# 1 Basic Concepts of Structural Analysis

**1.1 Introduction.** This book describes matrix methods for the analysis of framed structures with the aid of a digital computer. Both the flexibility and stiffness methods of structural analysis are covered, but emphasis is placed upon the latter because it is more suitable for computer programming. While these methods are applicable to discretized structures of all types, only framed structures will be discussed. After mastering the analysis of framed structures, the reader will be prepared to study the finite element method for analyzing discretized continua (see Refs. C-1 through C-8).

In this chapter various preliminary matters are considered in preparation for the matrix methods of later chapters. These subjects include descriptions of the types of framed structures to be analyzed and their deformations due to loads and other causes. Also discussed are the basic concepts of equilibrium, compatibility, determinacy, mobility, superposition, flexibility and stiffness coefficients, equivalent joint loads, energy, and virtual work.

**1.2 Types of Framed Structures.** All of the structures that are analyzed in later chapters are called *framed structures* and can be divided into six categories: beams, plane trusses, space trusses, plane frames, grids, and space frames. These types of structures are illustrated in Fig. 1-1 and described later in detail. These categories are selected because each represents a class of structures having special characteristics. Furthermore, while the basic principles of the flexibility and stiffness methods are the same for all types of structures, the analyses for these six categories are sufficiently different in the details to warrant separate discussions of them.

Every framed structure consists of members that are long in comparison to their cross-sectional dimensions. The *joints* of a framed structure are points of intersection of the members, as well as points of support and free ends of members. Examples of joints are points *A*, *B*, *C*, and *D* in Figs. 1-1a and 1-1d. Supports may be *fixed*, as at support *A* in the beam of Fig. 1-1a, or *pinned*, as shown for support *A* in the plane frame of Fig. 1-1d, or there may be *roller supports*, illustrated by supports $B$ and $C$ in Fig. 1-1a. In special instances the connections between members or between members and supports may be elastic (or semi-rigid). However, the discussion of this possibility will be postponed until later (see Arts. 6.9 and 6.15). *Loads* on a framed structure may be concentrated forces, distributed loads, or couples.

Consider now the distinguishing features of each type of structure shown in Fig. 1-1. A *beam* (Fig. 1-1a) consists of a straight member having one or more points of support, such as points $A$, $B$, and $C$. Forces applied to a beam are assumed to act in a plane which contains an axis of symmetry of the cross section of the beam (an axis of symmetry is also a principal axis of the cross section). Moreover, all external couples acting on the beam have their moment vectors normal to this plane, and the beam deflects in the same plane (the *plane of bending*) and does not twist. (The case of a beam which does not fulfill these criteria is discussed in Art. 6.17.) Internal stress resultants may exist at any cross section of the beam and, in the general case, may include an axial force, a shearing force, and a bending moment.

A *plane truss* (Fig. 1-1b) is idealized as a system of members lying in a plane and interconnected at hinged joints. All applied forces are assumed to act in the plane of the structure, and all external couples have their moment vectors normal to the plane, just as in the case of a beam. The loads may consist of concentrated forces applied at the joints, as well as loads that act on the members themselves. For purposes of analysis, the latter loads may be replaced by statically equivalent loads acting at the joints. Then the analysis of a truss subjected only to joint loads will result in axial forces of tension or compression in the members. In addition to these axial forces, there will be bending moments and shear forces in those members having loads that act directly upon them.

A *space truss* (see Fig. 1-1c) is similar to a plane truss except that the members may have any directions in space. The forces acting on a space truss may be in arbitrary directions, but any couple acting on a member must have its moment vector perpendicular to the axis of the member. The reason for this requirement is that a truss member is incapable of supporting a twisting moment.

A *plane frame* (Fig. 1-1d) is composed of members lying in a single plane and having axes of symmetry in that plane (as in the case of a beam). The joints between members (such as joints $B$ and $C$) are rigid connections. The forces acting on a frame and the translations of the frame are in the plane of the structure; all couples acting on the frame have their moment vectors normal to the plane. The internal stress resultants acting at any cross section of a plane frame member may consist in general of a bending moment, a shearing force, and an axial force.

A *grid* is a plane structure composed of continuous members that either intersect or cross one another (see Fig. 1-1e). In the latter case the connec-

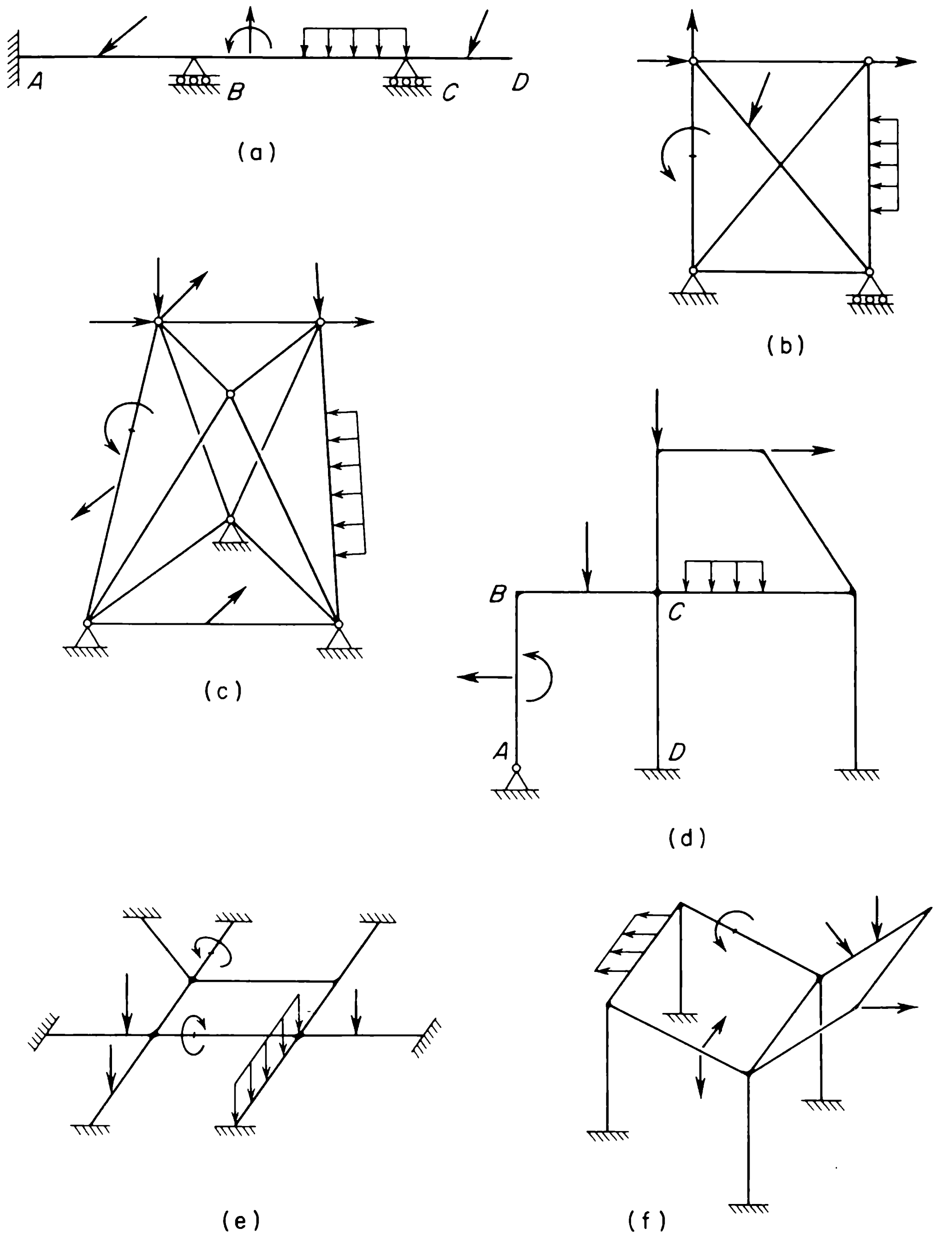


Fig. 1-1. Types of framed structures: (a) beam, (b) plane truss, (c) space truss, (d) plane frame, (e) grid, and (f) space frame.

tions between members are often considered to be hinged, whereas in the former case the connections are assumed to be rigid. While in a plane frame the applied forces all lie in the plane of the structure, those applied to a grid are normal to the plane of the structure; and all couples have their vectors in the plane of the grid. This orientation of loading may result in torsion as well as bending in some of the members. Each member is assumed to have two axes of symmetry in the cross section, so that bending and torsion occur independently of one another (see Art. 6.17 for a discussion of nonsymmetric members).

