

Chapter One: Introduction to Operations Research

Operations Research (OR) started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques. Following the war, numerous peacetime applications emerged, leading to the use of OR and management science in many industries and occupations.

Operations Research (OR) is the study of mathematical models for complex organizational systems. **Optimization** is a branch of OR which uses mathematical techniques such as linear and nonlinear programming to derive values for system variables that will optimize performance. Management science or operations research uses a logical approach to problem solving. This quantitative approach is widely employed in business. Areas of application include forecasting, capital budgeting, capacity planning scheduling inventory management, project management and production planning.

Significance of Operation Research

- ✓ It provides a tool for scientific analysis and provides solution for various business problems.
- ✓ It enables proper deployment and optimum allocation of scarce resources.
- ✓ It helps in minimizing waiting and servicing costs.
- ✓ It enables the management to decide when to buy and how much to buy through the technique of inventory planning.
- ✓ It helps in evaluating situations involving uncertainty.
- ✓ It enables experimentation with models, thus eliminating the cost of making errors while experimenting with reality.
- ✓ It allows quick and inexpensive examination of large numbers of alternatives.
- ✓ In general OR facilitates and improves the decision making process.

Areas of Application

- ✓ Inventory control
- ✓ Facility design (distribution decision)
- ✓ Product mix determination
- ✓ Portfolio analysis
- ✓ Allocation of scarce resources

- ✓ Investment decisions
- ✓ Project management.

CHAPTER 2: LINEAR PROGRAMING (LP)

2.1. Basic Concepts in LP

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equation and /or inequalities known as constraints.

- The objective function may be profit, cost, production capacity, or any other measure of effectiveness which is to be obtained in the best possible or optimal manner
- The constraints may be imposed by different resources such as market demand, production process and equipment, storage capacity, raw material availability, etc.
- By linearity is meant a mathematical expression in which the expressions among the variables are linear.

Definition:

LP is a mathematical modelling technique useful for economic allocation of “scarce” or “limited” resources (such as labor, material, machine, time, warehouse, space, capital, etc.) to several competing activities (such as products, services, jobs, new equipment, projects etc.) on the basis of a given criterion of optimality. All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. Supply of resources being limited, the management must find the best allocation of resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent.

Steps in formulation of LP model:

1. Identify activities (key decision variables).
 2. Identify the objective function as a linear function of its decision variables.
 3. State all resource limitations as linear equation or inequalities of its decision variables.
 4. Add non-negative constraints from the consideration that negative values of the decision variables do not have any valid physical interpretation.
 5. Use mathematical techniques to find all possible sets of values of the variables (unknowns) satisfying all the function
 6. Select the particular set of values of variables obtained in step five that leads to the achievement of the objective function.
- ❖ The result of the first four steps is called linear programming. The set of solutions obtained in step five is known as the set of feasible solutions and the solution finally selected in step six is called optimum (best) solution of the LP problem.
 - ❖ A typical linear programming has two step
 - ✓ The objective function
 - ✓ The constraints
 - Technical constraint, and
 - The non-negativity constraint

The objective function: is a mathematical representation of the overall goal of the organization stated as a linear function of its decision variables (X_j) to optimize the criterion of optimality.

- It is also called the measure of performance such as profit, cost, revenue, etc

The general form of the objective function is expressed as:

Optimize (Maximize or Minimize) $Z = \sum C_j X_j$

Where: Z is the measure of performance variable (profit/cost), which is the function of X_1, X_2, \dots, X_n . (Quantities of activities)

C_j ($C_1, C_2 \dots C_n$) (parameters or coefficients) represent the contribution of a unit of the respective variables X_1, X_2, \dots, X_n to the measure of performance Z (the objective function).

The constraints: the constraints must be expressed linear equalities or inequalities interms of decision variables.

The general form of the constraints functions are expressed as:

$$\sum a_{ij} x_j \quad (<, >, \leq, =, \geq) b_i$$

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n (<, >, \leq, =, \geq) b_1$$

$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2n}X_n (<, >, \leq, =, \geq) b_2$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$A_{m1}X_m + a_{m2}X_2 + \dots + a_{mn}X_n (<, >, \leq, =, \geq) b_n$$

- ❖ a_{ij} 's are called technical coefficients and measure the per unit consumption of the resources for executing one unit of unknown variable (activities) X_j .
 - a_{ij} 's can be positive, negative or zero in the given constraints.
- ❖ The b_i represents the total availability of the i^{th} resource.
 - It is assumed that $b_i \geq 0$ for all i. However, if any $b_i < 0$, then both sides of the constraint i can be multiplied by 1 to make $b_i > 0$ and reverse the inequality of the constraint.

Requirements for a linear programming problem: Generally speaking, LP can be used for optimization problems if the following conditions are satisfied.

1. There must be a well defined objective function which is to be either maximized or minimized and which can be expressed as a linear function of decision variables
2. There should be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities interms of variables.
3. There must be alternative courses of action. Fore e.g., a given product may be processed by two different machines and problem may be as to how much of the product to allocate which machine.
4. Decision variables should be interrelated and non-negative.
5. The resource should be in limited supply.

2.2. Assumptions of Linear Programming

A linear programming model is based on the following assumptions:

- 1. Proportionality assumption:** A basic assumption of LP is that proportionality exists in the objective function and the constraints. It states that the contribution of each activity to the value of the objective function Z is proportional to the level of the activity X_j , as represented by the C_jX_j term in the objective function. Similarly, the resource consumption of each activity in each functional constraint is proportional to the level of the activity X_j , as represented by the $a_{ij}X_j$ term in the constraint.
What happens when the proportionality assumption does not hold? In most cases we use nonlinear programming.
- 2. Additivity assumption:** States that every function in a linear programming model (whether the objective function or the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities.
What happens when the additivity assumption does not hold? In most cases we use nonlinear programming.
- 3. Divisibility assumption:** States that decision variables in linear programming model are allowed to have any values, including non-integer values, which satisfy the functional and non-negativity constraints. Thus, since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels.
In certain situations, the divisibility assumption does not hold because some of or all the decision variables must be restricted to integer values. For this restriction an integer programming model will be used.
- 4. Certainty assumption:** This assumption states that the various parameters (namely, the objective function coefficients, the coefficients in the functional constraints a_{ij} and resource values in the constraints b_i) are certainly and precisely known and that their values do not change with time. However, in real applications, the certainty assumption is seldom satisfied precisely. For this reason it is usually important to conduct sensitivity analysis after a solution is found that is optimal under the assumed parameter values.
- 5. Finiteness:** An LP model assumes that a finite (limited) number of choices (alternatives) are available to the decision-maker and that the decision variables are interrelated and non-negative. The non-negativity condition shows that LP deals with real-life situations as it is not possible to produce/use negative quantities.
- 6. Optimality:** In LP, the optimal solution always occurs at the corner point of the set of feasible solutions.

Important Definitions

- **Solution:** The set of values of decision variables X_i ($i=1, 2, \dots, n$) which satisfy the constraints of an LP problem is said to constitute a solution to that LP problem
- **Feasible solutions:** The set of values of decision variables X_i ($i=1, 2, \dots, n$) which satisfy all the constraints and non-negativity conditions of an LP simultaneously.
- **Infeasible solution:** The set of values of decision variables which do not satisfy all the constraints and non-negativity conditions of an LP simultaneously.

- **Optimum solution:** A feasible solution which optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.
- **Unbounded solution:** A solution which can increase or decrease the value of the objective function of LP problem indefinitely is called unbounded solution.

2.3. Methods of Solving Linear Programming Problems

A linear programming problem can be solved by graphic method or by applying algebraic method, called the simplex method.

1. **Graphic method:** it can be used when only two variables are involved. This method consists of the following steps:
 - i. Represent the given problem in mathematical form.
 - ii. Plot each of the constraint on the graph.
 - iii. Identify the feasible region (or solution space) that satisfies all the constraints simultaneously. Any point on or within the shaded region represents a feasible solution to the given problem.
 - iv. Locate the solution points (identify the corner points from the solution space).
 - v. Determine the optimal solution. A solution that optimizes the objective function.
 - vi. Interpret the results

a) Maximization Problem: This is the case of **Maximize Z** with inequalities of constraints in \leq form. Consider two models of color TV sets; **Model A** and **B**, are produced by a company to maximize profit. The profit realized is \$300 from A and \$250 from set B. The limitations are

- a. Availability of only 40 hrs of labor each day in the production department,
- b. A daily availability of only 45 hrs on machine time, and
- c. Ability to sale 12 set of model A.

How many sets of each model will be produced each day so that the total profit will be as large as possible?

Constraints	Resources used per unit		Maximum Available Hours
	Model A (x_1)	Model B (x_2)	
Labor Hours	2	1	40
Machine Hours	1	3	45
Marketing Hours	1	0	12
Profit	\$300	\$250	

Solution:

1. Formulation of mathematical model of LPP

$$\text{Max } Z = 300X_1 + 250X_2$$

St:

$$\left. \begin{array}{l} 2X_1 + X_2 \leq 40 \\ X_1 + 3X_2 \leq 45 \\ X_1 \leq 12 \\ X_1, X_2 \geq 0 \end{array} \right\} \text{LP Model}$$

2. Convert constraints inequalities into equalities

$$2X_1 + X_2 = 40$$

$$X_1 + 3X_2 = 45$$

$$X_1 = 12$$

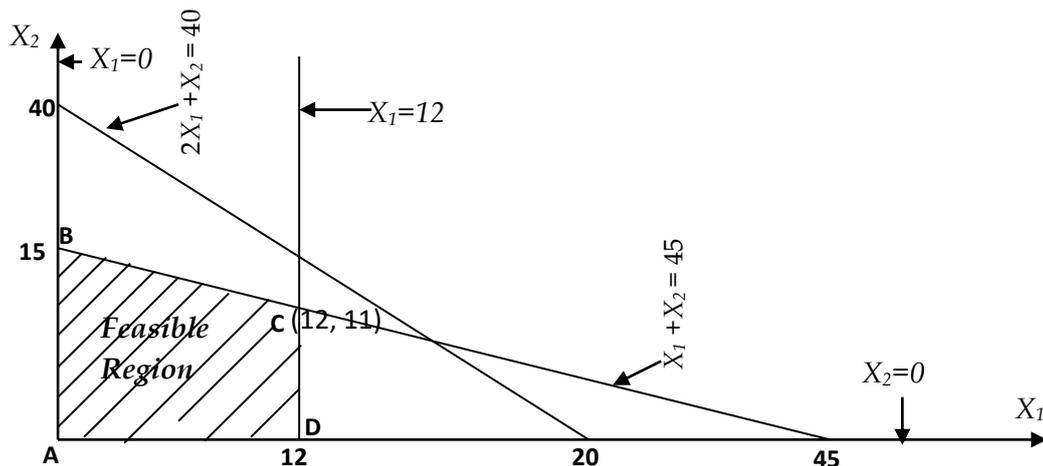
3. Draw the graph by finding out the x- and y- intercepts

$$2X_1 + X_2 = 40 \implies (0, 40) \text{ and } (20, 0)$$

$$X_1 + 3X_2 = 45 \implies (0, 15) \text{ and } (45, 0)$$

$$X_1 = 12 \implies (12, 0)$$

$$X_1, X_2 = 0$$



4. Identify the feasible area of the solution which satisfies all constraints. The shaded region in the above graph satisfies all the constraints and it is called *Feasible Region*.

