrt{F_h^2 + F_v^2} = \sqrt{(143.24)^2 + (117.13)^2} = \mathbf{185.03 \text{ kN}}$

If θ is the angle of inclination of resultant force with the horizontal, then

$$\tan\theta = \frac{F_v}{F_h} = \frac{117.13}{143.24} = 0.8177 \quad \therefore \quad \theta = \mathbf{39.27°}$$

Example 3.58. A cylindrical roller gate 3 m in diameter is placed on the dam in such a way that water is just going to spill (Fig. 3.54). If the length of gate is 6 m, calculate the magnitude and direction of the resultant force due to water acting on it.

Solution : The horizontal component of the resultant hydrostatic force acting on the gate is the horizontal force on the projected area of the curved surface on a vertical plane.

F_h = hydrostatic pressure force on the curved area *ACB* projected on the vertical plane *AOB*

$$= w A y_c$$

The projected area of the curved surface *ACB* on the vertical plane *AOB* is

$$A = \text{distance } AOB \times \text{length of gate}$$

$$= 3 \times 6 = 18 \text{ m}^2$$

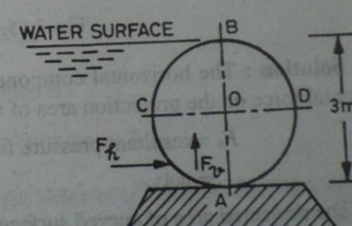

Fig. 3.54.

Depth of centroid of *AOB* is

$$y_c = \frac{1}{2} \times 3 = 1.5 \text{ m}$$

$$\therefore \quad F_h = 9810 \times 18 \times 1.5 = 264870 \text{ N} = \mathbf{264.87 \text{ kN}}$$

The vertical component of the resultant hydrostatic force is the weight of water supported by t[he] curved surface *ACB* which represents a semi-circle.

$F_v = w \times$ volume of curved surface ACB

$= w\left(\frac{\pi}{2} r^2 l\right)$

$= 9810 \times \frac{\pi}{2} (1.5)^2 \times 6 = 207920 \text{ N} = \mathbf{207.92 \text{ kN}}$

The vertical force act... he upward direction.

Resultant force $F = \sqrt{F_h^2 + F_v^2}$

$= \sqrt{(264.87)^2 + (207.92)^2} = \mathbf{336.73 \text{ kN}}$

If θ is the angle of inclination of resultant force F to horizontal, then

$\tan\theta = \frac{F_v}{F_h} = \frac{207.92}{264.87} = 0.785 \quad \therefore \quad \theta = \mathbf{38.13°}$

Example 3.59. Determine the magnitude, direction and location of total pressure exerted by water on the curved surface AB which is the quadrant of a circular cylinder (Fig. 3.55). Given that the radius of the surface is 2.5 m, the cylinder is 3 m long and water is 2 m above B.

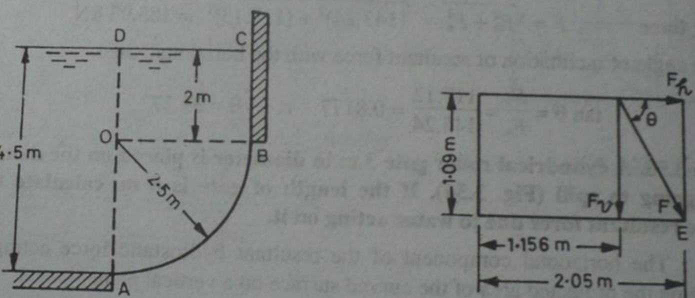

Fig. 3.55. *Force estimation on a curved surface*

Solution : The horizontal component of the resultant hydrostatic force acting on the cylinder is the horizontal force on the projection area of the curved surface on a vertical plane.

F_h = resultant pressure force on projected OA of the curved surface.

$= wAy_c$

The projected area of curved surface on the vertical plane OA is

A = distance $OA \times$ cylinder length $= 2.5 \times 3 = 7.5 \text{ m}^2$

Depth of centroid of OA, $y_c = 2 + \frac{2.5}{2} = 3.22$ m

∴ $F_h = 9.81 \times 7.5 \times 3.25 = \mathbf{239 \text{ kN}}$

The line of action of F_h is obtained from the relation,

$$y_{ph} = y_c + \frac{I_c \sin^2 \alpha}{A y_c} \quad ; \quad \sin \alpha = \sin 90^\circ = 1$$

$$I_c = \frac{bd^3}{12} = \frac{3 \times 2.5^3}{12} = 3.906 \text{ m}^4$$

$$\therefore \quad y_{ph} = 3.25 + \frac{3.906 \times 1}{7.5 \times 3.25} = 3.41 \text{ m}$$

Thus the horizontal component of the resultant pressure force is 239 kN and it acts $(4.5 - 3.41) = 1.09$ m vertically above *A*.

The vertical component of the resultant hydrostatic pressure is the weight of water above the trace *AB* of the curved surface.

$F_v = w$ [volume of rectangular portion *OBCD* + volume of region *OAB*]

The volume of region *OAB* which is the quadrant of circle equals $\left(\frac{1}{4}\pi r^2 \times l\right)$

$$\therefore \quad F_v = 9.81\left[2 \times 2.5 \times 3 + \frac{1}{4}\pi (2.5)^2 \times 3\right]$$

$$= 9.81\,[15 + 14.73] = 147.15 + 144.50 = 291.65 \text{ kN}$$

The line of action of F_v may be obtained by taking moments of its two components about the line *OA*

$$291.65 \times x = 147.15 \times \frac{2.5}{2} + 144.50 \times \frac{4 \times 2.5}{3\pi}$$

where the term $\frac{4 \times 2.5}{3\pi} = \frac{4r}{3\pi}$ represents the distance of centre of gravity from either straight line.

Solution gives : $x = 1.156$ m

Thus the vertical component of the resultant hydrostatic pressure is 291.65 kN and it acts 1.156 m to the right of *A*.

The resultant hydrostatic pressure force *F* is

$$F = \sqrt{F_h^2 + F_v^2} = \sqrt{(239)^2 + (291.65)^2} = \mathbf{377.15 \text{ kN}}$$

If θ is the angle of inclination of *F* to the horizontal,

$$\tan\theta = \frac{F_v}{F_h} = \frac{291.65}{239} = 1.220\,, \quad \therefore \quad \theta = 50^\circ - 40'$$

The distance of point *E*, where the resultant strikes the surface is :

$$= 1.156 + \frac{1.09}{\tan\theta} = 1.156 + \frac{1.09}{1.22} = 2.05 \text{ m from point } A$$

Example 3.60. Fig 3.56 shows the water level on the two sides of a cylinder that lies across the full width of an open channel. If the cylinder is 2 m in diameter and 1.5 m in length, determine :

(a) the magnitude and direction of the resultant hydrostatic force exerted on it.

(b) What should be the least weight of the cylinder so that it may not be lifted away from the floor of the channel ?

(c) Show that the resultant force passes through the centre of the cylinder.

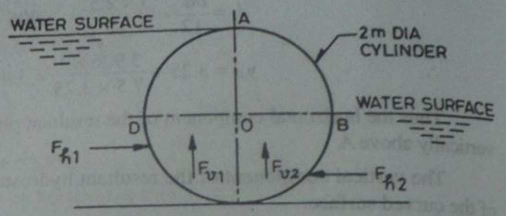

Fig. 3.56.

Solution : Horizontal component of the resultant hydrostatic force acting on a curved surface

= hydrostatic force acting on area projected on to a vertical plane.

$= w A y_c$

∴ Horizontal force exerted by water on the deeper side,

$$F_{h1} = w \times (\text{distance } AOC \times \text{cylinder length}) \times \frac{1}{2} \text{ of distance AOC}$$

$$= 9.81 \times (2 \times 1.5) \times \frac{1}{2}(2)$$

$$= 29.43 \text{ kN}$$

The line of action of force F_{h_1} acts at a distance

$$= \frac{1}{3} \text{ of length } AOC \text{ from point } C = \frac{1}{3} \times 2 = 0.667 \text{ m from point } C$$

$$= (1 - 0.667) = 0.333 \text{ m from point } O.$$

Horizontal force exerted by water on the shallow side,

$$F_{h_2} = w \times (\text{distance } OC \times \text{cylinder length}) \times \frac{1}{2} \text{ of distance } OC$$

$$= 9.81 \times (1 \times 1.5) \times \frac{1}{2}(1) = 7.36 \text{ kN}$$

The line of action of force F_{h_2} acts at a distance

$$= \frac{1}{3} \text{ of length } OC \text{ from point } C$$

$$= \frac{1}{3} \times 1 = 0.333 \text{ m from point } C$$

$$= (1 - 0.333) = 0.667 \text{ m from point } O.$$

Net horizontal force,

$$F_h = 29.43 - 7.36 = 22.07 \text{ kN}$$

The vertical component of the resultant hydrostatic force is the weight of water supported by the curved surface.

Upward force on ADC,

$F_{v_1} = w \times$ volume of curved surface ADC which represents a semi- circle.

$$= w \times \frac{1}{2} \pi r^2 l = 9.81 \times \frac{1}{2} \pi (1)^2 \times 1.5 = 23.10 \text{ kN}$$

and this force acts at a distance

$= \frac{4r}{3\pi}$ from point $O = \frac{4 \times 1}{3\pi} = 0.424$ m from point O.

Upward force on BC,

$F_{v_2} = w \times$ volume of curved surface BC which represents quadrant of a circle

$$= w \times \frac{1}{4} \pi r^2 l = 9.81 \times \frac{1}{4} \pi (1)^2 \times 1.5 = 11.55 \text{ kN}$$

and this force acts at a distance

$= \frac{4r}{3\pi}$ from point $O = \frac{4 \times 1}{3\pi} = 0.424$ m from point O

Net upward force, $F_v = 23.10 + 11.55 = 34.65$ kN

Resultant force $F = \sqrt{F_h^2 + F_v^2} = \sqrt{(22.07)^2 + (34.65)^2} = \mathbf{41.08\ kN}$

If θ is the inclination of the resultant force with the horizontal,

$$\tan \theta = \frac{F_v}{F_h} = \frac{34.65}{22.07} = 1.57 \quad \therefore \quad \theta = \mathbf{57.50°}$$

(b) The least weight of the cylinder should be equal to the net upward force of 34.65 kN so that it is not lifted from the floor of the channel.

(c) Taking moments about point O, we have the following moments :

(i) moment of $F_{h_1} = 29.43 \times 0.333 = 9.80$ kNm (anticlock wise)

(ii) moment of $F_{h_2} = 7.36 \times 0.667 = 4.91$ kNm (clock wise)

(iii) moment of $F_{c_1} = 23.10 \times 0.424 = 9.78$ kNm (clock wise)

(iv) moment of $F_{v_2} = 11.55 \times 0.424 = 4.89$ kNm (anticlock wise)

With little calculation errors, net moment about point O is zero, and obviously the resultant force passes through the point O, *i.e.*, the centre of the cylinder.

Example 3.61. For the tainter gate shown in Fig. 3.57, compute,

(i) **the total horizontal push of water on the gate,**

(ii) **the total vertical component of water pressure against the gate and**

(iii) **the resultant water pressure on the gate and its location.**

It may be assumed that the gate has a length of 0.8 m perpendicular to the plane of the paper.

Solution : From the given configuration, $AD = BC = 2.5$ m

(i) Horizontal component of pressure force,

$$F_h = w\, A\, y_c = 9.81 \times (2.5 \times 0.8) \times 1.25 = 24.53 \text{ kN}$$

Let y_{ph} be the depth of centre of pressure of F_h, then

$$y_{ph} = y_c + \frac{I_c \sin^2 \alpha}{A y_c} \; ; \; \sin \alpha = \sin 90° = 1$$

$$= 1.25 + \frac{\frac{1}{12} \times 0.8 \times 2.5^3 \times 1}{(2.5 \times 0.8) \times 1.25} = 1.67$$

(ii) Vertical component of hydrostatic pressure is the weight of imaginary volume of water on the curved surface *ABC*.

Area ABC = area of sector AOC – area of triangle OBC

The sector (included angle 30°) represents $\frac{1}{12}$th of a circle.

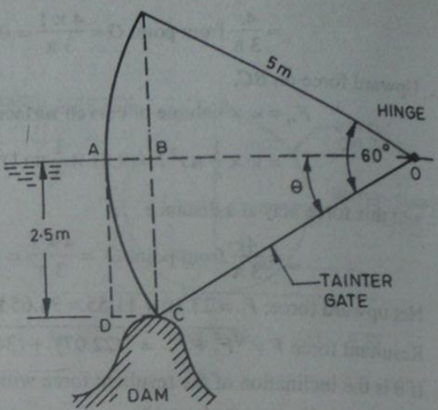

Fig. 3.57.

$$\therefore \text{Area } ABD = \frac{\pi}{12} r^2 - \frac{1}{2} \times OB \times BC$$

$$= \frac{\pi}{12} (5)^2 - \frac{1}{2} \times 5 \cos 30^\circ \times 2.5$$

$$= 6.54 - 5.41 = 1.13 \text{ m}^2$$

$\therefore$ F_v = specific weight × volume of water

$$= 9.81 \times (1.13 \times 0.8) = 8.87 \text{ kN}$$

Line of action of the vertical force passes through the centroid of area *ABC* and its distance *x* from the hinge can be obtained by taking moments about the hinge from *O*.

Area ABC × distance of its centroid from O

= (area OAC × centroid distance from O) – (area OBC × centroid distance from O)

$$1.13\, x = 6.54 \times \left(\frac{2 \times 5}{\pi}\right) - 5.41 \times \left(\frac{2}{3} \times 5 \cos 30^\circ\right)$$

$$1.13\, x = 5.229 , \quad x = 4.63$$

$\therefore$ Distance of line of action of F_v from BC

$$= 4.63 - OB = 4.63 - 5 \cos 30^\circ = 0.3 \text{ m}$$

(iii) Resultant hydrostatic pressure force F is,

$$F = \sqrt{(24.53)^2 + (8.87)^2} = \mathbf{26.2\ kN}$$

and its inclination to the horizontal is ;

$$\tan \theta = \frac{F_v}{F_h} = \frac{8.87}{24.53} = 0.3616 \ ; \quad \therefore \quad \theta = \mathbf{19^\circ - 54'}$$

Example 3.62. A tank is filled with water under pressure and the pressure gauge fitted at the top indicates a pressure of 15 kPa. The tank meausres 2.5 m perpendicular to the plane of the paper, and the curved surface AB of the top is the quarter of a circular cylinder of radius 2 m. Determine *(a)* horizontal and vertical components of fluid pressure on the curved surface AB, and *(b)* magnitude and direction of the resultant force.

Solution : The water head equivalent to the given pressure of 15 kPa is

$$h = \frac{15 \times 10^3}{9810} \approx 1.53 \text{ m}$$

Hence the free water surface can be imagined to be 1.53 m above the top of the tank.

Horizontal force, F_h

$= $ hydrostatic pressure force on vertical projection BC of the curved surface AB

$= w\,A\,y_c$

$$= 9810 \times (2 \times 2.5) \times \left(1.53 + \frac{2}{2}\right) = 124096 \text{ N} \simeq 124 \text{ kN} \rightarrow$$

Vertical component, F_v

= weight of volume of water above AB upto imaginary water surface *i.e.*, of volume $EABCD$

$$= \left\{1.53 \times 2 + \frac{1}{4}\pi\,(2)^2\right\} \times 2.5 \times 9810$$

$$= 152005 \text{ N} \simeq 152 \text{ kN} \uparrow$$

The resultant force

$$R = \sqrt{F_h^2 + F_v^2} = \sqrt{124^2 + 152^2} = 196 \text{ kN}$$

This resultant force is inclined at angle θ to the horizontal given by

$$\tan\theta = \frac{F_v}{F_h} = \frac{152}{124} = 1.225 \quad \therefore \quad \theta = 50.79°$$

Since the pressure is radial throughout, the resultant would pass through the centre of quadrant, *i.e.*, through point C.

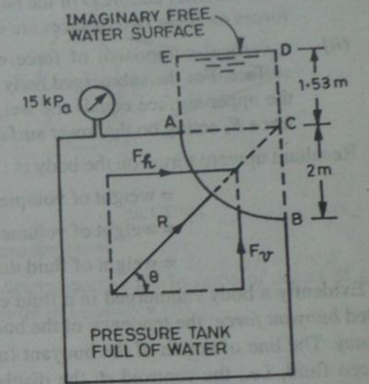

Fig. 3.58.

SECTION C : Buoyancy and Floatation

3.12. ARCHIMEDE'S PRINCIPLE

Fluid exerts pressure on the surfaces of a body which is either partially or completely submerged into it. We consider a potato- shaped irregular body $ABCD$ (Fig. 3.59) completely submerged in a fluid at rest. The body may be of any size, shape and weight distribution. The body considered is acted upon by two system of forces :

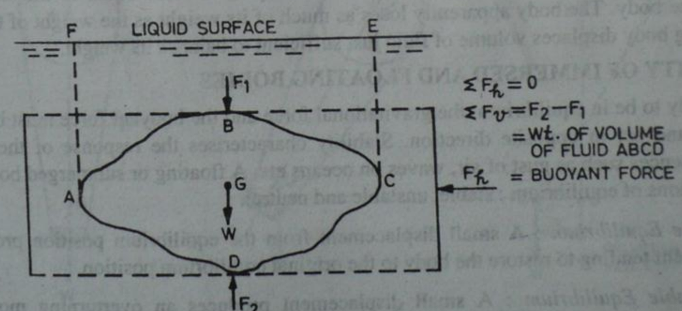

Fig. 3.59. *Buoyancy and Archimede's principle*

(i) a downward gravitational force due to weight of the body; this body force acts through the centre of mass or centre of gravity of the body.

(ii) an external pressure force acting all around the surface of the body.

Recalling that on the curved surface immersed in a liquid there is :

(i) a horizontal component of force equal to the force on a horizontal projection into a vertical plane. We consider any vertical plane drawn through the body. The projected areas of the two sections *ABD* and *BCD* of the body onto a vertical plane are equal. Consequently the horizontal forces on these two surfaces are equal and opposite, and so the net horizontal force is zero.

(ii) a vertical component of force equal to the weight of the fluid vertically above the curved surface. For the submerged body under consideration, a vertically downward force F_1 acting on the upper surface equals the weight of volume of fluid *ABCEF*. Likewise, a vertically upward force F_2 acting on the lower surface equals the weight of volume of fluid *ADCEF*.

Resultant upward thrust on the body is :

= weight of volume of fluid *ADCEF* – weight of volume of fluid *ABCEF*

= weight of volume of fluid *ABCD*

= weight of fluid displaced by the submerged body.

Evidently a body submerged in a fluid experiences an upward thrust due to fluid pressure. This force is called *buoyant force*; the tendency of the body to be lifted upward in a fluid due to buoyant force is called *buoyancy*. The line of action of the buoyant force is vertical and passes through the centre of gravity of the displaced fluid, *i.e.*, the centroid of the displaced volume. This centroid of the displaced fluid volume is called the *centre of buoyancy*.

Depending on the ratio of the weight W of a body and the buoyant force F_b, three cases are possible :

(i) $W > F_b$; the body tends to move downwards and eventually sinks

(ii) $W = F_b$; the body floats and is only partially submerged

(iii) $W < F_b$; the body is lifted upward and rises to the surface.

These aspects are known as Archimede's principle, after the well known Greek scientist Archimedes (287–212 B.C.) and may be summed up as :

"A body immersed in a fluid is buoyed or lifted up by a force equal to the weight of the fluid displaced by the body. The body apparently loses as much of its weight as the weight of the fluid displaced by it. A floating body displaces volume of fluid just sufficient to balance its weight."

