Addis Ababa Institute of Technology

Department of civil and Environmental Engineering

Hydraulics-II (CENG-2162)

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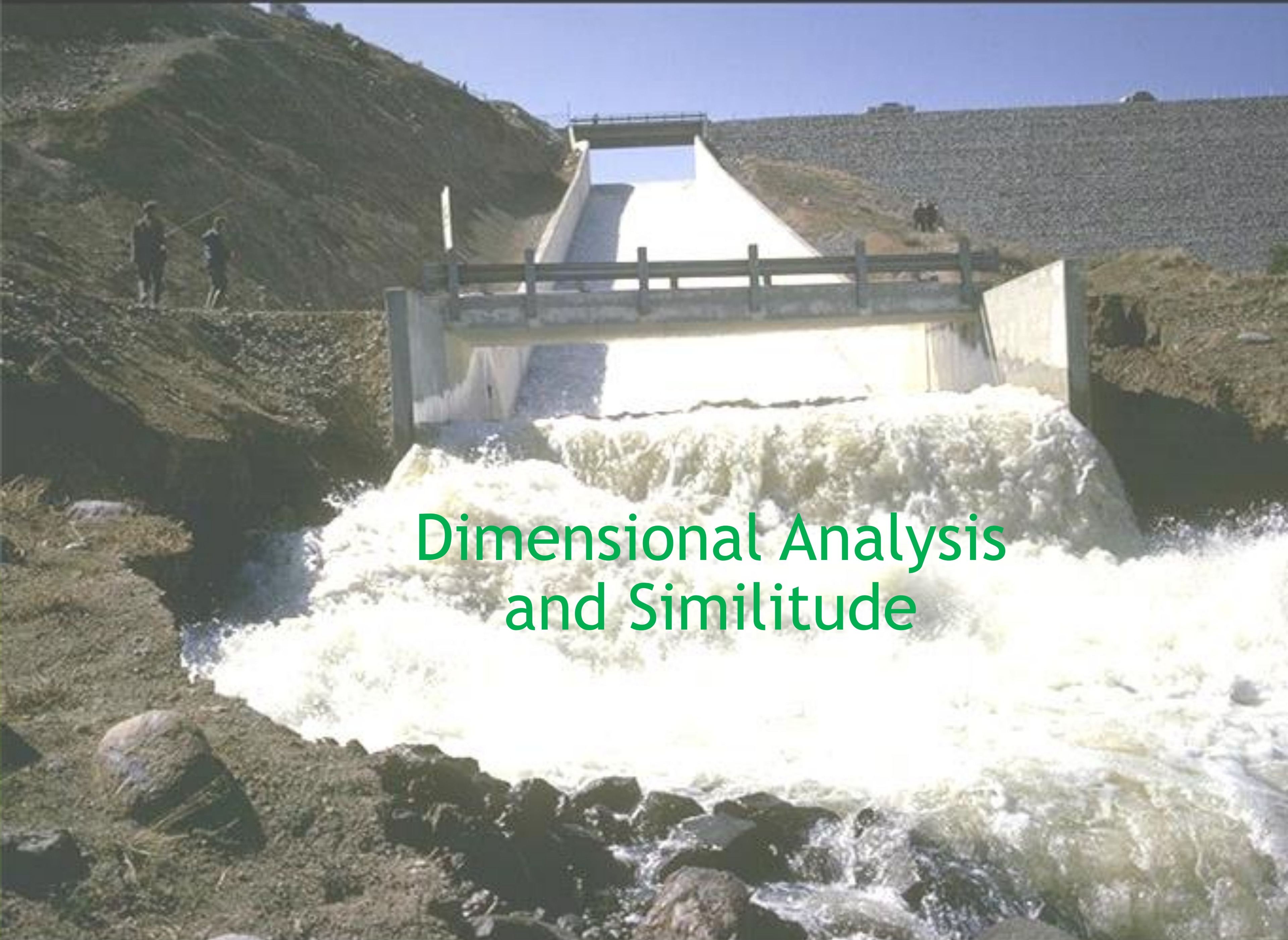


Dimensional Analysis and Similitude

Dimensional Analysis

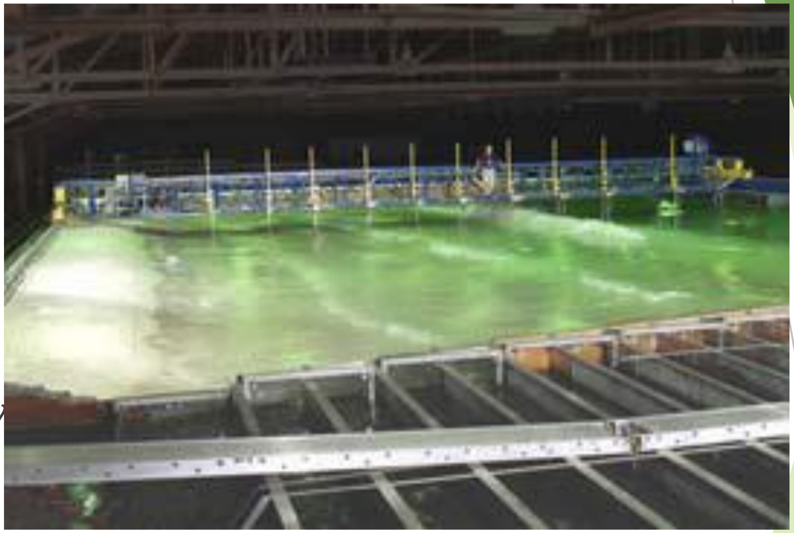
- Dimensions and Units
- Dimensional Analysis
- Rayleigh’s method
- Buckingham Pi Theorem
- Determination of Pi Terms
- Common Dimensionless Groups in Fluid Mechanics
- Correlation of Experimental Data
- Modeling and Similitude
- Typical Model Studies
- Application areas

