Chapter 2
Dimensional analysis, similitude and Hydraulic models

1. Introduction

Hydraulic engineering structures or machines can be designed using

i) pure theory
ii) empirical methods
iii) semi-empirical methods which are mathematical formulations based on theoretical concepts supported by suitably designed experiments or
iv) Physical models
v) Mathematical models.

The pure theoretical approach in hydraulic engineering is limited to a few cases of laminar flow, for example the Hagen Poisseuille equation for the hydraulic gradient in the laminar flow of an incompressible fluid in a circular pipeline. Empirical methods are based on corrections between observed variables affecting a particular physical system. Such relationships should only be used under similar circumstances to those under which the data were collected. Due to the inability to express the physical interaction of the parameters involved in mathematical terms some such methods are still in use. One well-known example is in the relationship between wave height, fetch, wind speed and duration for the forecasting of ocean wave characteristics.

A good example of semi-empirical relationship is the Colebrook – White equation for the friction factors in turbulent flow in pipes. This was obtained from theoretical concepts and experiments designed on the basis of dimensional analysis, it is universally applicable to all Newtonian fluids.

Dimensional analysis also forms the basis for the design and operation of physical scale models which are used to predict the behavior of their full sized counterparts called 'prototype'. Such models, which are generally geometrically similar to the prototype, are used in the design of aircraft, ships, submarines, pumps, turbines, harbours, breakwaters, river and estuary engineering works, spillways, etc.

While mathematical modeling techniques have progressed rapidly due to the advent of high-speed digital computers, enabling the equations of motion coupled with semi-empirical relationships to be solved for complex flow situations such as pipe network analysis, pressure transients in pipelines, unsteady flows in rivers and estuaries, etc., there are many cases, particularly where localized flow patterns cannot be mathematically modeled, when physical model are still needed.

Without the technique of dimensional analysis experimental and computational progress in fluid mechanics would have been considerably retarded.

1.1 Dimensional analysis

Any physical situation can be described by certain familiar properties e.g. length, velocity, area, volume, acceleration etc. These are all known as dimensions.
Of course dimensions are of no use without a magnitude being attached. We must know more than that something has a length. It must also have a standardised unit - such as a meter, a foot, a yard etc. Dimensions are properties which can be measured. Units are the standard elements we use to quantify these dimensions. In engineering the application of fluid mechanics in designs make much of the use of empirical results from a lot of experiments. This data is often difficult to present in a readable form. Even from graphs it may be difficult to interpret. Dimensional analysis provides a strategy for choosing relevant data and how it should be presented.
This is a useful technique in all experimentally based areas of engineering. If it is possible to identify the factors involved in a physical situation, dimensional analysis can form a relationship between them. The resulting expressions may not at first sight appear rigorous but these qualitative results converted to quantitative forms can be used to obtain any unknown factors from experimental analysis.

The basis of dimensional analysis is to condense the number of separate variables involved in a particular type of physical system into a smaller number of non-dimensional groups of the variables. The arrangement of the variables in the groups is generally chosen so that each group has a physical significance.

All physical parameters can be expressed in terms of a number of basic dimensions, in engineering the basic dimensions, mass (M), length (L) and time (T) are sufficient for this purpose. For example, velocity = distance/time (=LT^{-1}), discharge = volume/time (=L^3T^{-1}). Force is expressed using Newton's law of motion (Force = mass * acceleration). Hence Force = MLT^{-2}

A list of some physical quantities with their dimensional forms can be seen below

Physical Quantity	Symbol	Dimensional Form
Length	ℓ	L
Time	t	T
Mass	m	M
Velocity	V	LT^{-1}
Acceleration	a	LT^{-2}
Discharge	Q	L^3T^{-1}
Force	F	MLT^{-2}
Pressure	p	$ML^{-1}T^{-2}$
Power	P	ML^2T^{-3}
Density	ρ	ML^{-3}
Dynamic viscosity	μ	$ML^{-1}T^{-1}$
Kinematic viscosity	ν	L^2T^{-1}
Surface tension	σ	MT^{-2}
Bulk modulus of elasticity	K	$ML^{-1}T^{-2}$