5. Identify the corner points in the feasible region. Referring to the above graph, the corner points are in this case are:

A (0, 0), B (0, 15), C (12, 11) and D (12, 0)

6. Identify the optimal point.

Corners	Coordinates	Max $Z = 300 X_1 + 250 X_2$
A	(0, 0)	\$0
B	(0, 15)	\$3750
C	(12, 11)	\$ 6350 (Optimal)
D	(12, 0)	\$3600

7. Interpret the result. Accordingly, the highlighted result in the table above implies that 12 units of Model A and 11 units of Model B TV sets should be produced so that the total profit will be \$6350.

b) Minimization Problem: In this case, we deal with **Minimize Z** with inequalities of constraints in \geq form. Suppose that a machine shop has two different types of machines; Machine 1 and Machine 2, which can be used to make a single product. These machines vary in the amount of product produced per hr., in the amount of labor used and in the cost of operation. Assume that at least a certain amount of product must be produced and that we would like to utilize at least the regular labor force. How much should we utilize on each machine in order to utilize total costs and still meets the requirement?

Items	Resource Used		Minimum Required Hours
	Machine 1 (X_1)	Machine 2 (X_2)	
Product produced/hr	20	15	100
Labor/hr	2	3	15
Operation Cost	\$25	\$30	

Solution:

1. The LP Model:

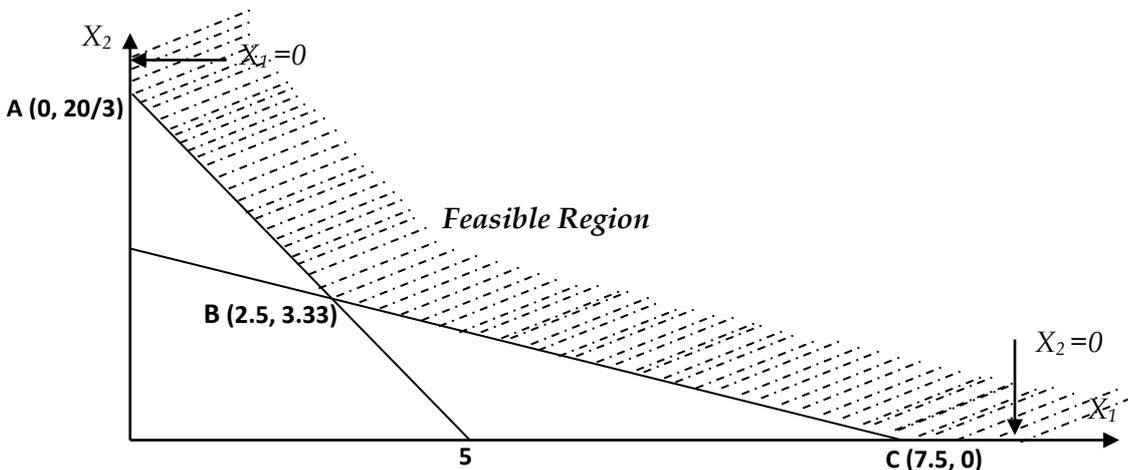
$$\begin{aligned} \text{Min. } Z &= 25X_1 + 30X_2 \\ \text{St:} \\ 20X_1 + 15X_2 &\geq 100 \\ 2X_1 + 3X_2 &\geq 15 \\ X_1, X_2 &\geq 0 \end{aligned}$$

LP Model

Dear student, can you graph the above constraints? See step 2 below.

2. The Graph of Constraint Equations:

$$\begin{aligned} 20X_1 + 15X_2 = 100 &\implies (0, 20/3) \text{ and } (5, 0) \\ 2X_1 + 3X_2 = 15 &\implies (0, 5) \text{ and } (7.5, 0) \\ X_1, X_2 = 0 & \end{aligned}$$



3. The Corners and Feasible Solution:

Corners	Coordinates	Min $Z = 25 X_1 + 30 X_2$
A	(0, 20/3)	200
B	(2.5, 3.33)	162.5 (Optimal)
C	(7.5, 0)	187.5

Since our objective is to minimize cost, the minimum amount (162.5) will be selected.

$$X_1 = 2.5$$

$$X_2 = 3.33 \text{ and}$$

$$\text{Min } Z = 162.5$$

Note:

- In maximization problems, our point of interest is looking the furthest point from the origin (Maximum value of Z).
- In minimization problems, our point of interest is looking the point nearest to the origin (Minimum value of Z).

Example3. A firm manufactures two products A and B on which the profits earned per unit are Birr 3 and Birr 4 respectively. Each product is processed on two machines M_1 and M_2 . Product A requires one minute of processing time on M_1 and two minutes on M_2 , while B requires one minute on M_1 and one minute on M_2 . Machine M_1 is available for not more than 7hrs. 30 minutes, while machine M_2 is available for 10hrs during any working day. Find the number of units of products A and B to be manufactured to get maximum profit.

Solution

Formulation of LPP model

Let x_1 and x_2 denote the number of units of products A and B to be produced per day.

Objective is to maximize the profit.

i.e. Maximize $Z = 3x_1 + 4x_2$

Constraints are on the time available for machines M_1 and M_2

i.e., for machine M_1 , $1x_1 + 1x_2 \leq 450$

for machine M_2 , $2x_1 + 1x_2 \leq 600$ **Ans: $x_1=0$, $x_2=450$ and $Z_{\max} = \text{Birr } 1800$**

Where $x_1, x_2 \geq 0$.

Example 4: The standard weight of a special purpose brick is 5kg and it contains two basic ingredients B_1 and B_2 . B_1 costs Birr 5/kg and B_2 costs Birr 8/kg. Strength considerations dictate that the brick contains not more than 4kg of B_1 and a minimum of 2kg of B_2 . Since the demand for the product is likely to be related to the price of the brick, find graphically the minimum cost of the brick satisfying the above conditions.

Solution

Formulation of LPP model

Let the quantities in kg of ingredients B_1 and B_2 to be used to make the bricks be x_1 and x_2 resp.

Objective is to minimize the cost of the brick.

i.e., Minimize $Z = 5x_1 + 8x_2$

Ans: $x_1=3\text{kg}$, $x_2= 2\text{kg}$, $Z_{\min}=\text{Birr } 31$

Constraints are

On the quantity of the ingredient B₁: $x_1 \leq 4$,

On the quantity of the ingredient B₂: $x_2 \geq 2$,

On the weight of the brick : $x_1 + x_2 = 5$

Where $x_1, x_2 \geq 0$.

Some Special Cases of Linear Programming

There are certain special cases in linear programming problem. Some of these cases are resulted from violations of the basic assumptions of LP. These include:

1. **Multiple optimal solutions:** This is the situation when more than one optimal solution arises. This will happen whenever the objective function equation represents a line parallel to some edge of the bounded solution space.

Example: Solve the following problems graphically

$$\text{Max } Z = 110x_1 + 110x_2,$$

Subject to:

$$x_1 + x_2 \leq 9,$$

$$x_1 \geq 2,$$

$$20x_1 + 50x_2 \leq 360,$$

$$x_2 \geq 3$$

$$x_1, x_2 \geq 0.$$

2. **Unbounded solution:** it exists when an LP problem has no limit on the constraints i.e., the feasible region is not bounded in any respect. Theoretically, the value of the decision variable increases infinitely without violating the feasibility and value of the objective function can increase to infinity. Mostly this situation occurs when the assumption of finiteness is violating.

Example: Solve the following problems graphically

$$\text{Max } Z = 30x_1 + 50x_2,$$

Subject to:

$$3x_1 + x_2 \geq 9,$$

$$x_1 + 2x_2 \geq 12,$$

$$x_1 + 2x_2 \geq 9,$$

$$x_1, x_2 \geq 0.$$

3. **Infeasible solution:** infeasibility is a condition that exists when there is no solution to an LP problem that satisfies all the constraints and non-negativity restrictions. It means that the constraints in the problem are conflicting and inconsistent. Infeasibility depends solely on the constraints and has nothing to do with the objective function

Example: Solve the following problems graphically

$$\text{Max } Z = 3x_1 + 2x_2,$$

Subject to:

$$-2x_1 + 3x_2 \leq 9,$$

$$3x_1 - 2x_2 \leq -20,$$

$$x_1, x_2 \geq 0.$$

Conditions for applications of simplex method

There are some conditions that must be fulfilled for the application of simplex method. These are:

1. The right hand side of each of the constraints b_i should be non-negative.
 - If the linear programming problem has a negative constraint, we should convert it to positive by multiplying both sides by -1. Forexample, if one of the constraints is given by $2x_1 - 4x_2 \geq -8$, we cannot apply simplex method as it is. So, we should convert it to positive by multiply both sides by -1. i.e., $-2x_1 + 4x_2 \leq 8$
2. Each of the decision variables of the problem should be non-negative. If one of the choice variables is not feasible, simplex method cannot be applied. Therefore, feasibility is a necessary condition for application of simplex method.
3. The inequality constraints of resources or any other activities must be converted in to equations.

Since simplex method cannot be applied in the case of inequalities, it is a mandatory that these inequalities should be converted in to equations. There are some variables that help us to convert these inequalities in to equations. These variables are: Slack and surplus variables.

Slack variables: are variables that can change the less than type inequality in to equations.

- Slack variables represent unused amount of a given resource by the activities or idle resources. For example, if one of the constraint is given as:
 - $x_1 \leq 4$ we can convert this in to equations by introducing a slack variable.
 - $x_1 + S_1 = 4$. The values of S_1 will range from zero to four. i.e., if $x_1=0$, $S_1=4$ meaning that all the available resources is not used. If $x_1=4$, $S_1=0$ meaning that all the available resource is fully utilized and there is no idle resource which is left over.

Surplus variables: if the constraint inequality is of greater than type, then we can convert it into equation by subtracting some variable called surplus variable. Surplus variables represent that the requirement of the resource is more than the availability.

- Surplus variables are the negative of the slack variables. For example, if one of the constraint is given as:
 $x_2 \geq 8$, this inequality can be converted to equation by introducing surplus variable as
 $x_2 - S_1 = 8$.

Note: Slack and surplus variables enter in to the objective function with zero coefficients.

Procedures of simplex method

Step 1: Express the problem in standard form.

The given problem is said to be expressed in standard form if the decision variables are non-negative, R.H.S of the constraints are non-negative and the constraints are expressed as equations. Non-negative slack variables are added to the L.H.S of the constraints to convert them in to equations.

Example: $\text{Max } Z = 3x_1 + 4x_2$,

Subject to:

$$x_1 + x_2 \leq 450,$$

$$2x_1 + x_2 \leq 600,$$

Where $x_1, x_2 \geq 0$.