The final type of structure is a *space frame* (Fig. 1-1f). This is the most general type of framed structure, inasmuch as there are no restrictions on the locations of joints, directions of members, or directions of loads. The individual members of a space frame may carry internal axial forces, torsional moments, bending moments in both principal directions of the cross section, and shearing forces in both principal directions. The members are assumed to have two axes of symmetry in the cross section, as explained above for a grid.

It will be assumed throughout most of the subsequent discussions that the structures being considered have prismatic members; that is, each member has a straight axis and uniform cross section throughout its length. Nonprismatic members are treated later by a modification of the basic approach (see Art. 6.12).

**1.3 Deformations in Framed Structures.** When a structure is acted upon by loads, the members of the structure will undergo *deformations* (or small changes in shape) and, as a consequence, points within the structure will be displaced to new positions. In general, all points of the structure except immovable points of support will undergo such displacements. The calculation of these displacements is an essential part of structural analysis, as will be seen later in the discussions of the flexibility and stiffness methods. However, before considering the displacements, it is first necessary to have an understanding of the deformations that produce the displacements.

To begin the discussion, consider a segment of arbitrary length cut from a member of a framed structure, as shown in Fig. 1-2a. For simplicity the bar is assumed to have a circular cross section. At any cross section, such as at the right-hand end of the segment, there will be stress resultants that in the general case consist of three forces and three couples. The forces are the axial force $N_x$ and the shearing forces $V_y$ and $V_z$; the couples are the twisting moment $T$ and the bending moments $M_y$ and $M_z$. Note that moment vectors are shown in the figure with double-headed arrows, in order to distinguish them from force vectors. The deformations of the bar can be analyzed by taking separately each stress resultant and determining its effect on an element of the bar. The element is obtained by isolating a portion of the bar between two cross sections a small distance $dx$ apart (see Fig. 1-2a).

The effect of the axial force $N_x$ on the element is shown in Fig. 1-2b. Assuming that the force acts through the centroid of the cross-sectional area, it is found that the element is uniformly extended, the significant strains in the element being normal strains in the $x$ direction. In the case of a shear force (Fig. 1-2c), one cross section of the bar is displaced laterally with respect to the other. There may also be some distortion of the cross sections, but this has a negligible effect on the determination of displacements and can be ignored. A bending moment (Fig. 1-2d) causes relative rotation of the two cross sections so that they are no longer parallel to one another. The resulting strains in the element are in the longitudinal direction of the bar, and they consist of contraction on the compression side and

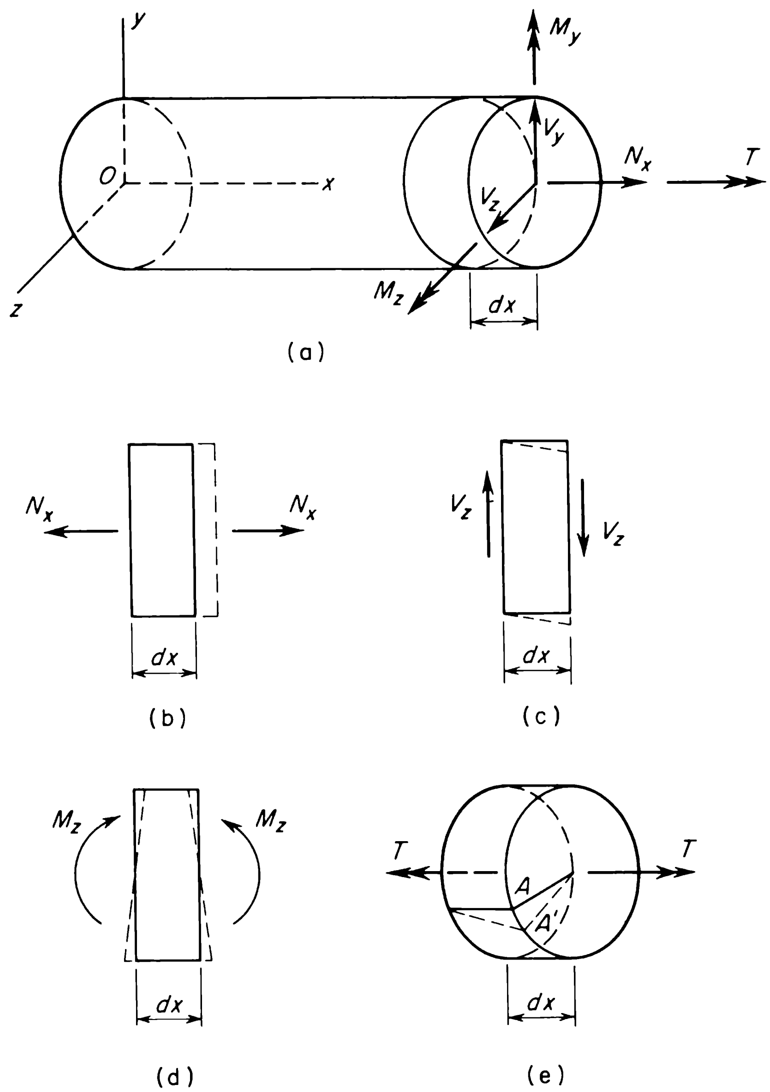


**Fig. 1-2.** Types of deformations: (b) axial, (c) shearing, (d) flexural, and (e) torsional.

extension on the tension side. Finally, the twisting moment $T$ causes a relative rotation of the two cross sections about the $x$ axis (see Fig. 1-2e) and, for example, point $A$ is displaced to $A'$. In the case of a circular bar, twisting produces only shearing strains; and the cross sections remain plane. For other cross-sectional shapes, distortion of the cross sections will occur.

The deformations shown in Figs. 1-2b, 1-2c, 1-2d, and 1-2e are called axial, shearing, flexural, and torsional deformations, respectively. Their evaluation is dependent upon the cross-sectional shape of the bar and the mechanical properties of the material. This book is concerned only with materials that are linearly elastic, that is, materials that follow Hooke's law. For such materials the various formulas for the deformations, as well as those for the stresses and strains in the element, are given for reference purposes in Appendix A, Art. A.1.

The *displacements* in a structure are caused by the cumulative effects of the deformations of all the elements. There are several ways of calculating these displacements in framed structures, depending upon the type of deformation being considered as well as the type of structure. For example, deflections of beams considering only flexural deformations can be found by direct integration of the differential equation for bending of a beam. Another method, which can be used for all types of framed structures including beams, trusses, grids, and frames, is the unit-load method, discussed later in Art. 1.14. In both of these methods, as well as others in common use, it is assumed that the displacements of the structure are small.