Motivation

- Verify if equation is always usable
- Predict nature of relationship between quantities (like friction, diameter etc)
- Minimize number of experiments
- Scale up / down
- Scale factors
- Often difficult to solve fluid flow problems by analytical or numerical methods. Also, data are required for validation
- The need for experiments
- Difficult to do experiment at the true size (prototype)
- so they are typically carried out at another scale (model)
- Develop rules for design of experiments and interpretation of measurement results

Fields of Application

- flow machinery (pumps, turbines)
- hydraulic structures
- rivers, estuaries
- sediment transport



Sediment transport facility

To solve practical problems, derive general relationships, obtain data for comparison with mathematical models

Basic Terminologies

- Any physical situation can be described by certain familiar properties e.g. length, velocity, area, volume, acceleration etc. These are all known as dimensions
- The expression for a derived quantity in terms of a basic quantities is called the dimension of the physical quantity
- Dimensions are properties which can be measured
- Units are the standard elements we use to quantify these dimensions
- In dimensional analysis we are only concerned with the nature of the dimension i.e. its quality not its quantity

Dimensional Analysis

- It is a mathematical technique used in research work for design and for conducting model tests
- It deals with the dimensions of the physical quantifies involved in the phenomenon
- provides a strategy for choosing relevant data and how it should be presented
- Enables scaling for different physical dimensions and fluid properties
- Length L, mass M and time T, temperature Θ are fixed dimensions which are of importance in Fluid Mechanics

Contd…

- Is there a possibility that the equation exists?
- Effect of parameters on drag on a cylinder
  - Choose important parameters
  - viscosity of medium, size of cylinder (dia, length?), density
  - velocity of fluid?
  - Choose monitoring parameter
  - drag (force)
- Are these parameters sufficient?
- How many experiments are needed?

Contd…

F and M related by F = Ma = MLT-2

$$\mu \equiv Pa - s \equiv M^1 L^{-1} T^{-1} \qquad D \equiv L^1$$

$$V \equiv L^1 T^{-1} \qquad \rho \equiv M^1 L^{-3} \qquad F \equiv M^1 L^1 T^{-2}$$

Determine the dimensions of the following quantities

i) Acceleration

ii) Discharge

iii) Specific weight

iv) Viscosity

v) Energy

vi) Pressure

Contd…

- The first step in the planning of an experiment to study this problem would be to decide on the factors, or variables, that will have an effect on the pressure drop.
- Pressure drop per unit length