Standard form of the above LP model is:

$$\text{Max } Z = 3x_1 + 4x_2 + 0S_1 + 0S_2,$$

Subject to:

$$x_1 + x_2 + S_1 = 450,$$

$$2x_1 + x_2 + S_2 = 600,$$

Where $x_1, x_2, S_1, S_2 \geq 0$

Step 2: Set up the initial solution

- In order to set up the initial solution, first we need to determine the basic and non-basic variables.
- Slack variables at initial table become basic variables and decision variables are non-basic variables. If we have n unknowns and m constraints, then m variables should be basic and the rest $(n-m)$ variables should be non-basic such that their value is zero.
- Basic variables are those variables whose values are obtained from the basic solution where as non-basic variables are those whose values are assumed as zero in the basic solution.

Example: The initial solution of the above standard form can be expressed in the following simplex table

Contribution/unit C_j		3	4	0	0	
		Body matrix		Identity matrix		
C_B	BV	x_1	x_2	s_1	s_2	b
0	s_1	1	1	1	0	450
0	s_2	2	1	0	1	600

Interpretation:

1. The first row indicates the coefficients C_j of the variables in the objective function and they are unchanged in the subsequent tables.
2. The first column C_B represents the coefficients of the current basic variables in the obj. fun. The second column is the basis column and it represents the basic variables of the current solution.
3. The body matrix (coefficient matrix) represents the coefficients of the constraints. These coefficients represent the amount of resource required to make a unit of decision variable.
4. The identity matrix represents the coefficients of slack variables in the constraints.
5. The b-column/solution values x_B column indicates the quantities of the variable resources or R.H.S values of the constraints or values of the basic variables, S_1 and S_2 in the initial basic solution.

Step 3: Perform optimality test

The test for optimality can be based on interms of the sign of the members of the index row.

Contribution/unit C_j		3	4	0	0	
		Body matrix		Identity matrix		
C_B	BV	x_1	x_2	s_1	s_2	b
0	S1	1	(1)	1	0	450 \Rightarrow key row
0	S2	2	1	0	1	600
	Z_j	0	0	0	0	0
	$\Delta_j = Z_j - C_j$	-3	-4	0	0	

↑K

Where: $Z_j = \sum C_{Bi} a_{ij}$ and $Z = \sum C_{Bi} x_{Bi}$

- ❖ Values in the Z_j represent the contribution lost per unit of the variables. i.e., the amounts by which contribution would be reduced if one of the corresponding variables $x_1, x_2 \dots$ was added.
- ❖ Δ_j is also called the index row. This row determines whether or not the current solution is optimal. Coefficients in this row represent the net profit (or net contribution or net marginal improvement) in the values of the objective function Z for each unit of the respective column variable introduced into the solution. For example, as x_1 increases by one unit Z will increase by 3 units.
- ❖ A positive coefficient in the $Z_j - C_j$ row indicates the amount by which the objective function will be decreased if a unit of the corresponding variable is introduced in to the solution where as a negative coefficient indicates the amount by which the profit will be increased if a unit of the corresponding variable is introduced in to the solution. These coefficients or elements are known as shadow prices or accounting values or imputed values of the resources.
- ❖ The solution is optimal if all the elements of the index row are +ve and zero. If at least one of the elements of the index row is -ve, the given solution is not optimal and goes to the next step. The next iteration will give as the next adjusted basic feasible solution.

Step 4: Iterate towards the optimal solution

In each iteration, the simplex method moves the current basic feasible solution to an improved basic feasible solution. This is done by replacing one current basic variable by a new non-basic variable as follows.

- ❖ **Pivot column:** The column which has the maximum -ve value of $Z_j - C_j$, is the one which should enter the solution (entering variable).
- ❖ **Pivot row:** The element lying at the intersection of key column and key row is called key /pivot element and is enclosed in (). This pivot element corresponds to the leaving variables. Leaving variable = $(x_{bi} / a_{ij}, a_{ij}) < 0$.

To find the new solution:

- If the key element (pivot number) is 1, then the row remains the same in the new simplex table
- If the key element (pivot number) is other than 1, then divide each element in the pivot row (including elements in x_b column) by the pivot number, to find the new values of the row in the new simplex table.

- The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero. i.e.,

No. in new row = No. in old row \pm (No. above or below key element) (corresponding No. in old row)

- ❖ If all ratios are negative or infinity, the solution is unbounded.
- ❖ The next solution will be feasible only if row containing minimum (non-negative) ratio is selected as key row; in case a row containing higher (non-negative) ratio is marked as key row, the next solution will become infeasible and there will be some negative value in the b-column which is illogical.

Contribution/unit		3	4	0	0	
C _j		Body matrix		Identity matrix		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b
4	X ₂	1	1	1	0	450
0	S ₂	1	0	-1	1	150
	Z _j	4	4	4	0	1800 \Rightarrow 2 nd f. solution
	$\Delta_j = Z_j - C_j$	1	0	4	0	

↑ Marginal loss value
↑ Shadow prices

Step 5: Perform optimality test for second feasible solution

If there is a negative number in the index row, then the given solution is not optimal. So we proceed to the step 4 again. However, values in the index row are non-negative the given solution is optimal. Hence the optimal solution is $x_1=0$, $x_2=450$, $Z_{\max}=1800$.

Remarks:

1. The optimal solution obtained above satisfies the non-negativity restrictions. It indicates that no other set of values of x_1 and x_2 results in as high value of Z .
2. Slack variable s_1 does not exist in the basis column. This means that the first resource is fully utilized. However, s_2 exists in the basis column and its value is 150. This means 150 units of the second resources remains unutilized.
3. Elements in the $Z_j - C_j$ row under slack variables indicate the shadow price (or accounting values or imputed values) of resources.
 - ✓ 4 and 0 are called the shadow prices of the first and the second resources respectively.
 - ✓ 4 under s_1 column indicate that if one more unit of the first resource is used, then the value of Z will increase by 4 units. Therefore, if we want to increase Z , it should increase the first resource.
 - ✓ Capacity of the second resource has not been utilized fully; there will be no use increasing it further.

Exercises: Solve the following problems by simplex method and interpret the results.

1. Max $Z = 40x_1 + 80x_2$

Subject to:

$$2x_1 + 3x_2 \leq 48,$$

$$x_1 \leq 15,$$

$$x_1 \leq 10,$$

$$x_1, x_2 \geq 0.$$

Ans. $x_1=9$ units, $x_2=10$ units, $Z_{\max}=1160$

$$s_1=20, s_2=0, s_3=20$$

2. Max $Z = 2x_1 + 5x_2$

Subject to:

$$x_1 + 4x_2 \leq 24, \quad s_1=0, s_2=0, s_3=1970/9$$

$$3x_1 + x_2 \leq 21,$$

$$x_1 + x_2 \leq 9,$$

$$x_1, x_2 \geq 0.$$

Ans. $x_1=0$ units, $x_2=380/9$ units, $x_3=470/3$ units, $Z_{\max}=3200/3$

2.4. Artificial Variables Technique

There are many linear programming problems where slack variables cannot provide a solution. In these problems at least one of the constraints is of (\geq) or (=) type. In such cases we introduce another type of variables called artificial variables.

- Artificial variables are fictitious and have no any physical or economic meaning. They assume the role of slack variables in the first iteration, only to be replaced at a later iteration.
- Artificial variables are merely a device to get the starting basic feasible solution and then the usual simplex procedure will be adopted until the optimal solution is obtained.
- Artificial variables are expressed by capital letter "A" and used for " \geq " type of inequalities and "=" equations.
- Artificial variables enter in to the objective function with "M" coefficient i.e., it inter in to the objective function with -ve value (-M) for maximization and +ve value (+M) for minimization, where $M > 0$.
- M denotes some huge positive number (very huge penalty), for that firm manipulates the artificial variables.

Steps of the big M-method algorithm

Step 1: Express the LP problem in the standard form by introducing slack variables.

- These variables are added to the L.H S of the constraints of (\leq) and subtracted from the constraints of (\geq) type.

Step 2: Add non-negative artificial variable (A) corresponding to constraints having " \geq " and "=" equations.

Step 3: Set up the initial solution

- In order to set up the initial solution, first we need to determine the basic and non-basic variables. In the initial solution, original variables have zero value, so they are non-basic variables.

Step 4: Solve the modified LPP by the simplex method.

While making iterations, using the simplex method, one of the following three cases may arise:

- If no artificial variable remains in the basis and the optimality condition is satisfied, then the solution is an optimal feasible solution.
- If at least one artificial variable appears in the basis at zero level and optimality condition is satisfied, then the solution is optimal feasible solution. But it is a degenerate solution. The constraints are consistent though redundancy may exist in them. Redundancy means that the problem has more than the required number of constraints.
- If at least one artificial variable appears in the basis at a non-zero level (with +ve value in b-column) and the optimality condition is satisfied, then the original problem has no feasible solution. Therefore, the original problem doesn't have feasible solution because it contains artificial variable (a very large penalty M).

Remarks:

- Slack variables are added to the constraints of (\leq) type and subtracted from the constraints of (\geq) type.
- Artificial variables are added to the constraints of (\geq) and ($=$) type. Equality constraints require neither slack nor surplus variables.
- Variables, other than the artificial variables, once driven out in iteration, may re-enter in a subsequent iteration.

Example:

$$\begin{aligned} \text{Max } Z &= 3x_1 - x_2 \\ \text{Subject to:} \\ 2x_1 + x_2 &\leq 2, \\ x_1 + 3x_2 &\geq 3, \\ x_2 &\leq 4, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Step 1: Set up the problem in standard form

$$\begin{aligned} \text{Max } Z &= 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1, \\ \text{Subject to:} \\ 2x_1 + x_2 + s_1 &= 2, \\ x_1 + 3x_2 - s_2 + A_1 &= 3, \\ x_2 + s_3 &= 4, \\ x_1, x_2, s_1, s_2, s_3, A_1 &\geq 0. \end{aligned}$$

Step 2: Find initial basic feasible solution

Substituting $x_1=x_2=s_2=0$, then the initial b.f.s obtained is: $s_1=2, A_1=3, s_3=4, Z=-3M$

Step 3: perform optimality test

Since the $Z_j - C_j$ is -ve under some variable columns the solution is not optimal and can be improved

Step 4: Iterate in to the optimal solution

Successive iteration yields the following tables

Table 1.

		C_j		3	-1	0	0	0	-M	
C_B	BV	x_1	x_2	s_1	s_2	S_3	A_1	b	r	
0	s_1	2	1	1	0	0	0	2	2	
-M	A_1	1	(3)	0	-1	0	1	3	1	
0	S_3	0	1	0	0	1	0	4	4	
	$Z_j = \sum C_B a_{ij}$	-M	-3M	0	M	0	-M	-3M	\Rightarrow i.s	
	$\Delta_j = Z_j - C_j$	-M-3	-3M+1	0	M	0	0			

↑K

Table 2.