1.2 The Need for Dimensional Analysis

As long as Dimensional analysis is a process of formulating fluid mechanics problems in terms of non-dimensional variables and parameters. It is useful for:
1. Reduction in Variables:
If F(A1, A2, … , An) = 0, Then F(Π1, Π1, … Πr < n)=0
F = functional form
Ai = dimensional variables
Πj = non dimensional parameters = Πj (Ai) i.e., Πj consists of non dimensional groupings of Ai's.
Thereby reduces number of experiments and/or simulations required to determine f vs. F
2. Helps in understanding physics
3. Useful in data analysis and modeling
4. Fundamental to concept of similarity and model testing
Enables scaling for different physical dimensions and fluid properties

1.3 Physical significance of non-dimensional groups

The main components of force which may act on a fluid element are those due to viscosity, gravity, pressure, surface tension and elasticity. The resultant of these components is called the inertial force and the ratio of this force to each of the force components indicates the relative importance of the force types in a particular flow system.

For example the ration of inertia force to viscous force,

$$\frac{F_i}{F_\mu} = \frac{\rho L^3 L T^{-2}}{\tau L^2}$$

Now $\tau = \mu \frac{dV}{dy} = \mu L T^{-1} L^{-1}$

Hence $\frac{F_i}{F_\mu} = \frac{\rho L^2 T^{-1}}{\mu} = \frac{\rho L V}{\mu} = \frac{\rho l V}{\mu}$

Where l is a typical length dimension of the particular system.

The dimensionless term $\frac{\rho l V}{\mu}$ is in the form of the Reynolds number.

Low Reynolds numbers indicate a significant dominance of viscous forces in the system which explains why this non-dimensional parameters may be used to identify the regime of flow, i.e. whether laminar or turbulent.
Similarly it can be shown that the Froude number is the ratio of inertial force to gravity force in the form

$F_r = \frac{V^2}{gl}$ (but usually expressed as $F_r = \frac{V}{\sqrt{gl}}$)

The Weber number, we is the ratio of inertial to surface tension force and is expressed by

$$\frac{V}{\sqrt{\sigma/\rho l}}$$

1.4 *Dimensional Homogeneity*

Any equation describing a physical situation will only be true if both sides have the same dimensions. That is it must be **dimensionally homogenous**.

For example the equation which gives for over a rectangular weir (derived earlier in this module) is,

$$Q = \frac{2}{3} B\sqrt{2g} H^{3/2}$$

The SI units of the left hand side are m^3s-1. The units of the right hand side must be the same. Writing the equation with only the SI units gives

$$\begin{aligned} m^3 s^{-1} &= m\left(m\, s^{-2}\right)^{1/2} m^{3/2} \\ &= m^3 s^{-1} \end{aligned}$$

i.e. the units are consistent.

To be more strict, it is the dimensions which must be consistent (any set of units can be used and simply converted using a constant). Writing the equation again in terms of dimensions,

$$\begin{aligned} L^3 T^{-1} &= L\left(L\, T^{-2}\right)^{1/2} L^{3/2} \\ &= L^3 T^{-1} \end{aligned}$$

Notice how the powers of the individual dimensions are equal, (for L they are both 3, for T both -1).

This property of dimensional homogeneity can be useful for:

1. Checking units of equations;
2. Converting between two sets of units;
3. Defining dimensionless relationships (see below).

1.5 Methods of Dimensional Analysis

1.5.1The Buckingham Π theorem

This states that the n quantities Q_1, Q_2, … Q_n, involved in a physical system can be arranged in (n – m) non–dimensional groups of the quantities where m is the number of basic dimensions required to express the quantities in dimensional form.

Thus f_1 (Q_1, Q_2, …, Q_n) = 0 can be expressed as f_2 ($\pi_1, \pi_2, ...\pi_{n-m}$) or Φ($\pi_1, \pi_2, ...\pi_{n-m}$) where 'f' means a function of …'. Each π term basically contains m repeated quantities which together contain the m basic dimensions together with one other quantity. In fluid mechanics m=3 and therefore each π term basically contains four of the quantity terms.

Choice of repeating variables

Repeating variables are those which we think will appear in all or most of the Π groups, and are a influence in the problem. Before commencing analysis of a problem one must choose the repeating variables. There is considerable freedom allowed in the choice.