In any particular structure under investigation, not all types of deformations will be of significance in calculating the displacements. For example, in beams the flexural deformations normally are the only ones of importance, and it is usual to ignore the axial deformations. Of course, there are exceptional situations in which beams are required to carry large axial forces, and under such circumstances the axial deformation must be included in the analysis. It is also possible for axial forces to produce a beam-column action which has a nonlinear effect on the displacements (see Art. 6.18).

For truss structures of the types shown in Figs. 1-1b and 1-1c, the analyses are made in two parts. If the joints of the truss are idealized as hinges and if all loads act only at the joints, then the analysis involves only axial deformations of the members. The second part of the analysis is for the effects of the loads that act on the members between the joints, and this part is essentially the analysis of simply supported beams. If the joints of a truss-like structure actually are rigid, then bending occurs in the members even though all loads may act at the joints. In such a case, flexural deformations could be important, and in this event the structure may be analyzed as a plane or space frame.

In plane frames (see Fig. 1-1d) the significant deformations are flexural and axial. If the members are slender and are not triangulated in the fashion of a truss, the flexural deformations are much more important than the axial ones. However, the axial contributions should be included in the analysis of a plane frame if there is any doubt about their relative importance.

In grid structures (Fig. 1-1e) the flexural deformations are always important, but the cross-sectional properties of the members and the method of fabricating joints will determine whether or not torsional deformations must be considered. If the members are thin-walled open sections, such as I-beams, they are likely to be very flexible in torsion, and large twisting moments will not develop in the members. Also, if the members of a grid are not rigidly connected at crossing points, there will be no interaction between the flexural and torsional moments. In either of these cases, only flexural deformations need be taken into account in the analysis. On the other hand, if the members of a grid are torsionally stiff and rigidly interconnected at crossing points, the analysis must include both torsional and flexural deformations. Usually, there are no axial forces present in a grid because the forces are normal to the plane of the grid. This situation is analogous to that in a beam having all its loads perpendicular to the axis of the beam, in which case there are no axial forces in the beam. Thus, axial deformations are not included in a grid analysis.

Space frames (Fig. 1-1f) represent the most general type of framed structure, both with respect to geometry and with respect to loads. Therefore, it follows that axial, flexural, and torsional deformations all may enter into the analysis of a space frame, depending upon the particular structure and loads.

Shearing deformations are usually very small in framed structures and hence are seldom considered in the analysis. However, their effects may always be included if necessary in the analysis of a beam, plane frame, grid, or space frame (see Art. 6.16).

There are other effects, such as temperature changes and prestrains, that may be of importance in analyzing a structure. These subjects are discussed in later chapters in conjunction with the flexibility and stiffness methods of analysis.

**1.4 Actions and Displacements.** The terms ''action'' and ''displacement'' are used to describe certain fundamental concepts in structural analysis. An *action* (sometimes called a generalized force) is most commonly a single force or a couple. However, an action may also be a combination of forces and couples, a distributed loading, or a combination of these actions. In such combined cases, however, it is necessary that all the forces, couples, and distributed loads be related to one another in some definite manner so that the entire combination can be denoted by a single symbol. For example, if the loading on the simply supported beam $AB$ shown in Fig. 1-3 consists of two equal forces $P$, it is possible to consider the combination of the two loads as a single action and to denote it by one symbol, such as $F$. It is also possible to think of the combination of the two loads plus the two reactions $R_A$ and $R_B$ at the supports as a single action, since all four forces have a unique relationship to one another. In a more general situation, it is possible for a very complicated loading system on a structure to be treated as a single action if all components of the load are related to one another in a definite manner.

In addition to actions that are external to a structure, it is necessary to deal also with internal actions. These actions are the resultants of internal stress distributions, and they include bending moments, shearing forces,

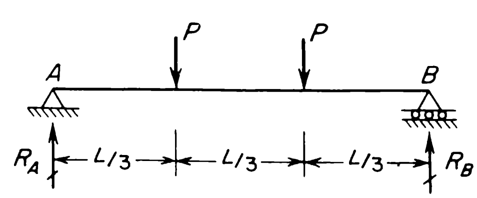


Fig. 1-3.

axial forces, and twisting moments. Depending upon the particular analysis being made, such actions may appear as one force, one couple, two forces, or two couples. For example, in making static equilibrium analyses of structures these actions normally appear as single forces and couples, as illustrated in Fig. 1-4a. The cantilever beam shown in the figure is subjected at end $B$ to loads in the form of actions $P_1$ and $M_1$. At the fixed end $A$ the reactive force and reactive couple are denoted $R_A$ and $M_A$, respectively. In order to distinguish these reactions from the loads on the structure, they are drawn with a slanted line (or slash) across the arrow. This convention for identifying reactions will be followed throughout the text (see also Fig. 1-3 for an illustration of the use of the convention). In calculating the axial force $N$, bending moment $M$, and shearing force $V$ at any section of the beam in Fig. 1-4a, such as at the midpoint, it is necessary to consider the static equilibrium of a portion of the beam. One possibility is to construct a free-body diagram of the right-hand half of the beam, as shown in Fig. 1-4b. In so doing, it is evident that each of the internal actions appears in the diagram as a single force or couple.

There are situations, however, in which the internal actions appear as two forces or couples. This case occurs most commonly in structural analysis when a "release" is made at some point in a structure, as shown in Fig. 1-5 for a continuous beam. If the bending moment is released at joint $B$ of the beam, the result is the same as if a hinge were placed in the beam at that joint (see Fig. 1-5b). Then, in order to take account of the bending moment $M_B$ in the beam, it must be considered as consisting of two equal and opposite couples $M_B$ that act on the left- and right-hand portions of the beam with the hinge, as shown in Fig. 1-5c. In this illustration the moment

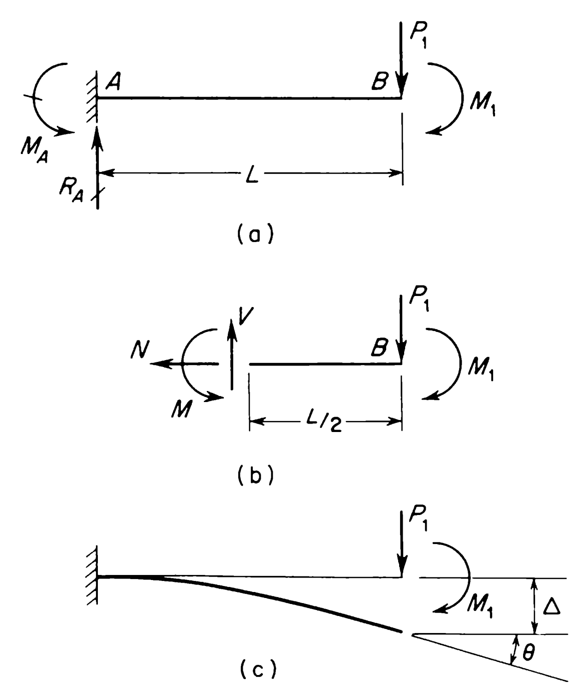


Fig. 1-4.

