CHAPTER 1. INELASTIC ANALYSIS OF CONTINUOUS BEAMS & MOMENT REDISTRIBUTION

1.1. INTRODUCTION

Reinforced concrete structures are generally analyzed by the conventional elastic theory. In flexural members, this is tantamount to assuming a linear moment-curvature relationship, even under factored loads. For under-reinforced sections, this assumption is approximately true, provided the reinforcing steel has not yielded at any section. Once yielding takes place(at any section), the behavior of a statically indeterminate structure enters an inelastic phase, and linear elastic structural analysis is strictly no longer valid.

For a proper determination of the distribution of bending moments for loading beyond the yielding stage at any section, inelastic analysis is called for. This is generally referred to as limit analysis, when applied to reinforced concrete framed structures, and ‘plastic analysis’ when applied to steel structures. In the special case of reinforced concrete slabs, the inelastic analysis usually employed is the ‘yield line analyses. The assumption generally made in limit analysis is that the moment-curvature relation is an idealized bilinear elasto-plastic relation [Figure 1-1]. This has validity only if the section is adequately under-reinforced and the reinforcing steel has a well-defined yield plateau. The ultimate moment of resistance (M_{uR}) of such sections, with specified area of steel, can be easily assessed.

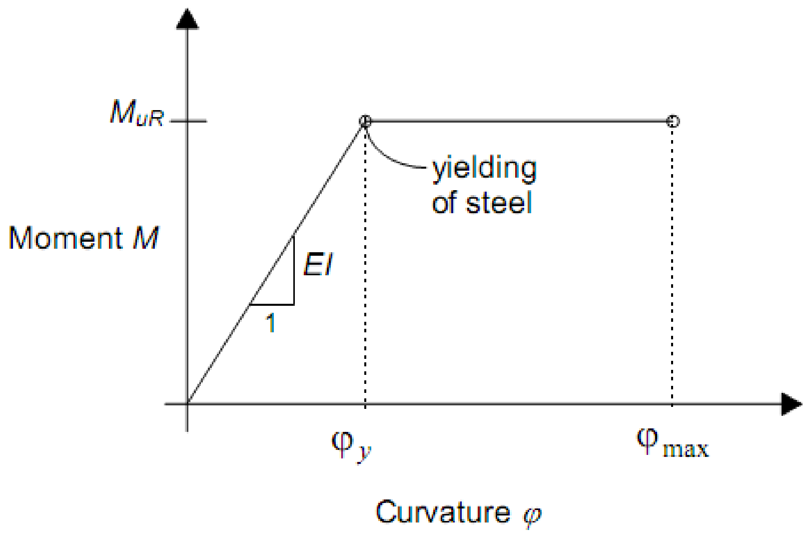


Figure 1-1: Idealized moment-curvature relation

1.2. METHODS OF ANALYSIS ALLOWED IN EBCS EN 2004:2014 (EUROCODE 2)

The methods of analysis provided in EC-2 are for the purpose to establish the distribution of either internal forces and moments, or stresses, strains and displacements, over the whole or part of a structure.

1. Linear Elastic Analysis

- Based on the theory of elasticity
- Suitable for both SLS and ULS
- Assumptions:
    i. Uncracked cross sections
    ii. Linear stress-strain relationships and,
    iii. Mean values of the elastic modulus [E]
- For thermal deformation, settlement and shrinkage effects at the (ULS), a reduced stiffness corresponding to the cracked sections may be assumed.
- For the (SLS) gradual evolution of cracking should be considered (e.g. rigorous deflection calculation).

2. Linear Elastic Analysis with Limited Redistribution

Although concrete structures only behave elastically under small loads while the sections remain uncracked, a linear elastic analysis may still be used for both the serviceability and strength limit states to determine the internal forces and moments, provided the structure has sufficient ductility to distribute moments from highly stressed regions to less highly stressed regions.

At ultimate limit state plastic rotations occur at the most stressed sections. These rotations transfer to other zones the effect of further load increase, thus allowing to take, for the design of reinforcement, a reduced bending moment δM, smaller than the moment M resulting from elastic linear design, provided that in the other parts of the structure the corresponding variations of load effects (viz. shear), necessary to ensure equilibrium, are considered.

- Suitable for ULS
- The moments at ULS calculated using a linear elastic analysis may be redistributed, provided that the resulting distribution of moments remains in equilibrium with the applied loads.
- In continuous beams or slabs which:
    a) Are predominantly subject to flexure and
    b) Have the ratio of the lengths of adjacent spans in the range of 0.5 to 2,

redistribution of bending moments may be carried out without explicit check on the rotation capacity, provided that:

- $\delta \geq k_1 + k_2 X_u / d$ for $f_{ck} \leq 50MPa$
- $\delta \geq k_3 + k_4 X_u / d$ for $f_{ck} > 50MPa$
- $\delta \geq k_5$ for reinforcement class B & C
- $\delta \geq k_6$ for reinforcement class A

Where

δ	Is the ratio of the redistributed moment to the elastic bending moment
x_u	Is the depth of the neutral axis at the ultimate limit state after redistribution
d	Is the effective depth of the section
ε_{cu1}	Is the ultimate strain for the section in accordance with Table 3.1

recommended value for k_1 is 0,44, for k_2 is 1,25(0,6+0,0014/ε_{cu2}), for k_3 = 0,54, for k_4 = 1,25(0,6+0,0014/ε_{cu2}), for k_5 = 0,7 and k_6 = 0,8

For the design of columns the elastic moments from frame action should be used without any redistribution.

3. Plastic Analysis

- Suitable ULS
- Suitable for SLS if compatibility is ensured
- When a beam yields in bending, an increase in curvature does not produce an increase in moment resistance. Analysis of beams and structures made of such flexural members is called plastic Analysis.
- This is generally referred to as limit analysis, when applied to reinforced concrete framed structures, and plastic analysis when applied to steel structures

4. Nonlinear analysis

Non-linear analysis is a procedure for calculation of action effects, based on idealizations of the non-linear behavior of materials [non-linear constitutive laws: for concrete and steel], of the elements and of the structure (cracking, second order effects), suitable for the nature of the structure and for the ultimate limit state under consideration.