		C_j		3	-1	0	0	0	
C_B	BV	x_1	x_2	s_1	s_2	S_3	b	r	
0	s_1	(5/3)	0	1	1/3	0	1	3/5	
-1	x_2	1/3	1	0	-1/3	0	1	3	
0	S_3	-1/3	0	0	1/3	1	3	-9	
	$Z_j = \sum C_B a_{ij}$	-1/3	-1	0	1/3	0	-1	\Rightarrow 2 nd b.f.s	
	$\Delta_j = Z_j - C_j$	-10/3	0	0	1/3	0			

↑K

Table 3.

		C_j		3	-1	0	0	0	
C_B	BV	x_1	x_2	s_1	s_2	S_3	b		
3	x_1	1	0	3/5	1/5	0	3/5		
-1	x_2	0	1	-1/5	-2/5	0	4/5		
0	S_3	0	0	1/5	2/5	1	16/5		
	$Z_j = \sum C_B a_{ij}$	3	-1	2	1	0	1		
	$\Delta_j = Z_j - C_j$	0	0	2	1	0			

Therefore, optimal solution is given by:

$X_1=3/5, x_2=4/5, Z_{max}=1$

Exercise: solve the f.f. LPP

Max $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to:

$x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Hint: the standard form is:

Max $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3$,

Subject to:

$x_1 + 2x_2 + 3x_3 + 0x_4 + A_1 + 0A_2 + 0A_3 = 15$

$$2x_1 + x_2 + 5x_3 + 0x_4 + 0A_1 + A_2 + 0A_3 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 + 0A_1 + 0A_2 + A_3 = 10$$

$$x_1, x_2, x_3, x_4, A_1, A_2, A_3 \geq 0$$

2.5. SPECIAL CASES IN THE SIMPLEX METHOD APPLICATION

1. Tie Breaking in Simplex Method

While operating the simplex iteration, tie may occur and we may face indifference in choosing the entry or the leaving variable. When tie may occur, the following are important to do:

Tie in the choice of entering variables: the non-basic variable that enters the basis is the one that gives the largest per unit improvement in the objective function.

- Variable having maximum –ve value in a maximization problem and the maximum +ve value in the minimization problem in $Z_j - C_j$ row is the entering variable. A tie in the entering variable exists when more than one variable has same largest (or smallest) value. To break the tie any one of them is selected arbitrarily as the entering variable.

Tie in the choice of leaving variables: This is the case when the ratio is the same.

- This tie can be resolved by choosing one of the variables arbitrarily.

2. No leaving basic variable (unbounded objective function)

- This is the case when all elements of the pivot column are –ve or zero that no variable qualifies to be a leaving basic variable.
- In this case no finite solution can be determined. Because one or more of the variables can be increased indefinitely without violating the feasibility.
- The interpretation is that the objective function is not constrained or the value of the objective function Z can increase indefinitely or the model is poorly constructed.

3. Multiple optimal solutions

- This is the case when the LPP have more than one optimal solution.
- In the simplex table if the $Z_j - C_j$ value is zero for a non-basic variable, it indicates that the existence of more than one optimal solution.

4. Infeasible solution

- If all the constraints are not satisfied simultaneously, the model has no feasible solution.
- In the case of big M method if at least one of the artificial variables has positive value and the solution is optimal, then this is the indication of no feasible solution.

2.6. MINIMIZATION CASE OF SIMPLEX METHOD

For most part of finding solution for minimization problem using simplex method are handled in the same fashion as maximization problem. The three key exceptions are:

- The coefficients of artificial variables (M) have positive sign in the objective function.

- The selection of pivot column (entering variables) is based on the largest positive number (value) in the index (Δ_j) row.
- The solution is optimal when all values in the index (Δ_j) row are non positive.

The other alternative method of solving minimization problem is by converting it to maximization. i.e., $\text{Min } Z = \text{Max } (-Z)$

Example

$$\text{Min } Z = 12x_1 + 20x_2$$

Subject to:

$$6x_1 + 8x_2 \geq 100,$$

$$7x_1 + 12x_2 \geq 120,$$

$$x_1, x_2 \geq 0.$$

Standard form:

$$\text{Min } Z = 12x_1 + 20x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

Subject to:

$$6x_1 + 8x_2 - s_1 + A_1 = 100,$$

$$7x_1 + 12x_2 - s_2 + A_2 = 120,$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0.$$

Cj		12	20	0	0	M	M		
C _B	BV	x ₁	x ₂	s ₁	s ₂	A ₁	A ₂	b	r
M	A ₁	6	8	-1	0	1	0	100	25/2
M	A ₂	7	(12)	0	-1	0	1	120	10
	Z _j = $\sum C_B a_{ij}$	13M	20M	-M	-M	M	M	220M	↑I.S
	$\Delta_j = Z_j - C_j$	13M-12	20M-20	-M	-M	0	0		

↑K

Cj		12	20	0	0	M	M		
C _B	BV	x ₁	x ₂	s ₁	s ₂	A ₁	A ₂	b	r
M	A ₁	4/3	0	-1	2/3	1	0	20	15
20	x ₂	7/12	1	0	-1/12	0	1	10	120/7
	Z _j = $\sum C_B a_{ij}$	-35/3+4/3M	20	-M	-5/3+2/3M	M	M	200+20M	
	$\Delta_j = Z_j - C_j$	4/3M-1/3	0	-M	2/3M-5/3	0	0		

↑K

Cj		12	20	0	0	
C _B	BV	x ₁	x ₂	s ₁	s ₂	b
12	x ₁	1	0	-3/4	1/2	15
20	x ₂	0	1	7/16	-3/4	5/4
	Z _j = $\sum C_B a_{ij}$	12	20	-1/4	-9	205
	$\Delta_j = Z_j - C_j$	0	0	-1/4	-9	

CHAPTER 3: DUALITY THEORY AND SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING

3.1. Duality in Linear Programming

In the context of LPP, duality implies that each linear programming problem can be analysed using two different ways which have equivalent solutions. The original problem (primal) can be converted into dual by transposing the rows and columns of the algebraic statement of the primal problem.

The various useful aspects of duality:

- ❖ If the primal contains large number of rows (constraints) and a smaller number of columns (variables), converting into dual reduces computational procedures.
- ❖ Gives additional information as to how the optimal solution changes as a result of changes in coefficients.
- ❖ Calculation of the dual checks the accuracy of the primal.
- ❖ Duality is used to solve LPP in which the initial solution is infeasible.

The original or primal problem

$$\text{Max } Z = \sum_{j=1}^n C_j X_j$$

Subject to

$$\sum_{j=1}^n a_{ij} X_j \leq b_i$$

$$X_j \geq 0$$

The dual problem

$$\text{Min } Z = \sum_{j=1}^n b_j y_j$$

Subject to

$$\sum_{j=1}^n a_{ji} y_i \geq c_j$$

$$y_i \geq 0$$

Rules of transformation of the primal to the dual

The following rules are important to note when we develop a dual problem:

1. If the primal contains n variables and m constraints, the dual will contain m variables and n constraints.
2. The maximization problem in the primal becomes the minimization problem in the dual and vice versa.
3. The maximization problem has (\leq) constraints while the minimization problem has (\geq) constraints.
4. Constraints of (\leq) type in the primal become (\geq) type in the dual and vice versa.
5. The coefficient matrix of the constraints of the dual is the transpose of the primal.
6. The constants c_1, c_2, \dots, c_n in the objective function of the primal appear in the constraints of the dual.
7. The constants b_1, b_2, \dots, b_m in the constraints of the primal appear in the objective function of the dual.

Example:

Primal

$$\text{Max } Z = 2x_1 + 3x_2$$

Subject to:

$$4x_1 + 3x_2 \leq 18,$$

$$5x_1 + 2x_2 \leq 19,$$

$$x_1, x_2 \geq 0.$$

Dual

$$\text{Min } Z' = 18y_1 + 19y_2$$

Subject to:

$$4y_1 + 5y_2 \geq 2,$$

$$3y_1 + 2y_2 \geq 3,$$

$$y_1, y_2 \geq 0.$$

Economic Interpretation of Dual Problem

a. The objective function of the dual problem is given by the equation:

$$Z' = b_1y_1 + b_2y_2 + b_3y_3 - - - + b_my_m$$

- Each b_iy_i is the current contribution to the objective function by having b_i units of resource i available for primal problem. Thus, y_i interpreted as the contribution to the objective function per unit of resource i when the current set of basic variables is used to obtain the primal solution.
- In other words, y_i 's are the shadow prices.

b. The constraint part which can be specified as: $\sum a_{ji}y_i \geq C_j$.

- Since each unit of activity j in the primal problem consumes a_{ji} unit of resource i , then $\sum a_{ji}y_i$ is interpreted as current contribution to the objective function of the mix of resources that would be consumed if one unit of activity j were produced.
 - C_j is interpreted as: per unit contribution to the objective function from activity j .
- ❖ Therefore, $\sum a_{ji}y_i \geq C_j$ can be interpreted as the current contribution to the objective function of resources must be at least as much as the unit contribution to the objective function from activity j , otherwise the use of resources would not be the best possible use. $Y_i \geq 0$, meaning that the current contribution to the objective function of resource i must be non-negative otherwise it is better not to use the resource at all. If we are using the resource i while its contribution to the objective function is negative, we are devastating the resource for nothing.

Solving a Dual Linear Programming

The methods of solving the dual linear programming are similar with that of the original linear programming problems. We can use either graphic method or simplex method.

- ❖ From the optimal simplex table of the dual problem, it is possible to read the optimal solution of the primal problem. That is, the shadow prices of the dual problem indicate the optimal values of original variables.

Example:

Primal

$$\begin{aligned} \text{Max } Z &= 2x_1 + 3x_2 \\ \text{Subject to:} \\ 4x_1 + 3x_2 &\leq 18, \\ 5x_1 + 2x_2 &\leq 19, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Dual

$$\begin{aligned} \text{Min } Z' &= 18y_1 + 19y_2 \\ \text{Subject to:} \\ 4y_1 + 5y_2 &\geq 2, \quad \text{OR} \quad 4y_1 + 5y_2 \geq 2, \\ 3y_1 + 2y_2 &\geq 18, \\ y_1, y_2 &\geq 0. \end{aligned}$$

Standardize the problem

$$\begin{aligned} \text{Max } Z &= 2x_1 + 3x_2 + 0s_1 + 0s_2 \\ \text{Subject to:} \\ 4x_1 + 3x_2 + s_1 &= 18, \\ 5x_1 + 2x_2 + s_2 &= 19, \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

$$\begin{aligned} \text{Max } (-Z') &= -18y_1 - 19y_2 + 0s_1 + 0s_2 - MA_1 - MA_2 \\ \text{Subject to:} \\ 4y_1 + 5y_2 - s_1 + A_1 &= 2, \\ 3y_1 + 2y_2 - s_2 + A_2 &= 18, \\ y_1, y_2, s_1, s_2, A_1, A_2 &\geq 0. \end{aligned}$$

Solution for the primal

$$\begin{aligned} x_1 &= 0, \quad x_2 = 6 \\ s_1 &= 0, \quad s_2 = 7 \end{aligned}$$

Solution for the dual

$$\begin{aligned} y_1 &= 1, \quad y_2 = 0 \\ s_1 &= 2, \quad s_2 = 0 \end{aligned}$$

Index row

$$s_1 = 1, s_2 = 0$$

$$x_1 = 2, x_2 = 0$$

$$y_1 = 0, y_2 = 7$$

$$s_1 = 0, s_2 = 6$$

3.2. Sensitivity Analysis

One of the assumptions of a LPP is the assumptions of certainty. This assumption implies that the coefficients of a LPP are completely known (determined) and do not change during the period being studied.