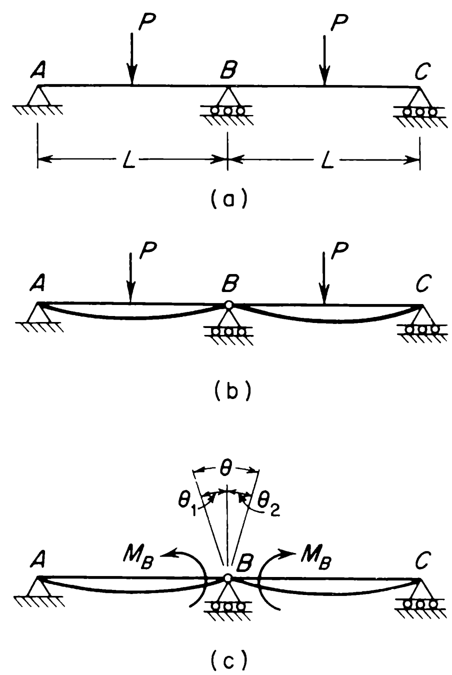


**Fig. 1-5.**

$M_B$ is assumed positive in the directions shown in the figure, signifying that the couple acting on the left-hand beam is counterclockwise and the couple acting on the right-hand beam is clockwise. Thus, for the purpose of analyzing the beam in Fig. 1-5c, the bending moment at point $B$ may be treated as a single action consisting of two couples. Similar situations are encountered with axial forces, shearing forces, and twisting moments, as illustrated later in the discussion of the flexibility method of analysis.

A second basic concept is that of a *displacement,* which is most commonly a translation or a rotation at some point in a structure. A translation refers to the distance moved by a point in the structure, and a rotation means the angle of rotation of the tangent to the elastic curve (or its normal) at a point. For example, in the cantilever beam of Fig. 1-4c, the translation $\Delta$ of the end of the beam and the rotation $\theta$ at the end are both considered as displacements. Moreover, as in the case of an action, a displacement may also be regarded in a generalized sense as a combination of translations and rotations. As an example, consider the rotations at the hinge at point $B$ in the two-span beam in Fig. 1-5c. The rotation of the right-hand end of the member $AB$ is denoted $\theta_1$, while the rotation of the left-hand end of member $BC$ is denoted $\theta_2$. Each of these rotations is considered as a displacement. Furthermore, the sum of the two rotations, denoted as $\theta$, is also a displacement. The angle $\theta$ can be considered as the relative rotation at point $B$ between the ends of members $AB$ and $BC$.

Another illustration of displacements is shown in Fig. 1-6, in which a plane frame is subjected to several loads. The horizontal translations $\Delta_A$, $\Delta_B$, and $\Delta_C$ of joints $A$, $B$, and $C$, respectively, are displacements, as also are the rotations $\theta_A$, $\theta_B$, and $\theta_C$ of these joints. Joint displacements of these types play an important role in the analysis of framed structures.

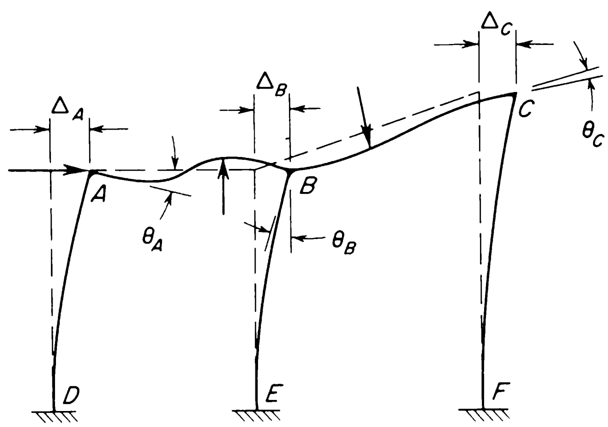


Fig. 1-6.

It is frequently necessary in structural analysis to deal with actions and displacements that *correspond* to one another. Actions and displacements are said to be corresponding when they are of an analogous type and are located at the same point on a structure. Thus, the displacement corresponding to a concentrated force is a translation of the structure at the point where the force acts, although the displacement is not necessarily caused by the force. Furthermore, the corresponding displacement must be taken along the line of action of the force and must have the same positive direction as the force. In the case of a couple, the corresponding displacement is a rotation at the point where the couple is applied and is taken positive in the same sense as the couple.

As an illustration, consider again the cantilever beam shown in Fig. 1-4a. The action $P_1$ is a concentrated force acting downward at the end of the beam, and the downward translation $\Delta$ at the end of the beam (see Fig. 1-4c) is the displacement that corresponds to this action. Similarly, the couple $M_1$ and the rotation $\theta$ are a corresponding action and displacement. It should be noted, however, that the displacement $\Delta$ corresponding to the load $P_1$ is not caused solely by the force $P_1$, nor is the displacement $\theta$ corresponding to $M_1$ caused by $M_1$ alone. Instead, in this example, both $\Delta$ and $\theta$ are displacements due to $P_1$ and $M_1$ acting simultaneously on the beam. In general, if a particular action is given, the concept of a corresponding displacement refers only to the definition of the displacement, without regard to the actual cause of that displacement. Similarly, if a displacement is given, the concept of a corresponding action will describe a particular kind of action on the structure, but the displacement need not be caused by that action.

As another example of corresponding actions and displacements, refer to the actions shown in Fig. 1-5c. The beam in the figure has a hinge at the middle support and is acted upon by the two couples $M_B$, which are considered as a single action. The displacement corresponding to the action $M_B$ consists in general of the sum of the counterclockwise rotation $\theta_1$ of the left-hand beam and the clockwise rotation $\theta_2$ of the right-hand beam. Therefore, the angle $\theta$ (equal to the sum of $\theta_1$ and $\theta_2$) is the displacement corresponding to the action $M_B$. This displacement is the relative rotation between the two beams at the hinge and has the same positive sense as $M_B$. If the angle $\theta$ is caused only by the couples $M_B$, then it is described as the displacement corresponding to $M_B$ and caused by $M_B$. This displacement can be found with the aid of the table of beam displacements given in Appendix A (see Table A-3, Case 5), and is equal to

$$\theta = \theta_1 + \theta_2 = \frac{M_B L}{3EI} + \frac{M_B L}{3EI} = \frac{2M_B L}{3EI}$$

in which $L$ is the length of each span and $EI$ is the flexural rigidity of the beam.

There are other situations, however, in which it is necessary to deal with a displacement that corresponds to a particular action but is caused by some other action. As an example, consider the beam in Fig. 1-5b, which is the same as the beam in Fig. 1-5c except that it is acted upon by two forces $P$ instead of the couples $M_B$. The displacement in this beam corresponding to $M_B$ consists of the relative rotation at joint $B$ between the two beams, positive in the same sense as $M_B$, but due to the loads $P$ only. Again using the table of beam displacements (Table A-3, Case 2), and also assuming that the forces $P$ act at the midpoints of the members, it is found that the displacement $\theta$ corresponding to $M_B$ and caused by the loads $P$ is

$$\theta = \theta_1 + \theta_2 = \frac{PL^2}{16EI} + \frac{PL^2}{16EI} = \frac{PL^2}{8EI}$$

The concept of correspondence between actions and displacements will become more familiar to the reader as additional examples are encountered in subsequent work. Also, it should be noted that the concept can be extended to include distributed actions, as well as combinations of actions of all types. However, these more general ideas have no particular usefulness in the work to follow.

In order to simplify the notation for actions and displacements, it is desirable in many cases to use the symbol $A$ for actions, including both concentrated forces and couples, and the symbol $D$ for displacements, including both translations and rotations. Subscripts can be used to distinguish between the various actions and displacements that may be of interest in a particular analysis. The use of this type of notation is shown in Fig. 1-7, which portrays a cantilever beam subjected to actions $A_1$, $A_2$, and $A_3$. The displacement corresponding to $A_1$ and due to all loads acting simultaneously is denoted by $D_1$ in Fig. 1-7a; similarly, the displacements corresponding to $A_2$ and $A_3$ are denoted by $D_2$ and $D_3$.