3.13. STABILITY OF IMMERSED AND FLOATING BODIES

For a body to be in equilibrium, the gravitational force and the buoyant force must be collinear, equal in magnitude and act in opposite direction. Stability characterises the response of the system to small disturbing influences such as gust of air, waves on oceans etc. A floating or submerged body can have three possible conditions of equilibrium : stable, unstable and neutral.

- *Stable Equilibrium* : A small displacement from the equilibrium position produces a righting moment tending to restore the body to the original equilibrium position.
- *Unstable Equilibrium* : A small displacement produces an overturning moment tending to displace the body further to a condition different from the initial equilibrium position.
- *Neutral Equilibrium* : The body remains at rest in any position to which it may be displaced. No

net force tends to return the body to its original state or to drive it further away from the original condition.

These three conditions of system equilibrium for a cone placed on a horizontal surface are reflected in Fig. 3.60.

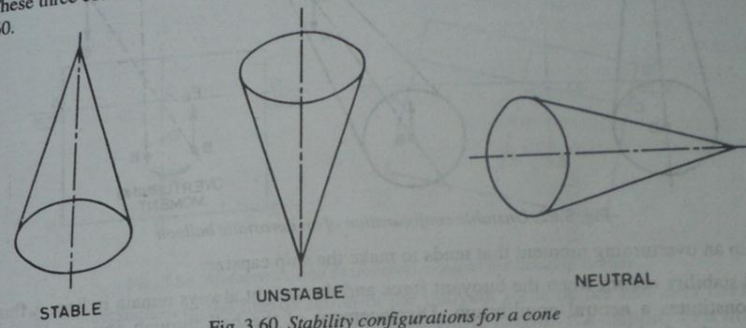

Fig. 3.60. *Stability configurations for a cone*

(i) Submerged Body : Stability of a submerged body is determined by the location of centre of gravity G with respect to centroid of the displaced volume, *i.e.*, the centre of buoyancy (B)

Stable : G is located below B

Neutral : G is located at B

Unstable : G is located above B

An aerostatic balloon and gondola system illustrated in Fig. 3.61 constitutes the stable equilibrium. The buoyant force F_b is located at the centre of balloon and acts vertically upwards. The load W is carried in the gondola and acts vertically downwards. Let the system be rolled through a small angular displacement by some disturbing force. This displacement from the normal position brings into action a restoring couple Wx that tends to bring the balloon to its initial upright position.

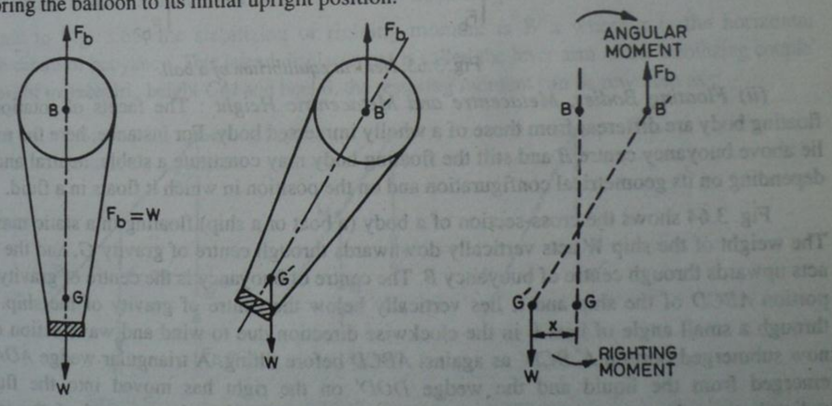

Fig. 3.61. *Stable configuration of an aerostatic balloon*

The aerostatic balloon considered above will be in unstable equilibrium when placed in the tilted position depicted in Fig. 3.62. Likewise a system comprising a ship with a small hull and a tall mast which is very heavily weighed at the top would constitute an unstable system. A small deflection from the vertical

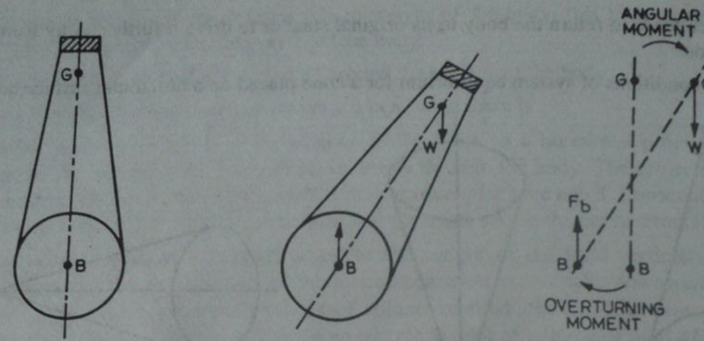

Fig. 3.62. *Unstable configuration of an aerostatic balloon*

would result in an overturning moment that tends to make the ship capsize.

Neutral stability occurs when the buoyant force and the weight always remain in line. A floating ball (Fig. 3.63) constitutes a neutral equilibrium. Displacement of the ball through any number of degrees changes neither the position nor the value of the forces acting and consequently no resulting moment occurs.

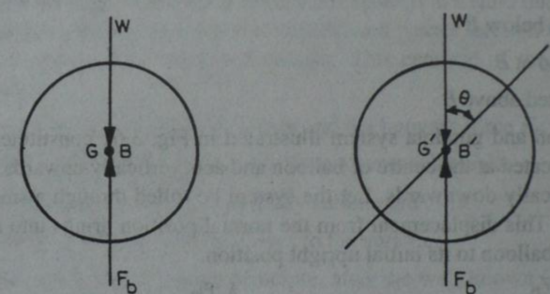

Fig. 3.63. *Neutral equilibrium of a ball*

(ii) Floating Bodies, Metacentre and Metacentric Height : The facets of rotational stability of a floating body are different from those of a wholly immersed body. For instance, here the mass centre G may lie above buoyancy centre B and still the floating body may constitute a stable, neutral and unstable system depending on its geometrical configuration and on the position in which it floats in a fluid.

Fig. 3.64 shows the cross-section of a body (a boat or a ship) floating in a static mass of fluid at rest. The weight of the ship W acts vertically downwards through centre of gravity G, and the buoyant force F_b acts upwards through centre of buoyancy B. The centre of buoyancy is the centre of gravity of the immersed portion $ABCD$ of the ship and it lies vertically below the centre of gravity of the ship. Let the ship tilt through a small angle of heel θ in the clockwise direction due to wind and wave action etc. The ship has now submerged portion $A'BCD'$ as against $ABCD$ before tilting. A triangular wedge AOA' on the left has emerged from the liquid and the wedge DOD' on the right has moved into the fluid. Due to this redistribution, the centre of buoyancy shifts from B to B' ; B' is the centroid of the immersed portion $A'BCD'$ of the ship. However, with a fixed cargo the relative position of the centre of gravity G remains unchanged.

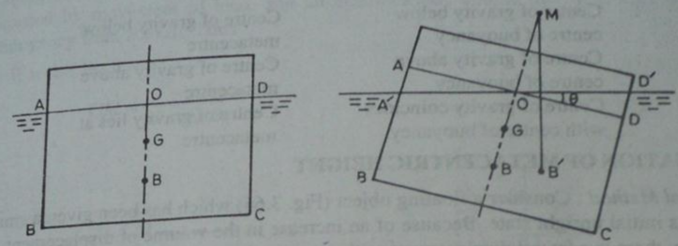

Fig. 3.64. *Metacentre and Metacentric height for floating bodies*

The point of intersection of the line of action of the buoyant force before and after the heel is called the *Metacentre (M)* and the distance *GM* is called the *Metacentric height*. The stability of a floating body is governed by the position of metacentre *M* relative to centre of gravity *G*.

- If *M* lies above *G*, then the metacentric height *GM* is regarded as positive and the system is in stable equilibrium. The couple $W x$ acting on the ship in its tilted position is the restoring couple, *i.e.*, the couple tries to bring the ship back to its original position. For proper design of ships, care is taken to ensure that the metacentre is above the centre of gravity for all angles of heel which may be encountered.
- If *M* lies below *G*, then the metacentric height *GM* is regarded as negative and the system is in unstable equilibrium. The couple acting on the body whilst in tilted position is the overturning couple, *i.e.*, the couple tries to tilt the body still further.
- When *M* and *G* coincide, then the body is in a state of neutral equilibrium.

Refering again to Fig. 3.65; the stabilizing or righting moment is $W x$ where x is the horizontal displacement of the centre of buoyancy. This lateral displacement is called the lever arm of the stabilizing couple. Expressing x in terms of metacentric height *GM* and heel θ, the restoring moment can be rewritten as :

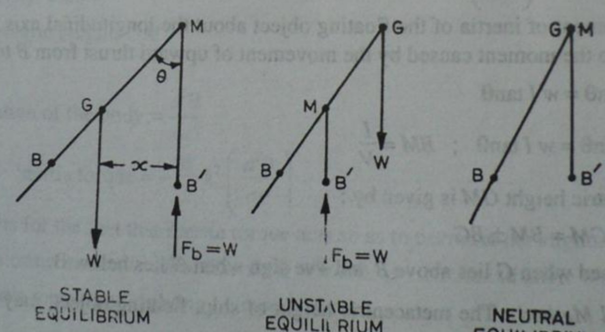

Fig. 3.65. *Position of metacentre for stability of floating bodies*

$$W\,GM \sin\theta \approx W\,GM\,\theta \approx W\,GM \tan\theta \qquad ...(3.27)$$

Stability criteria of submerged and floating bodies is summarised below :

		Submerged body	*Floating body*
Stable	:	Centre of gravity below centre of buoyancy	Centre of gravity below metacentre
Unstable	:	Centre of gravity above centre of buoyancy	Centre of gravity above metacentre
Neutral	:	Centre of gravity coincides with centre of buoyancy	Centre of gravity lies at metacentre

3.14. DETERMINATION OF METACENTRIC HEIGHT

- *Analytical Method* : Consider a floating object (Fig. 3.64) which has been given a small tilt angle θ from its initial upright state. Because of an increase in the volume of displacement on the right hand side, there is a lateral displacement of the centre of buoyancy from B to B_1. This shift results in a couple of moment $\approx W\,BM \tan\theta$ which tends to restore the object to its original upright position. Further recognising that the volume V of the liquid displaced by the object remains the same, area of the triangular wedge OAA' that emerges out of liquid is equal to that of wedge ODD' that has submerged into the liquid.

If l and b are the length (measured normal to the plane of the paper) and breadth of the object respectively, then the weight of each wedge shaped portion is :

$$\delta F_b = w \times \frac{1}{2}\left(\frac{b}{2} \times \frac{b}{2} \tan\theta\right) l = \frac{wb^2}{8}\, l \tan\theta \qquad ...(3.28)$$

Thus there acts a buoyant force δF_b upwards on the wedge ODD' and δF_b downwards on the wedge OAA', each at a distance $\frac{2}{3}\left(\frac{b}{2}\right) = \frac{b}{3}$ from the centre.

These two equal and opposite forces constitute a couple of magnitude

$$\delta M = \delta F_b \times \frac{2b}{3} = \frac{wb^2}{8}\, l \tan\theta \times \frac{2b}{3}$$

$$= w\left(\frac{lb^3}{12}\right)\tan\theta = w\, I \sin\theta \qquad ...(3.29)$$

where I is the moment of inertia of the floating object about the longitudinal axis. This moment must be equal and opposite to the moment caused by the movement of upward thrust from B to B_1. Thus :

$$W \times BM \tan\theta = w\, I \tan\theta$$

$$w\,V \times BM \tan\theta = w\, I \tan\theta \;;\; BM = \frac{I}{V} \qquad ...(3.30)$$

Then the metacentric height GM is given by :

$$GM = BM \pm BG \qquad ...(3.31)$$

The –ve sign is used when G lies above B and +ve sign when G lies below B.

- *Experimental Method* : The metacentric height of ship, floating object, may also be determined experimentally by :

(i) rolling off a moveable weight w across the deck through a distance x, and

(ii) observing the resulting small angular tilt θ after the body comes to rest in a new position of equilibrium. In the tilted position the ship is then under the action of a clockwise moment caused by movement of load, and an anticlockwise couple due to a shift in the centre of buoyancy from B to B_1. Thus :

$$W \times GM \tan\theta = wx$$

$$GM = \frac{wx}{W x \tan\theta} \qquad \text{...(3.32)}$$

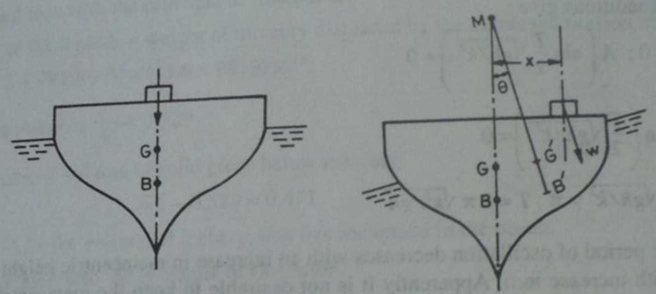

Fig. 3.66. *Experimental method for the determination of metacentric height*

The angular tilt θ is usually measured by suspending a plumb bob in the ship. If l is the length of the plump and d is its apparent deflection when the ship swings, then $\tan\theta = \frac{d}{l}$.

3.15. OSCILLATION OF A FLOATING BODY

When a floating body is given a tilt, it is set in a state of oscillation as if suspended at the metacentre in a manner similar to that a simple pendulum. Let

θ = angle of tilt at an instant of time t

k = radius of gyration of the body about a longitudinal axis passing through the centre of gravity, and

h = metacentric height, GM

Then :

angular acceleration of the body $= \dfrac{d^2\theta}{dt^2}$

$$\text{inertia torque} = -\frac{W}{g}k^2\left(\frac{d^2\theta}{dt^2}\right)$$

(–ve sign accounts for the fact that inertia torque acts so as to decrease the angular tilt θ)

righting or restoring moment $= W \times GM \sin\theta = W h \theta$ (for small angle θ measured in radians)

Equating the inertia torque and righting moment :

or $$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2}\theta = 0$$

Solution of this second order linear differential equation is :

$$\theta = A \sin\left(t\sqrt{gh/k^2}\right) + B\cos\left(t\sqrt{gh/k^2}\right) \qquad ...(3.33)$$

where A and B are the constants of integration. If T is the time period of oscillation in seconds, then the boundary conditions are :

$$\theta = 0 \text{ at } t = 0 \,;\, \theta = 0 \text{ at } t = \frac{T}{2}$$

These boundary conditions give :

$$B = 0\,;\; A\left(\sin\frac{T}{2}\sqrt{gh/k^2}\right) = 0$$

Since $A \neq 0$;

$$\sin\left(\frac{T}{2}\sqrt{gh/k^2}\right) = 0$$

$$\therefore \quad \frac{T}{2}\sqrt{gh/k^2} = \pi\,;\; T = 2\pi\sqrt{k^2/gh} \qquad ...(3.34)$$

Evidently the time period of oscillation decreases with an increase in metacentric height h; number of oscillations increase with increase in h. Apparently it is not desirable to keep the metacentric height very large to avoid large number of oscillations. Recalling that a large value of metacentric height corresponds to improved stable equilibrium, we find that the two requirements are contrary to each other. In actual practice, an optimum value of metacentric height is selected and specified for the ship. The normal values of metacentric height for different types of ships are :

— merchant ships	0.3 to 1 m
— sailing ships	0.45 to 1.25 m
— battle ships	1.0 to 1.5 m
— river crafts	up to 3.5 m

Example 3.63. A stone weighs 400 kN in air and when immersed in water it weighs 225 N. Calculate the volume of the stone and its relative density.

Solution : Buoyant force = 400 – 225 = 175 N

The buoyant force also equals the weight of the liquid dispaced. That is

Buoyant force = specific weight of water × volume of water displaced

$$17.5 = 9810 \times V\,;\; V = 0.0178 \text{ m}^3$$

This also represents the volume of stone immersed in water.

$$\text{Relative density} = \frac{\text{weight of stone}}{\text{weight of an equal volume of water}}$$

$$= \frac{400}{9810 \times 0.0178} = \mathbf{2.291}$$

Example 3.64. A solid metallic piece (relative density = 7.2) floats above the surface of mercury (relative density = 13.6) contained in a tank. What fraction of the volume of metallic piece lies above mercury surface ?

(b) Calculate the volume of iceberg (sp. wt. = 9 kN/m³) below the free surface of ocean if 2500 m³ of its volume is observed protruding above the free surface. Take specific weight of ocean water 10 kN/m³.

Solution : Let V be the total volume of the solid piece and V' represent its volume that lies immersed in mercury.

In accordance with the principle of floatation,

weight of solid piece = weight of mercury displaced by the immersed portion of the solid piece

$$(7.2 \times 9810) \times V = (13.6 \times 9810) \times V'$$

Ratio of volumes $\dfrac{V'}{V} = 0.529$

∴ Fraction of volume of solid piece below mercury,

$$= 1 - 0.529 = \mathbf{0.471}.$$

(b) Let V_b be the volume of iceberg, that lies immersed in the ocean.

In accordance with the principle of floatation,

weight of iceberg = buoyant force

= weight of volume of ocean water displaced by immersed portion of iceberg

= (specific weight of ocean water) × (volume of iceberg below the free surface)

$$9\,(V_b + 2500) = 10\,V_b$$

∴ Volume of iceberg below the free surface of ocean $V_b = \mathbf{22500\ m^3}$

Example 3.65. A wooden block floats in water with 5 cm height projecting above the water surface. The block is next launched in glycerine (relative density = 1.35) and projects 7.5 cm above the surface of glycerine. Make calculations for the height of block and its relative density.

Solution : Let A and h denote the cross-sectional area and height of the wooden block.

In accordance with the principle of floatation, the weight of each displaced liquid equals the total weight of the block. Thus :

$$\underset{(a)}{w_{wood} \times A \times h} = \underset{(b)}{w_{water}\, A\,(h - 0.05)} = \underset{(c)}{w_{glycerine}\, A\,(h - 0.075)}$$

From expression *(b)* and *(c)*,

$$h - 0.05 = \frac{w_{glycerine}}{w_{water}} \times (h - 0.075)$$

$$= 1.35\,(h - 0.075) = 1.35\,h - 0.10125$$

∴ height of block $h = \dfrac{0.10125 - 0.05}{0.35} = 0.146\ \text{m} = \mathbf{14.6\ cm.}$

From expressions *(a)* and *(b)*,

$$\frac{w_{wood}}{w_{water}} = \frac{h - 0.05}{h} = \frac{0.146 - 0.05}{0.146} = 0.657$$

$\therefore$ Relative density of wood is **0.657**

Example 3.66. A steel pipe (specific gravity 7.7) conveying a particular gas has been laid across a river bed. The pipe is completely submerged in water and is anchored at intervals 2.5 m along its length. If the pipe is 60 cm in internal diameter and 62.5 cm in external diameter, calculate the buoyant force per metre run and the upward force on each anchorage. Take specific weight of water = 9810 N/m³.

Solution : Buoyant force per metre run of pipe

$$= \text{weight of water displaced per metre run} = w \times \frac{\pi}{4} d^2\, l$$

$$= 9810 \times \frac{\pi}{4} (0.625)^2 \times 1 = 12032 \text{ N}$$

The anchors are provided at centres 2.5 metres apart

$\therefore$ Buoyant force on 2.5 m length of pipe

$$= 12032 \times 2.5 = 30080 \text{ N}$$

Weight of 2.5 m length of pipe

$$= \frac{\pi}{4} (0.625^2 - 0.60^2) \times 2.5 \times (9810 \times 7.7) = 18159 \text{ N}$$

$\therefore$ Upward force on the anchorage $= 30080 - 18159 = \mathbf{11921\ N}$

Example 3.67. *(a)* A rectangular box 7.5 m × 4 m × 5 m deep and having a mass of 500 kN is immersed in fresh water (sp. wt. 9.81 kN/m³). Calculate the depth of immersion of box.