Some rules which should be followed are

i. From the 2nd theorem there can be **n** (= 3) repeating variables.

ii. When combined, these repeating variables variable must contain all of dimensions (M, L, T) (That is not to say that each must contain M,L and T).

iii. A combination of the repeating variables must not form a dimensionless group.

iv. The repeating variables do not have to appear in all Π groups.

v. The repeating variables should be chosen to be measurable in an experimental investigation. They should be of major interest to the designer. For example, pipe diameter (dimension L) is more useful and measurable than roughness height (also dimension L).

In fluids it is usually possible to take r**, u** and **d** as the threee repeating variables.

This freedom of choice results in there being many different p groups which can be formed - and all are valid. There is not really a wrong choice.

Manipulation of the Π groups

Taking the defining equation as: Φ*(* Π*1* , Π *2* , Π *3* ... Π *m-n)* = 0, in this way it will be manipulated.

i. Any number of groups can be combined by multiplication or division to form a new group which replaces one of the existing. E.g. Π1 and Π 2 may be combined to form Π 1a = Π 1 / Π 2 so the defining equation becomes Φ(Π 1a , Π 2 , Π 3 ... Π m-n) = 0.

ii. The reciprocal of any dimensionless group is valid. So $\Phi(\Pi 1, 1/\Pi 2, \Pi 3 \ldots 1/\Pi m\text{-}n) = 0$ is valid.
iii. Any dimensionless group may be raised to any power. So $\Phi((\Pi 1)^2, (\Pi 2)^{1/2}, (\Pi 3)^3 \ldots \Pi m\text{-}n) = 0$ is valid.
iv. Any dimensionless group may be multiplied by a constant.
v. Any group may be expressed as a function of the other groups, e.g. $\Pi 2 = f(\Pi 1, \Pi 3 \ldots \Pi m\text{-}n)$

In general the defining equation could look like

$$\Phi(\Pi 1, 1/\Pi 2, (\Pi 3)^i \ldots 0.5\,\Pi m\text{-}n) = 0$$

1.5.2. Rayleigh Method

In this method of dimensional analysis a functional relationship of some variables is expressed in the form of an exponential equation which must be dimensionally homogenous.

If the number of independent out variables becomes more than four, then it is difficult to find expressions for the dependant variable

Let X be a function of independent variables x1, x2, x3. Then according to Rayleigh's method: X = f [x1, x2, x3]

This can be written as : $X = k x_1^a . x_2^b . x_3^c$

Where, k = constant

a, b ,c = arbitrary powers

The values of a, b , c are obtained by comparing the powers of fundamental dimension on both sides.

The Rayleigh's method of dimensional analysis becomes more laborious if the variables are more than the number of fundamental dimensions (M, L, and T). This difficulty is overcome by Buckingham's Π Theorem.

1.6. Similitude and Model studies

Similitude, or dynamic similarity, between two geometrically similar systems exist when the ratios of inertial force to the individual force components in the first system are the same as the corresponding points in space. Hence for absolute dynamic similarity the Reynolds, Froude and Weber numbers must be the same in the two systems. If this can be achieved the flow patterns will be geometrically similar, i.e. kinematic similarity exists.

In using physical scale models to predict the behaviour of prototype systems or designs it is rarely possible (except when only one force type is relevant) to achieve simultaneous equality of the predominant force , strict dynamic similarity is thus not achieved resulting in 'scale effect'.

Reynolds modeling is adopted for studies of flows without a free surface such as pipe flow and flow around submerged bodies, e.g. aircraft, submarines, vehicles and buildings. The Froude number becomes the governing parameter in flows with a free surface since gravitational forces are predominant. Hydraulic structures, including spillways, weirs and stilling basins, rivers and estuaries, hydraulic turbines and pumps and wave-making resistance of ships are modeled according to the Froude law.

Geometric similarity

Geometric similarity exists between model and prototype if the ratio of all corresponding dimensions in the model and prototype are equal.

$$\frac{Lp}{Lm} = \frac{Dp}{Dm} = \frac{Hp}{Hm} = Lr = \text{Length ratio}$$

Where, Lp and Lm = Length of the prototype and Length of the model

Dp and Dm = Diameter/Depth of the prototype and the diameter/depth of the model

Hp and Hm = Height of the prototype and the height of the model

$$\frac{Ap}{Am} = \frac{LpDp}{LmDm} = Lr^2$$

$$\frac{Vp}{Vm} = \frac{Lp\ Dp\ Hp}{Lm\ Dm\ Hm} = Lr^3$$

Where, Ap & Vp = Area and Volume of the prototype

Am & Vm = Area and Volume of the model

Kinematic similarity

Kinematic similarity is the similarity of time as well as geometry. It exists between model and prototype