Now consider the cantilever beam subjected to action $A_1$ only (see Fig. 1-7b). The displacement corresponding to $A_1$ in this beam is denoted by $D_{11}$. The significance of the two subscripts is as follows. The first subscript

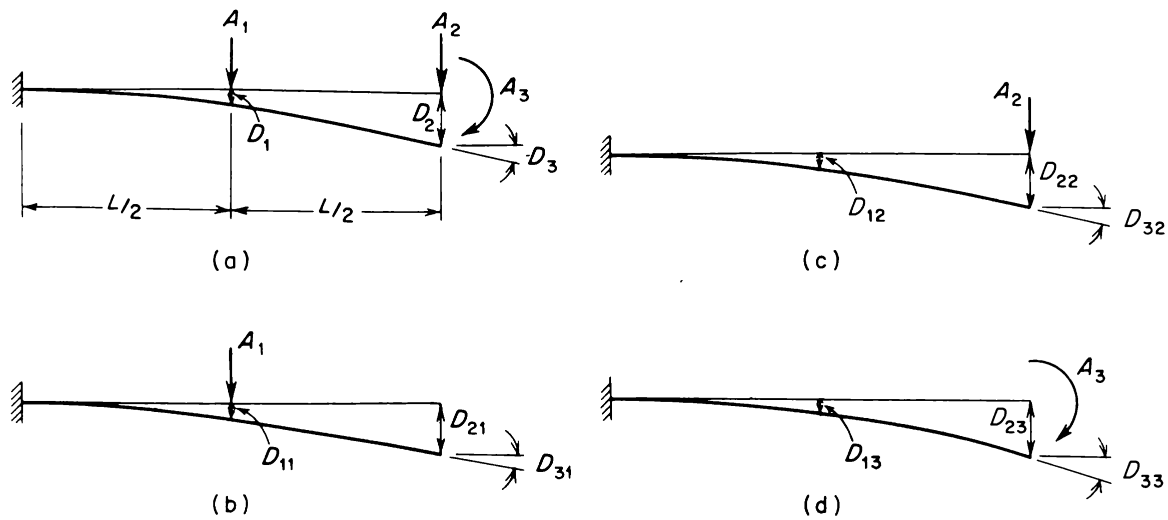


Fig. 1-7.

indicates that the displacement corresponds to action $A_1$, and the second indicates that the cause of the displacement is action $A_1$. In a similar manner, the displacement corresponding to $A_2$ in this beam is denoted by $D_{21}$, where the first subscript shows that the displacement corresponds to $A_2$ and the second shows that it is caused by $A_1$. Also shown in Fig. 1-7b is the displacement $D_{31}$, corresponding to the couple $A_3$ and caused by $A_1$.

The displacements caused by action $A_2$ acting alone are shown in Fig. 1-7c, and those caused by $A_3$ alone are shown in Fig. 1-7d. In each case the subscripts for the displacement symbols follow the general rule that the first subscript identifies the displacement and the second gives the cause of the displacement. In general, the cause may be a single force, a couple, or an entire loading system. Unless specifically stated otherwise, this convention for subscripts will always be used in later discussions.

For the beams pictured in Fig. 1-7 it is not difficult to determine the various displacements (see Table A-3, Cases 7 and 8). Assuming that the beam has flexural rigidity $EI$ and length $L$, it is found that the displacements for the beam in Fig. 1-7b are

$$D_{11} = \frac{A_1 L^3}{24EI} \qquad D_{21} = \frac{5A_1 L^3}{48EI} \qquad D_{31} = \frac{A_1 L^2}{8EI}$$

In a similar manner the remaining six displacements in Figs. 1-7c and d ($D_{12}$, $D_{22}$, . . . , $D_{33}$) can be found. Then the displacements in the beam under all loads acting simultaneously (see Fig. 1-7a) are determined by summation:

$$\begin{aligned} D_1 &= D_{11} + D_{12} + D_{13} \\ D_2 &= D_{21} + D_{22} + D_{23} \\ D_3 &= D_{31} + D_{32} + D_{33} \end{aligned}$$

These summations are expressions of the principle of superposition, which is discussed more fully in Art. 1.9.

**1.5 Equilibrium.** One of the objectives of any structural analysis is to determine various actions pertaining to the structure, such as reactions at the supports and internal stress resultants (bending moments, shearing forces, etc.). A correct solution for any of these quantities must satisfy all conditions of static equilibrium, not only for the entire structure, but also for any part of the structure taken as a free body.

Consider now any free body subjected to several actions. The resultant of all the actions may be a force, a couple, or both. If the free body is in static equilibrium, the resultant vanishes; that is, the resultant force vector and the resultant moment vector are both zero. A vector in three-dimensional space may always be resolved into three components in mutually orthogonal directions, such as the $x$, $y$, and $z$ directions. If the resultant force vector equals zero, then its components also must be equal to zero, and therefore the following equations of static equilibrium are obtained:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \tag{1-1a}$$

In these equations the expressions $\Sigma F_x$, $\Sigma F_y$, and $\Sigma F_z$ are the algebraic sums of the $x$, $y$, and $z$ components, respectively, of all the force vectors acting on the free body. Similarly, if the resultant moment vector equals zero, the moment equations of static equilibrium are

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0 \tag{1-1b}$$

in which $\Sigma M_x$, $\Sigma M_y$, and $\Sigma M_z$ are the algebraic sums of the moments about the $x$, $y$, and $z$ axes, respectively, of all the couples and forces acting on the free body. The six relations in Eqs. (1-1) represent the static equilibrium equations for actions in three dimensions. They may be applied to any free body such as an entire structure, a portion of a structure, a single member, or a joint of a structure.

When all forces acting on a free body are in one plane and all couples have their vectors normal to that plane, only three of the six equilibrium equations will be useful. Assuming that the forces are in the $x$-$y$ plane, it is apparent that the equations $\Sigma F_z = 0$, $\Sigma M_x = 0$, and $\Sigma M_y = 0$ will be satisfied automatically. The remaining equations are

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0 \tag{1-2}$$

and these equations become the static equilibrium conditions for actions in the $x$-$y$ plane.

In the stiffness method of analysis, the basic equations to be solved are those which express the equilibrium conditions at the joints of the structure, as described later in Chapter 3.

**1.6 Compatibility.** In addition to the static equilibrium conditions, it is necessary in any structural analysis that all conditions of compatibility be satisfied. These conditions refer to continuity of the displacements throughout the structure and are sometimes referred to as conditions of geometry. As an example, compatibility conditions must be satisfied at all points of support, where it is necessary that the displacements of the structure be consistent with the support conditions. For instance, at a fixed support there can be no translation and no rotation of the axis of the member.

Compatibility conditions must also be satisfied at all points throughout the interior of a structure. Usually, it is compatibility conditions at the joints of the structure that are of interest. For example, at a rigid connection between two members the displacements (translations and rotations) of both members must be the same.

In the flexibility method of analysis the basic equations to be solved are equations that express the compatibility of the displacements, as will be described in Chapter 2.

**1.7 Static and Kinematic Indeterminacy.** There are two types of indeterminacy that must be considered in structural analysis, depending upon whether actions or displacements are of interest. When actions are the unknowns in the analysis, as in the flexibility method, then *static indeterminacy* must be considered. In this case, indeterminacy refers to an excess of unknown actions as compared to the number of equations of static equilibrium that are available. The equations of equilibrium, when applied to the entire structure and to its various parts, may be used for the calculation of reactions and internal stress resultants. If these equations are sufficient for finding all actions, both external and internal, then the structure is statically determinate. If there are more unknown actions than equations, the structure is statically indeterminate. The simply supported beam shown in Fig. 1-3 and the cantilever beam of Fig. 1-4 are examples of statically determinate structures, since in both cases all reactions and stress resultants can be found from equilibrium equations alone. On the other hand, the continuous beam of Fig. 1-5a is statically indeterminate.