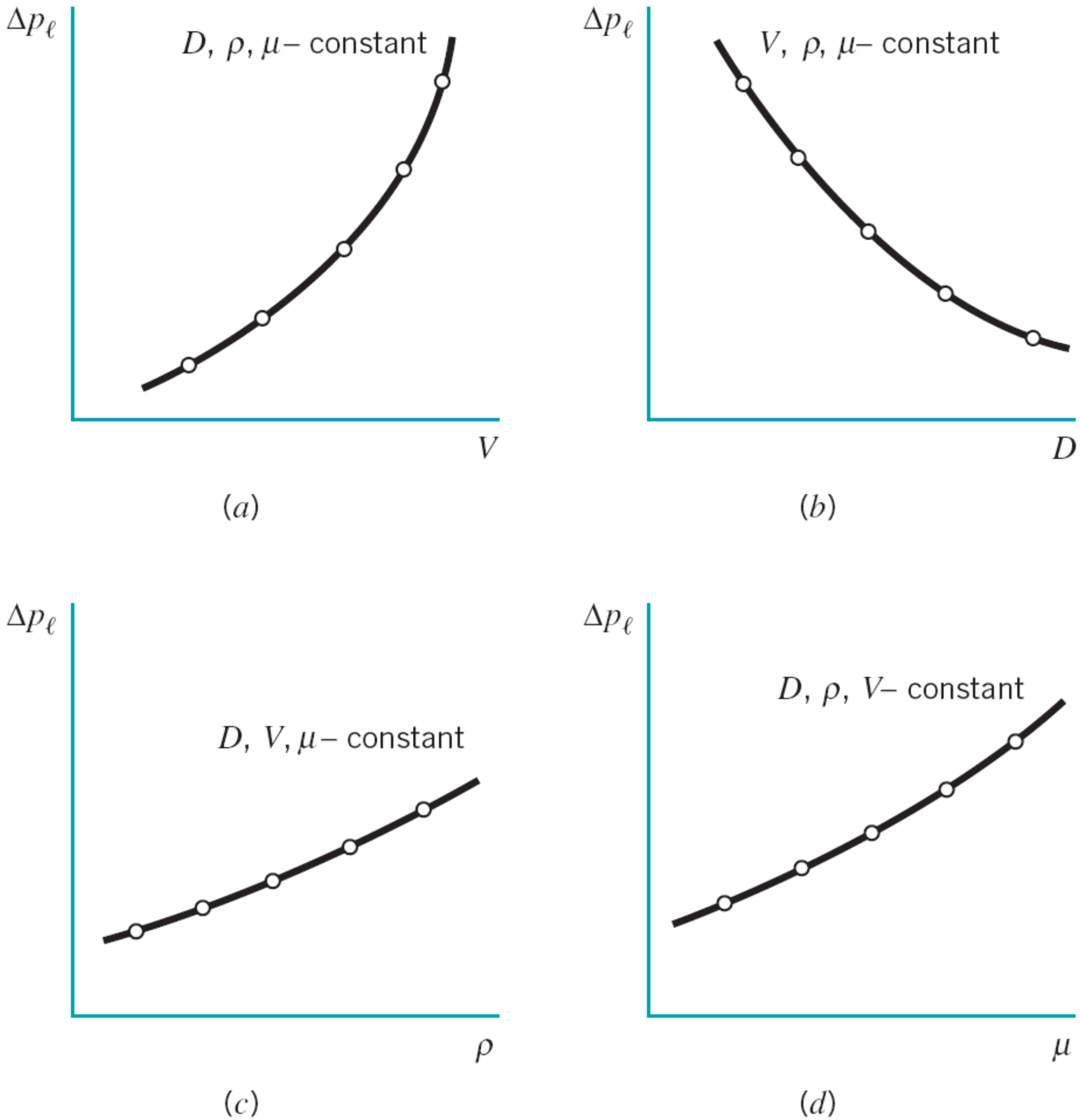
  $\Delta p = f\,(D,\rho,\mu,V)$

- ✓ *Pressure drop per unit length depends on FOUR variables:*

sphere size (D); speed (V); fluid density (ρ); fluid viscosity (m)

- To perform the experiments in a meaningful and systematic manner, it would be necessary to change on of the variable, such as the velocity, which holding all other constant, and measure the corresponding pressure drop.
- Difficulty to determine the functional relationship between the pressure drop and the various facts that influence it.

Series of Tests



Contd…

- Fortunately, there is a much simpler approach to the problem that will eliminate the difficulties described above.

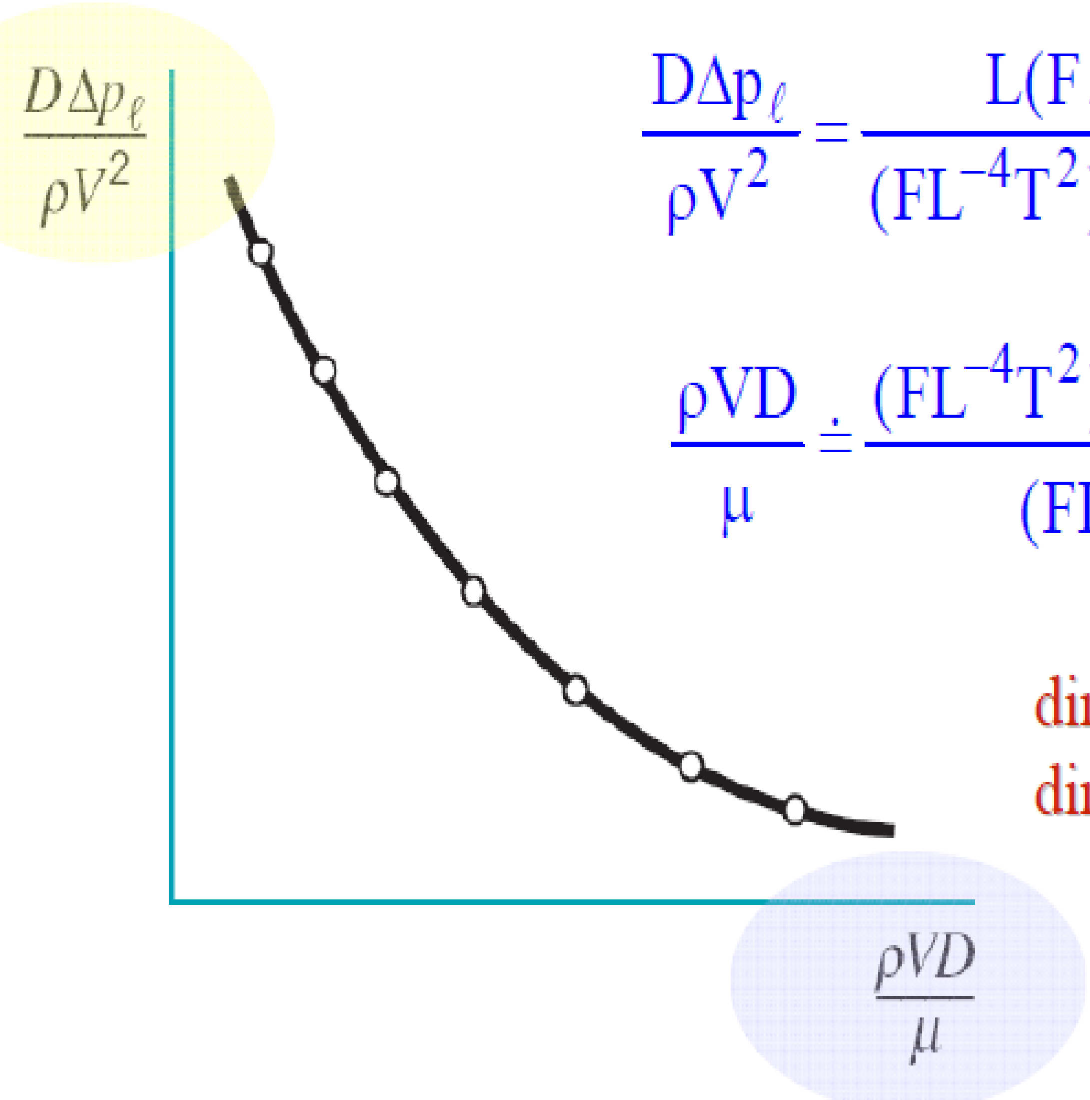
Collecting these variables into two non dimensional combinations of the variables (called dimensionless product or dimensionless groups)

$$\frac{D\Delta p_\ell}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Dependent variable