What mass of stone should be placed inside the box so that the box rests on the bottom of water which is 5 m deep, *i.e.*, the top of box coincides with the free surface of water ?

***(b)* An empty barge 5.5 m wide × 15.6 m long × 2.7 m high has a weight of 350 kN in air. What should be the draft *i.e.*, its depth below the water surface when it is loaded with 1500 kN of coal ?**

Solution : *(a)* Let h be the depth of immersion

In accordance with the principal of floatation :

weight of box = weight of displaced water

= specific weight of water × volume of water displaced

= specific weight of water × volume of box under water

$$500 = 9.81 \times (7.5 \times 4 \times h)$$

$\therefore$ Depth of immersion

$$h = \frac{500}{9.81 \times 7.5 \times 4} \approx 1.7\text{m}$$

weight of box + weight of stone = weight of water displaced

$$500 + W_s = 9.81 \times (7.5 \times 4 \times 5)$$

$\therefore$ Weight of stone $W_s = 9.81 \times (7.5 \times 4 \times 5) - 500 = \mathbf{971.5\ kN}$

(b) In accordance with the principle of floatation,

weight of loaded barge = buoyant force

= weight of water displaced

= (specific weight of water) × (volume of water displaced)

= (specific weight of water) × (volume of barge under water)

$$1500 + 350 = 9.81 \times (5.5 \times 15.6 \times h)$$

∴ draft, *i.e.*, depth below the water surface

$$h = \frac{1850}{9.81 \times 5.5 \times 15.6} = \mathbf{2.198\ m}$$

Example 3.68. A piece of wood (specific gravity = 0.6) of 10 cm square in cross-section and 2.5 m long floats in water. How much lead (specific gravity = 12) need to be fastened at the lower end of the stick so that it floats upright with 0.5 m length out of water ?

Solution : Weight of wooden stick

= specific weight × volume of stick

$= (0.6 \times 9810) \times (0.1 \times 0.1 \times 2.5) = 147.5$ N

Volume of stick in water $= 0.1 \times 0.1 \times (2.5 - 0.5) = 0.02\ \text{m}^3$

This is also the volume of water displaced by the immersed portion of the stick.

Let W be the weight of lead fastened at the bottom of the wooden stick.

$$\text{Volume of lead} = \frac{\text{weight}}{\text{sp . wt}} = \frac{W}{12 \times 9810} = \frac{W}{117720}$$

$$\text{Volume of water displaced by lead} = \frac{W}{117720}$$

∴ Total volume of water displaced by the lead and stick

$$= \left(0.02 + \frac{W}{117720}\right) \text{m}^3$$

$$\text{Weight of water displaced} = \left(0.02 + \frac{W}{117720}\right) \times 9810$$

For equilibrium,

weight of stick + weight of lead = weight of water displaced

$$147.5 + W = \left(0.02 + \frac{W}{117720}\right) \times 9810$$

Solution gives : Weight of lead = **53.50 N**

Example 3.69. *(a)* A wooden cylinder (sp. gr. 0.5) 60 cm in diameter has a concrete cylinder (sp. gr. 2.50) 50 cm long of the same diameter attached to one end. Determine the minimum length of wooden cylinder for the system to float in static equilibrium with axis vertical.

(*b*) A hollow vertical metal cylinder of 12.5 cm diameter is filled with a liquid (sp. wt. 8 kN/m³) upto a depth of 7.5 cm. Subsequently a 10 cm diameter solid cylinder of height 9.25 cm and weighing 3.8×10^{-3} N is immersed into the liquid contained in the cylinder. Determine the level at which the solid cylinder will float.

Solution : *(a)* Let l be the minimum length of wooden cylinder for the system to float in static equilibrium.

In accordance with the principle of floatation,

weight of the composite cylinder = weight of liquid

= displaced by the composite cylinder

$$0.5\, w\, Al + 2.5\, w\, A \times 0.5 = w\, A\, (l + 0.5)$$

where A is the cross-sectional area of the cylinder

$$0.5\, l + 1.25 = l + 0.5$$

$$l = \frac{1.25 - 0.5}{0.5} = \mathbf{1.5\ m}$$

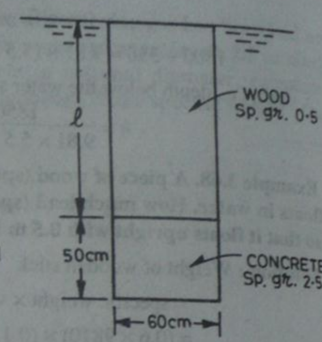

Fig. 3.67.

(b) Refer Fig. 3.68, let

x = distance the solid cylinder falls below the original liquid surface 0–0

y = distance the liquid rises above its original surface 0–0

From volume conservation,

volume of liquid depressed = volume of liquid risen

$$\frac{\pi}{4}(0.1)^2 \times x = \left[\frac{\pi}{4}(0.125)^2 - \frac{\pi}{4}(0.1)^2\right] y$$

$$(0.1)^2 x = \left[(0.125)^2 - (0.1)^2\right] y \; ; x = 0.5625 \quad \text{...(i)}$$

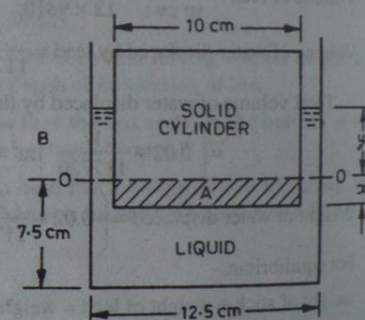

Fig. 3.68.

In accordance with principle of floatation,

weight of the solid cylinder = buoyant force

= weight of liquid displaced by submergence $(x + y)$ of the solid cylinder

= specific × volume of cylinder submergence

$$3.8 \times 10^{-3} = 8\left[\frac{\pi}{4} \times (0.1)^2 \times \left(\frac{x + y}{100}\right)\right]$$

$$x + y = 6.051 \quad \text{...(ii)}$$

From expressions (i) and (ii)

$$x = 2.178 \text{ cm and } y = 3.87 \text{ cm}$$

Apparently the bottom of solid cylinder will be (7.5 – 2.178) = **5.322 cm** above the bottom of hollow vertical cylinder.

Example 3.70. A log of wood, 1 m in diameter × 3 m in length floats in fresh water (specific weight = 9810 N/m³). If the log has a specific gravity of 0.5, work out the angle subtended at the centre and the depth of immersion.

Solution : Weight of log of wood = specific weight × volume of wood

$$= (0.5 \times 9810) \times \frac{\pi}{4}(1)^2 \times 3 = 11551 \text{ N}$$

This also represents the weight of water displaced

$\therefore$ Volume of water displaced $= \dfrac{11551}{9810} = 1.1775 \text{ m}^3$

Volume of wood under water $= 1.1775 \text{ m}^3$

If A represents the sectional area of log of wood under water, then

$$A = \frac{\text{volume of wood under water}}{\text{length of log of wood}} = \frac{1.1775}{3}$$
$$= 0.3925 \text{ m}^2$$

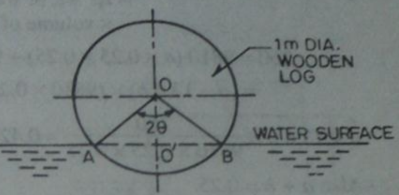

Fig. 3.69.

Let 2θ be the angle submerged at the centre by the immersed portion of the log of wood.

A = area of section OAB – area of triangles OAO' and OBO'

By symmetry : area of $\Delta\, OAO'$ = area of $\Delta\, OBO'$

The sector OAB (included angle 2θ) represents $\dfrac{2\theta}{2\pi}$th part of a circle.

$$\therefore \quad A = \text{area of sector } OAB - 2\,(\text{area of } \Delta\, OBO')$$
$$= \frac{2\theta}{2\pi}(\pi r^2) - 2\left(\frac{1}{2} \times OO' \times O'B\right)$$
$$= \frac{1}{2} r^2 (2\theta) - 2\left(\frac{1}{2} \times r\cos\theta \times r\sin\theta\right)$$
$$= \frac{1}{2} r^2 (2\theta) - \frac{r^2}{2}\sin 2\theta$$

$$\therefore \quad 0.3925 = \frac{1}{2}(0.5)^2(2\theta) - \frac{1}{2}(0.5)^2 \sin 2\theta$$

or $$2\theta - \sin 2\theta = 3.14$$

The above condition is satisfied when $\theta = \dfrac{\pi}{2}$ radians or $\theta = 90^\circ$

Evidently the floating log subtends an angle of **180°** at the centre. This also implies that the log is half immersed or the depth of immersion is **0.5 m.**

Example 3.71. *(a)* A metallic cube 25 cm on a side and weighing 260 N is lowered into a tank containing a two-fluid layer of water and mercury. Determine the position of block at mercury-water interface when it has reached equilibrium.

***(b)* In a two-layer fluid with specific gravities 1.2 and 0.9 is inserted a cube 60 cm on an edge. If the upper and lower halves of the cube are composed of materials with specific gravity 0.6 and 1.4 respectively, make calculations for the distance of the top of cube above the interface.**

Solution : *(a)* The metallic cube sinks beneath the water surface and comes to rest at the water-mercury interface :

In accordance with principle of floatation,

weight of cubical block = buoyant force

This also represents the weight of water displaced

$\therefore$ Volume of water displaced $= \dfrac{11551}{9810} = 1.1775 \text{ m}^3$

Volume of wood under water $= 1.1775 \text{ m}^3$

If A represents the sectional area of log of wood under water, then

$$A = \frac{\text{volume of wood under water}}{\text{length of log of wood}} = \frac{1.1775}{3} = 0.3925 \text{ m}^2$$

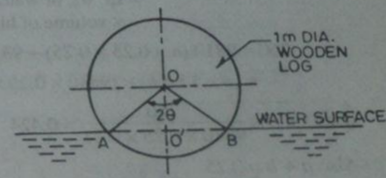

Fig. 3.69.

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$$\therefore \quad A = \text{area of sector } OAB - 2\,(\text{area of } \Delta\, OBO')$$

$$= \frac{2\theta}{2\pi}(\pi r^2) - 2\left(\frac{1}{2} \times OO' \times O'B\right)$$

$$= \frac{1}{2} r^2 (2\theta) - 2\left(\frac{1}{2} \times r\cos\theta \times r\sin\theta\right)$$

$$= \frac{1}{2} r^2 (2\theta) - \frac{r^2}{2}\sin 2\theta$$

$$\therefore \quad 0.3925 = \frac{1}{2}(0.5)^2(2\theta) - \frac{1}{2}(0.5)^2 \sin 2\theta$$

$$\text{or} \quad 2\theta - \sin 2\theta = 3.14$$

The above condition is satisfied when $\theta = \dfrac{\pi}{2}$ radians or $\theta = 90^\circ$

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Example 3.71. ***(a)*** **A metallic cube 25 cm on a side and weighing 260 N is lowered into a tank containing a two-fluid layer of water and mercury. Determine the position of block at mercury-water interface when it has reached equilibrium.**

(b) **In a two-layer fluid with specific gravities 1.2 and 0.9 is inserted a cube 60 cm on an edge. If the upper and lower halves of the cube are composed of materials with specific gravity 0.6 and 1.4 respectively, make calculations for the distance of the top of cube above the interface.**

Solution : *(a)* The metallic cube sinks beneath the water surface and comes to rest at the water-mercury interface :

In accordance with principle of floatation,

weight of cubical block = buoyant force

= weight of water and mercury displaced by the block

= sp. wt. of water × volume of block in water + sp. wt. of mercury × volume of block in mercury

$$260 = 9810\,(a \times 0.25 \times 0.25) + 9810 \times 13.6\,(b \times 0.25 \times 0.25)$$

$$= (a + 13.6\,b) \times (9810 \times 0.25 \times 0.25)$$

$$a + 13.6\,b = \frac{260}{9810 \times 0.25 \times 0.25} = 0.424 \quad ...(i)$$

Also $a + b = 0.25$ ...(ii)

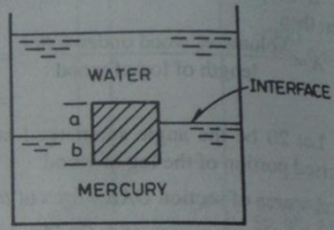

Fig. 3.70.

From expressions *(i)* and *(ii)*, the depth of cube below the water-mercury interface works out

$$b = \frac{0.424 - 0.25}{12.6} = 0.01381 \text{ m} = \mathbf{13.81\ mm}$$

(b) weight of cube

$= (1.4 \times 9.81 \times 0.6 \times 0.6 \times 0.3)$

$+ (0.6 \times 9.81 \times 0.6 \times 0.6 \times 0.3) = 2.119$ kN

Let h be the height of cube above the interface.

buoyant force = weight of lighter and heavier liquids displaced by the block

$= (0.9 \times 9.81) \times (0.6 \times 0.6 \times h)$

$+ (1.2 \times 9.81)\,[0.6 \times 0.6 \times (0.6 - h)]$

$= (-1.06\,h + 2.5428)$ kN

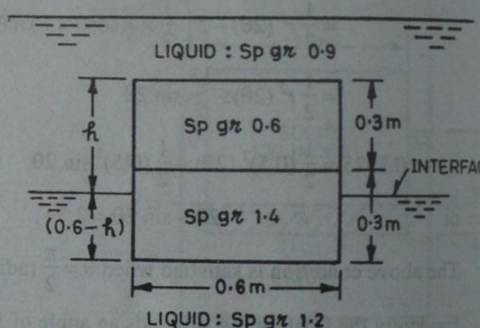

Fig. 3.71.

In accordance with the principle of floatation,

weight of block = buoyant force

$$2.119 = -1.06\,h + 2.5428$$

∴ height of cube above the interface,

$$h = \frac{2.5428 - 2.119}{1.06} = 0.4 \text{ m} = \mathbf{40\ cm}$$

Example 3.72. A wooden cylinder (specific gravity = 0.6) of circular cross section having length *l* and diameter *d* floats in water. Find the maximum permissible *l/d* ratio so that the cylinder may float in stable equilibrium with its axis vertical.

What would be the cylinder length (height) if it is 25 cm in diameter.

Solution : Weight of cylinder = specific weight × volume of cylinder

$$= sw \times \frac{\pi}{4} d^2\, l$$

where s is the specific gravity of the cylinder material.

This also represents the weight of water displaced.

Volume of water displaced

$$= \frac{sw \times \frac{\pi}{4} d^2 l}{w} = \frac{\pi}{4} sd^2 l$$

Volume of cylinder immersed in water

$$= \frac{\pi}{4} sd^2 l$$

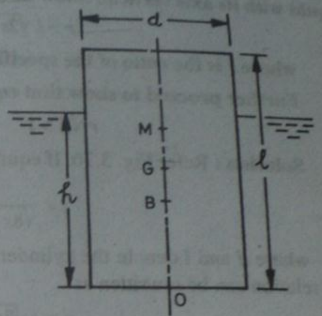

Fig. 3.72

$$\text{Depth of immersion} = \frac{\text{volume of cylinder in water}}{\text{sectional area of cylinder}}$$

$$h = \frac{\frac{\pi}{4} sd^2 l}{\frac{\pi}{4} d^2} = sl$$

Height of centre of buoyancy (B) above the base point O.

$$OB = \frac{h}{2} = \frac{sl}{2}$$

If M is the meta centre, then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2 h} = \frac{d^2}{16h} = \frac{d^2}{16sl}$$

$$OM = OB + BM = \frac{sl}{2} + \frac{d^2}{16sl}$$

Distance of centre of gravity (G) from the base point O : $OG = \frac{l}{2}$

For stable equilibrium, M should be at a level greater than G. That is

$$OM > OG$$

or $$\frac{sl}{2} + \frac{d^2}{16sl} > \frac{l}{2}$$

or $$\frac{l}{2} - \frac{sl}{2} < \frac{d^2}{16sl}$$

or $$\frac{l}{2}(1 - s) < \frac{d^2}{16sl}$$

or $$l^2 < \frac{d^2}{8s(1 - s)}$$

$\therefore$ $$l < \frac{d}{\sqrt{8s(1 - s)}}$$

Given $d = 25$ cm, $s = 0.6$

$$\therefore \quad l < \frac{25}{\sqrt{8 \times 0.6\,(1-0.6)}} \;;\; l < \mathbf{18.04\ m}$$

Example 3.73. A solid cylinder of length (height) l and radius r is floating in a homogenous liquid with its axis vertical. Show that for stable equilibrium of the cylinder.

$$r > l\sqrt{2s\,(1-s)}$$

where s is the ratio of the specific gravity of the cylinder material to that of the liquid.

Further proceed to show that equilibrium will be stable whatever be the value of s if

$$r\sqrt{2} > l$$

Solution : Refer Fig. 3.76. If equilibrium is to be stable with the cylinder axis vertical, then

$$l < \frac{d}{\sqrt{8s\,(1-s)}} \text{ and } h = sl \qquad \{\text{See Example 3.72}\}$$

where d and l denote the cylinder diameter and length respectively, and h is the depth of immersion. This relation can be rewritten as

$$d > l\sqrt{8s\,(1-s)}$$

or $$2r > 2l\sqrt{2s\,(1-s)}$$

or $$r > l\sqrt{2s\,(1-s)} \qquad ...(i)$$

which is the required expression.

A condition for stable equilibrium for any value of s is obtained by eleminating s from expression *(i)*. This is done by making the substitution : $s = h/l$. That is :

$$r^2 > l^2\,[2s\,(1-s)]$$

or $$r^2 > l^2\left[2\frac{h}{l}\left(1-\frac{h}{l}\right)\right]$$

or $$r^2 > (2hl - 2h^2) \qquad ...(ii)$$

Condition for maximum value of $(2lh - 2h^2)$ is now obtained by differentiating it with respect to h and setting the derivative to zero. That is

$$\frac{d}{dh}(2hl - 2h^2) = 0 \;;\; 2l - 4h = 0 \;;\; h = l/2$$

Thus the system will be in stable equilibrium if $h = \frac{l}{2}$ and the expression *(ii)* for stability becomes

$$r^2 > \left[2\frac{l}{2} \times l - 2\left(\frac{l}{2}\right)^2\right]$$

or $$r^2 > \left(l^2 - \frac{l^2}{2}\right)$$

or $$2r^2 > l^2 \quad \therefore \quad r\sqrt{2} > l.$$

Example 3.74. A hollow wooden cylinder of specific gravity 0.56 has an outer diameter of 60 cm and an inner diameter of 30 cm. It is required to float in oil of specific gravity 0.85. Calculate the maximum height of the cylinder so that it shall be stable when floating with its axis vertical. Also

calculate the depth to which it will sink.

Solution : Weight of cylinder = specific weight × volume of cylinder

$$= (9810 \times 0.56) \times \frac{\pi}{4}\left\{(0.6)^2 - (0.3)^2\right\} \times l = 1164.94\, l$$

where l is the length (height) of the cylinder.