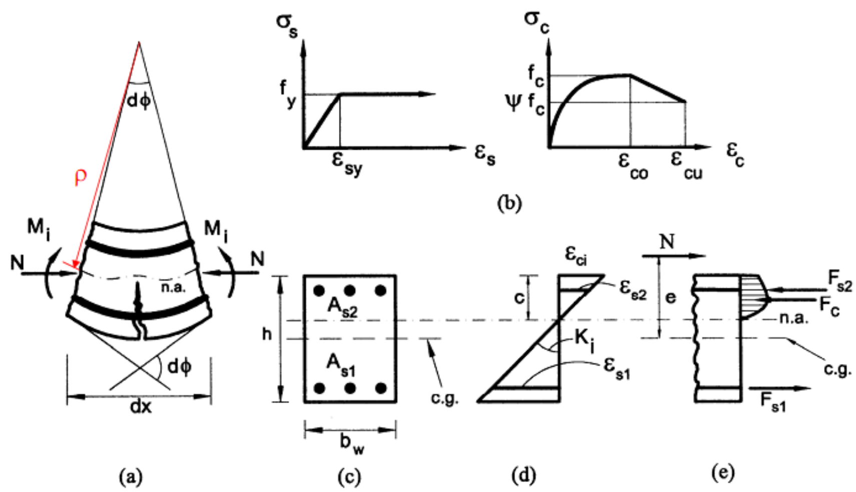
- may be used for both ULS and SLS, provided that equilibrium and compatibility are satisfied and an adequate non-linear behavior for materials is assumed.
- The non-linear analysis procedures are more complex and therefore very time consuming.
- The analysis maybe first or second order.

1.3. MOMENT CURVATURE RELATIONSHIP

Although it is not needed explicitly in ordinary design, the relation between moment applied to a given beam section and the resulting curvature, through the full range of loading to failure, is important in several contexts. It is basic to the study of member ductility, understanding the development of plastic hinges, and accounting for the redistribution of elastic moments that occurs in most reinforced concrete structures before collapse.

The flexural behavior of a reinforced concrete cross-section (a non-linear material) can best be studied by using its moment-curvature relationship. If the moment-curvature relationship is available, one can predict the strength and the stiffness, as well as the ductility characteristics of the cross-section.

1.3.1. CURVATURE



$$\frac{d^2y}{dx^2} = \frac{1}{\rho} = K = \frac{d\phi}{dx} \qquad K = \frac{\varepsilon_x}{y} = \frac{\varepsilon_{ci}}{c}$$

1.3.2. ELASTIC ANALYSIS OF BEAM SECTIONS

1.3.2.1. SECTION UN-CRACKED

As long as the tensile stress in the concrete is smaller than the tensile strength of concrete (f_{ctk}) the strain and stress is the same as in an elastic, homogeneous beam. The only difference is the presence of another material, i.e. the steel reinforcement. As it can be shown, in the elastic range, for any given value of strain, the stress in the steel is 'n' times that of the concrete, where $n = E_s/E_c$ is the modular ratio. In calculation the actual steel and concrete cross-section could be replaced by a fictitious section (transformed section) thought of as consisting of concrete only. In this section the actual steel area is replaced with an equivalent concrete area (nA_s) located at the level of the steel. Once the transformed section has been obtained, the beam is analyzed like an elastic homogeneous beam.

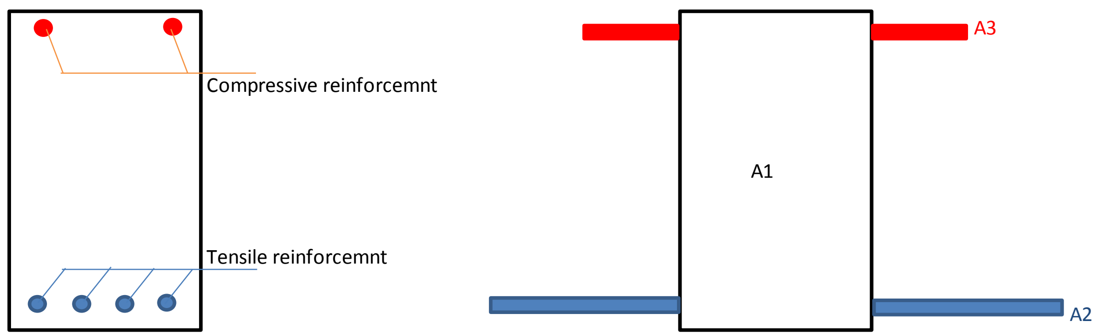


Figure 1-2 - Transformed Un-Cracked Section

1.3.2.2. SECTION CRACKED

When the tension stresses f_{ct} exceeds f_{ctk}, cracks form in the tension zone of the section. If the concrete compressive stress is smaller than approximately $0.5f_{ck}$ and the steel has not reached the yield strength, both materials continue to behave **elastically**.

At this stage, it is assumed that tension cracks have progressed all the way to the neutral axis and that sections that are plane before bending remain plane in the bent member. This situation of the section, strain and stress distribution is shown in the Figure 1-3 below.