- *Only one dependent and one independent variable*
- *Easy to set up experiments to determine dependency*
- *Easy to present results (one graph)*

Plot of Pressure Drop Data Using …



$$\frac{D\Delta p_\ell}{\rho V^2} = \frac{L(F/L^3)}{(FL^{-4}T^2)(FT^{-1})^2} \doteq F^0L^0T^0$$

$$\frac{\rho VD}{\mu} \doteq \frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)} = F^0L^0T^0$$

dimensionless product or
dimensionless groups

Dimensional Homogeneity

- Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal
- If the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation
- The powers of fundamental dimensions (i.e., L, M, T) on both sides of the equation will be identical for a dimensionally homogeneous equation
- In a dimensionally homogenous equations, only quantities with the same dimensions can be added, subtracted or equated

Contd…

Fourier principle of dimensional homogeneity: an equation which expresses a physical phenomena of fluid flow must be algebraically correct and dimensionally homogenous

$$t = 2\pi\sqrt{\frac{L}{g}} \quad \text{(time for swing of pendulum)}$$

$$Q = \frac{2}{3} C_d \sqrt{2g}\, LH^{3/2} \quad \text{(flow over rectangular weir)}$$

dimensionally – homogenous equations.

$$V = 1/nR^{2/3}\, S^{1/2} \quad \text{(Manning equation)}$$

dimensionally non – homogenous equation

It is always possible to reduce a dimensionally homogenous equation to a non dimensional equation

$$\left[\frac{t}{\sqrt{\frac{l}{g}}}\right] = 2\pi; \left[\frac{Q}{\sqrt{g}\, LH^{3/2}} = \frac{2\sqrt{2}}{3} C_d\right]$$

Methods of Dimensional Analysis

i. Rayleigh's method, and

ii. Buckingham's Π –theorem

Rayleigh's method

A functional relationship of some variables is expressed in the form of an exponential equation which must be dimensionally homogenous

Let X be a function of independent variables x1, x2, x3. Then according to Rayleigh's method: X = f [x1, x2, x3]

This can be written as : $X = kx_1^a . x_2^b . x_3^c$

Where, k = constant

a, b ,c = arbitrary powers

The values of a, b , c are obtained by comparing the powers of fundamental dimension on both sides

Buckingham's Π Theorem

- A fundamental question we must answer is how many

dimensionless products are required to replace the original list of variables ?

The answer to this question is supplied by the basic theorem of dimensional analysis that states

If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k-r independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

Pi terms

⇨ *Buckingham Pi Theorem*

Contd…

Given a physical problem in which the dependent variable is a function of k-1 independent variables.

$$u1 = f(u2, u3, \ldots, u k)$$

⟹ Mathematically, we can express the functional relationship in the equivalent form

$$g(u1, u2, u3, \ldots, u k) = 0$$

Where g is an unspecified function, different from f.

Contd…

- The Buckingham Pi theorem states that: Given a relation among k variables of the form

$$g(u_1, u_2, u_3, \ldots\ldots, u_k) = 0$$

- The k variables may be grouped into k-r independent dimensionless products, or Π terms, expressible in functional form by

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

or $$\bar{\phi}(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{k-r}) = 0$$

r ?? Π ??

Contd…

- The number r is usually, but not always, equal to the minimum number of independent dimensions required to specify the dimensions of all the parameters. Usually the reference dimensions required to describe the variables will be the basic dimensions M, L, and T.
- The theorem does not predict the functional form of φ or ϕ . The functional relation among the independent dimensionless products Π must be determined experimentally.
- The k-r dimensionless products Π terms obtained from the procedure are independent.
- A Π term is not independent if it can be obtained from a product or quotient of the other dimensionless products of the problem.

Determination of Pi Terms

Method of repeating variables

Step 1. List all the variables

- List all the dimensional variables involved.
- Keep the number of variables to a minimum, so that we can minimize the amount of laboratory work.
- All variables must be independent

$\gamma = \rho \times g$, that is, γ, ρ, and g are not independent.

Step 2. Express each of the variables in terms of basic dimensions.
Find the number of reference dimensions.

- Select a set of fundamental (primary) dimensions.

For example: MLT

Contd…

Step 3. Determine the required number of pi terms.

⟶ Let k be the number of variables in the problem.

⟶ Let r be the number of reference dimensions (primary dimensions) required to describe these variables.

⟶ The number of pi terms is k-r

- Example: For pressure drop per unit length k=5, r = 3,the number of pi terms is k-r=5-3=2.

Contd…

Step 4. Select a number of repeating variables, where the number required is equal to the number of reference dimensions.

Select a set of r dimensional variables that includes all the primary dimensions (repeating variables).

These repeating variables will all be combined with each of the remaining parameters. No repeating variables should have dimensions that are power of the dimensions of another repeating variable.

- Example: For pressure drop per unit length (r = 3) select ρ, V, D.