This also represents the weight of oil displaced.

$$\text{Volume of oil displaced} = \frac{1164.94\, l}{9810 \times 0.85} = 0.1397\, l$$

Volume of cylinder immersed in oil = $0.1397\, l$

$$\text{Depth of immersion } h = \frac{\text{volume of cylinder under oil}}{\text{cross–sectional area of cylinder}}$$

$$= \frac{0.1397\, l}{\frac{\pi}{4}(0.6^2 - 0.3^2)} = 0.659\, l$$

Height of centre of buoyancy (B) above the base point O,

$$OB = \frac{h}{2} = \frac{0.659\, l}{2} = 0.3295\, l$$

If M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64}\left[(0.6)^4 - (0.3)^4\right]}{0.1397\, l} = \frac{0.0427}{l}$$

$$OM = OB + OM = 0.3295\, l + \frac{0.0427}{l}$$

Distance of centre of gravity (G) from the base point O : $OG = \frac{l}{2} = 0.5\, l$

For stable equilibrium, M should be at a level greater than G. That is

$$OM > OG$$

$$0.3295\, l + \frac{0.0427}{l} > 0.5\, l$$

or $$\frac{0.0427}{l} > 0.1705\, l$$

or $$l < \left(\frac{0.0427}{0.1705}\right)^{1/2} \;;\; l < 0.50 \quad \therefore\; l_{max} = \mathbf{0.50\ m}$$

and $$h = 0.659\, l = 0.659 \times 0.50 = \mathbf{0.3295\ m}$$

Example 3.75. A hollow cylinder closed at both ends has an outside diameter of 1.25 m, length 3.5 m and specific weight 75 kN/m³. If the cylinder is to float just in stable equilibrium in sea water, find its minimum permissible thickness. Presume that sea water weighs 10 kN/m³.

Solution : Let t be the thickness of cylinder and x be its depth of immersion in sea water.

Weight of sea water displaced $= \frac{\pi}{4} d^2 x w$

$$= \frac{\pi}{4}(1.25)^2 x \times 10 = 12.26\, x$$

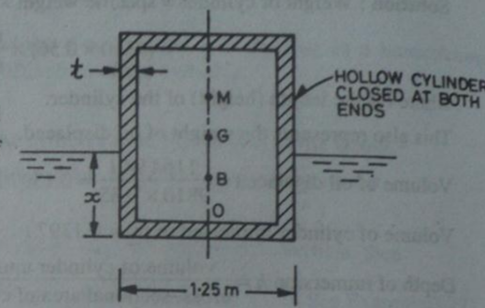

Fig. 3.73.

Weight of cylinder = volume of cylinder × specific weight of cylinder material

= (volume of two end sections + volume of circular portion) × specific weight

$$= \left[2 \times \frac{\pi}{4} d^2\, t + \frac{\pi}{4}\left\{ d^2 - (d-2t)^2 \right\} l \right] \times \text{sp. wt.}$$

$$= \left[2 \times \frac{\pi}{4} d^2\, t + \pi\, dtl \right] \times \text{sp. wt}$$

$$= \left[2 \frac{\pi}{4}(1.25)^2\, t + \pi \times 1.25\, t \times 3.5 \right] \times 75 = 1.21 \times 10^3\, t$$

Under equilibrium conditions, the weight of cylinder equals the weight of sea water displaced.

$$\therefore \qquad 12.26\, x = 1.21 \times 10^3\, t \quad ; \quad x = 98.69\, t$$

Volume of cylinder under water or volume of sea water displaced,

$$V = \frac{1.21 \times 10^3\, t}{10} = 121\, t$$

Let M be the meta centre. Then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64} \times 1.25^4}{121 t} = \frac{9.9 \times 10^{-4}}{t}$$

Depth of centre of buoyancy (B) from the base point O,

$$OB = \frac{x}{2} = \frac{98.69}{2}$$

Distance of centre of gravity (G) from the base point O,

$$OG = \frac{3.5}{2} = 1.75$$

$$BG = OG - OB = (1.75 - 49.56t)$$

For the cylinder to float just in stable equilibrium

$$BG = BM$$

$$1.75 - 49.56t = \frac{9.85 \times 10^{-4}}{t}$$

$$49.56t^2 - 1.75\, t + 9.85 \times 19^{-4} = 0$$

$$t = \frac{1.75 \pm \sqrt{(-1.75)^2 - 4 \times 49.56 \times 9.85 \times 10^{-4}}}{2 \times 49.56}$$

$$= \frac{1.75 \pm 1.69}{99.12} = 0.0347 \text{ m} \ \text{ or } \ 0.000605 \text{ m}$$

Obviously the minimum permissible thickness of the cylinder is **0.605 mm.**

Example 3.76. A cone of base radius R and height H floats in water with vertex downwards. Show that for stable equilibrium of the cone :

(i) $\sec^2\theta > \frac{H}{h}$ **(ii)** $H < \left[\frac{R^2 s^{1/3}}{1 - s^{1/3}}\right]^{\frac{1}{2}}$

where *h* is the depth of immersion, θ is the semi-vertex angle of the cone and *s* is the specific gravity of the cone material.

Solution : Distance of centre of gravity G from the vertex O,

$$OG = \frac{3}{4} H$$

Distance of centre of buoyancy B from the vertex O,

$$OB = \frac{3}{4} h$$

Let M denote the position of metacentre. Then

$$BM = \frac{I}{V}$$

where I = moment of inertia $= \frac{\pi}{4} r^4$

V = volume of water displaced

$$= \frac{1}{3} \pi r^2 h$$

$$\therefore \quad BM = \frac{\frac{\pi}{4} r^4}{\frac{1}{3} \pi r^2 h} = \frac{3}{4} \times \frac{r^2}{h}$$

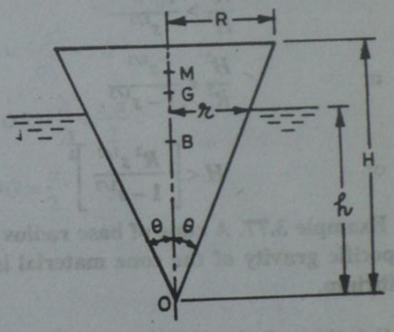

Fig. 3.74.

Substituting $r = h \tan\theta$, we get

$$BM = \frac{3}{4} \times \frac{(h \tan\theta)^2}{h} = \frac{3}{4} \tan^2\theta$$

$$OM = OB + BM = \frac{3}{4} h + \frac{3}{4} h \tan^2\theta$$

$$= \frac{3}{4} h (1 + \tan^2\theta) = \frac{3}{4} h \sec^2\theta$$

For stable equilibrium, M should be at a level higher than G. That is

$$OM > OG$$

$$\frac{3}{4} h \sec^2\theta > \frac{3}{4} H \text{ or } \sec^2\theta > \frac{H}{h}$$

(b) In accordance with the principle of floatation,

weight of cone = weight of water displaced

$$\frac{1}{3}\pi R^2 H w_c = \frac{1}{3}\pi r^2 h w_l$$

where w_l and w_c are specific weights of the liquid and cone material respectively.

Substituting $R = H\tan\theta$, $r = h\tan\theta$ and upon simplification, we obtain

$$h = H\left(\frac{w_c}{w_l}\right)^{1/3} = H s^{1/3}$$

where s is the specific gravity of the cone material with respect to the liquid.

From the relations $\sec^2\theta > \frac{H}{h}$ and $h = H s^{1/3}$ as outlined above,

$$\sec^2\theta > \frac{1}{s^{1/3}} \text{ or } (1+\tan^2\theta) > \frac{1}{s^{1/3}}$$

or $$\tan^2\theta > \frac{1}{s^{1/3}} - 1 \text{ or } \tan^2\theta > \frac{1-s^{1/3}}{s^{1/3}}$$

Substituting $\tan\theta = \frac{R}{H}$,

$$\frac{R^2}{H^2} > \frac{1-s^{1/3}}{s^{1/3}}$$

or $$\frac{H^2}{R^2} < \frac{s^{1/3}}{1-s^{1/3}}$$

or $$H < \left[\frac{R^2 s^{1/3}}{1-s^{1/3}}\right]^{\frac{1}{2}}.$$

Example 3.77. A cone of base radius R and height H floats in water with vertex downwards. If the specific gravity of the cone material is 0.8, find the least apex angle of the cone for just stable equilibrium.

Solution : Refer Fig. 3.74. From the relations, $\sec^2\theta > \frac{H}{h}$ and $h = H s^{1/3}$ as outlined in example 3.76,

$$\sec^2\theta > 1/s^{1/3} \text{ or } \cos^2\theta < s^{1/3}$$

For equilibrium to be just stable,

$$\cos^2\theta = s^{1/3} \text{ or } \cos\theta = s^{1/6}$$

Substituting $s = 0.8$

$$\cos\theta = (0.8)^{1/6} = 0.963 \;;\; \theta = 15.54°$$

∴ Apex angle $\quad 2\theta = 31.08°$

Example 3.78. A right solid cone with apex angle equal to 60° is of density relative to that of liquid in which it floats with apex downwards. Determine what range of s is compatible with stable equilibrium.

Solution : Refer Fig. 3.74. From the relations, $\sec^2\theta > H/h$ and $h = H s^{1/3}$ as outlined in example 3.76,

$$\sec^2\theta > 1/s^{1/3}$$

or
$$s^{1/3} > \frac{1}{\sec^2\theta} > \cos^2\theta$$

Given θ = half apex angle = 30°

$$s^{1/3} > \cos^2 30 > 0.75 \quad ; \quad s > \mathbf{0.4218}$$

Further if the body is not to sink then s must be less than unity. Hence the range $0.4218 < s < 1$ is compatible with stable equilibrium.

Example 3.79. A cube of side a and relative density s floats in water. Determine the conditions for its stability if it is given an angular tilt.

Solution : Weight of cube = specific weight × volume of cube = $s\,w\,a^3$

where w is the specific weight of water.

This also represents the weight of water displaced.

$$\text{Volume of water displaced} = \frac{s\,w\,a^3}{w} = s\,a^3$$

$\therefore$ Volume of cube immersed in water = $s\,a^3$

$$\text{Depth of immersion} = \frac{\text{volume of cube in water}}{\text{cross–sectional area of cube}} \quad ; \quad h = \frac{s\,a^3}{a^2} = s\,a$$

Height of centre of buoyancy B above the base point O, $OB = \dfrac{h}{2} = \dfrac{s\,a}{2}$

Distance of centre of gravity G from the base point O, $OG = \dfrac{a}{2}$

$$\therefore \quad BG = OG - OB = \frac{a}{2} - \frac{sa}{2} = \frac{a}{2}(1-s)$$

If M is the meta centre, then

$$BM = \frac{I}{V} = \frac{a \times a^3/12}{a^2 h} = \frac{a^2}{12sa} = \frac{a}{12s}$$

$$MG = BM - BG = \frac{a}{12s} - \frac{a}{2}(1-s)$$

Condition of stability is therefore :

(i) Stable if MG is +ve, *i.e.*, M lies above G

or
$$\frac{a}{12s} > \frac{a}{2}(1-s)$$

(ii) Neutral if $MG = 0$, *i.e.*, M coincides with G

or $\qquad \frac{a}{12s} = \frac{a}{2}(1-s)$

(iii) Unstable if MG is –ve, *i.e., M* lies below G

or $\qquad \frac{a}{12s} < \frac{a}{2}(1-s)$

Thus for stable/unstable equilibrium

$$\frac{a}{12s} \le \frac{a}{2}(1-s)$$

or $\qquad 6s^2 - 6s + 1 = 0$

or $\qquad s = \frac{6 \pm \sqrt{36-24}}{12} = \frac{1}{2} \pm \frac{\sqrt{3}}{6} = 0.5 \pm 0.289 = 0.789 \text{ or } 0.211$

Evidently the relative density of cube material should be greater than 0.789 or less than 0.211 for stability.

Example 3.80. A wooden block (specific gravity = 0.7) of width 15 cm × depth 30 cm and length 150 cm floats horizontally on the surface of sea water (sp. wt = 10 kN). Calculate the volume of water displaced, depth of immersion and the position of centre of buoyancy. Also find the metacentric height and the righting moment for a tilt of 5°. Comment on stability of the block.

Solution :

Weight of block = specific weight × volume of block = $(0.7 \times 9.81) \times (1.5 \times 0.3 \times 0.15) = 0.464$ kN

This also represents the weight of sea water displaced.

Volume of sea water displaced

$$= \frac{0.464}{10} = 0.0464 \text{m}^3$$

Volume of block under water

$$= 0.0464 \text{ m}^3$$

Depth of immersion

$$= \frac{\text{volume of block under water}}{\text{sectional area of the block}}$$

$$= \frac{0.0464}{1.5 \times 0.15} = \mathbf{0.206 \ m.}$$

Distance of centre of buoyancy *(B)* from the base point *O*,

$$OB = \frac{0.206}{2} = 0.103 \text{ m}$$

Let *M* be the metacentre. Then

$$BM = \frac{I}{V} = \frac{\frac{1}{12} \times 1.5 \times 0.15^3}{0.0464}$$

$$= 0.0091 \text{ m}$$

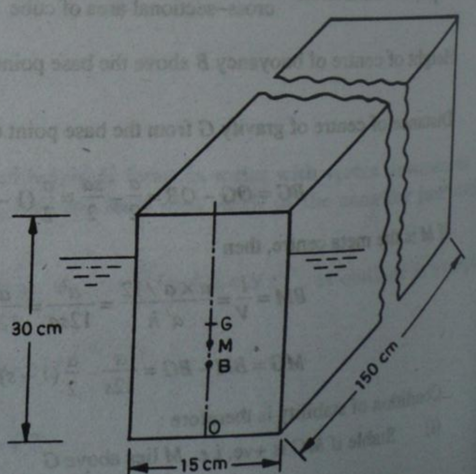

Fig. 3.75.

$$OM = OB + BM = 0.103 + 0.0091 = 0.1121 \text{ m}$$

Distance of centre of gravity *(G)*, from the base point *O*,

$$OG = \frac{0.30}{2} = 0.15 \text{ m}$$

Metacentric height $GM = OM - OG$

$$= 0.1121 - 0.15 = \mathbf{-0.0379 \text{ m}}$$

The metacentre *M* lies 3.79 cm below the centre of gravity ; hence the position is unstable and the block will turnover.

Over turning moment = $W \times GM\,\theta$ (for small tilt angle θ measured in radians)

$$= 0.464 \times 0.0379 \times \frac{5\pi}{180} = 1.53 \times 10^{-3} \text{ kN m}$$

Example 3.81. A rectangular pontoon 10 m long, 7.5 m wide and 2.5 m deep weighs 800 kN and floats in sea water (specific weight = 10 kN/m³). The pontoon carries on its upper deck a boiler 5 m in diameter and weighing 500 kN. The centre of gravity of each unit coincides with geometrical centre of the arrangement and lies in the same vertical line. Compute the metacentric height and comment on the stability of the system.

Solution : Weight of the arrangement, $W = 800 + 500 = 1300$ kN

This also represents the weight of water displaced

∴ Volume of sea water displaced by the arrangement,

$$\text{Volume } V = \frac{\text{weight of water displaced}}{\text{specific weight of water}} = \frac{1300}{10} = 130 \text{ m}^3$$

Volume of arrangement under water = 130 m³

Depth of immersion

$$= \frac{\text{volume of arrangement under water}}{\text{sectional area of the pontoon}}$$

$$= \frac{130}{10 \times 7.5} = 1.73 \text{ m}$$

Distance of centre of buoyancy *(B)* from the base point *O*,

$$OB = \frac{1.73}{2} = 0.865 \text{ m}$$

Let *M* be the metacentre. Then

$$BM = \frac{I}{V} = \frac{\frac{1}{12} \times 10 \times 7.5^3}{130} = 2.70 \text{ m}$$

$$OM = OB + BM = 0.865 + 2.7$$

$$= 3.565 \text{ m}$$

The position of the combined centre of gravity above the base point *O* can be

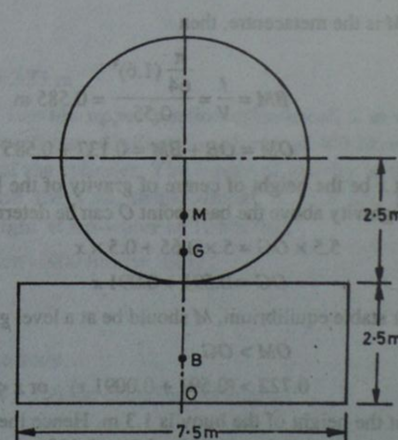

Fig. 3.76.

determined by taking moments about the point O.

$$1300 \times OG = 800 \times 1.25 + 500 \times 5$$

This gives distance of centre of gravity (G) from the base point O.

$$OG = \frac{800 \times 1.25 + 500 \times 5}{1300} = 2.69 \text{ m}$$

Since $OM > OG$, M is at a higher level than G. Obviously the arrangement is in stable equilibrium.

Metacentric height, $GM = OM - OG$

$$= 3.565 - 2.69 = \mathbf{0.875 \text{ m}}$$

Example 3.82. A cylindrical buoy, 1.6 m in diameter × 1.3 m in length and weighing 5 kN floats in sea water with its axis vertical. A 500 N load is placed centrally at the top of the buoy. If the buoy is to remain in stable equilibrium, find the maximum permissible height of the centre of gravity of the load above the top of the buoy. Specific weight of sea water = 10 kN/m^3.

Solution : Weight of buoy including that of load = 5.5 kN

This is also the weight of water displaced by the arrangement.

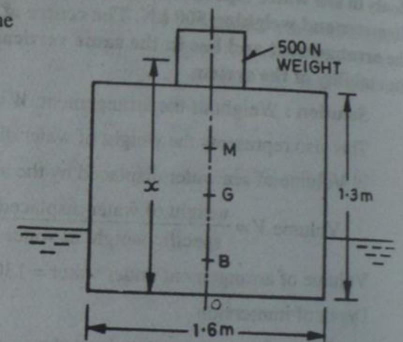

Fig. 3.77.

$$\text{Volume of water displaced} = \frac{5.5}{10} = 0.55 \text{ m}^3$$

Volume of buoy immersed in water = 0.55 m^3

$$\text{Depth of immersion} = \frac{0.55}{\frac{\pi}{4}(1.6)^2} = 0.274 \text{ m}$$

Height of centre of buoyancy (B) above base (O),

$$OB = \frac{0.274}{2} = 0.137 \text{ m}$$

If M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64}(1.6)^4}{0.55} = 0.585 \text{ m}$$

$$OM = OB + BM = 0.137 + 0.585 = 0.722 \text{ m}$$

Let x be the height of centre of gravity of the load above the base (O). The position of the combined centre of gravity above the base point O can be determined by taking moments about point O.

$$5.5 \times OG = 5 \times 0.65 + 0.5 \times x$$

$$OG = 0.591 + 0.091\, x$$

For stable equilibrium, M should be at a level greater than G. That is

$$OM > OG$$

or $\quad 0.722 > (0.591 + 0.0091\, x) \quad$ or $x < 1.439$ m

But the height of the buoy is 1.3 m. Hence the height of centre of gravity of the load above the buoy should not be greater than $(1.439 - 1.3) = \mathbf{0.139 \text{ m}}$.

Example 3.83. A rectangular pontoon weighing 2500 kN has a length of 25 metres. The centre of gravity is 30 cm above the centre of cross-section and for 10° angle of heel the metacentric height is 1.3 m. When the pontoon is vertical, the free board is stated to be not more than 0.65 m. Find the breadth and the height of pontoon if it is floating in fresh water (specific weight = 10 kN/m³).

Solution : Volume of water displaced $= \frac{2500}{10} = 250 \text{ m}^3$.

Let b be the breadth and d be the depth of pontoon. With a free board (height of portion lying outside water) of 0.65 m, the depth of immersion is $(d - 0.65 \text{ m})$.