The unknown actions in excess of those that can be found by static equilibrium are known as *static redundants,* and the number of such redundants represents the *degree* of static indeterminacy of the structure. Thus, the two-span beam of Fig. 1-5a is statically indeterminate to the first degree, since there is one redundant action. For instance, it can be seen that it is impossible to calculate all of the reactions for the beam by static equilibrium alone. However, after the value of one reaction is obtained (by one means or another), the remaining reactions and all internal stress resultants can be found by statics alone.

Other examples of statically indeterminate structures are shown in Fig. 1-8. The propped cantilever beam in Fig. 1-8a is statically indeterminate to the first degree, since there are four reactive actions ($H_A$, $M_A$, $R_A$, and $R_B$) whereas only three equations of equilibrium are available for the calculation of reactions (see Eqs. 1-2).

The fixed-end beam of Fig. 1-8b is statically indeterminate to the third degree because there are six reactions to be found in the general case. In

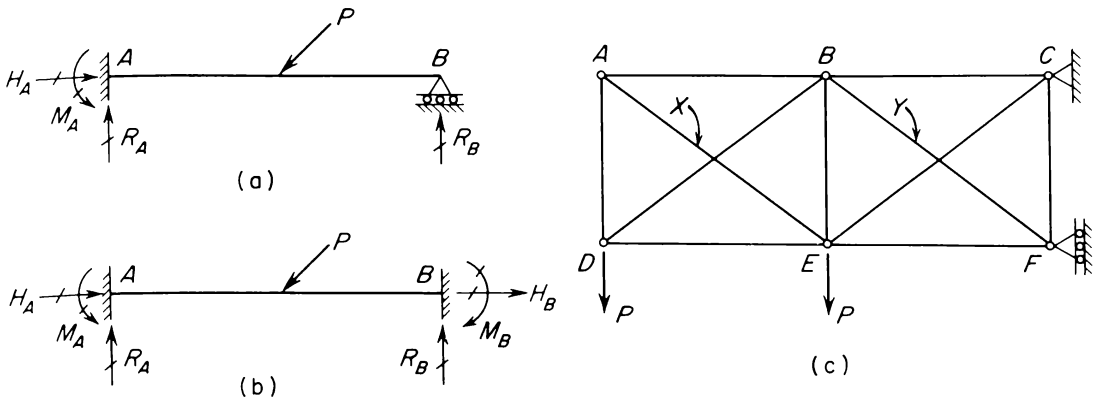


**Fig. 1-8.** Examples of statically indeterminate structures.

the special case when all the concentrated forces on a fixed-end beam act in a direction which is perpendicular to the axis of the beam, there will be no axial forces at the ends of the beam. In such a case the beam can be analyzed as if it were statically indeterminate to the second degree.

The plane truss in Fig. 1-8c is statically indeterminate to the second degree. This conclusion can be reached by cutting two bars, such as $X$ and $Y$, thereby releasing the forces in those bars. The truss with the cut bars is then statically determinate, since all reactions and bar forces can be found by a direct application of equations of equilibrium. Each bar that is cut represents one action, namely, the force in the bar, that is released from the truss. The number of actions that must be released in order to reduce the statically indeterminate structure to a determinate structure will be equal to the degree of indeterminacy. This method of ascertaining the degree of static indeterminacy is quite general and can be used with many types of structures.

As another example of this method for determining the degree of indeterminateness of a structure, consider the plane frame shown in Fig. 1-6. The object is to make cuts, or releases, in the frame until the structure has become statically determinate. If bars $AB$ and $BC$ are cut, the structure that remains consists of three cantilever portions (the supports of the cantilevers are at $D$, $E$, and $F$), each of which is statically determinate. Each bar that is cut represents the removal (or release) of three actions (axial force, shearing force, and bending moment) from the original structure. Because a total of six actions was released, the degree of indeterminacy of the frame is six.

A distinction may also be made between external and internal indeterminateness. External indeterminateness refers to the calculation of the reactions for the structure. Normally, there are six equilibrium equations available for the determination of reactions in a space structure, and three for a plane structure. Therefore, a space structure with more than six reactive actions, and a plane structure with more than three reactions, will usually be externally indeterminate. Examples of external indeterminateness can be seen in Fig. 1-8. The propped cantilever beam is externally indeterminate to the first degree, the fixed-end beam is externally indeterminate to the third degree, and the plane truss is statically determinate externally.

Internal indeterminateness refers to the calculation of stress resultants within the structure, assuming that all reactions have been found previously. For example, the truss in Fig. 1-8c is internally indeterminate to the second degree, although it is externally determinate, as noted above.

The total degree of indeterminateness of a structure is the sum of the external and internal degrees of indeterminateness. Thus, the truss in Fig. 1-8c is indeterminate to the second degree when considered in its entirety. The beam in Fig. 1-8a is externally indeterminate to the first degree and internally determinate, inasmuch as all stress resultants in the beam can be readily found after all the reactions are known. The plane frame in Fig. 1-6 has nine reactive actions, and therefore it is externally indeterminate to the sixth degree. Internally the frame is determinate since all stress resultants can be found if the reactions are known. Therefore, the frame has a total indeterminateness of six, as previously observed.

Occasionally, there are special conditions of construction that affect the degree of indeterminacy of a structure. The three-hinged arched truss shown in Fig. 1-9 has a central hinge at joint $B$ which makes it possible to calculate all four reactions by statics. For the truss shown, the bar forces in all members can be found after the reactions are known, and therefore the structure is statically determinate overall.

Several additional examples of statically indeterminate structures are given at the end of this article. These examples illustrate how the degree of indeterminacy can be obtained for many structures by intuitive reasoning. Other examples will be encountered in Chapter 2 in connection with the flexibility method of analysis. However, for large structures it is desirable to have more formalized methods of establishing static indeterminacy; such methods are discussed elsewhere and are not repeated here.*

In the stiffness method of analysis the displacements of the joints of the structure become the unknown quantities. Therefore, the second type of indeterminacy, known as *kinematic indeterminacy,* becomes important. In

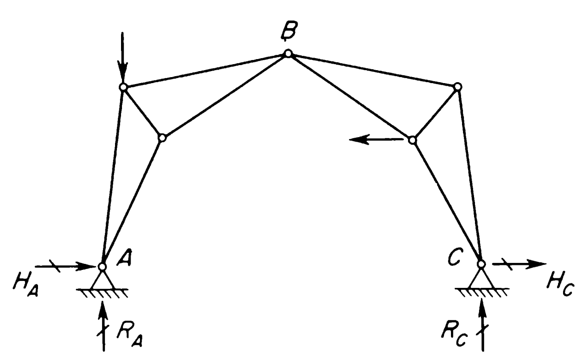


**Fig. 1-9.** Three-hinged arched truss.

*For a more complete discussion of static indeterminacy, see *Elementary Structural Analysis,* 3rd ed., by C. H. Norris, J. B. Wilbur, and S. Utku, McGraw-Hill, New York, 1976.

order to understand this type of indeterminacy, it should be recalled that joints in framed structures are defined to be located at all points where two or more members intersect, at points of support, and at free ends. When the structure is subjected to loads, each joint will undergo displacements in the form of translations and rotations, depending upon the configuration of the structure. In some cases the joint displacements will be known because of restraint conditions that are imposed upon the structure. At a fixed support, for instance, there can be no displacements of any kind. However, there will be other joint displacements that are not known in advance, and which can be obtained only by making a complete analysis of the structure. These unknown joint displacements are the kinematically indeterminate quantities, and are sometimes called kinematic redundants. Their number represents the degree of kinematic indeterminacy of the structure, or the number of *degrees of freedom* for joint displacement.