i. If the paths of moving particles are geometrically similar

ii. If the rations of the velocities of particles are similar

$$\frac{V_{P_1}}{V_{m_1}} = \frac{V_{P_2}}{V_{m_2}} = V_r$$

$$\frac{a_{P_1}}{a_{m_1}} = \frac{a_{P_2}}{a_{m_2}} = a_r$$

Where Vr is the velocity ratio, ar is the acceleration ratio.
Vp1 = Velocity of fluid at point 1 in prototype,
Vp2 = Velocity of fluid at point 2 in prototype,
ap1 = Acceleration of fluid at 1 in prototype,
ap2 = Acceleration of fluid at 2 in prototype, and
Vm1, Vm2 , am1, am2 = Corresponding values at the corresponding points of fluid Velocity and acceleration in the model
Also the directions of the velocities in the model and prototype should be same

Dynamic similarity

Dynamic similarity exists between geometrically and kinematically similar systems if the ratios of all forces in the model and prototype are the same.
Then for dynamic similarity, we have

$$\frac{(F_i)_P}{(F_i)_m} = \frac{(F_v)_P}{(F_v)_m} = \frac{(F_g)_P}{(F_g)_m} \dots = F_r$$

Fr = the force ratio
(Fi)p = Inertia force at a point in prototype,
(Fv)p = Viscous force at the point in prototype,
(Fg)p = Gravity force at the point in prototype,
(Fi)m, (Fv)m, (Fg)m = Corresponding values of forces at the corresponding point in model
Also the directions of the corresponding forces at the corresponding points in the model and prototype should be same

Model laws and similarity laws

Hence models are designed on the basis of ratio of the force, which is dominating in the phenomenon
The laws on which the models are designed for dynamic similarity are called **model laws or laws of similarity**
The followings are the model laws:
1) **Reynold's model law**: is the law in which models are based on Reynolds's number, which states that the Reynold number for the model must be equal to the Reynolds number for the prototype. Models based on Reynold's number include:
(i) Pipe flow
(ii) Resistance experienced by sub-marines, airplanes, fully immersed bodies etc.
fluid flow problems where viscous forces alone are predominant, the models are designed for dynamic similarity on Reynolds law. Then according to Reynold's model law,

$$\frac{\rho_r \cdot V_r \cdot L_r}{\mu_r} = 1 \qquad \text{where } \rho_r = \frac{\rho_P}{\rho_m}, V_r = \frac{V_P}{V_m} \text{ and } L_r = \frac{L_P}{L_m}, \frac{\mu_P}{\mu_m} = \mu_r$$

V_m = Velocity of fluid in model,
ρ_m = Density of fluid in model,
L_m = Length or linear dimension of the model,
μ_m = Viscosity or fluid in model,

and V_P, ρ_P, L_P and μ_P are the corresponding values of velocity, density, linear dimension and viscosity of fluid in prototype

ρ_r, V_r, L_r and μ_r are called the scale ratios for density, velocity, linear dimension and viscosity

The scale ratios for time, acceleration, force and discharge for Reynold's model law are obtained as

$$t_r = \text{Time scale ratio} = \frac{L_r}{V_r}$$

$$a_r = \text{Acceleration scale ratio} = \frac{V_r}{t_r}$$

$$F_r = \text{Force scale ratio} = (\text{Mass} \times \text{Acceleration})_r$$
$$= m_r \times a_r = \rho_r A_r V_r \times a_r$$
$$= \rho_r L_r^2 V_r \times a_r$$

Froude model law: the law in which the models are based on Froude number, which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal

- Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia
- Froude model law is applied in the following fluid flow problems:

i) Free surface flows such as flow over spillways, weirs, sluices, channels etc.,

ii) Flow of jet from an orifice or nozzle

iii) Where waves are likely to be formed on surface

iV) Where fluids of different densities flow over one another

- Hydraulic structures, including spillways, weirs and stilling basins, rivers and estuaries, hydraulic turbines and pumps and wave-making resistance of ships are modeled according to the Froude law.