$\therefore$ Height of centre of buoyancy above the base

$$OB = \frac{d - 0.65}{2}$$

and height of centre of gravity above the base

$$OG = \frac{d}{2} + 0.30$$

$$\therefore \quad BG = \left(\frac{d}{2} + 0.30\right) - \left(\frac{d - 0.65}{2}\right) = 0.625$$

Now if M is the metacentre, then

$$BM = \frac{I}{V} = \frac{\frac{1}{12} \times 25 \times b^3}{250} = \frac{b^3}{120}$$

$$GM = BM - BG = \frac{b^3}{120} - 0.625$$

$$\therefore \quad 1.3 = \frac{b^3}{120} - 0.625$$

and breadth $b = [120 \times 1.3 + 0.625]^{1/3} = \mathbf{5.30 \text{ m}}$

Also, volume $\quad V = b\,(d - 0.65) \times 25$

$$250 = 5.30\,(d - 0.65) \times 25\ ;\ d = \mathbf{2.53 \text{ m}}$$

Example 3.84. A buoy carrying a beam-light has the upper portion cylindrical, 2 m diameter and 1.25 m deep. The lower portion which is a curved one, displaces a volume of 400 litres and its centre of buoyancy is situated 1.3 m below the top of the cylinder. The centre of gravity of the whole buoy is 0.95 m below the top of the cylinder and the total displacement is 25.5 kN. Find the metacentric height of the body. Take the specific weight of sea-water as 10 kN/m³.

Solution : Volume displaced by the curved portion = 400 litres = 0.4 m³

Total displacement = 25.5 kN $= \frac{25.5}{10} = 2.55 \text{ m}^3$

This is also the volume of immersed portion of the body.

$\therefore \quad$ 2.55 = [volume of cylindrical part in water + volume of curved portion]

$$= \frac{\pi}{4} d^2 x + 0.4$$

where x is the depth of immersion of the cylindrical part in water

$$2.55 = \frac{\pi}{4}(2)^2 x + 0.4 \; ; x = 0.684 \text{ m}$$

Depth of centre of buoyancy B_1 of the curved portion : $OB_1 = 1.3$ m

Depth of centre of buoyancy B_2 of the cylindrical part

$$OB_2 = (1.25 - 0.684) + \frac{0.684}{2} = 0.908 \text{ m}$$

The distance of centre of buoyancy of the whole buoy from the top of the cylindrical part is given as :

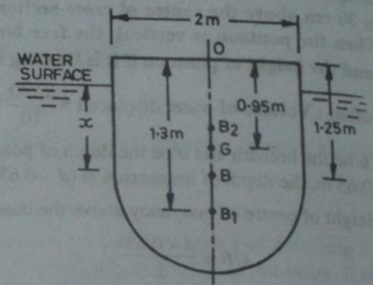

Fig. 3.78.

$$OB = \frac{(\text{volume of curved portion} \times OB_1) + (\text{volume of cylindrical portion} \times OB_2)}{\text{total volume of water displaced}}$$

$$= \frac{0.4 \times 1.3 + (2.55 - 0.4) \times 0.908}{2.55} = \mathbf{0.9695 \text{ m}}$$

If M denotes the metacentre, then

$$BM = \frac{I}{V}$$

where I = moment of inertia of the plane of the body at water surface

$$= \frac{\pi}{64} d^4 = \frac{\pi}{64}(2)^4 = 0.785 \text{ m}^4$$

V = Volume of the body under water = 2.6 m^3

$$\therefore \quad BM = \frac{0.785}{2.6} = 0.3019 \text{ m}$$

Distance of centre of gravity (G) from the top of the cylindrical part,

$OG = 0.95$ m ...(Given)

$BG = OB - OG = 0.9695 - 0.95 = 0.0195$ m

$\therefore$ Metacentric height $GM = BM - BG$

$= 0.3019 - 0.0195 = 0.2824$ m = **28.24 cm**

Example 3.85. A light metal stick of square section (12 cm × 12 cm × 3 cm) weighs 490 N and is oriented as shown in Fig. 3.79. Determine its angle of inclination when the free water surface is 2.5 m above the pivot. What minimum depth of water will be required to bring the metal stick in vertical position ?

Solution : Let y be the depth of water above the pivot when the stick makes an angle θ with horizontal.

Length of stick below water $= \dfrac{y}{\sin \theta}$

Volume of stick submerged in water

$$= 0.12 \times 0.12 \times \frac{y}{\sin\theta} = 0.0144\, y \operatorname{cosec}\theta$$

Buoyancy force, F_b = weight of liquid displaced

$$= 9810 \times 0.0144\, y \operatorname{cosec}\theta$$

$$= 141.26\, y \operatorname{cosec}\theta$$

For equilibrium conditions, the moment of the weight of stick must be equal to the moment of buoyant force about the pivot. That is

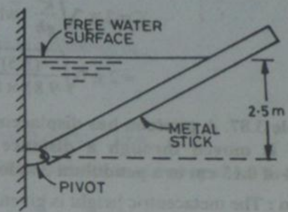

Fig. 3.79.

$$490 \times 2.5 \cos\theta = 141.26\, y \operatorname{cosec}\theta \left(\frac{1}{2} \times \frac{y}{\sin\theta}\right) \cos\theta$$

$$\frac{y}{\sin\theta} = \sqrt{\frac{490 \times 2.5 \times 2}{141.26}} = 4.165$$

(i) For water depth of 2.5 cm

$$\sin\theta = \frac{2.5}{4.165} = 0.6 \quad ; \quad \theta = 36.87^\circ$$

(ii) For vertical position of stick, $\theta = 90^\circ$

$$y = 4.165 \sin 90^\circ = 4.165 \text{ m}$$

For $y \geq 4.165$ the stick will stand vertically.

Example 3.86. A ship has a displacement of 50 MN. The second moment of area of the water line section about its fore-and-oft axis is 1.5×10^4 m^4 and the centre of buoyancy is 2.5 m below the centre of gravity. The radius of gyration of the ship is 3.5 m. Determine the metacentric height and the time period of oscillations if the ship floats in sea water of specific weight 10 kN/m^3.

Solution : Total displacement = 50 MN = 50×10^3 kN

This represents the weight of water displaced by the ship.

$$\text{Volume of water displaced} = \frac{\text{weight of water displaced}}{\text{specific weight of water}}$$

$$V = \frac{50 \times 10^3}{10} = 5000 \text{ m}^3$$

Distance between centre of buoyancy B and the metacentre M is given by

$$BM = \frac{I}{V} = \frac{1.5 \times 10^4}{5000} = 3 \text{ m}$$

$$BG = 2.5 \text{ m} \qquad \text{...(Given)}$$

$\therefore$ Metacentric height $GM = BM - BG$

$$= 3 - 2.5 = 0.5 \text{ m}$$

The metacentric height is **0.5 m** and it is positive; Metacentre *M* lies above the centre of gravity *G*.

(ii) Time period of oscillation,

$$T = 2\pi\sqrt{\frac{k^2}{gh}} \quad \text{where } h \text{ is the metacentric height.}$$

$$= 2\pi\sqrt{\frac{(3.5)^2}{9.81 \times 0.5}} = \mathbf{9.93\ s}$$

Example 3.87. A pontoon has displacement of 20 MN whilst floating in sea water. When a load of 0.25 MN is moved through a distance of 8 m across the deck, there occurs a horizontal displacement of 0.15 cm in a pendulum 3 m long. Compute the metacentric height of the pontoon.

Solution : The metacentric height is given by;

$$GM = \frac{w\,x}{W \tan\theta}$$

Given : $w = 0.25$ MN ; $x = 8$ m ; $W = 20$ MN ; $\tan\theta = \frac{0.15}{3} = 0.05$

$$GM = \frac{0.25 \times 8}{20 \times 0.05} = \mathbf{2\ m}$$

Example 3.88. A vessel has a length of 56 m, width 8 m and a displacement of 12×10^6 N when wholly submerged in sea water (mass density 1025 kg/m³). When a weight of 0.15×10^6 N is rolled off transversely across the deck through a distance of 5 m, the vessel tilts through 3°. The second moment of area of the water line section about its fore-and-oft axis is 75 percent of that of the circumscribing rectangle, and the centre of buoyancy is 1.6 m below the water line. Make calculations for the metacentric height and the position of centre of gravity of the vessel.

Solution : The metacentric height is given by :

$$GM = \frac{w\,x}{W \tan\theta}$$

$$= \frac{(0.15 \times 10^6) \times 5}{(12 \times 10^6) \times \tan 3^\circ} = 1.20 \text{ m}$$

Distance between the metacentric *M* and the centre of buoyancy *B* is,

$$BM = \frac{I}{V}$$

where I = second moment of water line area

$$= 0.75\left[\frac{56 \times 8^3}{12}\right] = 1792 \text{ m}^4$$

and V = volume of water displaced by the vessel

$$= \frac{\text{weight of the vessel}}{\text{specific weight of water}}$$

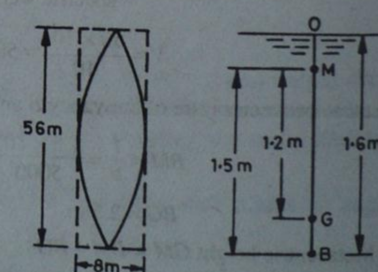

Fig. 3.80.

$$= \frac{12 \times 10^6}{1025 \times 9.81} = 1193.4 \text{ m}^3$$

$$\therefore \quad BM = \frac{1792}{1193.4} = 1.50 \text{ m}$$

Reference Fig. 3.84, the centre of gravity of the vessel is at a distance of :

$$OG = OM + MG$$
$$= (OB - MB) + MG$$
$$= (1.6 - 1.5) + 1.2 = 1.3 \text{ m below the water line.}$$

Example 3.89. (*i*) The metacentric height of a passenger ship is kept lower than that of a naval or cargo ship. Why ?

(*ii*) Why is it that containers of a liquid cargo are compartmentalised ?

A barge, 16 m long × 5 m wide, has a draft of 1.6 m when floating in upright position. What would be its metacentric height if its centre of gravity is 1.9 m above the bottom. Subsequently a 50 kN weight is shifted 3.5 m across the barge. To what distance would the water-line rise on the side ? Also calculate the corresponding righting moment.

Solution : (*i*) To decrease the frequency of rolling (*ii*) Otherwise the ship may become unstable.

If M denotes the metacentre, then

$$BM = I/V$$

where I = moment of inertia of the plane of the body at water surface

$$= \frac{bd^3}{12} = \frac{16 \times 5^3}{12} = 166.67 \text{ m}^4$$

V = volume of water displaced $= 16 \times 5 \times 1.6 = 128 \text{ m}^3$

$$\therefore \quad BM = \frac{166.67}{128} = 1.302 \text{ m}$$

Distance of centre of buoyancy (B) from the base point O,

$$OB = \frac{1.6}{2} = 0.8 \text{ m}$$

$$OM = OB + BM = 0.8 + 1.302 = 2.102 \text{ m}$$

Distance of centre of gravity from the base point O,

$$OG = 1.9 \text{ m (given)}$$

$\therefore$ Metacentric height $GM = OM - OG$

$$= 2.102 - 1.9 = 0.202 \text{ m}$$

Since M is at a level higher than G, the arrangement is in stable equilibrium.

Metacentric height is given by,

$$GM = \frac{w\,x}{W \tan \theta}$$

Weight of barge W = weight of water displaced

$= \text{specific weight} \times \text{volume of water displaced}$
$= 9.81 \times 128 = 1255.68 \text{ kN}$

By substitution,

$$\tan\theta = \frac{50 \times 3.5}{1255.68 \times 0.202} = 0.69$$

Rise of water-line on side $= \frac{b}{2}\tan\theta = \frac{2.5}{2} \times 0.69 = \mathbf{0.8625\ m}$

Righting moment for this tilt,

$$= W \times GM \tan\theta$$
$$= 1255.68 \times (0.202 \times 0.69) = 175 \text{ kN m}$$

Example 3.90. A cylindrical buoy is 2 m in diameter and 2.5 m long and weighs 21.5 kN. The specific weight of sea water is 10 kN/m³. Show that the buoy does not float with its axis vertical. What minimum pull should be applied to a chain attached to the centre of the base to keep the buoy vertical?

Solution : Weight of the buoy $W = 21.5$ kN

This also represents the weight of water displaced.

∴ Volume of sea water displaced by the buoy,

$$\text{Volume } V = \frac{\text{weight of water displaced}}{\text{specific weight of water}} = \frac{21.5}{10} = 2.15 \text{ m}^3$$

Volume of buoy immersed in water = 2.146 m³

$$\text{Depth of immersion} = \frac{\text{volume of buoy under water}}{\text{sectional area of the buoy}} = \frac{2.15}{\frac{\pi}{4}(2)^2} = 0.684 \text{ m}$$

Distance of centre of buoyancy (B) from the base point O,

$$OB = \frac{0.684}{2} = 0.342 \text{ m}$$

Let M be the metacentre. Then

$$BM = \frac{I}{V} = \frac{\frac{\pi}{64} \times (2)^4}{2.15} = 0.365 \text{ m}$$

$$OM = OB + BM = 0.342 + 0.365 = 0.707 \text{ m}$$

Distance of centre of gravity (G) from the base point O,

$$OG = \frac{2.5}{2} = 1.25 \text{ m}$$

Since $OM < OG$, M is at a level lower than G.

Obviously the buoy is unstable when floating with axis vertical.

Metacentric height, $GM = OM - OG$

$$= 0.707 - 1.25 = 0.543 \text{ m}$$

(ii) Let T in kN be the pull in a chain, anchored to the base of the buoy so as to keep it vertical.

Total downward force $R = (W + T) = (21.5 + T)$

Displaced volume of water $= \frac{R}{10}$ m^3

New depth of immersion $= \frac{R}{10 \times \frac{\pi}{4} \times (2)^2} = \frac{R}{31.4}$

$$OB' = \frac{1}{2} \times \frac{R}{31.4} = \frac{R}{62.8} \text{ m}$$

$$B'M' = \frac{I}{V} = \frac{\frac{\pi}{64} \times (2)^4}{\frac{R}{10}} = \frac{7.85}{R}$$

The new centre of gravity G' due to self weight acting at G and the tension T in the chain at O can be obtained by taking moments about point O.

$$R \times OG' = 21.5 \times 1.25 \;;\; OG' = \frac{26.875}{R}$$

$$B'G' = OG' - OB' = \frac{26.875}{R} - \frac{R}{62.8}$$

For stable equilibrium, M' must lie above G', *i.e.*,

$$B'M' > B'G'$$

$$\frac{7.85}{R} > \frac{26.875}{R} - \frac{R}{62.8}$$

Solving for the condition $B'M' = B'G'$; $R = 34.56$ kN

∴ Minimum pull in the chain to keep the buoy in vertical position,

$$T = 34.56 - 21.5 = \mathbf{13.06\ kN}$$

SECTION D : Liquids in Relative Equilibrium

Consider that a tank filled with a liquid is made to move with a constant acceleration. Initially the fluid particles will move relative to each other and to the boundaries of the tank. However, after a certain unsteady period, there will not be any relative motion between the fluid particles or between fluid particles and boundaries of the container. The entire fluid mass moves as a single unit. A similar situation occurs when a fluid mass is made to rotate with a uniform velocity. Fluids in such a motion are said to be in **relative equilibrium.** Because of no relative motion, the fluid is free from shearing forces. Further, the fluid pressure acts normal to the surface in contact with it. Under these conditions, an analysis of the fluid masses subjected to acceleration (or deceleration) can be made by the principles of hydrostatics with due allowance for the effects of accelerating (decelerating) forces.

3.16. LIQUID IN A CONTAINER SUBJECTED TO UNIFORM ACCELERATION IN THE HORIZONTAL DIRECTION

Consider that a tank filled with liquid is being accelerated horizontally to the right with uniform acceleration a_x. (The fuel tank on an airplane during take off is a good example). After initial sloshing of the liquid particles, the liquid motion stabilizes and the liquid moves as a solid mass under the action of acceleration force. The liquid redistributes itself in the tank and slopes as shown in Fig. 3.81; the slope is upwards in the direction opposite to that of horizontal acceleration.

An equation for the free liquid surface can be written by considering the equilibrium of a fluid particle *A* lying on the free surface. Since there is no relative motion between fluid particles, no shear stress exists and consequently the pressure force *P* exerted by the surrounding fluid on particle *A* is normal to the free surface. The particle *A* is then subjected to normal pressure force *P*, its weight *mg* acting vertically downwards and the accelerating force $m\,a_x$ acting in horizontal direction,

Resolving horizontally ; $P \sin \theta = m\,a_x$ ...(*i*)

Resolving vertically ; $P \cos \theta = mg$ ...(*ii*)

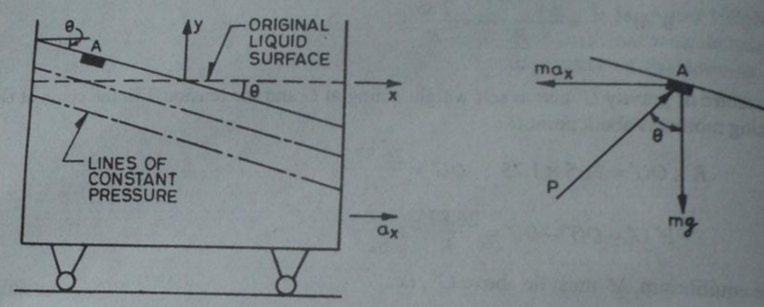

Fig. 3.81. *Liquid under constant linear acceleration in horizontal direction*

Dividing (*i*) by (*ii*) :

$$\tan \theta = \frac{a_x}{g} \qquad ...(3.35)$$

The term $\frac{a_x}{g}$ is constant at all points on the free liquid surface, hence $\tan\theta$ is constant and consequently the free surface is a straight plane inclined down at θ along the direction of acceleration (Fig. 3.81).

Considering the equilibrium of a fluid element at depth *h* from the free surface;

$$p\,dA = p_a\,dA + wh\,dA$$

where p_a is the atmospheric pressure and *dA* is the cross-sectional area of an elementary prism

$$p = p_a + wh\,;\; p = wh \text{ (gauge)} \qquad ...(3.36)$$

Thus pressure at any point in a liquid subjected to constant horizontal acceleration equals the head above that point. Consequently the lines of constant pressure will be parallel to the free liquid surface. The constant pressure lines and the variation in

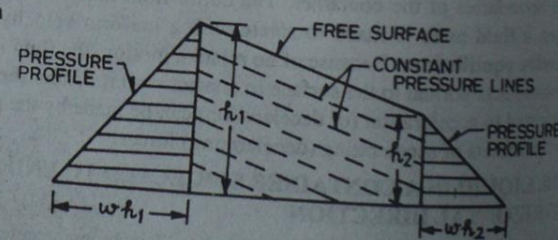

Fig. 3.82. *Pressure distribution for horizontally acceleration fluid*

liquid pressure on rear and front of the tank is depicted in Fig. 3.82. As the depth decreases in the direction of acceleration, the pressure along the bottom of the tank must also decrease.

For a tank completely filled with liquid and closed at the top, there is no preliminary adjustment in the surface elevation. The pressure builds up at the rear and is greater than that at the front. The slope of the constant pressure lines is still governed by the relation, $\tan\theta = \frac{a_x}{g}$.

It is good to remark that the shape of the container does not matter so long as the container provides a continuous connection in the liquid mass.

Example 3.91. A rectangular tank 6 m long, 2 m wide and 2 m deep contains water to a depth of 1 m. It is accelerated horizontally at 2.5 m/s² in the direction of its length. Determine

(*i*) slope of the free surface. (*ii*) maximum and minimum pressure intensities at bottom. (*iii*) total force due to water acting on each end of the tank.

Check the difference between these forces by calculating the inertia force of the accelerated mass.