To illustrate the concepts of kinematic indeterminacy, it is useful to consider again the examples of Fig. 1-8. Beginning with the beam in Fig. 1-8a, it is seen that end $A$ is fixed and cannot undergo any displacement. On the other hand, joint $B$ has two degrees of freedom for joint displacement, since it may translate in the horizontal direction and may rotate. Thus, the beam is kinematically indeterminate to the second degree, and there are two unknown joint displacements to be calculated in a complete analysis of this beam. In many practical analyses it would be permissible to neglect axial deformations of the beam; in such a case, joint $B$ would have only one degree of freedom (rotation), and the structure would be analyzed as if it were kinematically indeterminate to the first degree.

The second example of Fig. 1-8 is a fixed-end beam. Such a beam has no unknown joint displacements, and therefore is kinematically determinate. By comparison, the same beam was statically indeterminate to the third degree.

The third example in Fig. 1-8 is the plane truss that was previously shown to be statically indeterminate to the second degree. Joint $A$ of this truss may undergo two independent components of displacement (such as translations in two perpendicular directions) and hence has two degrees of freedom. Rotation of a joint of a truss has no physical significance because, under the assumption of hinged joints, rotation of a joint produces no effects in the members of the truss. Thus, the degree of kinematic indeterminacy of a truss is always found as if the truss were subjected to loads at the joints only. This philosophy is the same as in the case of static indeterminacy, wherein only axial forces in the members are considered as unknowns. The joints $B$, $D$, and $E$ of the truss in Fig. 1-8 also have two degrees of freedom each, while the restrained joints $C$ and $F$ have zero and one degree of freedom, respectively. Thus, the truss has a total of nine degrees of freedom for joint translation and is kinematically indeterminate to the ninth degree.

The rigid frame shown in Fig. 1-6 offers another example of a kinematically indeterminate structure. Since the supports at $D$, $E$, and $F$ of this frame are fixed, there can be no displacements at these joints. However, joints $A$, $B$, and $C$ each possess three degrees of freedom, since each joint may undergo horizontal and vertical translations and a rotation. Thus, the total number of degrees of kinematic indeterminacy for this frame is nine. If the effects of axial deformations are omitted from the analysis, the degree of kinematic indeterminacy is reduced. There would be no possibility for vertical displacement of any of the joints because the columns would not change length. Furthermore, the horizontal translations of joints $A$ and $B$ would be equal, and the horizontal translation of $C$ would have a known relationship to that of joint $B$. In other words, if axial deformations are neglected the only independent joint displacements are the rotations of joints $A$, $B$, and $C$ and one horizontal displacement (such as that of joint $B$). Therefore, the structure would be considered to be kinematically indeterminate to the fourth degree.

In summary, two simple rules can always be used to find the static and kinematic indeterminacy of a framed structure. First, to find the number of redundant actions, count the number of releases necessary to obtain a statically determinate structure. This can be done indirectly by finding the number of unknown actions in excess of those that can be found from static equilibrium equations. Second, to find the number of degrees of freedom for joint displacement, count the number of joint restraints that must be added to obtain a kinematically determinate structure (no joint displacements). Several examples involving both static and kinematic indeterminacy will now be given.

*Example 1.* The space truss shown in Fig. 1-10 has pin supports at $A$, $B$, and $C$. The degrees of static and kinematic indeterminacy for the truss are to be obtained.

In determining the degree of static indeterminacy, it can be noted that there are three equations of equilibrium available at every joint of the truss for the purpose of calculating bar forces or reactions. Thus, a total of 18 equations of statics is available. The number of unknown actions is 21, since there are 12 bar forces and 9 reactions (three at each support) to be found. The truss is, therefore, statically inde-

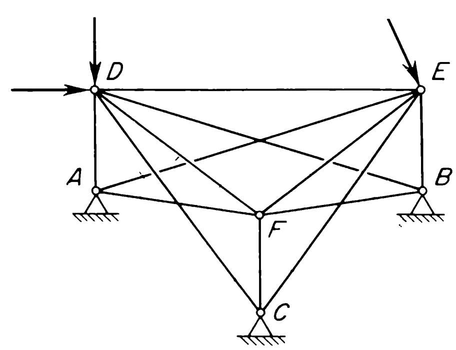


**Fig. 1-10.** Example 1.

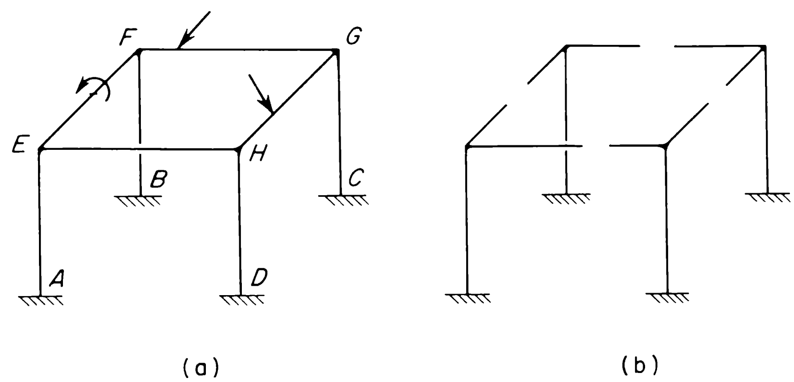


**Fig. 1-11.** Example 2.

terminate to the third degree. More specifically, the truss is externally indeterminate to the third degree, because there are nine reactions but only six equations for the equilibrium of the truss as a whole. The truss is internally determinate since all bar forces can be found by statics after the reactions are determined.

Each of the joints $D$, $E$, and $F$ has three degrees of freedom for joint displacement, because each joint can translate in three mutually orthogonal directions. Therefore, the truss is kinematically indeterminate to the ninth degree.

*Example 2.* The degrees of static and kinematic indeterminacy are to be found for the space frame shown in Fig. 1-11a.

There are various ways in which the frame can be cut in order to reduce it to a statically determinate structure. One possibility is to cut all four of the bars $EF$, $FG$, $GH$, and $EH$, thereby giving the released structure shown in Fig. 1-11b. Since each release represents the removal of six actions (axial force, two shearing forces, twisting couple, and two bending moments) the original frame is statically indeterminate to the 24th degree.

The number of possible joint displacements at $E$, $F$, $G$, and $H$ is six at each joint (three translations and three rotations); therefore, the frame is kinematically indeterminate to the 24th degree.

Now consider the effect of omitting axial deformations from the analysis. The degree of static indeterminacy is not affected, because the same number of actions will still exist in the structure. On the other hand, there will be fewer degrees of freedom for joint displacement. The columns will not change in length, thereby eliminating four joint translations (one each at $E$, $F$, $G$, and $H$). In addition, the four horizontal members will not change in length, thereby eliminating four more translations. Thus, it is finally concluded that the degree of kinematic indeterminacy is 16 when axial deformations are excluded from consideration.