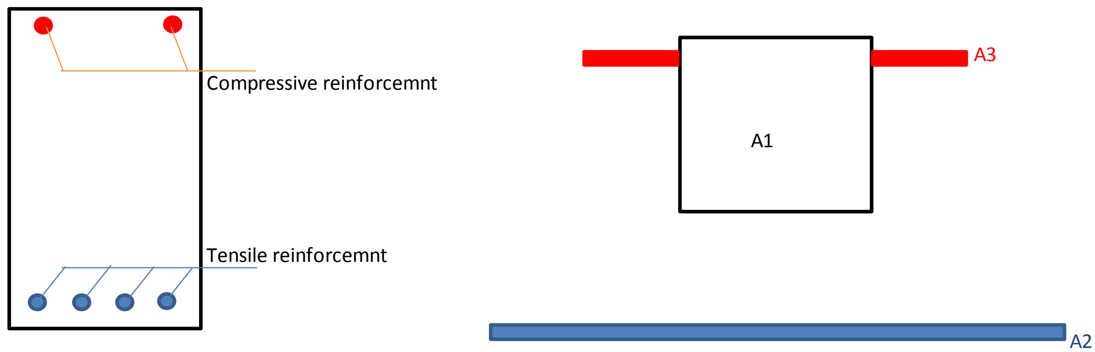


Figure 1-3 - Transformed Cracked Section

1.3.3. DRAWING THE MOMENT CURVATURE DIAGRAM

With the stress-strain relationships for steel and concrete, represented in idealized form and the usual assumptions regarding perfect bond and plane sections, it is possible to calculate the relation between moment and curvature for a typical under reinforced concrete beam section as follows.

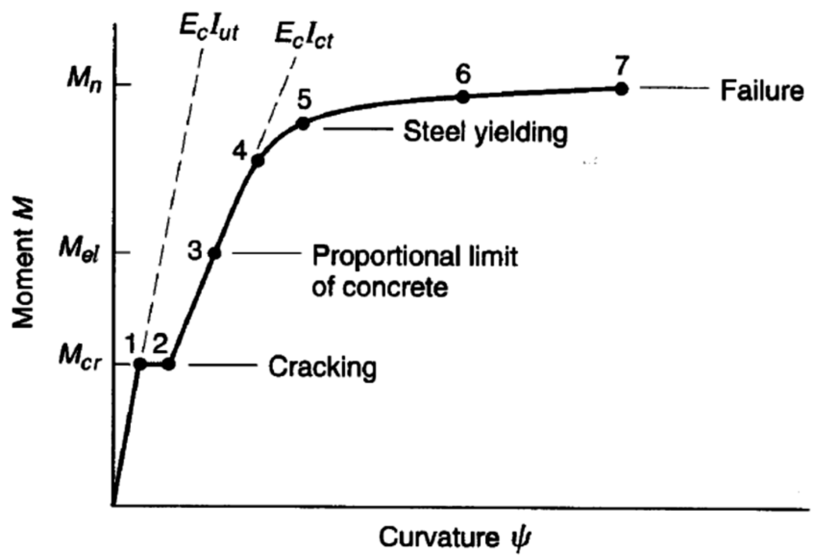


Figure 1-4 Moment-Curvature relationship for Reinforced Concrete Beam

A. Cracking point (point 1)

Figure 1-5 shows the transformed cross section of a rectangular, tensile reinforced beam in the uncracked elastic stage of loading, with steel represented by the equivalent concrete area. The neutral axis, a distance c_1 below the top surface of the beam, is easily found. In the limiting case, the concrete stress at the tension face is just equal to the modulus of rupture f_r and the strain is $\varepsilon_r = f_r / E_c$.

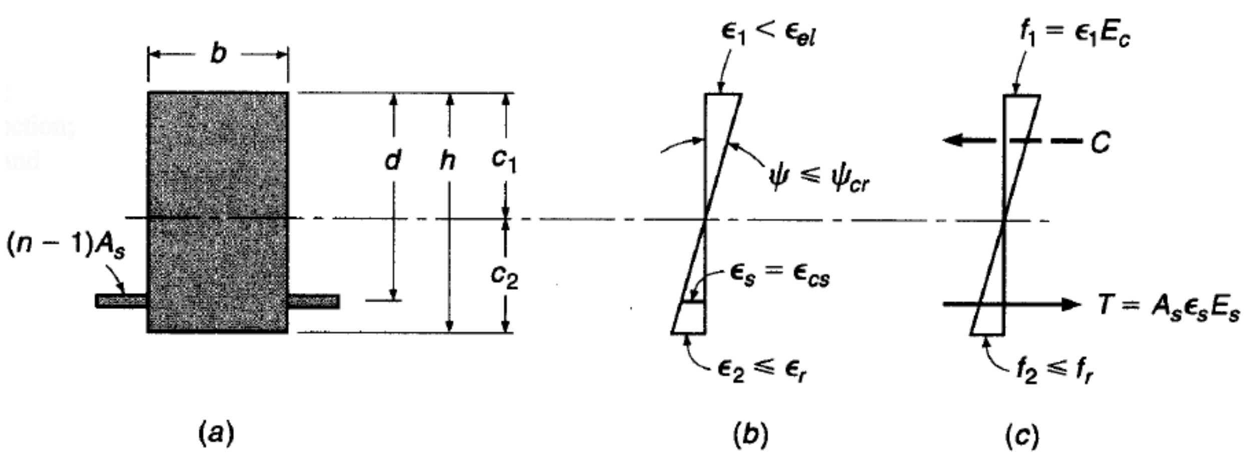


Figure 1-5 – Uncracked beam in the elastic range of loading

The steel is well below yield at this stage, which can be confirmed by computing, from the strain diagram, the steel strain. I is easily confirmed, also, that the maximum concrete compressive stress will be well below the proportional limit. The curvature is

$$\psi_{cr} = \frac{\varepsilon_1}{c_1} = \frac{\varepsilon_r}{c_2} \tag{1}$$

And the corresponding moment is

$$M_{cr} = \frac{f_r I_{ut}}{c_2} \tag{2}$$

Where I_{ut} is the moment of inertia of the uncracked transformed section.

These values (ϕ_{cr}, M_{cr}) provide information needed to plot point "1" of Figure 1-4.

B. Elastic limit (point 3)

When the tensile cracking occurs at the section, the stiffness is immediately reduced, and curvature increases to point "2" in Figure 1-4 with no increase in moment. The analysis now is based on the cracked transformed section of Figure 1-6 with steel represented by the transformed concrete area and tension concrete deleted. The cracked, elastic neutral axis distance $c_1 = kd$ is easily found by the usual methods.