Solution : (*i*) $a_x = 2.5\ \text{m/s}^2$; $g = 9.81\ \text{m/s}^2$

$$\tan\theta = \frac{a_x}{g} = \frac{2.5}{9.81} = 0.2548 \ ; \ \theta = 14° - 18'$$

Hence the free surface slopes at **14° – 18′** to the horizontal

(*ii*) Depth of liquid at front or shallow end,

$$h_{cd} = 1 - 3\tan\theta = 1 - 3 \times 0.2548 = 0.2356\ \text{m}$$

Depth of liquid at rear or deep end,

$$h_{ab} = 1 + 3\tan\theta = 1 + 3 \times 0.2548$$
$$= 1.7644\ \text{m}$$

Pressure intensity at point *C* (top of leading face)

$$p_c = 0$$

Pressure intensity at point *D* (bottom of leading face),

$$p_d = w \times h_{cd}$$
$$= 9810 \times 0.2356 = 2311\ \text{N/m}^2$$

Likewise the pressure intensities at the top (point *A*) and bottom (point *B*) of the trailing face are

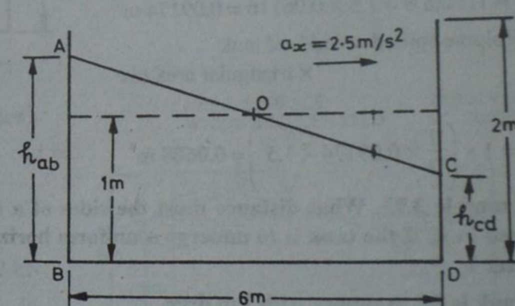

Fig. 3.83.

$$p_a = 0$$

$$p_b = 9810 \times 1.7644 = 17308\ \text{N/m}^2$$

(*iii*) Force on the leading face,

$$F_{cd} = \text{average pressure intensity} \times \text{area}$$

$$= \frac{p_c + p_d}{2} \times (h_{cd} \times \text{tank width}) = \frac{0 + 2311}{2} \times (0.2356 \times 2) = \mathbf{544.2\ N}$$

Like-wise force on the trailing face,

$$F_{ab} = \frac{0 + 17308}{2} \times (1.7644 \times 2) = \mathbf{30538\ N}$$

Resultant force, $F_{ab} - F_{cd}$

$$= 30538 - 544.2 = \mathbf{29993.8\ N}$$

An alternate solution is :

$$\text{Force} = \text{mass of water} \times \text{linear acceleration}$$

$$= (6 \times 2 \times 1 \times 1000) \times 2.5 = 30000 \text{ N}$$

The calculations, made for the resultant force, by the two methods are in close agreement.

Example 3.92. An open rectangular tank 1.5 m × 1 m × 1.2 m high is completely filled with water when at rest. Determine the volume spilled after the tank acquired a linear uniform acceleration of 0.6 m/s² in the horizontal direction.

Solution : The slope of the free surface *ac* after acceleration is given by :

$$\tan\theta = \frac{a_x}{g} = \frac{0.6}{9.81} = 0.06116$$

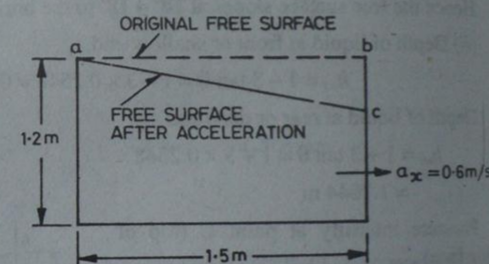

Fig. 3.84.

Water will be contained in the tank upto a height of 1.2 m on the rear edge and spilling will take place thereafter.

Drop in water level at the leading edge

$$= 1.5 \tan\theta = 1.5 \times 0.06116 = 0.09174 \text{ m}$$

Volume spilled = width of tank × triangular area *abc*

$$= 1 \times \left(\frac{1}{2} \times 0.09174 \times 1.5\right) = \mathbf{0.0688\ m^3}$$

Example 3.93. What distance must the sides of a tank be carried above the surface of water contained in it, if the tank is to undergo a uniform horizontal acceleration of 3 m/s² without spilling any water ?

Tank is 2.5 m square with 1 m deep.

Solution : The slope of the free surface of water after acceleration is given by :

$$\tan\theta = \frac{a_x}{g} = \frac{3}{9.81} = 0.3058$$

$$\text{Rise/fall in level of water} = \frac{2.5}{2} \times \tan\theta = \frac{2.5}{2} \times 0.3058 = 0.3822 \text{ m}$$

Therefore, the sides of the tank be carried 0.3822 m above the water surface when at rest. Evidently the tank must be at lead (1 + 0.3822) = 1.3822 m deep.

Example 3.94. An open tank 10 m long × 4 m wide × 2 m deep is filled with water upto a depth of 1.5 m. The tank is uniformly accelerated from rest to 12.5 m/s. Find the shortest time in which the tank may be accelerated without the water spilling over the edge.

Solution : Refer Fig. 3.85. The line *ab* denotes the original water level and *a′ b′* represents the water surface when the water is about to spill.

$$\tan\theta = \frac{bb'}{ob} = \frac{0.5}{5} = 0.1$$

Also, $\tan\theta = \frac{a_x}{g}$,

$$a_x = g\tan\theta = 9.81 \times 0.1 = 0.981\ \text{m/s}^2$$

Further $v = u + at$

$$\therefore \quad t = \frac{v-u}{a} = \frac{12.5-0}{0.981} = \mathbf{12.74\ s}$$

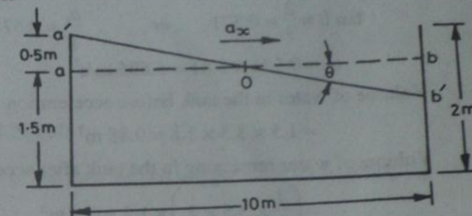

Fig. 3.85.

Example 3.95. An open rectangular tank 3.5 m long × 2 m deep × 1.8 m wide contains water to a depth of 1.5 m. Find the horizontal acceleration which may be imparted to the tank along its longer side so that :

(*a*) there is just no spilling of water from the tank, (*b*) the front bottom corner of the tank is just exposed, (*c*) the bottom of the tank is exposed upto its mid-point.

Proceed to calculate the total forces on each end of tank.

Solution : When there is just no spilling of water from the tank, the water can rise upto the rear top corner *D* of the tank.

$$\tan\theta = \frac{0.75}{1.75} = 0.2875$$

or $\frac{a_x}{g} = 0.2875$

$$\therefore \quad a_x = 0.2875 \times 9.81 = \mathbf{2.803\ m/s^2}$$

Pressure force acting on rear (trailing) end of the tank

$$F_1 = w A y_c$$

$$= 9.81 \times (2 \times 1.8) \times \left(\frac{2}{2}\right) = 35.32\ \text{kN}$$

Pressure force acting on leading end of the tank,

$$F_2 = 9.81 \times (1 \times 1.8) \times \left(\frac{1}{2}\right) = 8.83\ \text{kN}$$

Net force $F = F_1 - F_2 = 35.32 - 8.83$
$= \mathbf{26.49\ kN}$

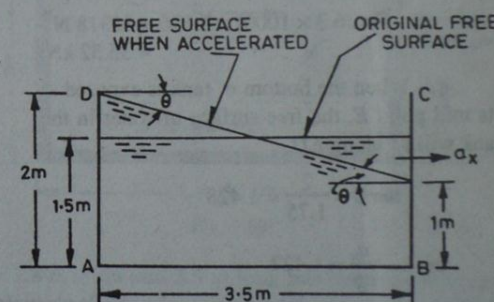

Fig. 3.86.

The resultant force equals the force necessary to accelerate the mass of water in tank

F = mass of water × linear acceleration

$= (3.5 \times 1.5 \times 1.8 \times 1000) \times 2.803 = 26488 \text{ N} \approx 26.49 \text{ kN}$

(*b*) For an open tank maximum level of water at the trailing end can be upto the top edge D of the tank. Accordingly when accelerated, the free surface of water in the tank will be along BD

$$\tan\theta = \frac{2}{3} = 0.571 \quad \text{or} \quad \frac{a_x}{g} = 0.571$$

$$\therefore \quad a_x = 0.571 \times 9.81 = \mathbf{5.606 \ m/s^2}$$

Volume of water in the tank before acceleration

$$= 1.5 \times 3.5 \times 1.8 = 9.45 \text{ m}^3$$

Volume of water remaining in the tank after acceleration

$$= \left(\frac{1}{2} \times 3.5 \times 2\right) \times 1.8 = 6.3 \text{ m}^3$$

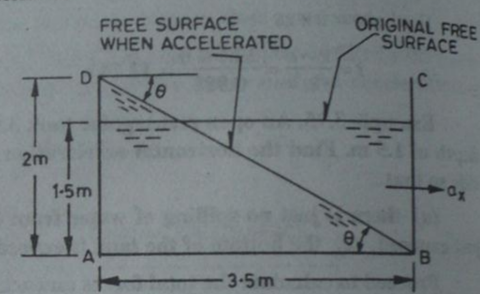

Fig. 3.87.

∴ Volume of water spilled out from the tank

$$= 9.45 - 6.3 = \mathbf{3.05 \ m^3}$$

Force acting on trailing end AD of the tank,

$$F_1 = w A y_c$$

$$= 9.81 \times (1.8 \times 2) \times \left(\frac{2}{2}\right) = 35.32 \text{ kN}$$

There is no water against the leading face BC of the tank and so the pressure force acting on this end is zero.

$$F_2 = 0$$

$$\therefore \quad \text{Net force } F = 35.32 - 0 = \mathbf{35.32 \ kN}$$

This net force equals the force necessary to accelerate the mass of water remaining in the tank.

$$F = (6.3 \times 1000) \times 5.606 = 35318 \text{ N} \approx 35.32 \text{ kN}$$

(*c*) When the bottom of tank is exposed at its mid point E, the free surface of water in the tank will be along ED.

$$\tan\theta = \frac{2}{1.75} = 1.428$$

$$\text{or} \quad \frac{a_x}{g} = 1.427$$

$$\therefore \quad a_x = 1.427 \times 9.81 = \mathbf{11.211 \ m/s^2}$$

Volume of water in the tank before acceleration,

$$= 1.5 \times 3.5 \times 1.8 = 9.45 \text{ m}^3$$

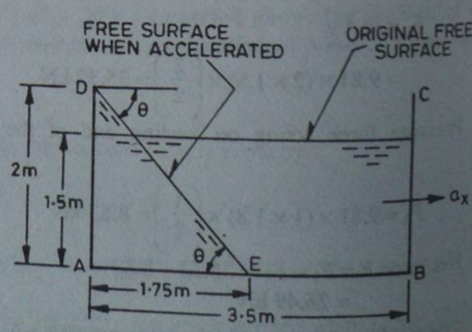

Fig. 3.88.

Volume of water remaining in the tank after acceleration

$$= \left(\frac{1}{2} \times 2 \times 1.75\right) \times 1.8 = 3.15 \text{ m}^3$$

$\therefore$ Volume of water spilled out from the tank

$$= 9.45 - 3.15 = \mathbf{6.30 \text{ m}^3}$$

Force acting on trailing end *AD* of the tank,

$$F_1 = w A y_c$$

$$= 9.81 \times (1.8 \times 2) \times \left(\frac{2}{2}\right) = 35.32 \text{ kN}$$

There is no water against the leading face *BC* of the tank and so the pressure force acting on this end is zero.

$$F_2 = 0$$

$\therefore$ Net force $F = 35.32 - 0 = \mathbf{35.32 \text{ kN}}$

This net force equals the force necessary to accelerate the mass of water remaining in the tank.

$$F = (3.15 \times 1000) \times 11.211 = 35315 \text{ N} \approx 35.32 \text{ kN}$$

Example 3.96. Fig. 3.89 shows an upright open U-tube accelerometer with vertical legs 30 cm apart. The cross-sectional area of the U tube is uniform throughout and it is partially filled with a liquid of specific gravity 4.5. The meter is installed in a car moving horizontally to the right at a constant uniform acceleration of a_x m/s². If the difference between the liquid surfaces in the two legs of the meter is 7.5 cm, calculate the acceleration of the automobile unit.

Solution : The free surface of the liquid in the two limbs is governed by the relation :

$$\tan \theta = \frac{a_x}{g} \; ; \; \frac{h_1 - h_2}{L} = \frac{a_x}{g}$$

Given :

$h_1 - h_2 = 7.5$ cm , $L = 30$ cm , $g = 981$ cm/s²

$$\therefore \quad \frac{7.5}{30} = \frac{a_x}{981}$$

or $$a_x = \frac{981 \times 7.5}{30} = \mathbf{245.25 \text{ cm/s}^2}$$

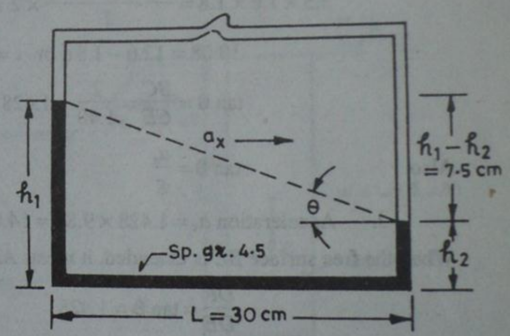

Fig. 3.89.

Example 3.97. A closed oil tanker 2 m deep × 1.8 m wide and 3.5 m long has been filled with an oil of specific gravity 0.8 upto a depth of 1.6 m. Calculate the acceleration which may be imparted to the tank in the direction of its length so that bottom front end of the tank is just exposed. Also calculate the net horizontal force acting on the tanker sides and show that this equals the force necessary to accelerate the liquid mass in the tanker. For water specific weight = 9807 N/m³.

Solution : It is to be noted that

(*i*) the tank is closed and as such the liquid cannot spill from it under any acceleration imparted to it.

The quantity of oil inside the tanker remains the same.

(ii) the oil surface which was initially horizontal (indicated by PQ) assumes the profile BED when the front bottom end B is just exposed :

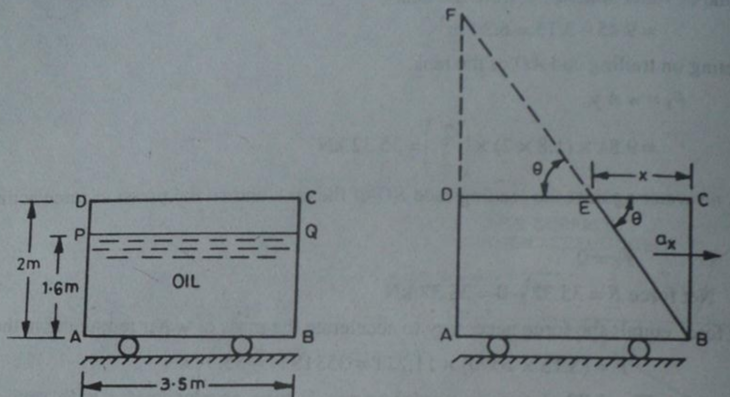

Fig. 3.90.

Equating volumes of oil before and after the motion,

Volume of rectangle $ABQP$ = volume of trapezium $ABED$

$$3.5 \times 1.6 \times 1.8 = \frac{3.5 + (3.5 - x)}{2} \times 2 \times 1.8$$

$$10.08 = 12.6 - 1.8x \text{ or } x = 1.40 \text{ m}$$

$$\tan\theta = \frac{BC}{CE} = \frac{2}{1.40} = 1.428$$

Also
$$\tan\theta = \frac{a_x}{g}$$

∴ Acceleration $a_x = 1.428 \times 9.81 = 14.01 \text{ m/s}^2$

When the free surface BE is extended, it meets AD produced at F.

$$\frac{DF}{DE} = \tan\theta = 1.428$$

$$\therefore \quad DF = (3.5 - 1.4) \times 1.428 = 3 \text{ m}$$

This represents an imaginary column of oil above point D.

Pressure intensity at point $D = w \times FD = 0.8 \times 9.81 \times 3 = 23.54 \text{ kN/m}^2$

Pressure intensity at point $A = w \times FA = 0.8 \times 9.81 \times (3 + 2) = 39.24 \text{ N/m}^2$

∴ Force on the trailing face AD

$$F_{ad} = \text{average pressure intensity} \times \text{area}$$

$$= \frac{23.54 + 39.24}{2} \times (2 \times 1.8) = 113 \text{ kN}$$

Alternately :

$$F_{ad} = w\,A\,y_c$$

$$= 0.8 \times 9.81 \times (2 \times 1.8) \times \left(3 + \frac{2}{2}\right) = 113 \text{ kN}$$

Since there is no oil against the end BC of the tank, the force acting on this face is zero.

$$F_{bc} = 0$$

$\therefore$ Net pressure force $= F_{ad} - F_{bc} = $ **113 kN**

The force needed to accelerate the liquid mass in the tank is

$$F = \text{mass of oil} \times \text{uniform linear acceleration}$$

$$= 0.8 \times 1000 \times 3.5 \times 1.6 \times 1.8) \times 14.01 = 112977 \approx 113 \text{ kN}$$

Obviously the difference between the forces on the two ends of the tank is equal to the force necessary to accelerate the liquid mass in the tank.

3.17. LIQUID IN A CONTAINER SUBJECTED TO UNIFORM ACCELERATION IN THE VERTICAL DIRECTION

Consider a tank containing liquid and moving vertically upwards with uniform acceleration a_y. (The water tank carried aboard a space capsule which is just leaving the launching pad is a good example of fluid mass subjected to vertical acceleration). The liquid in the tank will have a free horizontal surface but the pressure intensity at any point in the liquid will be different from what it would be when in a state of absolute rest. An equation for pressure distribution can be established by considering the equilibrium of forces acting on imaginary elementary prism of height h and cross-sectional area dA. Applying Newton's second law of motion.

$$\Sigma F_y = m\,a_y$$

The force ΣF_y also equals

$= $ pressure force acting upwards
$\quad -$ weight of prism acting downwards

$= $ (pressure intensity $p \times$ area dA)
$\quad - (w \times$ volume of prism)

$$= p\,dA - w\,h\,dA$$

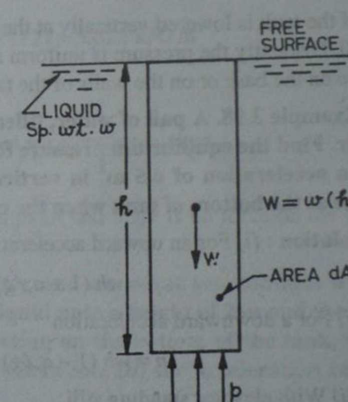

Fig. 3.91.

Also, mass $m = \dfrac{w}{g} \times$ volume of elementary prism

$$= \frac{w}{g} \times h dA$$

$$\therefore \quad pdA - wh\,dA = \frac{w}{g}(h\,dA)\,a_y$$

or

$$p = wh\left(1 + \frac{a_y}{g}\right) \qquad \text{...(3.37)}$$

Equation 3.37 reveals that *(i)* free liquid surface remains horizontal *(ii)* the pressure variation is linear in the vertical direction *(iii)* pressure intensity at any point is more than the static pressure wh by an amount

$wh\left(\frac{a_y}{g}\right)$ as shown in Fig. 3.92.

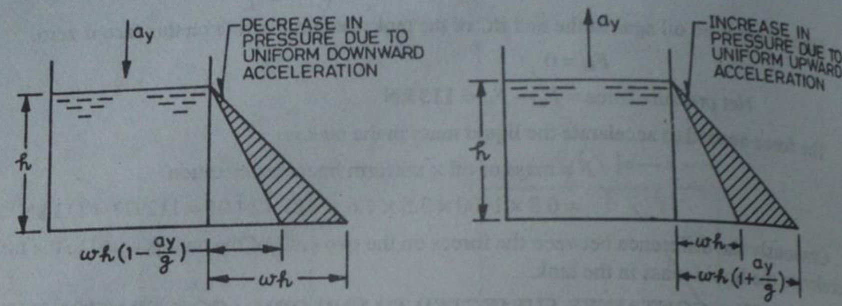

Fig. 3.92. *Pressure distribution for vertically accelerated fluids*

If the liquid mass is uniformly accelerated in vertically downward direction, a_y shall be –ve and then the equation 3.37 reduces to :

$$p = wh\left(1 - \frac{a_y}{g}\right) \quad ...(3.38)$$

i.e., pressure intensity at any point is less than the static pressure wh by an amount $wh\left(\frac{a_y}{g}\right)$.