- C_j - per unit (profit or cost) contribution of each decision variable
- b_i - availability of resources
- a_{ij} – technical coefficients (per unit resource consumptions or production of each decision variables) are constant and known with certainty. However, in reality, these coefficients are subject to change with time or error.

If such changes will be occurred, there should be a means to check for how long the present optimal solution continues as optimal. The method of evaluating the degree to which the present optimal solution is continued as optimal is called sensitivity analysis.

Sensitivity analysis is concerned with the study of ‘sensitivity’ of the optimal solution of LP problem with changes in parameters. In this case, we are going to determine the range (both lower and upper) over which the linear programming model parameters can change with out affecting the current optimal solution.

Changes in C_j of variables

One of the important parameters of linear programming problem is the coefficient of objective function. The test of the sensitivity of the coefficient of objective function involves finding the range of values within which each C_j can lie without changing the current optimal solution.

a. Changes in coefficient C_j of non-basic variable x_j

Among the variables included in the linear programming problem some are non-basic variables. If C_j is the coefficient of non-basic variable x_j :

1. it doesn't affect any of C_j values of basic variables.
2. it doesn't affect any of Z_j values.
3. it affects $(Z_j - C_j)$ values.

In addition, if optimality is to be maintained, for maximization LPP, coefficients in the index row should be non-negative $(Z_j - C_j) \geq 0$ for all j .

Example:

$$\text{Max } Z = 2x_1 + 3x_2$$

Subject to:

$$4x_1 + 3x_2 \leq 18,$$

$$5x_1 + 2x_2 \leq 19,$$

$$x_1, x_2 \geq 0.$$

		C _j		2	3	0	0		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b	r		
0	s ₁	4	(3)	1	0	18			
0	s ₂	5	2	0	1	19			
	Z _j =∑ C _B a _{ij}	0	0	0	0	0			
	Δ _j = Z _j - C _j	-2	-3	0	0				

		C _j		2	3	0	0		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b	r		
3	x ₂	4/3	1	1/3	0	6			
0	s ₂	7/3	0	-2/3	1	7			
	Z _j =∑ C _B a _{ij}	4	3	1	0	18			
	Δ _j = Z _j - C _j	2	0	1	0				

- In the final table, x₁ and s₁ are non-basic variable whereas x₂ and s₂ are basic variables.
- C₁=2 is the coefficient of non-basic variable x₁. Let c₁ is subject to change by Δc₁

$$C_1' = C_1 + \Delta C_1$$

To maintain the optimality condition: Δ_j (Z_j- C'_j) ≥ 0

$$(Z_j - C'_j) \geq 0$$

$$Z_j - (C_j + \Delta C_1) \geq 0$$

$$Z_1 - C_1 \geq \Delta C_1$$

$$4 - 2 \geq \Delta C_1$$

$$2 \geq \Delta C_1$$

$$\Delta c_1 \geq (-\infty, 2]$$

Therefore, the range over which the parameter (C₁) can change without affecting the current optimal solution (0, 6) is (-∞, 4].

Simply for non-basic variable x_k, the coefficient (C_j) can change without affecting the optimal solution if Z_j- C_j ≥ ΔC_k.

b. Change in Coefficient C_j of Basic Variable X_j

Change in the coefficients of basic variable can affect Z_j, Δ_j and the value of the objective function.

- ❖ For the above table (x₂ is basic variable). And suppose C₂ is changed by ΔC₂.

$$\text{i.e., } C_2' = 3 + \Delta C_2$$

		C _j		2	3	0	0		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b	r		
3+Δc ₂	x ₂	4/3	1	1/3	0	6			
0	s ₂	7/3	0	-2/3	1	7			
	Z _j =∑ C _B a _{ij}	4+4/3Δc ₂	3+Δc ₂	1+1/3Δc ₂	0	18+6Δc ₂			
	Δ _j = Z _j - C _j	2+4/3Δc ₂	0	1+1/3Δc ₂	0				

The current solution will be optimal if Δ_j (Z_j- c'_j) ≥ 0.

$$\text{i.e., } 2 + 4/3\Delta c_2 \geq 0 \Rightarrow \Delta c_2 \geq -3/2$$

$$1 + \frac{1}{3}\Delta c_2 \geq 0 \Rightarrow \Delta c_2 \geq -3$$

$$c_2' = 3 + (-3/2) = 1.5$$

The common value is $\Delta c_2 \geq -3 \Rightarrow \Delta c_2 = [-3/2, +\infty)$.

Therefore, the range over which the parameter (c_2) can change without affecting the current optimal solution (0, 6) is $[1.5, +\infty)$.

CHAPTER 4: INTEGER PROGRAMMING

In LPP all the decision variables were allowed to take any non-negative values as it is quite possible and appropriate to have fractional values in many situations. However, there are several conditions in business and industry that lead to planning models involving integer valued variables. That is, many practical problems involve decision variables that must be integer valued.

There are cases that the fractional values of variables may be meaningless in the context of the actual decision problem. For example:

- ❖ In production, manufacturing is frequently scheduled in terms of discrete quantities.
- ❖ Production of livestock involves discrete number of animals

Integer programming models require all of the assumptions implicit in linear programming models except that certain specific variables must be integer valued.

- ❖ When all the variables are constrained to be integers, it is called a pure-integer programming model, and in case only some of the variables are restricted to have integer values, the problem is said to be a mixed-integer programming problem.

There are two methods used to solve IPP, namely

- i. Gomory's fractional cut method
- ii. Branch and bound method (search method)

4.1. Gomory's Fractional Cut Method (Algorithm)

This method consists of first solving the IPP as an ordinary LPP by ignoring the restriction of integer values and then introducing a new constraint to the problem such that the new set of feasible solution includes all the original feasible integer solution, but does not include the optimum non-integer solution initially found. This new constraint is called Fractional Cut or Gomorian constraint. Then the revised problem is solved using simplex till an optimal integer solution is obtained.

Gomory's Fractional Cut Method involves the following steps:

1. Solve the IPP as an ordinary LPP by ignoring the restriction of integer values
2. Test the integrality of the optimum solution.
 - If all $Xb_i \geq 0$ and are integers, an optimum integer solution is obtained
 - If at least one b_i is not an integer then go to the next step
3. Select the source row
 - If only one Xb_i is non-integer, the row corresponding to non-integer solution will be the source row

- If more than one Xb_i is non-integer, choose the row having the largest fractional part (f_i) as a source row.
In this case write each Xb_i as $(Xb_i = x_{b_i} + f_i)$: Where x_{b_i} is integer part of Xb_i (solution) and f_i is the positive fractional part of Xb_i . $\Rightarrow 0 < f_i < 1$
 - If there is a tie when two or more rows have the same positive larger f_i , select arbitrarily.
4. Find the new constraint (Gomorian constraint) from the source row.
- i. $\sum a_{ij}X_j = Xb_i$
 - ii. $\sum ((a_{ij})+f_i)X_j = x_{b_i} + f_i$
 - iii. $\sum f_i X_j \geq f_i$
 - iv. $\sum (-f_i)X_j \leq -f_i$
 - v. $\sum (-f_i)X_j + G_i = -f_i \Rightarrow$ Gomorian constraint and G is Gomorian slack.
5. Add the new (Gomorian) constraint at the bottom of the simplex table obtained in step 1 and find the new feasible solution.
- Note that Gomorian slack enter in to the objective function with zero coefficient
6. Test the integrality of the new solution
- Again if at least one Xb_i (solution) is not an integer, go to step 3 and repeat the procedure until an optimal integer solution is obtained.
- Example: Find the optimal integer solution for the following LPP

Max $Z = x_1 + x_2$
 Subject to:
 $3x_1 + 2x_2 \leq 5,$
 $x_2 \leq 2,$
 $x_1, x_2 \geq 0$ and are integers

Solution

1. Solve the IPP as an ordinary LPP by ignoring the restriction of integer values

Max $Z = x_1 + x_2 + 0s_1 + 0s_2$
 Subject to:
 $3x_1 + 2x_2 + s_1 = 5,$
 $x_2 + s_2 = 2,$
 $x_1, x_2, s_1, s_2 \geq 0$ and are integers

		Cj		1	1	0	0		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b	r		
0	s ₁	3	2	1	0	5			
0	s ₂	0	1	0	1	2			
	$Z_j = \sum C_B a_{ij}$	0	0	0	0	0			
	$\Delta_j = Z_j - C_j$	-1	-1	0	0				

		Cj		1	1	0	0		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b	r		
1	x ₁	1	2/3	1/3	0	5/3			
0	s ₂	0	1	0	1	2			
	$Z_j = \sum C_B a_{ij}$	1	2/3	1/3	0	5/3			
	$\Delta_j = Z_j - C_j$	0	-1/3	1/3	0				

Cj		1	1	0	0		
C _B	BV	x ₁	x ₂	s ₁	s ₂	b	r
1	x ₁	1	0	1/3	-2/3	1/3	
1	x ₂	0	1	0	1	2	
	Z _j =∑ C _B a _{ij}	1	1	1/3	1/3	7/3	
	Δ _j = Z _j - C _j	0	0	1/3	1/3		

2. Test the integrality of the optimum solution.

The solution is $x_1 = 1/3$, $x_2 = 2$ and $Z = 7/3$. Since $x_1 = 1/3$ is non-integer solution so that the current feasible solution is not optimal integer solution.

3. Select the source row. \Rightarrow The row corresponding to non-integer solution.

$$x_1 + 1/3s_1 - 2/3s_2 = 1/3,$$

4. Find the new constraint (Gomorian constraint) from the source row.

$$x_1 + 1/3s_1 - 2/3s_2 = 1/3,$$

$(1+0)x_1 + (0+1/3)s_1 + (-1+1/3)s_2 = 0+1/3$, and eliminate the integer part

$$1/3s_1 + 1/3s_2 \geq 1/3,$$

$$-1/3s_1 - 1/3s_2 \leq -1/3$$

$-1/3s_1 - 1/3s_2 + G_1 = -1/3 \Rightarrow$ Gomorian constraint and G_1 is Gomorian slack.

5. Add the new (Gomorian) constraint at the bottom of the simplex table obtained in step 1 and find the new feasible solution.