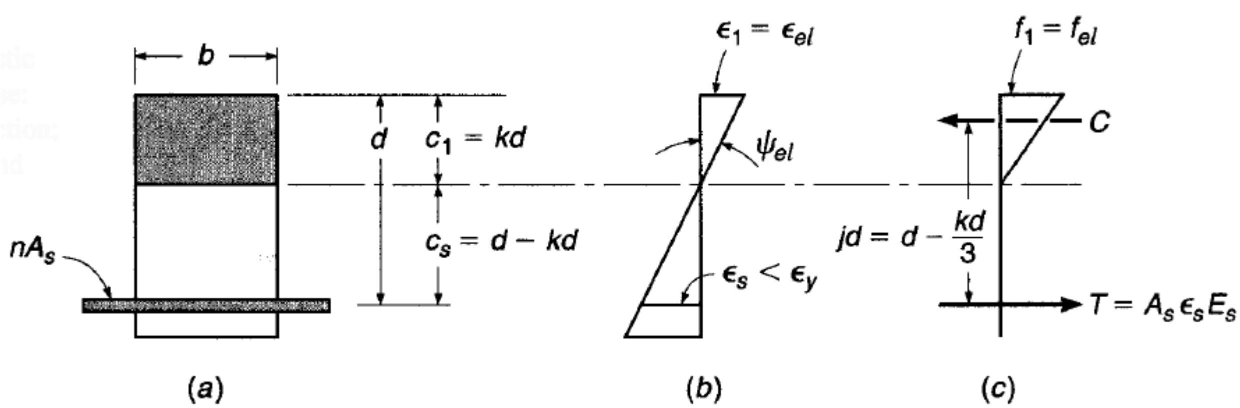


Figure 1-6 – Cracked beam in the elstic range of material response

In the limiting case, the concrete strain just reaches the proportional limit as shown in Figure 1-6(b), and typically the steel is still below the yield strain. The curvature is easily computed by

$$\psi_{el} = \frac{\varepsilon_1}{c_1} = \frac{\varepsilon_{el}}{c_1} \tag{3}$$

And the corresponding moment can be calculated using moment is

$$M_{el} = \frac{1}{2} f_{el} kjbd^2 \tag{4}$$

This provides point 3 in Figure 1-4. The curvature at point 2 can now be found from the ratio M_{cr}/M_{el} .

C. Inelastic zone (point 3 – 7)

Next, the cracked, inelastic stage of loading is shown in Figure 1-7. Here the concrete is well into the inelastic range, although the steel has not yielded. The neutral axis depth c_1 is less than the elastic kd and is changing with increasing load as the shape of the concrete stress distribution changes and the steel stress changes.

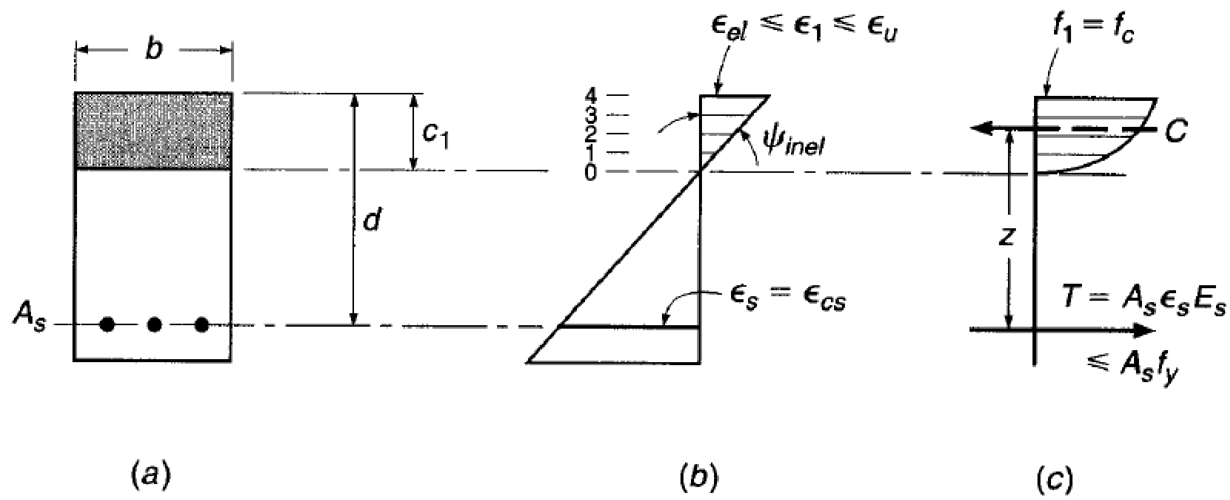


Figure 1-7 – Cracked beam with concrete in the inelastic range of loading

It is now convenient to adopt the equations of α_c and β_c and equilibrium of force and moment to find both the total concrete compressive force C and the location of its centroid, for any arbitrarily selected value of maximum concrete strain ε_1 in this range. The entire process can be summarized as follows:

1. Select any top face concrete strain ε_1 in the inelastic range, i.e., between ε_{el} and ε_u .

2. Assume the neutral axis depth, a distance c_1 below the top face.
3. From the strain diagram geometry, determine $\varepsilon_s = \varepsilon_{cs}$.
4. Compute $f_s = \varepsilon_s E_s \leq f_y$ and $T = A_s f_s$.
5. Determine C $\left(C = \alpha_c f_{cd} bd\right)$
6. Check to see if $C = T$. If not, the neutral axis must be adjusted upward or downward, for particular concrete strain that was selected in step 1, until equilibrium is satisfied. This determines the correct value of c_1

Curvature can then be found from

$$\psi_{inel} = \frac{\varepsilon_1}{c_1} \tag{5}$$

The internal lever arm z from the centroid of the concrete stress distribution to the tensile resultant, is calculated after which

$$M_{inel} = Cz = Tz \tag{6}$$

The sequence of steps 1 through 6 is then repeated for newly selected values of concrete strain ε_1. The end result will be a series of points, such as 4, 5, 6, and 7 in Figure 1-4. The limit of the moment –curvature plot is reached when the concrete top face strain equals ε_u , corresponding to point 7. The steel would be well past yield strain at this loading, and at the yield stress.