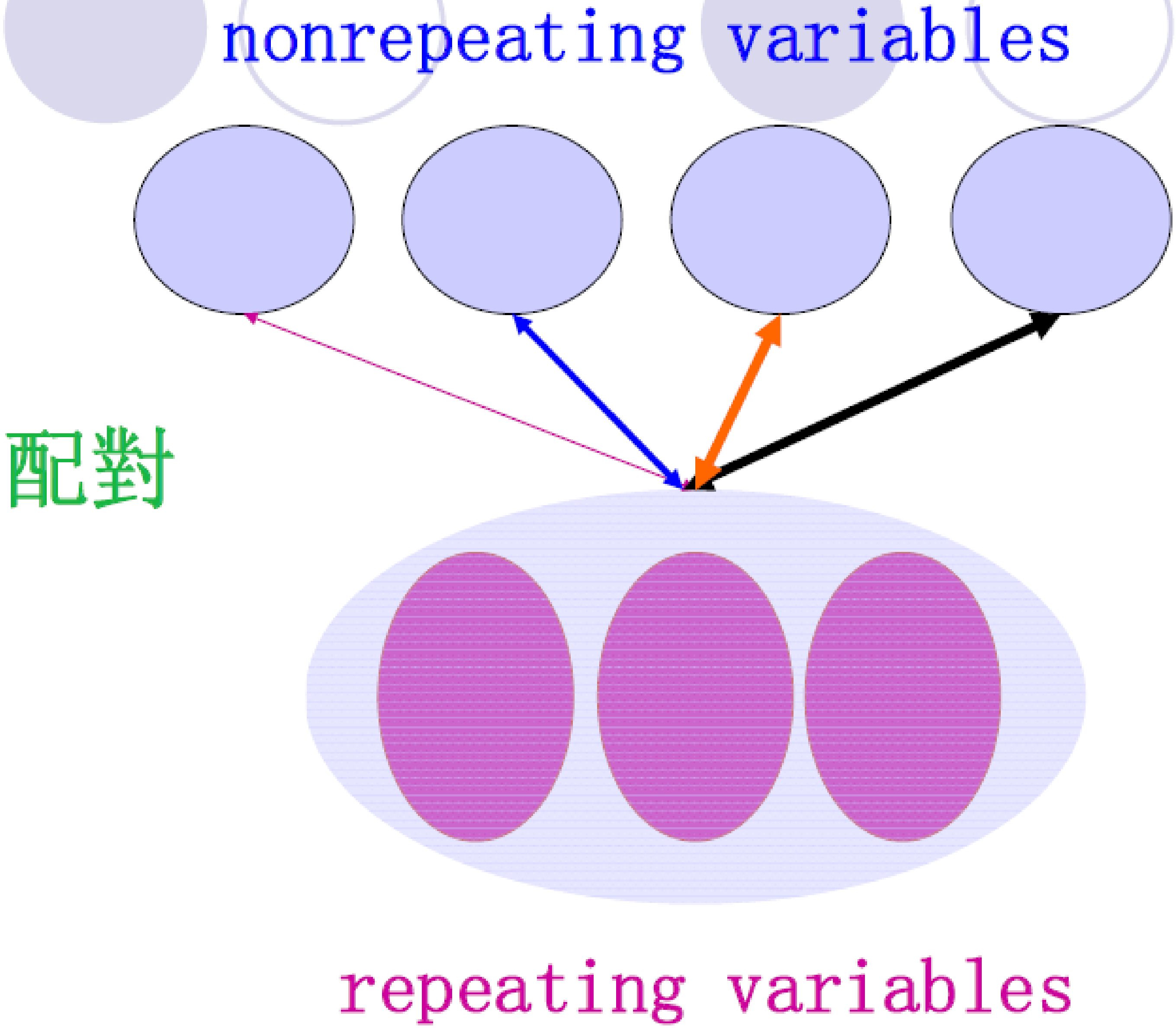
Contd…

Step 5. Form a pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless. 1

⟹ Set up dimensional equations, combining the variables selected in Step 4 with each of the other variables (nonrepeating variables) in turn, to form dimensionless groups or dimensionless product.

⟹ There will be k – r equations.

Example: For pressure drop per unit length



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Step 6. Repeat Step 5 for each of the remaining nonrepeating variables

Step 7. Check all the resulting pi terms to make sure they are dimensionless.

⟹ Check to see that each group obtained is dimensionless.

Step 8. Express the final form as a relationship among the pi terms, and think about what is means.

⟶ Express the result of the dimensional analysis.

$\Pi 1 = \varphi(\Pi 2 , \Pi 3 ,,,, \Pi k-r)$

Dimensionless parameters

- Reynolds Number $$R = \frac{\rho V l}{\mu}$$
- Froude Number $$F = \frac{V}{\sqrt{gl}}$$
- Weber Number $$W = \frac{V^2 l \rho}{\sigma}$$
- Mach Number $$M = \frac{V}{c}$$
- Pressure Coefficient $$C_p = \frac{-2\Delta p}{\rho V^2} \qquad C_d = \frac{2\text{Drag}}{\rho V^2 A}$$
  - (the dependent variable that we measure experimentally)

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Application of Dimensionless Parameters

- Pipe Flow
- Pump characterization
- Model Studies and Similitude
  - dams: spillways, turbines, tunnels
  - harbors
  - rivers
  - ships
  - ...

Example: Pipe Flow

- What are the important forces?
  Inertial, viscous. Therefore Reynolds number.
- What are the important geometric parameters? diameter, length, roughness height
  - Create dimensionless geometric groups
    l/D, ε/D
- Write the functional relationship

$$C_p = f\left(R, \frac{l}{D}, \frac{\varepsilon}{D}\right)$$

Contd…

- How will the results of dimensional analysis guide our experiments to determine the relationships that govern pipe flow?
- If we hold the other two dimensionless parameters constant and increase the length to diameter ratio, how will C_p change?