Cj		1	1	0	0	0	
C _B	BV	x ₁	x ₂	s ₁	s ₂	G ₁	b
1	x ₁	1	0	1/3	-2/3	0	1/3
1	x ₂	0	1	0	1	0	2
0	G ₁	0	0	-1/3	-1/3	1	-1/3
	Z _j =∑ C _B a _{ij}	1	1	1/3	1/3	0	7/3
	Δ _j = Z _j - C _j	0	0	1/3	1/3	0	

\Rightarrow The solution is not feasible because the solution ($G_1 = -1/3$) is -ve. Therefore, G_1 will leave the base.

➤ Select the new constraint as a pivot row. In order to select the entering variable, use:

$$\text{Max}\left(\frac{\Delta_j}{a_{ij}}, a_{ij} \leq 0\right)$$

➤ Therefore, $\text{Max}\left(\frac{\Delta_j}{a_{ij}}, a_{ij} \leq 0\right) = \text{Max}\left(\left(\frac{1}{3}, \frac{-1}{3}\right), \left(\frac{1}{-1}, \frac{-1}{3}\right)\right) = (-1, -1)$

Choose arbitrarily (Let s_1 is chosen as entering variable)

Cj		1	1	0	0	0	
C _B	BV	x ₁	x ₂	s ₁	s ₂	G ₁	b
1	x ₁	1	0	0	-1	1	0
1	x ₂	0	1	0	1	0	2
0	s ₁	0	0	1	1	-3	1
	Z _j = $\sum C_B a_{ij}$	1	1	0	0	1	2
	$\Delta_j = Z_j - C_j$	0	0	0	0	1	

Since all $\Delta_j \geq 0$ and all $Xb_i \geq 0$ and are integers, the current feasible solution is optimal integer solution.

Therefore, the solution is $x_1=0$, $x_2=2$ and $Z=2$

Exercises:

1. Max $Z = 4x_1 + 3x_2$

Subject to:

$$x_1 + 2x_2 \leq 4,$$

$$2x_1 + x_2 \leq 6,$$

$x_1, x_2 \geq 0$ and are integers

Ans: $x_1=3, x_2=0$ and $Z=12$

2. Max $Z = 3x_1 + 4x_2$

Subject to:

$$3x_1 + 2x_2 \leq 8,$$

$$x_1 + 4x_2 \leq 10,$$

$x_1, x_2 \geq 0$ and are integers

Ans: $x_1=1, x_2=2$ and $Z=11$

4.2. Branch and bound method (search method)

CHAPTER 5: SPECIAL TYPES OF LINEAR PROGRAMMING

5.1. Transportation Problems

One important application of linear programming is the area of physical distribution (transportation) of goods from several supply centers (origins) to several demand centers (destinations).

It is easy to express a transportation problem mathematically in terms of an LP model, which can be solved using the simplex method. However, since transportation problem involves a large number of variables (transportation/shipping routes) and constraints, it takes a long time to solve it. Therefore, other methods (transportation algorithm) have been developed for this purpose.

❖ The structure of transportation problem involves a large number of transportation /shipping routes from several supply origin to several demand destinations.

Objective: the objective is to determine the amount of commodities which should be transported from several sources to different destinations, at the minimum transportation cost and /or time.

- Sources or origins are the places where goods originate from (like plants, warehouses etc)
- Destinations are places where goods are to be shipped

The transportation algorithm applies to minimize the total cost of transporting a homogenous commodity (product) from supply origins to demand destinations. However, it can also be applied to the maximization of some total value or utility, in such a way that the profit is maximized.

General Transportation Problem Model

A transportation problem model, which has ‘m’ sending locations (origins) and ‘n’ receiving locations (destinations), provides a framework for presenting all relevant data. These are:

- Quantity supply of each origin (SS_i)
- Quantity demand of each destination (DD_i)
- Unit transportation cost from each origin to each destination (C_{ij})

		Destination				Total SS
		D ₁	D ₂	D _n	
Source or Origin	From \ To					
		S ₁	X ₁₁ c ₁₁		X _{1n} c _{1n}
	S ₂	X ₂₁ c ₂₁		X _{2n} c _{2n}	SS ₂
	• • •	• • •	• • •	• • •	• • •	• • •
	S _m	X _{m1} c _{mi}	X _{m2} c _{m2}	X _{mn} c _{mn}	SS _m
	Total DD	DD ₁	DD ₂		DD _m	ΣSS ΣDD

Where:

- SS_i is the total quantity of commodity available at origin i total supply of origin i)
- DD_i is total quantity of commodities needed at destinations j (total demand of destination)
- C_{ij} measures (represents) the costs of shipping (transporting) one unit of commodity from source i to destination j.
- X_{ij} is the quantity of commodities transported from ith origin to jth destination.

The transportation algorithm requires the assumption that:

- All goods are homogenous, so that any origin is capable of supplying to any destination.
- Transportation costs are a linear function of the quantity shipped over any route.
- All the assumptions of the general LPP also hold in the special case of the transportation problem.

The Linear Programming Representation of a Transportation Model

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} x_{ij}$$

Subject to :

$$\sum_{j=1}^n x_{ij} = SS_i, i = 1, 2, \dots, m \text{ (supply constraints)}$$

$$\sum_{i=1}^m x_{ij} = DD_j, j = 1, 2, \dots, n \text{ (demand constraint)}$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

Minimize (total transportation cost) $Z = C_{11}X_{11} + C_{12}X_{12} + C_{13}X_{13} + \dots + C_{mn}X_{mn}$

Subject to:

Capacity constraints (SS constraints)

$$X_{11} + X_{12} + \dots + X_{1n} = SS_1$$

$$X_{21} + X_{22} + \dots + X_{2n} = SS_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$X_{m1} + X_{m2} + \dots + X_{mn} = SS_m$$

Requirements constraints (DD constraint)

$$X_{11} + X_{21} + \dots + X_{m1} = DD_1$$

$$X_{12} + X_{22} + \dots + X_{m2} = DD_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$X_{n1} + X_{n2} + \dots + X_{nm} = DD_n$$

$$X_{ij} \geq 0 \text{ for all } i \text{ and } j$$

Before applying the transportation techniques (methods) to solve a specific problem, the problem should satisfy the following conditions.

1. Supplies (SS) and requirements (DD) must be expressed in the same unit

- This condition means that shipments received at any destination from different sources must be indistinguishable. In other words, all shipment must be measured in homogenous units

2. Total supply must equal to total demand $(\sum_{i=1}^m SS_i = \sum_{j=1}^n DD_j)$

- The problem satisfying this condition is called balanced transportation problem; otherwise it is known as unbalanced transportation problem.
- The condition $(\sum_{i=1}^m SS_i = \sum_{j=1}^n DD_j)$ is the necessary and sufficient condition for the existence of feasible solution to the transportation problem.

Methods of Solving Transportation Problem

The solution to the transportation problem can be obtained in two stages, namely initial solution and improved solution.

1. Methods to get the initial solution: even if there are different methods of such types, the following three common methods can be used.

- The North-West Corner Method (NWC)
- Minimum cost method (MCM), and
- The Vogel's Approximation Method (VAM).

2. Methods used to improve the solution: Among several available methods, in this section the following two methods are presented.

- The Stepping Stone Method
- The Modified Distribution Method

Any initial solution (to the transportation problem) can be accepted if it satisfies the following conditions:

1. It must satisfy the supply and demand constraints
 - The sum of allocations along each row equals the supply of each row and the sum of the allocations down each column equals the demand or requirement of each column. This condition is known as rim conditions.
2. It must be a basic feasible solution
 - A feasible solution is known as basic feasible solution if the number of allocation is equal to $m+n-1$ where m is the number of rows and n is the number of column in a transportation table.

The North-West Corner Method (NWC)

The North-West Corner Method (NWC) is a systematic approach for developing an initial feasible solution. Its chief advantage is that it is simple to use and easy to understand. Its chief drawback is that it does not take transportation cost into account. Consequently such a solution may require much additional effort to obtain the optimal solution.

The North-West Corner Method (NWC) gets its name because the starting point for the allocation process is the upper left-hand (northwest) corner of the transportation table.

Steps of solving transportation problem using North-West Corner Method (NWC)

1. Start with the cell at the upper left hand corner, and allocate as much as possible to that cell.
2. i) If $d_1 > s_1$, move down vertically to the second row and make the second allocation of the magnitude $X_{21} = \min(s_2, d_1 - X_{11})$ in cell (2, 1).
ii) If $d_1 < s_1$, move right horizontally to the second column and make the second allocation of the magnitude $X_{12} = \min(s_1 - X_{11}, d_2)$ in cell (1, 2).
iii) If $d_1 = s_1$, there is a tie for the second allocation because both the row and column are exhausted. Therefore, move diagonally to the next row and column and make the second allocation of the magnitude $X_{22} = \min(s_2, d_2)$ in cell (2, 2).

3. Repeat step 2 until all the rim requirements are satisfied.
4. Check to be sure that all rim conditions are satisfied and the numbers of occupied cells are equal to $m+n-1$.

Examples: a). Obtain the initial basic feasible solution of the following problem whose cost and rim requirement table is given below.

	D ₁	D ₂	D ₃	SS
S ₁	2	7	4	5
S ₂	3	3	1	8
S ₃	5	4	7	7
S ₄	1	6	2	14
DD	7	9	18	

b) A company has three production facilities S₁, S₂ and S₃ with production capacity of 7, 9 and 18 units per week of a product, respectively. These units are to be shipped to four warehouses D₁, D₂, D₃ and D₄ with requirements of 5, 8, 7 and 14 units per week, respectively. The transportation cost (in birr) per unit between the warehouses in the table below.

	D ₁	D ₂	D ₃	D ₄
S ₁	19	30	50	10
S ₂	70	30	40	60
S ₃	40	8	70	20

Obtain the initial basic feasible solution using NWC method.

- Total cost=Birr 1015

Minimum Cost Method (MCM)

Since the objective is to minimize the total transportation cost, we must try to transport as much as possible through those routes (cells) where the unit transportation cost is lowest.

- This method takes into account the minimum unit cost of transportation for obtaining initial solution and can be summarized as follows:

The steps involved in Minimum Cost Method are:

1. Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell, so that entire supply is exhausted or demand is satisfied.
 - In case the smallest unit cost cell is not unique, then select the cell where maximum allocation can be made.
2. After adjusting the supply and demand for all uncrossed-out rows or columns, repeat the procedure with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much possible to that cell.
3. Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied.
4. Check to be sure that all rim conditions are satisfied and that $m+n-1$ cells are allocated.

Example: find the initial basic feasible solution using MCM.

	D ₁	D ₂	D ₃	D ₄	SS
S ₁	19	30	50	10	7
S ₂	70	30	40	60	9
S ₃	40	8	70	20	18
DD	5	8	7	14	34

Ans: Total cost=Birr 814

The Vogel's Approximation Method (VAM)

It is a better method for obtaining an initial solution. However, it involves more computational effort than the NWC and MCM. This method recognizes the costs information to find the initial solution.