Consider next the effect of replacing the fixed supports at $A$, $B$, $C$, and $D$ by immovable pinned supports. The effect of the pinned supports is to reduce the number of reactions at each support from six to three. Therefore, the degree of static indeterminacy becomes 12 less than with fixed supports, or a total of 12 degrees. At the same time, three additional degrees of freedom for rotation have been added at each support, so that the degree of kinematic indeterminacy has been increased by 12 when compared to the frame with fixed supports. It can be seen that removing restraints at the supports of a structure serves to decrease the degree of statical indeterminacy, while increasing the degree of kinematic indeterminacy.

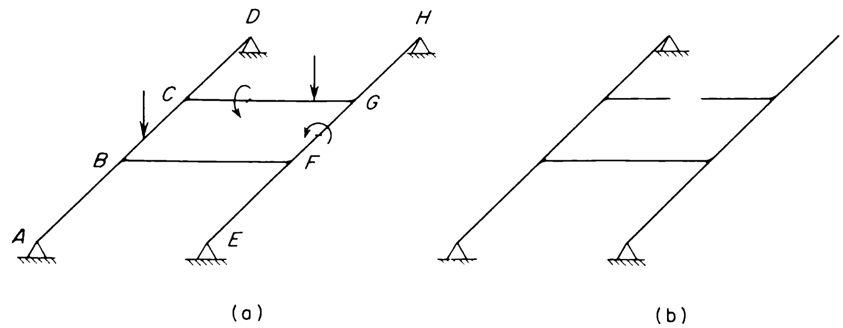


Fig. 1-12. Example 3.

*Example 3.* The grid shown in Fig. 1-12a lies in a horizontal plane and is supported at $A$, $D$, $E$, and $H$ by simple supports. The joints at $B$, $C$, $F$, and $G$ are rigid connections. What are the degrees of static and kinematic indeterminacy?

Because there are no axial forces in the members of a grid, only vertical reactions are developed at the supports of this structure. Therefore, the grid is externally indeterminate to the first degree, because only three equilibrium equations are available for the structure in its entirety, but there are four reactions. After removing one reaction, the grid can be made statically determinate by cutting one member, such as $CG$ (see Fig. 1-12b). The release in member $CG$ removes three actions (shearing force in the vertical direction, twisting moment, and bending moment). Thus, the grid can be seen to be internally indeterminate to the third degree and statically indeterminate overall to the fourth degree.

In general there are three degrees of freedom for displacement at each joint of a grid (one translation and two rotations). Such is the case at joints $B$, $C$, $F$, and $G$ of the grid shown in Fig. 1-12a. However, at joints $A$, $D$, $E$, and $H$ only two joint displacements are possible, inasmuch as joint translation is prevented. Therefore, the grid shown in the figure is kinematically indeterminate to the 20th degree.

**1.8 Structural Mobilities.** In the preceding discussion of external static indeterminacy, the number of reactions for a structure was compared with the number of equations of static equilibrium for the entire structure taken as a free body. If the number of reactions exceeds the number of equations, the structure is externally statically indeterminate; if they are equal, the structure is externally determinate. However, it was tacitly assumed in the discussion that the geometrical arrangement of the reactions was such as to prevent the structure from moving when loads act on it. For instance, the beam shown in Fig. 1-13a has three reactions, which is the same as the number of static equilibrium equations for forces in a plane. It is apparent, however, that the beam will move to the left when the inclined load $P$ is applied. A structure of this type is said to be *mobile* (or *kinematically unstable*). Other examples of mobile structures are the frame of Fig. 1-13b and the truss of Fig. 1-13c. In the structure of Fig. 1-13b the three reactive forces are concurrent (their lines of action intersect at point $O$). Therefore, the frame is mobile since it cannot support a general load, such as the force $P$, which does not act through point $O$. In the truss of Fig. 1-13c there are two bars which are collinear at joint $A$, and there is no other

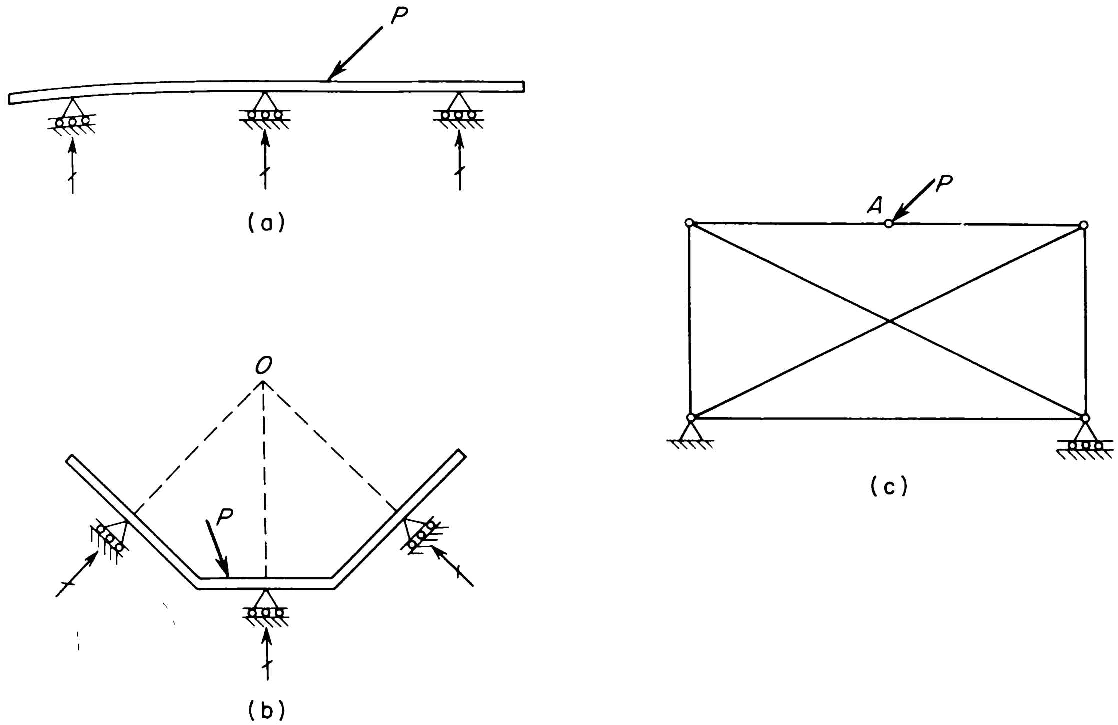


Fig. 1-13. Mobile structures.

bar meeting at that joint. Again, the structure is mobile since it is incapable of supporting the load $P$ in its initial configuration.

From the examples of mobile structures given in Fig. 1-13, it is apparent that both the supports and the members of any structure must be adequate in number and in geometrical arrangement to insure that the structure is not movable. Only structures meeting these conditions will be considered for analysis in subsequent chapters.

**1.9 Principle of Superposition.** The principle of superposition is one of the most important concepts in structural analysis. It may be used whenever linear relationships exist between actions and displacements (the conditions under which this assumption is valid are described later in this article). In using the principle of superposition it is assumed that certain actions and displacements are imposed upon a structure. These actions and displacements cause other actions and displacements to be developed in the structure. Thus, the former actions and displacements have the nature of causes, while the latter are effects. In general terms the principle states that the effects produced by several causes can be obtained by combining the effects due to the individual causes.

In order to illustrate the use of the principle of superposition when actions are the cause, consider the beam in Fig. 1-14a. This beam is subjected to loads $A_1$ and $A_2$, which produce various actions and displacements throughout the structure. For instance, reactions $R_A$, $R_B$, and $M_B$ are developed at the supports, and a displacement $D$ is produced at the midpoint of the beam. The effects of the actions $A_1$ and $A_2$ acting sepa-