C_p proportional to l

$$C_p \frac{D}{l} = f\left(\frac{\varepsilon}{D}, \mathbf{R}\right)$$

$$C_p = \frac{-2\Delta p}{\rho V^2}$$

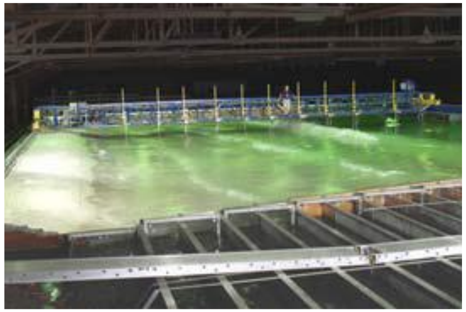
$$f = \left(C_p \frac{D}{l}\right) = f\left(\frac{\varepsilon}{D}, \mathbf{R}\right)$$

f is friction factor

Modeling and Similitude

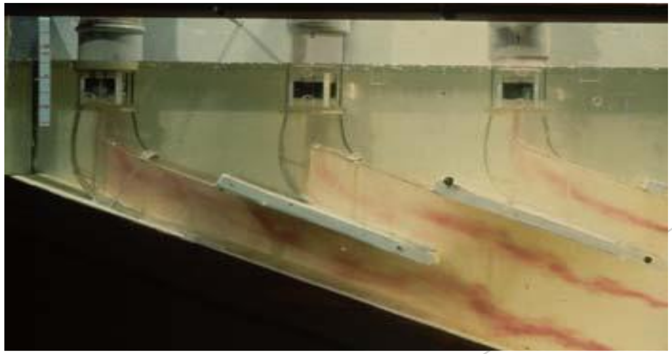
To develop the procedures for designing models so that the model and prototype will behave in a similar fashion…….

Spillway design



Sediment transport

hydropower station



Pump intake

Model vs. Prototype

- Model ? A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect.
- Mathematical or computer models may also conform to this definition, our interest will be in physical model.
- Prototype? The physical system for which the prediction are to be made.
- Models that resemble the prototype but are generally of a different size, may involve different fluid, and often operate under different conditions.
- Usually a model is smaller than the prototype.
- Occasionally, if the prototype is very small, it may be advantageous to have a model that is larger than the prototype

Contd…

- With the successful development of a valid model, it is possible to predict the behavior of the prototype under a certain set of conditions.
- There is an inherent danger in the use of models in that predictions can be made that are in error and the error not detected until the prototype is found not to perform as predicted.
- It is imperative that the model be properly designed and tested and that the results be interpreted correctly.

Similarity of Model and Prototype

- What conditions must be met to ensure the similarity of model and prototype?
- **Geometric Similarity**
  - ⟶ Model and prototype have same shape.
  - ⟶ Linear dimensions on model and prototype correspond within constant scale factor
- **Kinematic Similarity**
  - ⟶ Velocities at corresponding points on model and prototype differ only by a constant scale factor.
- **Dynamic Similarity**
  - ⟶ Forces on model and prototype differ only by a constant scale factor.

Validation of Models Design

- The purpose of model design is to predict the effects of certain proposed changes in a given prototype, and in this instance some actual prototype data may be available.

Validation of model design ?

- The model can be designed, constructed, and tested, and the model prediction can be compared with these data. **If the agreement is satisfactory**, then the model can be changed in the desired manner, and the corresponding effect on the prototype can be predicted with increased confidence.

Distorted Models

- In many model studies, to achieve dynamic similarity requires duplication of several dimensionless groups.
- In some cases, complete dynamic similarity between model and prototype may not be attainable. If one or more of the similarity requirements are not met, for example, if $\Pi 2 \neq \Pi 2m$, then it follows that the prediction equation is not true; that is,
- MODELS for which one or more of the similarity requirements are not satisfied are called **DISTORTED MODELS**.

Contd…

- Determine the drag force on a surface ship, complete dynamic similarity requires that both Reynolds and Froude numbers be duplicated between model and prototype.

$$Fr_m = \frac{V_m}{(g\ell_m)^{1/2}} = Fr_p = \frac{V_p}{(g\ell_p)^{1/2}}$$ **Froude numbers**

$$Re_m = \frac{V_m \ell_m}{\upsilon_m} = Re_p = \frac{V_p \ell_p}{\upsilon_p}$$ **Reynolds numbers**

- To match Froude numbers between model and prototype $\longrightarrow \dfrac{V_m}{V_p} = \left(\dfrac{\ell_m}{\ell_p}\right)^{1/2}$

Contd…

- To match Reynolds numbers between model and prototype

$$\frac{\upsilon_m}{\upsilon_p} = \frac{V_m}{V_p}\frac{\ell_m}{\ell_p} \Rightarrow \frac{\upsilon_m}{\upsilon_p} = \left(\frac{\ell_m}{\ell_p}\right)^{1/2} \quad \frac{\ell_m}{\ell_p} = \left(\frac{\ell_m}{\ell_p}\right)^{3/2}$$

If ℓ_m / ℓ_p equals 1/100(a typical length scale for ship model tests) , then υ_m / υ_p must be 1/1000.

>>> The kinematic viscosity ratio required to duplicate Reynolds numbers cannot be attained.