If the tank is lowered vertically at the gravitational acceleration, then $a_y = g$ and equation 3.37 reduces to $p = 0$. Evidently the pressure is uniform and equivalent to surrounding atmospheric pressure, and there is no force on the base or on the walls of the tank.

Example 3.98. A pail of water, filled with water to a depth of 20 cm, is placed on the floor of an elevator. Find the equilibrium pressure force at the bottom of the pail when the elevator moves with a uniform acceleration of 4.5 m² in vertically upward and downward directions. What would be the pressure at the bottom of tank when the elevator is standing still ?

Solution : *(i)* For an upward acceleration

$$p = wh\,(1 + a_y/g) = 9810 \times 0.2\,(1 + 4.5/9.81) = \mathbf{2862\ N/m^2}$$

(ii) For a downward acceleration

$$p = wh\,(1 - a_y/g) = 9810 \times 0.2\,(1 - 4.5/9.81) = \mathbf{1062\ N/m^2}$$

(iii) With elevator standing still

$$p = wh = 9810 \times 0.2 = \mathbf{1962\ N/m^2}$$

Example 3.99. A vessel containing 15 cm height of water is moved vertically downwards with a constant acceleration a_y. If water starts boiling at about pressure of 0.07 bar at the ambient temperature, find the value of a_y such that water just begins to boil.

Solution : The water starts boiling at a pressure of :

0.07 bar absolute = (0.07 —1.0132) bar gauge

When the vessel is moved vertically downwards,

$$p\text{ (gauge)} = wh\left(1 - \frac{a_y}{g}\right)$$

$$(0.07 - 1.0132) \times 10^5 = 9810 \times 0.15\left(1 - \frac{a_y}{g}\right)$$

or
$$-64.098 = \left(1 - \frac{a_y}{9.81}\right)$$

$\therefore$
$$a_y = (64.098 + 1) \times 9.81 = 638.61 \text{ m/s}^2$$

Example 3.100. A open cubical tank with each side 1.5 m contains oil of specific weight 7.5 kN/m³ upto a depth of 1.5 m. Find the force acting on side of the tank when it is being moved with an acceleration of g/2 m/s² in vertically upward and downward directions. What would be the pressure at the bottom of the tank when the acceleration rate is g m/s² vertically downwards ?

Solution : *(a)* When acceleration a_y is vertically upward

$$p = wh\left(1 + \frac{a_y}{g}\right) = 7.5 \times 1.5\left(1 + \frac{1}{2}\right) = 16.875 \text{ kN/m}^2$$

On a vertical side, the pressure intensity varies linearly from zero at the top to 16.875 kN/m² at the bottom.

Force on the side = average pressure intensity × area

$$= \frac{16.875}{2} \times (1.5 \times 1.5) = \mathbf{18.984 \text{ kN}}$$

(a) When acceleration a_y is vertically downward

$$p = wh\,(1 - a_y/g) = 7.5 \times 1.5\,(1 - 1/2) = 5.625 \text{ kN/m}^2$$

$$\text{Force on the side} = \frac{5.625}{2} \times (1.5 \times 1.5) = \mathbf{6.328 \text{ kN}}$$

(c) When the tank is lowered vertically at the gravitational acceleration g, then

$$p = wh\,(1 - g/g) = 0$$

i.e., the liquid remains at the atmospheric pressure throughout and there is no force on the base or on the walls of the tank.

Example 3.101. A cylindrical tank 2.5 m in diameter and closed at top contains a liquid of specific gravity 0.75. The tank is 2.5 m high and contains liquid upto a height of 2 m and the air above it is under a pressure of –3 kPa. Calculate *(i)* the force acting on the bottom of the tank, when it is accelerated vertically upward at half the gravitational acceleration, *(ii)* the acceleration required to have zero absolute pressure at the bottom.

Take atmospheric pressure as 100 kN/m².

Solution : The oil head equivalent to given negative pressure above the oil surface equals

$$h = \frac{-3 \times 10^3}{9810 \times 0.75} = 0.408 \text{ m of oil} \qquad (p = wh)$$

This negative pressure will reduce the oil level to (2 – 0.408) = 1.592 m. Subsequent calculations are thus to be made corresponding to oil level of 1.592 m.

(*i*) For the upward acceleration

$$p = wh\,(1 + a_y/g)$$

$$= 9810 \times 0.75 \times 1.592 \left(1 + \frac{g/2}{g}\right)$$

$$= 17564 \text{ N/m}^2$$

$$= \mathbf{17.564 \text{ kN/m}^2} \text{ (gauge)}$$

(*ii*) When there is zero absolute pressure at the bottom,

$$p_{abs} = p_{at} + p_g$$

$$0 = 100 \times 10^3 + 9810 \times 0.75 \times 1.592\,(1 + a_y/9.81)$$

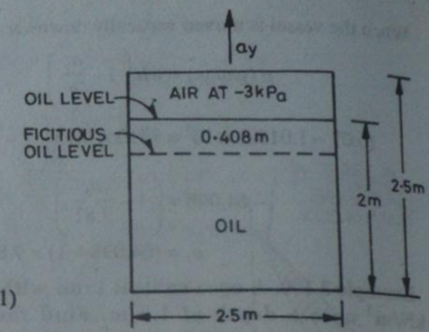

Fig. 3.93.

Solution gives : $a_y = \mathbf{-93.56 \text{ m/s}^2}$

The negative sign indicates that the container should be moved vertically ***downward*** with an acceleration of 93.56 m/s² so as to have zero absolute pressure at the bottom.

3.18. LIQUID IN A CONTAINER SUBJECTED TO UNIFORM ACCELERATION ALONG INCLINED PLANE

Consider that a tank filled with a liquid is being accelerated up an inclined plane with uniform acceleration *a*. If α is the inclination of the plane with horizontal then horizontal and vertical components of acceleration are

$$a_x = a \cos \alpha \quad \text{and} \quad a_y = a \sin \alpha$$

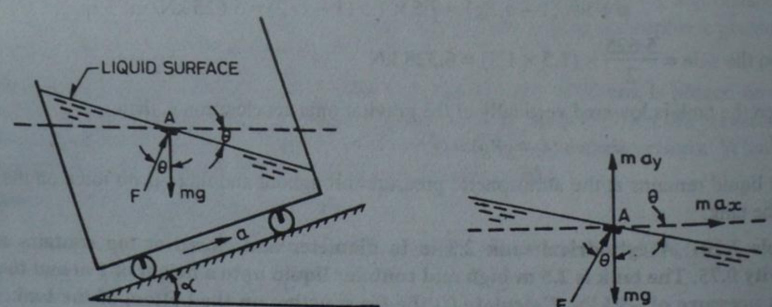

Fig. 3.94. *Acceleration of fluid mass along an upward slope*

After initial sloshing of the liquid particles, the liquid redistributes itself and the liquid surface makes an angle θ with the horizontal. The slope is upwards in the direction opposite to that of acceleration, *i.e.*, the liquid surface falls on the leading face and rises up on the trailing face. *A* fluid particle *A* of mass *m* lying on the liquid surface is in equilibrium under the action of following forces :

(*a*) weight *mg* acting vertically downward,

(b) pressure force F acting normal to the surface of the fluid element,

(c) accelerating force, $(m\,a)$ having a component $m\,a_x$ in the horizontal direction and a component $m\,a_y$ in the vertical direction.

Resolving horizontally : $F \sin\theta = m\,a_x$ ...(i)

Resolving vertically : $F \cos\theta = m\,a_y + mg$ ...(ii)

Dividing (i) by (ii) : $\tan\theta = \dfrac{a_x}{a_y + g}$ ...(3.39)

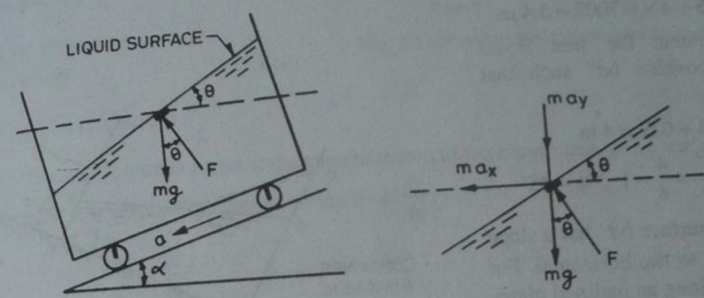

Fig. 3.95. *Acceleration of fluid mass along a downward slope*

If the fluid mass is subjected to acceleration down the slope, the above expression will become :

$$F \sin\theta = m\,a_x$$

$$F \cos\theta = mg - m\,a_y$$

$$\tan\theta = \frac{a_x}{g - a_y} \quad ...(3.40)$$

Example 3.102. An open rectangular tank 5 m long × 2 m wide is filled with water to a depth of 1.5 m. Find the slope of water surface when the tank moves with an acceleration of 3 m/s² *(a)* up a 30° inclined plane *(b)* down a 30° inclined plane.

Solution : Horizontal and vertical components of acceleration are :

$$a_x = a \cos\alpha = 3 \cos 30^\circ = 2.6 \text{ m/s}$$

$$a_y = a \sin\alpha = 3 \sin 30^\circ = 1.5 \text{ m/s}$$

Let θ be the slope of the free liquid surface

(a) When the tank moves with acceleration up the inclined plane

$$\tan\theta = \frac{a_x}{a_y + g} = \frac{2.6}{1.5 + 9.81} = 0.229$$

$\therefore \quad \theta = 12.89^\circ$

(b) When the tank moves with acceleration down the inclined plane

$$\tan\theta = \frac{a_x}{g - a_y} = \frac{2.6}{9.81 - 1.5} = 0.3128$$

$\therefore \quad \theta = 17.37^\circ$

Example 3.103. A rectangular container of cross-sectional area 4 m × 1.5 m rests on an inclined plane with liquid (bc = 0.6 m) as shown in Fig. 3.96. You are required to determine the acceleration 'a' along the plane so that the free surface at the edge touches the point b. It may be assumed that the sides of container are high enough so that there is no spilling of liquid.

Solution : When the container is at rest,

$$ad = bc + ab \tan 35^\circ$$
$$= 0.6 + 4 \times 0.7002 \approx 3.4 \text{ m}$$

When accelerated, the free liquid surface takes the position bd' such that $dd' = bc$

$\therefore \quad ad' = 3.4 + 0.6 = 4$ m

$$\tan\varphi = \frac{ad'}{ab} = \frac{4}{4} = 1 \;;\; \varphi = 45^\circ$$

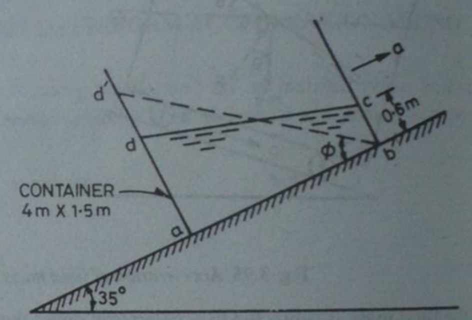

Fig. 3.96.

Hence the free surface bd' has a slope of $\theta = (45 - 35) = 10^\circ$ to the horizontal. For upward acceleration along an inclined plane, the inclination of the free surface with horizontal is given by $\tan\theta = \dfrac{a_x}{a_y + g}$ where a_x and a_y are the horizontal and vertical components of the acceleration. The acceleration a can be resolved into horizontal and vertical components :

$$a_x = a \cos 35^\circ = 0.8192\, a$$
$$a_y = a \sin 35^\circ = 0.5736\, a$$

$$\therefore \quad \tan 10^\circ = \frac{0.8192\, a}{0.5736\, a + 9.81}$$

$$0.176 = \frac{0.8192\, a}{0.5736\, a + 9.81} \;;\; \mathbf{a = 2.403\ m/s^2}$$

3.19. LIQUID IN A CONTAINER SUBJECTED TO CONSTANT ROTATION

Consider an open cylindrical reservoir partly filled with a liquid and being spun about its vertical axis with a constant angular velocity ω. Eventually the liquid in the reservoir rotates with the same angular velocity and hence as a single solid mass. Once the liquid adjusts itself to steady conditions, its free surface which was originally horizontal becomes concave or dished in form; it gets dispersed at the centre and rises near the walls. Reference Fig. 3.97, AA depicts the liquid surface before rotation and surface $A'A'$ shows how it appears a long period of time after a steady state has reached.

We focus our attention on a fluid element on the surface of fluid at a distance x from the axis of rotation. The particle will be in equilibrium under the action of its own weight W, the inertia force $R = \frac{W}{g}\omega^2 x$ which acts radially from the axis of rotation, and a resultant force F due to action of surrounding liquid particles which acts normal to the surface of the fluid element under consideration.

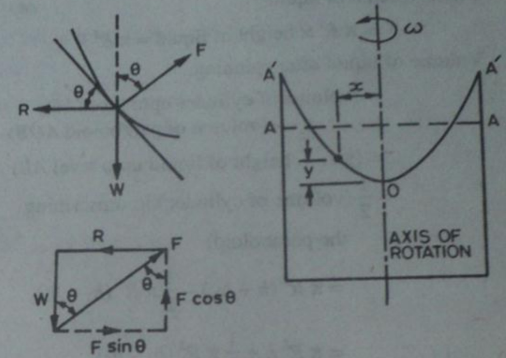

Fig. 3.97. *Rotating fluid in an open cylindrical reservoir*

Resolving horizontally :

$$F \sin\theta = \frac{W}{g}\omega^2 x \qquad ...(i)$$

Resolving vertically :

$$F \cos\theta = W \qquad ...(ii)$$

Dividing *(i)* by *(ii)*, $\tan\theta = \frac{\omega^2 x}{g}$ ...(3.41)

Now θ is the angle between x-axis and a tangent drawn to the curve (free water surface) at the location of the fluid element. The slope of this tangent is $\tan\theta$ or $\frac{dy}{dx}$.

Thus $$\frac{dy}{dx} = \frac{\omega^2 x}{g}$$

Integrating : $$y = \int_0^x \frac{\omega^2 x}{g}\, dx = \frac{\omega^2 x^2}{2g} + C$$

If the lowest point (vertex O) of the profile is taken as origin, then

$y = 0$ when $x = 0$, That gives $C = 0$

$$\therefore \qquad y = \frac{\omega^2 x^2}{2g} \qquad ...(3.42)$$

i.e., the shape of the surface of a free liquid in a rotating reservoir is that of a paraboloid of revolution. The concavity of the surface is independent of the liquid being spun and depends only on the angular speed of rotation ω and the distance x from the axis of rotation.

Example 3.104. Derive an expression for the depth of paraboloid formed by the surface of liquid contained in a cylindrical tank which is rotated at a constant angular velocity ω about its axis. Further proceed to show that the fall of liquid level at the axis of rotation is equal to the rise of liquid level at the ends.

Solution : Reter Fig. 3.98.

Let $a - a$ represent the initial level of liquid in a cylindrical container of radius R. When spun with constant angular velocity ω, the free liquid surface takes the form AOB which is paraboloid of revolution. Let the level of liquid at the axis of rotation drop by h_1 and that at walls rise by h_2. We are to show that $h_1 = h_2$.

Initial volume of liquid

$$= \pi R^2 \times \text{height of liquid} = \pi R^2 h$$

Volume of liquid after spinning,

= (volume of cylinder upto level AB)
  − (volume of paraboloid AOB)

= ($\pi R^2 \times$ height of liquid upto level AB)
 $-\frac{1}{2}$ (volume of cylinder circumscribing the paraboloid)

$$= \pi R^2 (h + h_2) - \frac{1}{2} \pi R^2 (h_1 + h_2)$$

$$= \pi R^2 h + \frac{1}{2} \pi R^2 (h_1 - h_2)$$

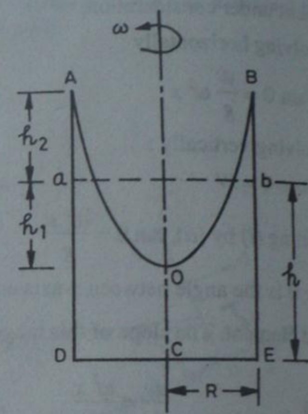

Fig. 3.98.

Equating the initial and final liquid values,

$$\pi R^2 h = \pi R^2 h + \frac{1}{2} \pi R^2 (h_1 - h_2)$$

or $\quad h_1 = h_2$

i.e., fall of liquid at centre = rise of liquid at the walls.

Thus the rise of liquid along the walls of the cylinder above the initial level is equal to depression of liquid at the axis of rotation.

Example 3.105. A cylindrical container with height 75 cm and radius 15 cm is filled with water to a depth of 60 cm. The cylinder is open at the top and is rotated uniformly about its axis. Calculate the height of paraboloid formed at the free surface of water.

(*b*) How fast should be container be rotated so that the free water surface reaches the upper rim of the container, *i.e.*, it is just on the point of spilling ? Also find the pressure on the side walls and bottom of the container.

Solution : Radius of cylinder $r = \frac{15}{2} = 7.5 \text{ cm} = 0.075 \text{ m}$

Angular velocity $\omega = \frac{2 \pi N}{60} = \frac{2 \pi \times 250}{60} = 26.17 \text{ rad/s}$

Equation to paraboloid gives :

$$y = \frac{\omega^2 x^2}{2g} \; ; \text{ where } x = r$$

$$= \frac{(26.17)^2 \times (0.075)^2}{2 \times 9.81} = 0.196 \text{ m} = \mathbf{19.6 \text{ cm}}$$

(*a*) When the water is just to spill over, its surface just touches the upper rim of the container at the walls.

Rise of water level at ends = 75 – 60 = 15 cm

Also the rise in the level at ends equals the fall in level at the centre.

Depression of water level at centre = 15 cm

∴ Height of paraboloid = 15 + 15 = 30 cm = 0.3 m

Equation to the paraboloid gives

$$0.3 = \frac{\omega^2 (0.075)^2}{2g}$$

$$\omega = \sqrt{\frac{2 \times 9.81 \times 0.3}{(0.075)^2}} = 32.35 \text{ rad/s}$$

Rotational speed $N = \dfrac{60\,\omega}{2\,\pi}$

$$= \frac{60 \times 32.35}{2\,\pi} = \mathbf{309.07\ rpm}$$

Total pressure on the bottom equals the total weight of water in the tank.

$$= w \times (\pi\, r^2\, l)$$

$$= 9810 \times (\pi \times 0.075^2 \times 0.6) = \mathbf{103.96\ N}$$

Total pressure on the side,

$$= w\, A\, y_c = w\,(2\,\pi\, rl) \times \frac{l}{2}$$

$$= 9810 \times (2\,\pi \times 0.075 \times 0.75) \times \frac{0.75}{2} = \mathbf{1299.52\ N}$$

Example 3.106. An open cylindrical tank 0.9 m high and 0.6 m in diameter is filled $\frac{2}{3}$rd with water when at rest. It is spun about its vertical axis with angular velocity ω radians per second, and the free liquid surface in the tank assumes the shape of a paraboloid of revolution. Determine the speed of rotation when

(a) the water just starts spilling over the sides of the tank

(b) the point at the centre of the base is just exposed. What would then be the percentage of water left in the tank ?

(c) Also estimate the pressure intensities at the centre of the tank bottom and at the bottom of walls if ω = 9 radians/second.