1.4. CONTINOUS BEAMS AND ONE WAY SLABS

Continuous beams and one-way slabs are indeterminate structures for which live load variation has to be considered. This is because dead load is always there but live load might vary during the life time of these structures.

One-way slabs transmit their load mainly in one direction (i.e., the direction. of span). A 1m strip is taken in the direction of span and treated similar to continuous beams.

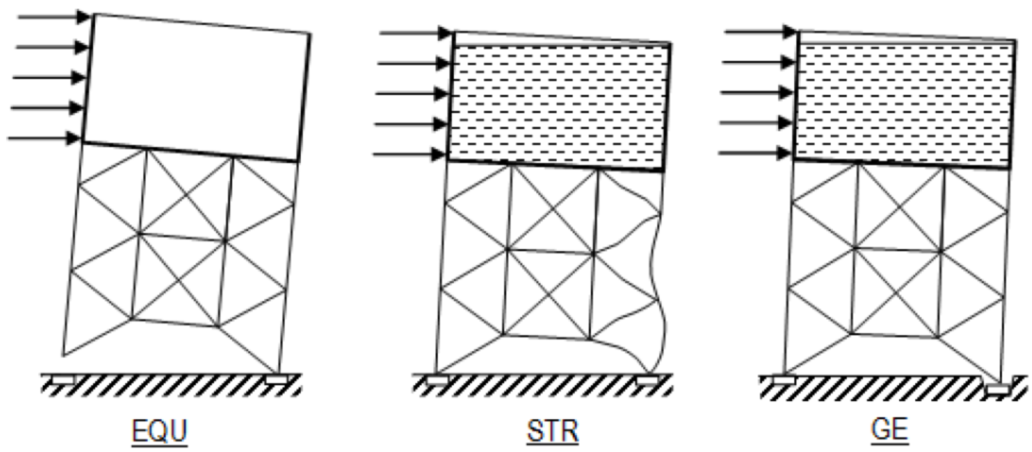
Elastic analysis such as slope-deflection, moment distribution and matrix method or plastic analysis or approximate method such as the use of moment coefficient or such methods as portal or cantilever can be used.

1.4.1. LOAD ARRANGEMENT OF ACTIONS

The process of designing concrete structures involves identifying the relevant design situations and limit states. These include persistent, transient or accidental situations. In each design situation the structures should be verified at the relevant limit states.

In the analysis of the structure at the limit state being considered, the maximum effect of actions should be obtained using a realistic arrangement of loads. Generally variable actions should be arranged to produce the most unfavorable effect, for example to produce maximum overturning moments in spans or maximum bending moments in supports.

For building structures, design concentrates mainly on the ULS, the ultimate limit state of strength (STR), and SLS, the serviceability limit state. However, it is essential that all limit states are considered. The limit states of equilibrium (EQU), strength at ULS with geotechnical actions (STR/GEO) and accidental situations must be taken into account as appropriate.



1.4.1.1. Load Arrangement of Actions: In relation to Influence Lines

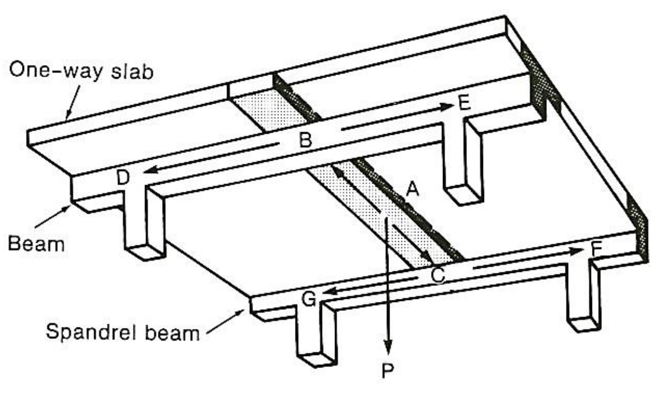


Figure 1-8 – One-way slab and continuous beam

The largest moment in continuous beams or one-way slabs or frames occur when some spans are loaded and the others are not. Influence lines are used to determine which spans should be loaded and which spans should not be to find the maximum load effect.

Figure 1-9a shows influence line for moment at B. The loading pattern that will give the largest positive moment at consists of load on all spans having positive influence ordinates. Such loading is shown in Figure 1-9b and is called ***alternate span loading*** or ***checkerboard loading.***

The maximum negative moment at C results from loading all spans having negative influence ordinate as shown in Figure 1-9d and is referred as an ***adjacent span loading***.