Contd…

- It is impossible in practice for this model/prototype scale of 1/100 to satisfy both the Froude number and Reynolds number criteria; **at best we will be able to satisfy only one of them.**
- If water is the only practical liquid for most model test of free-surface flows, a **full-scale test is required to obtain complete dynamic similarity**.
- In the study of open channel or free-surface flows. Typically in these problems both the Reynolds number and Froude number are involved

$$Fr_m = \frac{V_m}{(g_m \ell_m)^{1/2}} = Fr_p = \frac{V_p}{(g_p \ell_p)^{1/2}}$$ **Froude numbers**

$$Re_m = \frac{\rho_m V_m \ell_m}{\mu_m} = Re_p = \frac{\rho_p V_p \ell_p}{\mu_p}$$ **Reynolds numbers**

❖ To match Froude numbers between model and prototype $\rightarrow \frac{V_m}{V_p} = \left(\frac{\ell_m}{\ell_p}\right)^{1/2} = \sqrt{\lambda_\ell}$

Contd…

- To match Reynolds numbers between model and prototype

$$\frac{V_m}{V_p} = \frac{\mu_m}{\mu_p}\frac{\rho_p}{\rho_m}\frac{\ell_p}{\ell_m} \Rightarrow \frac{V_m}{V_p} = \sqrt{\frac{\ell_m}{\ell_p}} = \frac{\mu_m/\mu_p}{\rho_p/\rho_m}\frac{\ell_p}{\ell_m}$$

$$\lambda_\ell^{3/2} = \frac{\mu_m/\mu_p}{\rho_p/\rho_m} = \frac{\upsilon_m}{\upsilon_p}$$

If ℓ_m/ℓ_p equals 1/100(a typical length scale for ship model tests) , then υ_m/υ_p must be 1/1000.

>>>The kinematic viscosity ratio required to duplicate Reynolds numbers cannot be attained.

Scaling in Open Hydraulic Structures

- Examples
  - spillways
  - channel transitions
  - weirs

NCHRP Request For Proposal on "Effects of Debris on Bridge-Pier Scour"

- Important Forces
  - inertial forces
  - gravity: from changes in water surface elevation
  - viscous forces (often small relative to gravity forces)

$$F = \frac{V}{\sqrt{gl}}$$

- Minimum similitude requirements
  - geometric
  - Froude number

$$R = \frac{\rho V l}{\mu}$$

Froude similarity

Froude similarity

$$F = \frac{V}{\sqrt{gl}} \qquad F_m = F_p$$

- Froude number the same in model and prototype

$$\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$$

- difficult to change g

$$\frac{V_m^2}{L_m} = \frac{V_p^2}{L_p}$$

- define length ratio (usually larger than 1)

$$L_r = \frac{L_p}{L_m}$$

- velocity ratio $V_r = \sqrt{L_r}$
- time ratio $t_r = \frac{L_r}{V_r} = \sqrt{L_r}$
- discharge ratio $Q_r = V_r A_r = \sqrt{L_r}\, L_r L_r = L_r^{5/2}$
- force ratio $F_r = M_r a_r = \rho_r L_r^3 \frac{L_r}{t_r^2} = L_r^3$

Reynolds and Froude Similarity?