According to Froude model law:

$$(F_e)_{model} = (F_e)_{prototype} \qquad \text{or} \qquad \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_P}{\sqrt{g_P L_P}}$$

V_m = Velocity of fluid in model,
L_m = Linear dimension or length of model,

g_m = Acc. due to gravity at a place where model is tested, V_P, L_P and g_P are the corresponding values of the velocity, length and acceleration due to gravity for the prototype. Scale ratios for various physical quantities based on Froude model law are:

(a) Scale ratio for time

$$T_r = \frac{T_P}{T_m} = \frac{\left(\frac{L}{V}\right)_P}{\left(\frac{L}{V}\right)_m} = \frac{\frac{L_P}{V_P}}{\frac{L_m}{V_m}} = \frac{L_P}{L_m} \times \frac{V_m}{V_P} = L_r \times \frac{1}{\sqrt{L_r}}$$
$$= \sqrt{L_r}.$$

(b) Scale ratio for acceleration:
Acceleration is given by

$$= \frac{V}{T}$$
$$a_r = \frac{a_P}{a_m} = \frac{\left(\frac{V}{T}\right)_P}{\left(\frac{V}{T}\right)_m} = \frac{V_P}{T_P} \times \frac{T_m}{V_m} = \frac{V_P}{V_m} \times \frac{T_m}{T_P}$$
$$= \sqrt{L_r} \times \frac{1}{\sqrt{L_r}}$$
$$= 1.$$

(c) Scale ratio for discharge

$$Q = A \times V = L^2 \times \frac{L}{T} = \frac{L^3}{T}$$
$$Q_r = \frac{Q_P}{Q_m} = \frac{\left(\frac{L^3}{T}\right)_P}{\left(\frac{L^3}{T}\right)_m} = \left(\frac{L_P}{L_m}\right)^3 \times \left(\frac{T_m}{T_P}\right) = L_r^{\,3} \times \frac{1}{\sqrt{L_r}} = L_r^{\,2.5}$$

(d) Scale ratio for force

$$\text{Force} = \text{Mass} \times \text{Acc.} = PL^3 \times \frac{V}{T} = PL^2 . \frac{L}{T} . V = \rho L^2 V^2$$

$$F_r = \frac{F_P}{F_m} = \frac{\rho_P L_P^{\,2} V_P^2}{\rho_m L_m^{\,2} V_m^2} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2.$$

If the fluid used in model and prototype is same, then

$$\frac{\rho_P}{\rho_m} = 1 \text{ or } \rho_P = \rho_m$$

$$F_r = \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2 = L_r^{\,2} \times (\sqrt{L_r})^2 = L_r^{\,2} \,.\, L_r = L_r^{\,3}$$

(e) Scale ratio for pressure intensity

$$p = \frac{\text{Force}}{\text{Area}} = \frac{\rho L^2 V^2}{L^2} = \rho L^2$$

Pressure ratio:

$$p_r = \frac{p_P}{p_m} = \frac{\rho_P V_P^2}{\rho_m V_m^2}$$

$$\rho_P = \rho_m$$

$$p_r = \frac{V_P^2}{V_m^2} = \left(\frac{V_P}{V_m}\right)^2 = L_r.$$

(f) Scale ratio for work, energy, torque, moment etc.

$$\text{Torque} = \text{Force} \times \text{Distance} = F \times L$$

$$T_r^* = \frac{T_P^*}{T_m^*} = \frac{(F \times L)_P}{(F \times L)_m} = F_r \times L_r = L_r^{\,3} \times L_r = L_r^{\,4}$$

(g) Scale ratio for Power

$$\text{Power} = \text{Work per unit time}$$

$$= \frac{F \times L}{T}$$

$$\rho_r = \frac{\rho_P}{\rho_m} = \frac{\frac{F_P \times L_P}{T_P}}{\frac{F_m \times L_m}{T_m}} = \frac{F_P}{F_m} \times \frac{L_P}{L_m} \times \frac{1}{\frac{T_P}{T_m}}$$

$$= F_r \cdot L_r \cdot \frac{1}{T_r} = L_r^{\ 3} \cdot L_r \cdot \frac{1}{\sqrt{L_r}} = L^{3.5}.$$

3) **Euler model law:** is the law in which the models are designed on Euler's number which means for dynamic similarity between the model and prototype, the Euler number for model and prototype should be equal.