- It normally provides a solution which is optimal or very close to optimal

The steps involved in Vogel's Approximation Method (VAM) are:

1. For each row and column, compute the difference between the lowest cost element and the next lowest cost element of the row/column.
 - The difference between the lowest cost and the next lowest cost for a given row or column is said to be the penalty number for that row or column.
2. Select the row or column having the largest penalty number. If there is a tie it can be broken by selecting the cell where minimum cost of transportation is found.
3. Assign (allocate) the maximum number of units to the lowest cost cell in the corresponding row or column selected in step 2. And then eliminate a row or column whose supply is exhausted or requirement is satisfied.
4. Repeat step 1 to 3 until a complete initial solution is obtained.
5. Check to be sure that all rim conditions are satisfied and that $m+n-1$ cells are allocated.

Exercises:

- a). Find the initial basic feasible solution of the following problem using VAM

	D ₁	D ₂	D ₃	D ₄	SS
S ₁	19	30	50	10	7
S ₂	70	30	40	60	9
S ₃	40	8	70	20	18
DD	5	8	7	14	34

Ans: Total cost=Birr 779

- b). Determine an initial basic feasible solution for the following transportation problem by using NWC, MCM and VAM.

	D ₁	D ₂	D ₃	D ₄	SS
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
DD	200	225	275	250	

Methods for testing the optimality and improving the solution

Once an initial solution is obtained, the next step is to check its optimality. An optimal solution is a solution which is best among all feasible solution to achieve the objective function.

- It is the one where there is no other set of transportation routes (allocations) will further reduce the total transportation cost.

In order to test the optimality of the initial solution to a given transportation problem, we have to evaluate each unoccupied cell (unused route) in the transportation table in terms of an optimality to reduce total transportation cost.

There are several methods for testing and improving the initial solution of the transportation problem. The two important ones are:

- a) The Stepping-Stone Method, and
- b) The Modified-Distribution (MODI) Method

The Stepping-Stone Method

The Stepping-Stone Method is an iterative procedure that exchanges one variable that is in the basis (i.e. occupied cell), with other variable that is not in the basis (i.e. non-basic or empty or non-occupied cell) in such a way that the transportation cost is improved.

The procedure involved in this method will be as follows:

Step 1: Test the optimality

- In this case each variable with a value of zero empty cells is a candidate to be selected as entering variable. Now the question is “which non basic variables (empty cell) should be included in the new solution?”

To answer this question, the stepping-stone method involves (performs) the following:

- i. Construct the loop for each zero cell (empty cell). To construct a loop,
 - Start with the zero-cell, and then move horizontally and vertically with corner cell occupied. That means, each time move in a right angle to the last move.
 - Adding and subtracting the unit shipment cost in an alternating fashion until we end up at the zero cell.

Note:

1. There is only one loop for any given zero (empty)-cell
 2. The loop crosses itself as long as the intersection is at the right angle
 3. The loop can be traversed in either clock-wise or anti-clock wise direction
- ii. Compute the net contribution or improvement index for each zero cell
 - Improvement index indicates the opportunity of reducing total transportation cost.
 - If all values of improvement index are +ve, the total transportation cost is optimal and can't be improved. Therefore, the solution is optimal solution. However, if at least

one value of the improvement index is –ve, the total transportation cost can be improved.

Step 2: identifying the entering variable

- Select the cell with the most negative net contribution (improvement index) as entering variable.

Step 3: Identifying the leaving variable

- Examine the loop for entering variable, selected in the previous step, with minus sign and select the one with lowest (smallest) shipment (allocation) as leaving variable.

Step 4: Generate a new solution

- The new solution can be generated by adding the shipment amount in leaving variable cell to each plus sign cells and subtracting the amount from each minus cell in the selected loop.

Step 5: Check for optimality (back to step 1)

- a. Construct the loop for each zero-cell (empty cell)
- b. Compute the net contribution or improvement index for each zero cell

Example: obtain the optimum amount of shipments that minimize the transportation cost.

	D ₁	D ₂	D ₃	D ₄
S ₁	19	30	50	10
S ₂	70	30	40	60
S ₃	40	8	70	20

Ans: the minimum shipment cost is Birr 743

The Modified Distribution (MODI) Method

It is an efficient method for determining the entering variable from available non basic variable.

This method is based on the dual of the transportation problem or it uses shadow price U_i for the source (supply constraint) and V_j for the capacity (demand constraint).

- U_i is the relative implicit contribution (value) of an additional unit of capacity (SS) at sources i.
- V_j is the relative implicit contribution (value) of an additional unit of requirement (DD) at destination j.

Theorem (conditions): For balanced transportation problem,

- For occupied cell, $U_i + V_j = C_{ij}$
- For empty cells $C_{ij} - (U_i + V_j) = \Delta_{ij}$
 - ✓ $U_i + V_j$ represent the opportunity cost of shipping one unit from source i to requirement j and
 - ✓ $C_{ij} - (U_i + V_j)$ is the per unit net contribution to the objective function for opening a route from i to j which is denoted by Δ_{ij} .

Steps involved in MODI method to test and improve the solution are the followings

Step 1: From the initial basic solution, calculate U_i and V_j for each row and column using the occupied cells.

- To start with, assign zero for a particular U_i or V_j where there are maximum number of allocations in a row or column. If there is a tie select arbitrarily.
- Then, compute U_i 's and V_j 's for other rows and columns, respectively, using the relation $C_{ij} = U_i + V_j$ for all occupied cells (i, j).

Step 2: For unoccupied (empty) cells, calculate the net contribution by using the relation:

$$\Delta_{ij} = C_{ij} - (U_i + V_j) \text{ for all } i \text{ and } j$$

- i. Compute the sum of U_i and V_j and write that at the bottom left corner of that empty cell.
- ii. Compute the net contribution and write that at the bottom right corner of that empty cell.

Step 3: Examine the sign of Δ_{ij} for empty cell

- This step gives us the opportunity criteria

 - a. If $\Delta_{ij} \geq 0$ for all empty cell, the current basic feasible solution is optimum.
 - b. If $\Delta_{ij} = 0$ for some empty cell, then the current solution will remain optimum but an alternative solution exists.
 - c. If $\Delta_{ij} \leq 0$ for one or more empty cells, the current basic feasible solution is not optimum. So that an improved solution can be obtained by entering unoccupied cell (i, j) in the basis.

Step 4: Select an empty cell having the largest –ve number of Δ_{ij} as entering variable (cell).

Step 5: Identify the leaving variable (cell).

- The method (criteria) of identifying the leaving variable and determining the new solution is the same as the stepping-stone method.

 - a) Construct the loop (closed path) for entering variable (cell)
 - b) Assign +ve and –ve sign alternatively
 - c) Examine the loop with minus sign and select the cell with lowest (smallest) shipment (allocation) as leaving variable.

Step 6: Generate a new solution

- The new solution can be generated by adding the shipment (allocation) amount in leaving cell to each plus sign cells and subtracting the amount from each minus cell in the selected loop.

Step 7: Check for optimality of the current solution.

- If not optimal, repeat the steps from step 1 till an optimum basic feasible solution is obtained.
- The optimum solution is obtained when for all empty cell, $\Delta_{ij} \geq 0$

Exercises:

1. Find the optimum solution of the following problem using MODI

	D ₁	D ₂	D ₃	D ₄	SS
S ₁	19	30	50	10	7
S ₂	70	30	40	60	9
S ₃	40	8	70	20	18
DD	5	8	7	14	34

Ans: Total cost=Birr 743

2. A firm manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units, respectively during this month. The firm had made a commitment

to cell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. find the minimum possible transportation cost of shifting the manufactured product to the five customers. The net unit cost of transporting from the three plants to the five customers is given below:

		Customers				
		A	B	C	D	E
Plants	I	4	1	3	4	4
	II	2	3	2	2	3
	III	3	5	2	4	4

Ans: (I, B) =45, (I, E) =15, (II, A) =17, (II, D) =18, (III, A) =5, (III, C) =20, (III, E) =15, Total cost=290

Unbalanced Supply and Demand

For a feasible solution to exist, it is necessary that the total supply must equal to demand.

➤ That is, total supply =total demand $(\sum_{i=1}^m SS_i = \sum_{j=1}^n DD_j)$.

However, a situation may arise when the total available supply is not equal to the total demand. The following two cases may arise:

a) If total supply exceeds total demand, $(\sum_{i=1}^m SS_i > \sum_{j=1}^n DD_j)$

➤ An additional column (a dummy demand centre) can be added to the transportation table to absorb the excess supply. In this case, the unit transportation cost for the cell in this column is set equal to zero because these represent product items that are not being made (not being sent).

b) If total demand exceeds total supply, $(\sum_{i=1}^m SS_i < \sum_{j=1}^n DD_j)$

➤ A dummy row (a dummy supply centre) can be added to the transportation table to account for the excess demand. In this case, the unit transportation cost here also for the cells in the dummy row are equal to zero.

Example: A product is manufactured by four factories A, B, C and D. The unit production costs are Birr 2, 3, 1 and 5, respectively. Their production capacities are 50, 70, 30 and 50 units respectively. These factories supply the product to four stores, demands of which are 25, 35, 105 and 20 units respectively. Unit transportation cost in dollars from each factory to each store is given in the table below.

		Stores			
		I	II	III	IV
Factories	A	2	4	6	11
	B	10	8	7	5
	C	13	3	9	12
	D	4	6	8	3

Determine the extent of deliveries from each of the factories to each of the stores so that the total production and transportation cost is minimum.

Degeneracy and Its Resolution of Transportation Problems

A basic feasible for the general transportation problem must consist of exactly $m+n-1$ positive allocations in independent positions in the transportation table. A solution will be called degenerate when the number of occupied cells is less than the required number, $m+n-1$.

- In such cases, the current solution cannot be improved because it is not possible to draw a closed path for every occupied cell. In addition the values of dual values U_i and V_j which are used to test the optimality cannot be computed. Thus, we need to remove the degeneracy to test and improve the given solution.

The degeneracy in the transportation problems may occur when obtaining an initial solution or at any stage while moving towards optimal solution. This happens when two or more occupied cells with the same minimum allocation become unoccupied simultaneously.

How to resolve degeneracy

An initial stage, to resolve degeneracy, we proceed by allocating a very small quantity (ϵ) which is close to zero to one or more (if needed) unoccupied cells so as to get $m+n-1$ number of occupied cells.

- The amount is denoted by a Greek letter ϵ (epsilon). This quantity would not affect the total cost as well as supply and demand values. In minimization transportation problem it is better to allocate (ϵ) to unoccupied cells that have lowest transportation costs.

To resolve degeneracy which occurs during optimality test, the quantity (ϵ) may be allocated to one or more cells which have become unoccupied recently to have $m+n-1$ number of occupied cells in the new solution.

Example: Goods have to be transported from sources S_1 , S_2 and S_3 to destinations D_1 , D_2 and D_3 . The transportation cost per unit, capacities of the sources and requirements of the destinations are given in the following table.

		Supply			SS
		D1	D2	D3	
Demand	S1	2	4	6	120
	S2	10	8	7	80
	S3	13	3	9	80
	DD	150	80	50	

Determine a transportation schedule so that cost is minimized.