Reynolds

$$R = \frac{\rho V l}{\mu}$$

$$\frac{\rho_m V_m l_m}{\mu_m} = \frac{\rho_p V_p l_p}{\mu_p}$$

Water is the only practical fluid

$$V_m l_m = V_p l_p$$

$$\frac{V_p}{V_m} = \frac{l_m}{l_p}$$

$$V_r = \frac{1}{L_r}$$

Froude

$$F = \frac{V}{\sqrt{gl}}$$

$$V_r = \sqrt{L_r}$$

$$\frac{1}{L_r} = \sqrt{L_r}$$

$$L_r = 1$$

Closed Conduit Incompressible Flow

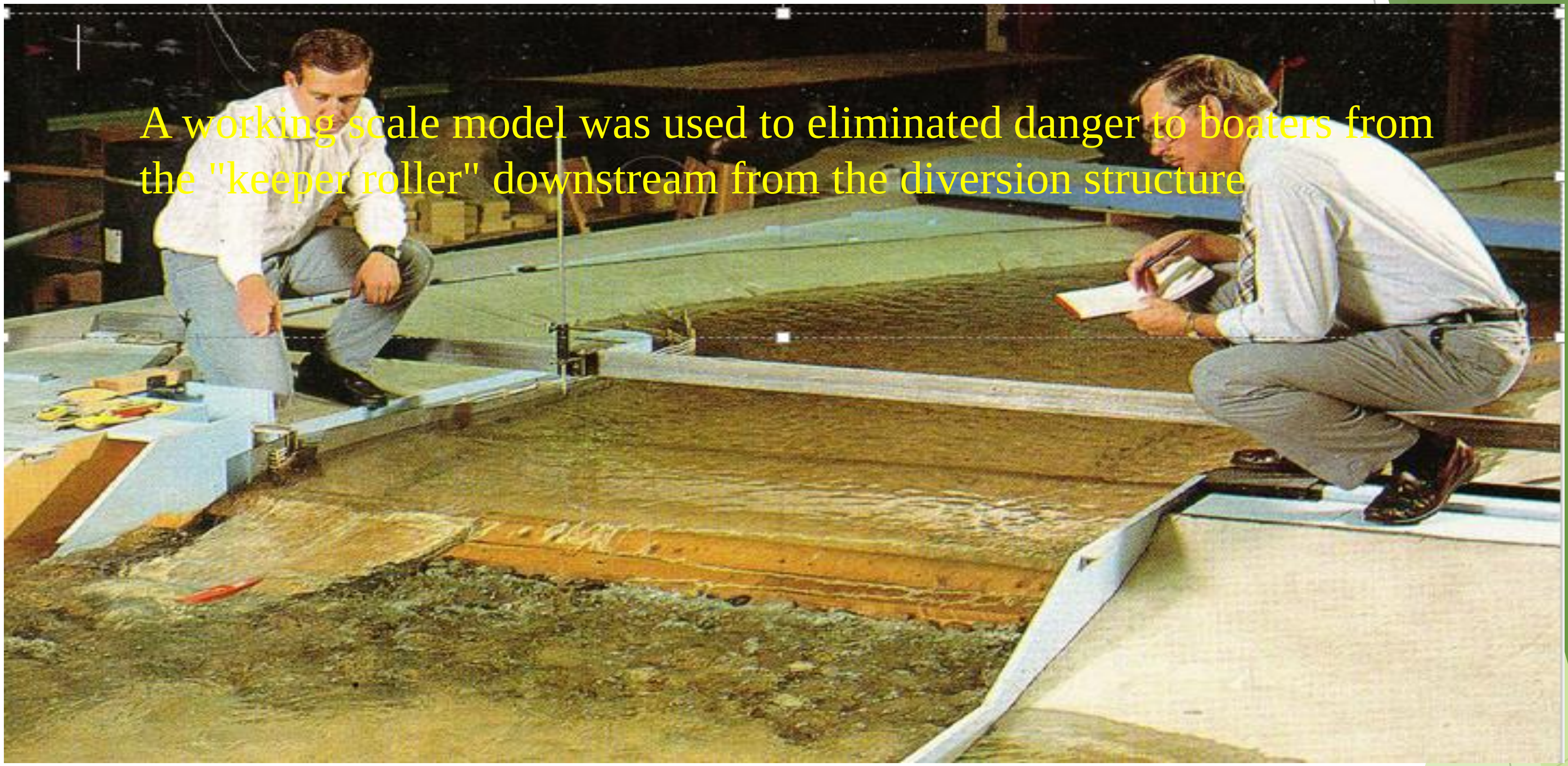
- Forces
  - _viscosity_
  - _inertia_
- If same fluid is used for model and prototype
  - *VD* must be the same
  - Results in high _velocity_ in the model
- High Reynolds number (**R**)
  - Often results are independent of **R** for very high **R**

Hydraulic Machinery: Pumps

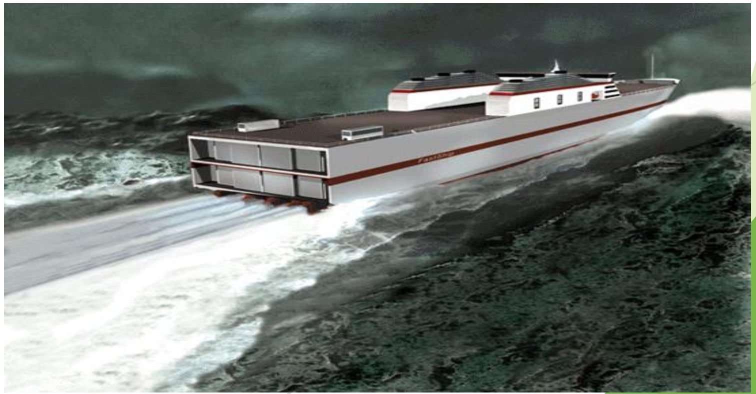
- Rotational speed of pump or turbine is an additional parameter
  - additional dimensionless parameter is the ratio of the rotational speed to the velocity of the water <u>streamlines must be geometrically similar</u>
  - homologous units: velocity vectors scale $V_r = l_r$
- Now we can't get same Reynolds Number!
  - Reynolds similarity requires $V_r = \frac{1}{l_r}$
  - Scale effects

Applications

Port Model



Ship’s Resistance



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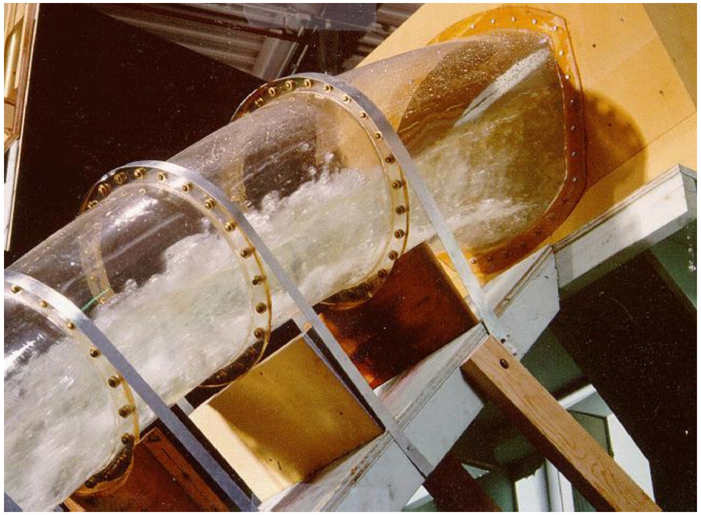
Hoover Dam Spillway

A 1:60 scale hydraulic model of the tunnel spillway at Hoover Dam for investigation of cavitation damage preventing air slots



Irrigation Canal Controls

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Spillways

Dams