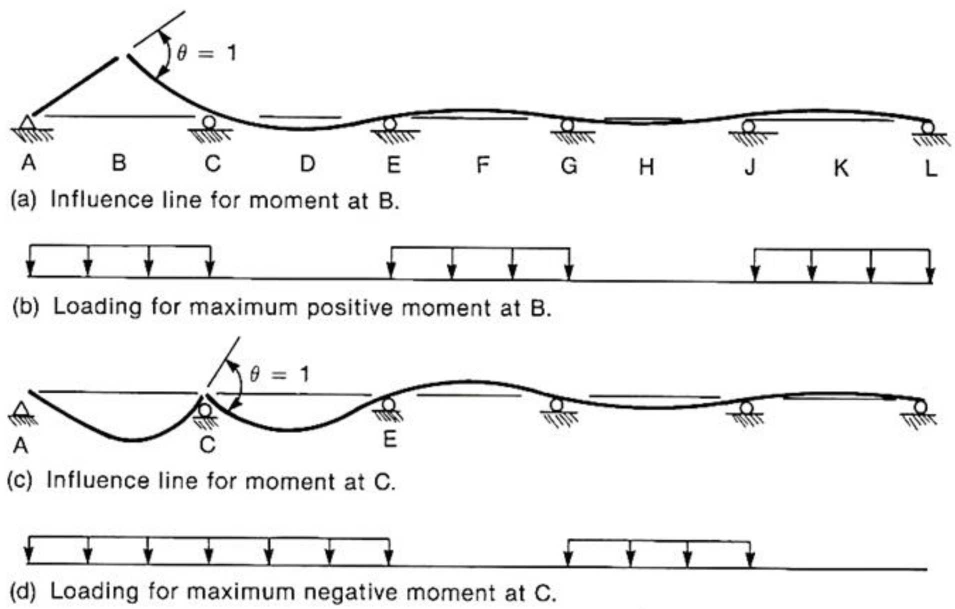
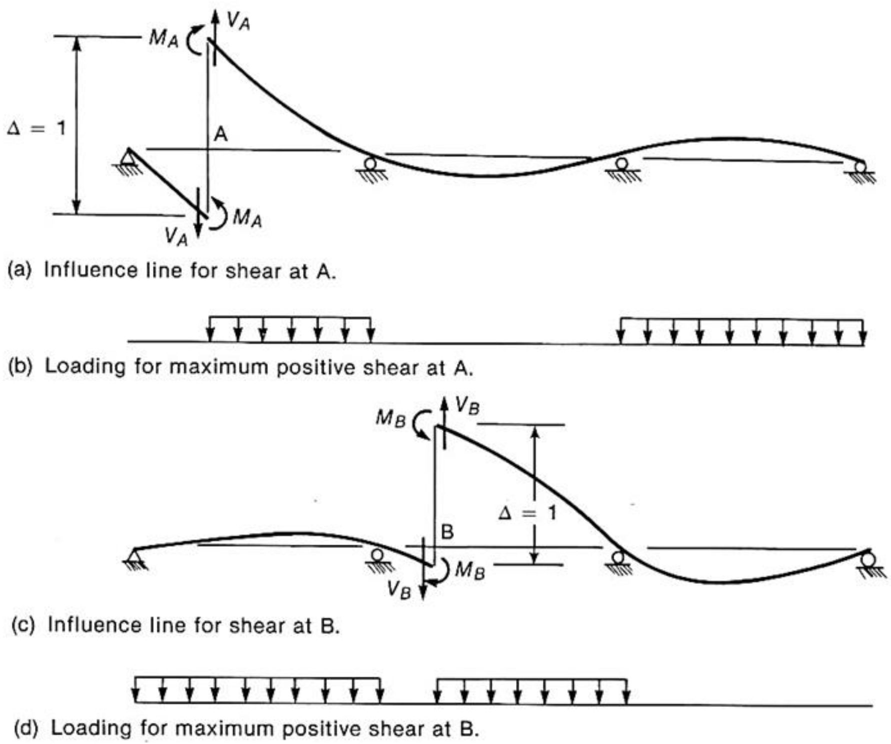


Figure 1-9 Influence line for moment and loading patterns

Similarly, loading for maximum shear may be obtained by loading spans with positive shear influence ordinate and are shown in Figure 1-10.



(a) Influence line for shear at A.

(b) Loading for maximum positive shear at A.

(c) Influence line for shear at B.

(d) Loading for maximum positive shear at B.

Figure 1-10 Influence line for shear

1.4.1.2. Load Arrangemnt of Actions: According Eurocode

In building structures, any of the following sets of simplified load arrangements may be used at ULS and SLS. (EBCES EN 2004:2014 Section 5.1.2)

- The more critical of:
  a) Alternative spans carrying $\gamma_G G_k + \gamma_Q Q_k$ with other spans loaded with $\gamma_G G_k$ and
  b) Any two adjacent spans carrying $\gamma_G G_{k+} \gamma_Q Q_k$
- Or the more critical of:
  a) Alternative spans carrying $\gamma_G G_k + \gamma_Q Q_k$ with other spans loaded with $\gamma_G G_k$ and
  b) Any two adjacent spans carrying $\gamma_G G_{k+} \gamma_Q Q_k$

1.4.2. DESIGN OF CONTINUOUS BEAMS AND ONE WAY SLABS

After obtaining the maximum load effects, the design of continuous beams and one way slabs is carried out as discussed in Reinforced Concrete Structures I course.

1.5. PLASTIC HINGES AND COLLAPSE MECHANISMS

If a short segment of a reinforced concrete beam is subjected to a bending moment, continued plastic rotation is assumed to occur after the calculated ultimate moment M_u is reached, with no change in applied moment. The beam behaves as if there were a hinge at that point. However, the hinge will not be “friction free”, but will have a constant resistance to rotation.

If such a plastic hinge forms in a determinate structure, as shown in Figure 1-11, an uncontrolled deflection takes place and the structure will collapse. The resulting system is referred to as a mechanism. This implies that a statically determinate system requires the formation of only one plastic hinge in order to become a mechanism.

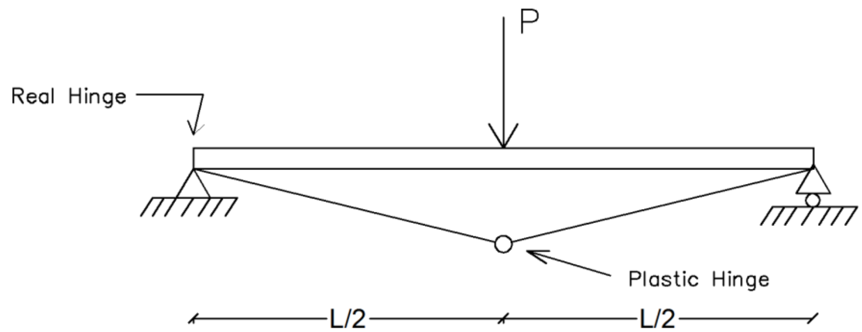


Figure 1-11 – Determinate structure

If the structure is statically indeterminate, it is still stable after the formation of a plastic hinge, and for further loading, it behaves as a modified structure with a hinge at the plastic hinge location (and one less degree of indeterminacy). It can continue to carry additional loading (with formation of additional plastic hinges) until the limit state of collapse is reached on account of one of the following reasons:

- formation of sufficient number of plastic hinges, to convert the structure (or a part of it) into a 'mechanism';
- Limitation in ductile behavior (i.e., curvature ϕ reaching the ultimate value ϕ_{max}, or, in other words a plastic hinge reaching its ultimate rotation capacity) at any one plastic hinge location, resulting in local crushing of concrete at that section.