Solution : For no spilling of water, the surface of the water should just touch the upper rim of the tank at *A* and *B*. The co-ordinates of point *A* with respect to vertex *O* are : $x = 0.3$ m and $y = 0.3 + 0.3 = 0.6$ m. Substitute these values in the equation of free liquid surface *AOB* which is a paraboloid of revolution.

$$y = \frac{\omega^2 x^2}{2g}\ ;\ 0.6 = \frac{\omega^2 \times (0.3)^2}{2 \times 9.81}$$

$$\therefore \omega = \left[\frac{2 \times 9.81 \times 0.6}{0.3}\right]^{1/2} = 11.46 \text{ radians/second}$$

Rotational speed $N = \dfrac{60\,\omega}{2\,\pi}$

$$= \frac{60 \times 11.46}{2\,\pi} = \mathbf{109.5\ rpm}$$

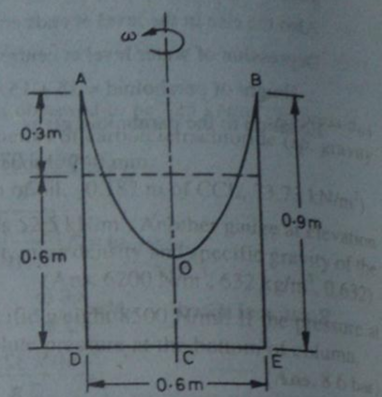

Fig. 3.99.

(*b*) When the point at the centre of the base is just exposed (*i.e.*, the axial depth becomes zero)

Depth of paraboloid

y = length (height) of the tank = 0.9 m

$$\therefore \quad 0.9 = \frac{\omega^2 \times (0.3)^2}{2g}\ ; \ \omega = 14.007 \text{ rad/s}$$

Rotational speed $N = \dfrac{60 \times 14.007}{2\,\pi} = \mathbf{133.82\ rpm}$

Original volume of water in the tank

$$= \pi \times (0.3)^2 \times 0.6 = 0.1696 \text{ m}^3$$

Amount of water thrown out = volume of paraboloid of revolution

$$= \frac{1}{2} \text{(volume of circumscribing cylinder)}$$

$$= \frac{1}{2}\,[\pi\,(0.3)^2 \times 0.9] = 0.1272 \text{ m}^3$$

Volume of water left in the tank

$$= 0.1696 - 0.1272 = 0.0424 \text{ m}^3$$

$$\% \text{ of water left} = \frac{0.0424}{0.1696} \times 100 = \mathbf{25\%}$$

(*c*) Depth of paraboloid $y = \dfrac{\omega^2 x^2}{2g}$

$$= \frac{9^2 \times (0.3)^2}{2 \times 9.81} = 0.3717 \text{ m}$$

At the axis, the water level drops by $\dfrac{y}{2}$ and at the walls it rises by $\dfrac{y}{2}$

$\therefore$ Height of water at the bottom centre of the tank, *i.e.*, at point *C*.

$$= 0.6 - \frac{0.3717}{2} = 0.4142 \text{ m}$$

Height of water at the bottom of walls, *i.e.*, at point *D*.

$$= 0.6 + \frac{0.3717}{2} = 0.7858$$

Pressure intensity at *C*, $= 9810 \times 0.4142 = \mathbf{4063.3\ N/m^2}$

Pressure intensity at $D = 9810 \times 0.7858 = \mathbf{7708.7\ N/m^2}$ $(p = wh)$

Example 3.107. An open cylindrical tank is spun at 105 rev/min with its axis vertical. The tank is 0.85 m high and 0.5 m diameter and is filled completely with water before spinning. Calculate the water left in the vessel when it has reached its full speed. Also calculate the slope of water surface at the point where it meets the rim of the tank.

Solution : $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 105}{60} = 11$ radians/second

Let y be the height of the paraboloid, *i.e.*, the depth of the vertex from the rim of the tank. Then :

$$y = \frac{\omega^2 x^2}{2g} = \frac{11^2 \times (0.25)^2}{2 \times 9.81} = 0.385 \text{ m}$$

Amount of water thrown out = volume of paraboloid of revolution

$$= \frac{1}{2} \text{(volume of circumscribing cylinder)}$$

$$= \frac{1}{2}[\pi \times (0.25)^2 \times 0.385] = 0.0378 \text{ m}^3$$

Original volume of water in the tank

$$= \pi \times (0.25)^2 \times 0.85 = 0.167 \text{ m}^3$$

$\therefore$ Volume of water left in the tank

$$= 0.167 - 0.0378 = \mathbf{0.1292 \text{ m}^3}$$

(*b*) Let θ be the angle with the horizontal which the tangent to the water surface makes at a point where it meets the rim of the tank. Then

$$\tan\theta = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\omega^2 x^2}{2g}\right)$$

$$= \frac{\omega^2 x^2}{g} = \frac{(11)^2 \times 0.25}{9.81} = 2.77 \qquad \therefore \ \theta = \mathbf{70.15^\circ}$$

REVIEW QUESTIONS

A : Pressure and its relationship with height

1. Comment upon the statements :
   (*i*) Without flow, pressure must be transmitted normal to solid boundaries and arbitrary surfaces in a fluid.
   (*ii*) Without flow, the pressure at a point is the same regardless of orientation of the area upon which it acts.
2. State and prove the Pascal's law and give some examples where this principle is applied.
3. What is meant by intensity of pressure ? How it varies with the depth of fluid ?
4. What is pressure head ? How pressure can be expressed in terms of height of a liquid column ?
5. State and prove the hydrostatic law and adapt it for incompressible as well as for compressible fluids.
6. Explain the absolute, gauge and vacuum pressures. For a water column of height 6 m, caculate the intensity of pressure in kPa and express the pressure in terms of cm of mercury, metre of water

absolute, kN/m^2 absolute and cm of mercury absolute. Take atmospheric pressure equivalent to 76 cm of mercury

(Ans. 58.86 kPa, 44.13 cm of Hg, 16.336 m of water absolute, 160.256 kN/m^2 absolute and 120.13 cm of mercury absolute)

7. At a point A in a fluid flow system, the pressure reading was observed to be -25 kN/m^2. Express this pressure in metres of water, metres of oil (sp. gravity 0.8), metres of carbon tetrachloride (sp. gravity 1.6) and absolute pressure in kN/m^2 if a mercury barometer records 740 mm.

(Ans. -2.548 m, -3.185 m of oil, -0.187 m of CCl_4, 73.73 kN/m^2)

8. A pressure guage at elevation 10 m on the side of a tank reads 52.5 kN/m^2. Another gauge at elevation 7.5 reads 68 kN/m^2. Make calculations for the specific weight, mass density and specific gravity of the fluid. **(Ans.** 6200 N/m^3, 632 kg/m^3, 0.632)

9. An oil refinery of column 60 m high contains a fluid of specific weight 8500 N/m^3. If the pressure at the top of column is 3.5 bar, make calculations for the absolute pressure at the bottom of column.

(Ans. 8.6 bar)

10. The barometric pressures at the sea level and at the top of mountain have been recorded to be 760 mm of mercury and 700 mm of mercury respectively. Workout this pressure difference in kPa and also calculate the height of mountain. Specific weight of air may be presumed as 11.85 N/m^3.

(Ans. 8.0 kPa, 675.5 m)

11. The specific weight of water in the ocean may be calculated by using the relation

$$w = w_0 + 8\,k\,\sqrt{h}$$

Set up an expression for the pressure at any point H metres below the surface and workout the pressure at a depth of 1.5 km. Given : height h is in metres, $k = 8.5 \times 10^{-5}$ and $w_0 = 10.3$ kN/m^3.

$$\left(\textbf{Ans. } p = w_0\,h + \frac{16}{3}\,kh^{3/2},\ 15.467 \text{ kPa}\right)$$

12. A closed tank contains 0.6 m of mercury, 1.5 of water, 2.5 m of oil of relative density 0.75 and empty space above is filled with air at a pressure of 150 kPa. What would be the reading of a pressure gauge installed at the bottom of tank ? **(Ans.** 263.16 kPa)

13. Given that the barometric pressure at ground level is p_0 when the temperature is 15°C, prove that the pressure p at height h is given by :

$$\log (p/p_0) = A \log (1 - Bh)$$

If the temperature of a quiescent atmosphere diminishes at a uniform rate of 0.0065 K/m and for air $p/\rho = 29.27\,T$, find the values of A and B.

14. The temperature lapse rate in the International standard atmosphere may be taken as 0.0065 K/m. Show that this corresponds to the law $\frac{p}{\rho^{1.235}}$ = constant.

Also determine the pressure on the top of mount Everest corresponding to an altitude of 8848 metres in this atmosphere. Given the atmospheric pressure at sea level as 760 mm of Hg, temperature 15°C and density 1.225 kg/m^3. For air $R = 287$ K/kg K.

15. Show that the ratios of pressures (p_2/p_1) and densities (ρ_2/ρ_1) for altitudes y_2 and y_1 in an isothermal atmosphere is given by

$$\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} = \exp\left[\frac{-g\,(y_2 - y_1)}{RT}\right]$$

At an elevation of 11000 m, the atmosphere has 20 kPa pressure and – 50° C temperature. Evaluate these parameters at an altitude of 17000 m if isothermal conditions are assumed to prevail on the given range of elevation. (**Ans.** – 50° C, 7.97 kPa)

16. The temperature of the earth's atmosphere drops about 5°C for every 1000 m of elevation above the earth's surface. If the air temperature at the ground level is 15°C and the pressure is 760 mm of Hg, at what elevation is the pressure 380 mm of H ?. Assume that air behaves as an ideal gas. (**Ans.** 5555 m)

B : Hydrostatic forces on submerged surfaces.

17. Define the term centre of pressure of the plane area immersed in a fluid. What relation has it got with the centre of gravity of the area ? Do the centre of pressure and centre of gravity ever coincide and if so under what conditions ?

18. Explain the term total pressure acting on a plane surface immersed in a fluid at any angle. Obtain an expression for this, and also for the corresponding depth of the centre of pressure.

19. Show that for a surface subjected to fluid pressure on one side, the depth of the centre of pressure is prescribed by the second moment divided by the first moment of the area of the surface. The moment is taken about the line of intersection of the plane surface and the free surface of the fluid.

20. Determine the force acting on one side of a 2.5 m high × 1.2 m wide concrete form used for pouring a basement wall. It may be presumed that freshly poured concrete exerts a hydrostatic force similar to a liquid of equal specific weight. The specific weight of concrete may be taken as 23.6 kN/m^3. (**Ans.** 88.5 kN)

21. A box of rectangular base 2.5 m × 4 m contains gasoline (specific gravity 0.8) upto a height of 6 m. Make calculations for the hydrostatic force on the base and on each vertical face. Also locate their lines of action. Take specific weight of water 10 kN/m^3 (**Ans.** 480 kN, centroid of the base, 360 kN and 576 kN, 4 m below the level)

22. A hollow circular plate of 2 m external and 1 m internal diameter is immersed vertically in water such that the centre of plate is 4 m deep from the water surface. Find the total pressure and the depth of centre of pressure. Take specific weight of water 10 kN/m^3. (**Ans.** 93.4 kN, 4.078 m)

23. A circular plate 3 metres diameter is submerged in water with its greatest and least depths below the surface being 2 metres and 1 metre respectively. Find *(i)* the total pressure on one face of the plate, and *(ii)* the position of the centre of pressure. (**Ans.** 104 kN, 1.542 m)

24. A triangular plate of base 3 m and height 3 m is immersed in water in such a way that plan of the plate makes an angle of 60 degree with the free surface. The base of the plate is parallel to water surface and at a depth of 2 m from water surface. Make calculations for the total pressure and depth of centre of pressure on the plate. (**Ans.** 126.5 kN, 3 m)

25. A vertical rectangular gate 4 m × 2 m (4 m side being vertical) is hinged at a point 10 cm below the centre of gravity of the gate. The depth of water is 6 m above the bottom of gate. What horizontal force must be applied at the bottom of the gate to keep it in vertical position ?

26. A rectangular tank 10 m × 5 m and 3 metres deep in divided by a partition wall parallel to the shorter wall of the tank. One of the compartments contains water to a depth of 3 metres and the other oil of specific gravity 0.75 to a depth of 2 metres. Find the resultant pressure on the partition. (**Ans.** 147.1 kN)

27. A hemispherical projection of diameter 1.5 m exists on one of the vertical sides of a tank (Fig. 3.100). If the tank contains water (sp. wt. 10 kN/m^3) to an elevation of 3.5 m above the centre of hemisphere, determine the horizontal and vertical forces acting on the projection.
(**Hint :** F_v = wt. of volume of water ABED – wt. of volume of water BCDE = wt. of water contained in the sphere ABC)
(**Ans.** $F_v = 707.4$ kN, $F_h = 247.27$ kN)

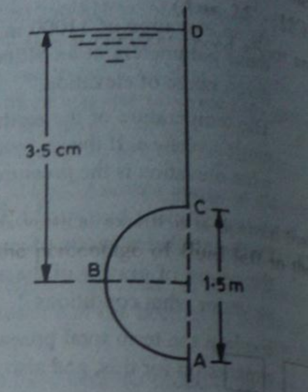

Fig. 3.100.

28. Make calculations for the horizontal and vertical components of total force acting on the outer surface of quadrant of a circular cylinder shown in Fig. 3.101. The surface has a width of 1 m in the direction perpendicular to the plane of the paper.
(**Ans.** $F_h = 37.15$ kN, $F_v = 27.97$ kN)

29. A circular gate in a vertical wall has a diameter of 4 metre. The water surface on the upstream side is 8 m above the top of the gate and on the downstream side is 1 m above the top of the gate. Find the forces acting on the two sides of the gate, the resultant force acting on the gate and the position at which it acts.

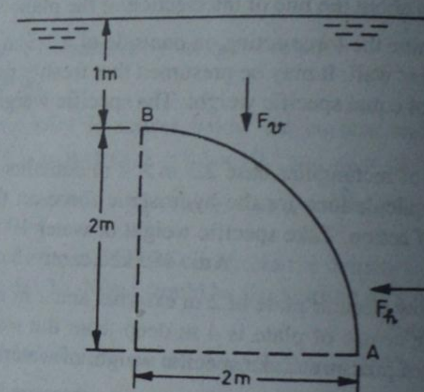

Fig. 3.101.

30. Each gate of a lock 6 m high and 2 m wide is supported on one side by two hinges, each located 60 cm from top and bottom respectively. When closed, the gates meet at an angle of 140°. The depths of water on two sides of the gates are 5 m and 1.5 m respectively. Determine
   (i) the magnitude of resultant pressure force due to water pressure on each gate,
   (ii) the magnitude of reaction between the gates, and
   (iii) the magnitude of force on each hinge.

It may be presumed that the gate reaction acts in the same horizontal plane as the resultant water pressure. Take specific weight of water 10 kN/m^3.
(**Ans.** 227.53 kN, 329 kN, $R_T = 106$ kN, $R_B = 223$ kN)

C : Buoyancy and stability of floating bodies

31. State and explain Archimede's principle and mention some of its practical applications.
   (b) State and explain the principle of floatation. How does it differ from the principle of buoyancy?

32. An object which has a volume of 0.2 m^3 requires a force of 250 N to keep it immersed in water (sp. wt. 10 kN/m^3). If a force of 150 N is required to keep it immersed in another liquid, calculate the specific gravity of the object and that of the liquid.
(**Ans.** 0.875, 0.95)

33. Show that if B is the centre of buoyancy and M is the metacentre of a partially immersed floating body then $BM = I/V$ where I is the second moment of area of the surface of floatation about the longitudinal axis and V is the immersed volume.

34. State and explain the conditins that should be fulfilled for the stable equilibrium of a body.

A wooden block (relative density 0.7) measures 2 m width × 1 m depth and has a length of 2.5 m. If it is placed horizontally on the surface of water (sp. wt. 9.81 kN/m^3), calculate its metacentric height and comment on its stability. **(Ans.** 0.32 m; stable**)**

35. Explain the terms centre of buoyancy, metacentre and metacentric height.

A cylindrical buoy 2 m in diameter, 2.5 m high, weighs 18 kN. Examine whether the buoy will or will not float with its axis vertical in sea water of specific gravity 1.025.

(Ans. metacentric height = – 0.527 m; unstable**)**

36. A circular cylinder 50 cm diameter and 25 cm long floats in mercury with its axis vertical, and depth of immersion is 15 cm. Find the meta-centric height. If water is poured over mercury till the whole of the cylinder is immersed (partly in mercury and partly in water), find its depth of immersion in the mercury.

37. A body has the cylindrical upper portion 2 m diameter and 1.2 m deep. The lower portion which is curved displaces a volume of 0.4 m^3 of water, and its centre of buoyancy is situated 1.3 m below the top of the cylinder. The centre of gravity of the whole unit is 80 cm below the top of the cylinder and the total displacement is 2.6 m^3. Find the metacentric height. **(Ans.** 0.182 m**)**

38. Derive an expression for the period of oscillation of rolling of ship and then discuss its stability and comfort characteristics.

A pontoon 6 m × 2.4 m × 1.2 m is a hollow rectangular box with walls 12.5 m thick. The pontoon is made of steel and weighs 76 kN/m^3. Calculate the period of oscillation about the longitudinal axis when floating in fresh water. You may neglect the secondary effects due to thickness. **(Ans.** 1.88 s**)**

D : Liquids in relative equilibrium

39. Explain why the fluid mass under constant linear acceleration or constant angular velocity is equivalent to a fluid mass at rest.

A rectangular tank containing water is imparted a uniform acceleration a_x along a straight horizontal path. The water goes through a complex motion at first but eventually orients itself into a fixed shape relative to the container. Show that $\tan\theta = \frac{a_x}{g}$ where θ is the inclincation of the free surface after uniform acceleration has been given to the tank. Further show that the pressure variation along a vertical line from the free surface is hydrostatic in nature.

40. An oil tanker 4.5 m long, 3 m wide and 1.5 m deep is half filled with oil. What should be the linear horizontal acceleration in the direction of tank movement so that after attainment of relative equilibrium, the depth at the forward edge is zero ? **(Ans.** 3.27 m/s^2**)**

41. A rectangular tank 2 m wide × 4 m long × 3 m deep contains water to a depth of 2 m. It is accelerated horizontally at 4 m/s^2 in the direction of its length. Calculate the depth of water and the total force on each end of the tank. Check the difference between these factors by calculating the inertia force of the accelerated mass.

42. An open tank 30 metres long and 2 metres deep is filled with 1.5 metres of oil of specific gravity 0.82. The tank is accelerated uniformly from rest to a speed of 20 metres per second. What is the shortest

time in which this speed may be attained without spilling any oil ? **(Ans.** 20.4 s)

43. An open container of liquid accelerates at 4.2 m/s^2 down a 30-degree inclined plane. Calculate the slope of free surface. **(Ans.** 25° – 15′)

44. Show that when the liquid contained in an open vessel is made to rotate at constant speed about an axis, the free surface approximates to a paraboloid of revolution.

An open cylindrical vessel 1 m in diameter and containing 750 litres of a liquid is rotated about its vertical axis. Find the smallest height the vessel should have so that it can be rotated at 100 rpm without spilling any liquid over the sides. **(Ans.** 1.65 m)

45. A cylindrical container of 0.3 m diameter and 0.6 m long is filled two-third with a fluid of specific gravity 0.8. Determine the speed of rotation of the cylinder about its vertical axis if it is desired that the point at the centre of the base is just exposed. Also compute the percentage of fluid left in the container.

□ □

time in which this speed may be attained without spilling any oil ? **(Ans. 20.4 s)**

43. An open container of liquid accelerates at 4.2 m/s^2 down a 30-degree inclined plane. Calculate the slope of free surface. **(Ans. 25° – 15′)**

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