- Euler's model law is applicable when the pressure forces are alone predominant in addition to the inertia force. According to this law:

$$(E_u)_{\text{model}} = (E_u)_{\text{prototype}}$$

$$\frac{V_m}{\sqrt{p_m/\rho_m}} = \frac{V_P}{\sqrt{p_P/\rho_P}} ,$$

If fluid is same in model and prototype, then the above equation becomes:

$$\frac{V_m}{\sqrt{p_m}} = \frac{V_P}{\sqrt{p_P}}$$

Where,

$(E_u)_{\text{model}} = (E_u)_{\text{prototype}}$

V_m = Velocity of fluid in model,

p_m = Pressure of fluid in model,

ρ_m = Density of fluid in model,

V_P, p_P, ρ_P = Corresponding values in prototype

Euler's model law is applied for fluid flow problems where flow is taking place in a closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension force is absent. This law is also used where the phenomenon of cavitations takes place

4) **Weber model law:** is the law in which models are based on Weber's number, which is the ratio of the square root of inertia force to surface tension force

Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between the model and prototype is obtained by equating the Weber number of the model and its prototype. Hence, according to this law:

$$(W_e)_{\text{model}} = (W_e)_{\text{prototype}}$$

$$\frac{V_m}{\sqrt{\sigma_m/\rho_m L_m}} = \frac{V}{\sqrt{\sigma_P/\rho_P L_P}}$$

V_m = Velocity of fluid in model,

σ_m = Surface tensile force in model,

ρ_m = Density of fluid in model,

L_m = Length of surface in model,

$V_P, \sigma_P, \rho_P, L_P$ = Corresponding values of fluid in prototype

Weber model law is applied in following cases:

1. Capillary rise in narrow passages,
2. Capillary movement of water in soil,
3. Capillary waves in channels,
4. Flow over weirs for small heads

5) Mach model law: is the law in which models are designed on Mach number, which is the ratio of the square root of inertia force to elastic force of a fluid

Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the Mach number of the model and its prototype

Hence according to this law:

$$(M)_{\text{model}} = (M)_{\text{prototype}}, \quad = \frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V}{\sqrt{K_P/\rho_P}}$$

where M = Mach number = $\frac{V}{\sqrt{K/\rho}}$

V_m = Velocity of fluid in model,

K_m = Elastic stress for model,

ρ_m = Density of fluid in model,

Mach model law is applied in the following cases

1. Flow of aero plane and projectile through air at supersonic speed, i.e., at a velocity more than the velocity of sound,
2. Aerodynamic testing,
3. Under water testing of torpedoes,
4. Water-hammer problems

Classification of models

The hydraulic models are classified as:

1. Undistorted models: Are models which are geometrically similar to their prototypes or in other words if the scale ratio for the linear dimensions of the model and its prototype is same. The behavior of the prototype can be easily predicted from the results of undistorted model

2. Distorted models: A model is said to be distorted if it is not geometrically similar to its prototype. For a distorted model different scale ratios for the linear dimensions are adopted

For example, in case of rivers, harbors, reservoirs etc. two different scale ratios, one for horizontal dimensions and other for vertical dimensions are taken

Thus the models of rivers, harbors and reservoirs will become as distorted models. If for the river, the horizontal and vertical scale ratios are taken to be same so that the model is undistorted, then the depth of water in the model of the river will be very-very small which may not be measured accurately

The followings are the advantages of distorted models:

I) The vertical dimensions of the model can be measured accurately
II) The cost of the model can be reduced
III) Turbulent flow in the model can be maintained

Though there are some advantages of the distorted model, yet the results of the distorted model cannot be directly transferred to its prototype. But sometimes from the distorted models very useful information can be obtained.

Examples

Example 1

Obtain an expression for the pressure gradient in a circular pipeline, of effective roughness, k, conveying an incompressible fluid of density, ρ, dynamic viscosity, μ, at a mean velocity, V, as a function of non-dimensional groups.

By comparison with the Drcy-Weisbach equations show that the friction factor is a function of relative roughness and the Reynolds number.

Solution

In full pipe flow gravity and surface tension forces do not influence the flow.

Let $\Delta p = pressure drop in a length L$

Then f1 $\left(\Delta p, L, \rho, V, D, \mu, k\right) = 0$

The repeating variables will be ρ, V and D. Δp clearly is not to be repeated since this variable is required to be expressed in terms of the other variables. If μ or k were to be repeated the relative effect of the parameter would be hidden.

f2($\pi_1, \pi_2, \pi_3, \pi_4$) = 0

Where $\alpha, \beta \, and \, \gamma$ are indices to be evaluated.