For illustration let us see the behavior of an indeterminate beam of Figure 1-12. It will be assumed for simplicity that the beam is symmetrically reinforced, so that the negative bending capacity is the same as the positive. Let the load P be increased gradually until the elastic moment at the fixed support, 3PL/16 is just equal to the plastic moment capacity of the section, M_u. This load is

$$P = P_{el} = \frac{16M_u}{3L} = 5.33\frac{M_u}{3L} \qquad (1.1)$$

At this load the positive moment under the load is $\frac{5}{32}$ PL, as shown in Figure 1-12.

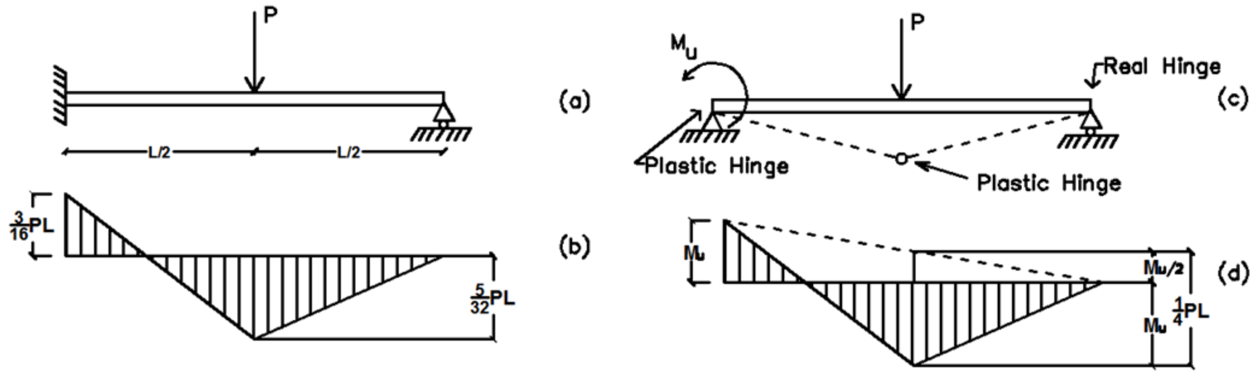


Figure 1-12 – Indeterminate Structures

The beam still responds elastically everywhere but at the left support. At that point the actual fixed support can be replaced for purpose of analysis with a plastic hinge offering a known resisting moment M_u, which makes the beam statically determinate.

The load can be increased further until the moment under the load also becomes equal to M_u, at which load the second hinge forms. The structure is converted into a mechanism, as shown in Figure 1-12(c), and collapse occurs. The moment diagram at collapse is shown in Figure 1-12(d).

The magnitude of the load causing collapse is easily calculated from the geometry of Figure 1-12(d).

$$M_u + \frac{M_u}{2} = \frac{PL}{4} \qquad (7)$$

From which

$$P = P_u = \frac{6M_u}{L} \qquad (8)$$

By comparison, it is evident that an increase of 12.5% is possible beyond the load which caused the formation of the first plastic hinge, before the beam will actually collapse. Due to the formation of plastic hinges, a redistribution of moments has occurred such that, at failure, the ratio between positive moment and negative moment is equal to that assumed in reinforcing the structure.

1.6. ROTATION CAPACITY

It may be evident that there is a direct relation between the amount of redistribution desired and the amount of inelastic rotation at the critical sections of a beam required to produce the desired redistribution. In general, the greater the modification of elastic-moment ratio, the greater the required rotation capacity to accomplish that change. Thus the designer adopting the limit/plastic analysis in concrete must calculate the inelastic rotation capacity it undergoes at plastic-hinge locations.

One way to calculate this rotation capacity is making use of the moment-curvature relationship established for a given section. But his plastic rotation is not confined to one cross section but is distributed over a finite length referred to as the hinging length $\left(l_p\right)$.

The total inelastic rotation θ_s can be found by multiplying the average curvature by the hinging length:

$$\theta_s = \left(k_{ult} - k_{yd}\frac{M_{ult}}{M_{yd}}\right)l_p \qquad (9)$$

where

k_{ult}	Curvature at the ultimate point of the moment curvature diagram
k_{yd}	Curvature at the yield point of the moment curvature diagram
M_{ult}	moment at the ultimate point of the moment curvature diagram
M_{yd}	moment at the yield point of the moment curvature diagram
l_p	$=1.2h$ In which z is the distance from the point of maximum moment to the nearest point of zero moment

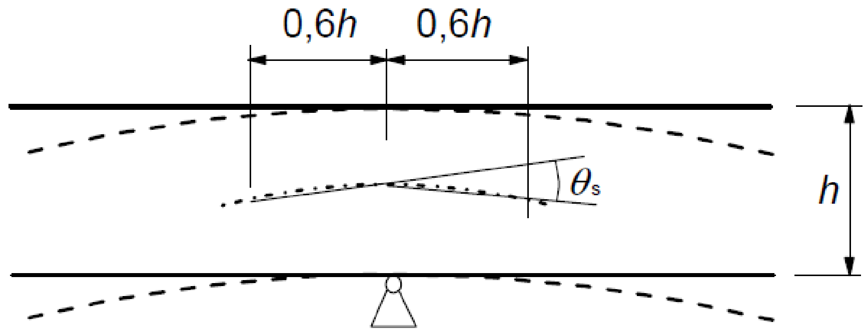


Figure 1-13 – Plastic rotation of θ_s of reinforced concrete sections for continuous beams and continuous one way spanning slabs

According to section 5.6.3 of EBCS EN 2004:2014, verification of the plastic rotation in the ultimate limit state is considered to be fulfilled, if it is shown that under the relevant action, the calculated rotation, θ_s, is less than or equal to the allowable plastic rotation, $\theta_{pl,d}$

In the simplified procedure, the allowable plastic rotation may be determined by multiplying the basic value of allowable rotation by a correction factor k_λ that depends on the shear slenderness. The recommended basic value of allowable rotation, for steel classes B and C (the use of Class A steel is not recommended for plastic analysis) and concrete strength classes less than or equal to C50/60 and C90/105 are given in Figure 1-14.