Alternative Optimal Solutions

The existence of alternative optimal solution can be determined by an inspection of the opportunity costs (net contribution of empty cells), Δ_{ij} for the occupied cells.

- If an occupied cell in an optimal solution has opportunity cost of zero, an alternative optimal solution can be formed with another set of allocations without increasing the total transportation cost.

Example: Consider optimal solution of the given table.

	A	B	C	D	Supply	U_i
W	4 +4	8 76	8 +8	0 +16	76	$U_i=-16$
X	16 0	24 21	16 41	0 20	82	$U_i=0$
Y	8 72	16 5	24 +16	0 +8	77	$U_i=-8$
Demand	72	102	41	20	235	
V_j	$V_1=16$	$V_2=24$	$V_3=16$	$V_4=0$		

The net contributions of all unoccupied cells are positive except for the cell (X, A) which has a zero. This means if cell (X, A) is entered in to the basis, no change in the transportation cost would occur.

To determine this alternative solution, form a closed path for cell (X, A) as shown in table below.

	A	B	C	D	Supply	U_i
W	4 +4	8 76	8 +8	0 +16	76	$U_i=-16$
X	16 (+) 0	24 21(-)	16 41	0 20	82	$U_i=0$
Y	8 (-) 72	16 5(+)	24 +16	0 +8	77	$U_i=-8$
Demand	72	102	41	20	235	
V_j	$V_1=16$	$V_2=24$	$V_3=16$	$V_4=0$		

The maximum quantity which can be allocated to cell (X, A) is 21. After this change, the new solution is:

	A	B	C	D	Supply	U _i
W	4 +4	8 76	8 +8	0 +16	76	U _i =-16
X	16 21	24 0	16 41	0 20	82	U _i =0
Y	8 51	16 26	24 +16	0 +8	77	U _i =-8
Demand	72	102	41	20	235	
V _j	V ₁ =16	V ₂ =24	V ₃ =16	V ₄ =0		

Since all Δ_{ij} values are positive or zero, the solution given the table is optional with a minimum total transportation cost of 2424 which is the same as in the previous solution.

Maximization Transportation Problem

In general, transportation model is used for cost minimization problems. However, it is also used to solve problems in which the objective is to maximize total value or benefit. That is, instead of unit cost c_{ij} , the unit profit or payoff p_{ij} associated with each route, (i, j) is given. The objective is to maximize the total profit for which the profit matrix is given. The objective is to maximize the total profit for which the profit matrix is given.

$$\text{Maximize } Z = \sum_{i=1}^m \sum_{j=1}^n p_{ij} x_{ij}$$

For this we have to convert the maximization problem in to minimization problem (profit matrix in to loss matrix) by subtracting all the elements (profit) from the highest element (profit) in the given transportation table. This modified minimization problem can be solved in the usual manner.

- i.e the algorithm for solving modified problem is the same as that for the minimization problem.

Example: Solve the following transportation problem to maximize the profit.

		Destination				
		A	B	C	D	Supply
Source	1	15	51	42	33	23
	2	80	42	26	81	44
	3	90	40	66	60	33
	Demand	23	31	16	30	100

Total profit=90x70+5x50+9x90+11x40+15x10+1x70+7x90=Birr 4580

5.2. Assignment Problems

Definitions: Given n facilities and n jobs and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so as to optimize given measure of effectiveness.

The assignment problems are characterized by a need to pair items in one group with items in another group in a one-to-one matching. There are many situations where the assignment of people or machines and so on to different jobs.

The assignment is a problem because people possess varying abilities for performing different jobs and, therefore, the costs of performing the jobs by different people are different. Obviously, all persons could do a job in the same time or at the same cost then it would not matter who of them is assigned the job.

Thus, in an assignment problem, the question is how should the assignments be made in order that the total cost of performing the job is minimum (or the total value is maximized when pay-offs are given in terms of, say profits).

The assignment problem has many applications in allocations and scheduling: Examples,

- In assigning salesmen to different regions
- Vehicles and drivers to different routes
- Products to factories
- Jobs to machines
- Contracts to binders, etc

Mathematical formulation of the assignment problem

The assignment model can be expressed as follows:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n C_{ij} x_{ij}$$

Subject to :

$$\sum_{j=1}^n x_{ij} = 1, i = 1, 2, \dots, n \text{ (one job is assigned to the } i^{\text{th}} \text{ facility)}$$

$$\sum_{i=1}^m x_{ij} = 1, j = 1, 2, \dots, n \text{ (only one person should be assigned to the } j^{\text{th}} \text{ job)}$$

$$x_{ij} = \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ person is assigned to the } j^{\text{th}} \text{ job} \\ 0, & \text{if not} \end{cases}$$

Comparison with transportation and assignment problem

Transportation problem	Assignment problem
i. No. of sources and destinations need not be equal. Hence the cost matrix is not necessarily a square matrix.	No. of workers and jobs need not be equal. Hence cost matrix is a square matrix.
ii. X_{ij} : indicates the quantity to be transported from i^{th} origin to j^{th} destination and take any possible value.	X_{ij} : indicates the j^{th} job to be assigned to i^{th} person and it can take either 1 or 0.
iii. The row and column sum is equal to $\sum s_i$ and $\sum d_j$.	The row and column sum is exactly 1
iv. The problem is unbalanced if $\sum s_i \neq \sum d_j$.	The problem is unbalanced if the cost matrix is not a square matrix.

Although an assignment problem can be solved using simplex method as it is a special case of linear programming problems, it can be easily solved manually using a special-purpose algorithm known as the assignment method/ Hungarian Method.

Methods of solving the assignment problem (Hungarian Method)

Once we have the cost matrix of a given problem, the optimum assignments (solutions) using this method can be obtained through the following major steps:

Step 1: Subtract the smallest number in each cost matrix (table) row from every number in that row.

Step 2: In the new cost table resulted from step 1 subtract the smallest number in each column from every number in that column. As a result of this action, we get a table called reduced cost table in which there is at least one zero in each row and each column.

Step 3: In the reduced cost table draw the minimum number of vertical and horizontal straight lines necessarily to cover all zero elements in the table.

If the number of lines equals either the number of columns in the table then the solution is optimum and one may proceed to step 5. If the number of lines drawn is smaller than number of rows or columns, proceed to step 4.

Step 4: Subtract the smallest number not covered by a line from every uncovered number and add the same number to any number(s) lying at the intersection of any two lines. And then return to step 3 and continue until an optimum assignment is reached.

Step 5: Once it is confirmed that the optimum solution has been obtained as indicated in step 3, make the optimum assignments (indicated by the zero elements), as explained below;

- a. Beginning with row 1 of the reduced table, examine all the rows, until a row with exactly one unmarked zero is found. (A zero is marked if it is either bolded (0) or has a symbol x over it). Bold the unmarked zero to represent an assignment and cross out (x) all other unmarked zeros in that column. Repeat this process until, each row has no unmarked zeros or has at least two zeros.
- b. Beginning with column 1, examine all the columns, until a column with exactly one unmarked zero is found. Bold the unmarked zero to represent an assignment and cross out (x) all other unmarked zeros in that row. Repeat this process until each column has no unmarked zeros or has at least two zeros.
- c. If a complete assignment is found then stop, otherwise go to step 4.

Examples:

1. A machine tool company decides to make four subassemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in the following table.

		Contractors			
		I	II	III	IV
Subassemblies	1	15	13	14	17
	2	11	12	15	13
	3	13	12	10	11
	4	15	17	14	16

Assign the different subassemblies to contractors so as to minimize the total cost?

2. Using the following cost matrix , determine:
 - a. Optimal job assignment and
 - b. The cost of assignments

Machinist	Job				
	1	2	3	4	5
A	10	3	3	2	8
B	9	7	8	2	7
C	7	5	6	2	4
D	3	5	8	2	4
E	9	10	9	6	10

Ans: The optimal job assignment and cost associated with the assignment pattern is:

Workers	Job	Cost
A	2	3
B	4	2
C	5	4
D	1	3
E	3	9

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Unbalanced assignment problems

The Hungarian method of solving an assignment problem requires that the number of columns should be equal to the number of rows. When they are equal, the problem is balanced problem, and if not, it is unbalanced problem. In such situations, dummy column(s)/ row(s), whichever is smaller in number, are inserted with zeros as the cost elements. After introducing dummy columns/rows, the problem is solved in the usual manner.

Example: A company has 4 machines to do 3 jobs. Each job can be assigned to one machine. The cost of doing the job in different machine is given below. Determine the optimal job assignment which minimizes the total cost.

Jobs	Machine			
	A	B	C	D
1	18	24	28	32
2	8	13	17	18
3	10	15	19	22

Ans: The optimal job assignment and cost associated with the assignment pattern is:

Machine	Job	Cost
1	A	18
2	C	15
3	B	17

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UNIT 6: NETWORK ANALYSIS WITH CPM AND PERT

The techniques of operations research used for planning, scheduling and controlling large and complex projects are often referred as network analysis, network planning or scheduling techniques. All these techniques are based on the representation of the project as a network of activities.

A network is a graphical plan consisting of a certain configuration of arrow and nodes which indicates the logical sequence of various activities to be performed to achieve the objectives of a project. The two well known techniques are PERT and CPM.

PERT (Program Evaluation and Review Technique)

- It was developed in 1956-58 by research team to help in the planning and scheduling of the US navy's Polaris Nuclear Submarine Missile Project which involved thousands of activities. The technique has proved to be useful for all jobs or projects which have an element of uncertainty in the estimation of duration, as is the case with new type of projects the like of which have never been taken up before.

CPM (Critical Path Method)

- Was developed by E.I. DuPont Company along with Remington Rand Corporation. The aim behind its development was to provide a technique for control of maintenance of company's chemical plants. In course of time, use of CPM got extended to the field of cost and resource allocation.

6.1 Basic difference between PERT and CPM

PERT

- I. It assumes a probability distribution for the duration of each activity. Thus, completion time estimates for all the activities are needed.
- II. To perform PERT analysis on a project the emphasis is given on the completion of a task rather than the activities required to perform to reach to a particular event or task. Thus, it is also called an event-oriented technique.
- III. It is used for one-time projects involving activities of non-repetitive nature (i.e. activities which may never have been performed before) in which time estimates are uncertain.
- IV. It helps in identifying critical areas in a project so that necessary adjustment can be made to meet the scheduled completion date of the project.

CPM

- I. This technique was developed in connection with a construction and maintenance project in which duration of each activity was known with certainty.
- II. It is suitable for establishing a trade-off for optimum balancing between schedule time and cost of the project.

III. It is used for completion of projects involving activities of repetitive nature.

6.2. Network Components and Precedence Relationships

Drawing a network diagram is the first step in network (PERT/CPM) analysis. A network diagram represent project milestones, such as the start or the completion of activity (task) or activities, and occur at a particular instant of time at which some specific part of the project has been or is to be achieved. CPM/PERT network diagram consists of two major components: Events and Activities.