In dimensional form:

$\pi_1 = (ML^{-3})^{\alpha} L^{\beta} (LT^{-1})^{\gamma} ML^{-1}T^{-2}$

The sum of the indices of each dimension must be zero.

Thus for M, 0 = $\alpha + 1$, hence $\alpha = -1$;

For T, 0 = $-\gamma - 1$, hence $\gamma = -2$;

And for L, $0 = -3\alpha + \beta + \gamma - 1, hence \beta = 0$

Therefore, $\pi_1 = \dfrac{\Delta p}{\rho V^2}$

$$\pi_2 = \rho^{\alpha} D^{\beta} V^{\gamma} L$$

The π terms are dimensionless and hence D and L have the same dimensions the solution is

$$\pi_2 = \frac{L}{D}$$

Similarly, $\pi_3 = \dfrac{k}{D}$

$$\pi_4 = \rho^{\alpha} D^{\beta} V^{\gamma} \mu$$

$$\pi_4 = (ML^{-3})^{\alpha} L^{\beta} (LT^{-1})^{\gamma} ML^{-1}T^{-1}$$

Indices of M: 0 = $\alpha + 1$ $\alpha = -1$

Indices of T: 0 = $-\gamma - 1$ $\gamma = -1$

Indices of L: $0 = -3\alpha + \beta + \gamma - 1$; $\beta = -1$

Therefore, $\pi_4 = \dfrac{\mu}{\rho D V}$

Therefore, f2$\left(\dfrac{\Delta p D}{L \rho V^2}, \dfrac{k}{D}, \dfrac{\mu}{\rho D V}\right) = 0$

Hence $\dfrac{\Delta p}{L} = \dfrac{\rho V^2}{D} \Phi\left[\dfrac{k}{D}, R_e\right]$

Where Φ means 'a function of' the form of which is to be obtained experimentally.

The hydraulic gradient

$$\frac{\Delta h}{L} = \frac{\Delta p}{\rho g L}$$

Hence $\dfrac{\Delta h}{L} = \dfrac{V^2}{gD} \Phi\left[\dfrac{k}{D}, R_e\right]$

Comparing the above equation with the Darcy-Weisbach equation,

$\frac{h_f}{L} = \frac{\lambda V^2}{2gD}$, it is seen that λ is dimensionless and that

$$\lambda = \phi\left[\frac{k}{D}, R_e\right]$$

Example 2

An underwater missile, diameter 2m and length 10m is tested in a water tunnel to determine the forces acting on the real prototype. A 1/20th scale model is to be used. If the maximum allowable speed of the prototype missile is 10 m/s, what should be the speed of the water in the tunnel to achieve dynamic similarity?

For dynamic similarity the Reynolds number of the model and prototype must be equal:

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\left(\frac{\rho u d}{\mu}\right)_m = \left(\frac{\rho u d}{\mu}\right)_p$$

So the model velocity should be

$$u_m = u_p \frac{\rho_p}{\rho_m} \frac{d_p}{d_m} \frac{\mu_m}{\mu_p}$$

As both the model and prototype are in water then, $\mu_m = \mu_p$ and $\rho_m = \rho_p$ so

$$u_m = u_p \frac{d_p}{d_m} = 10 \frac{1}{1/20} = 200 m/s$$

Note that this is a **very** high velocity. This is one reason why model tests are not always done at exactly equal Reynolds numbers. Some relaxation of the equivalence requirement is often acceptable when the Reynolds number is high. Using a wind tunnel may have been possible in this example. If this were the case then the appropriate values of the r and m ratios need to be used in the above equation.

Exercises

1) Show that the discharge of a liquid through a rotodynamic pump having an impeller of diameter, D, and width, B, running at speed, N, when producing a total head, H, can be expressed in the form

$$Q = ND^3 \phi\left[\frac{D}{B}, \frac{N^2 D^2}{gH}, \frac{\rho N D^2}{\mu}\right]$$

2) A model aeroplane is built at 1/10 scale and is to be tested in a wind tunnel operating at a pressure of 20 times atmospheric. The aeroplane will fly at 500km/h. At what speed should the wind tunnel operate to give dynamic similarity between the model and prototype? If the drag measure on the model is 337.5 N. what will be the drag on the plane?