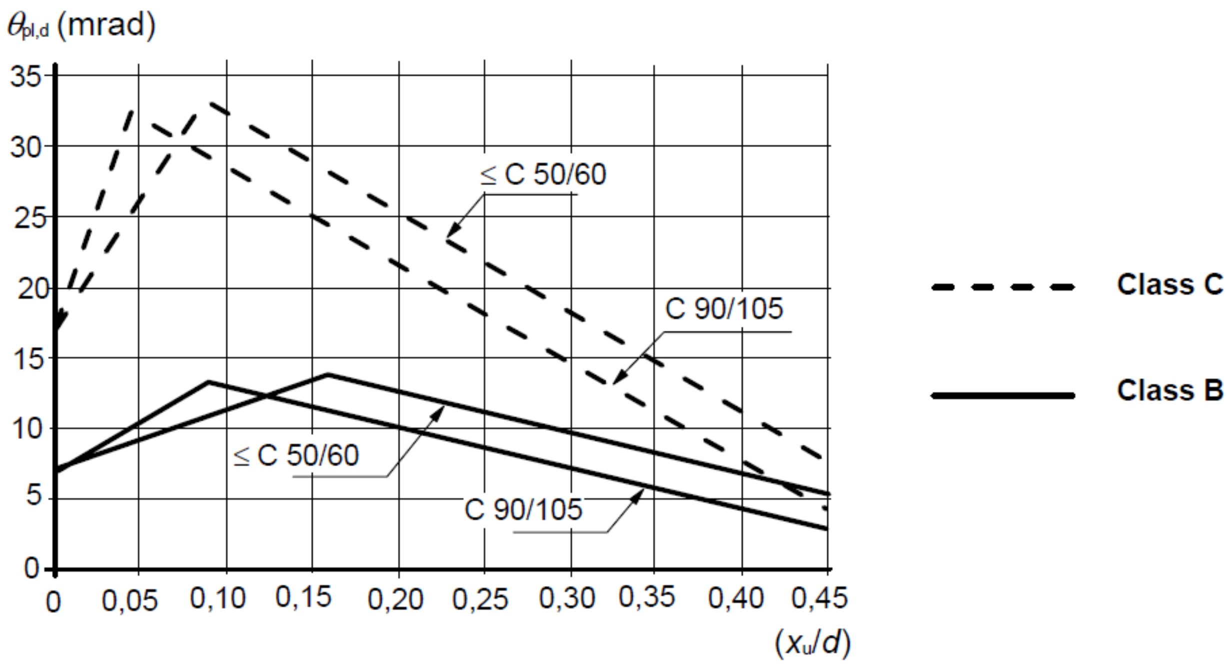


Figure 1-14 – Allowable plastic rotation, $\theta_{pl,d}$, of reinforced concrete sections for Class B and C reinforcement.

The values in Figure 1-14 apply for a shear slenderness $\lambda = 3.0$. For different values of shear slenderness $\theta_{pl,d}$ should be multiplied by k_λ.

$$k_\lambda = \sqrt{\lambda/3} \tag{10}$$

Where:

λ is the ratio of the distance between point zero and maximum moment after redistribution and effective depth, d. As a simplification λ may be calculated for the concordant design values of the bending moment and shear. $\lambda = M_{sd}/(V_{sd}d)$

1.7. MOMENT REDISTRIBUTION

Statically indeterminate structures made of reinforced concrete like fixed ended one span beams, continuous beams and frames are designed considering internal forces like bending moment, shear force and axial thrust obtained from structural analysis. Either one or several sections of these structures may have peak values of the internal forces, which are designated as critical sections. These sections are dimensioned and reinforced accordingly. Flexural members, however, do not collapse immediately as soon as the loads at a particular section cause bendding moment exceeding the maximum resisting moment capacity of that section. Instead, that section starts rotating at almost constant moment. This is known as formation of plastic hinge at that section reaching its maximum resisting moment capacity. The section then transfers loads to other sections if the applied loads are further increased. This process continues till the structures have plastic hinges at sufficient sectins to form a failure mechanism when it actually collapses. However, significant transfer of loads has occurred before the collapse of the structure. This transfer of loads after the formation of first plastic hinge at section having the higest bending moment till the collapse of the structure is known as redistribution of moments. By this process, therefore, the structure continues to accommodate higher loads before it collapses.

The elastic bending moment diagram prior to the formation of first plastic hinge and the final bending moment diagram just before the collapse are far different. The ratio of the negative to positive elastic bending moments is no more valid. The development of plastic hinges depends on the available plastic moment capacity at critical sections. It is worth mentioning that the redistribution of moment is possible if the section forming the plastic hinge has the ability to rotate at constant moment, which depends on the amount of reinforcement actually provided at that section. The section must be under-reinforced and should have sufficient ductility.

This phenomenon is well known in steel structures. However, the redistribution of moment has also been confirmed in reinforced concrete structure by experimental investigations. It is also a fact that reinforced concrete structures have comparatively lower capacity to rotate than steel structures. yet, this phenomenon is drawing the attention of the designers. Presently, design codes of most of the countries allow the redistribution up to a maximum limit because of the following advantages:

1) It gives a more realistic picture of the actual load carrying capacity of the indeterminate structure.
2) Structures designed considering the redistribution of moment (though limited) would result in economy as the actual load capacity is higher than that we determine from any elastic analysis.
3) The designer enjoys the freedom of modifying the design bending moments within limits. These adjustments are sometimes helpful in reducing the reinforcing bars, which are crowded, especially at locations of high bending moment.

The choice of the bending moment diagram after the redistribution should satisfy the equilibrium of internal forces and external loads. Moreover, it must ensure the following:

1. The plastic rotations required at the critical sections should not exceed the amount the sections can sustain.
2. The extent of cracking or the amount of deformation should not make the performance unsatisfactory under service loads.